Derivation of symmetrized $\frac{\delta F}{\delta E^*_{ij}}$ in terms of E_{ij} and its derivatives

1 Initial setup

1.1 Functional derivatives and independent variables

Here I assume all the components of $\underline{\underline{E}}$ are independent, except symmetry, and that $\underline{\underline{E}}$ and $\underline{\underline{E}}^*$ are independent as usual with Wirtinger derivatives. The reason for this is essentially as symmetry is easy to include, unlike the others which will be enforced for the dynamics using Lagrange multipliers, at least for now.

This can be achieved by a suitable choice of $\frac{\partial E_{ij}}{\partial E_{ab}}$ (and respectively for gradients), in the previous derivation I used $\delta_{ia}\delta_{jb}$. Here something of the form $\frac{\partial E_{ij}}{\partial E_{ab}} = \kappa(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})$ seems like a logical choice, with κ being there due to the following problem.

This form however can't quite be made right for all i, j, a, b. Consider $i = j$, then the only non-zero choice for a, b is $a = b = i = j$, this then gets a contribution of 2κ, but it should be 1 as clearly $\frac{\partial E_{ii}}{\partial E_{ii}} = \frac{\partial x}{\partial x} = 1$ (no sums). Thus to get the $i = j$ terms right we would need $\kappa = \frac{1}{2}$.

However then for any $i \neq j$ we would get two non-zero options, either $a = i, b = j$ or $a = j, b = i$, for each of these the formula above gives κ. However for each of these we should get $\frac{\partial E_{ij}}{\partial E_{ij}} = \frac{\partial E_{ij}}{\partial E_{ij}} = 1$, this means κ would have to be 1.

Essentially the diagonal and off-didagonal elements of $\underline{\underline{E}}$ would have to be treated differently, all of this can be nicely summarized by table 1.

	11	22	33	12	21	13	31	23	32
11	1	0	0	0	0	0	0	0	0
22	0	1	0	0	0	0	0	0	0
33	0	0	1	0	0	0	0	0	0
12	0	0	0	1	1	0	0	0	0
21	0	0	0	1	1	0	0	0	0
13	0	0	0	0	0	1	1	0	0
31	0	0	0	0	0	1	1	0	0
23	0	0	0	0	0	0	0	1	1
32	0	0	0	0	0	0	0	1	1

Table 1: The results of $\frac{\partial E_{ij}}{\partial E_{ab}}$ for a symmetric $\underline{\underline{E}}$, rows and columns correspond to the ij and ab tuples and the table values to the partial derivative, clearly there is a difference between the $i = j$ and $i \neq j$ terms.

So in this work I adopt using $\frac{\partial E_{ij}}{\partial E_{ab}} = \kappa(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})$ and $\frac{\partial E_{ij,k}}{\partial E_{abc}} = \kappa(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})\delta_{kc}$ and leave κ unspecified, hoping that we can then set it to 1 or $\frac{1}{2}$ depending on if $i = j$ in $\frac{\delta F}{\delta E^*_{ij}}$

1.2 Free energies and projection operators

All of this except the extra term in eq. (3) are directly from Jack's work, as are both forms for $\underline{\underline{\Pi}}$, though only eq. (7) was used.

$$F = \int f_{\text{bulk}} + f_{\text{comp}} + f_{\text{curv}} \, \mathrm{d}V = F_{\text{bulk}} + F_{\text{comp}} + F_{\text{curv}} \tag{1}$$

$$f_{\text{bulk}} = \frac{A}{2} E_{ij} E^*_{ij} + \frac{C}{4} (E_{ij} E^*_{ij})^2 \tag{2}$$

$$f_{\text{comp}} = b_1^{\parallel} \Pi_{kl} E_{ij,k} E^*_{ij,l} + b_1^{\perp} T_{kl} E_{ij,k} E^*_{ij,l} \quad \text{maybe try adding} \quad b_1^d E_{ij,j} E^*_{ik,k} \quad \text{later too} \tag{3}$$

