Example

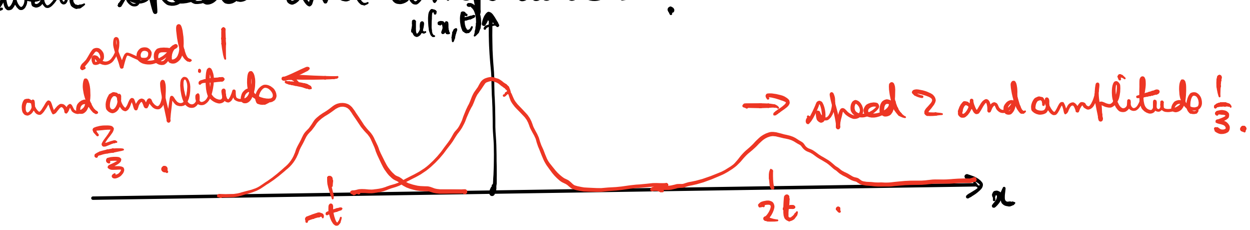
$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x \partial t} - 2 \cdot \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } -\infty < x < \infty \text{ and } t > 0$$

where $u(x,0) = e^{-x^2}$ and $\frac{\partial u}{\partial t}(x,0) = 0$ for all x.

General solution

$$u = \frac{1}{3} e^{-(x-2t)^2} + \frac{2}{3} e^{-(x+t)^2}.$$

Plotting this shows that the parts have different wave speeds and amplitudes.



An alternative way of finding the general solution of:

$$A \cdot \frac{\partial^2 u}{\partial t^2} + B \cdot \frac{\partial^2 u}{\partial x \partial t} + C \cdot \frac{\partial^2 u}{\partial x^2} = 0 \qquad (**).$$

is to follow the procedure for the wave equation, and introduce v such that

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x}$$

then it can be shown that satisfy $(**)$ then.

$$A \cdot \frac{\partial u}{\partial t} + B \cdot \frac{\partial u}{\partial x} + C \cdot \frac{\partial v}{\partial x} = 0.$$

The linear PDE can therefore be written as,

$$\frac{\partial \underline{u}}{\partial t} = \hat{A} \cdot \frac{\partial \underline{u}}{\partial x}$$

where $\underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix}$ and $\tilde{A} = \begin{bmatrix} -\frac{B}{A} & -\frac{C}{A}. \\ 1 & 0 \end{bmatrix}$.

If the eigenvalues λ_1 & λ_2 of $\tilde{A}$ are real and distinct, then they have corresponding eigenvectors $\underset{\sim}{v}^{(1)}$, $\underset{\sim}{v}^{(2)}$. Then introducing

$$\underset{\sim}{u} = P\underset{\sim}{u}' = [\underset{\sim}{v}^{(1)}\ \underset{\sim}{v}^{(2)}]\,\underset{\sim}{u}'$$

the equation for $\underset{\sim}{u}$ becomes

$$\frac{\partial \underset{\sim}{u}'}{\partial t} = P^{-1}AP\,\frac{\partial \underset{\sim}{u}'}{\partial x}.$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \frac{\partial \underset{\sim}{u}'}{\partial x} \qquad (***).$$

which is in diagonal form, and each component on $\underset{\sim}{u}'$ is a linear combination of u, v.

Each component of $(***)$ is linear advection equation with characteristic slope:

$$\frac{dx}{dt} = -\lambda_1 \quad \text{and} \quad \frac{dx}{dt} = -\lambda_2 \quad \text{respectively}.$$

For the wave equation we used $\eta = \frac{u+v}{2}$ and $\xi = \frac{u-v}{2}$ to derive.

$$\frac{\partial \eta}{\partial t} = a\frac{\partial \eta}{\partial x}, \qquad \frac{\partial \xi}{\partial t} = -a.\frac{\partial \xi}{\partial x}$$

so that $\lambda_1 = -a$ and $\lambda_2 = a$ here, and the combinations of $u, \underset{\sim}{v}$ for η, ξ are given by the eigenvectors of $\tilde{A}$ in this case.

