Derivation of $\frac{\delta F}{\delta E^*_{ij}}$ in terms of E_{ij} and its derivatives – Old version using $\frac{\delta E_{ij}}{\delta E_{ab}} = \delta_{ai}\delta_{bj}$, most probably wrong!

1 Initial setup

$$
\begin{aligned}
F &= \int f_{\text{bulk}} + f_{\text{comp}} + f_{\text{curv}}\,\mathrm{d}V = F_{\text{bulk}} + F_{\text{comp}} + F_{\text{curv}} && (1)\\
f_{\text{bulk}} &= \frac{A}{2}E_{ij}E^*_{ij} + \frac{C}{4}(E_{ij}E^*_{ij})^2 && (2)\\
f_{\text{comp}} &= b_1^{\parallel}\Pi_{kl}E_{ij,k}E^*_{ij,l} + b_1^{\perp}T_{kl}E_{ij,k}E^*_{ij,l} \quad \text{maybe try adding} \quad b_1^d E_{ij,j}E^*_{ik,k} \quad \text{later too} && (3)\\
f_{\text{curv}} &= \ldots \quad \text{for later} \quad \ldots && (4)\\
& && (5)
\end{aligned}
$$

where

$$
\underline{\underline{\Pi}} = \underline{N}\,\underline{N} \qquad \text{and} \qquad \underline{\underline{T}} = \underline{\underline{\delta}} - \underline{\underline{\Pi}} \tag{6}
$$

are the projection operators. We need to express these using $\underline{\underline{E}}$ as well, there are 2 options which I quote here

$$
\begin{aligned}
\underline{\underline{\Pi}} &= \frac{d-1}{d-2}\left(\frac{\underline{\underline{E}}\cdot\underline{\underline{E}}^*}{\underline{\underline{E}}:\underline{\underline{E}}^*} - \frac{\underline{\underline{\delta}}}{d(d-1)}\right) \quad \text{or} && (7)\\
\underline{\underline{\Pi}} &= \sqrt{\frac{d-1}{d\underline{\underline{E}}:\underline{\underline{E}}}}\,\underline{\underline{E}} + \frac{\underline{\underline{\delta}}}{d} \quad \text{which has a complex square root} && (8)
\end{aligned}
$$

$\underline{\underline{T}}$ just being calculated from $\underline{\underline{\Pi}}$.

2 $\frac{\delta F_{\mathbf{comp}}}{\delta E^*_{ij}}$ using eq. (7) for $\underline{\underline{\Pi}}$ and the Euler-Lagrange derivative

As $\frac{\delta F_{\text{bulk}}}{\delta E^*_{ij}}$ has been calculated before and it is the easier one, I move straight to the compression contribution. There the function in the integral only depends on the first derivatives of $\underline{\underline{E}}$ so we can use the usual Euler-Lagrange expression for calculating the functional.

$$
\begin{aligned}
\frac{\delta}{\delta\phi}\int f(\phi,\boldsymbol{\nabla}\phi)\,\mathrm{d}V &= \frac{\partial f}{\partial\phi} - \boldsymbol{\nabla}\cdot\frac{\partial f}{\partial(\boldsymbol{\nabla}\phi)} \quad \text{or for us} && (9)\\
\frac{\delta}{\delta E^*_{ij}}\int f_{\text{comp}}(\underline{\underline{\Pi}}(\underline{\underline{E}},\underline{\underline{E}}^*),\boldsymbol{\nabla}\underline{\underline{E}},\boldsymbol{\nabla}\underline{\underline{E}}^*)\,\mathrm{d}V &= \frac{\partial f_{\text{comp}}}{\partial E^*_{ij}} - \boldsymbol{\nabla}\cdot\frac{\partial f_{\text{comp}}}{\partial(\boldsymbol{\nabla}E^*_{ij})} && (10)
\end{aligned}
$$

Here I use eq. (7) as the expression for $\underline{\underline{\Pi}}$ furhter using $p = \frac{d-1}{d-2}$ and $q = \frac{1}{d(d-1)}$ as shorthands. I repeat the equations here for convenience

$$
\Pi_{kl} = p\left(\frac{E_{kj}E^*_{lj}}{E_{ij}E^*_{ij}} - q\delta_{kl}\right) \quad \text{and} \quad f_{\text{comp}} = b_1^{\parallel}\Pi_{kl}E_{ij,k}E^*_{ij,l} + b_1^{\perp}(\delta_{kl}-\Pi_{kl})E_{ij,k}E^*_{ij,l} \tag{11}
$$

