Dynamic Game of Discrete Choice Model

Kodai Yasuda

Graduate School of Economics, Hitotsubashi University

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Preface

- Formerly, we reviewed the dynamic discrete choice model **with a single agent**.
- Now, we will consider the model **with multiple agents**. i.e., the dynamic games.
- In the first section, we consider the concept of **Markov perfect equilibrium** (MPE) and define some functions for the estimation.
- In the second section, we learn the algorithms for estimating the MPE.
  - 2-step algorithm (Bajari, Benkard and Levin, 2007)
  - The nested pseudo-likelihood algorithm (NPL, Aguirregabiria and Mira (2007))
- The notation is based on Aguirregabiria and Mira (2007).

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The setting of dynamic games

- For simplicity, we only consider the oligopoly case.
- The state variable follows a stable Markov process.
- Then we assume that firm $i = 1, 2$ have Markov strategies.
  - An agent i has a Markov strategy $\sigma_i(x_t)$ if her action a_i depends only on the state x_t; i.e., for all time t and t', we have

$$x_t = x_{t'} \Rightarrow \sigma_i(x_t) = \sigma_i(x_{t'}).$$

  - In this case, we can interpret the Markov strategy as a function $\sigma_i : X \to A_i$.

Conditional Choice Probability (CCP)

- The CCP in dynamic games is the probability that firm i chooses an action a_i given the state s defined as

$$
\begin{aligned}
P_i^\sigma(a_i \mid s) &\stackrel{\text{def}}{=} \Pr(a_i = \sigma_i(s, \varepsilon_i) \mid s) \\
&= \int \mathbb{1}\left(a_i = \sigma_i(s, \varepsilon_i)\right) g(\varepsilon_i) d\varepsilon_i.
\end{aligned}
$$

- Recall that the CCP in the model with a single agent is defined as

$$
\begin{aligned}
&P(a_j \mid x, \theta) \\
&\quad = \int \mathbb{1}\left(a_j = \arg\max_a \left\{u(a, x; \theta) + \varepsilon(a) + \beta EV(a, x; \theta)\right\}\right) p(x \mid a, x, \theta) dx.
\end{aligned}
$$

Expected payoffs and transition probabilities

- Then the expected payoff of firm i at state s is

$$\pi_i^\sigma(a_i, s) = \sum_{a_{-i} \in A^{I-1}} \left(\prod_{j \neq i} P_j^\sigma(a_{-i}[j] \mid s) \right) \tilde{\pi}_i(a_i, a_{-i}, s).$$

- Note that a_{-i} denotes the vector of actions of all firms except firm i $(\{1, 2, \cdots, i-1, i+1, \cdots, I\})$.
- Hence, the expected transition probability from state s to s' conditional on firms following the set of strategy strategy σ is

$$f^\sigma(s' \mid s) = \sum_{a_{-i} \in A^{I-1}} \left(\prod_{j \neq i} P_j^\sigma(a_{-i}[j] \mid s) \right) f(s' \mid s, a_i, a_{-i}).$$

The ex-ante value function

- The Bellman equation is

$$\tilde{V}_i(s, \varepsilon_i; \sigma) = \max_{a_i \in A} \Bigg\{ \pi_i^\sigma(a_i, s) \\ + \varepsilon_i(a_i) + \beta \sum_{s' \in \mathcal{S}} \left[\int \tilde{V}_i(s', \varepsilon_i'; \sigma) g_i(\varepsilon_i') d\varepsilon_i' \right] f_i^\sigma(s' \mid s, a_i) \Bigg\}.$$

- We define the ex-ante value function as the integrated Bellman equation:

$$V_i(s, \sigma) = \int \max_{a_i \in A} \Bigg\{ \pi_i^\sigma(a_i, s) \\ + \beta \sum_{s' \in \mathcal{S}} V_i(s, \sigma) f_i^\sigma(s' \mid s, a_i) + \varepsilon_i(a_i) \Bigg\} g_i(d\varepsilon_i).$$

- Hereinafter we denote $v_i^\sigma(s, a_i) = \pi_i^\sigma(a_i, s) + \beta \sum_{s' \in \mathcal{S}} V_i(s, \sigma) f_i^\sigma(s' \mid s, a_i)$.

Markov perfect equilibrium

- Then we define the equilibrium of this dynamic game as follows:

Definition 1 (Markov perfect equilibrium (Aguirregabiria and Mira, 2007))

A stationary Markov perfect equilibrium (MPE) in this game is a set of strategy functions σ^* such that for any firm i and for any $(s, \varepsilon_i) \in \mathcal{S} \times \mathbb{R}^{J+1}$,

$$\sigma_i^*(s, \varepsilon_i) = \underset{a_i \in A}{\arg\max} \left\{ v_i^{\sigma^*}(s, a_i) + \varepsilon_i(a_i) \right\}.$$

- Since the strategies of firms are characterized by the CCPs, we can define the MPE in probability space.

