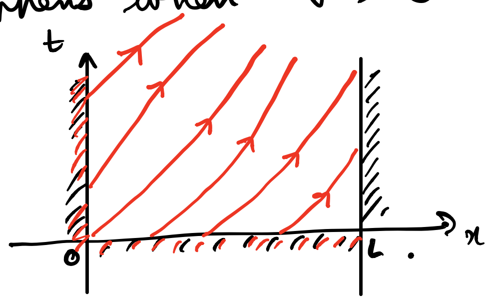
2.4 The advection equation in a finite domain.

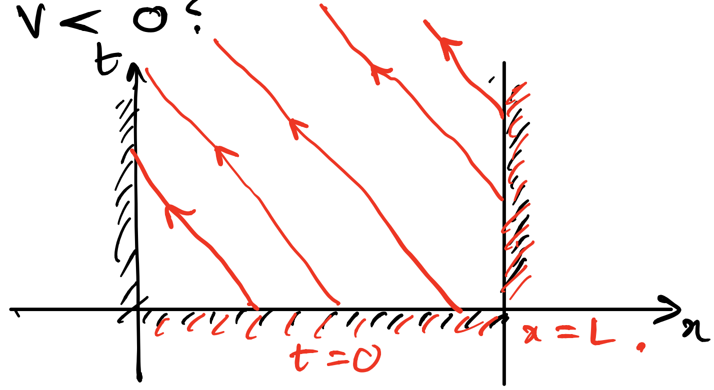
Lets go back to the PDE.

$$\frac{\partial u}{\partial t} + V(x,t) \cdot \frac{\partial u}{\partial x} = 0 .$$

in a finite spatial domain $0 < x < L$ for $t > 0$.
What conditions do we need to specify on u on the boundaries of the domain?
What happens when $V > 0$ everywhere.



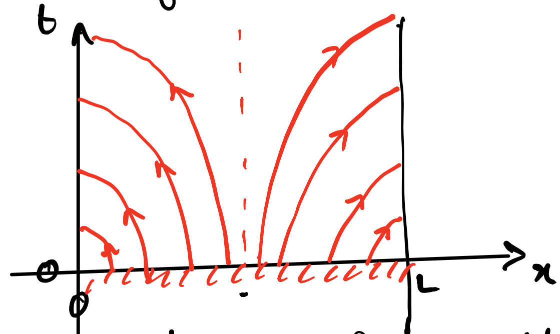
We can apply the IC on $t = 0$, but where do we apply the BCs?
All characteristics have $\frac{dx}{dt} = V > 0$ so have they start from either $t = 0$ or $x = 0$ here, and determine the solution $u(x,t)$ everywhere in the domain including at $x = L$.
Hence only need to specify conditions on $x = 0$ & $t = 0$.
What if $V < 0$?



$$\frac{dx}{dt} = V < 0$$

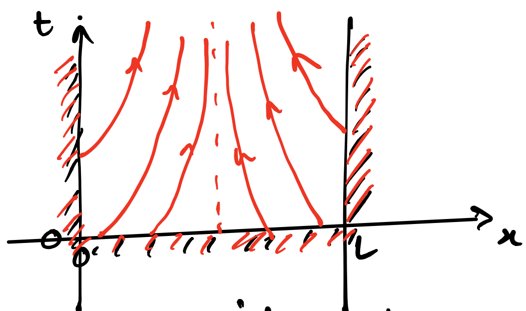
In this case we specify the conditions on $x = L$ and $t = 0$ to uniquely determine the solution throughout.

What if V changes sign in the domain, with $V < 0$ on the left and $V > 0$ on the right?



Here all characteristics leave the domain at $x = 0$ or $x = L$ so we only need to specify u on $t = 0$.

What if $V > 0$ on the left and $V < 0$ on the right?



Here we must specify the conditions on u on $t = 0$ and both $x = 0$ & $x = L$.

To have a well-posed problem, we need to be aware about where characteristics leave from on the boundaries of the domain.

With first-order PDEs this leads to the concept of 'Cauchy initial data' or appropriate initial curves on which solution must be specified to determine a <u>unique</u>

solution everywhere in the required domain.

2.5 Nonlinear first-order PDEs.

Characteristics do not always behave as simply as in the previous examples, where we can track a point (x,t) back to a value of $t=0$ and obtain a unique solution. Consider for example

$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0 \quad \text{over } -\infty < x < \infty \text{ and } t > 0.$$

which is like the advection, but with $V(x,t)$ replaced by $u(x,t)$. This is a simple model for more complicated behaviour e.g. for surface waves or gas dynamics, although in the cases usually have more than one dependent variable,

On the characteristics $x(t)$ such that.

$$\frac{dx}{dt} = u$$

then the above PDE reduces to $\frac{du}{dt} = 0$, so u is constant on each characteristic, and

$$x(t) = ut + c \quad \text{for any constant } c.$$

The general $u = F(c)$ for any function F, then becomes

$$u(x,t) = F(x - u(x,t)t).$$

for any function F. This is an 'implicit equation for $u(x,t)$ at any (x,t) once F is known.