$$f_{\text{curv}} = b_2^{\parallel} \Pi_{kl} E_{ij,lk} \Pi_{mn} E^*_{ij,nm} + b_2^{\perp} T_{kl} E_{ij,lk} T_{mn} E^*_{ij,nm} + b_2^{\parallel\perp} (\Pi_{kl} E_{ij,lk} T_{mn} E^*_{ij,nm} + T_{kl} E_{ij,lk} \Pi_{mn} E^*_{ij,nm}) \tag{4}$$

where

$$\underline{\underline{\Pi}} = \underline{N}\,\underline{N} \qquad \text{and} \qquad \underline{\underline{T}} = \underline{\underline{\delta}} - \underline{\underline{\Pi}} \tag{5}$$

are the projection operators. We need to express these using $\underline{\underline{E}}$ as well, there are 2 options which I quote here

$$\underline{\underline{\Pi}} = \frac{d-1}{d-2} \left(\frac{\underline{\underline{E}} \cdot \underline{\underline{E}}^*}{\underline{\underline{E}} : \underline{\underline{E}}^*} - \frac{\underline{\underline{\delta}}}{d(d-1)} \right) \quad \text{or} \tag{6}$$

$$\underline{\underline{\Pi}} = \sqrt{\frac{d-1}{d \underline{\underline{E}} : \underline{\underline{E}}}} \underline{\underline{E}} - \frac{\underline{\underline{\delta}}}{d} \quad \text{which has a complex square root} \tag{7}$$

$\underline{\underline{T}}$ just being calculated from $\underline{\underline{\Pi}}$.

2 F_{bulk}

2.1 Using κ

$$\frac{\delta F_{\text{bulk}}}{\delta E^*_{ij}} = \frac{A\kappa}{2}(E_{ij} + E_{ji}) + \frac{B\kappa}{4} 2(E_{ab}E^*_{ab})(E_{ij} + E_{ji}) \tag{8}$$

$$= \kappa(A + BE_{ab}E^*_{ab})E_{ij} \quad \text{using symmetry of } \underline{\underline{E}} \tag{9}$$

2.2 Treating the $i = j$ and $i \neq j$ separately, as according to table 1

With **no sum on** $\boldsymbol{i}$:

$$\frac{\delta F_{\text{bulk}}}{\delta E^*_{ii}} = \frac{A}{2}E_{ii} + \frac{B}{4} 2(E_{ab}E^*_{ab})E_{ii} \tag{10}$$

$$= (A + BE_{ab}E^*_{ab})\frac{E_{ii}}{2} \tag{11}$$

and clearly

$$\frac{\delta F_{\text{bulk}}}{\delta E^*_{ij}} = (A + BE_{ab}E^*_{ab})E_{ij} \tag{12}$$

3 $F_{\mathbf{comp}}$ using eq. (7)

(This lead to a symmetric result even without requiring symmetry of $\underline{\underline{E}}$, so this should lead to the same result)

To simplify use $s = \sqrt{\frac{d-1}{d}}$ to get

$$\Pi_{kl} = s(E_{ab}E_{ab})^{-\frac{1}{2}} E_{kl} - \frac{\delta_{kl}}{d} \quad \text{and recall} \quad f_{\text{comp}} = (b_1^{||} - b_1^{\perp})\Pi_{kl} E_{ij,k} E^*_{ij,l} + b_1^{\perp} E_{ij,k} E^*_{ij,k} \tag{13}$$

So to get $\frac{\delta F_{\text{comp}}}{\delta E^*_{ij}}$, the first term in the Euler-Lagrange equations will be 0 again as only gradients of $\underline{\underline{E}}$ appear directly in the form of f_{comp} and $\underline{\underline{\Pi}}$ only has non-conjugated elements of $\underline{\underline{E}}$ appear. Thus

$$\frac{\delta F_{\text{comp}}}{\delta E^*_{ij}} = -\partial_k \frac{\partial f_{\text{comp}}}{\partial E_{ij,k}} = -\partial_k \Big((b_1^{||} - b_1^{\perp}) \Pi_{cd} E_{ab,c} \kappa (\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}) \delta_{kd} + b_1^{\perp} E_{ab,c} \kappa (\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}) \delta_{kc} \Big) \tag{14}$$

