CHAPTER

2

DISCRETE-TIME SIGNALS

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2.1 INTRODUCTION

In this modern age of microelectronics, signals and systems play very vital roles. It is an extraordinary subject with diverse applications in areas of science and technology such as circuit design, seismology, communications, biomedical engineering, energy generation and distribution, speech processing etc. Therefore, it is essential that every practising engineer and designer must have a thorough knowledge of this subject. Understanding of signals and systems is also must for study of other parts of engineering such as signal processing and control systems.

2.2 SIGNALS

A signal may be a function of time, temperature, position, pressure, distance etc. Some signals in our daily life are music, speech, picture and video signals. Systematically, we can define a signal as "A function of one or more independent variables which contains some information is called a signal".

In electrical sense, the signal can be voltage or current. The voltage or current is the function of time as an independent variable.

In daily life, we come across several electric signals such as Radio Signal, T.V. Signal, Computer Signal etc.

Many signals that we come across are naturally generated signals. However, few signals are also generated synthetically.

The signals can be one-dimensional or multidimensional.

1. One Dimensional Signals

When the function depends on a single variable, the signal is said to be one dimensional. Example of one dimensional signal is speech signal whose amplitude varies with time.

2. Multidimensional Signals

When the function depends on two or more variables, the signal is said to be multidimensional. The example of a multidimensional signal is an image because it is a two dimensional signal with horizontal and vertical coordinates.

2.3 CLASSIFICATION OF SIGNALS

Based upon their nature and characteristics in the time domain, the signals may be broadly classified as under :

(1) Continuous-time signals

(2) Discrete-time signals

1. Continuous-time Signals

A continuous-time signal may be defined as a mathematical continuous function. This function is defined continuously in the time domain. For continuous-time signals, the independent variable is time t. A continuous-time signal is represented by x(t). Figure 2.1 shows a continuous-time signal.

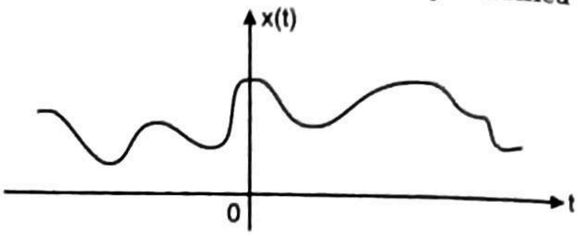


Fig. 2.1. A continuous-time signal

Thus, we can say that a signal of continuous amplitude and time is known as a continuous signal or an analog signal. This signal will have some value at every instant of time. The electrical signals derived in proportion with the physical quantities such as temperature, pressure, sound etc. are generally continuous signals. The other examples of continuous signals are sine wave, cosine wave, triangular wave etc. The continuous time signals are represented by x(t) where x represents the shape of the signal and t shows that the variable is time.

2. Discrete-time Signals

A discrete-time signal is defined only at certain time-instants. For discrete-time signal, the amplitude between two time instants is just not defined. For discrete-time signals, the independent variable is time n. A discrete time signal is represented by x(n). This means that to represent a discrete-time signal, we shall enclose the independent variable 'n' in brackets ( ). Figure 2.2 shows a discrete-time signal.

Thus, we can say that if the signal is represented only at discrete instants of time, then it is known as a discrete-time signal and also the discrete time signals have values only at certain instants of time.

If we take the blood pressure readings of a patient after every one hour and plot the graph, then the resultant signal will be a discrete time signal.

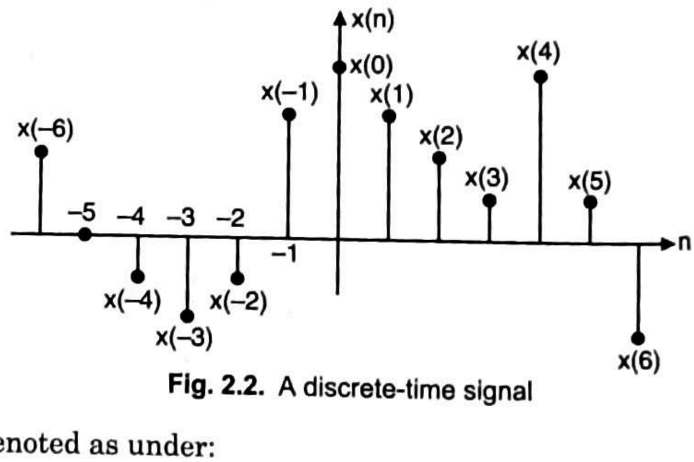


Fig. 2.2. A discrete-time signal

Mathematically, a discrete-time signal is denoted as under:

$$x(n) = \{\dots 0, 0, 1, \underset{\uparrow}{2}, 0, -1, 1, 2, 0, 0, \dots\}$$

where the arrow indicates the value of x(n) at n = 0.

We may further classify both continuous-time and discrete-time signals as under :

(i) Deterministic and non-deterministic (random) signals

(ii) Periodic and non-periodic signals
(iii) Even and odd signals
(iv) Energy and power signals

2.4 DETERMINISTIC AND NON-DETERMINISTIC (RANDOM) SIGNALS

Deterministic signals are those signals which can be completely specified in time. The pattern of this type of signal is regular and can be characterized mathematically. Also, the nature and amplitude of such a signal at any time can be predicted.

Few examples of deterministic signals are :

(i) $x(t) = bt$
This is a ramp signal whose amplitude increases linearly with time and the slope is b.

(ii) $x(t) = A \sin \omega t$
For this signal, the amplitude varies sinusoidally with time and its maximum amplitude is A.

(iii) $x(n) = \begin{cases} 2 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

This is a discrete-time signal whose amplitude is 2 for the sampling instants $n \geq 0$ and for all other samples, the amplitude is zero.

Hence, for all the above signals, it is clear that the amplitude at any time instant can be predicted in advance. Therefore, all the above signals are deterministic signals.

On the other hand, a non-deterministic signal is one whose occurrence is always random in nature. The pattern of such a signal is quite irregular. Non-deterministic signals are called **random signals**.

A typical example of non-deterministic signals is thermal noise generated in an electric circuit. Such a noise signal has probabilistic behaviour. Figure 2.3 shows deterministic and random signals.

Albert Michelson

Albert Michelson (of Michelson-Morley fame) was an intense, practical man who developed ingenious physical instruments of extraordinary precision, mostly in the field of optics, harmonic analyzer, developed in 1898, could compute the first 80 coefficients of the Fourier series of a signal x(t) specified by any graphical description. The instrument could also be as a harmonic synthesizer, which could plot a function x(t) generated by summing the first harmonics (Fourier components) of arbitrary amplitudes and phases. This analyzer, then had the ability of self-checking its operation by analyzing a signal x(t) and the adding resulting 80 components to see whether the sum yielded a close approximation of x(t).

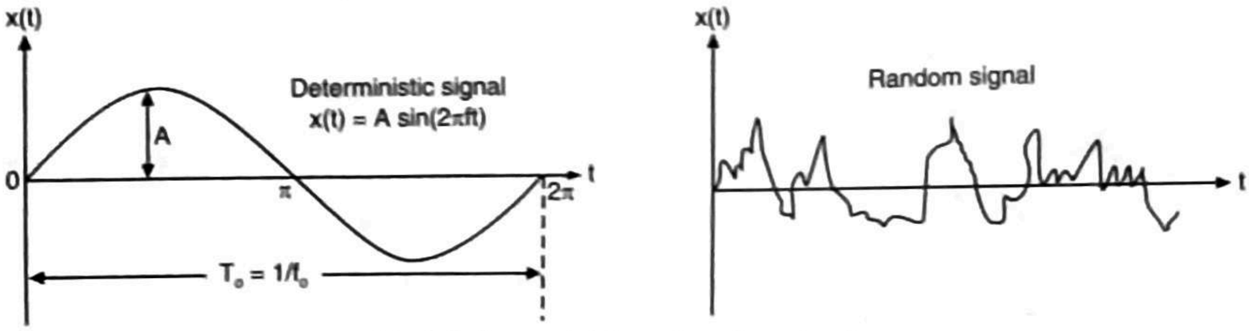


Fig. 2.3. Deterministic and random signals

2.5 PERIODIC AND NON-PERIODIC SIGNALS

1. Periodic Signal

A signal which repeats itself after a fixed time period is called as a periodic signal. The periodicity of a signal can be defined mathematically as under :

$$x(t) = x(t + T_0) : \text{Condition of periodicity}$$

where T_0 is called as the period of signal x (t). In other words, signal x (t) repeats itself after a period of T_0 sec.

Examples of periodic signals are sine wave, cosine wave, square wave etc. Figure 2.4. (a) shows a sine wave which is periodic because it repeats itself after a period T_0.

2. Non-periodic Signal

A signal which does not repeat itself after a fixed time period or does not repeat at all is called as a non-periodic or a aperiodic signal.

The non-periodic signals do not satisfy the condition of periodicity.

Thus, for a non-periodic signal, we have $x(t) \neq x(t + T_0)$ ...(2.1)

Sometimes it is said that an aperiodic signal has a period $T_0 = \infty$. Figure 2.4. (b) shows a decaying exponential signal. This exponential signal is non-periodic but it is deterministic because we can mathematically express it as $x(t) = e^{-\alpha t}$.

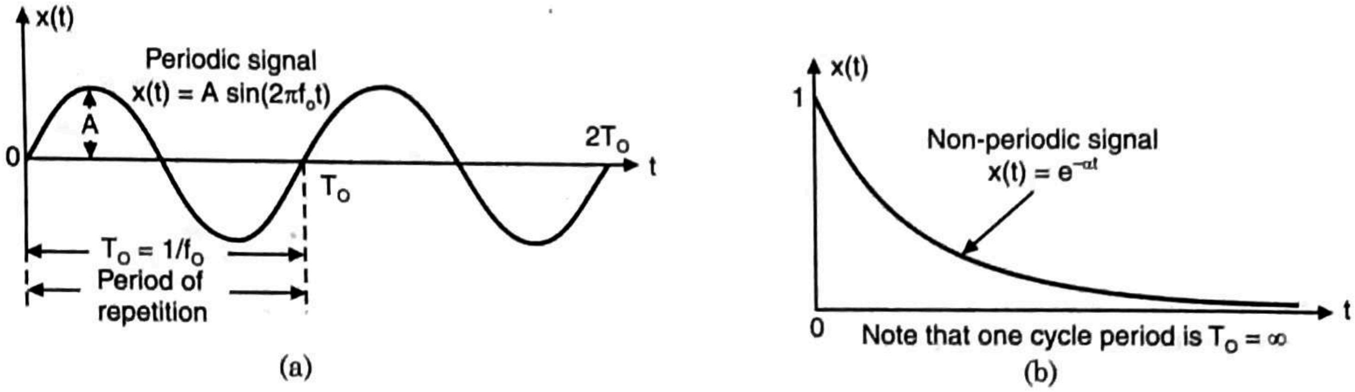


Fig. 2.4. Periodic and non-periodic signals

3. Periodic Discrete-Time Signal

For the discrete time signal, the condition of periodicity is,

$$x(n) = x(n + N) \quad \text{...(2.2)}$$

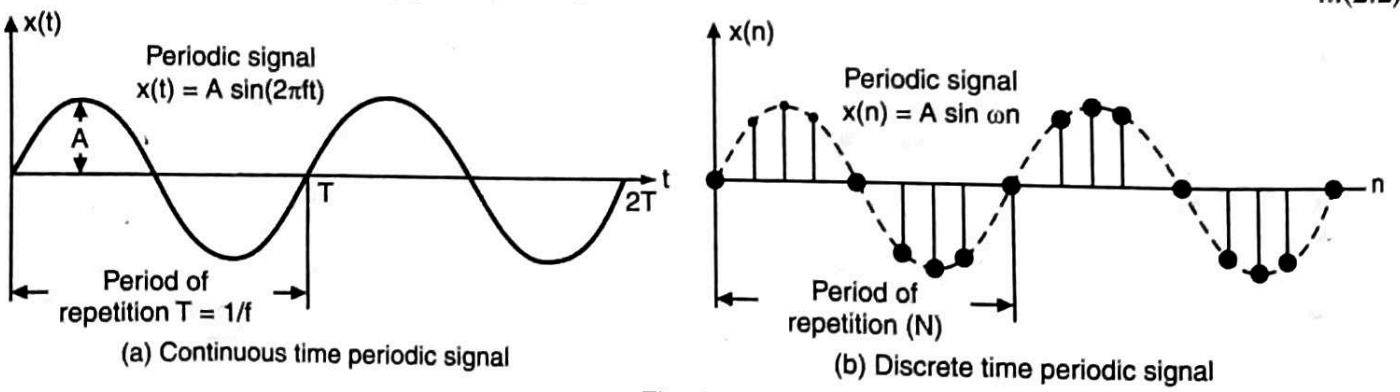


Fig. 2.5.

Here, number N is the period of signal. The smallest value of N for which the condition of periodicity exists is called as fundamental period.

Periodic signals are shown in figure 2.5 (a) and (b).

4. Non-periodic Signal

A signal which does not repeat itself after a fixed time period or does not repeat at all is called as non-periodic or aperiodic signal. Thus, mathematical expression for non-periodic signal is,

$$x(t) \neq x(t + T_0)$$

and $$x(n) \neq x(n + N)$$

Sometimes it is said that non-periodic signal has a period T = ∞ as shown in figure 2.6. This is an exponential signal having period, T = ∞. It can be mathematically expressed as

$$x(t) = e^{-\alpha t}$$

The other examples of non periodic signals are rectangular signal, dc signal, unit step signal etc.

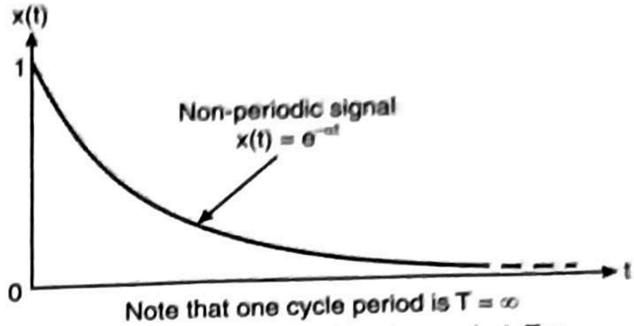


Fig. 2.6. A periodic signal having period, T = ∞

5. Non-periodic Discrete-Time Signal

Figure 2.7 shows a discrete-time non-periodic signal, for which

$$x(n) \neq x(n + N)$$

6. Important condition for periodicity of a discrete-time signal

A discrete time sinusoidal signal is periodic only if its frequency f_0 is rational. This means that frequency f_0 should be in the form of ratio of two integers.

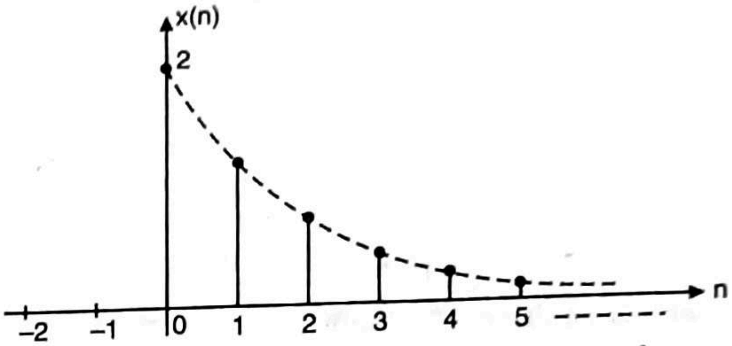


Fig. 2.7 Non-periodic discrete-time signal.

Proof

For the discrete-time signal, the condition of periodicity is given by

$$x(n + N) = x(n) \quad ...(2.3)$$

Let x (n) be the cosine wave. Hence, it can be expressed as,

$$x(n) = A \cos(2\pi f_0 n + \theta) \quad ...(2.4)$$

DO YOU KNOW?
An important subclass of aperiodic signals is the singularity functions.

Here, A = Amplitude

and θ = Phase shift

Now, the equation of x (n + N) can be obtained by replacing n by n + N in equation (2.4).

Therefore, we have

$$x(n + N) = A \cos[2\pi f_0 (n + N) + \theta] \quad ...(2.5)$$

According to the condition of periodicity i.e., equation (2.3), we can equate equations (2.4) and (2.5) as under :

$$A \cos[2\pi f_0 (n + N) + \theta] = A \cos(2\pi f_0 n + \theta)$$

or $$A \cos(2\pi f_0 n + 2\pi f_0 N + \theta) = A \cos(2\pi f_0 n + \theta) \quad ...(2.6)$$

To satisfy this equation, we must have

$$2\pi f_0 N = 2\pi k \quad ...(2.7)$$

where k is an integer

Therefore, we have $$f_0 = \frac{k}{N} \quad ...(2.8)$$

Here, k and N both are integers. Thus, discrete-time signal is periodic if its frequency f_0 is rational.

7. Periodicity Condition for $x(n) = x_1(n) + x_2(n)$

Here, input sequence is expressed as summation of two discrete time sequences. We can calculate the values of f_1 and f_2 corresponding to $x_1(n)$ and $x_2(n)$. Let $x_1(n)$ and $x_2(n)$ both be periodic discrete time signals (sequences).

Therefore, according to condition of periodicity, we have

$$f_1 = \frac{k_1}{N_1}$$

and

$$f_2 = \frac{k_2}{N_2}$$

The resultant signal x (n) is periodic if $\frac{N_1}{N_2}$ is ratio of two integers. The period of x(n) will be least common multiple of N_1 and N_2.

Similarly, if continuous time signal is,

$$x(t) = x_1(t) + x_2(t)$$

Then, we can calculate the values of T_1 and T_2 corresponding to $x_1(t)$ and $x_2(t)$. Then the resultant signal x(t) is periodic if $\frac{T_1}{T_2}$ is ratio of two integers. The fundamental period of x (t) will be least common multiple of T_1 and T_2.

EXAMPLE 2.1. Prove that the sine wave shown in figure 2.5 (a) is a periodic signal.

Solution: The sine wave shown in the figure 2.5 (a) can be mathematically represented as,

$$x(t) = A \sin \omega_0 t \quad \text{...(i)}$$

Now, let us test if it satisfies the condition for periodicity i.e., if

$$x(t) = x(t + T_0) \quad \text{...(ii)}$$

So, let us find the expression for $x(t + T_0)$ i.e.,

$$x(t + T_0) = A \sin \omega_0 (t + T_0) = A \sin [\omega_0 t + \omega_0 T_0] \quad \text{...(iii)}$$

But

$$\omega_0 = 2\pi f_0$$

and

$$T_0 = \frac{1}{f_0}.$$

Therefore,

$$\omega_0 T_0 = 2\pi f_0 \times \frac{1}{f_0} = 2\pi.$$

Substituting this in equation (iii), we obtain

$$x(t + T_0) = A \sin [\omega_0 t + 2\pi] = A [\sin \omega_0 t \cos 2\pi + \cos(\omega_0 t) \sin 2\pi]$$

or

$$x(t + T_0) = A \sin \omega_0 t = x(t)$$

Therefore, the sine wave shown in figure 2.5 (a) is a periodic signal. **Hence proved.**

DO YOU KNOW?

If the ratio of their periods can be expressed as a rational number or their frequencies are commensurable, their sum will be a periodic signal.

EXAMPLE 2.2. Prove that the exponential signal shown in figure 2.6 is non-periodic.

Solution: The exponential signal shown in figure 2.6 is expressed mathematically as,

$$x(t) = e^{-\alpha t}$$

Substituting $t = (t + T_0)$, we get,

$$x(t + T_0) = e^{-\alpha(t + T_0)} = e^{-\alpha t} \cdot e^{-\alpha T_0}$$

But

$$T_0 = \infty$$

> **DO YOU KNOW?**
>
> The sum of two or more sinusoids may or may not be periodic, depending on the relationships between their respective periods or frequencies.

Therefore, $e^{-\alpha T_0} = e^{-\infty} = 0$

or $x(t + T_0) = e^{-\alpha t} \cdot 0 = 0$

or $x(t) \neq x(t + T_0)$

Hence, the exponential signal shown in figure 2.6 is a non-periodic signal. **Hence proved.**

EXAMPLE 2.3. What is the fundamental frequency of the waveform shown in figure 2.8 in Hz and rad/sec.?

Solution: One cycle corresponds to 0.2 sec.

Hence $T_0 = 0.2$ sec.

Therefore, Frequency $T_0 = \frac{1}{T_0} = \frac{1}{0.2} = 5$ Hz. **Ans.**

Frequency in rad/sec. $\omega_0 = 2\pi f_0$

$= 2 \times 3.14 \times 5$

$= 31.4$ rad/s. **Ans.**

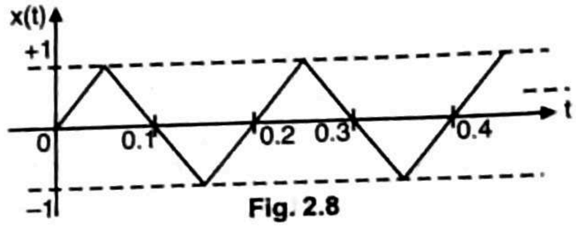


Fig. 2.8

EXAMPLE 2.4. What is the fundamental frequency of the discrete time square wave shown in figure 2.9.

