The solution becomes multivalued on the curve when the slope becomes vertical or where $\frac{\partial u}{\partial x} \to \infty$. Using the exact solution

$$u(x,t) = h\left(x - u(x,t)\,t\right)$$

then:

$$\frac{\partial u}{\partial x} = h'(x - u(x,t)t)\left[1 - \frac{\partial u}{\partial x}\cdot t\right]$$ by the Chain rule.

collecting $\frac{\partial u}{\partial x}$ terms:

$$\frac{\partial u}{\partial x}\left[1 + h'(x-ut).\,t\right] = h'(x-ut).$$

and so

$$\frac{\partial u}{\partial x} = \frac{h'(x-ut)}{1 + h'(x-ut).\,t}.$$

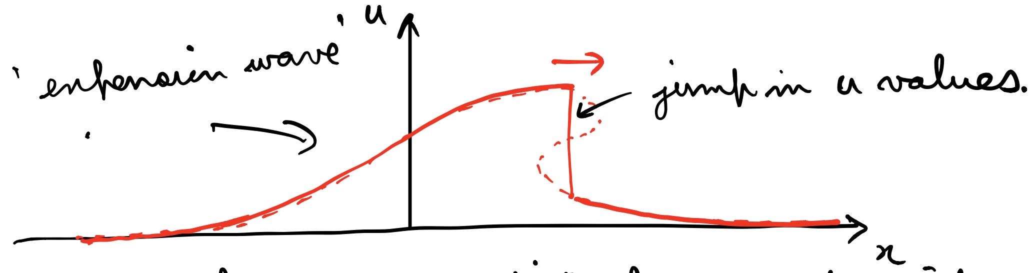
The denominator vanishes when

$$t_c = -\frac{1}{h'(x-ut)}$$ along the curve.

Therefore discontinuity occurs at the point of maximum absolute slope of u.

For an increasing function, where $h' > 0$, the denominator never vanishes and so there is no 'singularity', but for a decreasing function, $h' < 0$, there will be a time $t_c > 0$ where $\frac{\partial u}{\partial x} \to \infty$.

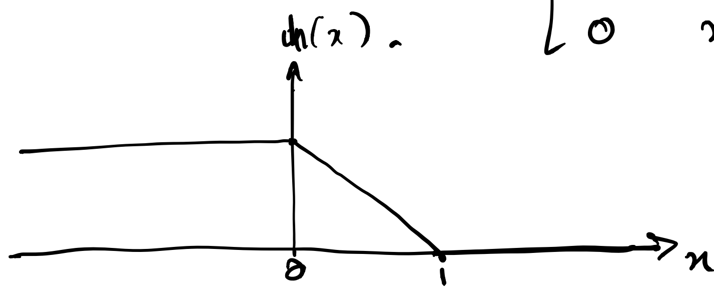
For $t > t_c$ we say that a 'shock' has formed, and the solution $u(x,t)$ is discontinuous at a point x. As t increases, the discontinuity moves and can increase in amplitude.



On the LHS above, the disturbance spreads out – which is called an expansion wave (or rarefaction in gas dynamics).

Example

Consider $u(x,0) = h(x) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 < x < 1. \\ 0 & x \ge 1. \end{cases}$



As above the solution is $u(x,t) = h(x-ut)$, which gives the following three cases.

(a) $(x-ut) \le 0$ then $h(\eta) = 1$ and $u = 1$ everywhere and hence $(x-ut) \le 0 \Rightarrow \boxed{x \le t.}$

(b) $(x-ut) \ge 1$ then $h(\eta) = 0$ and $u = 0$ everywhere and hence $x - ut \ge 1 \Rightarrow \boxed{x \ge 1}$

(c) $0 < (x-ut) < 1$ then $h(\eta) = 1 - \eta$ so that

$$u = h(x-ut) = 1 - (x-ut) \text{ for } 0 \le c \le 1.$$

For given (x,t) in this region

$$u(x,t) = \frac{1-x}{1-t}.$$

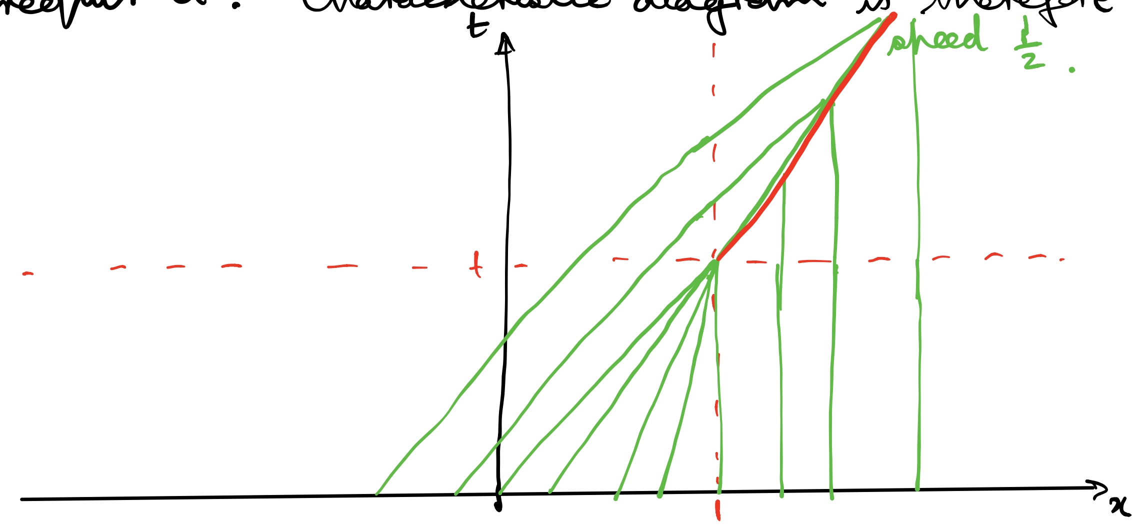
Hence $c = x - ut = x - \left(\frac{1-x}{1-t}\right) \cdot t$.

