Basic Theory of Dynamic Discrete Choice Models

Kodai Yasuda

Graduate School of Economics, Hitotsubashi University

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Static Situation to Dynamic Situation

- Formerly, we considered the static situation.
  - Consumers choose one product from the set of products.
  - Firms choose one price from the set of prices.
- Now, we consider the dynamic situation. i.e.,
  - Consumers choose one product from the set of products **over time**.
  - Firms choose one price from the set of prices **over time**.
- To consider the dynamic situation, we need to consider the **timing** of the decision.
- So we need to introduce the **dynamic programming** to the discrete choice models.
  - In this section, we review the theory of dynamic programming to prepare for the next section.

Setting of Dynamic Programming

- Dynamic programming is a method to solve the dynamic optimization problem.
- The typical formulation of the dynamic optimization problem is as follows:

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t f(x_t)$$

  where
  - x_t is the state variable at time t,
  - $f(x_t)$ is the instantaneous payoff at time t,
  - $\beta \in (0,1)$ is the discount factor.
- The state variable x_t follows the **Markov process**.
  - The state variable x_t depends **only on** the state variable x_{t-1}.
  - Hence, the probability of the state variable x_t can be reduced to

$$\Pr(x_t | x_{t-1}, x_{t-2}, \ldots, x_0) = \Pr(x_t | x_{t-1}).$$

Bellman Equation

- The dynamic optimization problem can be transformed into the static optimization problem for each time period.
- Before the transformation, we define the **value function** $V(x_t)$ as follows:

$$V(x_t) = \max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t f(x_t).$$

- Then, the dynamic optimization can be transformed by using the following **Bellman equation**:

$$V(x) = \max_{x} \left\{ f(x) + \beta V(x') \right\}.$$

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Application of Dynamic Programming

- We have learned the basic theory of dynamic programming.
- Now, we apply the dynamic programming to the discrete choice model.
- For the simplicity, we consider the discrete choice model **with a single agent**.
- Notations and transformations are based on 上武 et al. 2022.

Model Setting

- A single agent faces the following utility maximization problem:

$$\max_{(a_t)_{t=1}^{\infty}} E \sum_{t=1}^{\infty} \beta^{t-1} u(a_t, x_t, \varepsilon_t; \theta)$$

  where
  - $(a_t)_{t=1}^{\infty}$: the sequence of actions
  - $x_t(\text{observed}), \varepsilon_t(\text{unobserved})$: the state variable of time t
    - $(x_t)_{t=1}^{\infty}$ follows the Markov process.
  - θ: the parameter vector of the utility function
- This UMP can be transformed into the following dynamic programming problem:

$$V(x_t, \varepsilon_t; \theta) = \max_{(a_t)_{t=1}^{\infty}} E_t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(a_t, x_t, \varepsilon_t; \theta) \,\middle|\, x_t \right]$$

$$\Rightarrow V(x,\varepsilon;\theta) = \max_a \left\{u(a,x,\varepsilon;\theta) + \beta E\left[V(x',\varepsilon';\theta) \mid \ a,x,\varepsilon\right]\right\}$$

Letting $EV(a,x,\varepsilon;\theta) \overset{\text{def}}{=} E\left[V(x',\varepsilon';\theta) \mid \ a,x,\varepsilon\right]$ yields

$$= \max_a \left\{u(a,x,\varepsilon;\theta) + \beta EV(a,x,\varepsilon;\theta)\right\} \quad (1)$$

- Note that we have

$$E\left[V(x',\varepsilon';\theta) \mid \ a,x,\varepsilon\right] = \iint V(x',\varepsilon';\theta) p\left(x',\varepsilon' \mid \ a,x,\varepsilon;\theta\right) dx' d\varepsilon' \quad (2)$$

  where $p\left(x',\varepsilon' \mid \ a,x,\varepsilon;\theta\right)$ denotes the transition probability of the state x',ε' given a,x,ε.
- Solving the equation (1), we can obtain the optimal policy function $a^*(x,\varepsilon;\theta)$.

