Example 7

Consider the PDE

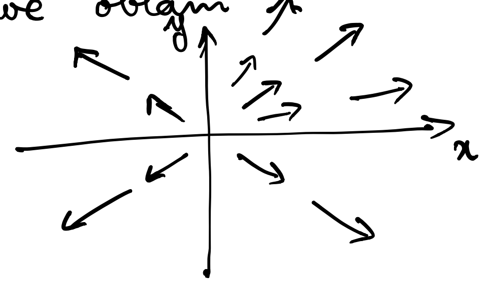
$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = x$$

over $-\infty < x < \infty$, for $y > 1$, with $u(x,1) = 0$ on $y = 1$.

We seek the characteristic curves $y(x)$ with

$$\frac{dy}{dx} = \frac{y}{x} = V(x,y).$$

Plot $V(x,y)$ we obtain



Then solve for $y(x)$ as it separable $\Rightarrow$

$$\int \frac{dy}{y} = \int \frac{dx}{x}.$$

and hence $\ln|y| = \ln|x| + k$ for any constant k.

Taking exponentials gives:

$$|y| = e^k |x|.$$

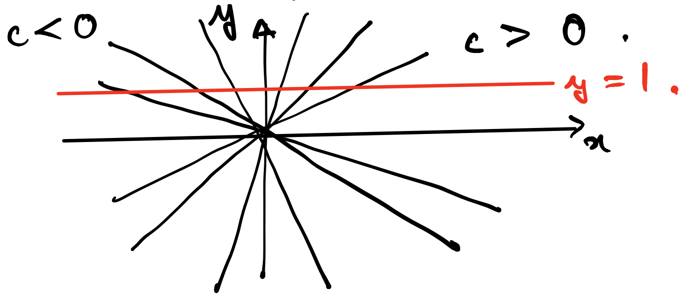
so that $y = (\pm e^k)x$ where k is any real number.

Since e^k is any positive real number and $\pm e^k$ is any non-zero real number. We can write.

$$y = cx \quad \text{for any constant } c.$$

Note that this also satisfies the ODE when $c = 0$.

Plot characteristics obtain:



Want to solve:

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = x.$$

along each of the curves $y = cx$. Using the chain rule we obtain.

$$\frac{d}{dx} u(x, y(x)) = \frac{S(x,y,u)}{a(x,y)} = \frac{x}{x} = 1.$$

On these curves

$$\frac{du}{dx} = 1 \quad \text{and so} \quad u = x + G(c)$$

where G is any function of c. Writing in terms of (x, y) and using $c = y/x$ gives

$$u(x, y) = x + G\left(\frac{y}{x}\right) \quad \text{for any function } G.$$

This is the general solution of the PDE.
To determine G we apply the initial condition $u(x,1) = 0$ which we evaluate at $y = 1$, hence:

$$u(x, 1) = 0 = x + G\left(\frac{1}{x}\right).$$

for any $x \neq 0$.
To find G, let $\eta = \frac{1}{x}$ then:

$$\frac{1}{\eta} + G(\eta) = 0 \quad \text{for all } \eta \neq 0.$$

so $$G(\eta) = -\frac{1}{\eta}.$$ Hence $u(x, y)$ is:

$$u(x, y) = x + \left(-\frac{1}{y/x}\right) = x - \frac{x}{y}.$$

$$= x\left(\frac{y-1}{y}\right).$$

which satisfies $u(x, 1) = 0$ as required.

<u>Example 3</u>

Consider $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x + y \quad \text{for } x > 0.$$

with $u(0, y) = y$ on $x = 0$ for $-\infty < y < \infty$.

Characteristics are $y(x) = x + c$ for any c (see example 1).

On these characteristics have:

$$\frac{du}{dx} = x + y(x). = x + (x + c.)$$

along $y(x) = x + c$.

Integrating ODE gives

$$u = x^2 + xc + H(c) \quad \text{for any function } H.$$

or in terms of (x, y).

$$u = x^2 + x(y - x) + H(y - x).$$

$$= xy + H(y - x) \quad \text{for any function } H.$$

Consider

$$\frac{\partial u}{\partial x} = y - H'(y - x).$$

$$\frac{\partial u}{\partial y} = x + H'(y - x).$$

therefore

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x + y$$

as required.