$$= -\kappa \partial_k \Big((b_1^{||} - b_1^{\perp}) \Pi_{ck} (E_{ij,c} + E_{ji,c}) + (E_{ij,k} + b_1^{\perp} E_{ji,k}) \Big) \tag{15}$$

$$= -2\kappa \partial_k \Big((b_1^{||} - b_1^{\perp}) \Pi_{ck} E_{ij,c} + b_1^{\perp} E_{ij,k} \Big) \tag{16}$$

$$= -2\kappa \Big((b_1^{||} - b_1^{\perp}) \Pi_{ck,k} E_{ij,c} + (b_1^{||} - b_1^{\perp}) \Pi_{ck} E_{ij,ck} + b_1^{\perp} E_{ij,kk} \Big) \tag{17}$$

So we need

$$\Pi_{ck,k} = \partial_k \left(s(E_{ab}E_{ab})^{-\frac{1}{2}} E_{ck} - \frac{\delta_{ck}}{d} \right) = s \partial_k \Big((E_{ab}E_{ab})^{-\frac{1}{2}} E_{ck} \Big) \tag{18}$$

$$= s \left((E_{ab}E_{ab})^{-\frac{1}{2}} E_{ck,k} - \frac{1}{2} (E_{ab}E_{ab})^{-\frac{3}{2}} E_{ck} 2 E_{ab} E_{ab,k} \right) \tag{19}$$

$$= s \Big((E_{ab}E_{ab})^{-\frac{1}{2}} E_{ck,k} - (E_{ab}E_{ab})^{-\frac{3}{2}} E_{ck} E_{ab} E_{ab,k} \Big) \tag{20}$$

$$= \frac{s}{\sqrt{E_{ab}E_{ab}}} \left(E_{ck,k} - \frac{E_{ab}E_{ab,k}}{E_{ab}E_{ab}} E_{ck} \right) \tag{21}$$

so together we get

$$\frac{\delta F_{\text{comp}}}{\delta E^*_{ij}} = -2\kappa \left(\frac{s(b_1^{||} - b_1^{\perp})}{\sqrt{E_{ab}E_{ab}}} \left(E_{ck,k} - \frac{E_{ab}E_{ab,k}}{E_{ab}E_{ab}} E_{ck} \right) E_{ij,c} + (b_1^{||} - b_1^{\perp}) E_{ij,ck} + b_1^{\perp} E_{ij,kk} \right) \tag{22}$$

4 The extra term in $F_{\mathbf{comp}}$

Should be easy so might as well

$$\frac{\delta}{\delta E^*_{ij}} \; b_1^d E_{ab,b} E^*_{ac,c} = b_1^d \partial_d E_{ab,b} \frac{\partial E^*_{ac,c}}{\partial E^*_{ij,d}} \tag{23}$$

$$
\begin{aligned}
&= b_1^d \partial_d E_{ab,b} \kappa (\delta_{ia}\delta_{jc} + \delta_{ic}\delta_{ja}) \delta_{cd} && (24)\\
&= b_1^d \kappa \partial_d (E_{ib,b}\delta_{jc}\delta_{cd} + E_{jb,b}\delta_{ic}\delta_{cd}) && (25)\\
&= b_1^d \kappa \partial_d (E_{ib,b}\delta_{jd} + E_{jb,b}\delta_{id}) && (26)\\
&= b_1^d \kappa (E_{ib,bd}\delta_{jd} + E_{jb,bd}\delta_{id}) && (27)\\
&= b_1^d \kappa (E_{ib,bj} + E_{jb,bi}) && (28)\\
& && (29)
\end{aligned}
$$

Agress with the result for unconstrained $\underline{\underline{E}}$ except κ, as expected.

5 F_{curv} using eq. (7)

Starting from

$$\Pi_{kl} = s(E_{ab}E_{ab})^{-\frac{1}{2}}E_{kl} - \frac{\delta_{kl}}{d} \quad \text{and} \tag{30}$$

$$f_{\text{curv}} = b_2^{\|}\Pi_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm} + b_2^{\perp}T_{kl}E_{ij,lk}T_{mn}E^*_{ij,nm} + b_2^{\|\perp}(\Pi_{kl}E_{ij,lk}T_{mn}E^*_{ij,nm} + T_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm}) \tag{31}$$

from section 1.2, where $s = \sqrt{\frac{d-1}{d}}$.

