2. Badając pochodne jednostronne rozstrzygnij, czy istnieją pochodne podanych funkcji we wskazanych punktach:

a) $f(x) = |x^2 - x|$, $x_0 = 1$; b) $f(x) = \sin x \cdot \mathrm{sgn}(x)$, $x_0 = 0$;

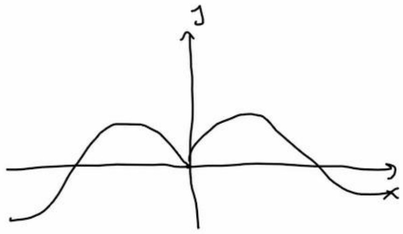
c) $f(x) = \min\{x^2, 4\}$, $x_0 = 2$.

Naszkicuj wykresy tych funkcji.

$f'_-(1) \neq f'_+(1)$ $\quad f'(1)$ nie istnieje

b) $f(x) = \sin x \cdot \mathrm{sgn}(x)$, $x_0 = 0$

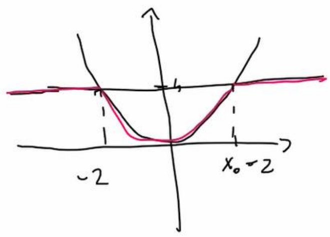
$$f(x) = \begin{cases} \sin x & \text{dla } x > 0 \\ 0 & \text{dla } x = 0 \\ -\sin x & \text{dla } x < 0 \end{cases}$$



$$\left.\begin{array}{ll} f'_-(x) = -\cos x & f'_-(0) = -1 \\ f'_+(x) = \cos x & f'_+(0) = 1 \end{array}\right\} f'(0) \text{ nie istnieje}$$

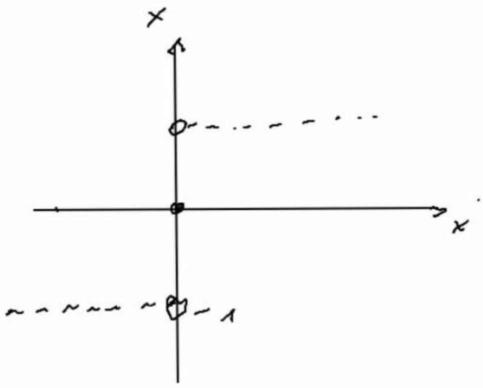
c) $f(x) = \min\{x^2, 4\}$ $\quad x_0 = 2$

$$f(x) = \begin{cases} 4 & \text{gdy } x \in (-\infty, -2) \cup (2, \infty) \\ x^2 & \text{gdy } x \in (-2, 2) \end{cases}$$



$$\left.\begin{array}{ll} f'_-(x) = 2x & f'_-(2) = 4 \\ f'_+(x) = 0 & f'_+(2) = 0 \end{array}\right\} f'(2) \text{ nie istnieje}$$

Komentarz Gusin



$\mathrm{sign}(x) = f(x)$

$\Downarrow$

$$\underline{\delta(x)} = \begin{cases} \infty & \text{dla } x = 0 \\ 0 & \text{dla } x \neq 0 \end{cases}$$

a

3. Korzystając z reguł różniczkowania oblicz pochodne funkcji:

a) $3\sin x + \cot x$; b) $e^x(x2 - x + 1)$; c) $\frac{x^2+2}{x-2}$; d) $e^{-x}(3x+1)^2$;

e) $e^{1/x}\arctan(3-x)$; f) $\ln(x^2+1)\tan\sqrt{x}$; g) $\ln(\cos^2 x + 1)$; h) $\sqrt{\arccos(x^2)}$;

i) $\frac{\sqrt{5}}{(x^2+1)^3}$; j) $\frac{3^{\sin^2 x}}{2^{\cos^2 x}}$.

3,

a) $y = 3\sin x + \cot x$

$y' = 3\cos x - (1+\cot^2 x) = 3\cos x + \left(\frac{-1}{\sin^2 x}\right)$

b) $e^x(x^2-x+1)$

$y' = e^x(x^2-x+1) + e^x(2x-1) = e^x(x^2-x+1+2x-1) = e^x(x^2+x)$

c) $y = \frac{x^2+x}{x-2}$

$y' = \frac{(2x+1)(x-2)-(x^2+x)}{(x-2)^2} = \frac{2x^2+x-4x-2-x^2-x}{(x-2)^2} =$

$= \frac{x^2-4x-2}{(x-2)^2}$

d)