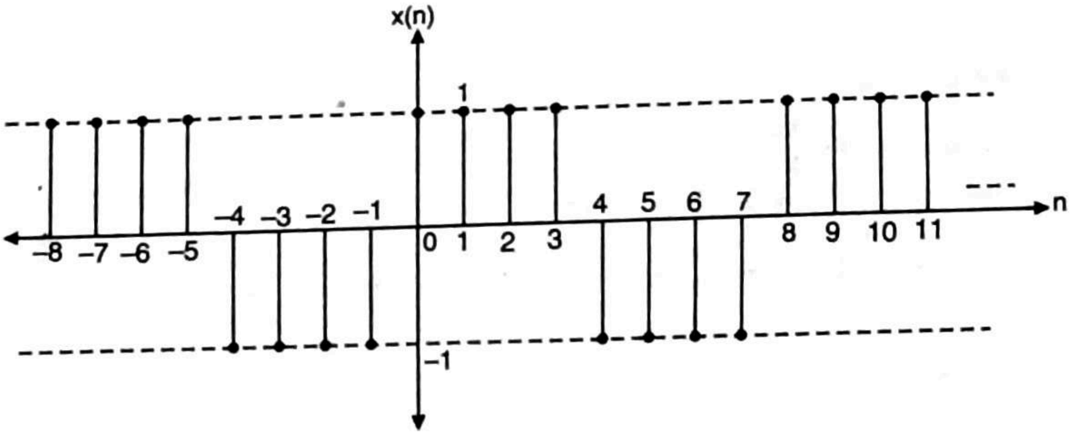


Fig. 2.9.

Solution: The fundamental angular frequency or simply fundamental frequency of x(n) is given by

$$\Omega = \frac{2\pi}{N}$$

where N = a positive integer indicating number of samples in one cycle.

For the given signal, N = 8.

Therefore, $\Omega = \frac{2\pi}{8} = \frac{\pi}{4}$ radius **Ans.**

EXAMPLE 2.5. Determine whether the following discrete-time signals are periodic or not ? If periodic, determine fundamental period.

(i) $\cos(0.01\pi n)$

(ii) $\cos(3\pi n)$

(iii) $\sin 3n$

(iv) $\cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

(v) $\cos\left(\frac{n}{8}\right)\cos\left(\frac{n\pi}{8}\right)$

(vi) $\sin(\pi + 0.2n)$

(vii) $e^{\left(j\frac{\pi}{4}\right)n}$

Solution: (i) Given that **x (n) = cos 0.01 πn**

Comparing above expression with $x(n) = \cos 2\pi fn$, we have

$$2\pi fn = 0.01\pi n$$

or
$$f = \frac{0.01}{2} = \frac{1}{200} = \frac{k}{N}$$

Here, f is expressed as ratio of two integers with k = 1 and N = 200. Hence, the signal is periodic with N = 200.

(ii) Given that **x(n) = cos 3πn**

Comparing above expression with $x(n) = \cos 2\pi fn$, we have

$$2\pi fn = 3\pi n$$

or
$$f = \frac{k}{N} = \frac{3}{2} \text{ i.e., ratio of two integers.}$$

Hence, this signal is periodic with N = 2

(iii) Given that **x(n) = sin 3n**

Comparing above expression with $x(n) = \cos 2\pi fn$, we have

$$2\pi fn = 3n$$

or
$$f = \frac{k}{N} = \frac{3}{2\pi}$$

which is not the ratio of two integers.

Hence, this signal is non periodic.

(iv) Given that $\mathbf{x(n) = \cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}}$

Comparing above expression with, $x(n) = \cos 2\pi f_1 n + \cos 2\pi f_2 n$, we have

$$2\pi f_1 n = \frac{2\pi n}{5}$$

or
$$f_1 = \frac{1}{5} = \frac{k_1}{N_1},$$

or
$$N_1 = 5$$

and
$$2\pi f_2 n = \frac{2\pi n}{7}$$

or
$$f_2 = \frac{1}{7} = \frac{k_2}{N_2},$$

or
$$N_1 = 7$$

Here, since $\frac{N_1}{N_2} = \frac{5}{7}$ is the ratio of two integers, the sequence is periodic. The period of x(n) is least common multiple of N_1 and N_2. Here, least common multiple of $N_1 = 5$ and $N_2 = 7$ is 35. Therefore, this sequence is periodic with N = 35.

(v) Given that
$$\mathbf{x(n) = \cos\left(\frac{n}{8}\right)\cos\left(\frac{n\pi}{8}\right)}$$

Here,
$$2\pi f_1 n = \frac{n}{8}$$

or
$$f_1 = \frac{1}{16\pi}, \text{ which is not rational}$$

and
$$2\pi f_2 n = \frac{n\pi}{8}$$

or
$$f_1 = \frac{1}{16}, \text{ which is rational}$$

Thus, $\cos\left(\frac{n}{8}\right)$ is non-periodic and $\cos\left(\frac{n\pi}{8}\right)$ is periodic. Hence, x(n) is non-periodic since it is the product of periodic and non-periodic signal.

(vi) Given that $\quad x(n) = \sin(\pi + 0.2\,n)$

Comparing above expression with $x(n) = \sin(2\pi f n + \theta)$, we have

$\theta = \pi$ i.e., phase shift

and $\quad 2\pi fn = 0.2n$

or $\quad f = \frac{0.2}{2\pi} = \frac{1}{10\pi}$ which is not rational.

Hence, this signal is non periodic

(vii) Given that, $\quad x(n) = e^{\left(j\frac{\pi}{4}\right)n}$

Simplifying, we get $\quad x(n) = \cos\frac{\pi}{4}n + j\sin\frac{\pi}{4}n$

Comparing above expression with $x(n) = \cos 2\pi fn + j\sin 2\pi fn$, we have

$$2\pi fn = \frac{\pi}{4}n$$

or $\quad f = \frac{1}{8} = \frac{k}{N'}$ which is rational

Hence, the given signal is periodic

2.6 SYMMETRICAL (EVEN) OR ANTISYMMETRICAL (ODD) SIGNALS

1. Symmetrical Signal (Continuous Time)

A signal x (t) is said to be symmetrical or even if it satisfies the following condition:

Condition for symmetry : $x(t) = x(-t)$ ...(2.9)

where $\quad x(t)$ = value of signal for positive t

and $\quad x(-t)$ = value of the signal for negative t

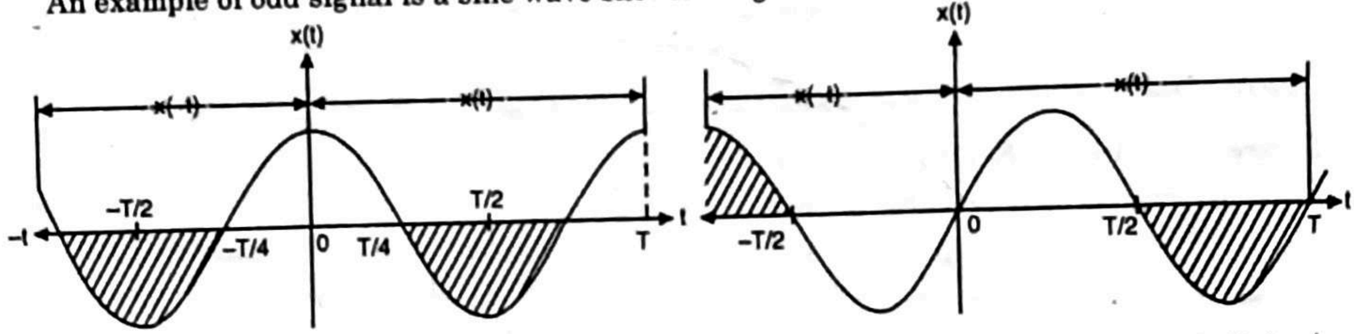
An example of symmetrical signal is a cosine wave shown in figure 2.10 (a).

2. Antisymmetrical signal (Continuous time)

A signal x (t) is said to be antisymmetrical or odd if it satisfies the following condition:

Condition for antisymmetry : $x(t) = -x(-t)$

An example of odd signal is a sine wave shown in figure 2.10 (b).



(a) Cosine wave x(t) = x(–t) symmetrical or even signal (b) Sine wave x(t) = – x(–t) antisymmetrical or odd signal

Fig. 2.10. Symmetrical and antisymmetrical signals

3. Even and Odd Discrete Time Signals

(i) Even (Symmetric) Discrete-time Signals

A discrete time real valued signal is said to be symmetric (even) if it satisfies the following condition:

Condition of symmetry

$$x(n) = x(-n) \quad \text{...(2.10)}$$

Examples : Figures 2.11 (a) and (b) are examples of discrete-time even signals.

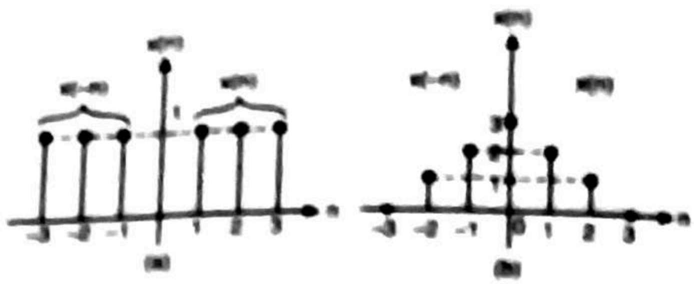


Fig. 2.11. Discrete-time symmetric (even) signals

(ii) Odd (Antisymmetric) Discrete-Time Signal

A discrete time signal x (n) is said to be antisymmetric or odd if it satisfies the following condition:

Condition of antisymmetry : $x(n) = -x(-n)$

Figure 2.12 shows the antisymmetric discrete time signals.

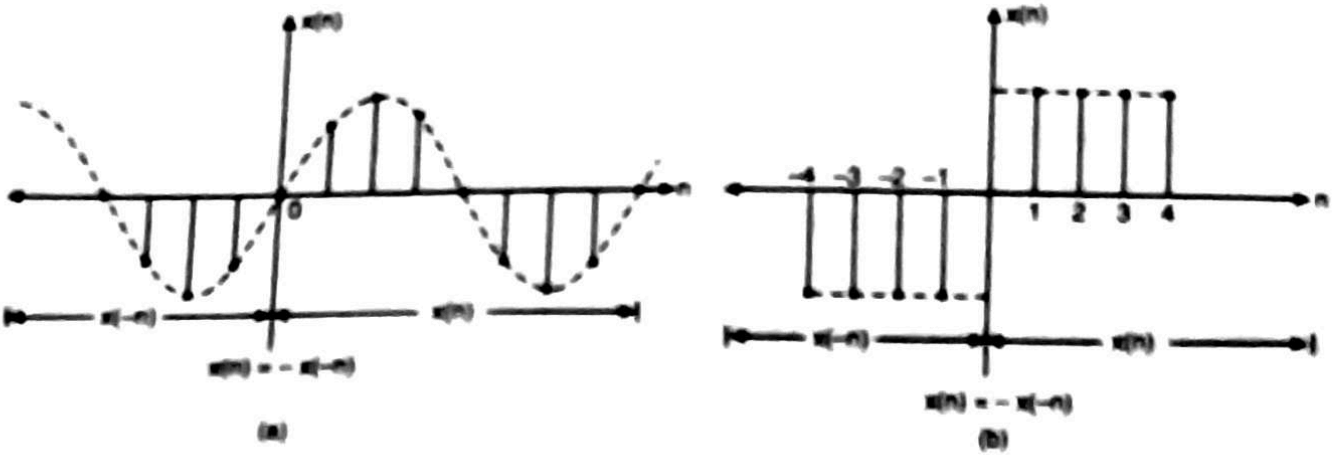


Fig. 2.12. Antisymmetric (odd) discrete-time signals

4. Decomposing a Signal into Even and Odd Parts

Any continuous or discrete time signal can be expressed as the summation of even part and odd part.

$$x(t) = x_e(t) + x_o(t) \quad \text{...(2.11)}$$

Hence $x_e(t)$ = Even component of signal x (t)

and $x_o(t)$ = Odd component of signal x (t)

substituting $t = -t$ in equation (2.11), we obtain,

$$x(-t) = x_e(-t) + x_o(-t) \quad \text{...(2.12)}$$

Let us now obtain the expressions for the even and odd part, $x_e(t)$ and $x_o(t)$.

5. Expression for the Even Part $x_e(t)$

For the even signal, we have,

$$x_e(t) = x_e(-t) \quad \text{...(2.13)}$$

And for odd signal, we have,

$$x_o(-t) = -x_o(t) \quad \text{...(2.14)}$$

Substituting equations (2.13) and (2.14) in equation (2.12), we obtain

$$x(-t) = x_e(t) - x_o(t) \quad \text{...(2.15)}$$

Adding equations (2.11) and (2.15), we have

$$x(t) + x(-t) = 2x_e(t)$$

or
$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \quad ...(2.16)$$

Equation (2.16) gives even component of x (t).

6. Expression for the Odd Part $x_0(t)$

Now, subtracting equation (2.15) from equation (2.11), we obtain

$$x(t) - x(-t) = 2x_0(t)$$

or
$$x_0(t) = \frac{1}{2}[x(t) - x(-t)] \quad ..(2.17)$$

Equation (2.17) gives odd components of x (t).

7. Even and Odd Components for a discrete-time Signal x (n)

The discrete time signal x (n) can be expressed in terms of its even and odd components as follows, by replacing t by n.

Therefore, expression for the even component $x_e(n)$ is given by

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \quad ...(2.18)$$

Also, equation of odd component is,

$$x_0(n) = \frac{1}{2}[x(n) - x(-n)] \quad ...(2.19)$$

EXAMPLE 2.6. Function x(t) is shown in figure 2.13. Draw even and odd parts of x(t).

Solution: The even part is given by

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

Also, the odd part is given by,

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$$

where x (– t) represents the folded version of x (t).

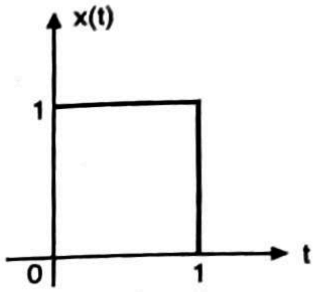


Fig. 2.13. Given function x(t)

8. Steps to be Followed :

Step 1 : First, we draw the signal x (t).

Step 2 : Then, we draw the folded version x (– t)

Step 3 : Next, we add x (t) and x (– t) or subtract x (– t) from x (t).

Step 4 : Finally, we divide the addition or subtraction by 2 to get $x_e(t)$ and $x_0(t)$.

These steps are followed in figure 2.14 to obtain $x_e(t)$ and $x_0(t)$.

The above definitions of even and odd signals assume that the signals are real valued.

If the signals are complex valued, then we have to take in terms of conjugate symmetry.

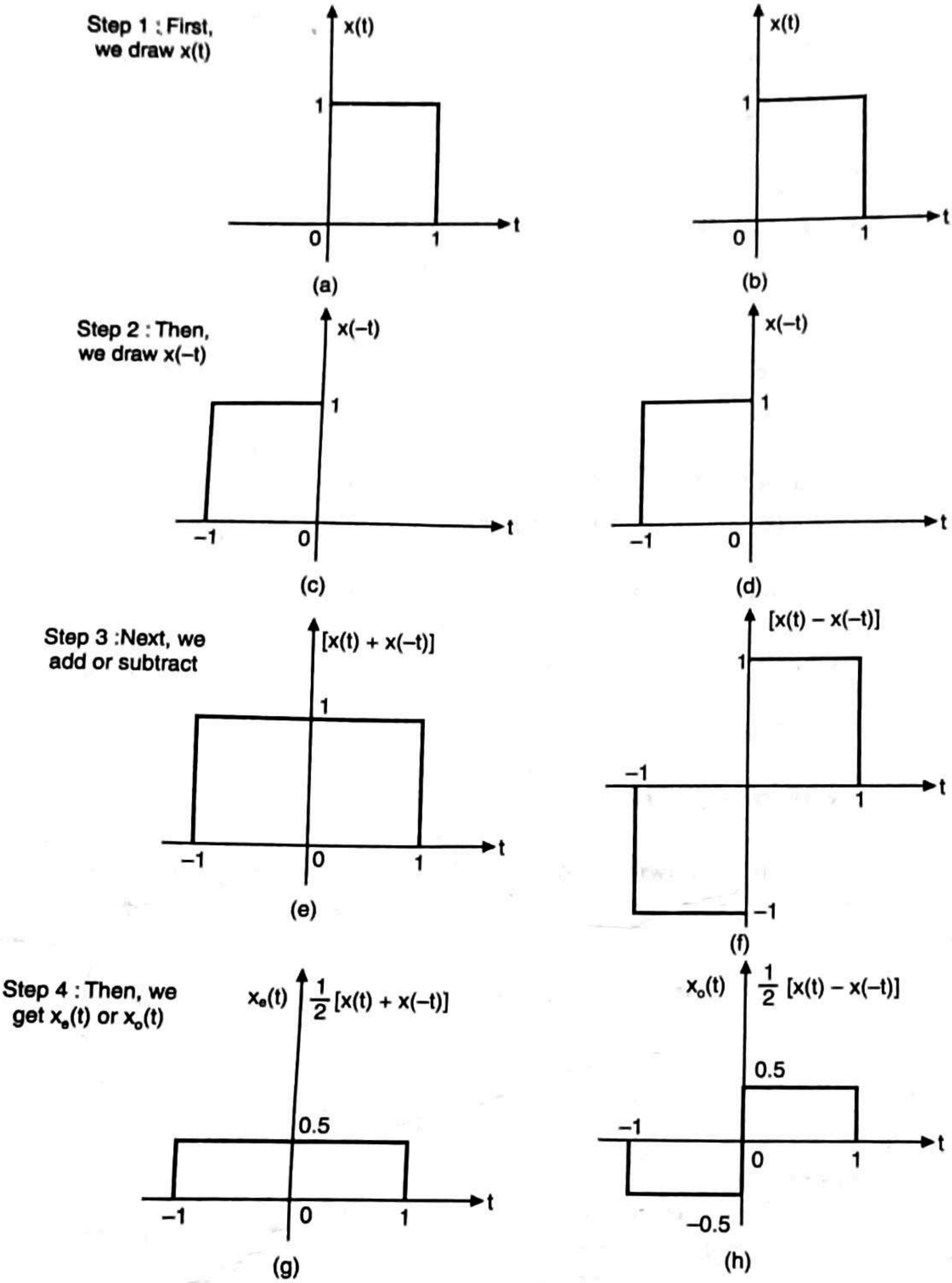


Fig. 2.14.

A complex valued signal is said to be having a conjugate symmetry if it satisfies the following condition:

$$x(-t) = x^*(t)$$

Remember that if $x(t) = a + jb$

then its complex conjugate is given by,

$$x^*(t) = (a - jb).$$

Therefore, the complex valued signal x(t) is conjugate symmetric if its real part is even and its imaginary part is odd.

EXAMPLE 2.7. Consider the signal shown in figure 2.15 and state which of these signals are even and which are odd.

Solution: The signal shown in figure 2.15(a) is an odd signal, that shown in figure 2.15 (b) is an even signal whereas the one shown in figure 2.15(c) is neither even nor odd.

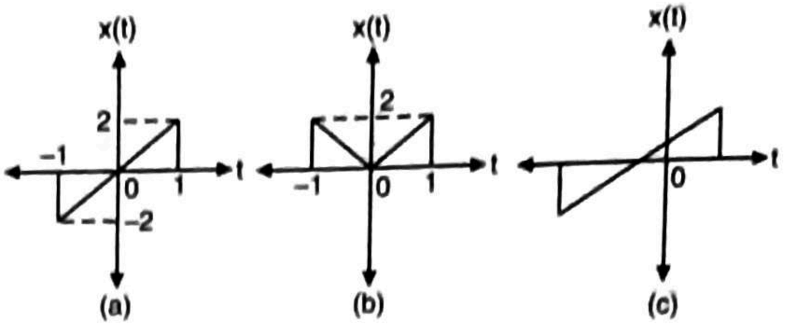


Fig. 2.15.

EXAMPLE 2.8. The even and odd parts of a signal are shown in figure 2.16. Draw the signal x (t).

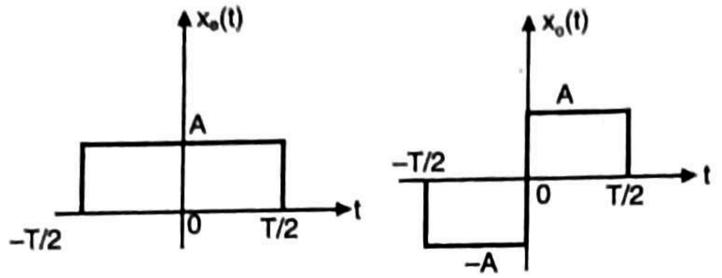


Fig. 2.16

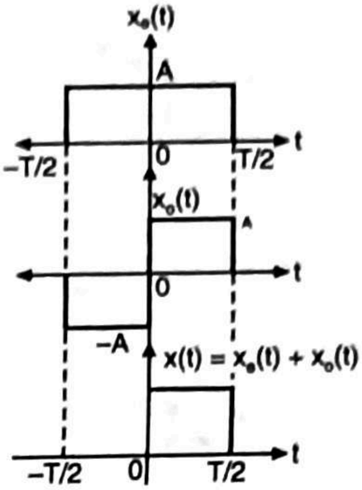


Fig. 2.17

Solution: The signal x(t) is given by

$$x(t) = x_e(t) + x_0(t)$$

The addition of $x_e(t)$ and $x_0(t)$ is shown in figure 2.17.