MPE in probability space

- Let P^* be the probabilities induced by the MPE σ^*. By definition, we have

$$P_i^*(a_i \mid s) = \int \mathbb{1}\left(a_i = \sigma_i^*(s, \varepsilon_i)\right) g(\varepsilon_i) d\varepsilon_i.$$

- Hence, let the right-hand side be the function of P, denoted by $\Lambda_i(a_i \mid s; P_{-i})$:

$$\Lambda_i(a_i \mid s; P_{-i}) = \int \mathbb{1}\left(a_i = \sigma_i^*(s, \varepsilon_i)\right) g(\varepsilon_i) d\varepsilon_i.$$

- Then the MPE is characterized by the fixed point of the function $\Lambda(P) = \{\Lambda_i(a_i \mid s; P_{-i})\}$.

Another definition of MPE in probability space

- It is known that that definition has a computational problem.
- Then, another approach of the MPE is used in the literature.
- Let $V_i^{P^*}(s)$ be the value function induced by P^*. Then it can be written as

$$\begin{aligned} V_i^{P^*}(s) = &\sum_{a_i \in A} P_i^*(a_i \mid s) \left[\pi_i^{P^*}(a_i, s) + e_i^{P^*}(a_i, s) \right] \\ &+ \beta \sum_{s' \in \mathcal{S}} V_i^{P^*}(s) f^{P^*}(s' \mid s) \end{aligned} \tag{1}$$

where

$$\begin{aligned} e_i^{P^*}(a_i, s) &= E\left[\varepsilon_i(a_i) \mid s, a_i = \sigma_i^*(s, \varepsilon_i) \right], \\ f^{P^*}(s' \mid s) &= \sum_{a \in A^I} \left(\prod_{j=1}^{I} P_j^*(a_j \mid s) \right) f(s' \mid s, a). \end{aligned}$$

- Given the probabilities P^*, $V_i^{P^*}$ can be interpreted as the solution value of the Bellman equation.
- Then solving the Bellman equations (1) and expressing them in vector form yields

$$
\begin{aligned}
(I - \beta F^{P^*}) V_i^{P^*} &= \sum_{a_i \in A} P_i^*(a_i) \bigodot \left[\pi_i^{P^*}(a_i) + e_i^{P^*}(a_i) \right] \\
V_i^{P^*} &= (I - \beta F^{P^*})^{-1} \sum_{a_i \in A} P_i^*(a_i) \bigodot \left[\pi_i^{P^*}(a_i) + e_i^{P^*}(a_i) \right].
\end{aligned}
$$

- Then let $\Gamma_i(P^*) \stackrel{\text{def}}{=} \{\Gamma_i(s; P^*) : s \in \mathcal{S}\}$ be the solution of the Bellman equations, such that $\Gamma_i(s; P^*) = V_i^{P^*}(s)$.
- Note that $\Gamma_i(P) \stackrel{\text{def}}{=} (I - \beta F^P)^{-1} \sum_{a_i \in A} P_i^*(a_i) \bigodot \left[\pi_i^P(a_i) + e_i^P(a_i) \right]$.

- Finally we get the contraction mapping $\Psi(P) \stackrel{\text{def}}{=} \{\Psi_i(a_i \mid s; P)\}$

$$\Psi_i(a_i \mid s; P) = \int \mathbb{1}\Bigg(a_i = \underset{a \in A}{\arg\max} \Bigg\{ \pi_i^P(a, s) + \varepsilon_i(a)$$

$$\beta \sum_{s' \in \mathcal{S}} \Gamma_i(s'; P) f_i^P(s' \mid s, a) \Bigg\} \Bigg) g(\varepsilon_i) d\varepsilon_i.$$

  and the MPE is characterized by the fixed point of $\Psi(P)$.
- Under some assumptions, the fixed points of Λ and Ψ are identical (See Aguirregabiria and Mira (2007)).

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Algorithms for estimating MPE

- In the previous section, we defined the MPE in multiple ways.
- In this section, we learn the two algorithms for estimating the MPE:
  - 2-step algorithm (Bajari et al., 2007)
  - The nested pseudo-likelihood algorithm (NPL, Aguirregabiria and Mira (2007))

2-step algorithm (Bajari et al., 2007)

- The 2-step algorithm is a simple and intuitive method for estimating the MPE.
- The algorithm is applicable for dynamic games with both continuous and discontinuous choice space.
- The algorithm consists of the following two steps:
  1. Calculate the CCP and the transition probability by the observed data.
  2. Given the CCP and the transition probability, estimate the expected value conditional on firms behaving according to the equilibrium strategy.

The nested pseudo-likelihood algorithm (NPL, Aguirregabiria and Mira (2007))

- The NPL algorithm is a method for estimating the MPE in dynamic games with discontinuous choice space.
- The algorithm runs faster than the 2-step algorithm.
- The algorithm uses the nested pseudo-likelihood function defined as

$$Q_M(\theta, P) = \frac{1}{M} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{I} \ln \Psi_i(a_{imt} \mid s_{imt}; \theta, P).$$

- The algorithm consists of the following steps:
  1. Take the initial guess of P as P^0.
  2. Get the $\theta^k = \arg\max_{\theta \in \Theta} Q_M(\theta, P^{k-1})$.
  3. Get the $P^k = \Psi(\theta^k, P^{k-1})$
  4. For $k = 1, 2, \cdots$, repeat steps 2 and 3 until the convergence.

Reference I

Aguirregabiria, Victor and Pedro Mira (2007) "Sequential Estimation of Dynamic Discrete Games," *Econometrica*, Vol. 75, No. 1, pp. 1–53, DOI: https://doi.org/10.1111/j.1468-0262.2007.00731.x.

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