3.3 Waves in moving medium.

Seen previously that in a fluid with background density ρ_0 and moving with velocity V, then one-dimensional perturbations satisfy

$$\frac{\partial \rho'}{\partial t} + V \cdot \frac{\partial \rho'}{\partial x} + \rho_0 \frac{\partial u'}{\partial x} = 0 \qquad (1)$$

and

$$\rho_0 \left(\frac{\partial u'}{\partial t} + V \cdot \frac{\partial u'}{\partial x} \right) + a^2 \cdot \frac{\partial \rho'}{\partial x} = 0 \qquad (2)$$

where a is the speed of sound in the fluid.
Using (1) we can write

$$\rho_0 \frac{\partial u'}{\partial x} = - \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) \rho'$$

and (2) is

$$\rho_0 \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) u' = - a^2 \frac{\partial \rho'}{\partial x}$$

Take $\frac{\partial}{\partial x}$ of equation (1) we obtain.

$$\begin{aligned} \rho_0 \frac{\partial^2 u'}{\partial x^2} &= - \frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) \rho' \\ &= - \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) \frac{\partial \rho'}{\partial x}. \\ &= - \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) \left(- \frac{\rho_0}{a^2} \right) \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) u' \end{aligned}$$

hence we have that

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right)^2 u' = a^2 \cdot \frac{\partial^2 u'}{\partial x^2}. \qquad (3)$$

Note that if $V = 0$ we obtain the wave equation.
Can write the above equation (3) as:

$$\frac{\partial^2 u'}{\partial t^2} + 2V \cdot \frac{\partial^2 u'}{\partial x \partial t} + V^2 \frac{\partial^2 u'}{\partial x^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}.$$

or equivalently

$$\frac{\partial^2 u'}{\partial t^2} + 2V \cdot \frac{\partial^2 u'}{\partial x \partial t} = (a^2 - V^2) \cdot \frac{\partial^2 u'}{\partial x^2} .$$

and typically $a^2 >> V^2$, hence $a^2 - V^2 > 0$.

Then we have $A = 1$, $B = 2V$ and $C = V^2 - a^2$

and

$$B^2 - 4AC = 4V^2 - 4(V^2 - a^2) = 4a^2 > 0$$

and therefore the equation is hyperbolic.

One way to find the solution of (3) is to follow a similar process as for the wave equation by introducing

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) \phi = a \frac{\partial u'}{\partial x} .$$

in terms of the new variable ϕ we must have that

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) u' = a \cdot \frac{\partial \phi}{\partial x} .$$

Adding these two equations we obtain:

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) (u' + \phi) = a \frac{\partial}{\partial x} (u' + \phi)$$

or

$$\left(\frac{\partial}{\partial t} + (V - a) \frac{\partial}{\partial x} \right) (u' + \phi) = 0$$

Subtracting the two equations we obtain:

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) (u' - \phi) = a \frac{\partial}{\partial x} (\phi - u')$$

or

$$\left(\frac{\partial}{\partial t} + (V + a) \frac{\partial}{\partial x} \right) (u' - \phi) = 0 .$$

Hence the characteristic curves are

$$\frac{dx}{dt} = V - a \quad \text{and} \quad \frac{dx}{dt} = V + a$$

and therefore the general solution is

$$u(x,t) = F(x - Vt + at) + G(x - Vt - at)$$

where F, G are arbitrary functions.

As for the wave equation, this has the form

$$u(x,t) = F(x - \alpha_1 t) + G(x - \alpha_2 t)$$

where α_1, α_2 are constants.

3.4 Other hyperbolic "wave-like" PDEs of 2nd order.

Consider

$$A \cdot \frac{\partial^2 u}{\partial t^2} + B \cdot \frac{\partial^2 u}{\partial x \partial t} + C \frac{\partial^2 u}{\partial x^2} = 0$$

where A, B, C are constants and $B^2 > 4AC$ and therefore the system is hyperbolic. Assume that this has a solution of the form

$$u(x,t) = F(x - \alpha t)$$

where F is arbitrary and α is constant which is to be determined.

