4) Given a function $f(x,y)$, we seek a solution $u(x,y)$ of:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y).$$

at points (x,y) in some spatial domain D.
This is known as Poisson's equation for given f, or the Laplace equation when $f \equiv 0$.
Examples of elliptic equations.

1.2. Order of PDE.

This is the order of the highest derivative with respect to any or all independent variables.

$$\frac{\partial u}{\partial t} = K.\frac{\partial^2 u}{\partial x^2}$$ is a second-order PDE $(k>0)$

$$\frac{\partial u}{\partial t} + V.\frac{\partial u}{\partial x} = k.\frac{\partial^2 u}{\partial x^2}$$ — " — $(k>0)$

$$\frac{\partial u}{\partial t} + V.\frac{\partial u}{\partial x} = 0$$ is a first-order PDE.

$$\frac{\partial^2 u}{\partial x.\partial y} = 0$$ is a second-order PDE.

1.3. Linear or nonlinear PDEs.

A PDE is linear if it doesn't involves products or nonlinear functions or either u or its derivatives.

Examples: $\frac{\partial u}{\partial t} = K.\frac{\partial^2 u}{\partial x^2}$ is linear when K is constant.

$$\frac{\partial u}{\partial t} + u.\frac{\partial u}{\partial x} = 0$$ is a nonlinear PDE.

$$e^u \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(e^u) = 0$$ is a nonlinear PDE in u.

$$\frac{\partial^2 u}{\partial t^2} - c^2.\frac{\partial^2 u}{\partial x^2} = \sin(u)$$ is a nonlinear PDE.

In general a PDE $\mathcal{L}\{u\} = f$ say, where $\mathcal{L}$ is a differential operator, is linear iff.

$$\mathcal{L}\{u+v\} = \mathcal{L}\{u\} + \mathcal{L}\{v\}.$$

$$\mathcal{L}\{cu\} = c\,\mathcal{L}\{u\}.$$

1.4 Homogeneous and nonhomogeneous PDEs.

A linear PDE is said to be <u>homogeneous</u> if every term depends on u <u>or</u> one of its derivatives, e.g.

$$a_1 \frac{\partial^2 u}{\partial t^2} + a_2 \frac{\partial u}{\partial t} + a_3 \frac{\partial^2 u}{\partial x^2} + a_4 \frac{\partial u}{\partial x} + a_5 u = 0$$

is homogeneous & linear if $a_1, \ldots, a_5$ are constants or functions on (x,t).

For example

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -u.$$

is homogeneous linear PDE of first order.

If a linear PDE is not homogeneous then it is nonhomogeneous or inhomogeneous, e.g.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 1.$$

1.5. Initial conditions and boundary conditions

For a second-order ODE.

$$\alpha \cdot \frac{d^2 u}{dx^2} + \beta \cdot \frac{du}{dx} + \gamma \cdot u = S(x), \quad \alpha, \beta, \& \gamma \text{ are constants}$$

on $0 < x < L$. To obtain a unique solution, we need to specify two conditions, as the general solution has two arbitrary constants.

If both conditions are specified at $x = 0$ (say), then this is an initial value problem (IVP), for example by specifying

$u(0) = u_0$, and $\dfrac{du}{dx}(0) = u_0'$ initial conditions

when u_0, u_0' are given constants.

Alternatively, we could specify boundary conditions on the problem

e.g. $u(0) = u_0$, and $u(L) = u_1$ say.

This is a boundary value problem (BVP).

Under some conditions this may lead to one, many or no solutions, depending on the ODE and the conditions.

For PDEs, most types of PDEs may involve a mixture of initial conditions and boundary conditions, depending on whether the independent variable is "spatial" or "temporal". For example, the heat equation.

$$\frac{\partial u}{\partial t} = K \cdot \frac{\partial^2 u}{\partial x^2}, \quad K > 0 \text{ and constant.}$$

has an initial condition at $t = 0$ (say) and boundary conditions at two different values of x.

A well-posed PDE has all of the following:

1) a solution to exist ("existence")
2) no more than one solution ("uniqueness").
3) that the solution does not greatly depend on the accuracy of the initial & boundary conditions ("stability").

1.6. Some simple PDEs and their solutions.

Using methods from ODEs:

Example

1) Solve $\frac{\partial u}{\partial x} = 0$ for the most general solution $u(x,y)$.

Here $\frac{du}{dx} = 0$ is an ODE with general solution $u(x) = c$ for any constant c, so if y is correspondingly kept constant.

$\frac{\partial u}{\partial x} = 0$ has general solution $u(x,y) = c(y)$ with c an arbitrary function of y.

2). Solve $\frac{\partial^2 u}{\partial x \partial y} = 0$ for $u(x,y)$.

Let $v = \frac{\partial u}{\partial y} \Rightarrow \frac{\partial v}{\partial x} = 0$ so

$$\frac{\partial u}{\partial y} = v = F(y) \text{ for } \underline{\text{any}} \text{ function } F(y).$$

then

$$u(x,y) = \int F(y)\,dy + G(x) \text{ for any functions } F(y) \text{ \& } G(x).$$

Equivalently

$$u(x,y) = H(y) + G(x) \text{ for any functions } G(x) \text{ \& } H(y)$$

which are differentiable

3) Find the general solution $u(x,y)$ of the PDE

$$\frac{\partial u}{\partial x} + u = 1 \quad \text{for all } (x,y). \qquad (*)$$

Treat y as a parameter and solve.

$$\frac{du}{dx} + u = 1$$

which has the solution $u(x) = 1 + Ae^{-x}$, for any constant A.

Hence the PDE $(*)$ has the general solution

$$u(x,y) = 1 + A(y)e^{-x}$$

for any function $A(y)$.