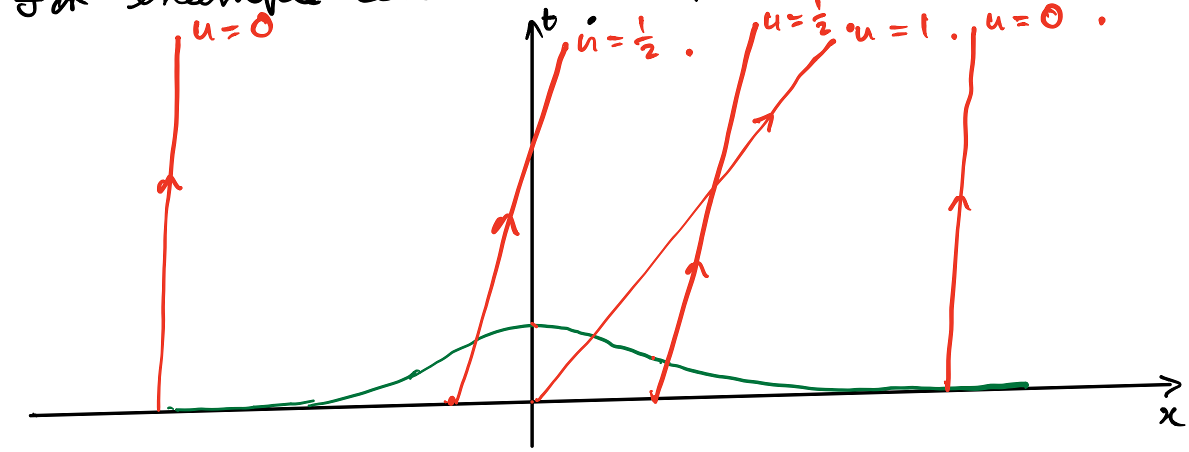
For a given initial condition $u(x,0) = h(x)$ say, where $h(x)$ is given then.

$$u(x,0) = F(x-0) = h(x) \text{ so } F(x) = h(x) \text{ for all } x$$

and

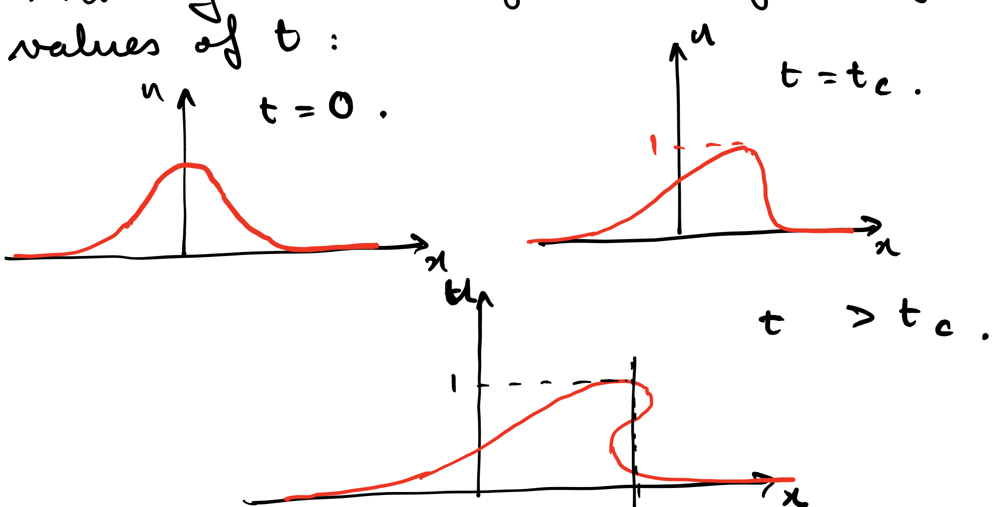
$$u(x,t) = h(x - u(x,t)t) \text{ where } h \text{ is given.}$$

For example consider $u(x,0) = h(x) = e^{-x^2}$.



and following the characteristics, say $u = 0, \frac{1}{2}, 1$, at $t = 0$ we can see that these can intersect for those initially having $x \geq 0$.

Plotting u as a function of x for several values of t:



Notice that this curve steepens as t increases and some t it has infinite at slope at a point. Then beyond this point it becomes multi-valued.