$e^{-x}(3x+1)^2$ $y' = (e^{-x})'(3x+1)^2 + e^{-x}[(3x+1)^2]' =$

$u = -x$ $3x+1 = z$ $= -e^{-x}(3x+1) + e^{-x} 6(3x+1)$

$y = e^u$ $y = z$

$\frac{dy}{du} = e^u$ $\frac{dy}{dz} = 2z$

$\frac{du}{dx} = -1$ $\frac{dz}{dx} = 3$

$\frac{dy}{dx} = -e^{-x}$ $\frac{dy}{dx} = 6(3x+1)$

e) $e^{\frac{1}{x}} \arctan(3-x)$

$$y' = \left(e^{\frac{1}{x}}\right)' \arctan(3-x) + e^{\frac{1}{x}} \left(\arctan(3-x)\right)'$$

$\frac{1}{x} = u$ $\qquad\qquad$ $3-x = z$

$y = e^u$ $\qquad\qquad$ $y = \arctan z$

$\frac{dy}{du} = e^u$ $\qquad\qquad$ $\frac{dy}{dz} = \frac{1}{1+z^2}$

$\frac{du}{dx} = \frac{1}{x^2}$ $\qquad\qquad$ $\frac{dz}{dx} = -1$

$$\frac{dy}{dx} = -\frac{1}{x^2} \cdot e^{\frac{1}{x}}$$

$$y' = -\frac{1}{x^2} e^{\frac{1}{x}} \cdot \arctan(3-x) + e^{\frac{1}{x}} \cdot \frac{-1}{1+(3-x)^2} =$$

komentarz Gusina

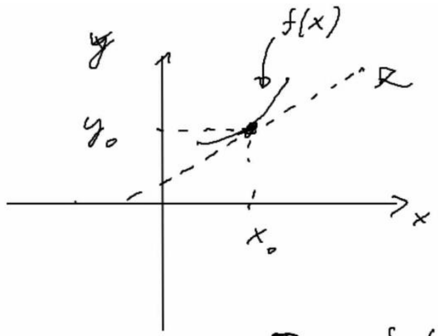
$$f(x) = \frac{3^{\sin^2 x}}{2^{\cos^2 x}} \Rightarrow \ln f = \sin^2 x \ln 3 - \cos^2 x \ln 2 \Rightarrow$$

$$\frac{d}{dx}(\ln f) = \frac{1}{f}\frac{df}{dx} = 2 \sin x \cos x \ln 3 + 2 \cos x \sin x \ln 2 = \sin(2x) \ln(2 \cdot 3) \Rightarrow$$

$$\frac{df}{dx} = f(x) \sin(2x) \ln 6 = \sin(2x) \frac{3^{\sin^2 x}}{2^{\cos^2 x}} \ln 6 .$$

$$f(x) = x^x, \quad x > 0 \quad \Rightarrow \ln f = x \ln x \Rightarrow$$

$$\frac{1}{f} f' = \ln x + 1 \Rightarrow \quad f' = (1 + \ln x)\, x^x .$$



$$y = Ax + B \qquad f(x_0) = y_0$$

$$\boxed{\frac{y - y_0}{x - x_0} = f'(x_0)} \, ;$$

$$\begin{cases} \Gamma_f = \{(x,y) \in \mathbb{R}^2 \mid y = f(x)\} \subset \mathbb{R}^2 , \\ \Gamma_f \cap \Gamma_y = \{(x_0, f(x_0))\} . \end{cases}$$

4. Napisz równania stycznych do wykresów podanych funkcji we wskazanych punktach:

a) $f(x) = \arctan x$, $(1, f(1))$; b) $f(x) = \ln(x^2 + e)$, $(0, f(0))$;

$$f(x) = \operatorname{arc\,tg} x \qquad (1, f(1))$$

$$(\operatorname{arc\,tg} x)' = \frac{1}{1+x^2} \qquad y = \frac{1}{2}x + \frac{\pi - 2}{4}$$

$$\operatorname{tg} x = 1 \equiv x = \frac{\pi}{4} \Rightarrow \operatorname{arc\,tg}(1) = \frac{\pi}{4}$$

$$x_0 = 1 \qquad y_0 = \frac{\pi}{4} \qquad f'(x_0) = \frac{1}{2}$$

$$\frac{(y - y_0)}{x - x_0} = f'(x_0)$$

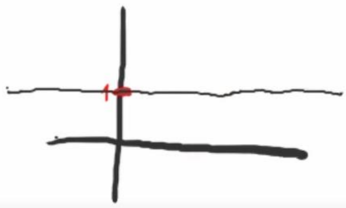
$$y = f'(x_0)(x - x_0) + y_0 = \frac{1}{2}(x-1) + \frac{\pi}{4}$$