2.1 The first term

Noticing that the only dependence that f_{comp} only depends on $\underline{\underline{E}}$ through $\underline{\underline{\Pi}}$ lets first get

$$\frac{\partial \Pi_{kl}}{\partial E^*_{ij}} = p\left(\frac{E_{kc}\delta_{li}\delta_{cj}}{E_{ab}E^*_{ab}} - \frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}E_{ab}\delta_{ai}\delta_{bj}\right) \tag{12}$$

$$= p\left(\frac{E_{kj}\delta_{li}}{E_{ab}E^*_{ab}} - \frac{E_{kc}E^*_{lc}E_{ij}}{(E_{ab}E^*_{ab})^2}\right) \tag{13}$$

Okay so here it got a little sketchy - the problem is that I always do $\frac{\partial E^*_{ij}}{\partial E^*_{ab}} = \delta_{ia}\delta_{jb}$, but I'm not sure this is compatible with $\underline{\underline{E}}$ being symmetric - then we should really have $\frac{\partial E^*_{ij}}{\partial E^*_{ab}} = \delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}$ right? HOWEVER, I'm pretty sure we want to treat the different components of $\underline{\underline{E}}$ as independent, just "coincidentally" having the same values everywhere (and so also gradients), which is maintained by the Lagrange multiplier.
Lets also try to do this using the real components of $E_{ij} = X_{ij} + iY_{ij}$

$$\frac{\partial \Pi_{kl}}{\partial E^*_{ij}} = \frac{p}{2}\left(\frac{\partial}{\partial X_{ij}} + i\frac{\partial}{\partial Y_{ij}}\right)\frac{(X_{kc}+iY_{kc})(X_{lc}-iY_{lc})}{(X_{ab}+iY_{ab})(X_{ab}-iY_{ab})} \tag{14}$$

$$= \frac{p}{2}\left(\left(\frac{\delta_{ki}\delta_{cj}(X_{lc}-iY_{lc}) + (X_{kc}+iY_{kc})\delta_{li}\delta_{cj}}{E_{ab}E^*_{ab}} + \frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}\delta_{ai}\delta_{bj}((X_{ab}-iY_{ab}) + (X_{ab}+iY_{ab}))\right)\right. \tag{15}$$

$$\left. + i\left(\frac{i\delta_{ki}\delta_{cj}(X_{lc}-iY_{lc}) - i(X_{kc}+iY_{kc})\delta_{li}\delta_{cj}}{E_{ab}E^*_{ab}} + \frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}\delta_{ai}\delta_{bj}(i(X_{ab}-iY_{ab}) - i(X_{ab}+iY_{ab}))\right)\right) \tag{16}$$

$$= \frac{p}{2}\left(\left(\frac{\delta_{ki}(X_{lj}-iY_{lj}) + (X_{kj}+iY_{kj})\delta_{li}}{E_{ab}E^*_{ab}} + \frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}2X_{ij}\right)\right. \tag{17}$$

$$\left. + \left(\frac{-\delta_{ki}(X_{lj}-iY_{lj}) + (X_{kj}+iY_{kj})\delta_{li}}{E_{ab}E^*_{ab}} - \frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}(-2iY_{ij})\right)\right) \tag{18}$$

$$= \frac{p}{2}\left(\frac{\delta_{ki}(X_{lj}-iY_{lj}) + (X_{kj}+iY_{kj})\delta_{li}}{E_{ab}E^*_{ab}} + 2X_{ij}\frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}\right. \tag{19}$$

$$\left. + \frac{-\delta_{ki}(X_{lj}-iY_{lj}) + (X_{kj}+iY_{kj})\delta_{li}}{E_{ab}E^*_{ab}} + 2iY_{ij}\frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}\right) \tag{20}$$

$$= \frac{p}{2}\left(\frac{2(X_{kj}+iY_{kj})\delta_{li}}{E_{ab}E^*_{ab}} + 2(X_{ij}+iY_{ij})\frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}\right) \tag{21}$$

$$= p\left(\frac{(X_{kj}+iY_{kj})\delta_{li}}{E_{ab}E^*_{ab}} + (X_{ij}+iY_{ij})\frac{E_{kc}E^*_{lc}}{(E_{ab}E^*_{ab})^2}\right) = p\left(\frac{E_{kj}\delta_{li}}{E_{ab}E^*_{ab}} + \frac{E_{kc}E^*_{lc}E_{ij}}{(E_{ab}E^*_{ab})^2}\right) \tag{22}$$

