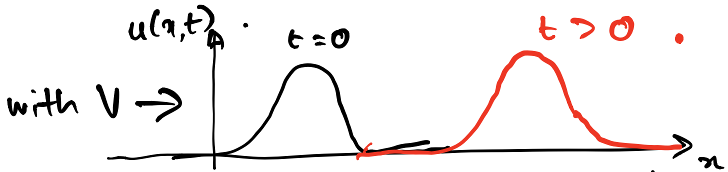
2. First-order PDEs.

2.1 Motion with constant speed.



What happens for $t > 0$ if there no diffusion of material?
Here $u(x,t)$, can be written mathematically as:

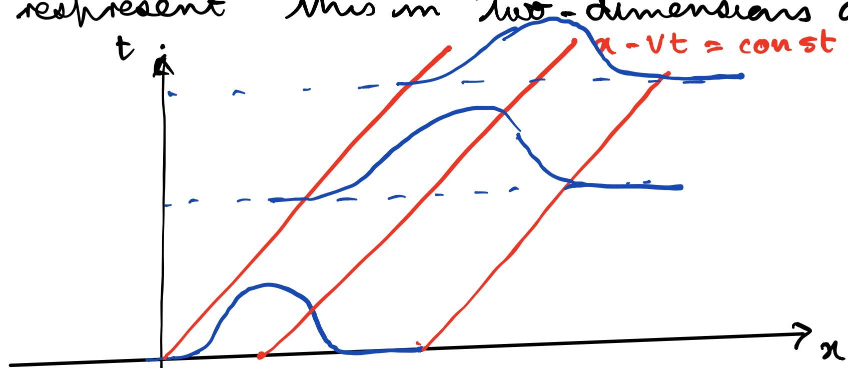
$$u(x,t) = F(x - Vt)$$

where $F(x)$ is a given function. Such that at $t=0$

$$u(x,0) = F(x).$$

i.e. $F(x)$ represents the initial shape of u.

Can represent this in two-dimensions as:



Characteristic diagram.

Along each of the parallel lines $\frac{dx}{dt} = V$ we have characteristics (or characteristic curves), where the value of u is a constant.

But what PDE is satisfied?

Well, note that $u(x,t) = F(x - Vt)$ then.

$$\frac{\partial u}{\partial t}(x,t) = F'(x - Vt) \cdot \frac{d}{dt}(x - Vt) = -V \cdot F'(x - Vt).$$

$$\frac{\partial u}{\partial x}(x,t) = F'(x-Vt).\frac{d}{dx}(x-Vt) = F'(x-Vt).$$

Hence we note that.

$$\frac{\partial u}{\partial t} + V.\frac{\partial u}{\partial x} = -V.F'(x-Vt) + VF'(x-Vt) = 0$$

for all (x,t). Therefore the solution $F(x-Vt)$ corresponds to solving the advection equation

$$\frac{\partial u}{\partial t} + V.\frac{\partial u}{\partial x} = 0 \qquad \text{for } -\infty < x < \infty \text{ and } t > 0.$$

with initial condition $u(x,0) = F(x)$ for some given function and V constant. This works for $V > 0$ (plotted above) or $V < 0$ or $V = 0$.

2.2 A more general PDE.

Consider

$$\frac{\partial u}{\partial t} + V(x,t).\frac{\partial u}{\partial x} = 0$$

not really the advection equation!.

where V depends on (x,t). How might we solve this?.

Consider some curves $x(t)$ on which u is constant. On these curves have.

$$\frac{d}{dt}u(x(t),t) = 0.$$

where u remains constant. Using the chain rule

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial x}(x(t),t).\frac{dx}{dt} + \frac{\partial u}{\partial t}(x(t),t).$$

$$= \frac{\partial u}{\partial t} + \frac{dx}{dt}.\frac{\partial u}{\partial x} \quad \text{on } (x(t),t).$$

$= 0$ when $x(t)$ is a characteristic.

But we know that $u(x,t)$ satisfies the PDE.

$$\frac{\partial u}{\partial t} + V(x,t) \cdot \frac{\partial u}{\partial x} = 0 .$$

and u is constant on curves $x(t)$ when these curves have that

$$\frac{dx}{dt} = V(x,t).$$

For a given function $V(x,t)$ this is a first-order ODE for $x(t)$.

For example if V is constant, the characteristic curves have

$$\frac{dx}{dt} = V$$

so $x(t) = V \cdot t + c$ for any constant c.
Here c corresponds to where the curve or line commences at $t=0$, and can be any real number.

