School of Mathematical Sciences
MTH3011
2018 [1]

MTH3011 Partial differential equations

Exercise sheet 3 — The heat equation

1. **Properties of solutions of the heat equation:** Show that if $u(x,t)$ is a solution of the heat equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad \text{over} \quad -\infty < x < \infty, \quad \text{for} \quad t > 0$$

with constant thermal conductivity $K > 0$, then:

(a) $w(x,t) = u(-x,t)$, $\dfrac{\partial u}{\partial t}(x,t)$, $\dfrac{\partial u}{\partial x}(x,t)$ and $u(x-x_0, t-t_0)$ are also solutions of the heat equation.

(b) $\int_{-\infty}^{\infty} u(x,t)\,dx$ is independent of t provided u approaches zero as $x \to \pm\infty$.

(c) $\int_{-\infty}^{\infty} u^2(x,t)\,dx$ is a decreasing function of t provided u approaches zero as $x \to \pm\infty$ (and *strictly* decreasing unless u is zero everywhere).

2. **Linearity of solutions of the heat equation:** Show that if $u_1(x,t)$ and $u_2(x,t)$ are both solutions of the heat equation in question 1 for all x over $-\infty < x < \infty$, then the linear combination

$$u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t)$$

is also solution of that equation, for any values of the constants c_1 and c_2.

3. **A solution of the heat equation:** Show by differentiation and evaluation that the function

$$u(x,t) = \frac{1}{\sqrt{4Kt+1}} \exp\left[-\frac{x^2}{4Kt+1}\right]$$

satistifes the heat equation for all x over $-\infty < x < \infty$, *and* the initial condition $u(x,0) = \exp(-x^2)$.

4. **Solutions of the heat equation with given initial conditions:** Use the general solution

$$u(x,t) = \frac{1}{\sqrt{4\pi Kt}} \int_{-\infty}^{\infty} g(\bar{x}) \exp\left[-(\bar{x}-x)^2/4Kt\right] d\bar{x}$$

over $-\infty < x < \infty$, or otherwise, to determine the solutions that satisfy the following initial conditions:

(a) $u(x,0) = 1$; (b) $u(x,0) = x$; (c) $u(x,0) = ax + b$ for constants a, b;

given that for any $p > 0$ then $\int_{-\infty}^{\infty} \exp(-p^2\bar{x}^2)\,d\bar{x} = \sqrt{\pi}/p$ and $\int_{-\infty}^{\infty} \bar{x}\exp(-\bar{x}^2)\,d\bar{x} = 0$.

[*Answers*: (a) change the variable of integration to $\bar{x}' = \bar{x} - x$ and then use $p = 1/\sqrt{4Kt}$ in the given integral to obtain that $u(x,t) = 1$, (b) similarly $u(x,t) = x$, (c) using linearity, $u(x,t) = ax + b$.]

5. Use appropriate changes of variable in the general solution for $u(x,t)$ in question 4 to show that:

   (a) If g is an even function of x, with $g(x) = g(-x)$, then $u(x,t)$ is also an even function of x.

   (b) If g is an odd function of x, with $g(x) = -g(-x)$, then $u(x,t)$ is also an odd function of x.

6. **Solutions of the heat equation with given initial conditions:** As in question 4 above, use the general solution of the heat equation to determine the solutions that satisfy the following initial conditions:

   (a) $u(x,0) = \exp(-x)$;
   (b) $u(x,0) = \exp(x)$;
   (c) $u(x,0) = \sinh x$;
   (d) $u(x,0) = \exp(-\frac{1}{2}x^2)$;
   (e) $u(x,0) = \sin x$;

   given that for any $p > 0$ then $\int_{-\infty}^{\infty} \exp(-p^2\bar{x}^2 \pm q\bar{x})\, d\bar{x} = \sqrt{\pi}/p \exp(\frac{1}{4}q^2/p^2)$ and $\int_{-\infty}^{\infty} \cos(q\bar{x}) \exp(-p^2\bar{x}^2)\, d\bar{x} = \sqrt{\pi}/p \exp(-\frac{1}{4}q^2/p^2)$ and noting that since sine is an odd function, then $\int_{-\infty}^{\infty} \sin(q\bar{x}) \exp(-p^2\bar{x}^2)\, d\bar{x} = 0$.