omg, so it's actually the same...okay I'm pretty sure this is correct then

then we get

$$\frac{\partial f_{\text{comp}}}{\partial E^*_{ij}} = b_1^{\parallel}\frac{\partial \Pi_{kl}}{\partial E^*_{ij}}E_{\alpha\beta,k}E^*_{\alpha\beta,l} - b_1^{\perp}\frac{\partial \Pi_{kl}}{\partial E^*_{ij}}E_{\alpha\beta,k}E^*_{\alpha\beta,l} \tag{23}$$

$$= (b_1^{\parallel} - b_1^{\perp})\frac{\partial \Pi_{kl}}{\partial E^*_{ij}} E_{\alpha\beta,k}E^*_{\alpha\beta,l} \tag{24}$$

$$= p(b_1^{\parallel} - b_1^{\perp})\left(\frac{E_{kj}\delta_{li}}{E_{ab}E^*_{ab}} - \frac{E_{kc}E^*_{lc}E_{ij}}{(E_{ab}E^*_{ab})^2}\right)E_{\alpha\beta,k}E^*_{\alpha\beta,l} \tag{25}$$

$$= p(b_1^{\parallel} - b_1^{\perp})\left(\frac{E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i}}{E_{ab}E^*_{ab}} - \frac{E_{kc}E^*_{lc}E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l}}{(E_{ab}E^*_{ab})^2}\right) \tag{26}$$

$$= \frac{p(b_1^{\parallel} - b_1^{\perp})}{(E_{ab}E^*_{ab})^2}\left(E_{ab}E^*_{ab}E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i} - E_{kc}E^*_{lc}E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l}\right) \tag{27}$$

$$= \frac{p(b_1^{\parallel} - b_1^{\perp})}{Y_{aa}^2}\left(Y_{aa}E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i} - Y_{kl}E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l}\right) \quad \text{with} \quad Y_{ij} = E_{ik}E^*_{jk} \tag{28}$$

which is suspicious to me as it's the first thing I encountered that is not symmetric. I also tried to first substitute in to $\underline{\underline{\Pi}}$ and then do the derivative and I got the same result, I think it is correct. Coming back to eq. (13) the first term there really encapsulates the problem, on the LHS both $\underline{\underline{\Pi}}$ and $\underline{\underline{E}}$ are symmetric but the first term on the RHS is not invariant under switching i and j or k and l. I am currently hoping that maybe it cancels with a part of the second term

2.2 The second term

Let's start working on the second term of eq. (10), firstly as the projection operators only depend on the $\underline{\underline{E}}$ and not on any of its derivatives we have

$$\frac{\partial f_{\text{comp}}}{\partial(E^*_{ab,c})} = b_1^{\parallel}\Pi_{kl}E_{ij,k}\delta_{ia}\delta_{jb}\delta_{lc} + b_1^{\perp}T_{kl}E_{ij,k}\delta_{ia}\delta_{jb}\delta_{lc} \tag{29}$$

$$= b_1^{\parallel}\Pi_{kc}E_{ab,k} + b_1^{\perp}(\delta_{kc} - \Pi_{kc})E_{ab,k} \tag{30}$$

As the next step is to take a divergence of the above, lets first calculate

$$\partial_\alpha \Pi_{\beta\gamma} = p\,\partial_\alpha\,\frac{E_{\beta j}E^*_{\gamma j}}{E_{ij}E^*_{ij}} \tag{31}$$

$$= p\left(\frac{E_{\beta j,\alpha}E^*_{\gamma j}}{E_{ij}E^*_{ij}} + \frac{E_{\beta j}E^*_{\gamma j,\alpha}}{E_{ij}E^*_{ij}} - \frac{E_{\beta j}E^*_{\gamma j}}{(E_{ij}E^*_{ij})^2}\left(E_{ij,\alpha}E^*_{ij} + E_{ij}E^*_{ij,\alpha}\right)\right) \qquad \text{or} \tag{32}$$

$$\partial_\alpha \Pi_{\beta\gamma} = p\left(\frac{Y_{\beta\gamma,\alpha}}{Y_{ii}} - \frac{Y_{\beta\gamma}Y_{ii,\alpha}}{Y_{ii}^2}\right) \quad \text{using} \quad Y_{ij} = E_{ik}E^*_{jk} \tag{33}$$

which leads to

$$\partial_c\frac{\partial f_{\text{comp}}}{\partial(E^*_{ab,c})} = \left(b_1^{\parallel}\Pi_{kc,c}E_{ab,k} - b_1^{\perp}\Pi_{kc,c}E_{ab,k}\right) + \left(b_1^{\parallel}\Pi_{kc}E_{ab,kc} + b_1^{\perp}(\delta_{kc} - \Pi_{kc})E_{ab,kc}\right) \tag{34}$$