$f(x) = \ln(x^2 + e)$ $\quad\quad (0, f(0))$

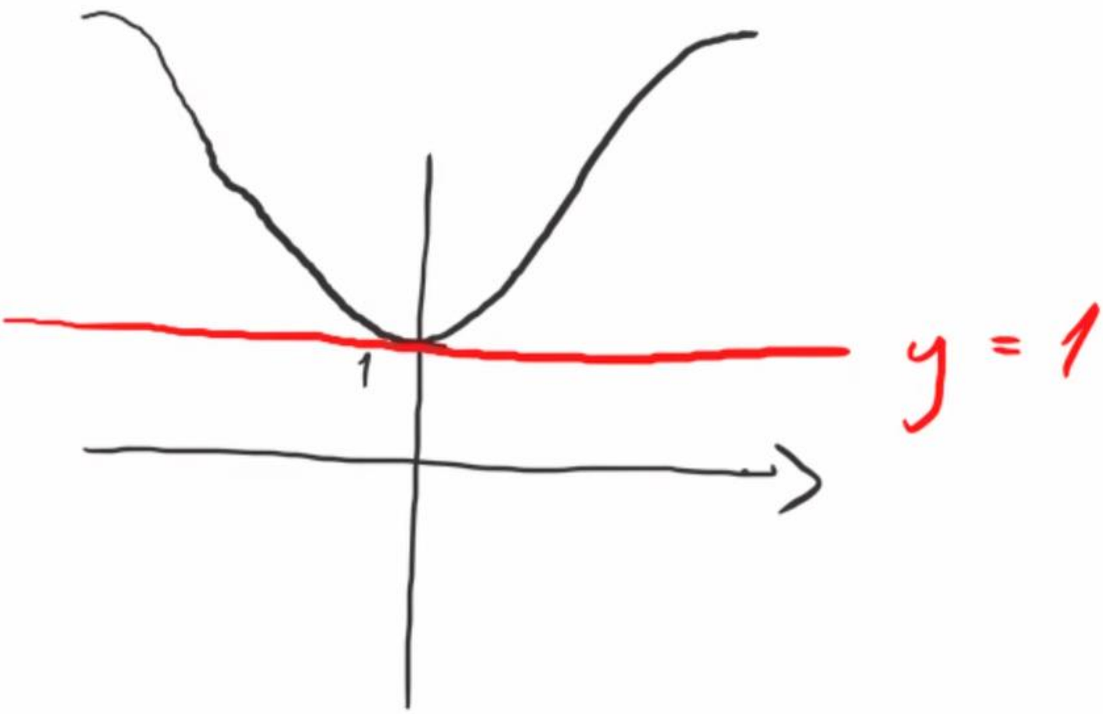
$$f'(x) = \frac{1}{x^2+e} \cdot 2x = \frac{2x}{x^2+e}$$

$x_0 = 0$ $\quad f'(x_0) = 0$ $\quad y_0 = 1$

$y = f'(x_0)(x - x_0) + y_0$



$y = 1$



5. a) Napisz równanie stycznej do wykresu funkcji $f(x) = x^4 - 2x + 5$, która jest równoległa do prostej $y = 2x + 3$.

b) Wyznacz styczną do wykresu funkcji $f(x) = \sqrt{x}$, która tworzy kąt $\frac{\pi}{4}$ z osią Ox.

c) Znajdź równanie stycznej do wykresu funkcji $f(x) = x \ln x$, która jest prostopadła do prostej $2x + 6y - 1 = 0$.

d) Znajdź równanie stycznej do wykresu funkcji $f(x) = x \arctan \frac{1}{x}$, w punkcie jego przecięcia z prostą $\pi x = 4y$.

e) Znajdź równanie stycznej do wykresu funkcji $f(x) = \sin 2x - \cos 3x$ w punkcie jego przecięcia z osią Oy.