$$= \underline{x(1-t) - (1-x)t}$$

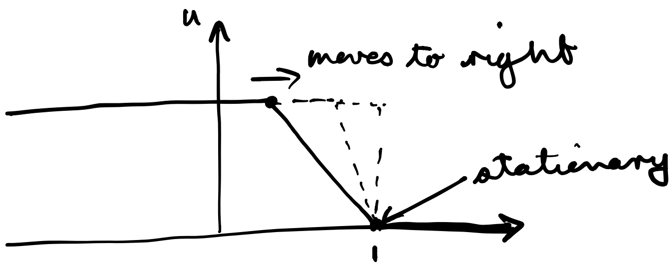
$1 - t$.

$$= \frac{x - t}{1 - t}.$$

which has $0 \le c \le 1$ since $0 \le x \le 1$ as required. Characteristic diagram is therefore :

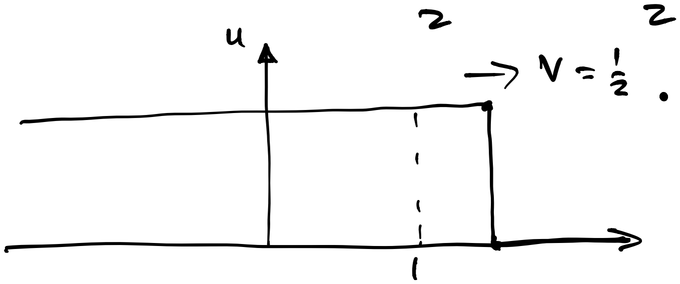


for $0 < t < 1$.



Beyond $t = 1$ there is a discontinuity in solution of amplitude 1 which moves with the average value of u on either side, i.e.

$$V = \frac{1 + 0}{2} = \frac{1}{2}.$$



2.6 Nondimensionalization

All the equations we have considered so far have had simple coefficients, e.g.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -u$$

rather than involving physical constants such as $T(x,t)$ satisfying

$$\frac{\partial T}{\partial t} + V \cdot \frac{\partial T}{\partial x} = -k(T - T_0).$$

where for example V, k, T_0 are constant.
To simplify the latter we can introduce new dependent and independent variables, e.g. $X = \dfrac{x}{V/k}$ where x is in metres, and V is in ms^{-1} and k is s^{-1}.
Similarly

$$\tau = kt$$ where t is measured in s.

Also introduce a new dependent variable

$$\phi = \frac{T - T_0}{T_0}$$ where T and T_0 are measured in $°K$ (say).

Each of ϕ, X & τ are dimensionless variables which are independent on the particular system.
The equivalent PDE for $\phi(X, \tau)$ can be found by introducing the change of variables.

Hence :
$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x} \cdot \frac{\partial}{\partial X} + \frac{\partial \tau}{\partial x} \cdot \frac{\partial}{\partial \tau} .$$
$$\frac{\partial}{\partial t} = \frac{\partial X}{\partial t} \cdot \frac{\partial}{\partial X} + \frac{\partial \tau}{\partial t} \cdot \frac{\partial}{\partial \tau} .$$
From above have :
$$\frac{\partial X}{\partial x} = \frac{1}{v/k} = \frac{k}{v} \qquad \frac{\partial X}{\partial t} = 0 .$$
$$\frac{\partial \tau}{\partial x} = 0 \qquad \frac{\partial \tau}{\partial t} = k .$$
hence :
$$\frac{\partial}{\partial x} = \frac{k}{v} \cdot \frac{\partial}{\partial X} \quad \text{and} \quad \frac{\partial}{\partial t} = k \cdot \frac{\partial}{\partial \tau} .$$
Using $T(x, t) = T_0 \left[1 + \phi(X, \tau) \right]$ from above obtain .
$$\frac{\partial T}{\partial x} = \frac{k}{v} \cdot T_0 \cdot \frac{\partial \phi}{\partial X} .$$
and
$$\frac{\partial T}{\partial t} = k \cdot T_0 \cdot \frac{\partial \phi}{\partial \tau} .$$
Note that the LHS only involves dimensional quantities & RHS only involves dimensionless quantities .

The PDE therefore becomes .
$$k \cdot T_0 \cdot \frac{\partial \phi}{\partial \tau} + v \cdot \frac{k}{v} \cdot T_0 \cdot \frac{\partial \phi}{\partial X} = \underbrace{- k T_0 \cdot \phi}_{- k(T - T_0)} .$$
or in terms of (X, τ) .
$$\frac{\partial \phi}{\partial \tau} + \frac{\partial \phi}{\partial X} = - \phi .$$

The BCs & ICs for $T(x,t)$ also have to be converted to $\phi(X,\tau)$.
e.g. $t=0$ corresponds $\tau=0$ and can use

$$\phi = \frac{T - T_0}{T_0}$$ as earlier.