EXAMPLE 2.9. Find and sketch the even and odd components of the following:

(i) $x(n) = e^{-(n/4)} u(n)$ **(ii)** $x(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 \leq t \leq 2 \end{cases}$

(iii) $x(t) = \cos^2\left(\frac{\pi t}{2}\right)$ **(iv)** $x(n) = \text{Im}[e^{jn\pi/4}]$

Solution: (i) $x(n) = e^{-(n/4)} u(n)$

Even and odd parts of the sequence x(n) are expressed as under:

Even part, $$x_e(n) = \frac{1}{2}\{x(n) + x(-n)\}$$

and, odd part, $$x_0(n) = \frac{1}{2}\{x(n) - x(-n)\}$$

Figure 2.18 shows the steps to obtain even and odd parts for the above expressions.

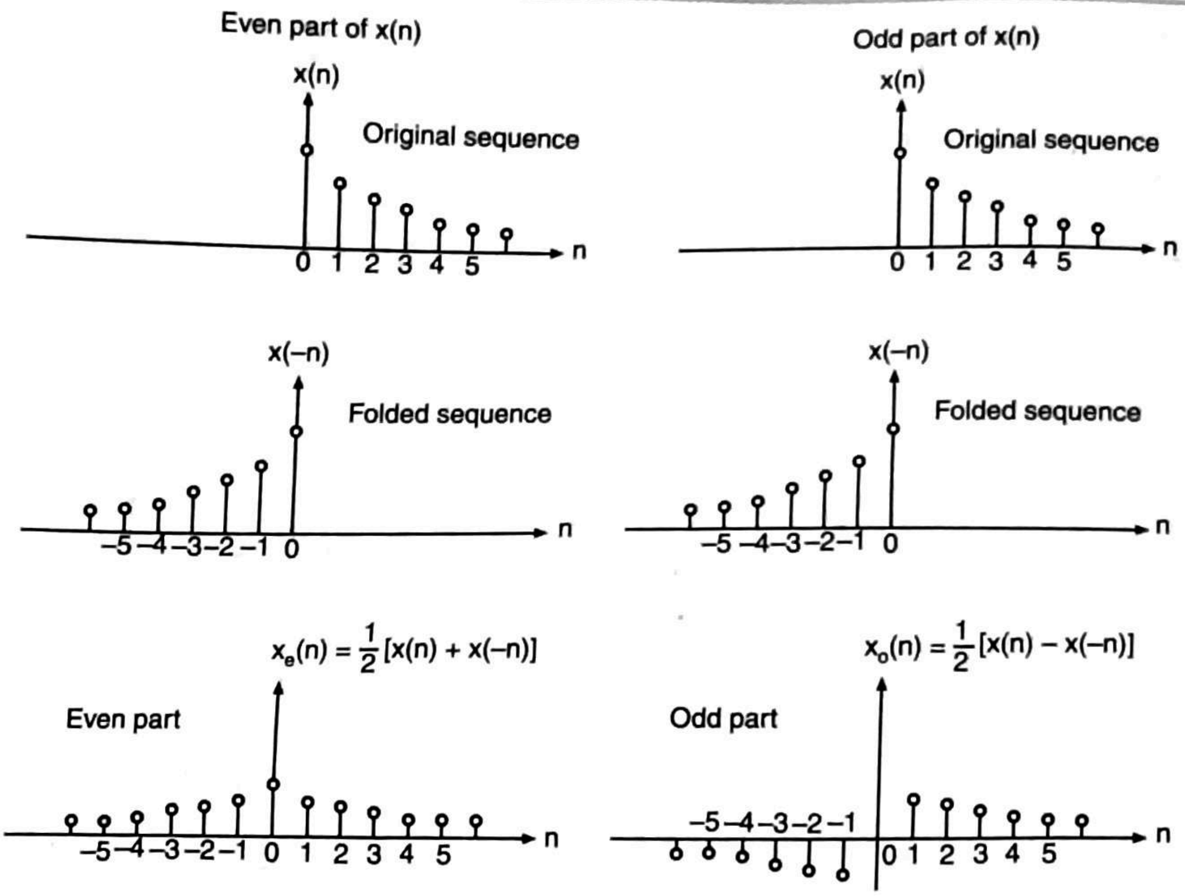


Fig. 2.18. Odd and even parts of x(n)

(ii)
$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2 - t & 0 \le t \le 1 \end{cases}$$

This is a triangular pulse as shown in figure 2.19. The even and odd parts of the signal x(t) are given as follows :

Even part $\quad x_e(t) = \frac{1}{2}\{x(t) + x(-t)\},$

and, odd part $\quad x_0(t) = \frac{1}{2}\{x(t) - x(-t)\}$

Figure 2.19 shows the steps to obtain even and odd parts.

(iii) Given that $x(t) = \cos^2\left(\frac{\pi}{2}t\right) = \frac{1 + \cos \pi t}{2}^{*} = \frac{1}{2} + \frac{1}{2}\cos \pi t$

Comparing the above expression with $x(t) = A + A \cos 2\pi\, ft$, we have, $2\pi\, ft = \pi t$

or $\quad f = \frac{1}{2}$ or $T = 2$.

Figure 2.20 shows the waveform of x(t). Even and odd parts may be expressed as under:

Even part, $\quad x_e(t) = \frac{1}{2}\{x(t) + x(-t)\},$

and, odd part, $\quad x_0(t) = \frac{1}{2}\{x(t) - x(-t)\}$

(iv) Given that $x(n) = \text{Im}[e^{jn\pi/4}]$

Simplifying, we get

* Since $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$$x(n) = \text{Im}\left[\cos\frac{n\pi}{4} + j\sin\frac{n\pi}{4}\right]^{*}, = \sin\frac{n\pi}{4}$$

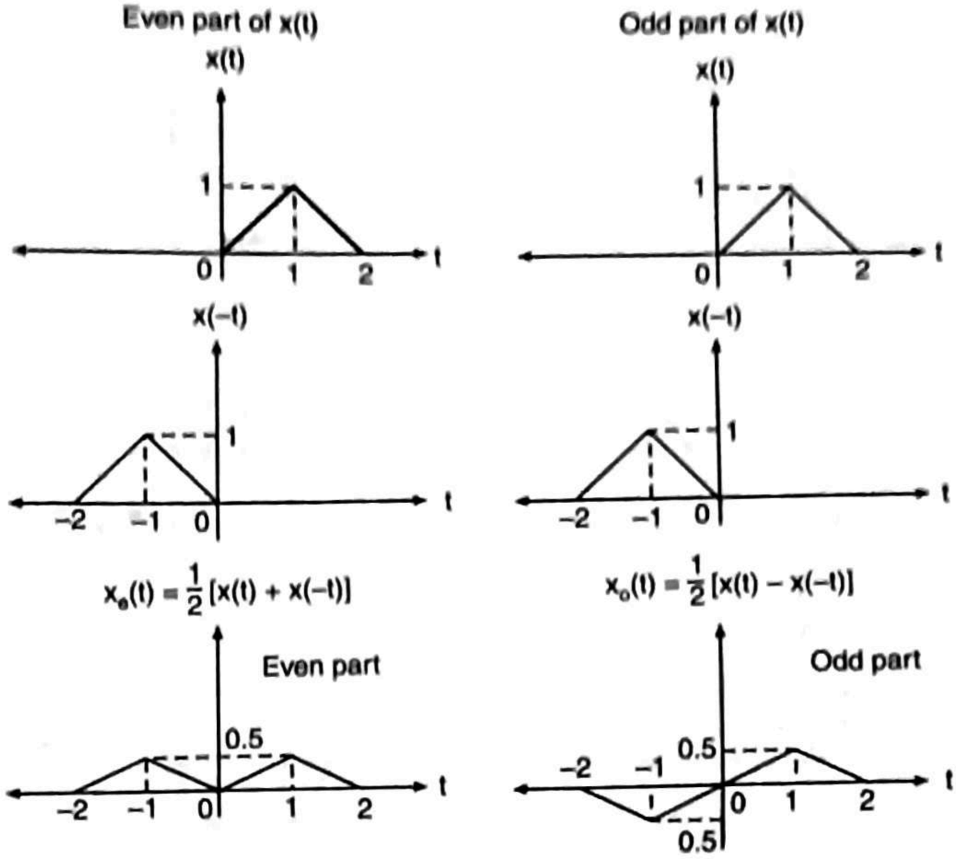


Fig. 2.19. Even and odd parts of x(t)

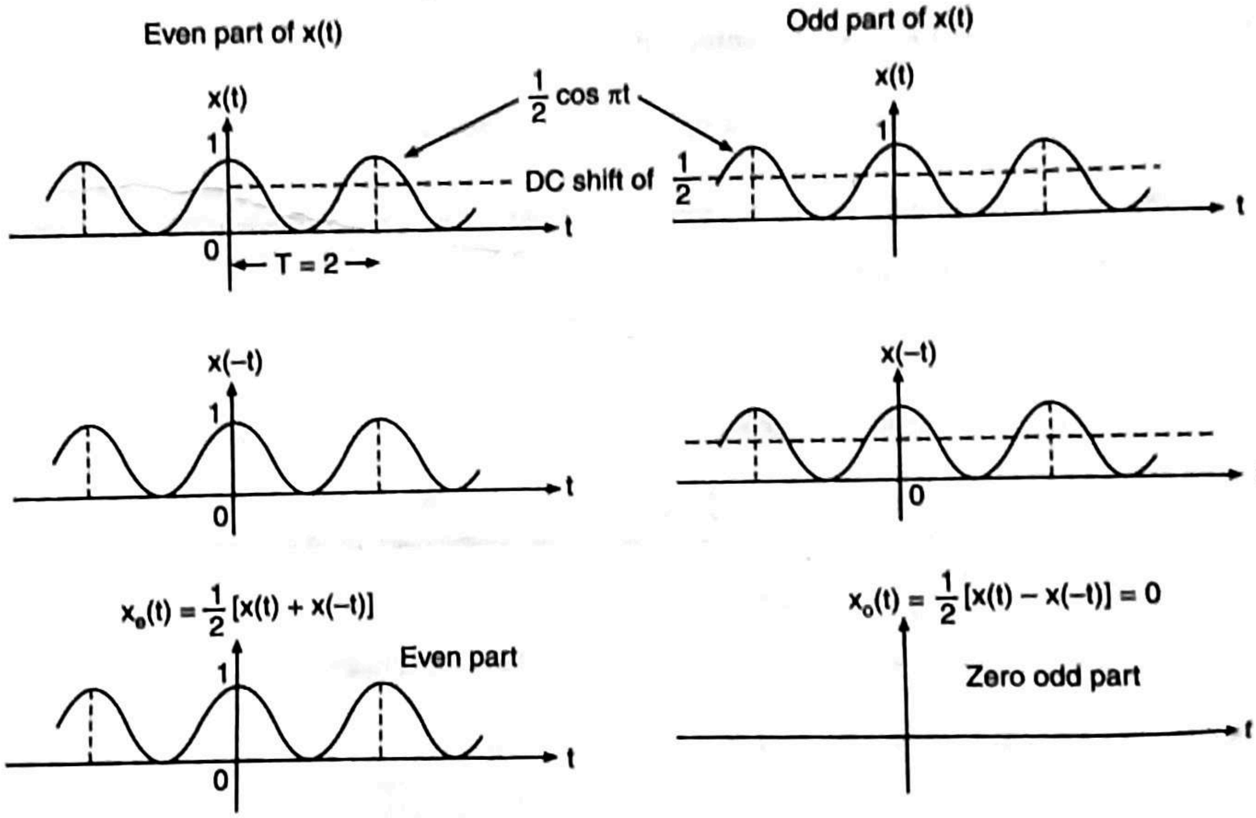


Fig. 2.20. Even and odd parts of x(t)

* Since $e^{j\theta} = \cos\theta + j\sin\theta$

Comparing above expression with $x(n) = \sin 2\pi fn$, we have, $2\pi fn = \frac{n\pi}{4}$

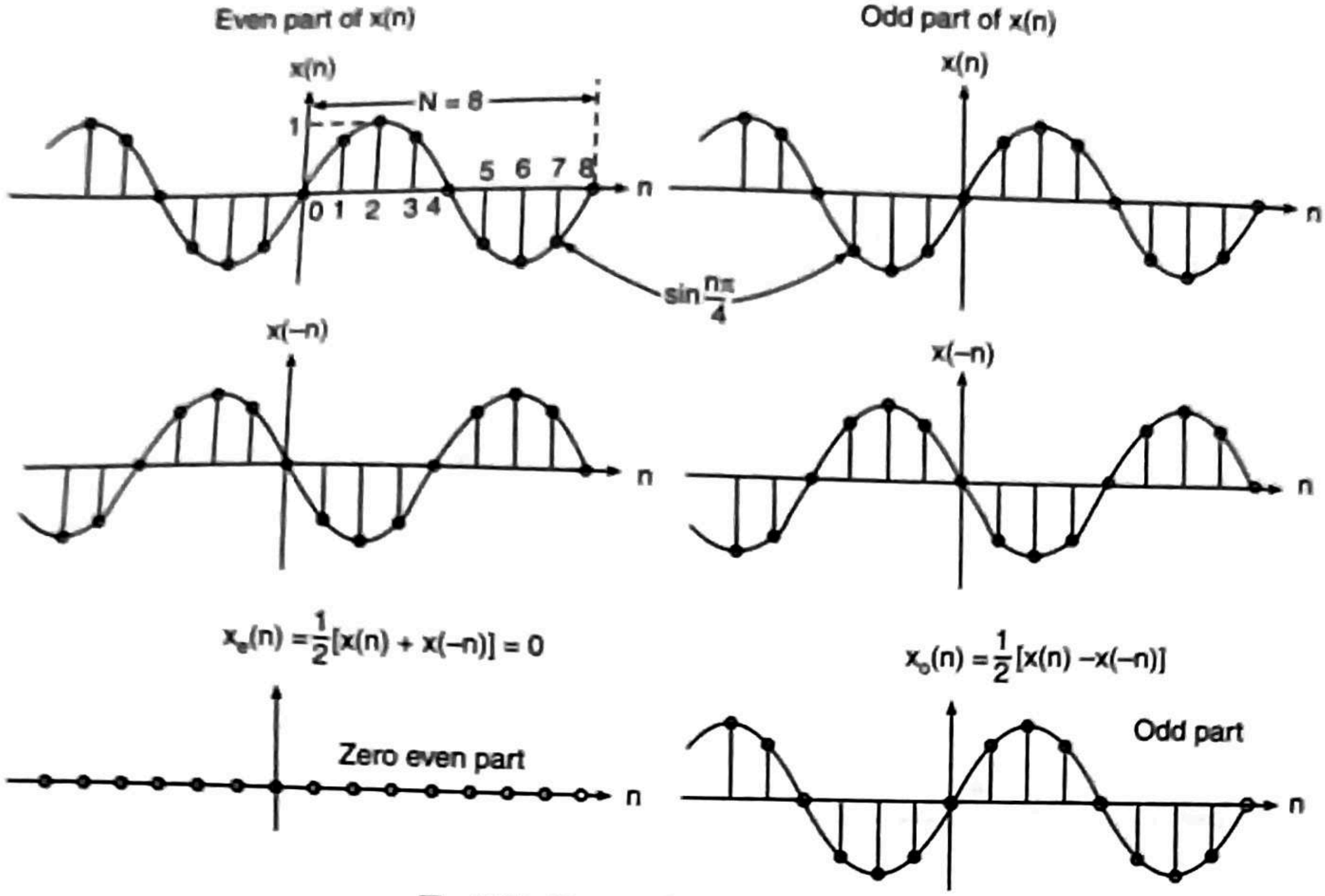


Fig. 2.21. Even and odd parts of x(n)

or $$f = \frac{1}{8} \text{ cycles/sample.}$$

Since $f = \frac{k}{N} = \frac{1}{8}$, there will be 8 samples in one period of discrete-time sine wave.

Figure 2.21 shows the waveform of $x(n) = \sin \frac{n\pi}{4}$ and its even and odd parts are also shown. Even and odd parts are expressed as under:

Even part, $x_e(n) = \frac{1}{2}\{x(n) + x(-n)\}$,

and odd part, $x_0(n) = \frac{1}{2}\{x(n) - x(-n)\}$.

2.7 ENERGY AND POWER SIGNALS

Signals may also be classified as energy and power signals. However, there are some signals which can neither be classified as energy signals nor power signals.

The energy signal is one which has finite energy and zero average power.

Hence, $x(t)$ is an energy signal, if :

$$0 < E < \infty \text{ and } P = 0$$

where, E is the energy and P is the power of the signal $x(t)$.

The power signal, is one which has finite average power and infinite energy.

Hence, x(t) is a power signal, if :

$$0 < P < \infty \text{ and } E = \infty$$

However, if the signal does not satisfy any of the above two conditions, then it is neither an energy signal nor a power signal.

2.7.1 Signal Energy and Power

As a matter of fact, the signal energy does not indicate the actual energy of the signal because the signal energy depends not only on the signal but also on the load resistor. Here, energy is interpreted as the energy dissipated in a normalized load of a 1 ohm resistor.

Now, if a voltage x(t) is applied across a R ohm resistor, the current through the resistor R is $\frac{x(t)}{R}$ as shown in figure 2.22.

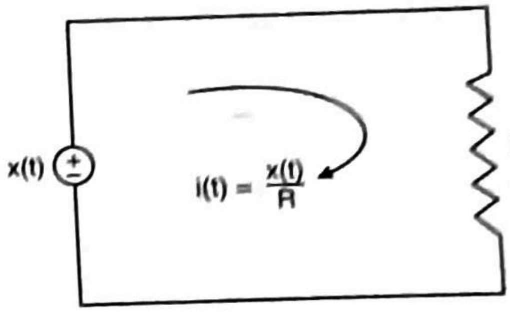


Fig. 2.22.

Hence, the instantaneous power in a circuit is given by

$$p(t) = v(t) \,.\, i(t) \qquad ...(2.20)$$

Substituting $v(t) = x(t)$ and $i(t) = \frac{x(t)}{R}$ in equation (2.20), we get

$$p(t) = x(t).\frac{x(t)}{R} = \frac{x^2(t)}{R} \qquad ...(2.21)$$

Further, we know that total energy dissipated is the integral of the instantaneous power

Therefore, the energy dissipated $= \int_{-\infty}^{\infty} \frac{x^2(t)}{R} dt$

Now, substituting $R = 1$, we get

$$\text{Energy dissipated} = \int_{-\infty}^{\infty} x^2(t)\, dt = E$$

Hence, if $R = 1$, the energy dissipated in the resistor is E. Therefore, the signal energy E may be interpreted as the energy dissipated in a unit resistor if a voltage x(t) is applied across this unit resistor.

For real signal, energy is :

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt \qquad ...(2.22)$$

For a complex valued signal, energy is :

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \qquad ...(2.23)$$

Similarly, the energy in a discrete-time signal x(n) is given as :

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad ...(2.24)$$

DO YOU KNOW ?

Quite often the particular representation used for a signal depends on the type of signal involved. It is therefore convenient to introduce a method for signal classification. It is useful to classify signals as those having finite energy and those having finite average power. Some signals have neither finite average power nor finite energy.

The signal energy must be finite for it to be meaningful. For the energy to be finite, the signal amplitude x(t) or x(n) must go to zero [$x(t)$ or $x(n) \to 0$] as $|t| \to \infty$. Otherwise, the integral in equation (2.23) or sum in equation (2.24) will not converge.

Figure 2.23 shows an example of energy signal.

For the signal shown in figure 2.23, the signal amplitude becomes zero as $|t| \to \infty$. Hence, this is an energy signal.

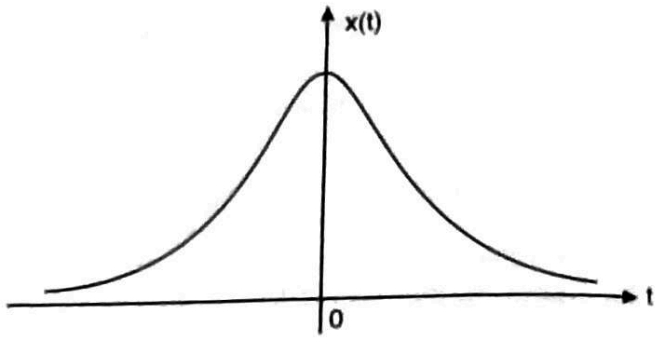


Fig. 2.23. An energy signal or a signal with finite energy

From above discussion, it is clear that almost all the practical non-periodic signals which are defined over finite time (also called time-limited signals) are energy signals.

If the signal amplitude does not become zero as $|t| \to \infty$, the signal energy will be infinite.

Consider any periodic signal. Its cycle repeats over and the signal amplitude of this signal does not become zero as $|t| \to \infty$. Hence, the energy for this signal will be infinite.

In such cases, we define another parameter called **average power,** which is the time average of energy. This is denoted by letter P.

For a real signal, average power P is given by :

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt \qquad \text{...(2.25)}$$

For a complex valued signal average power P is

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\,dt$$

Similarly, for a discrete-time signal (Periodic)

$$P = \lim_{N \to \infty} \frac{1}{N} \cdot \sum_{n=-N/2}^{N/2} |x(n)|^2 \qquad \text{...(2.26)}$$

From equation (2.26), it is obvious that the signal power P is also equal to the **mean-square value** of x(t).