5.1 Starting off

Now to get $\frac{\delta F_{\text{curv}}}{\delta E^*_{ij}}$, we can no longer use the same Euler-Lagrange equation as in the previous sections as here our "Lagrangian" depends on second derivatives. If we use the "integral/delta function" method, integrate by parts and take all boundary terms to zero, it follows that

$$\frac{\delta}{\delta\psi(\underline{r})}\int f(\psi(\underline{r'}), \underline{\nabla}\psi(\underline{r'}), \underline{\nabla}\underline{\nabla}\psi(\underline{r'}))dV' = \frac{\partial f}{\partial\psi} - \underline{\nabla}\cdot\frac{\partial f}{\partial\underline{\nabla}\psi} + \underline{\nabla}\cdot\underline{\nabla}\cdot\frac{\partial f}{\partial\underline{\nabla}\underline{\nabla}\psi} \tag{32}$$

$$\text{or} \quad = \frac{\partial f}{\partial\psi} - \partial_\alpha\cdot\frac{\partial f}{\partial(\partial_\alpha\psi)} + \partial_\alpha\partial_\beta\frac{\partial f}{\partial(\partial_\alpha\partial_\beta\psi)} \tag{33}$$

$$\tag{34}$$

Adapting to $\underline{\underline{E}}$, we get

$$\frac{\delta F_{\text{curv}}}{\delta E^*_{ij}} = \frac{\partial f_{\text{curv}}}{\partial E^*_{ij}} - \partial_k\frac{\partial f_{\text{curv}}}{\partial E^*_{ij,k}} + \partial_k\partial_l\frac{\partial f_{\text{curv}}}{\partial E^*_{ij,kl}} \tag{35}$$

where the first two terms are 0, so we just need the last one, but start by simplifying f_{curv} to only have $\underline{\underline{\Pi}}$

$$f_{\text{curv}} = b_2^{\|}\Pi_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm} + b_2^{\perp}T_{kl}E_{ij,lk}T_{mn}E^*_{ij,nm} + b_2^{\|\perp}(\Pi_{kl}E_{ij,lk}T_{mn}E^*_{ij,nm} + T_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm}) \tag{36}$$

$$= b_2^{\|}\Pi_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm} + b_2^{\perp}(\delta_{kl} - \Pi_{kl})E_{ij,lk}(\delta_{mn} - \Pi_{mn})E^*_{ij,nm} \tag{37}$$

$$+ \; b_2^{\|\perp}(\Pi_{kl}E_{ij,lk}(\delta_{mn} - \Pi_{mn})E^*_{ij,nm} + (\delta_{kl} - \Pi_{kl})E_{ij,lk}\Pi_{mn}E^*_{ij,nm})$$

$$= (b_2^{\|} + b_2^{\perp})\Pi_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm} + b_2^{\perp}(E_{ij,kk}E^*_{ij,mm} - \Pi_{kl}E_{ij,kl}E^*_{ij,mm} - \Pi_{mn}E_{ij,kk}E^*_{ij,mn}) \tag{38}$$

$$+ \; b_2^{\|\perp}(\Pi_{kl}E_{ij,lk}E^*_{ij,mm} + \Pi_{mn}E_{ij,kk}E^*_{ij,nm} - 2\Pi_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm})$$

$$= (b_2^{\|} + b_2^{\perp} - 2b_2^{\|\perp})\Pi_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm} \tag{39}$$

$$+ \; (b_2^{\|\perp} - b_2^{\perp})(\Pi_{kl}E_{ij,lk}E^*_{ij,mm} + \Pi_{mn}E_{ij,kk}E^*_{ij,nm}) + b_2^{\perp}E_{ij,kk}E^*_{ij,mm}$$

