FACULTY OF MATHEMATICS,
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3D Vision

Lecture 1: Introduction - Mathematical Background

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Contents

Traffic Camera Calibration

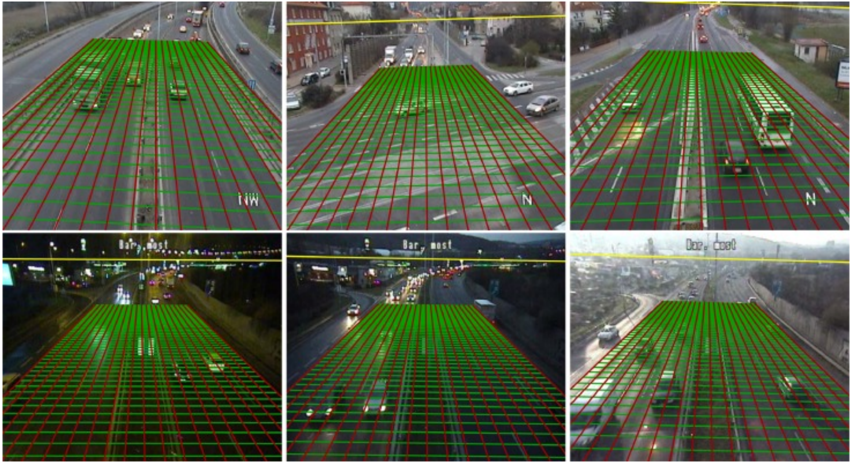

Image adopted from: Jakub Sochor, Roman Juránek, and Adam Herout. "Traffic surveillance camera calibration by 3d model bounding box alignment for accurate vehicle speed measurement." In: *Computer Vision and Image Understanding* 161 (2017), pp. 87–98

Monocular 3D Object Detection

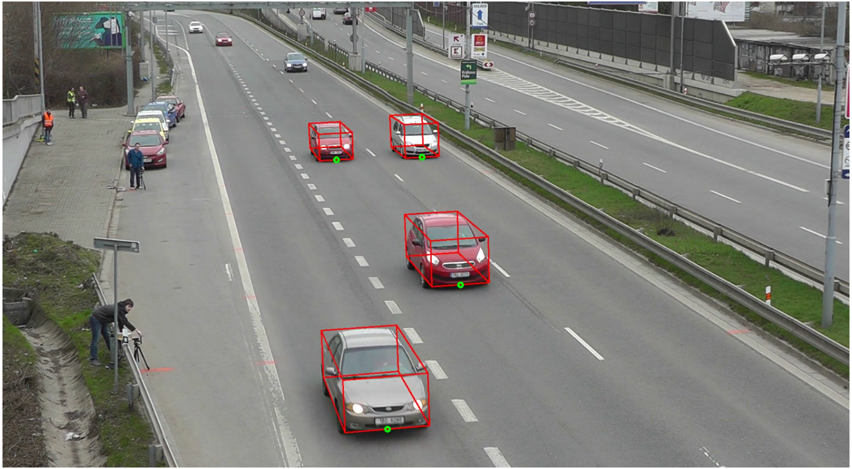

Image adopted from: Viktor Kocur and Milan Ftáčnik. “Detection of 3D bounding boxes of vehicles using perspective transformation for accurate speed measurement.” In: *Machine Vision and Applications* 31.7 (4), pp. 1–15

6DoF Object Pose Estimation

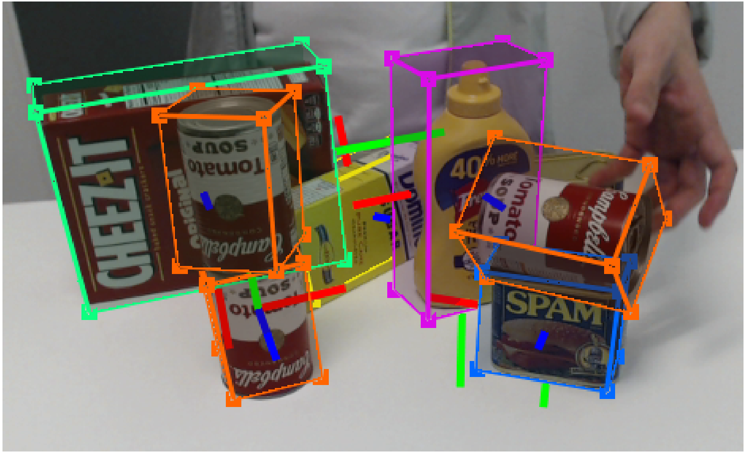

Image adopted from: Jonathan Tremblay et al. “Deep object pose estimation for semantic robotic grasping of household objects.” In: *arXiv preprint arXiv:1809.10790* (2018)

