3. The wave equation

3.1 Classification of second-order PDEs

Consider a general PDE of second-order for $u(x_1, x_2)$ which is linear, then:

$$A \cdot \frac{\partial^2 u}{\partial x_1^2} + B \cdot \frac{\partial^2 u}{\partial x_1 \partial x_2} + C \cdot \frac{\partial^2 u}{\partial x_2^2} + a \frac{\partial u}{\partial x_1} + b \cdot \frac{\partial u}{\partial x_2} + cu = d(x_1, x_2).$$

where A, B, C, a, b, c are given functions of (x_1, x_2) or constants.

Case 1 $\Delta = B^2 - 4AC > 0$ then the PDE is hyperbolic. One example is the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2} \qquad a \text{ is wave speed.}$$

where $A = 1$, $B = 0$, $C = -a^2$ so

$$\Delta = 0^2 - 4 \times (1) \times (-a^2) = 4a^2 > 0.$$

Case 2: $\Delta = 0$ then the PDE is parabolic.

One example of this is the heat equation.

$$\frac{\partial u}{\partial t} = K \cdot \frac{\partial^2 u}{\partial x^2}, \qquad K > 0$$

$A = -K$, $B = C = 0$ then.

$$\Delta = 0 - 4 \times (0) \times (-K) = 0$$

Case 3: $\Delta < 0$ then the PDE is elliptic, and one example is the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1$, $B = 0$, $C = 1$ so $\Delta = 0 - 4 \times 1 \times 1 = -4 < 0$

3.2 The wave equation

This when $u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}$$ where $a > 0$ is "wave speed"

which is a typical example of a hyperbolic PDE.

In a finite spatial domain, say $0 < x < L$, we need to specify two BCs, e.g.

$u(0,t)$ and $u(L,t)$ are specified for $t > 0$.

as well as two initial conditions

$u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$ for $0 < x < L$ and $t = 0$

This was solved in MTH2032 using separation of variables, here we commence with the solution in the infinite domain $-\infty < x < \infty$.

3.3 Solution on the infinite line.

In the case where $-\infty < x < \infty$ we do not specify precise conditions at $x = \pm\infty$, except say "nothing happens there". We do however specify the initial conditions, e.g.

$$u(x,0) = f(x) \qquad -\infty < x < \infty.$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

and f and g are given functions.

To solve

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}.$$

for $u(x,t)$ we introduce an new dependent variable $v(x,t)$ such that

$$\frac{\partial u}{\partial t} = a \cdot \frac{\partial v}{\partial x} \qquad (1).$$

and

$$\frac{\partial v}{\partial t} = a \cdot \frac{\partial u}{\partial x} \qquad (2).$$

Substituting the second equation into $\frac{\partial}{\partial t}(1)$ then

$$\frac{\partial^2 u}{\partial t^2} = a \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = a \frac{\partial}{\partial x}\left(a \cdot \frac{\partial u}{\partial x}\right)$$

$$= a^2 \cdot \frac{\partial^2 u}{\partial x^2}. \qquad (3)$$

so that u (and v) satisfy the wave equation.
So solving the coupled equations (1) & (2) together is equivalent to solving the second-order PDE (3).

Introducing $\eta = \frac{1}{2}(u+v)$ then adding (1) & (2) obtain:

$$\frac{\partial}{\partial t}(u+v) = a \frac{\partial}{\partial x}(u+v).$$

and dividing by 2 gives

$$\frac{\partial \eta}{\partial t} = a \cdot \frac{\partial \eta}{\partial x} \quad \text{or} \quad \frac{\partial \eta}{\partial t} - a \cdot \frac{\partial \eta}{\partial x} = 0$$

which is the advection equation with "speed" $-a$. The general solution is

$$\eta = F(x+at)$$

for any function F.

If we also introduce $\xi = \frac{1}{2}(u-v)$ then taking the difference of (1) & (2) gives

$$\frac{\partial}{\partial t}(u-v) = a \frac{\partial}{\partial x}(v-u) = -a \frac{\partial}{\partial x}(u-v).$$

Dividing this equation by 2 gives

$$\frac{\partial \xi}{\partial t} = -a\frac{\partial \xi}{\partial x} \quad \text{or} \quad \frac{\partial \xi}{\partial t} + a.\frac{\partial \xi}{\partial x} = 0$$

which is the advection equation with speed "a". This has the solution

$$\xi = G(x - at)$$

for any function G.

Noting that

$$\eta + \xi = \tfrac{1}{2}(u+v) + \tfrac{1}{2}(u-v) = u$$

then

$$\begin{aligned} u(x,t) &= \eta(x,t) + \xi(x,t) \\ &= F(x+at) + G(x-at) \end{aligned}$$

for any functions F + G.

This is the <u>general solution</u> of the wave equation on the infinite line $-\infty < x < \infty$ and is known as D'Alembert's solution.

To determine F & G in each case we need to specify f & g in the initial conditions.

<u>Example</u>

Consider the solution of the wave equation in $-\infty < x < \infty$ and $t \geq 0$ which satisfies

$$u(x,0) = e^{-x^2}.$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

The general solution is

$$u(x,t) = F(x+at) + G(x-at) \text{ for any } F, G.$$

and here we want

$$u(x,0) = e^{-x^2} = F(x+a.0) + G(x-a.0)$$
$$= F(x) + G(x)$$

for all x in $-\infty < x < \infty$. Hence F, G satisfy

$$F(x) + G(x) = e^{-x^2}$$

Using the second initial condition

$$\frac{\partial u}{\partial t} = F'(x+at).a + G'(x-at).(-a).$$

hence

$$\frac{\partial u}{\partial t}(x,0) = a F'(x+a.0) - a G'(x-a.0)$$
$$= a F'(x) - a G'(x).$$

and therefore since $\frac{\partial u}{\partial t}(x,0) = 0$ then,

$$a. F'(x) - a G'(x) = 0$$

or $$F'(x) - G'(x) = 0 \text{ for all } x.$$

Integrating this once gives

$$F(x) = G(x) + c$$

where c is an arbitrary constant.