Therefore, we can also conclude that the square root of power P is the root mean square (rms) value of x(t).

Figure 2.24 shows an example of power signal.

Almost all the practical periodic signals are power signals since their average power is finite and non-zero.

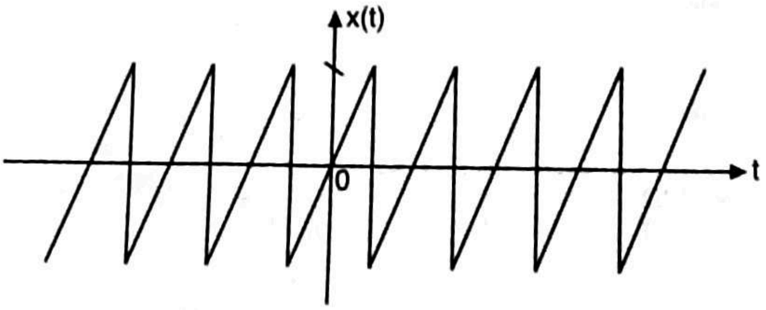
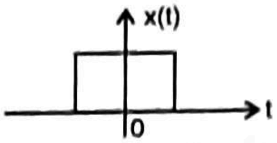
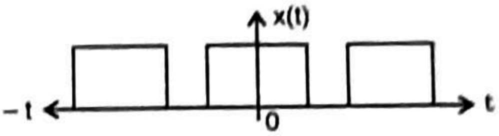


Fig. 2.24. A power signal

2.7.2 Performance Comparison of Energy and Power Signal

Table 2.1. Performance Comparison of Energy and Power Signal

S.No.	Energy Signal	Power Signal
1.	Total normalized energy is finite and non zero.	1. The normalized average power is finite and non zero.
2.	The energy is obtained by $E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 \, dt$.	2. The average power is obtained by $P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \lvert x(t)\rvert^2 \, dt$
3.	Non periodic signals are energy signals.	3. Practical periodic signals are power signals.
4.	These signals are time limited.	4. These signals can exist over infinite time.
5.	Power of energy signal is zero.	5. Energy of the power signal is infinite.
6.	For example a single rectangular pulse. 	6. For example a periodic pulse train. 

EXAMPLE 2.10. Sketch the following signal
$x(t) = A \sin t$ for $-\infty < t < \infty$
Also check whether the above signal is a power signal or an energy signal or neither.

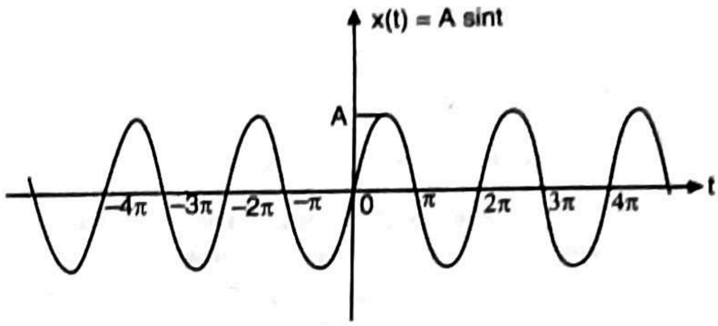


Fig. 2.25.

Solution : Figure 2.25 shows the given signal.

From figure it is clear that x(t) is periodic with period $T = 2\pi$ and therefore it is a power signal.

We know that power of a signal is expressed as

$$P = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} A^2 \sin^2 t \, dt$$

or $$P = \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}(1-\cos 2t)\,dt = \frac{A^2}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2t\right)dt$$

or $$P = \frac{A^2}{2\pi}\left[\frac{t}{2}-\frac{\sin 2t}{4}\right]_0^{2\pi} = \frac{A^2}{8\pi}[2t-\sin 2t]_0^{2\pi}$$

or $$P = \frac{A^2}{8\pi}\,[(4\pi - \sin 4\pi)] - (0-0)] = \frac{A^2}{2}$$ **Ans.**

EXAMPLE 2.11. Sketch the following signal
$x(t) = A\,[u(t+a) - u(t-a)]$. for $a > 0$
Also determine whether the given signal is a power signal or an energy signal or neither.
(Expected)

Solution : Figure 2.26 shows the given signal.

From figure, it is clear that x(t) is a finite duration signal and therefore x(t) is an energy signal.

We know that energy of a signal is expressed as

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-a}^{a} A^2\,dt$$

since $x(t) = A$ for $-a < t < a$

or $$E = 2\int_0^a A^2\,dt = 2A^2[t]_0^a = 2A^2 a = 2aA^2 \quad \textbf{Ans.}$$

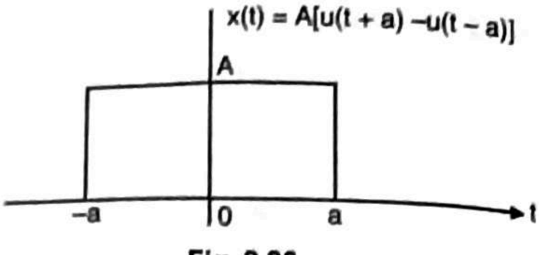


Fig. 2.26.

EXAMPLE 2.12. Sketch the following signal

$$x(t) = e^{-a|t|} \text{ for } a > 0$$

Also determine whether the signal is a power signal or an energy signal or neither.

Solution : The given signal is

$$x(t) = e^{-a|t|} \qquad \text{for } a > 0$$

This given signal may be expressed as

$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & \text{for } t > 0 \\ e^{at} & \text{for } t < 0 \end{cases}$$

This signal may be sketched as shown in figure 2.27.

From figure, it is obvious that the amplitude of $x(t) \to 0$ as $t \to \infty$, therefore, $x(t)$ is an energy signal.

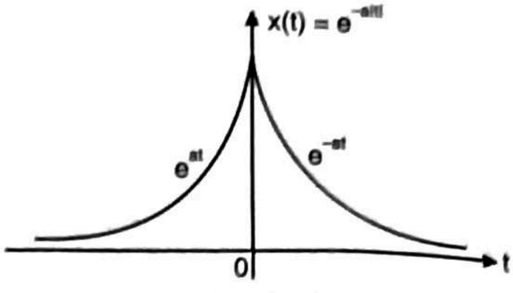


Fig. 2.27.

We know that the energy of a signal is expressed as

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} e^{-2a|t|}\,dt = \int_{-\infty}^{0} e^{-2a(-t)}\,dt + \int_0^{\infty} e^{-2a(t)}\,dt$$

or $$E = \int_{-\infty}^{0} e^{2at} + \int_0^{\infty} e^{-2at}\,dt = \int_0^{\infty} e^{-2at}dt + \int_0^{\infty} e^{-2at}\,dt = 2\int_0^{\infty} e^{-2at}\,dt = 2\left[\frac{e^{-2at}}{-2a}\right]_0^{\infty}$$

or $$E = -\frac{1}{a}\left[e^{-\infty} - e^0\right] = -\frac{1}{a}[0-1] = \frac{1}{a} \quad \textbf{Ans.}$$

Now, since, $E = \frac{1}{a} < \infty$ (a finite value), therefore $x(t)$ is an energy signal.

EXAMPLE 2.13. Figure 2.28 shows the signal x(t). Determine whether the signal is an energy signal or a power signal or neither. Also determine its energy or power.

Solution: From figure 2.28, it is obvious that the signal amplitude $|x(t)| \to 0$ as $|t| \to \infty$. Hence, the given signal is the energy signal.

The energy E of this signal is given as

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-1}^{0} (2)^2\,dt + \int_0^{\infty} \left(2e^{-t/2}\right)^2 dt$$

or $$E = \int_{-1}^{0} 4\,dt + \int_0^{\infty} 4\,e^{-t}\,dt = 4 + 4 = 8 \quad \textbf{Ans.}$$

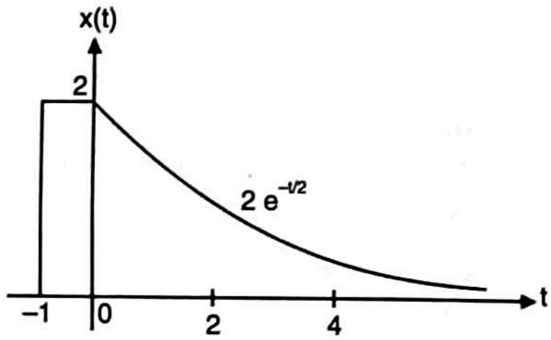


Fig. 2.28.

EXAMPLE 2.14. Find the power and the rms value of the signal

$$x(t) = A\cos(\omega_0 t + \theta).$$

Solution : The given signal $x(t) = A\cos(\omega_0 t + \theta)$ is a periodic signal having period T_0,

$$T_0 = \frac{2\pi}{\omega_0}$$

Hence, this is a power signal.

The power of a signal is expressed as

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$$

Putting the value of $x(t)$, we get

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} A^2\cos^2(\omega_0 t+\theta)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{A^2}{2}\left[2\cos^2(\omega_0 t+\theta)\right]dt$$

or
$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{A^2}{2}\left[1+\cos(2\omega_0 t+2\theta)\right]dt$$

or
$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{A^2}{2}dt + \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{A^2}{2}\cos(2\omega_0 t+2\theta)\,dt$$

or
$$P = \lim_{T\to\infty}\frac{A^2}{2T}\int_{-T/2}^{T/2}dt + \lim_{T\to\infty}\frac{A^2}{2T}\int_{-T/2}^{T/2}\cos(2\omega_0 t+2\theta)\,dt$$

or
$$P = \lim_{T\to\infty}\frac{A^2}{2T}.T + \lim_{T\to\infty}\frac{A^2}{2T}\int_{-T/2}^{T/2}\cos(2\omega_0 t+2\theta)\,dt \quad ...(i)$$

or
$$P = \frac{A^2}{2}+0 = \frac{A^2}{2} \quad \textbf{Ans.}$$

The second term in equation (i) is zero because in this term, the integral represents an area under a sinusoid over a time-interval T with $T\to\infty$. At most, this area could be equal to the area of half the cycle due to the cancellations of the positive and negative areas of a sinusoid.

Hence, this area multiplied by factor $A^2/2T$ with $T\to\infty$ will be zero since $\frac{A^2}{2T}\to 0$ as $T\to\infty$.

Therefore, $P = \frac{A^2}{2}$

This means that any sinusoid of amplitude A has a power $\frac{A^2}{2}$ irrespective of the value of its frequency ω_0 ($\omega_0 \neq 0$) and phase angle θ.

We know that the power of a signal is the square of its rms value.

Hence, the rms value of the given signal will be

$$\sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}} \quad \textbf{Ans.}$$

EXAMPLE 2.15. Figure 2.29 shows the signal x(t). Determine whether the signal is an energy signal or a power signal. Also determine its energy or power.

Solution : From figure 2.29, it is clear that the signal amplitude does not $\to 0$ as $|t|\to\infty$ and the signal is periodic. Therefore it is a power signal.

The power of a signal is expressed as

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}x^2(t)\,dt$$

We know that averaging $x^2(t)$ over an infinitely large time interval is identical to averaging it over one period (i.e., 2 secs in this case).

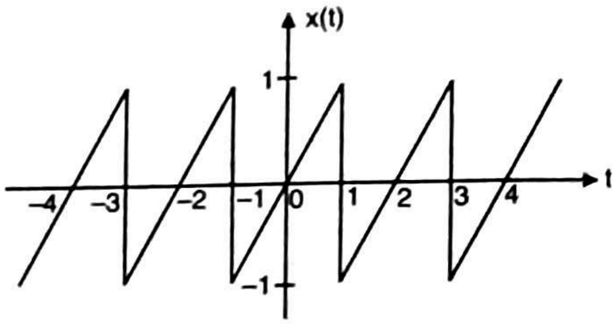


Fig. 2.29.

Therefore, power

$$P = \frac{1}{T}\int_{-T/2}^{T/2}x^2(t)\,dt = \frac{1}{2}\int_{-1}^{1}x^2(t)\,dt = \frac{1}{2}\int_{-1}^{1}t^2\,dt = \frac{1}{2}\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{1}{6}t^3 = \frac{1}{6}.2 = \frac{1}{3} \quad \textbf{Ans.}$$

EXAMPLE 2.16. What is the total energy of the rectangular pulse shown in figure 2.30.

Solution: Energy $\quad E = \int_{-\infty}^{\infty}x^2(t)\,dt$

$$= \int_{-T_1/2}^{T_1/2} A^2\,dt = A^2[t]_{-T_1/2}^{T_1/2}$$

$$= A^2\left[\frac{T_1}{2} - \left(-\frac{T_1}{2}\right)\right]$$

Therefore, we have $E = A^2 \times T_1 = A^2 T_1$ **Ans.**

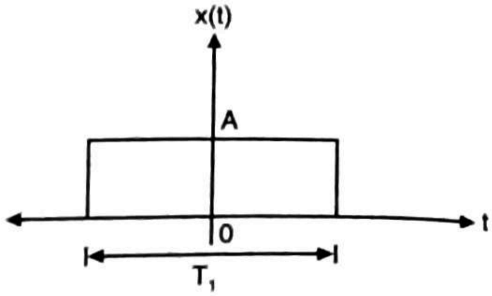


Fig. 2.30.

EXAMPLE 2.17. What is the average power of the square wave shown in figure 2.31.

Solution: Given signal is periodic. Hence, let us consider one cycle from 0 to T.

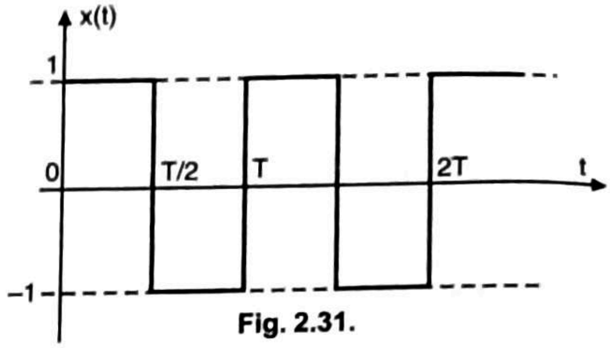


Fig. 2.31.

Hence, we have $P = \frac{1}{T}\int_0^T x^2(t)$

But
$$x(t) = \begin{cases} 1 & \text{for } 0 \le x(t) \le \frac{T}{2} \\ -1 & \text{for } \frac{T}{2} \le x(t) \le T \end{cases}$$

Therefore, we write

$$P = \frac{1}{T}\left[\int_0^{T/2}(1)^2 dt + \int_{T/2}^{T}(-1)^2 dt\right] = \frac{1}{T}\left[1(t)_0^{T/2} + 1(t)_{T/2}^{T}\right] = \frac{1}{T}\left[\frac{T}{2} + \frac{T}{2}\right] = 1 \quad \textbf{Ans.}$$

EXAMPLE 2.18. Calculate the average power of the triangular wave shown in figure 2.32.

Solution: This signal is mathematically represented as under :

$$x(t) = \begin{cases} 20t - 1 & \text{for } 0 \le t \le 0.1 \\ -20t + 3 & \text{for } 0.1 \le t \le 0.2 \end{cases}$$

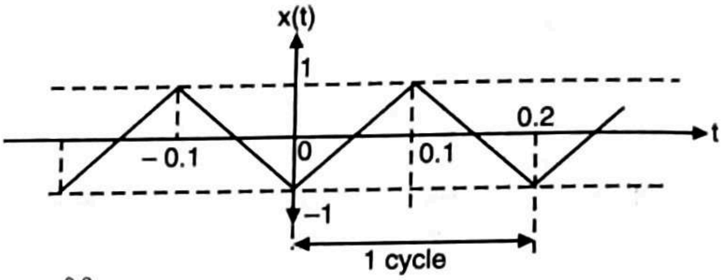


Fig. 2.32

Hence, the signal power is given by

$$P = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{0.2}\int_0^{0.1}(20t-1)^2 dt + \frac{1}{0.2}\int_{0.1}^{0.2}(-20t+3)^2 dt$$

or
$$P = \frac{1}{0.2}\int_0^{0.1}(400t^2 - 40t + 1)dt + \frac{1}{0.2}\int_{0.1}^{0.2}(400t - 120t + 9)\,dt$$

or
$$P = 5\left\{400\left[\frac{t^3}{3}\right]_0^{0.1} - 40\left[\frac{t^2}{2}\right]_0^{0.1} + [t]_0^{0.1} + 400\left[\frac{t^3}{3}\right]_{0.1}^{0.2} - 120\left[\frac{t^2}{2}\right]_{0.1}^{0.2} + 9[t]_{0.1}^{0.2}\right\}$$

or
$$P = 5\{0.1333 - 0.2 + 0.1 + 0.9333 - 1.8 + 0.9\}$$

or
$$P = 0.333 \text{ or } \frac{1}{3} \quad \textbf{Ans.}$$

EXAMPLE 2.19. Prove the following:

(i) The power of the energy signal is zero over infinite time.

(ii) The energy of the power signal is infinite over infinite time.

Solution: **(i) Power of the energy signal**

Let x(t) be an energy signal. We know that the power is given by

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = \lim_{T\to\infty}\frac{1}{T}\left[\lim_{T\to\infty}\int_{-T/2}^{T/2}|x(t)|^2 dt\right] = \lim_{T\to\infty}\frac{1}{T}\left[\int_{-\infty}^{\infty}|x(t)|^2 dt\right]$$

Here, limits are changed from $-\infty$ to ∞ as $T \to \infty$

Thus, we have $$P = \lim_{T \to \infty} \frac{1}{T} E$$

Since quantity inside brackets is E, therefore, we have

$$P = 0 \times E = 0. \qquad \left[\text{since } \lim_{T \to \infty} \frac{1}{T} = 0\right]$$

Thus, Power of energy signal is zero over infinite time

(ii) Energy of the power signal

Energy of the signal is given by the following expression

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Let us change the limits of integration as $-\frac{T}{2}, \frac{T}{2}$ and take $\lim_{T \to \infty}$. This will not change meaning of above expression.

Therefore, we have

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

Rearranging, we get $$E = \lim_{T \to \infty} \left[T . \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \right]$$

or $$E = \lim_{T \to \infty} T . \left[\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \right]$$

or $$E = \lim_{T \to \infty} T . P \quad \text{(since quantity inside brackets is P).}$$

or $$E = \infty \quad \text{(By taking limits as } T \to \infty)$$

Thus, Energy of the power signal is infinite over infinite time

EXAMPLE 2.20. Determine whether the following signals are energy signals or power signals and calculate their energy or power.

(i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$ **(ii)** $x(t) = \text{rect}\left(\frac{t}{T_0}\right)$

(iii) $x(t) = \cos^2 \omega_0 t$ **(iv)** $x(t) = \text{rect}\left(\frac{t}{T_0}\right) \cos \omega_0 t$

(v) $x(n) = u(n)$ **(vi)** $x(t) = A\, e^{-\alpha t} . u(t), \alpha > 0$

Solution: When solving such examples, we do not know whether the signal has finite power or finite energy. Hence, we follow the following steps :

Important Steps

(i) First, we observe the given signal carefully. If it is periodic and of infinite duration, then, it can be power signal. Hence, we calculate its power directly.

(ii) If the signal is periodic and of finite duration, then, it can be energy signal. Hence, we calculate its energy directly.

(iii) If the signal is not periodic, then it can be energy signal. Hence, we calculate its energy directly.

(i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$

This signal is not periodic. Hence as per step (iii), we calculate its energy directly.

Thus, we have $$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty}\left[\left(\frac{1}{2}\right)^n\right]^2 = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n*}$$

Here, let us use, $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$. Then, the above expression will be

$$E = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

Since energy is finite and non-zero, therefore, it is energy signal with $E = \frac{4}{3}$. **Ans.**

(ii) $\mathbf{x(t) = rect\left(\frac{t}{T_0}\right)}$

We know that the $\text{rect}\left(\frac{t}{T_0}\right)$ function can be expressed as under:

$$\text{rect}\left(\frac{t}{T_0}\right) = \begin{cases} 1 & \text{for } -\frac{T_0}{2} \le t \le \frac{T_0}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \text{...(i)}$$

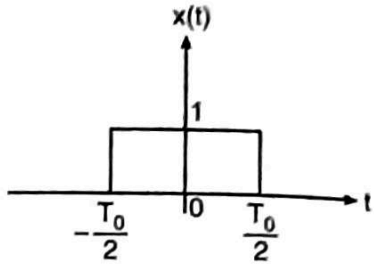


Fig. 2.33. A rect function

Figure 2.33 shows this function.
It is a non-periodic function. Hence, it can be energy signal. Therefore, we calculate its energy directly.

Therefore, we have $$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

From figure 2.33, we have $$E = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (1)^2\, dt = [t]_{-T_0/2}^{T_0/2} = T_0$$

The energy is finite and non-zero. Hence, it is energy signal with $E = T_0$.

(iii) $\mathbf{x(t) = \cos^2 \omega_0 t}$

This is squared cosine wave, hence it is periodic. Therefore, this can be periodic signal. As per step (i), we calculate power of this signal directly.

