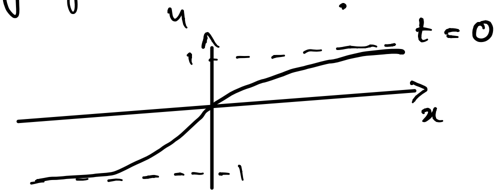
$$\frac{\partial u}{\partial t} - x \cdot \frac{\partial u}{\partial x} = 0 \quad \text{over} \quad -\infty < x < \infty, \quad t > 0.$$

has the solution.

$$u(x,t) = F(xe^t) \quad \text{for any function } F.$$

Consider the initial condition

$$u(x,0) = \tanh x = \frac{\sinh x}{\cosh x}$$



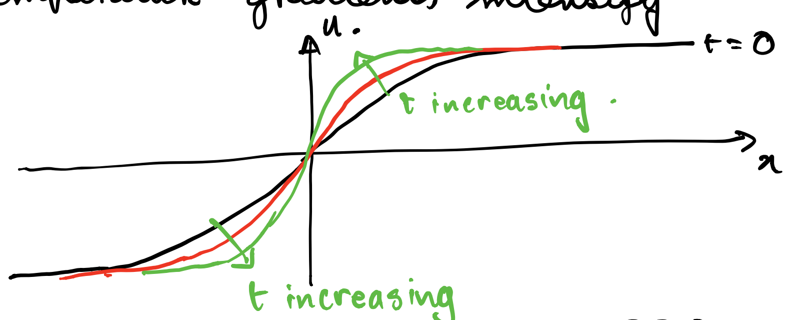
For the above PDE we want that.

$$u(x,0) = F(xe^0) = \tanh x \quad \text{at } t=0, \text{ for all } x$$

and therefore $F(x) = \tanh x$ in this case. Hence the solution of the PDE with $u(x,0) = \tanh x$. for $-\infty < x < \infty$ is

$$u(x,t) = \tanh xe^t.$$

One example of this system is 'frontogenesis', where temperature gradients intensify



2.4. A more general first-order PDE.

Consider a PDE of the form

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = S(x,y,u).$$

for an unknown u on a given domain D.

Here a, b are given functions of x, y and the "source term" $S(x,y,u)$ is also given and can depend on u.

First consider the case $S \equiv 0$ everywhere and seek a solution $u^*(x,y)$ such that

$$a(x,y)\,\frac{\partial u^*}{\partial x} + b(x,y)\cdot\frac{\partial u^*}{\partial y} = 0. \qquad (*).$$

Assuming $a \neq 0$ anywhere, divide by a so.

$$\frac{\partial u^*}{\partial x} + \frac{b(x,y)}{a(x,y)}\cdot\frac{\partial u^*}{\partial y} = 0.$$

and seek curves $y(x)$ such that

$$\frac{dy}{dx} = \frac{b(x,y(x))}{a(x,y(x))}.$$

Then u^* along these curves has:

$$\frac{\partial u^*}{\partial x} + \frac{b(x,y)}{a(x,y)}\cdot\frac{\partial u^*}{\partial y} = \frac{\partial u^*}{\partial x} + \frac{dy}{dx}\cdot\frac{\partial u^*}{\partial y} = 0$$

so that

$$\frac{d}{dx}\,u^*(x,y(x)) = 0 \qquad (**).$$

or u^* is constant along the curves $y(x)$.
These are called the <u>characteristics</u> of this PDE $(*)$. The key feature of these curves is that PDE has be reduced to ODE $(**)$

If we now use the same process with non-zero source term:

$$a(x,y)\,\frac{\partial u}{\partial x} + b(x,y)\cdot\frac{\partial u}{\partial y} = S(x,y,u).$$

then assuming $a \neq 0$

$$\frac{\partial u}{\partial x} + \frac{b(x,y)}{a(x,y)}\cdot\frac{\partial u}{\partial y} = \frac{S(x,y,u)}{a(x,y)}.$$

Following the same process as for the homogeneous equation we must have that this reduces to the ODE

$$\frac{d}{dx}\left[u(x, y(x))\right] = \frac{s(x, y, u(x, y(x)))}{a(x, y(x))}.$$

along the characteristics

$$\frac{dy}{dx} = \frac{b(x, y(x))}{a(x, y(x))}.$$

We then solve the ODE for u along the curves:

$$\frac{du}{dx} = \frac{s(x, y(x), u(x, y(x)))}{a(x, y(x))}.$$

which (in principle) can be solved for $u(x, y(x))$.

Process is, first solve for $y(x)$ using characteristic equation, then solve for u.

Could also solve for $x(y)$ by solving

$$\frac{dx}{dy} = \frac{a(x(y), y)}{b(x(y), y)}.$$

then solve $\displaystyle \frac{du}{dy} = \frac{s(x(y), y, u(x(y), y))}{b(x(y), y)}$.

Or could solve for parametrisation $x(t), y(t)$.

<u>Examples</u>

1) Consider the PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1 \quad \text{for } -\infty < x < \infty \text{ and } y > 0$$

and initial condition $u(x, 0) = e^{-x^2}$ when $y = 0$.

Here the characteristics $y(x)$ have

$$\frac{dy}{dx} = \frac{1}{1} = 1$$

and hence $y(x) = x + c$ for any constant c.

On <u>each line</u> we can write that.

$$\frac{d}{dx} u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{dy}{dx} \cdot \frac{\partial u}{\partial y} \quad \text{by the chain Rule.}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1 \quad \text{from the PDE.}$$

Hence on each curve

$$\frac{du}{dx} = 1.$$

so $\quad u = x + F(c)$ for any function F.

This can be a different 'constant' for each curve (which is defined by c), and hence is an unknown function of c.

For any given point (x, y) in our domain $c = y - x$ so

$$u(x, y) = x + F(y - x).$$

which consists of general solution of homogeneous and particular solution of non-homogeneous equation. This applies for any F corresponds to general solution of PDE.

For a given initial condition $u(x, 0) = e^{-x^2}$ then for each x in $(-\infty, \infty)$

$$u(x, 0) = x + F(0 - x) = e^{-x^2}.$$

and hence

$$F(-x) = e^{-x^2} - x \quad \text{for all } x.$$

Here $(-x)$ is a dummy variable, let $\eta = -x$ for any $\eta \in \mathbb{R}$ and hence

$$F(\eta) = e^{-(-\eta)^2} - (-\eta).$$

$$= e^{-\eta^2} + \eta.$$

for any real value of η. Therefore.

$$
\begin{aligned}
u(x,y) &= x + F(y-x). \\
&= x + e^{-(y-x)^2} + y - x. \\
&= y + e^{-(y-x)^2}.
\end{aligned}
$$

which satisfies both the PDE and the initial condition.