The invariance of the heat equation under the transform

$$x' = \sqrt{\lambda}\, x \;, \quad t' = \lambda t \;.$$

for some constant $\lambda > 0$, suggests that the we look for a similarity solution of the form

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right) = f\left(\frac{x'}{\sqrt{t'}}\right).$$

So we will seek a trial solution of

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}$$

and determine the function f.

So need to determine $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} f\left(\frac{x}{\sqrt{t}}\right) = f'\left(\frac{x}{\sqrt{t}}\right) \cdot \frac{\partial}{\partial t}\left(\frac{x}{\sqrt{t}}\right) = -\frac{x}{2\sqrt{t^3}} f'\left(\frac{x}{\sqrt{t}}\right)$$

then

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f\left(\frac{x}{\sqrt{t}}\right) = f'\left(\frac{x}{\sqrt{t}}\right) \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{t}} f'\left(\frac{x}{\sqrt{t}}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{t}} f'\left(\frac{x}{\sqrt{t}}\right)\right)$$

$$= \frac{1}{\sqrt{t}} \cdot f''\left(\frac{x}{\sqrt{t}}\right) \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{t}}\right)$$

$$= \frac{1}{t} \cdot f''\left(\frac{x}{\sqrt{t}}\right)$$

and this form of solution satisfies the heat equation when f satisfies

$$-\frac{x}{2\sqrt{t^3}} f'\left(\frac{x}{\sqrt{t}}\right) = k \cdot \frac{1}{t} \cdot f''\left(\frac{x}{\sqrt{t}}\right)$$

for all (x,t).

Rearranging

$$-\frac{x}{2\sqrt{t}} \cdot f'\left(\frac{x}{\sqrt{t}}\right) = k \cdot f''\left(\frac{x}{\sqrt{t}}\right)$$

or $\quad -\frac{\eta}{2} f'(\eta) = k \cdot f''(\eta) \quad$ where $\eta = \frac{x}{\sqrt{t}}$

is satisfied for any value of the dummy variable η.

Let $g = f'$ then g satisfies

$$-\frac{\eta g}{2} = k \cdot \frac{dg}{d\eta}.$$

This has separable form so write as

$$\int \frac{dg}{g} = -\frac{1}{2k} \int \eta \, d\eta.$$

Therefore

$$\ln|g| = -\frac{1}{2k}\frac{\eta^2}{2} + C \quad \text{for any constant } C.$$

and

$$|g| = e^{-\eta^2/4k + C}$$

or $\quad g = A e^{-\eta^2/4k} \quad$ where $A = \pm e^{C}$.

for any constant $A \neq 0$. But this also works for $A = 0$ so the general solution is

$$g(\eta) = A e^{-\eta^2/4k}.$$

Therefore since $f'(\eta) = g$ then

$$f(\eta) = A \int_0^{\eta} \exp\{-(\eta')^2/4k\} \, d\eta' + B$$

for any constants A, B.

Hence our similarity solution of the heat equation is

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right) = A\int_0^{x/\sqrt{t}} \exp\left(-\frac{(\eta')^2}{4k}\right).d\eta' + B.$$

Since $\frac{\partial u}{\partial x}$ will also be a solution of the heat equation, we can differentiate this solution with respect to x using Leibniz's rule:

$$\frac{\partial u}{\partial x} = A\exp\left(-\left(\frac{x}{\sqrt{t}}\right)^2 \cdot \frac{1}{4k}\right)\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{t}}\right)$$

$$= \frac{A}{\sqrt{t}}\exp\left(-\frac{x^2}{4kt}\right)$$

which is the fundamental solution of the heat equation.

4.3 A more general solution.

Note that if

$$u(x,t) = \frac{A}{\sqrt{t}}e^{-x^2/4kt}.$$

is a solution of the heat equation, then

$$u(x,t) = \frac{A}{\sqrt{t}}e^{-(x-\bar{x})^2/4kt}$$

is also a solution of the heat equation.

Now consider a set of points $\bar{x}_k = k\Delta x$ say $(k = 1, 2, 3, \ldots, n)$ with corresponding solution at each point

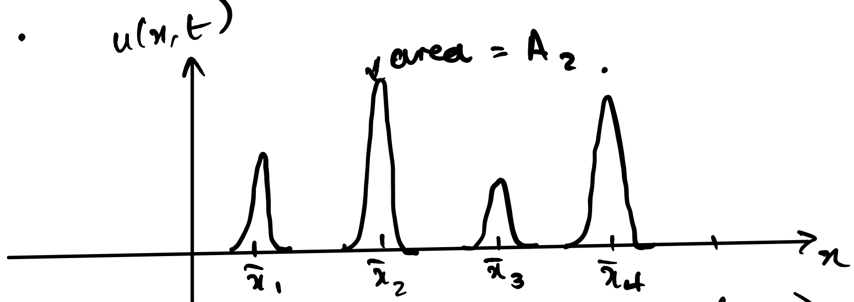
$$\bar{u}_k(x,t) = \frac{A_k}{\sqrt{4\pi kt}}\exp\left(-\frac{(x-\bar{x}_k)^2}{4kt}\right)$$

at each $\bar{x}_k$, where Δx is the spacing of the points

Since the PDE is linear then

$$u(x,t) = \sum_{k=1}^{n} \frac{A_k}{\sqrt{4\pi K t}} \exp\left(-\frac{(x-\bar{x}_k)^2}{4Kt}\right)$$

is also a solution. Then A_k is the 'local' strength or amplitude of the solution at each point.



If we also assume that $A_k = g(\bar{x}_k).\Delta x$ for each point, where $g(\bar{x}_k)$ represents the 'density' of the solutions, then

$$u(x,t) = \sum_{k=1}^{n} \frac{g(\bar{x}_k)\Delta x}{\sqrt{4\pi K t}} \exp\left(-\frac{(x-\bar{x}_k)^2}{4Kt}\right).$$

If we use points for $k = -n, \dots, 0, \dots, n$, then solution becomes

$$u(x,t) = \sum_{k=-n}^{n} \frac{g(\bar{x}_k).\Delta x}{\sqrt{4\pi K t}} \exp\left(-\frac{(x-\bar{x}_k)^2}{4Kt}\right)$$

assuming that $g(\bar{x}_k) \to 0$ as $k \to \infty$. Now let $\Delta x \to 0$ and evaluate gives

$$u(x,t) = \lim_{\Delta x \to 0} \left\{ \underbrace{\sum_{k=-\infty}^{\infty} \frac{g(\bar{x}_k).\Delta x}{\sqrt{4\pi K t}} \exp\left(-\frac{(x-\bar{x}_k)^2}{4Kt}\right)}_{\text{Riemann Sum}} \right\}$$

$$= \int_{-\infty}^{\infty} \frac{g(\bar{x})}{\sqrt{4\pi K t}} \exp\left(-\frac{(x-\bar{x})^2}{4Kt}\right) d\bar{x}.$$

since the integral is defined as the limit of Riemann sums (see VCE).

This will be a solution of the heat equation, since PDE is linear, and since $g(\bar{x})$ can be any specified function over $-\infty < \bar{x} < \infty$, then we call this the general solution of the heat equation i.e.

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} g(\bar{x}) \exp\left(-\frac{(x-\bar{x})^2}{4kt}\right) d\bar{x}.$$

and satisfies

$$u(\bar{x},0) = g(\bar{x}).$$