Depth Estimation

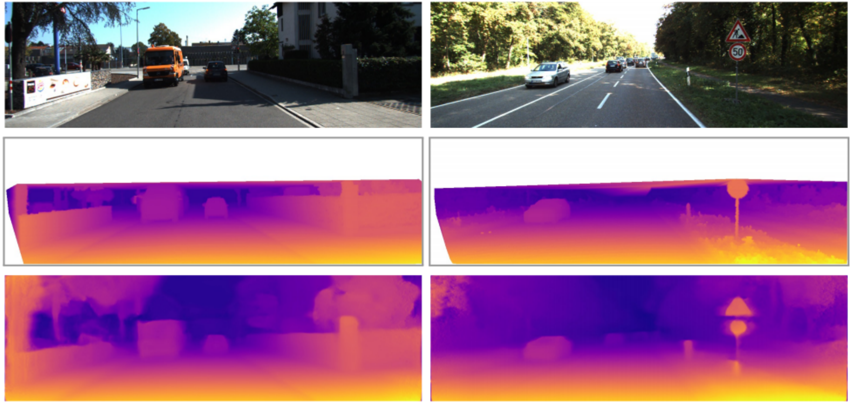

Image adopted from: Clément Godard, Oisin Mac Aodha, and Gabriel J Brostow. "Unsupervised monocular depth estimation with left-right consistency." In: *Proceedings of the IEEE conference on computer vision and pattern recognition*. 2017, pp. 270–279

Motivation - Capturing Reality

- City-wide 3D model - Banská Štiavnica
  - https://www.youtube.com/watch?v=K7onvPSwhbQ
- Creating 3D models with a phone
  - https://www.youtube.com/watch?v=fXLOMOWWBJQ

Body Pose Estimation

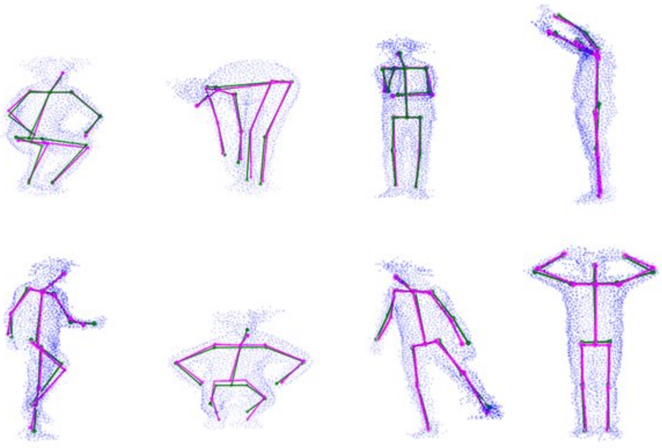

Image adopted from: Dana Skorvankova and Martin Madaras. “Human Pose Estimation using Per-Point Body Region Assignment.” In: *Computing and Informatics* 32 (July 2021), pp. 1001–1020

Course Information

The course information and resources can be found on Github:
`https://github.com/kocurvik/edu`.

Available Resources

Two good textbooks:

- Hartley, R. and Zisserman, A., 2003. *Multiple view geometry in computer vision*. Cambridge university press.
- Ma, Y., Soatto, S., Koěcká, J. and Sastry, S., 2004. *An invitation to 3-d vision: from images to geometric models*. Springer.

Lecture: Multiple View Geometry (Prof. D. Cremers): `https://www.youtube.com/playlist?list=PLTBdjV_4f-EJn6udZ34tht9EVIW7lbeo4`

Syllabus

- Mathematical Background
- Single-view Geometry
- Two-view Geometry
- Structure from Motion
- Advanced Estimation and Optimization
- Deep Learning Approaches

Definition of a Vector Space - I

A set V is called a **vector space** (*vektorový priestor*) over a field (*pole*) F with addition $V \times V \mapsto V$ and multiplication $F \times V \mapsto V$ if the following axioms are satisfied (cont. on next frame):

1. Closure under addition: $\forall \boldsymbol{u}, \boldsymbol{v} \in V$, $\boldsymbol{u} + \boldsymbol{v} \in V$.
2. Closure under field multiplication: $\forall a \in F$ and $\boldsymbol{v} \in V, a\boldsymbol{v} \in V$
3. Associativity of addition: $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$, $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.
4. Commutativity of addition: $\forall \boldsymbol{u}, \boldsymbol{v} \in V$, $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
5. Existence of additive identity: $\exists \mathbf{0} \in V$ such that $\forall \boldsymbol{v} \in V$, $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$.