Assumptions for the Estimation

Assumption 1 (Additive Separability)

The utility function $u(a, x, \varepsilon; \theta)$ is **additively separable** in ε. That is,

$$u(a, x, \varepsilon; \theta) = u(a, x; \theta) + \varepsilon(a).$$

Assumption 2 (Conditional Independence)

The transition of the state variable $\{x_t, \varepsilon_t\}$ follows the Markov process. Also, the transition probability $p\left(x', \varepsilon' \mid a, x, \varepsilon; \theta\right)$ satisfies

$$p\left(x', \varepsilon' \mid a, x, \varepsilon; \theta\right) = p\left(x' \mid a, x; \theta\right) p\left(\varepsilon' \mid x'; \theta\right).$$

Assumption 3 (Conditional Logit)

The shock ε is i.i.d. with the type I extreme value distribution and does not depend on x. That is, for type I extreme value distribution $p(\cdot)$, we have

$$p(\varepsilon \mid x, \theta) = p(\varepsilon).$$

Transformation of the Bellman Equation

- By Assumption 2, the equation (2) can be written as

$$
\begin{aligned}
& EV(a, x, \varepsilon; \theta) \\
&\quad = \iint V(x', \varepsilon'; \theta) p\left(x', \varepsilon' \mid a, x, \varepsilon; \theta\right) dx' d\varepsilon' \\
&\quad = \int \left\{ \int V(x', \varepsilon'; \theta) p\left(\varepsilon' \mid x'; \theta\right) d\varepsilon' \right\} p(x' \mid a, x, \theta) dx' \\
&\quad \overset{\text{def}}{=} EV(a, x; \theta)
\end{aligned} \tag{3}
$$

- By Assumption 1, the Bellman equation (1) can be written as

$$
\begin{aligned}
V(x, \varepsilon; \theta) &= \max_a \left\{ u(a, x, \varepsilon; \theta) + \beta EV(a, x; \theta) \right\} \\
&= \max_a \left\{ u(a, x; \theta) + \varepsilon(a) + \beta EV(a, x; \theta) \right\}
\end{aligned}
$$

Then, by taking the expectation of the equation above w.r.t. ε, we have

$$
\begin{aligned}
& \int V(x, \varepsilon; \theta) p(\varepsilon \mid x; \theta) d\varepsilon \\
& = \int \max_a \left\{ u(a, x; \theta) + \varepsilon(a) + \beta EV(a, x; \theta) \right\} p(\varepsilon \mid x; \theta) d\varepsilon
\end{aligned} \tag{4}
$$

- Let
$$W(\{\nu(a)\}_a; x, \theta) \stackrel{\text{def}}{=} \max_a \{\nu(a) + \varepsilon(a)\}\, p(\varepsilon \mid x; \theta) d\varepsilon.$$
- Substituting the equation (4) into the equation (3), we have
$$\begin{aligned} &EV(a, x; \theta) \\ &\quad = \int W(\{u(a, y; \theta) + \varepsilon(a) + \beta EV(a, y; \theta)\}_a; y, \theta) p(y \mid a, x, \theta) dy \end{aligned} \tag{5}$$
where
$$\begin{aligned} &W(\{u(a, y; \theta) + \varepsilon(a) + \beta EV(a, y; \theta)\}_a; y, \theta) \\ &\quad = \max_a \{u(a, y; \theta) + \varepsilon(a) + \beta EV(a, y; \theta)\}\, p(\varepsilon \mid y; \theta) d\varepsilon. \end{aligned}$$
- Note that this equation is recursive with respect to $EV(a, x; \theta)$.

Conditional Choice Probability

- By Assumption 3, $W(\cdot)$ can be written as

$$W(\{\nu(a)\}_a ; x, \theta) = \log\left(\sum_a \exp(\nu(a))\right) + \gamma \tag{6}$$

  where γ is the Euler's constant (recall that this is called the log-sum formula).
- Then, the probability of choosing a_j given x is

$$\begin{aligned}
&\Pr(a_j \mid x, \theta) \\
&\quad = \int \mathbb{1}\left(a_j = \arg\max_a \{u(a, x; \theta) + \varepsilon(a) + \beta EV(a, x; \theta)\}\right) p(x \mid a, x, \theta) dx \\
&\quad = \frac{\exp(u(a_j, x; \theta) + \beta EV(a_j, x; \theta))}{\sum_a \exp(u(a, x; \theta) + \beta EV(a, x; \theta))},
\end{aligned}$$

  which is called the **conditional choice probability**.

