The initial conditions give that $F(x)$ & $G(x)$ satisfy

$$F(x) + G(x) = e^{-x^2}$$

and

$$F'(x) - G'(x) = 0 .$$

Integrating the second equation gives .

$$F(x) = G(x) + c \quad \text{where } c \text{ is arbitrary} .$$

and so

$$2G(x) + c = e^{-x^2} .$$

or

$$G(x) = \tfrac{1}{2}\left(e^{-x^2} - c\right).$$

Then

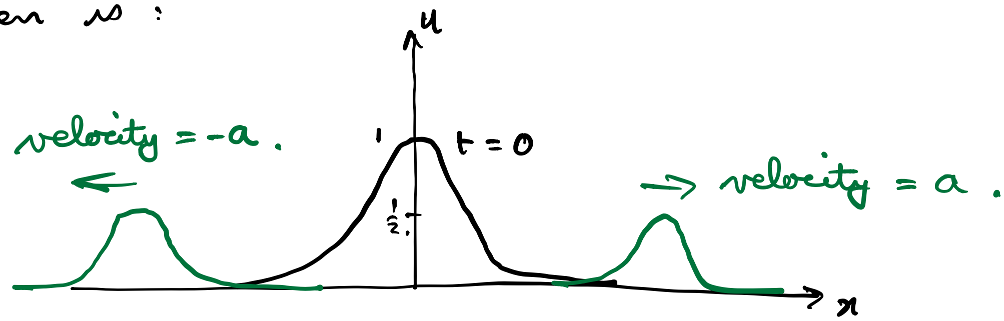
$$\begin{aligned} F(x) &= G(x) + c . \\ &= \tfrac{1}{2}\left(e^{-x^2} - c\right) + c . \\ &= \tfrac{1}{2}\left(e^{-x^2} + c\right). \end{aligned}$$

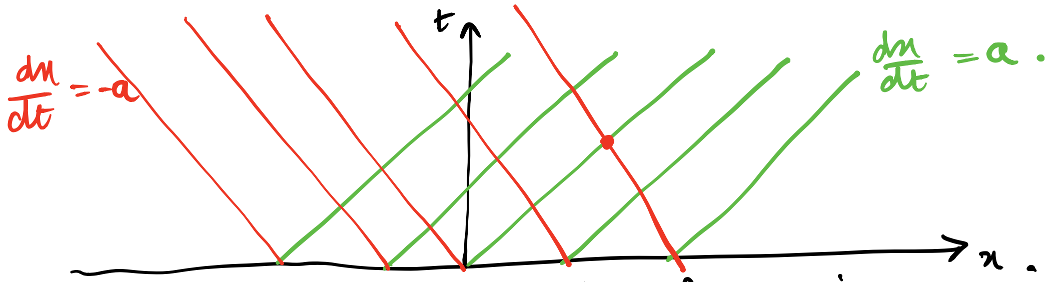
The general solution is therefore

$$\begin{aligned} u(x,t) &= F(x - at) + G(x + at) . \\ &= \tfrac{1}{2}\left(e^{-(x-at)^2} + c\right) + \tfrac{1}{2}\left(e^{-(x+at)^2} - c\right). \\ &= \tfrac{1}{2}\left(e^{-(x-at)^2} + e^{-(x+at)^2}\right) \end{aligned}$$

for all $-\infty < x < \infty$ and $t > 0$:

Solution is :

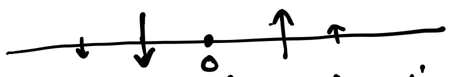




Another example is for the initial conditions

$$u(x,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 2ax e^{-x^2}.$$

$$\text{for } -\infty < x < \infty.$$



so that the medium is disturbed at $t=0$ using the second condition, rather than the first. General solution is still

$$u(x,t) = F(x-at) + G(x+at)$$

where F, G are unknown functions, which satisfy

$$u(x,0) = F(x) + G(x) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = -aF'(x) + aG'(x) = 2ax\, e^{-x^2}.$$

for $-\infty < x < \infty$. Substituting the first equation $(F(x) = -G(x))$ into the second gives