Then

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= F''(x - \alpha t) \cdot \left(\frac{\partial}{\partial t}(x - \alpha t)\right)^2 \\ &= (-\alpha)^2 F''(x - \alpha t) \\ &= \alpha^2 F''(x - \alpha t).\end{aligned}$$

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x}\left[F'(x-\alpha t) . \frac{\partial}{\partial t}(x-\alpha t)\right].$$
$$= -\alpha . F''(x-\alpha t).$$
and finally
$$\frac{\partial^2 u}{\partial x^2} = F''(x-\alpha t) . \left(\frac{\partial}{\partial x}(x-\alpha t)\right)^2$$
$$= F''(x-\alpha t).$$
The PDE above will be satisfied for a solution of this form when.
$$A\alpha^2 F''(x-\alpha t) + B.(-\alpha)F''(x-\alpha t) + CF''(x-\alpha t) = 0$$
<u>for all</u> (x,t). Hence this requires that
$$[A\alpha^2 - B\alpha + C] F''(x-\alpha t) = 0$$
in order for this to be true for any function F. Therefore we must $A\alpha^2 - B\alpha + C = 0$ or
$$\alpha = \frac{B \pm (B^2 - 4AC)^{1/2}}{2A}.$$
These are real and distinct when $B^2 - 4AC > 0$, which is as assumed here.
If we write the two solutions as α_+, α_- say, then
$$u(x,t) = F(x-\alpha_+ t) + G(x-\alpha_- t)$$
where F & G are arbitrary functions. This is then the <u>general solution</u> for this PDE.

__Example__ The wave equation has $A = 1$, $B = 0$ and $C = -a^2$ ($a > 0$ is the wave speed).

Hence:

$$\alpha_{\pm} = \frac{0 \pm (0^2 - 4.1.(-a^2))^{1/2}}{2}.$$

$$= \pm \frac{(4a^2)^{1/2}}{2}.$$

$$= \pm a.$$

which is as obtained previously.

__Exercise__: consider waves in a moving medium where

$$A = 1, \quad B = 2V, \quad C = V^2 - a^2.$$

__Example__ Consider

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x \partial t} - 2.\frac{\partial^2 u}{\partial x^2} = 0 \qquad (*)$$

for $-\infty < x < \infty$ and $t > 0$

where $u(x,0) = e^{-x^2}$ and $\frac{\partial u}{\partial t}(x,0) = 0$

for $-\infty < x < \infty$.

Seek a solution

$$u(x,t) = F(x - \alpha t).$$

then

$$\frac{\partial^2 u}{\partial x^2} = F''(x - \alpha t), \quad \frac{\partial^2 u}{\partial x \partial t} = -\alpha . F''(x - \alpha t)$$

and $$\frac{\partial^2 u}{\partial t^2} = \alpha^2 . F''(x - \alpha t).$$

Hence for this solution to satisfy (*) we must have:

$$\alpha^2 - \alpha - 2 = 0$$

or

$$(\alpha - 2)(\alpha + 1) = 0$$

which has the solutions $\alpha = 2$ and $\alpha = -1$.
Then the general solution of (*) is

$$u(x,t) = F(x - 2t) + G(x + t).$$

and these must satisfy

$$u(x,0) = F(x) + G(x) = e^{-x^2}. \quad \text{for all } x$$

and

$$\frac{\partial u}{\partial t}(x,0) = -2F'(x) + G'(x) = 0 \quad \text{for all } x.$$

Hence have the two equations.

$$\left.\begin{aligned} F(x) + G(x) &= e^{-x^2}. \\ -2F'(x) + G'(x) &= 0 \end{aligned}\right\} \text{ for all } x.$$

Integrating the second equation obtain

$$G(x) = 2F(x) + c.$$

for some constant c. Substituting into the first equation gives

$$F(x) + 2F(x) + c = 3F(x) + c = e^{-x^2}.$$

hence

$$F(x) = \tfrac{1}{3}\left(e^{-x^2} - c\right).$$

Then

$$\begin{aligned} G(x) &= 2F(x) + c \\ &= \tfrac{2}{3}\left(e^{-x^2} - c\right) + c \end{aligned}$$

$$= \frac{2}{3} e^{-x^2} + \frac{1}{3} c.$$

The solution to our problem for all x and $t > 0$ is therefore

$$u(x,t) = \frac{1}{3}\left(e^{-(x-2t)^2} - c\right) + \frac{2}{3}\left(e^{-(x+t)^2}\right) + \frac{1}{3} c.$$

$$= \frac{1}{3} e^{-(x-2t)^2} + \frac{2}{3} e^{-(x+t)^2}.$$