Therefore, we have $$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

The given signal $x(t) = \cos^2 \omega_0 t$ has some period T_0 and it is real signal. Hence, above expression will be

$$P = \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} [\cos^2 \omega_0 t]^2\, dt$$

Here, $[\cos^2 \omega_0 t]^2 = \cos^4 \omega_0 t$. It can be expanded by standard trigonometric relations as under :

$$P = \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{8}[3 + 4\cos 2\omega_0 t + \cos 4\,\omega_0 t]\, dt$$

or $$P = \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{3}{8}\, dt + \underbrace{\lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} 4\cos 2\omega_0 t\, dt}_{\text{This term will be zero since it is integration of cosine wave over "full cycles"}} + \underbrace{\lim_{T_0\to\infty} \int_{-T_0/2}^{T_0/2} \cos 4\,\omega_0 t\, dt}_{\text{This term will be also zero since it is integration of cosine wave over "full cycles"}}$$

* Since $u(n) = 1$ for $n = 0$ to ∞.

Therefore, we write $$P = \lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8} \cdot [t]_{-T_0/2}^{T_0/2} + 0 + 0 = \lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8} \cdot T_0 = \frac{3}{8}$$

The power of the signal is finite and non-zero, hence it is power signal with $P = \frac{3}{8}$.

(iv) $x(t) = \text{rect}\left(\frac{t}{T_0}\right) \cos \omega_0 t$

The given function is the product of cosine wave and rect function. Figure 2.34 shows how x(t) is derived.

(i) $\cos \omega_0 t$ is periodic and infinite duration signal.

(ii) Basically it is power signal.

(iii) $\cos \omega_0 t$ is multiplied with the rectangular pulse. Hence, the resultant signal is a cosine wave of duration $-\frac{T_0}{2} \le t \le \frac{T_0}{2}$.

(iv) It is assumed that there are multiple number of cycles of cosine wave in $-\frac{T_0}{2} \le t \le \frac{T_0}{2}$.

The final signal is periodic but of finite duration. Hence, it can be energy signal. Therefore, we calculate as per step (ii) its energy directly.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-T_0/2}^{T_0/2} |\cos^2 \omega_0 t|^2 \, dt$$

$$= \int_{-T_0/2}^{T_0/2} \left(\frac{1 + \cos 2\omega_0 t}{2} \right) dt$$

or $$E = \frac{1}{2} \int_{-T_0/2}^{T_0/2} dt + \underbrace{\frac{1}{2} \int_{-T_0/2}^{T_0/2} \cos 2\omega_0 t}_{\substack{\text{This term will be zero} \\ \text{since it is integration of} \\ \text{cosine wave over "full cycles"}}} = \frac{T_0}{2}$$

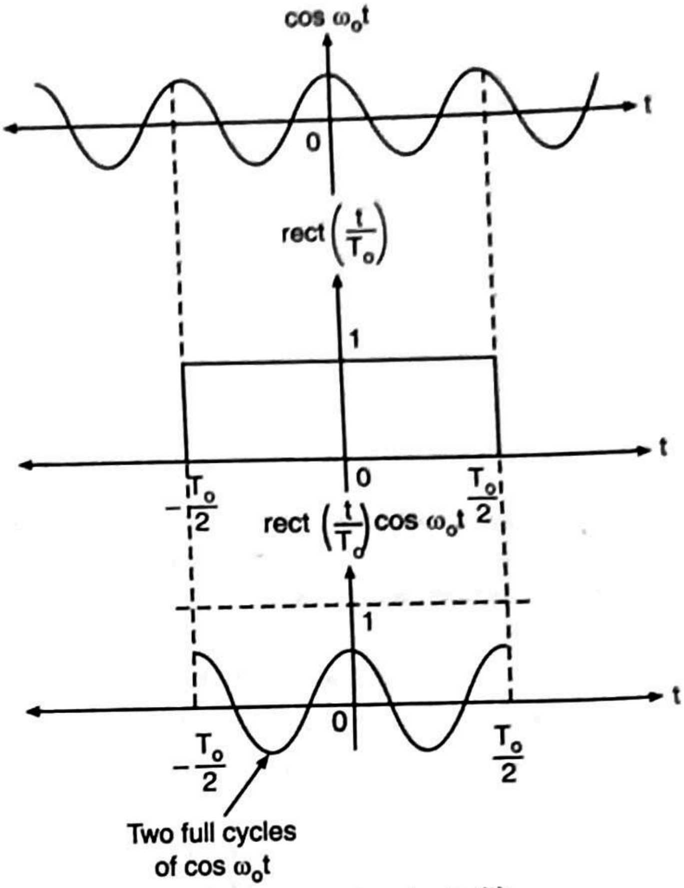


Fig. 2.34. Sketch of x(t)

Here, energy is finite and non-zero, hence, it is Energy signal with $E = \frac{T_0}{2}$

(v) $x(n) = u(n)$

This signal is periodic (since u(n) repeats after every sample) and of infinite duration. Hence, it may be power signal. Therefore, let us calculate its power directly.

Thus, we have $$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} (1)^2 \quad \text{since } u(n) = 1 \text{ for } 0 \le n \le 8$$

Here, $\sum_{n=0}^{N} (1)^2$ means 1 + 1 + 1 + 1 + 1 .... for n = 0 to N. In other words, we have

1 + 1 + 1 + 1 ...... (N + 1) times = (N + 1). Therefore, above expression will be,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \cdot (N+1) = \lim_{N \to \infty} \frac{N+1}{2N+1} = \lim_{N \to \infty} \frac{1+\frac{1}{N}}{2+\frac{1}{N}} = \frac{1}{2}$$

The power is finite and non-zero, hence unit step function is power signal with $P = \frac{1}{2}$.

(vi) $x(t) = Ae^{-\alpha t}u(t), a > 0$

This signal is non periodic and of infinite duration. It can be energy signal. Therefore, we calculate its energy directly as per step (iii).

Thus, we have $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{0}^{\infty} [A\, e^{-\alpha t}]^2 \, dt^* = A^2 \int_{0}^{\infty} e^{-2\alpha t} \, dt = A^2 \cdot \left[\frac{e^{-2\alpha t}}{-2\alpha}\right]_0^{\infty} = \frac{A^2}{2\alpha}$

The energy is finite and non-zero, hence the given signal is energy signal with $E = \frac{A^2}{2\alpha}$

Important Points:

(i) As $\alpha \to 0$, $e^{-\alpha t} \to 1$, x(t) becomes unit step and $E \to \infty$.

(ii) Unit step is a power signal whose energy becomes infinite as stated above. This confirms to our earlier statement that *energy of power signal is infinite over infinite time.*

2.8 MULTICHANNEL AND MULTIDIMENSIONAL SIGNALS

1. Multichannel signals

As the name indicates, multichannel signals are generated by multiple sources or multiple sensors. The resultant signal is the vector sum of signals from all channels.

Example

A common example of multichannel signal is ECG waveform. To generate ECG waveform, different leads are connected to the body of a patient. Each lead is acting as individual channel. Since there are n number of leads ; the final ECG waveform is a result of multichannel signal. Mathematically, final wave is expressed as,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \text{ ; If three leads are used.}$$

2. Multidimensional Signals

If a signal is a function of single independent variable, the signal is called as one-dimensional signal. On the other hand, if the signal is a function of multi (many) independent variables then it is called as multidimensional signal.

A good example of multidimensional signal is the picture displayed on the TV screen. To locate a pixel (a point) on the TV screen, two coordinates namely X and Y are required.

Similarly this point is a function of time also. So, to display a pixel, minimum three dimensions are required, namely x, y and t. Thus, this is multidimensional signal. Mathematically, it can be written as P (x, y, t).

DO YOU KNOW?
Important examples of singul-arity functions are the unit impulse, the unit step, and the unit ramp.

3. Performance Comparison of Multichannel and Multidimensional Signal

S.No.	Multichannel Signal	Multidimensional signal
1.	Such signals are generated by multiple sources or multiple sensors.	Such signals are function of many independent variables.
2.	Example : ECG signal. Mathematically it is represented by, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$	Example : Picture displayed on TV screen. Mathematically, it is represented by P(x,y,t)

2.9 SOME COMMONLY USED SIGNALS : SOME STANDARD SIGNALS

In signals and systems, we need to use some standard signals. In this section, we shall show some important standard signals graphically and express them mathematically.

Some of the standard continuous time and discrete time signals are :

(i) A dc signal
(ii) Sinusoidal signal
(iii) Unit step signal
(iv) Signum function
(v) A rectangular pulse
(vi) Delta or unit impulse function
(vii) Unit Ramp signal
(viii) Exponential signals
(ix) Sinc function.

Let us understand them one-by-one

2.9.1 DC Signal

1. Continuous Time DC Signal

A dc signal is shown in figure 2.35. As seen from the waveform, the amplifier A of a direct current (dc) signal remains constant and independent of time. The dc signal can be represented mathematically as under :

A dc signal : $x(t) = A$ for $-\infty < t < \infty$ ...(2.28)

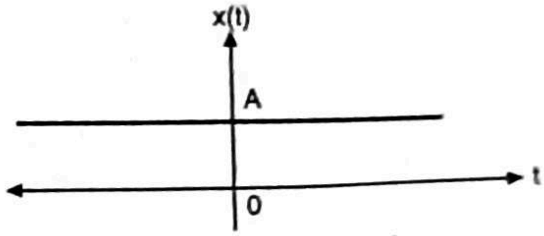


Fig. 2.35. A dc signal

2. Discrete Time DC Signal

Figure 2.36 shows the discrete time dc signal. It is a sequence of samples each of amplitude A and extending from $-\infty < n < \infty$.

This signal can be mathematically represented as under :

$$x(n) = A \quad \text{for } -\infty < n < \infty$$

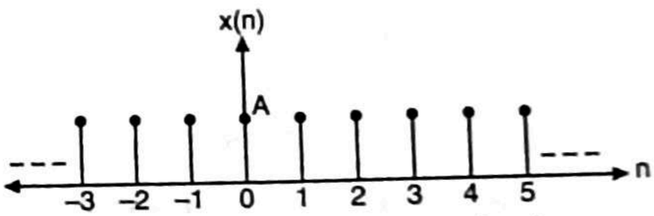


Fig. 2.36. A discrete time dc signal

Or it can be represented in the infinite sequence form as under :

$$x(n) = \{\ldots\ldots A, A, \underset{\uparrow}{A}, A, A, \ldots.)$$

2.9.2 Sinusoidal Signals

The sinusoidal signals include sine and cosine signals. They are shown in the figure 2.37.

Mathematically, they can be represented as under :

A sine signal : $x(t) = A \sin \omega t = A \sin (2\pi ft)$ ...(2.29)

A cosine signal : $x(t) = A \cos \omega t = A \cos (2\pi ft)$ ...(2.30)

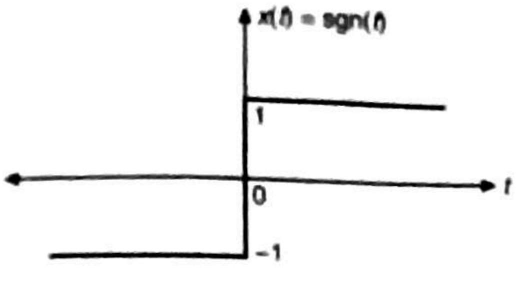


Fig. 2.41. Signum function

DO YOU KNOW?

The signum function is closely related to the unit step function. But the name signum is more common because of the confusion which can result between the homonyms sign and sine.

Mathematically, signum function is represented as under:

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases} \quad ...(2.32)$$

The signum function is an odd or antisymmetric function.

Discrete Time Signum Function

A discrete-time signum function can be obtained by sampling the continuous time signum function.

It is a train of samples of values + 1 for positive n and − 1 for negative n as shown in figure 2.42.

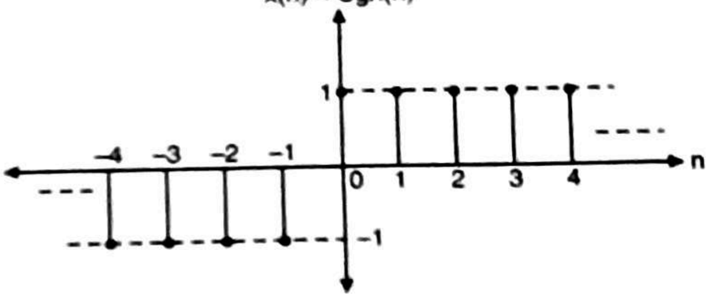


Fig. 2.42. A discrete time signum function

2.9.5 Rectangular Pulse

A rectangular pulse of unit amplitude and duration is as shown in figure 2.43.

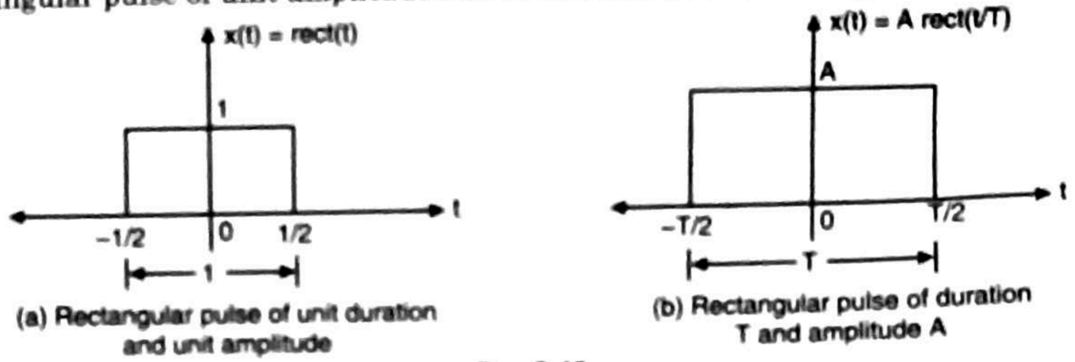


Fig. 2.43.

1. Rectangular Pulse of Unit Amplitude and Unit Duration

The unit rectangular pulse is centered about the y-axis i.e. about t = 0.

Mathematically, it is represented as under:

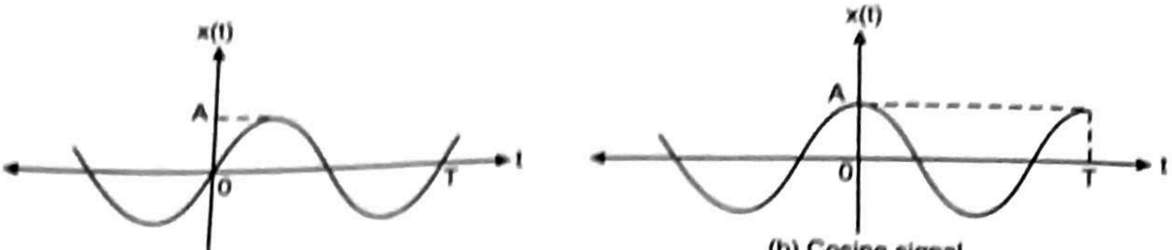
$$\text{rect}(t) = \begin{cases} 1 & \text{for } -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad ...(2.33)$$

2. Rectangular Pulse of Amplitude A and duration T

The other general type of rectangular pulse having an amplitude of A over a duration of T is as shown in figure 2.43.

This pulse is also centered about t = 0. It is, mathematically, represented as,

$$A\,\text{rect}\left[\frac{t}{T}\right] = \begin{cases} A & \text{for } -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \quad ...(2.34)$$



(a) Sine signal

(b) Cosine signal

Fig. 2.37. Sinusoidal Signal

Discrete Time Sinusoidal Wave

A discrete time sinusoidal waveform is denoted by,

$$x(n) = A \sin \omega n$$

Here A = Amplitude

ω = Angular Frequency = $2\pi f$.

This waveform is shown in figure 2.38.

Similarly, a discrete time cosine wave is expressed as,

$$x(n) = A \cos \omega n$$

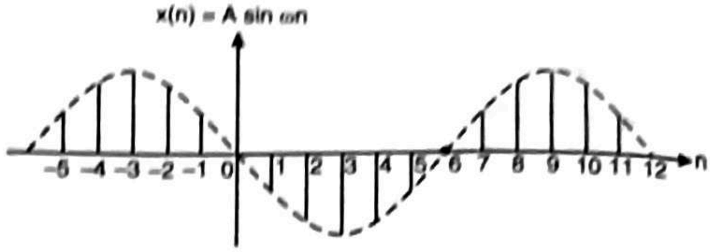


Fig. 2.38. Discrete time sinusoidal waveform

2.9.3 Unit Step Signal

1. Continuous Unit Step Signal

The unit step signal is shown in figure 2.39. It has a constant amplitude of unity for the zero or positive values of time t. Whereas it has a zero value for negative values of t.

Mathematically, the unit step signal is represented as,

$$\text{Unit step signal: } u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \qquad \text{...(2.31)}$$

2. Discrete-Time Unit Step Signal

A discrete time unit step signal is denoted by u(n). Its value is unity for all positive values of n. This means that its value is one for $n \geq 0$. While for other values of n, its value is zero.

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

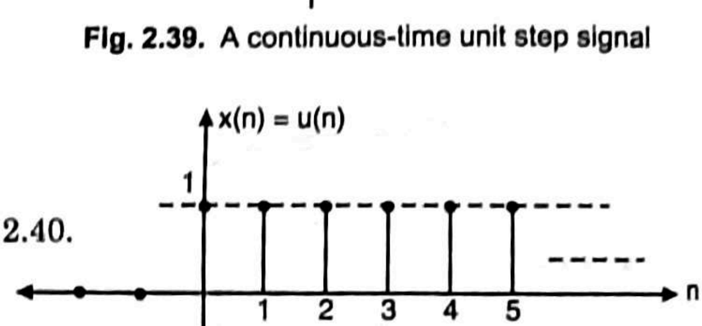


Fig. 2.39. A continuous-time unit step signal

In the form of sequence, it can be written as under:

$$u(n) = \{\underset{\uparrow}{1}, 1, 1, 1, ...\}$$

This signal is graphically represented in figure 2.40.

Fig. 2.40. A discrete time unit step signal

2.9.4 Signum Function

The signum function is shown in figure 2.41.

In this expression, T shows that it is a function of time and T represents the width of the rectangular pulse, and A is the amplitude. Rectangular pulse is an even or symmetrical function. Hence, we have

$$x(t) = \underbrace{A}_{\text{Amplitude}}\ \text{rect}\left[\frac{\overbrace{t}^{\text{Variable is time } t}}{\underbrace{T}_{\text{Width of the pulse}}}\right]$$

It is possible to represent the rectangular signal in frequency domain as under :

$$X(f) = A\ \text{rect}\left[\frac{f}{2f_m}\right]$$

and it is graphically represented as shown in figure 2.44.

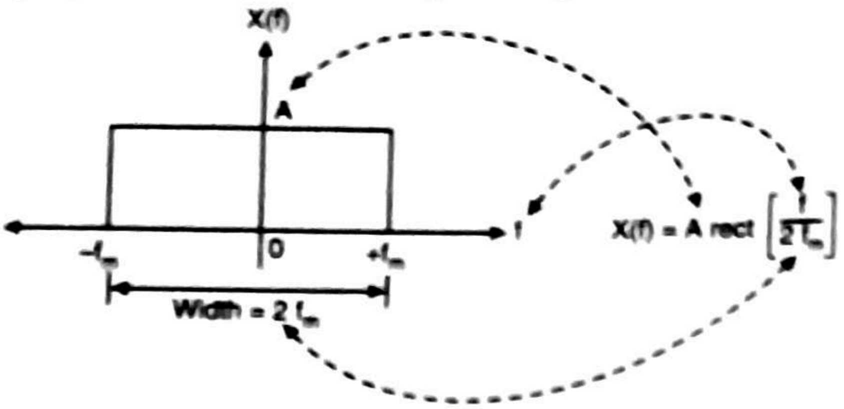


Fig. 2.44.

2.9.6 Delta or Unit Impulse Function δ(t)*

1. Basic Concepts

The delta function δ (t) is an extremely important function used for the analysis of communication systems. The impulse response of a system is its response to a delta function applied at the input.

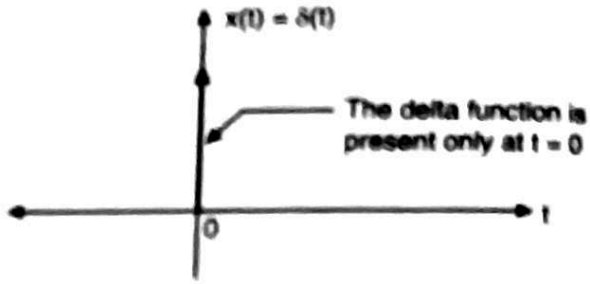


Fig. 2.45. Delta function

The delta function is as shown in figure 2.45. It is present only at t = 0, its width tends to 0 and its amplitude at t = 0 is infinitely large so that the area under the pulse is unity (i.e. 1). Due to unity area, it is called as a unit impulse function.

Delta function : $$\delta(t) = \begin{cases} 1 & \text{For } t = 0 \\ 0 & \text{For } t \neq 0 \end{cases} \qquad ...(2.35)$$

The area under the unit impulse is given as,

Area under unit impulse :

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad ...(2.36)$$

---

* Discuss the significance of :
(i) Impulse function (ii) Unit step function (iii) Sinc function
(WBUT, Kolkata, Sem. Exam. 2003-04) (08 marks)

2. Important Properties of a Delta Function

Here, let us understand two important properties of the delta function. They are :

(i) Shifting property.

(ii) Replication property.

> **DO YOU KNOW?**
>
> The area under an impulse is called its strength, or sometimes its weight. An impulse with a strength of one is called a unit impulse.