5.2 Dealing with derivatives of second order derivatives

Next we want to start taking the derivatives with respect to $E^*_{ij,kl}$, however we have another problem, we have found what this is in section 1.1, the first guess would be $\frac{\partial E_{ij,kl}}{\partial E_{ab,cd}} = \kappa(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})\delta_{kc}\delta_{ld}$, however this has the same problem as the i, j indices. As derivatives commute we have say $E_{ij,12} = E_{ij,21}$, however the mentioned derivative gives 0. We can symmetrize it as before, introducing another κ to fix the $c = d$ or $c \neq d$ issue, however this κ is not the same as the other one at that point, it depends on cd, not ij so I introduce $\kappa^{ij} = \frac{1}{1+\delta_{ij}}$ (this gives the right values as discussed earlier) and I up the indices upstairs as they sadly make a mess of the summation convention, this has nothing to do with co/contra variance, it's just for me to keep track of them in a clear way without confusing which indices are to be summed over. With this we have $\frac{\partial E_{ij,kl}}{\partial E_{ab,cd}} = \kappa^{ab}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})\kappa^{cd}(\delta_{kc}\delta_{ld} + \delta_{kd}\delta_{lc})$ with no sums over any variables.

I first take the partial derivative of the terms in eq. (39) separately, start with

$$\frac{\partial}{\partial E^*_{ab,cd}}\Pi_{kl}E_{ij,lk}\Pi_{mn}E^*_{ij,nm} = \kappa^{ab}\kappa^{cd}\Pi_{kl}E_{ij,lk}\Pi_{mn}(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja})(\delta_{nc}\delta_{md} + \delta_{nd}\delta_{mc}) \tag{40}$$

$$= \kappa^{ab}\kappa^{cd}\Pi_{kl}(E_{ab,lk} + E_{ba,lk})(\Pi_{cd} + \Pi_{dc}) \tag{41}$$

$$= 4\kappa^{ab}\kappa^{cd}\Pi_{kl}E_{ab,lk}\Pi_{cd} \quad \text{as } \underline{\underline{E}} \text{ and } \underline{\underline{\Pi}} \text{ are symmetric} \tag{42}$$

The next one raises some doubts, but I believe blindly applying the form above works

$$\frac{\partial}{\partial E^*_{ab,cd}}\Pi_{kl}E_{ij,lk}E^*_{ij,mm} = \kappa^{ab}\kappa^{cd}\Pi_{kl}E_{ij,lk}(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja})(\delta_{mc}\delta_{md} + \delta_{md}\delta_{mc}) \tag{43}$$

$$= 2\kappa^{ab}\kappa^{cd}\Pi_{kl}E_{ab,lk}2\delta_{cd} \tag{44}$$

$$= 2\kappa^{ab}\Pi_{kl}E_{ab,lk}\delta_{cd} \quad \text{given the form of } \kappa^{ij} \tag{45}$$

to justify the result, consider $\frac{\partial}{\partial\partial_i\partial_j\psi}\nabla^2\psi = \frac{\partial}{\partial\partial_i\partial_j\psi}\sum_k \partial_k\partial_k\psi$ which is clearly 1 if $i = j$ and 0 otherwise, so $\frac{\partial}{\partial\partial_i\partial_j\psi}\nabla^2\psi = \delta_{ij}$, the result above is analogous to this. Continuing we also have

$$\frac{\partial}{\partial E^*_{ab,cd}}\Pi_{mn}E_{ij,kk}E^*_{ij,nm} = \kappa^{ab}\kappa^{cd}\Pi_{mn}E_{ij,kk}(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja})(\delta_{nc}\delta_{md} + \delta_{nd}\delta_{mc}) \tag{46}$$

$$= 4\kappa^{ab}\kappa^{cd}\Pi_{cd}E_{ab,kk} \tag{47}$$

and

$$\frac{\partial}{\partial E^*_{ab,cd}}E_{ij,kk}E^*_{ij,mm} = \kappa^{ab}\kappa^{cd}E_{ij,kk}(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja})(\delta_{mc}\delta_{md} + \delta_{md}\delta_{mc}) \tag{48}$$