   [*Answers*: (a) collect coefficient of $\bar{x}$ and $\bar{x}^2$ in the power of e, define p and q appropriately, then use the first given integral to evaluate the result, obtaining that $u(x,t) = e^{Kt-x}$, (b) similarly $u(x,t) = e^{Kt+x}$, or note that $u(-x,t)$ is also a solution and so the sign of x can be changed in the answer to (a), (c) using linearity, $u(x,t) = e^{Kt}\sinh x$, (d) similarly $u(x,t) = e^{-x^2/(4Kt+2)}/\sqrt{2Kt+1}$, (e) use the second given integral (and sine summation formula) to obtain that $u(x,t) = e^{-Kt}\sin x$.]

7. **Solutions of the heat equation with one 'insulated' boundary:** Consider the solution of the heat equation in a semi-infinite domain $0 < x < \infty$ with the 'insulating' boundary condition

   $$\frac{\partial u}{\partial x}(0,t) = 0$$

   for all $t > 0$. Write down the corresponding function $g(x)$ for $-\infty < x < \infty$ that must be used in the general solution of the heat equation for each of the following initial conditions:

   (a) $u(x,0) = 1$;
   (b) $u(x,0) = \cos x$;
   (c) $u(x,0) = x$ for $x > 0$.

   Use this (along with your results from questions 4–6) to determine $u(x,t)$ for all $t > 0$. Confirm that the boundary condition above is satisfied by this solution.

   [*Answers*: (a) $u(x,t) = 1$, (b) $u(x,t) = e^{-Kt}\cos x$, (c) $u(x,t) = \frac{x}{\sqrt{\pi K t}} \int_0^x e^{-\bar{x}^2/4Kt} d\bar{x} + 2\sqrt{\frac{Kt}{\pi}} e^{-x^2/4Kt}$.]

8. **The heat equation in a finite domain:** The general solution of the heat equation

   $$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \quad \text{over} \quad 0 < x < L \quad \text{for} \quad t > 0,$$

   with constant diffusivity $K > 0$, that satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ is

   $$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \exp\left[-K\left(\frac{n\pi}{L}\right)^2 t\right].$$

   Derive this solution without using your lecture notes. Use this general solution to determine the solution that satisfies each of the initial conditions for $0 < x < L$:

(a) $u(x,0) = \sin\left(\frac{\pi x}{L}\right)$;

(b) $u(x,0) = 2\sin\left(\frac{3\pi x}{L}\right) - 4\sin\left(\frac{5\pi x}{L}\right)$;

(c) $u(x,0) = 1$;

(d) $u(x,0) = x(L-x)$.

[*Answers*: (a) $u(x,t) = \sin\left(\frac{\pi x}{L}\right)\exp\left[-K\left(\frac{\pi}{L}\right)^2 t\right]$, (b) deduce that $A_3 = 2$, $A_5 = -4$ and other $A_n = 0$, so $u(x,t) = 2\sin\left(\frac{3\pi x}{L}\right)\exp\left[-K\left(\frac{3\pi}{L}\right)^2 t\right] - 4\sin\left(\frac{5\pi x}{L}\right)\exp\left[-K\left(\frac{5\pi}{L}\right)^2 t\right]$, (c) $A_n = 4/(n\pi)$ for n odd, $A_n = 0$ for n even, (d) $A_n = 8L^2/(n\pi)^3$ for n odd, $A_n = 0$ for n even.]

9. **The heat equation with insulating boundary conditions:** Use the method of 'separation of variables' to show that the general solution of the heat equation in question 8 that satisfies the insulating boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

is $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\exp\left[-K\left(\frac{n\pi}{L}\right)^2 t\right]$. Indicate how the coefficients A_n are determined when the initial conditions are $u(x,0) = f(x)$ and then determine the solution for each of the following cases:

(a) $u(x,0) = 1$;

(b) $u(x,0) = \cos\left(\frac{2\pi x}{L}\right)$;

(c) $u(x,0) = 1 - \cos\left(\frac{2\pi x}{L}\right)$;

(d) $u(x,0) = \sin\left(\frac{\pi x}{L}\right)$.