3. Shifting property

The shifting property of delta function states that

$$\int_{-\infty}^{\infty} x(t)\,\delta(t - t_m)\,dt = x(t_m) \qquad ...(2.37)$$

where $\delta(t - t_m)$ represents the time shifted delta function. This delta function is present only at $t = t_m$. The RHS of equation (2.37) represents the value of x (t) at $t = t_m$. This result indicates that the area under the product of a function with an impulse $\delta(t)$ is equal to the value of that function at the instant where the unit impulse is located. This property is also known as the sampling property. The other property of delta function is the replication property. It states that,

4. Replication property

$$x(t) * \delta(t) = x(t) \qquad ...(2.38)$$

This property states that the convolution of any function x(t) with delta function yields the same function. The sign * in equation (2.38) represents convolution.

5. Unit Sample Sequence

The discrete time version of unit impulse signal is the unit sample sequence.

A discrete time unit impulse function is denoted by $\delta(n)$. Its amplitude is 1 at n = 0 and for all other values of n, its amplitude is zero.

Therefore,
$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

In the sequence form, it can be represented as under:

$$\delta(n) = \{..., 0, 0, 0, \underset{\uparrow}{1}, 0, 0, 0\}$$

or
$$\delta(n) = \{1\}$$

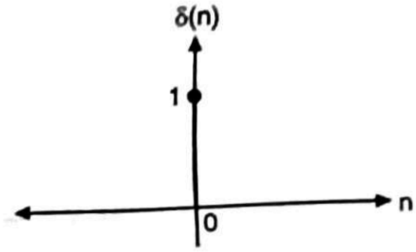


Fig. 2.46. Unit sample signal δ(n)

The graphical representation of unit sample signal is as shown in figure 2.46.

6. Relation Between Discrete-time Unit Impulse and Discrete-time Unit Step Signals

Figure 2.47 (a) shows how to obtain a discrete-time unit impulse signal δ(n) from a discrete-time unit step signal u(n). Mathematically, this relation can be expressed as

$$\delta(n) = u(n) - u(n-1) \qquad ...(2.39)$$

We can obtain a discrete-time unit step signal u(n) by taking the sum of unit samples. This is expressed as,

$$u(n) = ... + \delta(n) + \delta(n-1) + \delta(n-2) + ...$$

or
$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) \qquad ...(2.40)$$

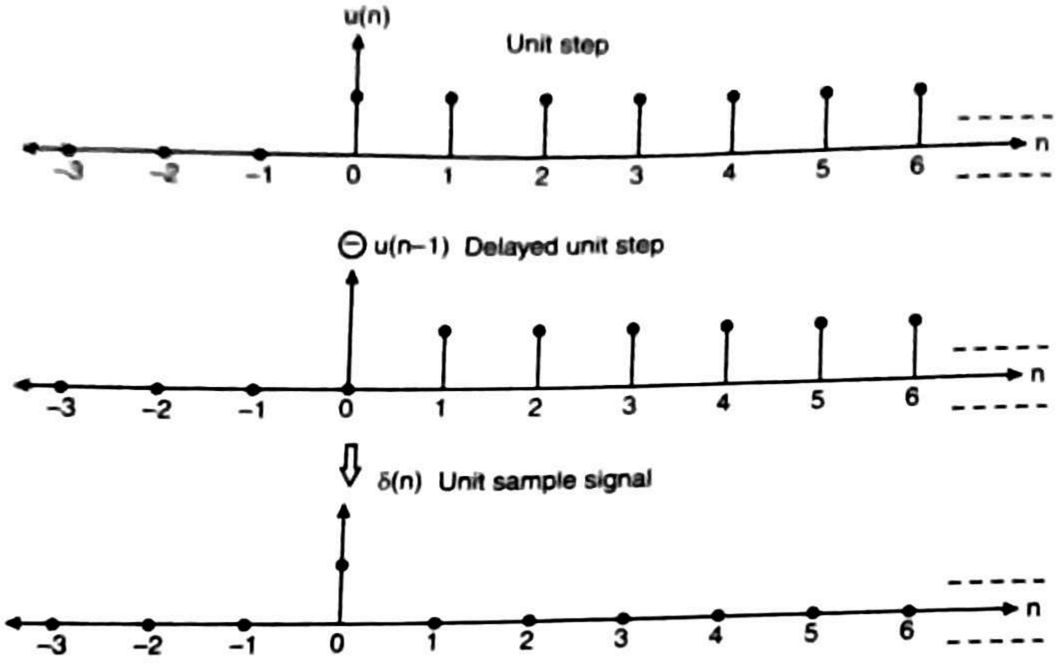


Fig. 2.47. How to obtain a unit sample signal from a unit step signal

The summation of unit impulses to obtain a unit step function is shown in figure 2.48.

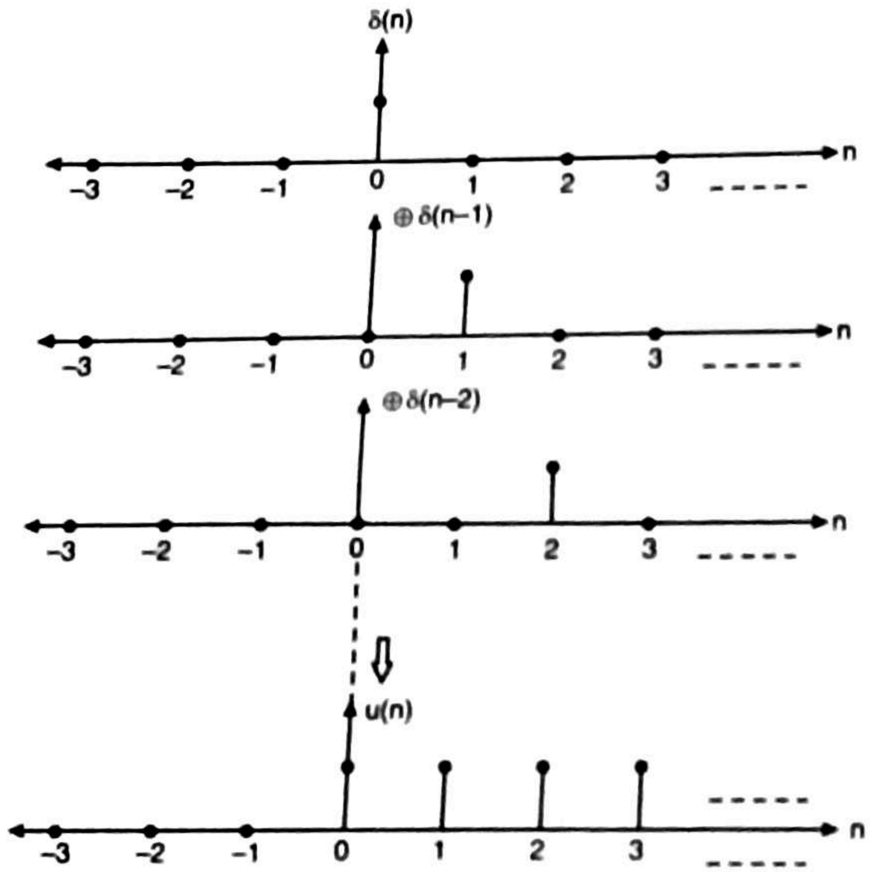


Fig. 2.48. How to obtain a discrete-time unit step from unit impulses

7. Sampling Using a Unit Sample Signal

The unit sample sequence can be used to sample the value of a signal.

If signal x (n) is multiplied by a unit sample δ (n), then we get the value of x (n) at n = 0 as the product. This means that

$$x(n)\,\delta(n) = x(0) \qquad ...(2.41)$$

This happens because

$$\delta(n) = 1 \text{ only at } n = 0$$

Similarly, we can write that,

$$x(n)\,\delta(n - n_0) = x(n_0) \qquad ...(2.42)$$

2.9.7 Unit Ramp Signal

A discrete time unit ramp signal is denoted by $u_r(n)$. Its value increases linearly with sample number n. Mathematically, it is defined as

$$u_r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Graphically, it is represented as shown in figure 2.49 (a).

A continuous time ramp signal is denoted by r (t). Mathematically, it is expressed as,

$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

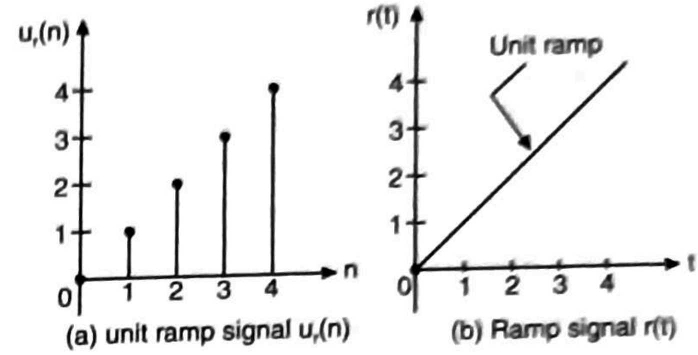


Fig. 2.49.

It has been shown in figure 2.49 (b).

2.9.8 Continuous-Time Complex Exponential Signals

The continuous time complex exponential signal is of the following form:

$$x(t) = Ce^{\alpha t} \qquad ...(2.43)$$

Here, C and α are in general complex numbers. Depending on the values of these parameters, the complex exponent can have several different characteristics.

Types of Complex Exponential Signal

Depending on the type of C and α, the complex exponential signal can be classified as under:

1. Real exponential signals
2. Periodic complex exponential signals.

1. Real Exponential Signals

If C and α both are real, then the corresponding exponential signal is called as the real exponential signal.

The exponential functions are also used extensively in the signal analysis. There are two types of exponential functions viz., rising and decaying exponential functions as shown in the figure 2.50.

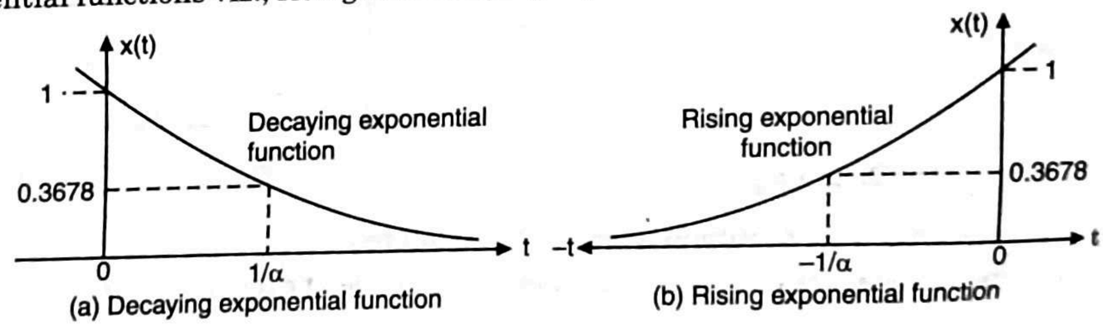


Fig. 2.50.

They are mathematically represented as under

(i) Decaying exponential function

$$x(t) = e^{-\alpha t}$$ ...(2.44)

(ii) Rising exponential function

$$x(t) = e^{\alpha t}$$ ...(2.45)

Note that we have assumed C = 1 for both the equations stated above

2. Periodic Complex Exponential Signals

This is the second important type of complex exponential signals. For this type, α is assumed to be purely imaginary. Such as exponential is mathematically expressed as under:

$$x(t) = e^{j\omega_0 t}$$ ...(2.46)

The most important property of this signal is that it is a periodic signal. Applying the condition of periodicity, we can write that,

$$x(t) = e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$
$$e^{j\omega_0 t} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$$ ...(2.47)

The above expression will be true if and only if,

$$e^{j\omega_0 T} = 1$$

So the conclusion is that for $e^{j\omega_0 t}$ to be periodic, $e^{j\omega_0 t}$ has to be equal to 1.

If $\omega_0 = 0$,

then $x(t) = 1$ for any value of T.

If $\omega_0 \neq 0$ then the fundamental period T_0 of x (t) that is the smaller positive value of T for which $e^{j\omega_0 T} = 1$ is

$$T_0 = \frac{2\pi}{|\omega_0|}$$ ...(2.48)

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ will have the same fundamental period of T_0.

Discrete Time Exponential Signals

A discrete time exponential signal is expressed as

$$x(n) = a^n$$ ...(2.49)

Here, a is some real constant. If a is the complex number then x (n) is written as,

$$x(n) = r\, e^{j\theta}$$ ...(2.50)

Depending on the values of a, we have four different cases as under:

(i) Case 1 : a > 1

(ii) Case 2 : 0 < a < 1

(iii) Case 3 : a < – 1

(iv) Case 4 : – 1 < a < 0.

> **DO YOU KNOW?**
> The differentiation process or integration process can be carried out indefinitely to obtain a doubly infinite class of singularity functions.

Case 1 : For a > 1

Let a = 3.

Thus, we have,

$$x(n) = a^n = 3^n$$

Graphically, such signal is represented as shown in figure 2.51.

Since the signal is exponentially growing : it is called as rising exponential signal.

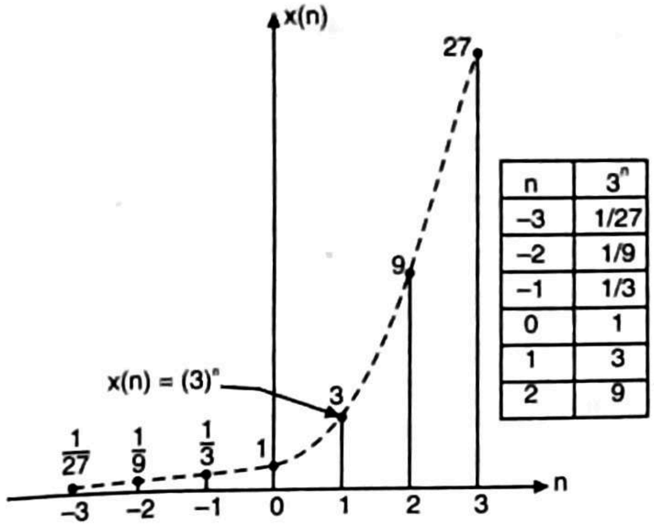
Case 2 : For 0 < a < 1 :

In this case, we will get decaying exponential sequence.

Let, $a = \frac{1}{3}$

Therefore, we have $\quad x(n) = a^n = \left(\frac{1}{3}\right)^n$

Graphically, such signal is represented as shown in figure 2.52.



n	3^n
–3	1/27
–2	1/9
–1	1/3
0	1
1	3
2	9

Fig. 2.51. Rising exponential signal (a > 1)

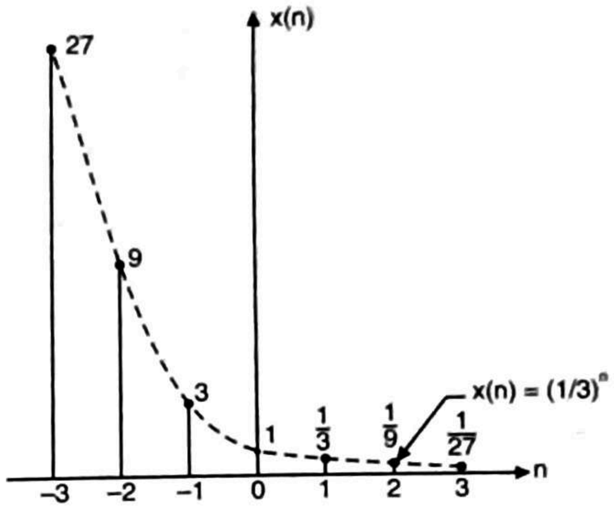


Fig. 2.52. Decaying exponential signal

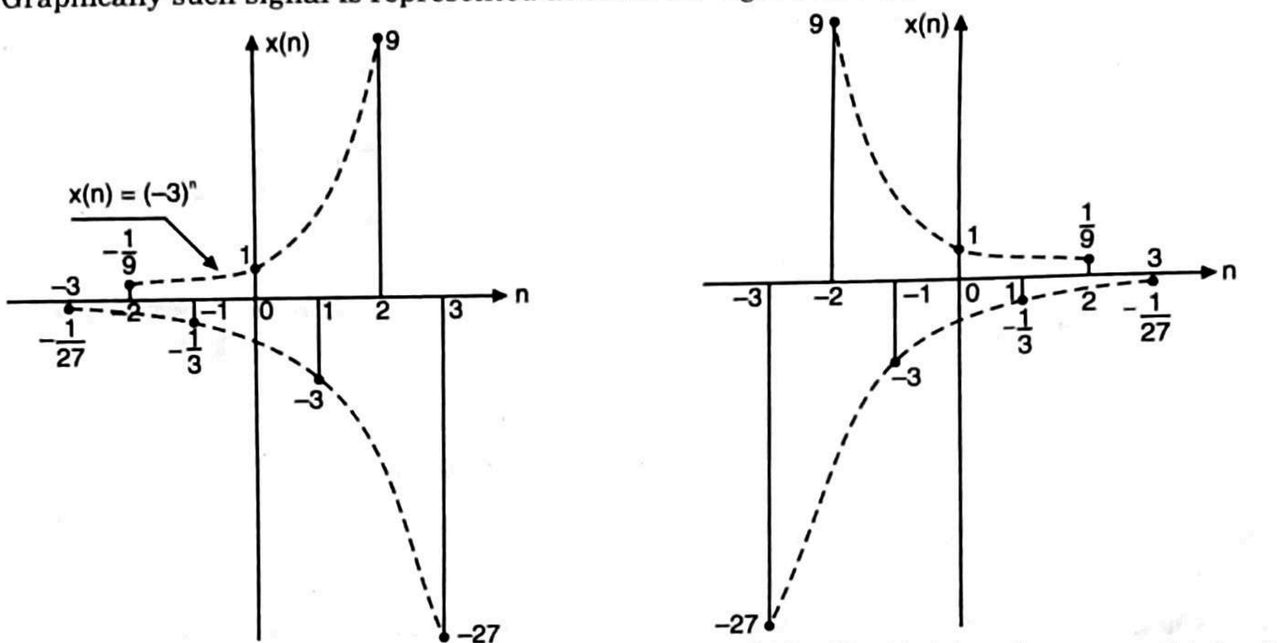
Case 3 : For a < – 1

In this case, we will get double sided rising exponential signal.

Let $\quad a = -3$

Therefore, $\quad x(n) = (-3)^n$

Graphically such signal is represented as shown in figure 2.53 (a).



(a) Double sided growing exponential signal

(b) Double sided decaying exponential signal

Fig. 2.53.

Case 4 : For – 1 < a < 0

In this case, we will get double sided decaying exponential signal.

Let $$a = -\frac{1}{3}$$

Therefore, we have $$x = a^n = \left(-\frac{1}{3}\right)^n$$

Graphically, such signal is as shown in figure 2.53 (b).

Relation between the Complex Exponential and Sinusoidal Signals

The continuous time complex exponential signal is given by

$$x(t) = e^{j\omega_0 t} \quad ...(2.51)$$

By Euler's relation, the complex exponential signal can be written in terms of sinusoidal signal as under:

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t \quad ...(2.52)$$

General Complex Exponential Signal

The general complex exponential signal is given by

$$x(t) = C e^{\alpha t}$$

If we express C in the polar form and α in the rectangular form, then we have

$$C = |C| e^{j\theta}$$

and $$\alpha = r + j\omega_0$$

Substituting, we get

$$C e^{\alpha t} = |C| e^{j\theta} e^{(r + j\omega_0)t}$$

$$= |C| e^{rt} e^{j(\omega_0 t + \theta)} \quad ...(2.53)$$

Using the Euler's expression, we can expand this equation as under :

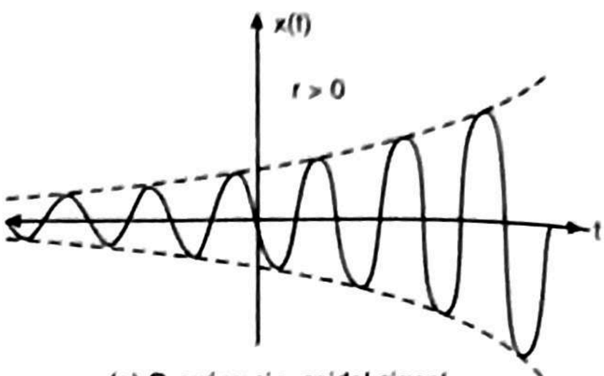
$$C e^{\alpha t} = |C| e^{rt} \cos(\omega_0 t + \theta) + j\,|C|\, e^{rt} \sin(\omega_0 t + \theta) \quad ...(2.54)$$

If $r = 0$ then $e^{rt} = 1$

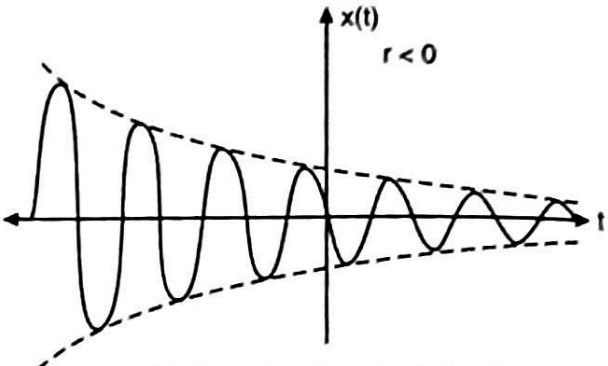
Therefore, we have

$$Ce^{\alpha t} = |C| \cos(\omega_0 t + \theta) + j\,|C| \sin(\omega_0 t + \theta) \quad ...(2.55)$$

Thus, for $r = 0$, the real and imaginary parts of the complex exponential are sinusoidal.