Definition of a Vector Space - II

6. Existence of additive inverses: $\forall \boldsymbol{v} \in V$, $\exists -\boldsymbol{v} \in V$ such that $\boldsymbol{v} + (-\boldsymbol{v}) = (-\boldsymbol{v}) + \boldsymbol{v} = \mathbf{0}$.
7. Compatibility of scalar multiplication with field multiplication: $\forall a, b \in F$ and $\forall \boldsymbol{v} \in V$, $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$.
8. Distributivity of scalar multiplication with respect to vector addition: $\forall a \in F$ and $\forall \boldsymbol{u}, \boldsymbol{v} \in V$, $a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$.
9. Distributivity of scalar multiplication with respect to field addition: $\forall a, b \in F$ and $\forall \boldsymbol{v} \in V$, $(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$.
10. Existence of multiplicative identity: $\exists 1 \in F$ such that for any vector $\boldsymbol{v} \in V$, $1\boldsymbol{v} = \boldsymbol{v}$.

We will mostly work with vectors from $\mathbb{R}^n$ - n-tuples of real numbers. In text we may write them as $\boldsymbol{x} = (x_1, x_2, ...x_n)^T$, we include the transpose symbol, because when multiplied by matrices we consider the vector to be in a column:

$$\boldsymbol{x} = \begin{pmatrix} 2.8 \\ -4.2 \\ 115.2 \end{pmatrix}$$

$$\boldsymbol{x}^T = (2.8, -4.2, 115.2)$$

Definition of Vector Subspace

Let V be a vector space and let W be a non-empty subset of V. Then, W is called a **vector subspace** (*podpriestor*) of V if it satisfies the following conditions:

1. Closure under addition: $\forall \boldsymbol{u}, \boldsymbol{v} \in W$, their sum $\boldsymbol{u} + \boldsymbol{v} \in W$.
2. Closure under scalar multiplication: $\forall a \in F$ and $\forall \boldsymbol{v} \in W$, $a\boldsymbol{v} \in W$.
3. Contains the zero vector: $\mathbf{0} \in W$.

Let V be a vector space and let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a finite set of vectors in V. The set of all linear combinations of these vectors, i.e., the set of all vectors of the form:

$$\boldsymbol{x} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_2,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, is called the **linear span** (*lineárny obal*) of the set of vectors and is denoted by $\mathrm{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$.

Definition of Linear Independence

Let V be a vector space and let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a finite set of vectors in V. The set of vectors is called **linearly independent** (*lineárne nezávislý*) if the only way to write the zero vector as a linear combination of these vectors is by having all coefficients equal to zero. Mathematically, this means that $\exists \alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n = \boldsymbol{0}$ if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

This also means that a set of vectors is linearly independent if it is impossible to express one vector from the set as a linear combination of the remaining vectors!

Definition of a Basis of a Vector Space

Let V be a vector space. A **basis** for V is a linearly independent set of vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$ such that every vector in V can be expressed as a unique linear combination of these basis vectors. Mathematically, this means that $\forall \boldsymbol{v} \in V$, there exist unique coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that

$$\boldsymbol{v} = \alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2 + \cdots + \alpha_n \boldsymbol{b}_n.$$

Some Properties of Vector Bases

1. Existence: Every finite-dimensional vector space has a basis.
2. Uniqueness: Given two different bases for the same vector space, each basis vector of one basis can be expressed as a unique linear combination of the basis vectors of the other basis.
3. Cardinality: The number of vectors in a basis for a vector space is called the **dimension** of the space and is denoted by $\dim(V)$.
4. Linear Span: The linear span of a basis for a vector space is equal to the entire vector space.

Basis Transformations

Given two bases $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$ and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$ for the same vector space V, there exists a unique linear transformation $T : V \to V$ such that $\boldsymbol{c}_i = T(\boldsymbol{b}_i)$ for all $i = 1, 2, \ldots, n$. This transformation T is called a **basis transformation**.

Given a vector $\boldsymbol{v} \in V$ with coordinates $(v_1, v_2, \ldots, v_n)$ with respect to the basis $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$, its coordinates with respect to the basis $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$ are given by $(v'_1, v'_2, \ldots, v'_n)$, where $v'_i = \langle \boldsymbol{v}, \boldsymbol{c}_i \rangle$.

Inner Product

An **inner product** (*vnútorný/skalárny súčin*) on a vector space V is a mapping $\langle \cdot, \cdot \rangle : V \times V \mapsto F$ that satisfies the following properties for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and scalars $a, b \in F$:

1. Conjugate symmetry: $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$.
2. Linearity in the first argument: $\langle a\boldsymbol{u} + b\boldsymbol{v}, \boldsymbol{w} \rangle = a \langle \boldsymbol{u}, \boldsymbol{w} \rangle + b \langle \boldsymbol{v}, \boldsymbol{w} \rangle$.
3. Positive definiteness: $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$ with equality if and only if $\boldsymbol{v} = \mathbf{0}$.

If $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$ then $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal. We will mostly use the standard scalar product:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$$

Norm and Metric Induced by an Inner Product

Given a vector space V with an inner product $\langle \cdot, \cdot \rangle$, the induced **norm** of a vector $\boldsymbol{v} \in V$ is defined as $|\boldsymbol{v}| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$. The norm is a nonnegative real number that measures the length of a vector.