To determine $H(y-x)$ apply the initial condition $u(0,y) = y$ on $x = 0$:

$$u(0,y) = H(y) = y$$

and therefore $H(y) = y$, and the solution is:

$$u(x,y) = xy + (y - x).$$

Example 4

Solve $\dfrac{\partial u}{\partial x} + y \cdot \dfrac{\partial u}{\partial y} = -u$ for $x > 0, -\infty < y < \infty$

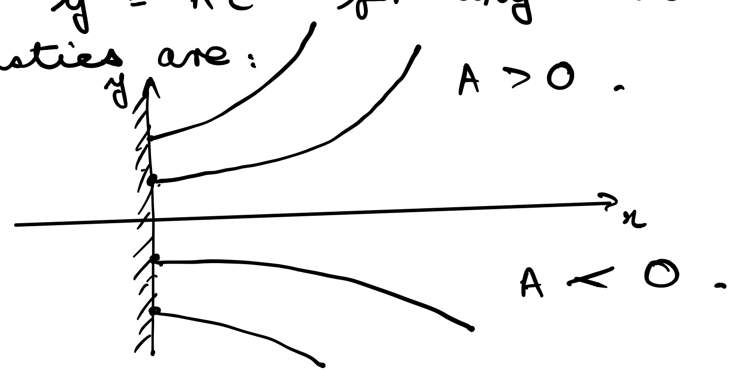
with initial condition $u(0,y) = y$ on $x = 0$.

To determine characteristics, solve

$$\frac{dy}{dx} = y.$$

so that $y = Ae^{x}$ for any constant A.

Characteristics are:



On these curves the PDE becomes.

$$\frac{du}{dx}(x, y(x)) = -u.$$

or $\dfrac{du}{dx} = -u$ on each curve (or for each A).

This has the general solution

$$u = f(A)\, e^{-x}.$$

where f is any function of A. Therefore

$$u = f(y e^{-x}) . e^{-x} .$$ for any function f.

To satisfy $u(0,y) = y$ at $x = 0$ to determine f, we require.

$$u(0,y) = f(y.1).1 = f(y) = y.$$

for any $-\infty < y < \infty$.

Hence $f(y) = y$ and the general solution is

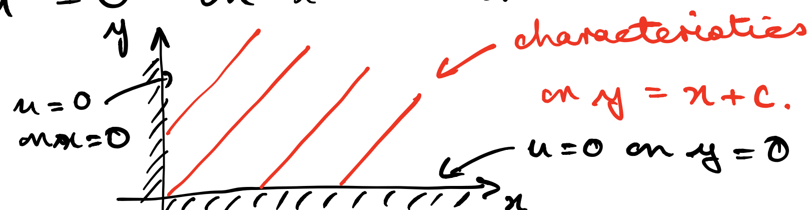
$$u(x,y) = (y e^{-x}).e^{-x}.$$
$$= y e^{-2x}.$$

for all $x \geq 0$ and $-\infty < y < \infty$.

<u>Example 5</u>

Consider $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 1.$

for $x > 0$, $y > 0$ with initial/boundary conditions of $u = 0$ on $x = 0$ and $u = 0$ on $y = 0$.



Hence boundaries will uniquely determine solution.

From previous details we know that

$$u(x,y) = x + F(y-x)$$

for any function F.

To apply the condition $u = 0$ on $y = 0$ when $x > 0$, we use that

$$u(x,0) = x + F(-x) = 0 \quad \text{for } x > 0.$$

hence this gives that

$$F(-x) = -x \quad \text{for } x > 0$$

and therefore

$$F(\eta) = \eta \quad \text{for } \eta < 0 \quad (\text{where } \eta = -x)$$

On $x = 0$ we have $u = 0$ for $y \geq 0$ and so

$$u(0,y) = 0 + F(y) = 0 \quad \text{for } y \geq 0.$$

and hence

$$F(\eta) = 0 \quad \text{for } \eta \geq 0.$$

Therefore F is the piecewise function.

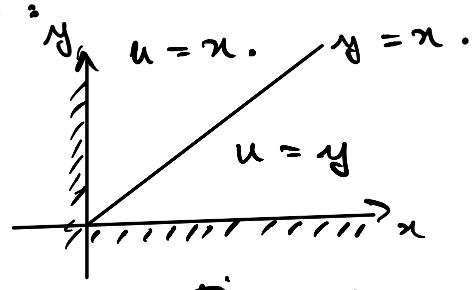
$$F(\eta) = \begin{cases} 0 & \text{for } \eta \geq 0 \\ \eta & \text{for } \eta < 0 \end{cases}$$

and hence $u(x,y)$ is:

$$u(x,y) = x + \begin{cases} 0 & \text{for } y \geq x, \\ y - x & \text{for } y < x. \end{cases}$$

$$= \begin{cases} x & \text{for } y \geq x, \\ y & \text{for } y < x. \end{cases}$$

Hence solution is:



Note that solution is continuous on $y = x$, but it has a discontinuous gradient on $y = x$.