(a) Growing sinusoidal signal



(b) Decaying sinusoidal signal

Fig. 2.54.

For $r > 0$, we have

If $r > 0$, then e^{rt} in equation (2.54) will be a growing exponential signal.

Hence, in equation (2.55), the cosine and sine terms are multiplied by a growing exponential signal.

Hence, we get a growing sinusoidal signal as shown in figure 2.54 (a).

For $r < 0$

If r is negative ($r < 0$), then e^{rt} in equation (2.54) will be a decaying exponential.

Hence, such sinusoidal signal in equation (2.55) is multiplied by a decaying exponential. So, we get a decaying sinusoidal signal as shown in figure 2.54 (b).

Damped Sinusoidal

Sinusoidal signals multiplied by decaying exponentials are called as damped sinusoidal.

2.9.9 Sinc Function

The sinc function or sinc pulse is mathematically expressed as under:

Sinc function : $\text{sinc}(x) = \dfrac{\sin(\pi x)}{(\pi x)}$ for $x = 0$ ...(2.56)

where x is the independent variable. The procedure of plotting the sinc function has been explained earlier.

It has been proved that

$\text{sinc}(x) = 1$ at $x = 0$

and $\text{sinc}(x) = 0$ at $x = \pm 1, \pm 2, \pm 3 \ldots$

Hence, the graphical representation of a sinc function is as shown in figure 2.55.

Figure 2.55 shows that sinc function has the shape of a sine wave. Its amplitude goes on decreasing as the value of $|x|$ increases. Thus, sinc $x \to 0$ when $|x| \to \infty$

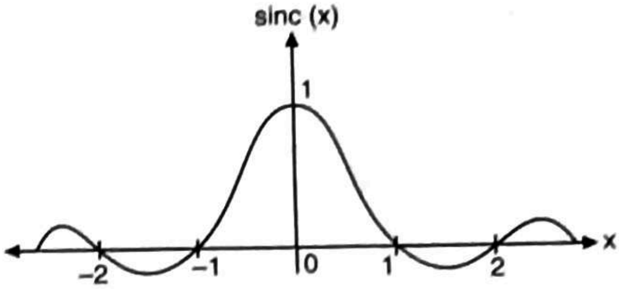


Fig. 2.55. Sinc function

2.9.10 Relationship Between Step, Ramp and Delta Functions

In this subsection, let us establish relationship between step, ramp and delta functions.

(i) Relation between unit step and unit ramp function : The relationship between unit step and unit ramp function can be written as below:

$$\frac{d}{dt} r(t) = u(t) \quad \text{or} \quad \int u(t)\, dt = r(t)$$

(ii) Relation between unit step and delta functions : The relationship between the unit step and delta function can be written as below :

$$\frac{d}{dt} u(t) = \delta(t)$$

or $$\int \delta(t)\, dt = u(t)$$

Hence, on integrating an unit impulse function, we get an unit step function.

(iii) Relation between unit ramp and delta functions : The relationship between unit ramp and delta functions can be written as below :-

$$r(t) = \iint \delta(t)\, dt$$

or $$\frac{d^2}{dt^2} r(t) = \delta(t)$$

Thus, on summarizing points (i), (ii) and (iii), we get

$$\delta(t) \xrightarrow{\text{Integrate}} u(t) \xrightarrow{\text{Integrate}} r(t)$$

or $$r(t) \xrightarrow{\text{Differentiate}} u(t) \xrightarrow{\text{Differentiate}} \delta(t)$$

> **DO YOU KNOW?**
>
> The unit step is the derivative of the unit ramp, and the unit impulse is the derivative of the unit step. Likewise, a unit doublet could be defined as the derivative of the unit impulse, and a unit parabola could be defined as the integral of the unit ramp.

EXAMPLE 2.21. Prove the following :

(i) $\delta(n) = u(n) - u(n-1)$

(ii) $u(n) = \sum_{k=-\infty}^{n} \delta(k)$

(iii) $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$

(GTU, Gujrat, Semester Exam. 2008-2009) (05 marks)

Solution : (i) Given : $\delta(n) = u(n) - u(n-1)$

We know that

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

so that

$$u(n-1) = \begin{cases} 1 & \text{for } n \geq 1 \\ 0 & \text{for } n < 1 \end{cases}$$

Therefore, we have

$$u(n) - u(n-1) = \begin{cases} 0 & \text{for } n \geq 1 \text{ i.e., } n > 0 \\ 1 & \text{for } n = 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Note that the above equation is nothing but $\delta(n)$.

This means that

$$u(n) - u(n-1) = \delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

Hence Proved.

(ii) Given $u(n) = \sum_{k=-\infty}^{n} \delta(k)$

We know that $\sum_{k=-\infty}^{n} \delta(k) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$

Note that the right hand side of above equation is an unit sample sequence u(n).
Therefore, the given equation is proved.

(iii) Given $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$

We know that

$$\sum_{k=0}^{\infty} \delta(n-k) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

Note that the right hand side of above equation is an unit sample sequence u(n).

Therefore, the given equation is proved.

EXAMPLE 2.22. Prove the following relationship between functions:

(i) $\delta(n) = u(n) - u(n-1)$

(ii) $u(n) = \sum_{k=-\infty}^{n} \delta(k)$

(iii) $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$

Solution : (i) $\delta(n) = u(n) - u(n-1)$

We know that

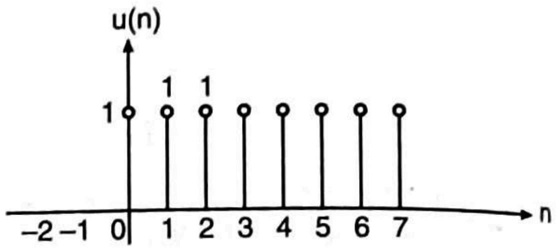
$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

and

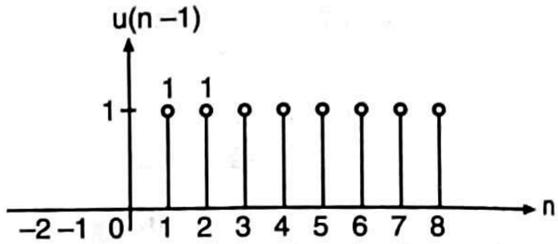
$$u(n-1) = \begin{cases} 1 & \text{for } n \geq 1 \\ 0 & \text{for } n < 1 \end{cases}$$

The waveforms of u(n) and u(n – 1) are shown in figure 2.56. The subtraction u(n) – u(n – 1) of figure 2.56. (a) and (b) is shown in figure 2.56 (c). All the samples become zero except n = 0. Hence, the signal of figure 2.56 (c) is nothing but δ(n). i.e.,

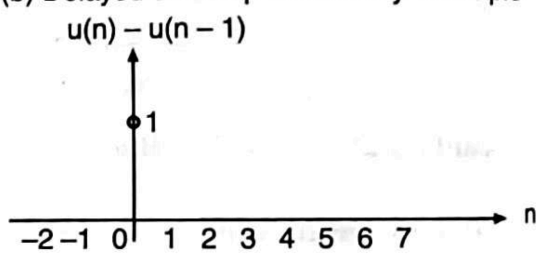
$$u(n) - u(n-1) = \delta(n)$$



(a) Unit step function



(b) Delayed unit step function by 1 sample



(c) Subtraction of above two produces δ(n)

Fig. 2.56. u(n) – u (n – 1) gives unit sample function

e.g.: If $x(n) = \{1, 2, \underset{\uparrow}{0}, 1, 2\}$

Then, we can write,

$$x(n) = \{0, 0, 1, 2, \underset{\uparrow}{0}, 1, 2\} \text{ or } x(n) \{1, 2, \underset{\uparrow}{0}, 1, 2, 0, 0\}$$

Remember that the position of pointer (arrow) does not change.

EXAMPLE 2.23. Represent the following signals grahpically:

(i) $x(n) = \{1, 2, \underset{\uparrow}{0}, -1, 1\}$ **(ii)** $x(n) = \{0, 0, -1, 2, \underset{\uparrow}{3}\}$

(iii) $x(n) = \{\underset{\uparrow}{0}, 1, -1, 1, -1\}$

Solution: These signals have been shown in figures 2.58 (a), (b) and (c) respectively.

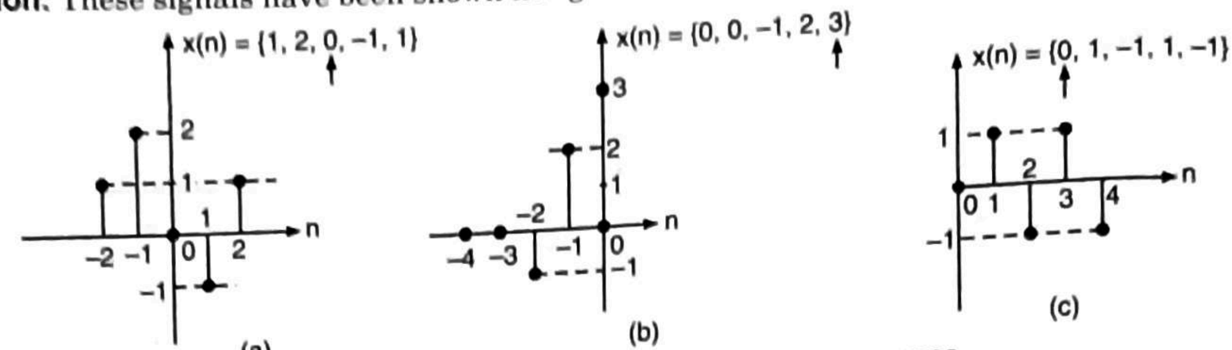


Fig. 2.58. Graphical representation of given sequences

2.11 OPERATIONS ON CONTINUOUS-TIME AND DISCRETE-TIME SIGNALS

2.11.1 Transformation in Independent Variable of Signal

As a matter of fact, independent variable 't' or 'n' can be manipulated by

(i) Delay/Advancing; (ii) Time folding;

(iii) Time scaling

2.11.1.1 Time Delay/Advancing

Figure 2.59 shows the delay and advancing operations on unit step signal. The discrete time signals can also be delayed/advanced similarly. Figure 2.60 shows time delay/advance operations on discrete-time unit step signal.

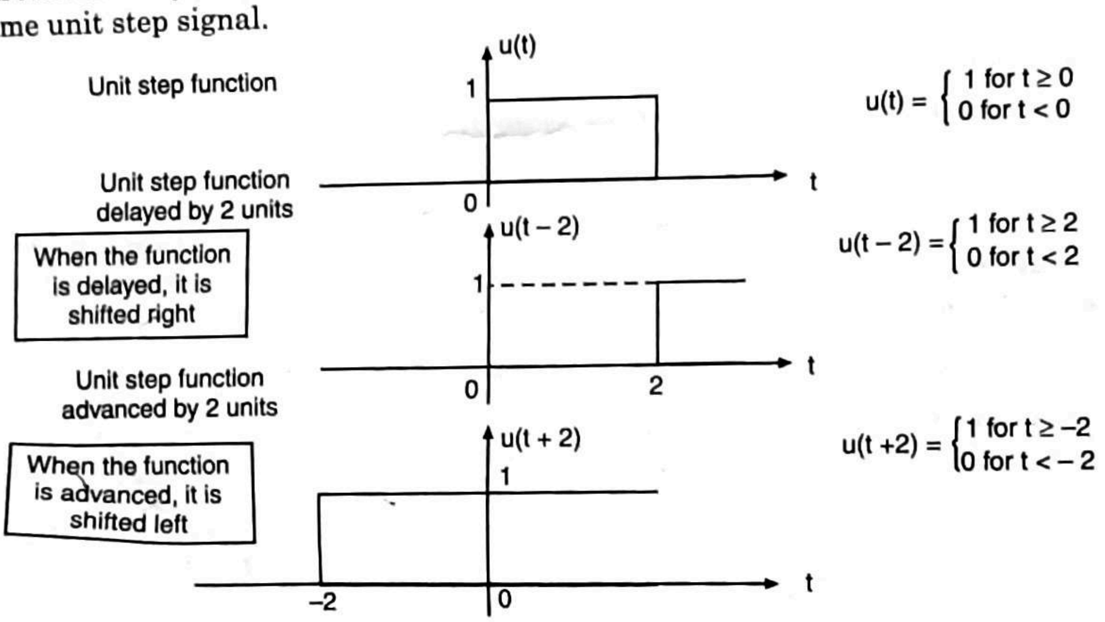


Fig. 2.59. Time delay/advance operations

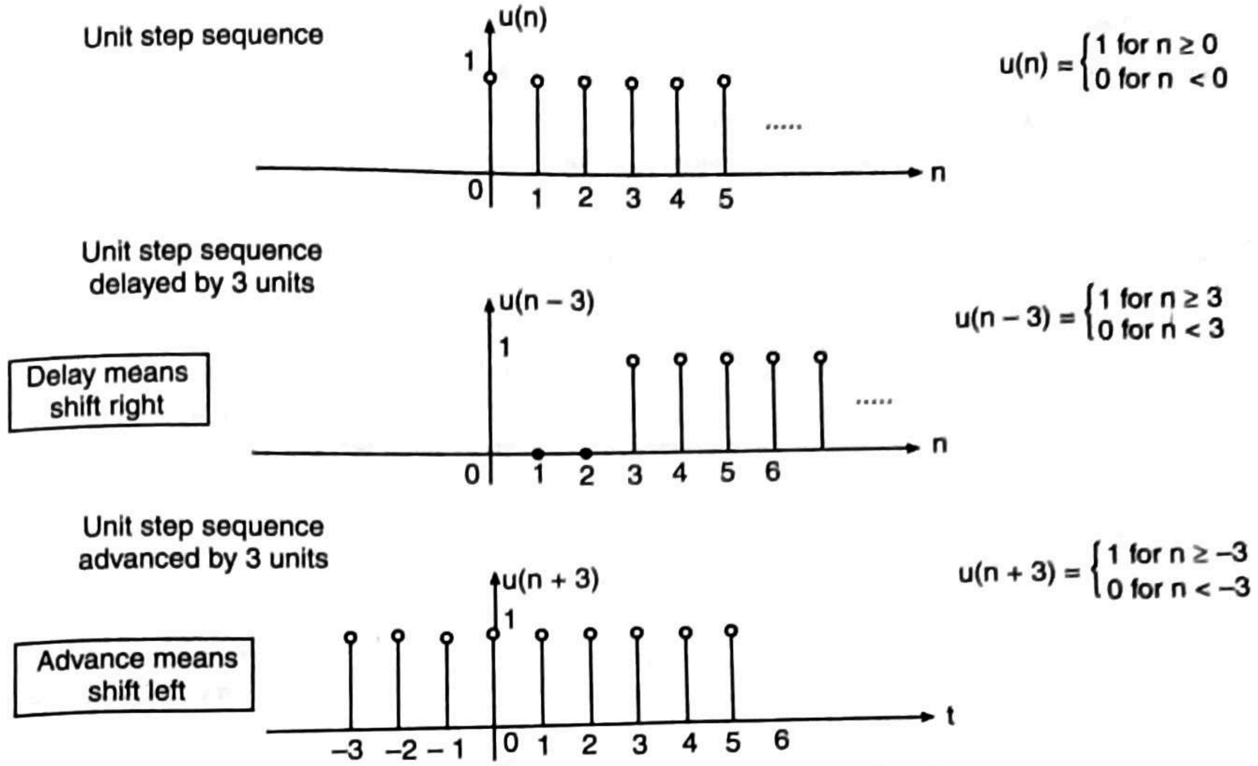


Fig. 2.60. Delayed/advanced discrete time signals

2.11.1.2. Time Folding

As a matter of fact, the time folding operation is used in convolution. Let us consider the continuous time signal x(t). Then its time folded signal is obtained by replacing t with –t, i.e.,

$$y(t) = x(-t)$$

Here, y(t) is time folded or reflected version of x(t). Let us consider the unit step function u(t). Its folded signal will be u(–t). Figure 2.61 shows these signals. Folding is done at t = 0. Hence, u(–t) is mirror image of u(t) at t = 0. Time folding can be done at any value of t.

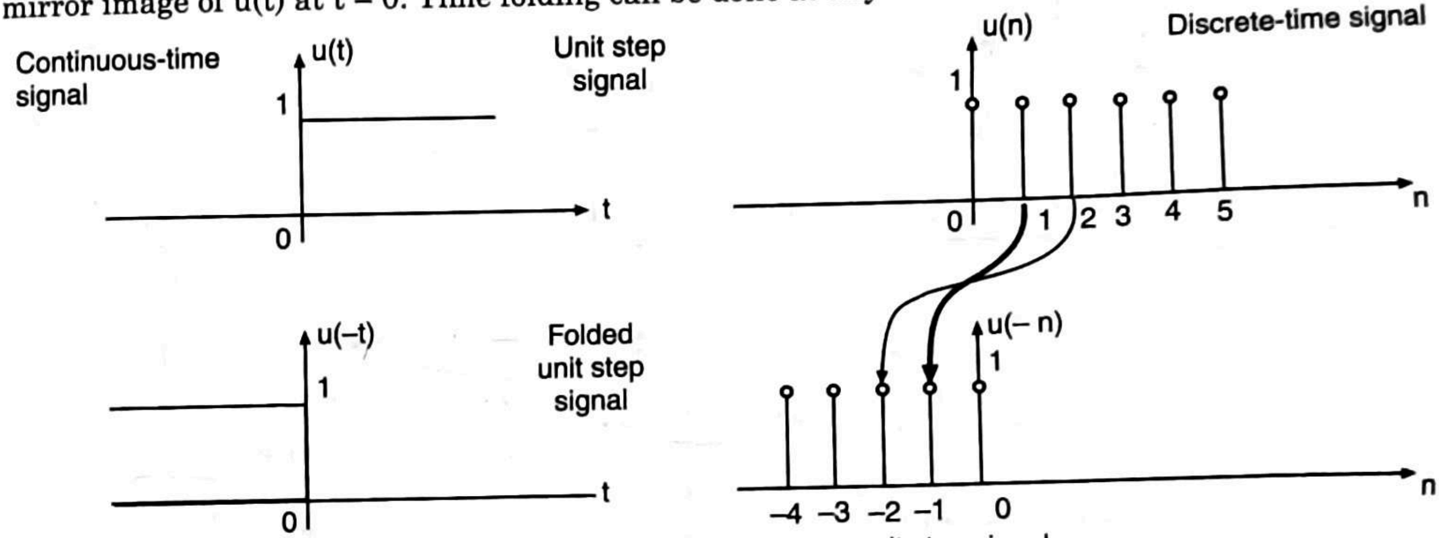


Fig. 2.61. Time folding operation on unit step signal

Further, folding operation can be performed on discrete-time signals also. Let us consider the signal x(n). Then, its folded signal can be obtained by replacing n with –n. i.e.,

$$y(n) = x(-n)$$

Figure 2.61 shows the sketch of x(n) and x(–n). Here, x(n) is considered as unit step sequence u(n).

2.11.1.3 Time Scaling

Basically, there are following two types of time scaling :

1. **Time compression** : The time axis is compressed. For example, $y(t) = x(2t)$. Here, $y(t)$ will be compressed in time.

2. **Time expansion** : The time axis is expanded. For example, $y(t) = x\left(\frac{t}{2}\right)$. Here, $y(t)$ will be expanded in time.

Figure 2.62 shows the compressed and expanded signals.

3. **Compression of Discrete-time Signal:** Let us consider the discrete time triangular pulse as shown in figure 2.62. Let this pulse be represented by $x(n)$. And let,

$$y(n) = x(2n)$$

Then, we have,

$$n = 0 \Rightarrow y(0) = x(0) = 0$$
$$n = 1 \Rightarrow y(1) = x(2) = 1$$
$$n = 2 \Rightarrow y(2) = x(4) = 2$$

Thus, the signal is compressed. Alternate samples of $x(n)$ are skipped. This is also called as subsampling.