The induced **distance** between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as the norm of the difference vector $\boldsymbol{u} - \boldsymbol{v}$. This defines a **metric** on V, denoted by $d(\boldsymbol{u}, \boldsymbol{v}) = |\boldsymbol{u} - \boldsymbol{v}|$. The metric satisfies the properties of non-negativity, symmetry, and the triangle inequality.

Calculating Angles using the Inner Product

The inner product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ can be used to calculate the angle between them.

We can calculate the angle θ using the following equation:

$$\cos(\theta) = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{|\boldsymbol{u}|\,|\boldsymbol{v}|}. \tag{1}$$

A **linear transformation** from a vector space V to a vector space W is a mapping $T : V \to W$ that satisfies the following properties for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and scalars $a, b \in F$:

1. Additivity: $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$.
2. Homogeneity: $T(a\boldsymbol{v}) = aT(\boldsymbol{v})$.

Given a basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ for V and a basis $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_m\}$ for W, the **matrix representation** (*maticová reprezentácia*) of a linear transformation $T : V \to W$ is an $m \times n$ matrix A with entries a_{ij} defined by $T(\boldsymbol{e}_j) = \sum_{i=1}^{m} a_{ij}\boldsymbol{f}_i$. The matrix representation of T with respect to the given bases is unique and can be used to compute $T(\boldsymbol{v})$ for any $\boldsymbol{v} \in V$.

This means that we can calculate $T(\boldsymbol{v}) = A\boldsymbol{v}$ using the standard matrix-vector product:

$$(A\boldsymbol{v})_i = \sum_{i=1}^{n} a_{ij}v_j \tag{2}$$

Definition of a Group

A **group** (*grupa*) is a set G with a binary operation $\circ : G \times G \mapsto G$ that satisfies the following properties:

1. Closure: For all $a, b \in G$, $a \circ b \in G$.
2. Associativity: For all $a, b, c \in G$, $(a \circ b) \circ c = a \circ (b \circ c)$.
3. Identity element: There exists an element $e \in G$ such that for all $a \in G$, $a \circ e = e \circ a = a$.
4. Inverse element: For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$.

Examples of groups include the set of real numbers under addition, the set of non-zero real numbers under multiplication, and the set of all rotations and translations of a two-dimensional plane.

The **General Linear Group** $\mathrm{GL}(n)$, is the group of all invertible $n \times n$ matrices. The group operation is matrix multiplication.

The **Special Linear Group** $\mathrm{SL}(n)$ is the subgroup of $\mathrm{GL}(n)$ consisting of all matrices with determinant equal to 1. This means that the scale of the vectors in the subspace represented by the matrix is preserved under the group operation.

These groups are important in linear algebra, as they provide a way to study and describe linear transformations that preserve certain properties, such as volume, orientation, and area.

Matrix Representation of a Group

A **matrix representation** of a group is a map from the group elements to matrices, such that the group operation is preserved. That is, if A and B are matrices representing group elements a and b, respectively, then the matrix representation of $a \circ b$ is equal to the product AB.

Matrix representations provide a way to study and visualize the properties of groups in terms of matrices. They are also useful for computational purposes, as many algorithms for studying groups are most easily expressed in terms of matrices.

For example, the special linear group $\mathrm{SL}(2)$ can be represented by 2×2 matrices, where each matrix represents a linear transformation of the plane.

The **Affine Group** $\mathrm{A}(n)$ is the group of all affine transformations of $\mathbb{R}^n$, i.e. transformations of the form $\boldsymbol{x} \mapsto S\boldsymbol{x} + \boldsymbol{b}$, where $S \in \mathrm{GL}(n)$ and $\boldsymbol{b} \in \mathbb{R}^n$ is a translation. The group operation is composition of transformations, and the identity element is the identity transformation $\boldsymbol{x} \mapsto \boldsymbol{x}$.

The affine group is important in mathematics and computer graphics, as it provides a way to describe and study transformations that preserve parallelism and ratios of distances.

Example of Affine Transformation

Consider a two-dimensional space, where a point is represented by a vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Let's consider an affine transformation of the form $T(\boldsymbol{x}) = S\boldsymbol{x} + \boldsymbol{b}$, where

$$S = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{3}$$

Then, for a point $\boldsymbol{x} = (3, 4)^T$, the image of the point under the transformation is

$$T(\boldsymbol{x}) = S\boldsymbol{x} + \boldsymbol{b} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 13 \end{pmatrix}. \tag{4}$$

Thus, the affine transformation transforms the point $(3, 4)$ to the point $(3, 13)$.

Affine Matrix on Homogeneous Coordinates

T defined on the previous slide is not a linear map unless $\boldsymbol{b} = \mathbf{0}$. We can introduce **homogeneous coordinates** (*homogénne súradnice*) by representing a point from a two-dimensional space by a vector as $\boldsymbol{x} = \left(x_1, x_2, 1\right)^T$. T then becomes a linear mapping on $\mathbb{R}^{2+1}$.

