Showing that requiring normality, symmetry and eigenvalues gives a form for $\underline{\underline{E}}$

As motivated in Jack's thesis, if we **take the eigenvalues of $\underline{\underline{E}}$ to be proportional to ψ, add to zero** (to make $\underline{\underline{E}}$ traceless) **and all except one be equal** (nearly fully degenerate) to choose one "special" axis. Then they must all be $\frac{-\psi}{d}$ except the special one which is $\frac{(d-1)\psi}{d}$ (up to scaling). If we further **require $\underline{\underline{E}}$ to be normal**, then it can be diagonalised using a unitary matrix (this I got from wikipedia), such that

$$\underline{\underline{E}} = \underline{\underline{U}}^\dagger \begin{pmatrix} \frac{-\psi}{d} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{(d-1)\psi}{d} \end{pmatrix} \underline{\underline{U}} \quad \text{where } \underline{\underline{U}} \text{ is unitary} \tag{1}$$

$$= \frac{\psi}{d} \underline{\underline{U}}^\dagger \begin{pmatrix} -1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d-1 \end{pmatrix} \underline{\underline{U}} = \frac{\psi}{d} \underline{\underline{U}}^\dagger \left(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d \end{pmatrix} - \underline{\underline{\delta}} \right) \underline{\underline{U}} \tag{2}$$

$$= \frac{\psi}{d} \left(\underline{\underline{U}}^\dagger \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d \end{pmatrix} \underline{\underline{U}} - \underline{\underline{\delta}} \right) \tag{3}$$

$$\tag{4}$$

now switching to index notation

$$E_{ij} = \psi \left(U^\dagger_{id} U_{dj} - \frac{\delta_{ij}}{d} \right) \quad d \text{ is the dimension here, not a dummy index} \tag{5}$$

$$= \psi \left(U^*_{di} U_{dj} - \frac{\delta_{ij}}{d} \right) \tag{6}$$

So only a single (the last) row of $\underline{\underline{U}}$ has any effect. To move further, recall that the **rows (or columns) or a unitary matrix for a complex orthonormal basis**. Thus, if we are only considering a single row, the only constraint that applies is that it's nor has to be 1. I also switch to **specifically considering $d = 3$** here to make it simpler, but I expect it's general. Parametrize this last row as follows

$$\underline{U_3} \leftrightarrow \begin{pmatrix} R_a e^{i\phi_a} & R_b e^{i\phi_b} & R_c e^{i\phi_c} \end{pmatrix} \quad \text{with} \quad R_a, R_b, R_c \geq 0 \quad \text{and} \quad R_a^2 + R_b^2 + R_c^2 = 1 \tag{7}$$

and

$$\underline{U_3^*} \leftrightarrow \begin{pmatrix} R_a e^{-i\phi_a} & R_b e^{-i\phi_b} & R_c e^{-i\phi_c} \end{pmatrix} \tag{8}$$

this gives $\underline{\underline{E}}$ as

$$\underline{\underline{E}} \leftrightarrow \psi \left(\begin{pmatrix} R_a^2 & R_a R_b e^{i(\phi_b - \phi_a)} & R_a R_c e^{i(\phi_c - \phi_a)} \\ R_a R_b e^{-i(\phi_b - \phi_a)} & R_b^2 & R_b R_c e^{i(\phi_c - \phi_b)} \\ R_a R_c e^{-i(\phi_c - \phi_a)} & R_b R_c e^{-i(\phi_c - \phi_b)} & R_c^2 \end{pmatrix} - \frac{\underline{\underline{\delta}}}{d} \right) \tag{9}$$

Finally, **requiring $\underline{\underline{\boldsymbol{E}}}$ to be symmetric** means each of the phase differences above must be an integer multiple of π (as $e^{i\theta} = e^{-i\theta}$ iff θ is an integer multiple of π). Using $\phi_b - \phi_a = k\pi$ and $\phi_c - \phi_a = n\pi$ gives $\phi_c - \phi_b = (n-k)\pi$. Focusing on the matrix in eq. (9), noting that the terms in it only depend on whether each of n, k, $n-k$ are odd or even we end up with 4 options for $\underline{\underline{E}}$, each equivalent to $\psi\left(\underline{R}\underline{R} - \frac{\underline{\underline{\delta}}}{d}\right)$ with a different $\underline{R}$.

k	n	$n-k$		The term appearing in $\underline{\underline{E}}$	$\underline{R}$
even	even	even	$\leftrightarrow$	$\begin{pmatrix} R_a^2 & R_aR_b & R_aR_c \\ R_aR_b & R_b^2 & R_bR_c \\ R_aR_c & R_bR_c & R_c^2 \end{pmatrix}$	$\begin{pmatrix} R_a & R_b & R_c \end{pmatrix}$
even	odd	odd	$\leftrightarrow$	$\begin{pmatrix} R_a^2 & R_aR_b & -R_aR_c \\ R_aR_b & R_b^2 & -R_bR_c \\ -R_aR_c & -R_bR_c & R_c^2 \end{pmatrix}$	$\begin{pmatrix} R_a & R_b & -R_c \end{pmatrix}$
odd	even	odd	$\leftrightarrow$	$\begin{pmatrix} R_a^2 & -R_aR_b & R_aR_c \\ -R_aR_b & R_b^2 & -R_bR_c \\ R_aR_c & -R_bR_c & R_c^2 \end{pmatrix}$	$\begin{pmatrix} R_a & -R_b & R_c \end{pmatrix}$
odd	odd	even	$\leftrightarrow$	$\begin{pmatrix} R_a^2 & -R_aR_b & -R_aR_c \\ -R_aR_b & R_b^2 & R_bR_c \\ -R_aR_c & R_bR_c & R_c^2 \end{pmatrix}$	$\begin{pmatrix} -R_a & R_b & R_c \end{pmatrix}$

Table 1: The 4 options for $\underline{\underline{E}}$

Finally recalling that each of the $R_?$ components are positive and that $R_a^2 + R_b^2 + R_c^2 = 1$, we thus get that any $\underline{\underline{E}}$ must be of the form $\psi\left(\underline{N}\underline{N} - \frac{\underline{\underline{\delta}}}{d}\right)$ for some unit vector $\underline{N}$ pointing into one of the 4 quadrants where at most 1 Cartesian component is negative, which spans exactly the rotational symmetry we require of the system.