$$= \left(b_1^{\parallel} - b_1^{\perp}\right)\Pi_{kc,c}E_{ab,k} + \left(b_1^{\perp}E_{ab,cc} + \left(b_1^{\parallel} - b_1^{\perp}\right)\Pi_{kc}E_{ab,kc}\right) \tag{35}$$

bracketing to separate the first and second order derivatives. Turns out we only need

$$\Pi_{\alpha\beta,\beta} = \partial_\beta\Pi_{\alpha\beta} = p\left(\frac{Y_{\alpha\beta,\beta}}{Y_{ii}} - \frac{Y_{\alpha\beta}Y_{ii,\beta}}{Y_{ii}^2}\right) \tag{36}$$

leading to

$$\partial_c \frac{\partial f_{\text{comp}}}{\partial (E^*_{ab,c})} = p\left(b_1^{\|} - b_1^{\perp}\right)\left(\frac{Y_{kc,c}}{Y_{ii}} - \frac{Y_{kc}Y_{ii,c}}{Y_{ii}^2}\right)E_{ab,k} + \left(b_1^{\perp}E_{ab,cc} + \left(b_1^{\|} - b_1^{\perp}\right)\Pi_{kc}E_{ab,kc}\right) \tag{37}$$

$$= \frac{p\left(b_1^{\|} - b_1^{\perp}\right)}{Y_{ii}^2}(Y_{ii}Y_{kc,c} - Y_{kc}Y_{ii,c})E_{ab,k} + \left(b_1^{\perp}E_{ab,cc} + \left(b_1^{\|} - b_1^{\perp}\right)\Pi_{kc}E_{ab,kc}\right) \tag{38}$$

2.3 Combining the terms

$$\frac{\delta f_{\text{comp}}}{\delta E^*_{ij}} = \frac{p(b_1^{\|} - b_1^{\perp})}{Y_{aa}^2}(Y_{aa}E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i} - Y_{kl}E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l}) \tag{39}$$

$$- \frac{p(b_1^{\|} - b_1^{\perp})}{Y_{aa}^2}(Y_{aa}Y_{kc,c} - Y_{kc}Y_{aa,c})E_{ij,k} - (b_1^{\perp}E_{ij,cc} + (b_1^{\|} - b_1^{\perp})\Pi_{kc}E_{ij,kc}) \tag{40}$$

$$= \frac{p(b_1^{\|} - b_1^{\perp})}{Y_{aa}^2}\Big(Y_{aa}E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i} - Y_{kl}E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l} + (Y_{kc}Y_{aa,c} - Y_{aa}Y_{kc,c})E_{ij,k}\Big) \tag{41}$$

$$- S_{ij} \quad \text{with} \quad S_{ij} = (b_1^{\perp}E_{ij,cc} + (b_1^{\|} - b_1^{\perp})\Pi_{kc}E_{ij,kc}) \quad \text{containing the second order terms} \tag{42}$$

$$= \frac{p(b_1^{\|} - b_1^{\perp})}{Y_{aa}^2}\Big(Y_{aa}(E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i} - Y_{kc,c}E_{ij,k}) + Y_{kl}(Y_{aa,l}E_{ij,k} - E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l})\Big) \tag{43}$$

$$- S_{ij} \tag{44}$$

$$= \frac{p(b_1^{\|} - b_1^{\perp})}{Y_{aa}^2}\Big(Y_{aa}(E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i} - (E_{kd,c}E^*_{cd} + E_{kd}E^*_{cd,c})E_{ij,k}) \tag{45}$$

$$+ Y_{kl}((E_{ad,l}E^*_{ad} + E_{ad}E^*_{ad,l})E_{ij,k} - E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l})\Big) \tag{46}$$

$$- S_{ij} \tag{47}$$

$$= \frac{p(b_1^{\|} - b_1^{\perp})}{Y_{aa}^2}\Big(Y_{aa}(E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i} - E^*_{cd}E_{ij,k}E_{kd,c} - E_{kd}E_{ij,k}E^*_{cd,c}) \tag{48}$$

$$+ Y_{kl}(E^*_{ad}E_{ij,k}E_{ad,l} + E_{ad}E_{ij,k}E^*_{ad,l} - E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l})\Big) \tag{49}$$

$$- S_{ij} \tag{50}$$

$$\frac{\delta f_{\text{comp}}}{\delta E^*_{ij}} = \frac{p(b_1^{\|} - b_1^{\perp})}{Y_{aa}^2}\Big(E_{af}E^*_{af}(E_{kj}E_{\alpha\beta,k}E^*_{\alpha\beta,i} - E^*_{cd}E_{ij,k}E_{kd,c} - E_{kd}E_{ij,k}E^*_{cd,c}) \tag{51}$$