We can represent the mapping using a matrix of the following form:

$$A = \begin{pmatrix} S & \boldsymbol{b} \\ \mathbf{0}^T & 1 \end{pmatrix}, \tag{5}$$

which is called the affine matrix and we get $T(\boldsymbol{x}) = A\boldsymbol{x}$

Affine Matrix Example

The previous example will then have the form $T(\boldsymbol{x}) = A\boldsymbol{x}$, where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

Then, for a point $\boldsymbol{x} = (3, 4, 1)^T$, the image of the point under the transformation is

$$T(\boldsymbol{x}) = A\boldsymbol{x} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 13 \\ 1 \end{pmatrix}. \tag{7}$$

Thus, the affine transformation transforms the point $(3, 4)$ to the point $(3, 13)$.

The Orthogonal Group

The **orthogonal group**, denoted $\mathrm{O}(n)$, is the set of all $n \times n$ orthogonal matrices, i.e. matrices A that satisfy

$$A^T A = AA^T = I, \tag{8}$$

where I is the identity matrix and A^T is the transpose of A.

Orthogonal matrices preserve the lengths of vectors, i.e. for any vector $\boldsymbol{x}$, we have $|A\boldsymbol{x}| = |\boldsymbol{x}|$. They also preserve the angles between vectors, i.e. if $\boldsymbol{x}$ and $\boldsymbol{y}$ are two vectors, then the angle between $A\boldsymbol{x}$ and $A\boldsymbol{y}$ is equal to the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$. From (8) it can be shown that $\det(A) \in \{-1, 1\}$.

The Special Orthogonal Group

The **special orthogonal group**, denoted $\mathrm{SO}(n)$, is the set of all $n \times n$ orthogonal matrices with a determinant of 1. In other words,

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det(A) = 1\}. \tag{9}$$

Like the orthogonal group, the special orthogonal group preserves the lengths of vectors and the angles between vectors. It additionally preserves the orientation of the axes when applied as a transformation. In particular the $\mathrm{SO}(3)$ is the group of all 3-dimensional **rotation matrices**.

The (Special) Euclidean Group

The **Euclidean group**, denoted $\mathrm{E}(n)$, is the group of all transformations of $\mathbb{R}^n$ that preserve the Euclidean metric. It can be represented as the semi-direct product of the orthogonal group $\mathrm{O}(n)$ and the additive group of translations $\mathbb{R}^n$.

In other words, an element of the Euclidean group can be written as a pair $(A, \boldsymbol{t})$, where $A \in \mathrm{O}(n)$ represents a rotation or reflection and $\boldsymbol{t} \in \mathbb{R}^n$ represents a translation.

If we restrict A to be from $\mathrm{SO}(n)$ (e.g. a rotation) then we have the **Special Euclidean Group** $\mathrm{SE}(n)$. In particular $\mathrm{SE}(3)$ represents the rigid-body motions.

Rigid Body Motion

Consider a vector $\boldsymbol{v} = \left(x, y, z\right)^T$ in $\mathbb{R}^3$. A **rigid body motion** (*pohyb tuhého telesa*) is a transformation from $\mathrm{SE}(3)$. The transformed vector $\boldsymbol{v}'$ can be obtained by multiplying the original vector with a rotation matrix $R \in SO(3)$ and adding a translation vector $\boldsymbol{t}$:

$$\boldsymbol{v}' = R\boldsymbol{v} + \boldsymbol{t}. \tag{10}$$

Example: Rigid Body Motion

For example, let's consider a rotation by 90° around the z-axis and a translation along the y-axis by 2 units. The corresponding rotation matrix is:

$$R = \begin{pmatrix} \cos(90^\circ) & -\sin(90^\circ) & 0 \\ \sin(90^\circ) & \cos(90^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

and the translation vector is $\boldsymbol{t} = (0, 2, 0)^T$. Applying the rigid body motion to the vector $\boldsymbol{v} = (1, 0, 0)^T$ gives us the transformed vector:

$$\boldsymbol{v}' = R\boldsymbol{v} + \boldsymbol{t} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}. \tag{12}$$

This shows that the vector has been rotated by 90° in the xy-plane and translated along the y-axis by 2 units.

Similarly to affine transformation we can utilize the homogeneous coordinates by representing a point from a three-dimensional space by a vector as $\boldsymbol{v} = \left(x, y, z, 1\right)^T$.

We can represent the transformation using a matrix of the following form:

$$T = \begin{pmatrix} R & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix}. \tag{13}$$

We can then simply apply the transformation as $\boldsymbol{v}' = T\boldsymbol{v}$.