$f(x) = x^4 - 2x + 5$ $\qquad$ $y = \underset{\smile}{2}x + 3$

$y = f'(x_0)(x - x_0) + f(x_0)$ $\qquad$ $a = 2 = f'(x_0)$

$f'(x) = 4x^3 - 2$

$4x_0^3 - 2 = 2$ $\qquad$ $4x_0^3 - 4 = 0$ $\qquad$ $x_0^3 = 1$

$4(x_0^3 - 1) = 0$ $\qquad$ $x_0 = 1$

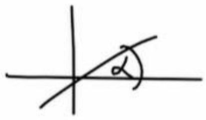
$f'(1) = 4 - 2 = 2$

$y = 2(x - 1) + 1 - 2 + 5$

$y = 2x + 2$

$y = 2x + 2$

b) $f(x) = \sqrt{x}$



$\operatorname{tg}\alpha = a = f'(x_0)$

$f'(x_0) = \operatorname{tg}\frac{\pi}{4} = 1$

$f'(x_0) = \dfrac{1}{2\sqrt{x_0}} = 1$

$\sqrt{x_0} = \dfrac{1}{2}$ $\qquad$ $x_0 = \dfrac{1}{4}$

$y = x - \dfrac{1}{4} + \dfrac{1}{2} = x + \dfrac{1}{4}$

c) $f(x) = x \ln x$

$f'(x_0) = 3$

$f'(x) = \ln x + 1$

$\ln x_0 = 2$

$x_0 = e^2$

$2x + 6y - 1 = 0$

$6y = -2x + 1$

$y = -\frac{1}{3}x + \frac{1}{6}$

$a = 3$

$y = 3(x - e^2) + e^2 \ln e^2$

$y = 3x - 3e^2 + 2e^2 = 3x - e^2$

d) $f(x) = x \arctan\frac{1}{x}$ $\quad \pi x = 4y \quad y = \frac{\pi}{4}x$

$\cancel{x}\arctan\frac{1}{x} = \frac{\pi}{4}\cancel{x}$

$\arctan\frac{1}{x} = \frac{\pi}{4}$

$\tan\frac{\pi}{4} = \frac{1}{x} \qquad x_0 = 1$

$f'(x) = \arctan\frac{1}{x} + \frac{x}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) = \arctan\frac{1}{x} - \frac{1}{x+\frac{1}{x}} = \arctan\frac{1}{x} - \frac{x}{x^2+1}$

$f'(1) = \arctan 1 - \frac{1}{1+1} = \frac{\pi}{4} - \frac{1}{2}$

$y = \left(\frac{\pi}{4} - \frac{1}{2}\right)(x-1) + \frac{\pi}{4} = \left(\frac{\pi}{4} - \frac{1}{2}\right)x + \frac{1}{2}$

$y = \left(\frac{\pi}{4} - \frac{1}{2}\right)(x-1) + \frac{\pi}{4} = \left(\frac{\pi}{4} - \frac{1}{2}\right)x + \frac{1}{2}$

e) $f(x) = \sin 2x - \cos 3x$ $\qquad x_0 = 0$

$f'(x) = 2\cos 2x + 3\sin 3x$

$f'(0) = 2$

$y = 2x - 1$

6. Korzystając z twierdzenia o pochodnej funkcji odwrotnej oblicz $(f^{-1})'(y_0)$, jeżeli:
a) $f(x) = x + \ln x$, $y_0 = e + 1$; b) $f(x) = \cos x - 3x$, $y_0 = 1$;
c) $f(x) = \sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x}$, $y_0 = 3$; d) $f(x) = x^3 + 3x$, $y_0 = 4$.

a) $f(x) = x + \ln(x)$; $y_0 = e+1$

$y_0 = f(x_0)$

$f'(x) = 1 + \frac{1}{x}$

$x_0 + \ln(x_0) = e + 1$

$x_0 = e$

$f'(x_0) = 1 + \frac{1}{e}$

$(f^{-1}(y_0))' = \frac{1}{f'(x_0)} = \frac{1}{1+\frac{1}{e}} = \frac{e}{e+1}$

b) $f(x) = \cos x - 3x$ $y_0 = 1$

$f(x_0) = 1$ $f'(x) = -\sin x - 3$

$\cos x_0 - 3x_0 = 1$ $(f^{-1}(y_0))' =$

$x_0 = 0$

$(f^{-1}(y_0))' = \frac{1}{f'(x_0)} = -\frac{1}{3}$

c) $f(x) = \sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x}$, $y_0 = 3$

$f(x_0) = 3$ $\qquad$ $f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

$\sqrt[3]{x_0} + \sqrt[5]{x_0} + \sqrt[7]{x_0} = 3$

$x_0 = 1$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} + \frac{1}{5} x^{-\frac{4}{5}} + \frac{1}{7} x^{-\frac{6}{7}}$$

$$f'(x_0) = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{71}{105}$$

$$\left(f^{-1}(y_0)\right)' = \frac{1}{\frac{71}{105}} = \frac{105}{71}$$

d) $f(x) = x^3 + 3x$ $\qquad$ $y_0 = 4$

$x_0^3 + 3x_0 = 4$ $\qquad$ $f'(x) = 3x^2 + 3$

$x_0 = 1$ $\qquad$ $f'(x_0) = 6$

$$\left(f^{-1}(y_0)\right)' = \frac{1}{6}$$