3.5. The wave equation with reflection at boundaries

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}$$

in a domain $-\infty < x < 0$ with a boundary at $x = 0$.
We seek a solution for $t > 0$ with given initial conditions

$$\left.\begin{aligned} u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= g(x) \end{aligned}\right\} \text{ for } x \leq 0.$$

and a stated boundary condition at $x = 0$ for $t > 0$.
We know that the general solution of the wave equation is

$$u(x, t) = F(x + at) + G(x - at)$$

for any functions F, G.
Applying the initial conditions at $t = 0$ gives.

$$\left.\begin{aligned} u(x, 0) &= F(x) + G(x) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= aF'(x) - aG'(x) &= g(x) \end{aligned}\right\} \text{ for } x \leq 0.$$

and hence we find $F(x)$ and $G(x)$ for given f, g when $x \leq 0$.
Once $t > 0$, notice that $x + at > 0$ when $t > -\frac{x}{a}$ for some negative x. Hence $u(x, t)$ is undetermined when $x + at > 0$ and F has some positive argument.
If we specify a boundary condition at $x = 0$, e.g.

$$u(0, t) = 0 \quad \text{for all } t > 0$$

then from the general solution.

$$F(0+at) + G(0-at) = 0 \quad \text{for all } t > 0.$$

Let $\eta = at$ then this tells us that.

$$F(\eta) = -G(-\eta) \quad \text{for all } \eta > 0,$$

and hence the argument of F is known for all positive values.

Example

Consider when $u(x,0) = f(x)$ for any given f when $x \leq 0$ and $\frac{\partial u}{\partial t}(x,0) = 0$ for all $x \leq 0$, with $u(0,t) = 0$ for $t > 0$.

From earlier, we showed that.

$$F(x) = \tfrac{1}{2} f(x) \text{ and } G(x) = \tfrac{1}{2} f(x)$$

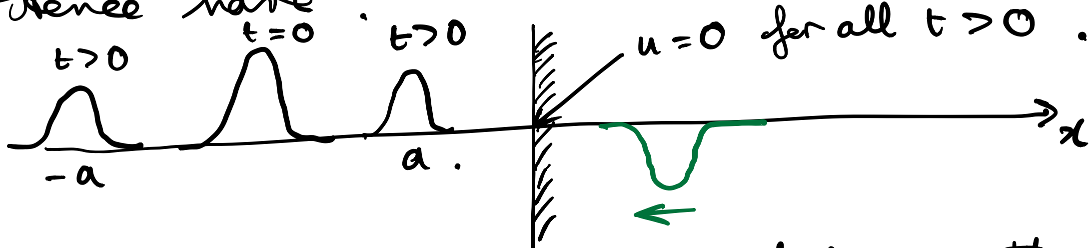
when $x \leq 0$.

and hence

$$u(x,t) = \tfrac{1}{2} f(x+at) + \tfrac{1}{2} f(x-at)$$

provided that $x + at = 0$.

Hence have.



Once $x + at > 0$ we also need to use that $u(0,t) = 0$ so that

$$u(0,t) = F(at) + G(-at) = 0 \quad \text{for } t > 0$$

and hence

$$F(\eta) = -G(-\eta) = -\tfrac{1}{2} f(-\eta) \text{ for } \eta > 0.$$

Hence

$$F(\eta) = \begin{cases} \frac{1}{2} f(\eta) & \text{for } \eta \leq 0 \\ -\frac{1}{2} f(-\eta) & \text{for } \eta > 0 . \end{cases}$$

with $G(\eta) = \frac{1}{2} f(\eta)$ for all $\eta < 0$. This leads to a second component of the solution which travels to left (i.e. F part) which becomes important once its argument is positive. This corresponds to the reflection of the G part of the solution in the boundary $x = 0$.