[*Answers*: $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ and $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$ for $n = 1, 2, 3, \ldots$ in general. (a) $u(x,t) = 1$, (b) $u(x,t) = \cos\left(\frac{2\pi x}{L}\right)\exp\left[-K\left(\frac{2\pi}{L}\right)^2 t\right]$, (c) $u(x,t) = 1 - \cos\left(\frac{2\pi x}{L}\right)\exp\left[-K\left(\frac{2\pi}{L}\right)^2 t\right]$, (d) $A_0 = 2/\pi$, $A_n = 4/(\pi(1-n^2))$ for n even and nonzero, and $A_n = 0$ for n odd.]

10. **The heat equation with mixed boundary conditions:** Use the method of 'separation of variables' to determine the (series) form of the general solution $u(x,t)$ of the heat equation in question 8 that satisfies the *mixed* boundary conditions with

$$u(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0.$$

Indicate how the coefficients A_n are determined for initial conditions $u(x,0) = f(x)$. Find the coefficients A_n for each of the cases: (a) $u(x,0) = \sin(\frac{\pi x}{2L})$, and (b) $u(x,0) = 1$.

[*Answers*: (a) as for 8(a) with L replaced by $2L$, (b) as for 8(c) with L replaced by $2L$.]

11. **The heat equation with a source:** Consider the solution of the heat equation with a source term

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 \quad \text{over} \quad 0 < x < 1 \quad \text{for} \quad t > 0$$

that satisfies the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = 0$.

Determine the solution as $t \to \infty$ by assuming that the system reaches a steady state solution $U(x)$ and solve for that function which satisfies the given boundary conditions.

Now write $u(x,t) = U(x) + \bar{u}(x,t)$ and show that $\bar{u}(x,t)$ satisfies the heat equation without a source, but with initial condition $\bar{u}(x,0) = -U(x)$. Find the solution for $\bar{u}(x,t)$ and hence for $u(x,t)$. Sketch the form of $u(x,t)$ for several values of t.

[*Answer*: $u(x,t) = \frac{1}{2}x(1-x) - \frac{4}{\pi^3}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^3}\sin[(2m+1)\pi x]\exp[-(2m+1)^2\pi^2 t]$, using 8(d).]

12. **The heat equation with nonzero boundary conditions:** Consider the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{over} \quad 0 < x < 1 \quad \text{for} \quad t > 0$$

that satisfies the initial condition $u(x, 0) = 0$, the boundary condition $u(0, t) = 0$ and the nonzero boundary condition $u(1, t) = 1$. As in question 11, determine the solution as $t \to \infty$ by assuming that the system reaches a steady state solution $U(x)$ which satisfies the given boundary conditions. Use a similar process to that in question 11 to define $\bar{u}(x, t)$ and determine it for all t. Hence determine the full unsteady solution $u(x, t)$ and sketch it for several values of t.

[*Answer*: Use integration by parts $A_n = 2(-1)^{n+1}/n\pi$,
so $u(x, t) = x + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) \exp(-n^2\pi^2 t)$.]

13. **For experts — other parabolic PDEs:** Consider a constant-coefficient linear *parabolic* PDE with:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t},$$

where A, B, C, a and b are constants with $A > 0$, and show that it can be written in the form

$$A^2\frac{\partial^2 u}{\partial x^2} + AB\frac{\partial^2 u}{\partial x \partial t} + \frac{1}{4}B^2\frac{\partial^2 u}{\partial t^2} = Aa\frac{\partial u}{\partial x} + Ab\frac{\partial u}{\partial t}.$$

Write this as

$$\left(A\frac{\partial}{\partial x} + \frac{1}{2}B\frac{\partial}{\partial t}\right)^2 u = a\left(A\frac{\partial}{\partial x} + \frac{1}{2}B\frac{\partial}{\partial t}\right) + (Ab - \frac{1}{2}Ba)\frac{\partial u}{\partial t},$$

introduce the new independent variables (ξ, τ) such that $x = A\xi$ and $t = \frac{1}{2}B\xi + (Ab - \frac{1}{2}Ba)\tau$ and show that the corresponding solution $U(\xi, \tau)$ satisfies the *advection-diffusion equation* of the form

$$\frac{\partial U}{\partial \tau} + a\frac{\partial U}{\partial \xi} = \frac{\partial^2 U}{\partial \xi^2}$$

when $(Ab - \frac{1}{2}Ba) > 0$. Given some solution $U(\xi, \tau)$ of this equation, rewrite this as a solution $u(x, t)$ in terms of the original variables (x, t).

With a different choice of (x, t) in terms of (ξ, τ), could this become a *heat equation* for $U(\xi, \tau)$?

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