$$= 4\kappa^{ab}\kappa^{cd}E_{ab,kk}\delta_{cd} \tag{49}$$

$$= 2\kappa^{ab}E_{ab,kk}\delta_{cd} \tag{50}$$

5.3 Putting it all together

$$\begin{aligned}
\frac{\partial f_{\rm curv}}{\partial E^*_{ab,cd}} &= 4(b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})\kappa^{ab}\kappa^{cd}\Pi_{kl}E_{ab,lk}\Pi_{cd} && (51)\\
&+ (b_2^{\|\perp}-b_2^{\perp})(2\kappa^{ab}\Pi_{kl}E_{ab,lk}\delta_{cd}+4\kappa^{ab}\kappa^{cd}\Pi_{cd}E_{ab,kk})+2b_2^{\perp}\kappa^{ab}E_{ab,kk}\delta_{cd}\\
&\quad \text{expecting the } \kappa^{cd} \text{ to be a problem use the following}\\
\frac{\partial f_{\rm curv}}{\partial E^*_{ab,cd}} &= 2\kappa^{ab}\delta_{cd}((b_2^{\|\perp}-b_2^{\perp})\Pi_{kl}E_{ab,lk}+b_2^{\perp}E_{ab,kk}) && (52)\\
&+ 4\kappa^{ab}\kappa^{cd}\Pi_{cd}((b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})\Pi_{kl}E_{ab,lk}+(b_2^{\|\perp}-b_2^{\perp})E_{ab,kk}) && (53)
\end{aligned}$$

so

$$\begin{aligned}
\frac{\delta F_{\rm curv}}{\delta E^*_{ab}} &= \partial_c\partial_d\frac{\partial f_{\rm curv}}{\partial E^*_{ab,cd}} && (54)\\
&= 2\kappa^{ab}\partial_c\partial_c((b_2^{\|\perp}-b_2^{\perp})\Pi_{kl}E_{ab,lk}+b_2^{\perp}E_{ab,kk}) && (55)\\
&+ 4\kappa^{ab}\kappa^{cd}\partial_c\partial_d(\Pi_{cd}((b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})\Pi_{kl}E_{ab,lk}+(b_2^{\|\perp}-b_2^{\perp})E_{ab,kk})) && (56)\\
&= 2\kappa^{ab}\partial_c((b_2^{\|\perp}-b_2^{\perp})(\Pi_{kl,c}E_{ab,lk}+\Pi_{kl}E_{ab,lkc})+b_2^{\perp}E_{ab,kkc}) && (57)\\
&+ 4\kappa^{ab}\kappa^{cd}\partial_c(\Pi_{cd}((b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})(\Pi_{kl,d}E_{ab,lk}+\Pi_{kl}E_{ab,lkd})+(b_2^{\|\perp}-b_2^{\perp})E_{ab,kkd}) && (58)\\
&\quad +\Pi_{cd,d}((b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})\Pi_{kl}E_{ab,lk}+(b_2^{\|\perp}-b_2^{\perp})E_{ab,kk})) && (59)\\
&= 2\kappa^{ab}((b_2^{\|\perp}-b_2^{\perp})(\Pi_{kl,cc}E_{ab,lk}+2\Pi_{kl,c}E_{ab,lkc}+\Pi_{kl}E_{ab,lkcc})+b_2^{\perp}E_{ab,kkcc}) && (60)\\
&+ 4\kappa^{ab}\kappa^{cd}(\Pi_{cd}((b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})(\Pi_{kl,dc}E_{ab,lk}+\Pi_{kl,d}E_{ab,lkc}+\Pi_{kl,c}E_{ab,lkd}+\Pi_{kl}E_{ab,lkdc}) && (61)\\
&\quad +(b_2^{\|\perp}-b_2^{\perp})E_{ab,kkdc}) && (62)\\
&\quad +\Pi_{cd,c}((b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})(\Pi_{kl,d}E_{ab,lk}+\Pi_{kl}E_{ab,lkd})+(b_2^{\|\perp}-b_2^{\perp})E_{ab,kkd}) && (63)\\
&\quad +\Pi_{cd,d}((b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})(\Pi_{kl,c}E_{ab,lk}+\Pi_{kl}E_{ab,lkc})+(b_2^{\|\perp}-b_2^{\perp})E_{ab,kkc}) && (64)\\
&\quad +\Pi_{cd,cd}((b_2^{\|}+b_2^{\perp}-2b_2^{\|\perp})\Pi_{kl}E_{ab,lk}+(b_2^{\|\perp}-b_2^{\perp})E_{ab,kk})) && (65)\\
& && (66)
\end{aligned}$$