Example with Homogeneous Coordinates

For example, let's consider again a rotation by 90° around the z-axis and a translation along the y-axis by 2 units. The corresponding transformation matrix is:

$$T = \begin{pmatrix} \cos(90^\circ) & -\sin(90^\circ) & 0 & 0 \\ \sin(90^\circ) & \cos(90^\circ) & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

Applying the rigid body motion to the vector $\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}$ gives us the transformed vector:

$$\boldsymbol{v}' = T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}. \tag{15}$$

Linear System and Matrix Representation

A linear system is a set of linear equations that can be written in the form:

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{16}$$

where $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$.

The matrix representation of a linear system is obtained by stacking the equations of the system vertically, i.e., concatenating the coefficients of each equation into a single matrix A and the right-hand side terms into a single vector $\boldsymbol{b}$.

Linear System Example

For example, consider the linear system:

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 6 \end{aligned}$$

Its matrix representation is given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}. \tag{17}$$

The solution to a linear system is a vector $\boldsymbol{x}$ that satisfies the equations of the system.

Linear System Example

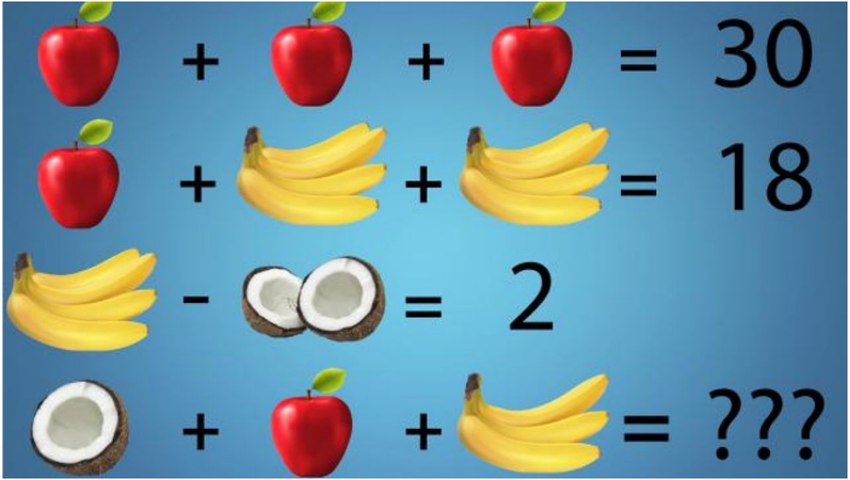

Image adopted from: Chris Jager. “Can You Solve This Children’s Maths Puzzle?” In: *LifeHacker Australia* (18). URL: https://www.lifehacker.com.au/2016/02/can-you-solve-this-childrens-maths-puzzle-2/

Given a matrix A of size $m \times n$, the **range** or span (*obraz*) of A, denoted by $\mathcal{R}(A)$, is the set of all possible output vectors $\boldsymbol{y}$ that can be obtained by multiplying A with a vector $\boldsymbol{x} \in \mathbb{R}^n$. Mathematically,

$$\mathcal{R}(A) = \{\boldsymbol{y} = A\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n\}\,. \tag{18}$$

The range of a matrix is a subspace of the target space. The system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if $\boldsymbol{b} \in \mathcal{R}(A)$.

Matrix Kernel

The null-space or **kernel** (*jadro*) of a matrix A of size $m \times n$, denoted as $\ker(A)$, is the set of all vectors $\boldsymbol{x} \in \mathbb{R}^n$ that satisfy the equation $A\boldsymbol{x} = \mathbf{0}$.

In other words, the kernel of a matrix is the set of all vectors that are mapped to the zero vector under the linear transformation represented by the matrix.

The dimension of the kernel of a matrix is known as the nullity, and it represents the number of independent solutions to the equation $A\boldsymbol{x} = \mathbf{0}$.

Matrix Rank

Given a matrix A of size $m \times n$, the **rank** (*hodnost'*) of the matrix A, denoted as $\mathrm{rank}(A)$, is defined as the maximum number of linearly independent columns in the matrix. Rank is also the dimension of $\mathcal{R}(A)$. Following also hold:

$$\mathrm{rank}(A) = n - \dim(\ker(A)) \tag{19}$$

$$0 \leq \mathrm{rank}(A) = \min\{m, n\} \tag{20}$$

For all non-singular (regular) $C \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{n \times n}$:

$$\mathrm{rank}(CAD) = \mathrm{rank}(A). \tag{21}$$

Sylvester's inequality - consider $B \in \mathbb{R}^{n \times k}$:

$$\mathrm{rank}(A) + \mathrm{rank}(B) - n \leq \mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\} \tag{22}$$

Eigenvalues and Eigenvectors

A (right) **eigenvector** (*vlastný vektor*) of a square matrix $A \in \mathbb{C}^{n \times n}$ is a non-zero vector $\boldsymbol{v} \in \mathbb{C}^n$ such that when A is multiplied by $\boldsymbol{v}$, the result is a scalar multiple of $\boldsymbol{v}$:

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \tag{23}$$

where $\lambda \in \mathbb{C}$ is a scalar known as the **eigenvalue** (*vlastné číslo*) corresponding to $\boldsymbol{v}$.

