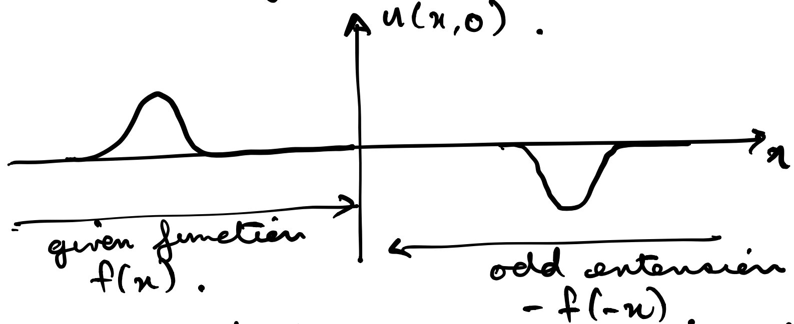
3.5 Wave equation with reflection at boundaries.

An alternative way of solving the problem is to solve over $-\infty < x < \infty$ but with an odd extension of the initial condition for $x > 0$.

i.e.



It turns out that this problem satisfies $u = 0$ at $x = 0$ and hence boundary condition. This is known as 'the method of images', and works for simple symmetries of the solution.

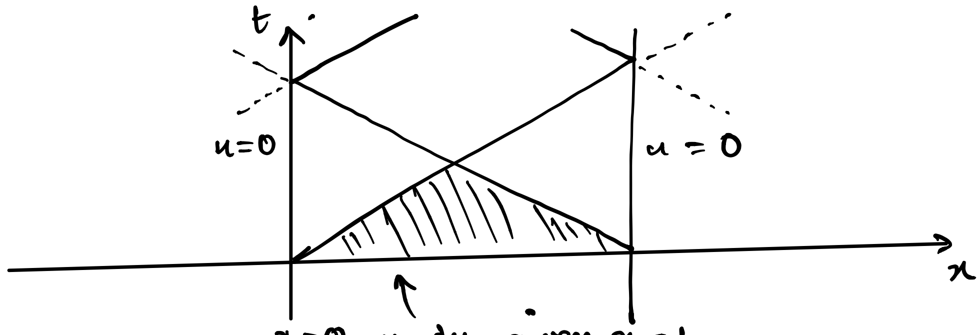
For $\frac{\partial u}{\partial x} = 0$ at $x = 0$ for $t > 0$, for example, we can use the even extension instead.

3.6. Wave equation on a finite domain.

Similarly using the general solution

$$u(x,t) = F(x+at) + G(x-at).$$

in a finite domain $0 < x < L$, say, can yield a solution, but the solution becomes very complicated due to the infinite number of reflections at the boundaries as the characteristics reach $x = 0$ and $x = L$.



This can be done in principle, but F, G become very complicated. Instead the problem is usually by 'separation of variables' as covered in MTH2032 [please revise these notes].

Note that the separation of variable solution can be written as

$$u(x,t) = F(x+at) + G(x-at).$$

3.7. Invariants

These are properties of a PDE, for example the energy of the wave equation in $-\infty < x < \infty$

$$I = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}a^2\left(\frac{\partial u}{\partial x}\right)^2 \right] dx$$

can be shown to be independent of t for any solution of

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}. \qquad (*)$$

To show that this is independent of time consider

$$\frac{dI}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}a^2\left(\frac{\partial u}{\partial x}\right)^2 \right] dx$$

when u satisfies (*) and assuming $u \to 0$ as $t \to \pm\infty$.

Liebniz's rule states that

$$\frac{d}{dt} \int_{a(t)}^{b(t)} u(x,t)\, dx$$

$$= \int_{a(t)}^{b(t)} \frac{\partial u}{\partial t} \cdot dx + u(b(t),t) \cdot b'(t) - u(a(t),t) a'(t).$$

Use Liebniz's rule

$$\frac{dI}{dt} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{a^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right] dx.$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial u}{\partial t} \cdot \frac{\partial^2 u}{\partial t^2} + a^2 \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial t\, \partial x} \right] dx \quad \text{by the Chain Rule,}$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial u}{\partial t} \cdot a^2 \cdot \frac{\partial^2 u}{\partial x^2} + a^2 \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial t\, \partial x} \right] dx \quad \text{since } u \text{ satisfies wave equation}$$

$$= \int_{-\infty}^{\infty} a^2 \left[\frac{\partial u}{\partial t} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial t\, \partial x} \right] dx.$$

$$= \int_{-\infty}^{\infty} a^2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} \right) dx.$$

$$\frac{dI}{dt} = a^2 \left[\frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} \right]_{-\infty}^{\infty}$$

$$= 0 \text{ for all } t > 0$$

assuming that $u \to 0$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$.

Hence $I(t)$ is independent of t and determined by the initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x)$$

for given functions $f(x)$ and $g(x)$. Therefore

$$I(0) = \int_{-\infty}^{\infty} \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + a^2\left(\frac{\partial u}{\partial x}\right)^2 \right]_{t=0} \cdot dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}\left[(g(x))^2 + a^2 [f'(x)]^2 \right] dx.$$

Hence $I(t)$ is equal to $I(0)$ for all t.

For example if $f(x) = 0$ and $g(x) = 0$ for all x (i.e. zero ICs) then

$$I(0) = 0.$$

and $I(t) = 0$ for all $t > 0$.

Since the integrand of $I(t)$ is non-negative and $I(t) = 0$ then this implies that

$$\frac{\partial u}{\partial t} = 0 \quad \text{and} \quad \frac{\partial u}{\partial x} = 0$$

everywhere for all $t > 0$ and hence that $u = 0$ everywhere for all $t > 0$. This enables us to prove (see Ex Sheet 2) that the solution of the wave equation is <u>unique</u>.

Invariants can also be used for

(a) checking solutions,

(b) validating the accuracy (or otherwise) of numerical solutions.