$$+ E_{kf}E^*_{lf}(E^*_{ad}E_{ij,k}E_{ad,l} + E_{ad}E_{ij,k}E^*_{ad,l} - E_{ij}E_{\alpha\beta,k}E^*_{\alpha\beta,l})\Big) \tag{52}$$

$$- (b_1^{\perp}E_{ij,cc} + (b_1^{\|} - b_1^{\perp})\Pi_{kc}E_{ij,kc}) \tag{53}$$

$$\tag{54}$$

I don't see any nicer or more intuitive way to write this as of now, also I do not see it being symmetric in exchanging i and j as I thought it would. For now I am leaving this as it is, $\underline{\underline{\Pi}}$ and Y_{aa} can readily be expressed in terms of $\underline{\underline{E}}$ for computation.

3 $\frac{\delta F_{\textbf{comp}}}{\delta E^*_{ij}}$ using eq. (8) for $\underline{\underline{\Pi}}$ and the Euler-Lagrange derivative

We can still use the Euler-Lagrange version of the functional derivative as before, the only additional thing to consider is how to deal with the root – I will essentially ignore any problems with branch cuts etc. until the end, and just treat it like a real root until then.

So lets use

$$\Pi_{kl} = s\frac{E_{kl}}{\sqrt{E_{ab}E_{ab}}} + \frac{\delta_{kl}}{d} \quad \text{and} \quad f_{\text{comp}} = b_1^{\|}\Pi_{kl}E_{ij,k}E^*_{ij,l} + b_1^{\perp}(\delta_{kl} - \Pi_{kl})E_{ij,k}E^*_{ij,l} \tag{55}$$

with $s = \sqrt{\frac{d-1}{d}}$

Note that this gives $\frac{\partial \Pi_{kl}}{\partial E^*_{ij}} = 0$ as $\underline{\underline{\Pi}}$ is independent of the conjugated $\underline{\underline{E}}$! This means that reusing eq. (23) we get that the first term of the Euler-Lagrange like expression (eq. (10)) is 0. So lets get the second term, for that we can reuse eq. (35) to get

$$\frac{\delta F_{\text{comp}}}{\delta E^*_{ij}} = -\partial_c\frac{\partial f_{\text{comp}}}{\partial E^*_{ij,c}} = -\left(b_1^{\|} - b_1^{\perp}\right)\Pi_{kc,c}E_{ij,k} - \left(b_1^{\perp}E_{ij,cc} + \left(b_1^{\|} - b_1^{\perp}\right)\Pi_{kc}E_{ij,kc}\right) \tag{56}$$

so we need

$$\partial_c\Pi_{kc} = s\left(\frac{E_{kc,c}}{\sqrt{E_{ab}E_{ab}}} - \frac{E_{kc}(E_{ab,c}E_{ab} + E_{ab}E_{ab,c})}{2(E_{ab}E_{ab})^{\frac{3}{2}}}\right) \tag{57}$$

$$= \frac{s}{\sqrt{E_{ab}E_{ab}}}\left(E_{kc,c} - \frac{E_{kc}E_{ab}E_{ab,c}}{E_{ab}E_{ab}}\right) \tag{58}$$

$$\tag{59}$$

giving

$$\frac{\delta F_{\text{comp}}}{\delta E^*_{ij}} = -\frac{s}{\sqrt{E_{ab}E_{ab}}}\left(b_1^{\|} - b_1^{\perp}\right)\left(E_{kc,c} - \frac{E_{kc}E_{ab}E_{ab,c}}{E_{ab}E_{ab}}\right)E_{ij,k} - \left(b_1^{\perp}E_{ij,cc} + \left(b_1^{\|} - b_1^{\perp}\right)\Pi_{kc}E_{ij,kc}\right) \tag{60}$$

4 The divergence term of $f_{\textbf{comp}}$

This turned out a lot simpler. I do this using the Euler-Lagrange functional derivative (eq. (10)) again, though here I was a little unsure when taking the grad of a Kronecker delta, so I also tried acting it directly on the derivative and using $\frac{\delta E^*_{ab}(\underline{r})}{\delta E^*_{ij}(\underline{r}')} = \delta_{ai}\delta_{bj}\delta(\underline{r} - \underline{r}')$ and it gave the same, so I don't present it.

$$\frac{\delta}{\delta E^*_{ij}} \; b_1^d E_{ab,b}E^*_{ac,c} = b_1^d\partial_d E_{ab,b}\delta_{ia}\delta_{jc}\delta_{dc} = b_1^d E_{ib,bj} \tag{61}$$