Similarly, $\boldsymbol{v}$ is a **left** eigenvector of A if $\boldsymbol{v}^T A = \lambda \boldsymbol{v}^T$.

The **spectrum** of matrix A, denoted as $\sigma(A)$, is the set of all eigenvalues.

Some Properties of Eigenvalues and Eigenvectors

- The eigenvalues of a matrix are the roots of the characteristic polynomial of the matrix $\det(A - \lambda I) = 0$.
- This also means that the product of eigenvalues (some included multiple times) is equal to $\det(A)$.
- For any matrix A with a right eigenvector $\mathbf{x}$ and corresponding eigenvalue λ, there exists a corresponding left eigenvector $\mathbf{y}$ with the same eigenvalue.

Some Properties of Eigenvalues and Eigenvectors

- The eigenvectors of a matrix corresponding to distinct eigenvalues are linearly independent.
- The eigenvectors of a matrix corresponding to the same eigenvalue are linearly dependent.
- If $B = PAP^{-1}$ for some non-singular matrix P then $\sigma(A) = \sigma(B)$.
- If $\lambda \in \mathbb{C}$ is an eigenvalue then its conjugate $\overline{\lambda}$ is also an eigenvalue. Thus $\sigma(A) = \overline{\sigma(A)}$ for real matrices.

Symmetric Matrices

A real square matrix A is said to be *symmetric* if $A^T = A$. Consider a $S \in \mathbb{R}^n$ to be symmetric. Then:

- All eigenvalues of S are real.
- Eigenvectors corresponding to distinct eigenvalues are orthogonal.
- There always exists n orthonormal eigenvectors of S which form a basis of $\mathbb{R}^n$. Let $V = (\boldsymbol{v}_1, \boldsymbol{v}_2, ...\boldsymbol{v}_n) \in \mathrm{O}(n)$ be the orthogonal matrix composed of these vectors and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ...\lambda_n)$ the diagonal matrix of eigenvalues. Then we have $S = V\Lambda V^T$.

Skew-Symmetric Matrices

A real square matrix A is said to be **skew-symmetric** (*antisymetrická*) if its transpose is equal to its negative, i.e., $A^T = -A$. A 2×2 skew symmetric matrix is of the form:

$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}. \tag{24}$$

The eigenvalues of a skew-symmetric are either zero or purely imaginary. The non-zero eigenvalues come in pairs of conjugates. This means that the rank of skew-symmetric matrices is always even.

3×3 Skew-Symmetric Matrices

3×3 skew-symmetric matrices can be used to represent cross products as matrix multiplications. Consider $\boldsymbol{a} = (a_1, a_2, a_3)^T$ then defining the matrix:

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{25}$$

the cross product with a vector $\boldsymbol{b} \in \mathbb{R}^3$ can be written as $\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b}$. Also this means that $[\boldsymbol{a}]_\times \boldsymbol{a} = \mathbf{0}$ and $\boldsymbol{a}$ spans the null-space of A.

Every matrix $A \in \mathbb{R}^{n \times n}$ can be decomposed into a symmetric matrix:

$$A_{sym} = \frac{1}{2}\left(A + A^T\right) \tag{26}$$

and a skew-symmetric matrix:

$$A_{skew} = \frac{1}{2}\left(A - A^T\right) \tag{27}$$

Induced 2-Norm of a Matrix

The **induced 2-norm** of a matrix is a scalar that measures the size of the matrix. It is also known as the spectral norm or the operator norm and is defined as:

$$||A||_2 = \max_{|\boldsymbol{x}|_2=1} |A\boldsymbol{x}|_2 \tag{28}$$

where A is a matrix and x is a vector with $|\boldsymbol{x}|_2 = 1$, which means the Euclidean norm of $\boldsymbol{x}$ is 1. The induced 2-norm of a matrix represents the maximum amplification factor of the matrix with respect to the Euclidean norm. It is the largest singular value of the matrix, which is the square root of the largest eigenvalue of the positive semidefinite matrix $A^T A$.

Frobenius Norm of a Matrix

The **Frobenius norm** of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{i,j}|^2} \tag{29}$$

where $a_{i,j}$ are the elements of A.

Note that since we are in a finite dimensional space the norm equivalence holds, but the norms are still significantly different. We will discuss this more in a few slides.

Singular values can be used to calculate the Frobenius norm:

$$||A||_F = \sqrt{\sigma_1^2 + \ldots + \sigma_2^2}, \tag{30}$$

where σ_i is the i-th singular value. Singular values can also be used for the induced 2-norm:

$$||A||_2 = \sigma_{max}, \tag{31}$$

where σ_{max} is the largest singular value of A.