Example

Consider the PDE.

$$\frac{\partial u}{\partial t} - x \cdot \frac{\partial u}{\partial x} = 0 .$$

over $-\infty < x < \infty$, with $t > 0$ and $u(x,0)$ given.
Seek $x(t)$ such that

$$\frac{dx}{dt} = -x \quad (= V(x,t)) .$$

on these curves $u(x,t)$ is constant
Solving the first-order ODE note that it is in separable form and can be written as:

$$\frac{dx}{x} = -dt.$$

hence integrating gives.

$$\ln|x| = -t + k \quad \text{for any constant } k.$$

Take exponentials of both sides

$$|x| = e^{-t+k}.$$

or $\quad x = (\pm e^{k}) e^{-t}.\quad$ where k is any real number

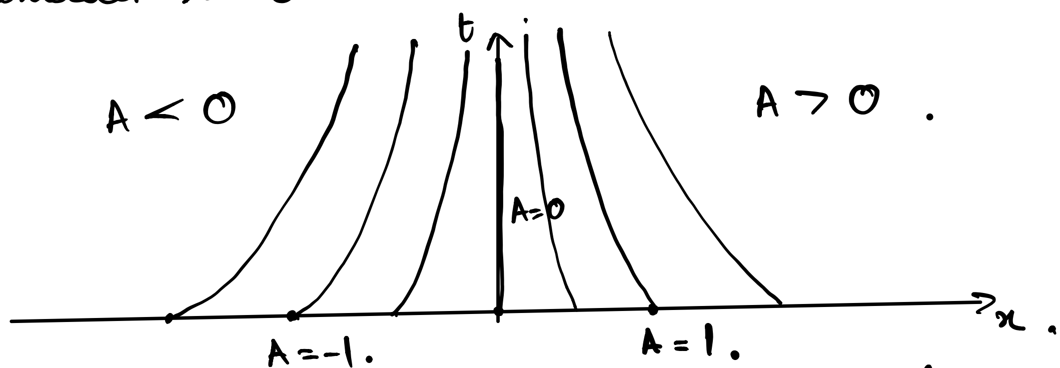
$= A e^{-t}$ for any real number $A \neq 0$.

Therefore all the characteristic curves have the form

$$x(t) = A e^{-t} \quad \text{for any real number } A.$$

and satisfy $\frac{dx}{dt} = -x$ (note also works for $A = 0$)

Consider these characteristic curves



and on these curves u is a constant, i.e.

$$\frac{du}{dt} = 0.$$

so u does not change as we follow the curves.

Note that all of these curves approach $x = 0$ as t increases for any value A.

For any given (x, t), we can find the corresponding value of A on the characteristic curve since

$$A = x e^{t} .$$

Then the value of $u(x,t)$ is obtained from the initial condition on the same curve.

Since u is constant on each curve with a given A, but differs for different A, it follows that

$$u = F(A) \quad \text{for any function } F.$$

Here F is determined from the initial condition and since $A = x e^{t}$ then

$$u = F(x e^{t}) \quad \text{for any function } F.$$

We can check this for $u(x,t) = F(x e^{t})$.

$$\frac{\partial u}{\partial t} = F'(x e^{t}) \frac{\partial}{\partial t}(x e^{t}) = x e^{t} F'(x e^{t})$$

$$\frac{\partial u}{\partial x} = F'(x e^{t}) \frac{\partial}{\partial x}(x e^{t}) = e^{t} F'(x e^{t}) .$$

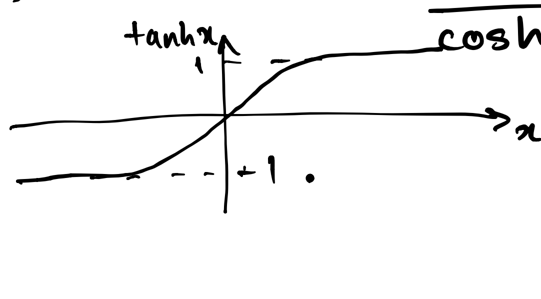
hence

$$\frac{\partial u}{\partial t} - x . \frac{\partial u}{\partial x} = x e^{t} F'(x e^{t}) - x e^{t} F'(x e^{t})$$

$$= 0 .$$

Consider the initial condition

$$u(x,0) = \tanh x = \frac{\sinh x}{\cosh x}$$



For the PDE above we want that.

$$u(x,0) = F(x e^{0}) = F(x) = \tanh x.$$

at $t=0$, for all x.

$\Rightarrow \quad F(x) = \tanh x$. Hence the solution of the PDE with $u(x,0) = \tanh x$ is

$$u(x,t) = \tanh(x e^{t})$$

for $-\infty < x < \infty$
and $t > 0$.