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Introduction

- Our goal is to estimate the structural parameters θ.
- In the estimation, some estimation algorithms are proposed.
  1. **Nested Fixed Point Algorithm** (NFXP; Rust (1987))
     - The primitive method for the estimation of dynamic discrete choice models.
     - It takes a long time to converge due to the curse of dimensionality.
     - To derive the structural parameters, we use the contraction mapping $EV(\cdot)$.

     $\rightarrow$ The identification problem will be discussed in the next chapter.
  2. **2-step Algorithm** (Hotz and Miller (1993), Arcidiacono and Miller (2011), Bajari, Benkard, and Levin (2007))
     - The modification of NFXP; it needs not to solve the Bellman equation.
     - It can be implemented in a short time, but it has a problem of the identification.
- In this chapter, we only review the NFXP briefly (Other algorithms will be discussed in the next chapter).

Procedure of NFXP

- The procedure of NFXP consists of two loops.
- Inner loop
  - Given the parameter vector θ, we solve the Bellman equation.
  - Derive the conditional choice probability $P(a \mid x; \theta)$.
- Outer loop
  - Given the conditional choice probability $P(a \mid x; \theta)$, we derive the log-likelihood.
  - Find the new parameter vector θ' that make the log-likelihood higher than θ.

The inner loop

- In the inner loop, we use the contraction mapping $EV(\cdot)$ derived from the equation (5) and (6).

$$
\begin{aligned}
&EV(a, x; \theta) \\
&\quad = \int \log \left\{ \sum_{a'} \exp(u(a', x'; \theta) + \beta EV(a', x'; \theta)) \right\} p(x' \mid a, x, \theta) dx'
\end{aligned}
$$

- The procedure of the inner loop is as follows:
  1. Guess the initial value of $EV(a, x)$, say $EV_{t=0}(a, x) = 0, \forall a, x$.
  2. Calculate the new value of $EV_{t+1}(a, x)$ by using the contraction mapping $EV(\cdot)$:

$$
EV_{t+1}(a, x) = \int \log \left\{ \sum_{a'} \exp(u(a', x'; \theta) + \beta EV_t(a', x')) \right\} p(x' \mid a, x) dx'
$$

  3. Repeat the step 2 until the convergence. For example, for $\varepsilon = 0.00001$,

$$
\max_{a, x} |EV_{t+1}(a, x) - EV_t(a, x)| < \varepsilon.
$$

The outer loop

- In the outer loop, we derive the log-likelihood by using derived conditional choice probability $P(a \mid x; \theta)$.

$$
\begin{aligned}
&\mathcal{L}(a_1, \cdots, a_T, x_1, \cdots, x_T; \theta \mid a_0, x_0; \theta) \\
&\quad = \sum_{t=1}^{T} \log \left\{ \Pr(a_t, x_t \mid a_0, x_0, \cdots, a_{t-1}, x_{t-1}; \theta) \right\} \\
&\quad = \sum_{t=1}^{T} \log \left\{ \Pr(a_t, x_t \mid a_{t-1}, x_{t-1}; \theta) \right\} \quad (\because (x_t)_{t=1}^{\infty} \text{: Markov}) \\
&\quad = \sum_{t=1}^{T} \log \left\{ \Pr(a_t \mid x_t; \theta) \right\} + \sum_{t=1}^{T} \log \left\{ \Pr(x_t \mid a_{t-1}, x_{t-1}; \theta) \right\}
\end{aligned}
$$

The outer loop

- The procedure of the outer loop is as follows:
  1. Guess the initial parameter vector θ_0.
  2. Calculate the conditional choice probability $P(a \mid x; \theta)$ by using the contraction mapping $EV(\cdot)$ (the inner loop).
  3. Derive the log-likelihood $\mathcal{L}(\theta)$ by using the conditional choice probability $P(a \mid x; \theta)$.
  4. Find the new parameter vector θ' that make the log-likelihood higher than θ. That is,
  $$\theta' \; s.t. \; \mathcal{L}(\theta') > \mathcal{L}(\theta).$$
  5. Repeat the step 2 to 4 until the convergence. For example, for $\varepsilon = 0.00001$,
  $$\max_{\theta} \left| \mathcal{L}(\theta') - \mathcal{L}(\theta) \right| < \varepsilon.$$