Singular Value Decomposition

Given a matrix $A \in \mathbb{R}^{m \times n}$, the **singular value decomposition (SVD)** (*singulárny rozklad*) is a factorization of A into three matrices U, S, V^T such that:

$$A = USV^T \tag{32}$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices and $S \in \mathbb{R}^{m \times n}$ is a diagonal matrix with non-negative entries, called the singular values of A.

The columns of U and V are called the left and right singular vectors of A, respectively, and S contains the singular values of A along its diagonal.

SVD - Properties

- The columns of U and V are orthonormal bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.
- If $\text{rank}(A) = p$. Then there will be exactly p non-zero singular values (diagonal entries of S).
- The right singular vectors (columns of V) corresponding to zero singular values form a basis of the matrix null-space.
- The left singular vectors (columns of U) corresponding to non-zero singular values form a basis of the matrix range.
- Algorithms commonly output S such that the singular values are ordered from the largest to the smallest.

SVD - Symmetric matrices

- The right singular vectors (columns of V) are eigenvectors of $A^T A$.
- The left singular vectors (columns of U) are eigenvectors of AA^T.
- The non-zero singular values are the square roots of non-zero eigenvalues of $A^T A$ or AA^T.
- If A is square and symmetric then SVD is also its eigenvalue decomposition - singular vectors are its eigenvectors and singular values are its eigenvalues.

SVD Example

Consider a matrix A:

$$\begin{pmatrix} 4 & -4 & 2 \\ 3 & 6 & 6 \end{pmatrix} \tag{33}$$

Then its decomposition will be:

$$A = USV^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{34}$$

SVD Example - null space

$$A = \begin{pmatrix} 4 & -4 & 2 \\ 3 & 6 & 6 \end{pmatrix} = USV^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{35}$$

Then we can see that for the last row of V^T denoted as $\boldsymbol{v} = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)^T$:

$$A\boldsymbol{v} = \begin{pmatrix} 4 & -4 & 2 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \mathbf{0} \tag{36}$$

SVD Example - null space

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 6 & 6 \end{pmatrix} = USV^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{37}$$

Then we can see that also for the second form last row of V^T denoted as $\boldsymbol{v} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)^T$:

$$A\boldsymbol{v} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \mathbf{0} \tag{38}$$

SVD - Geometrical Interpretation

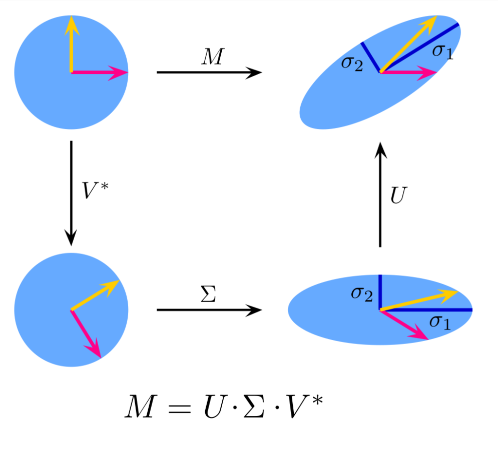

Image adopted from: Wikipedia. *Singular value decomposition — Wikipedia, The Free Encyclopedia*. `http://en.wikipedia.org/w/index.php?title=Singular%20value%20decomposition&oldid=1136343944`. [Online; accessed 10-February-2023]. 2023

Pseudoinverse

For square matrices with $\det(A) \neq 0$ we can define its inverse as A^{-1} such that $A^{-1}A = I$. Using SVD we define a **pseudoinverse matrix** for a singular square or a non-square matrix $A \in \mathbb{R}^{m \times n}$ denoted as $A^{\dagger}$.

$$A^{\dagger} = VS^{\dagger}U^{T}, \tag{39}$$

where $S^{\dagger} = \text{diag}_{m \times n}(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n})$.

We can use $A^{\dagger}$ in a linear system $A\boldsymbol{x} = \boldsymbol{b}$ which may have multiple or no solutions. If we set $\boldsymbol{x}_{min} = A^{\dagger}\boldsymbol{b}$ then $\boldsymbol{x}_{min}$ is the $\boldsymbol{x}$ which is a global minimum of $|A\boldsymbol{x} - b|^2$ with the smallest norm $|x|$.

SVD - Some Remarks

Implementations of SVD for $A \in \mathbb{R}^{m \times n}$ has a time complexity of $O(mn^2 + n^2 m + n^3)$. This is quite expensive for larger matrices or multiple SVD loops on smaller matrices. It is also common for the matrix S to be output as a vector of size $\min(m, n)$ instead of the full matrix.

It is worth noting that SVD sometimes denotes a different version which we will call compact SVD. It is very similar and also produces three matrices U, S and V, but U and V are non-square semi-unitary and S is square.