4. Diffusion and the heat equation.

4.1 Introduction

A typical example of a 'parabolic PDE' is the 'heat equation' (or 'diffusion equation').

$$\frac{\partial u}{\partial t} = \kappa . \frac{\partial^2 u}{\partial x^2}, \qquad \kappa > 0$$

and we will first concentrate on an infinite domain $-\infty < x < \infty$ and $t > 0$.

[Recall that $\kappa < 0$ gives an 'ill-posed' problem].

Typical initial conditions are

$$u(x, 0) = f(x) \qquad \text{for} \quad -\infty < x < \infty$$

at $t = 0$, assuming $u \to 0$ as $x \to \pm\infty$ (although other cases such as $u \to$ const are also possible).

4.2 A fundamental solution.

It can be shown that

$$u(x,t) = \frac{A}{\sqrt{t}} e^{-x^2/4\kappa t}$$

is a solution of the equation for all (x, t) when $t > 0$.

[See question 3 to prove this].

The properties of this solution 'include':

1). $u \to 0$ as $x \to \pm\infty$ provided $t > 0$ (and $\kappa > 0$).

2). it is an even function (symmetric about $x = 0$).

3). it does not exist at $t = 0$, although it can be evaluated for small t.

4) at $x = 0$ it has a maximum value of $u(0, t) = \frac{A}{\sqrt{t}}$ and u decreases on either

side of this (and the maximum $\to 0$ as $t \to \infty$).

5). The width of the function is proportional to $\sqrt{4kt}$, so the function widens as t increases and gets narrower as t is decreased towards zero.

6). It can be shown that

$$\int_{-\infty}^{\infty} u \, dx = \text{const}$$

For the heat equation must have that

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} \cdot dx = k \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} \cdot dx.$$

$$= k \left[\frac{\partial u}{\partial x} \right]_{-\infty}^{\infty}$$

$$= 0$$

if $u \to 0$ as $x \to \pm\infty$.

Hence must have that.

$$\frac{d}{dt} \int_{-\infty}^{\infty} u \, dx = 0.$$

and $\int_{-\infty}^{\infty} u \, dx$ is therefore an 'invariant' of the heat equation.

Hence the area of the curve under $u(x,t)$ is independent of t (for $t > 0$).

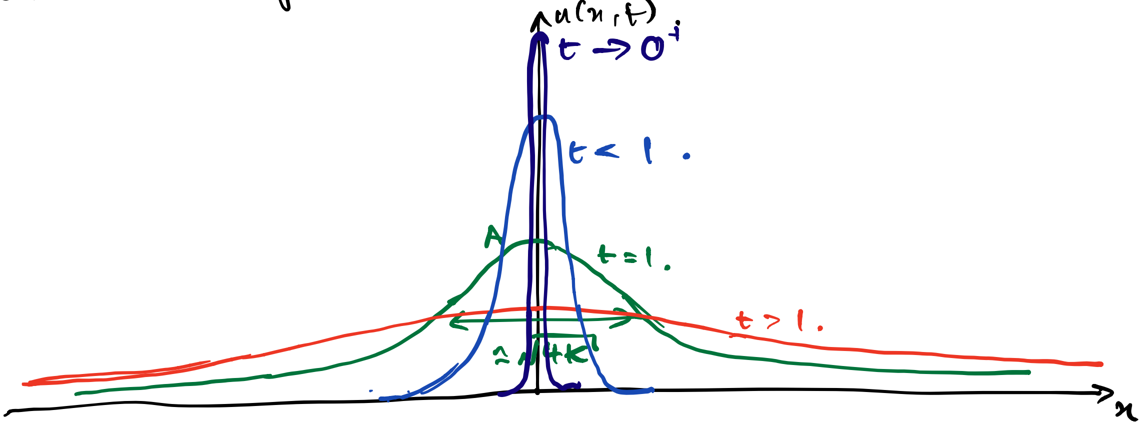
7). For very small $t \to 0^+$ the maximum value at $x = 0$ becomes large as $\frac{A}{\sqrt{t}} \to \infty$, but its width $\sqrt{4kt}$ becomes small.

For any $x \neq 0$ then

$$u(x,t) = \frac{A}{\sqrt{t}} e^{-x^2/4kt}.$$

will tend to zero as $e^{-x^2/4kt} \to 0$ as $t \to 0^+$ when $x \neq 0$.

Solution therefore looks like.



The area under each of these curves remains the same.

We have stated that have constant area of $\int_{-\infty}^{\infty} u \, dx$ for all t. We can 'normalize' this area to be equal 1 for a particular choice of A.

Consider $\int_{-\infty}^{\infty} u(x,t)\, dx$

$$= \int_{-\infty}^{\infty} \frac{A}{\sqrt{t}} \exp\left\{-\frac{x^2}{4kt}\right\} dx. \qquad \text{for any } t>0$$

$$= \int_{-\infty}^{\infty} \frac{A}{\cancel{\sqrt{t}}} \exp\left\{-\eta^2\right\} \cdot \sqrt{4k\cancel{t}}\, d\eta. \qquad \text{Let } \eta = \frac{x}{\sqrt{4kt}} \Rightarrow dx = \sqrt{4kt}\, d\eta.$$

$$= A\sqrt{4K} \int_{-\infty}^{\infty} e^{-\eta^2} \, d\eta.$$

$$= A\sqrt{4K} \cdot \sqrt{\pi}$$

since $\int_{-\infty}^{\infty} e^{-p^2\eta^2} d\eta = \frac{\sqrt{\pi}}{p}$. or see MTH2010

$$= A\sqrt{4K\pi}.$$

If we want that $\int_{-\infty}^{\infty} u\,dx = 1$ for our 'normalized solution' then

$$A\sqrt{4\pi k} = 1 \quad \text{or} \quad A = \frac{1}{\sqrt{4\pi k}}.$$

Hence

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-\frac{x^2}{4kt}}.}$$

is called the <u>normalized fundamental solution</u> of the heat equation.

Where does this solution come from? Consider the following transformation

$$x' = \alpha x \quad \text{and} \quad t' = \beta t.$$

Then the heat equation becomes.

$$\beta \cdot \frac{\partial u}{\partial t'} = \alpha^2 \cdot k \cdot \frac{\partial^2 u}{\partial x'^2}.$$

or

$$\frac{\partial u}{\partial t'} = \frac{\alpha^2}{\beta} \cdot k \cdot \frac{\partial^2 u}{\partial x'^2}.$$

Therefore if $\beta = \alpha^2$ then the heat equation is 'invariant' under this transformation. Let.

$$\lambda = \beta \quad \text{and} \quad \alpha = \sqrt{\lambda}.$$

and the invariant transformation is

$$x' = \sqrt{\lambda}\, x, \quad t' = \lambda t.$$

The term in the exponential of the fundamental solution is

$$\frac{x^2}{4kt} = \left(\frac{x'}{\cancel{\sqrt{\lambda}}}\right)^2 \cdot \frac{\cancel{\lambda}}{4kt'} = \frac{(x')^2}{4kt'}$$