$$+aG'(x) + aG'(x) = 2ax\, e^{-x^2}.$$

or

$$G'(x) = x e^{-x^2} = \frac{d}{dx}\left\{-\tfrac{1}{2} e^{-x^2}\right\}.$$

Hence we obtain

$$G(x) = -\tfrac{1}{2} e^{-x^2} + c \quad \text{where } c \text{ is arbitrary}$$

and

$$F(x) = +\tfrac{1}{2} e^{-x^2} - c.$$

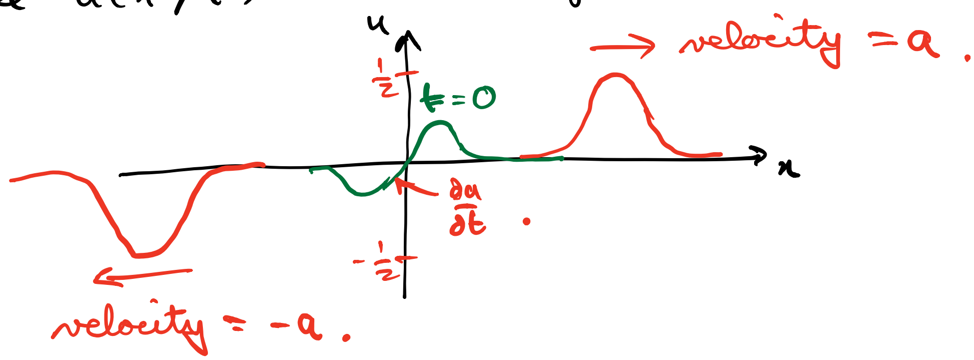
Hence the full solution is:

$$u(x,t) = F(x-at) + G(x+at).$$

$$= \frac{1}{2}\left(e^{-(x-at)^2}\right) - \cancel{c} - \frac{1}{2}e^{-(x+at)^2} + \cancel{c}.$$

$$= \frac{1}{2}\left(e^{-(x-at)^2} - e^{-(x+at)^2}\right).$$

Hence $u(x,t)$ has the form:



Since the problem is linear, then the solution of the wave equations with initial conditions

$$u(x,0) = e^{-x^2} \quad \text{and} \quad \frac{du}{dt} = 2ax\,e^{-x^2}$$

for all $-\infty < x < \infty$, is:

$$u(x,t) = \frac{1}{2}\left(e^{-(x-at)^2} + e^{-(x+at)^2}\right)$$
$$+ \frac{1}{2}\left(e^{-(x-at)^2} - e^{-(x+at)^2}\right).$$
$$= e^{-(x-at)^2}$$

Which only has a Gaussian wave propagating to the right, and no leftward propagating wave.

For general initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

over $-\infty < x < \infty$.

Given the general solution

$$u(x,t) = F(x-at) + G(x+at)$$

we must have that

$$u(x,0) = F(x) + G(x) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = -a F'(x) + a G'(x) = g(x)$$

for $-\infty < x < \infty$. Integrating the second equation gives

$$G(x) = \frac{1}{a} \int_0^x g(x')\,dx' + F(x) + c$$

where c is arbitrary.

Hence

$$2F(x) + \frac{1}{a}\int_0^x g(x')\,dx' + c = f(x).$$

or

$$F(x) = \frac{1}{2}\left(f(x) - \frac{1}{a}\int_0^x g(x')\,dx' - c \right).$$

then

$$G(x) = \frac{1}{2}\left(f(x) + \frac{1}{a}\int_0^x g(x')\,dx' + c \right).$$

and the general solution is

$$\begin{aligned}
u(x,t) &= F(x-at) + G(x+at) \\
&= \frac{1}{2}\left(f(x-at) - \frac{1}{a}\int_0^{x-at} g(x')\,dx' - c \right). \\
&\quad + \frac{1}{2}\left(f(x+at) + \frac{1}{a}\int_0^{x+at} g(x')\,dx' + c \right). \\
&= \frac{1}{2}\left(f(x-at) + f(x+at) \right) \\
&\quad + \frac{1}{2a}\int_{x-at}^{x+at} g(x')\,dx'.
\end{aligned}$$

for all $-\infty < x < \infty$ and $t > 0$.
The solution is known as <u>D'alembert's solution</u>

<u>3.3 Waves in a moving medium.</u>

Consider sound waves travelling through a compressible medium such as air or water. If the medium is moving with speed V say, where $|V| << a$ and a is the speed of sound, and we assume that the disturbance is "small", then the density can be written as

$$\rho = \underbrace{\rho_0}_{\text{mean density}} + \underbrace{\rho'}_{\text{small perturbation to the density}}.$$

Then conservation of mass gives the density and velocity satisfy

$$\frac{\partial \rho'}{\partial t} + V . \frac{\partial \rho'}{\partial x} + \rho_0 . \frac{\partial u'}{\partial x} = 0$$

where $u = V + u'$ and u' is a small perturbation to the velocity. Conservation of momentum gives that they also satisfy

$$\rho_0 \left\{ \frac{\partial u'}{\partial t} + V . \frac{\partial u'}{\partial x} \right\} + a^2 . \frac{\partial \rho'}{\partial x} = 0 .$$

Eliminating ρ' can show:

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right)^2 u' = a^2 . \frac{\partial^2 u'}{\partial x^2} .$$