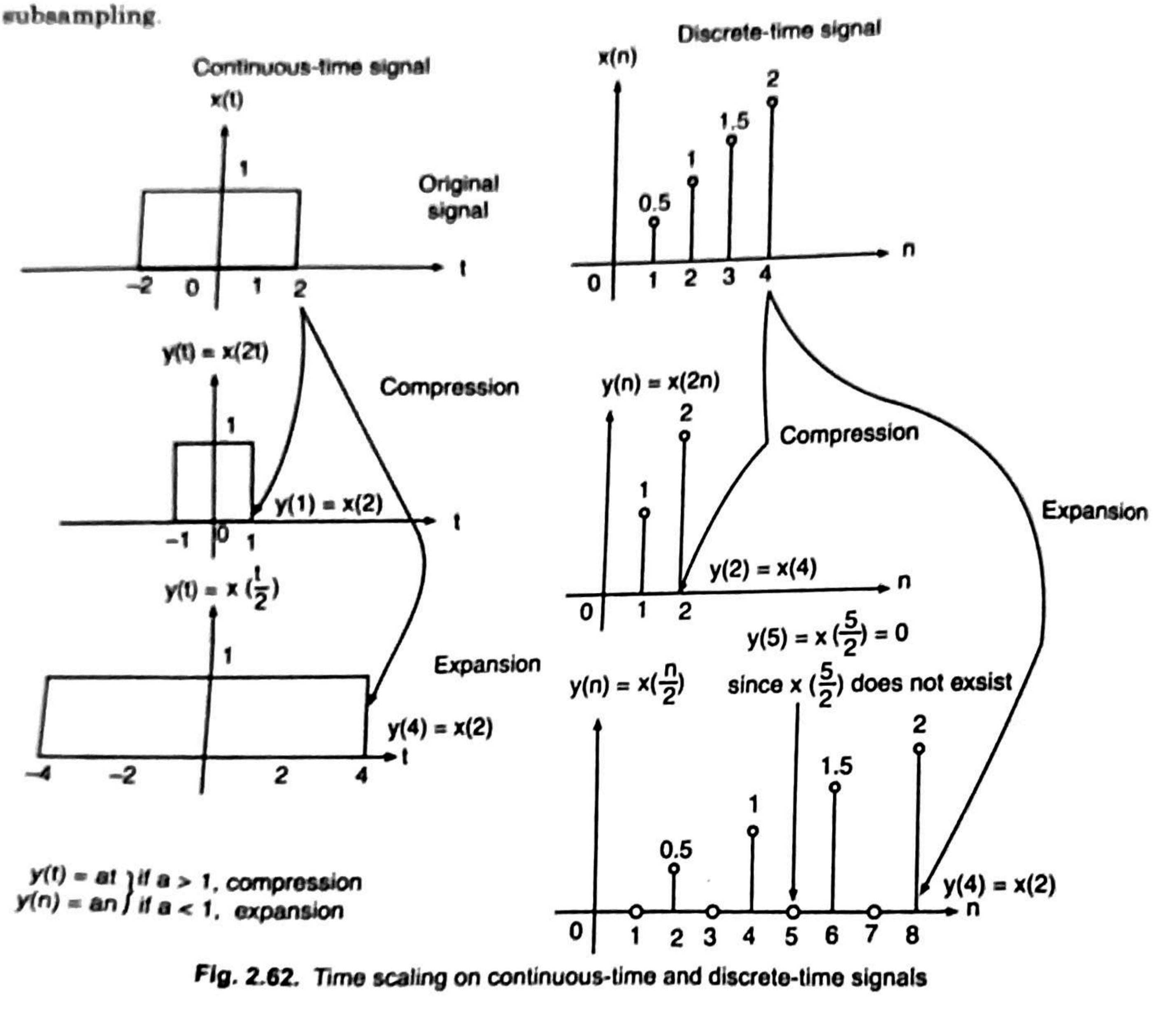


Fig. 2.62. Time scaling on continuous-time and discrete-time signals

4. Expansion of discrete-time signal: The discrete time signal can be expanded if $a < 1$. Let us consider,

$$y(n) = x\left(\frac{n}{2}\right)$$

Then, we have,

$$y(0) = x(0)$$

$$y(2) = x\left(\frac{2}{2}\right) = x(1)$$

$$y(4) = x\left(\frac{4}{2}\right) = x(2)$$

$$y(6) = x\left(\frac{6}{2}\right) = x(3)$$

Here, it may be noted that $y(1) = x\left(\frac{1}{2}\right)$, which is not available. Hence, y(1) can be considered zero. Similarly, y(3) and y(5) can be considered zero. When the signals are practically expanded, such values are interpolated from other values.

2.11.1.4 Important Rules for Time Shifting and Time Scaling

Let us consider the following transformation:

$$y(t) = x(a\,t - b)$$

with t = 0, we have $y(0) = x(-b)$

with $t = \frac{b}{a}$, we have $y\left(\frac{b}{a}\right) = x(0)$

Rules

(i) First, we do the shifting operation.

(ii) Then, we do the time scaling operation

Example

Let us consider $y(t) = x(-2t + 3)$. The x(t) is a rectangular pulse of amplitude 1 and duration $-1 \le t \le 1$.

(i) First, we shift x(t) to left by 3, to get x(t + 3). Figure 2.63 shows this plot of x(t + 3)

(ii) Then, we compress x(t + 3) by 2 to get x(2t + 3). Figure 2.64 (a) shows the plot of x(2t + 3).

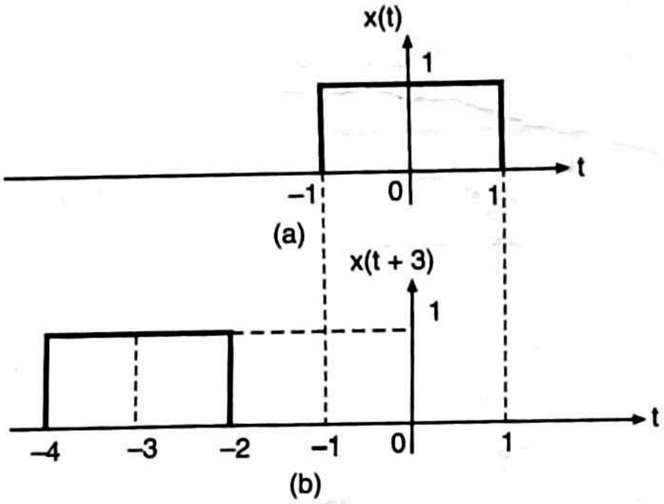


Fig. 2.63. Plot of x(t) and x(t + 3)

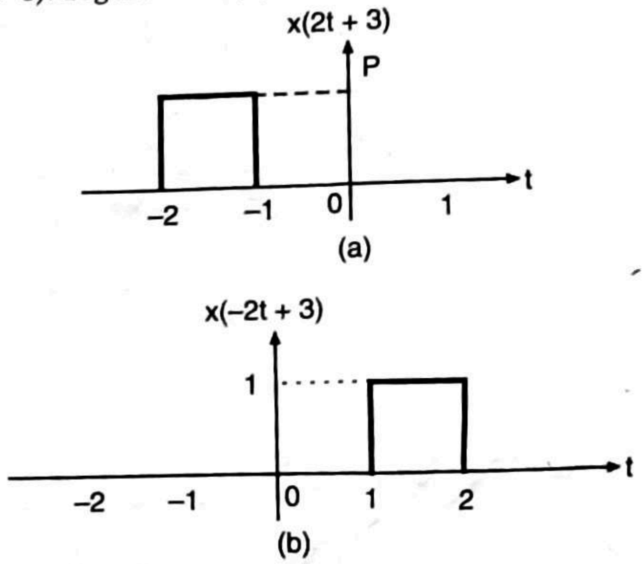


Fig. 2.64. Plot of x(2t + 3) and x(–2t + 3)

(iii) The $x(2t + 3)$ of figure 2.64 (a) is folded in time to get $x(-2t + 3)$. This is shown in figure 2.64 (b).

2.11.2 Transformations on Amplitude of the Signal

2.11.2.1 Amplitude Scaling

The amplitude of the signal can be changed with amplitude scaling. Let us consider the unit step function $u(t)$. Let,

$$y(t) = 2\,u(t)$$

Here, amplitude of unit step function is 2. This function is sketched in figure 2.65 (b). It may be observed that the amplitude of step function is '2'. Similarly, negative amplitudes are also possible. Let us consider,

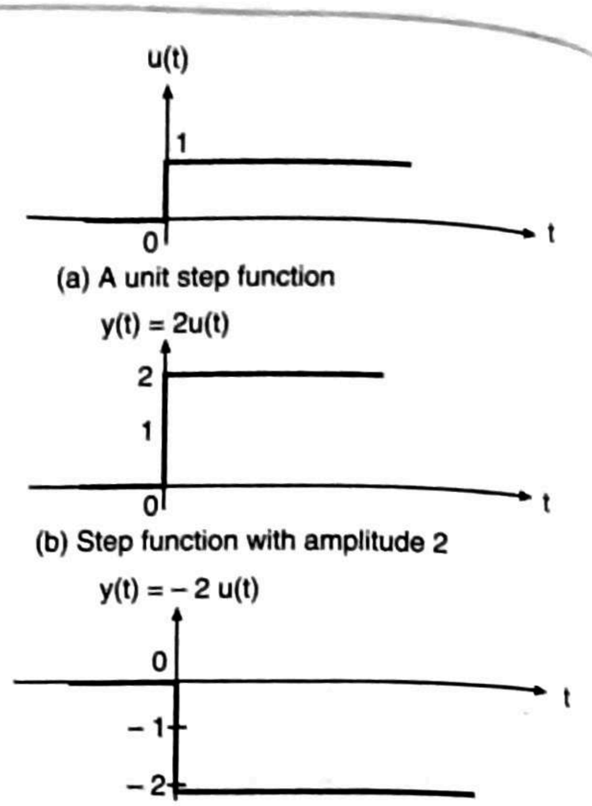
$$y(t) = -2\,u(t)$$

This function is sketched in figure 2.65 (c). It may be observed that the step function has negative amplitude, i.e., '–2'. Amplitude scaling can also be performed on discrete time signals. Let us consider the unit step sequence $u(n)$.

Let, $$y(n) = 2\,u(n)$$

This increases the amplitude of unit step sequence by '2'. Figure 2.66 (b) shows the sketch of this function. Similarly, negative amplitudes are also possible. Let us consider,

$$y(n) = -2\,u(n)$$

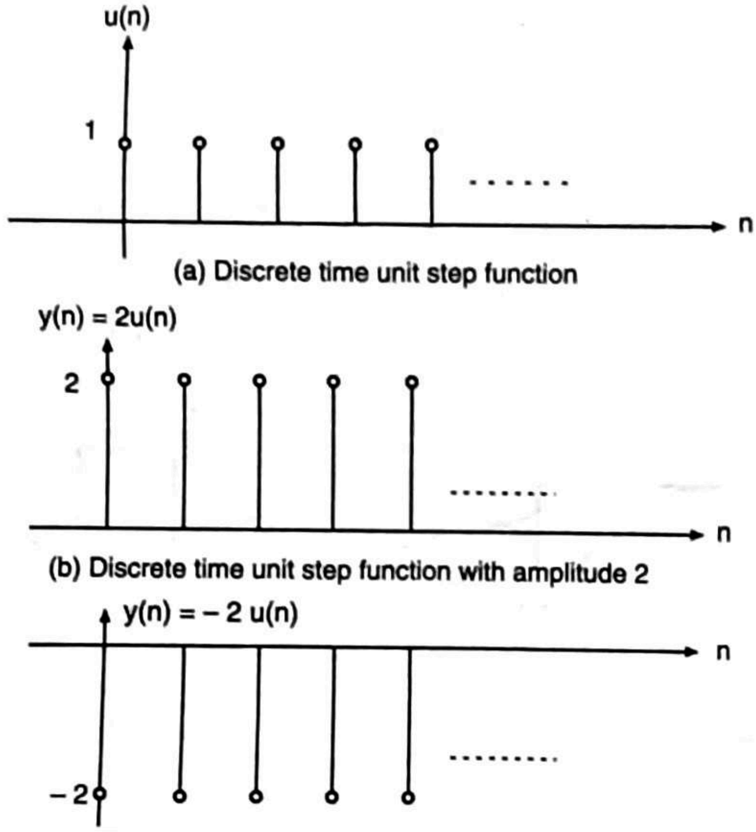


(a) A unit step function

(b) Step function with amplitude 2

(c) Step function with amplitude – 2

Fig. 2.65. Amplitude scaling operation on continuous time signals



(a) Discrete time unit step function

(b) Discrete time unit step function with amplitude 2

(c) Discrete time unit step function with amplitude –2

Fig. 2.66. Amplitude scaling operation on discrete time signals

This function is sketched in figure 2.66 (c). It may be observed that the samples of step function have the amplitude of '–2'.

2.11.2.2 Addition and Subtraction

Let $x_1(t)$ and $x_2(t)$ be the two continuous time signals. Then addition of $x_1(t)$ and $x_2(t)$ can be given as,

$$y(t) = x_1(t) + x_2(t)$$

Similarly, the subtraction of $x_1(t)$ and $x_2(t)$ is given as,

$$y(t) = x_1(t) - x_2(t)$$

Similarly, let $x_1(n)$ and $x_2(n)$ be two discrete time signals. Their addition is given as,

$$y(n) = x_1(n) + x_2(n)$$

Here, it may be noted that the addition is made from sample to sample basis. Similarly subtraction is given as,

$$y(n) = x_1(n) - x_2(n)$$

2.11.2.3 Multiplication and Division

Let $x_1(t)$ and $x_2(t)$ be the two continuous time signals. Then, their multiplication is given as,

$$y(t) = x_1(t) \cdot x_2(t)$$

And, their division is given as,

$$y(t) = \frac{x_1(t)}{x_2(t)}$$

Similarly, let $x_1(n)$ and $x_2(n)$ be two discrete time signals. They are multiplied as,

$$y(n) = x_1(n) \cdot x_2(n)$$

and their division is given as,

$$y(n) = \frac{x_1(n)}{x_2(n)}$$

Here, it may be noted that the multiplication and division is made on sample to sample basis.

2.11.2.4 Differentiation and Integration

Let $x(t)$ be the continuous time signal. Then, its differentiation with respect to time is given as,

$$y(t) = \frac{d}{dt} x(t)$$

Here, $y(t)$ is the derivative of $x(t)$. Differentiation is useful to represent voltage developed across the inductor. Let the current $i(t)$ is flowing through an inductor L, then the voltage across it will be,

$$v(t) = L\frac{d}{dt} i(t)$$

Similarly, integration of $x(t)$ can be expressed as,

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Here, $y(t)$ is integration of $x(t)$. Integration is used to represent voltage across the capacitor C, when charging current $i(t)$ is flowing through it, i.e.,

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau$$

Here, v(t) is the voltage across the charging capacitor.

Integration and differentiation do not exist directly for discrete time signals. But difference and accumulation operations exists. For example, the difference operation is given as,

$$y(n) = x(n) - x(n-1)$$

Similarly, the accumulation operation is given as,

$$y(n) = \sum_{k=-\infty}^{n} x(n)$$

These operations do not represent differentiation or integration but they are used similarly.

EXAMPLE 2.24. A discrete time signal is as shown in figure 2.67:

Sketch the following

(i) x(n – 3) (ii) x(3 – n) (iii) x(2n)

(iv) x(n)u (3 – n)

(v) x[(n – 1)²]

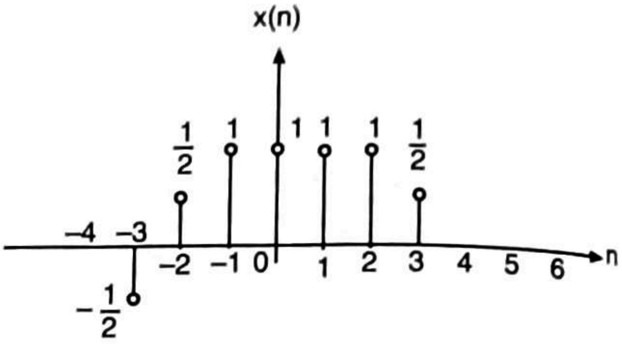


Fig. 2.67. Discrete-time signal for example 2.24

Solution: (i) x(n – 3)

This indicates that the signal x(n) is delayed by 3 samples. Hence, all the samples are shifted right by three sample positions. This is shown in figure 2.68. In this figure, it may be observed that x(3) is shifted to x(6). Similarly, x(–3) is shifted to x(0) and so on.

(ii) x(3 – n)

This can be written as,

$$\begin{aligned} x(3-n) &= x(-n+3) \\ &= x[-(n-3)] \end{aligned}$$

This means that x(n – 3) is folded in time. Hence, if we fold x(n – 3) of figure 2.68 in time, we will get x(3 – n). This is shown in figure 2.69. In this figure, it may be observed that all the samples of x(n – 3) of figure 2.68 are folded in time. Above figure is mirror image of figure 2.69 at n = 0.

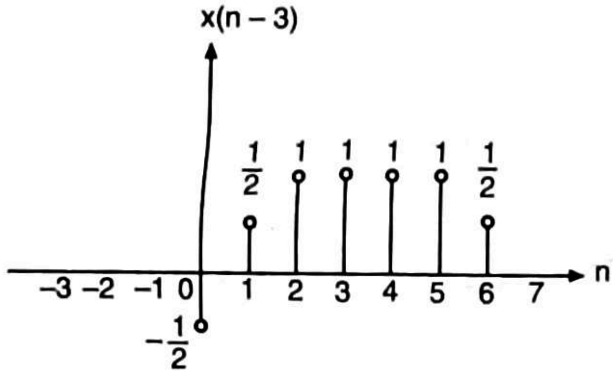


Fig. 2.68. x(n) delayed by 3 samples

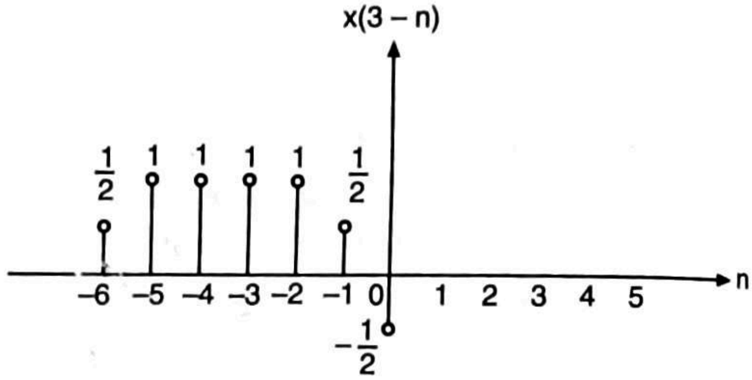


Fig. 2.69. Sketch of x(3 – n)

(iii) y(n) = x (2n)

This is compression of the signal x(n). This can be verified easily as under :

$$n = 0 \Rightarrow y(0) = x(0) = 1$$

$$n = 1 \Rightarrow y(1) = x(2) = 1$$

$$n = 2 \Rightarrow y(2) = x(4) = 0$$

$$n = 3 \Rightarrow y(3) = x(6) = 0$$

Similarly

$$n = -1 \Rightarrow y(-1) = x(-2) = \frac{1}{2}$$

$$n = -2 \Rightarrow y(-2) = x(-4) = 0$$

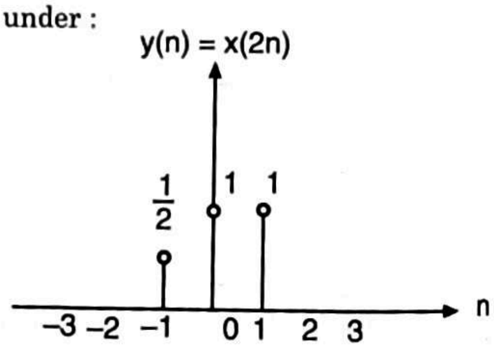


Fig. 2.70. Sketch of y(n) = x(2n)

This signal is sketched in figure 2.70.

Here, it may be observed that the signal has only three samples. It is time compressed version of x(n).

$$E = 4\left[\frac{e^{-2t}}{-2}\right]_0^{\infty} - 36\left[\frac{e^{-4t}}{-4}\right]_0^{\infty} = \frac{4}{-2}[e^{-\infty} - e^{0}] + \frac{36}{4}[e^{-\infty} - e^{0}]$$

or $$E = -2(-1) + 9(-1) = -7$$

But, we have to consider magnitude.

Therefore, $E = 7$ Joules.

EXAMPLE 2.39. Examine whether the following signal :

$x(n) = \cos\left(\frac{n}{10}\right)\cos\left(\frac{n\pi}{10}\right)$ is a periodic signal or not.

Solution:

The signal is given by,

$$x(n) = \cos\left(\frac{n}{10}\right)\cos\left(\frac{n\pi}{10}\right) \quad \text{...(i)}$$

Let $$x_1(n) = \cos\left(\frac{n}{10}\right) \quad \text{...(ii)}$$

and $$x_2(n) = \cos\left(\frac{n\pi}{10}\right) \quad \text{...(iii)}$$

The a standard equation can be expressed as,

$$x(n) = \cos \omega n = \cos 2\pi f n \quad \text{...(iv)}$$

Comparing equations (ii) and equation (iii) with equation (i), we get

$$2\pi f_1 n = \frac{n}{10}$$

or $$f_1 = \frac{1}{20\pi}$$

and $$2\pi f_2 n = \frac{n\pi}{10}$$

or $$f_2 = \frac{1}{20}$$

Since, f_1 is not a rational number; $x_1(n)$ is non-periodic.

Thus, $x(n)$ is also non periodic. **Ans.**

2.12 CORRELATION OF CONTINUOUS-TIME (CT) ENERGY SIGNALS

As a matter of fact, the correlation function is used to indicate the degree of similarity between two signals. Higher the value of correlation function is, more is the degree of similarity. The correlation function is of two types:

(i) Auto-correlation function

(ii) Cross-correlation function

2.12.1 Auto-Correlation Function for the Continuous-Time (CT) Energy Signals

The auto-correlation function for an energy signal $x(t)$ provides the measure of similarity between the signal $x(t)$ and its time delayed version $x(t-\tau)$. The auto-correlation function is denoted by $R(\tau)$ and it is defined as under :

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\cdot x(t-\tau)dt \quad \text{If } x(\tau) \text{ is real valued} \quad \text{...(2.57)}$$

The auto-correlation function can also be defined as under:

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\cdot x(t)dt \quad \text{If } x(t) \text{ is real valued} \quad \text{...(2.58)}$$

If the signal x(t) is complex valued, then the auto-correlation function for it is defined as under:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\cdot x^*(t-\tau)dt \qquad \text{If } x(t) \text{ is complex} \qquad ...(2.59)$$

Figure 2.97 shows the block diagram of a system used for measuring the auto-correlation function $R(\tau)$.

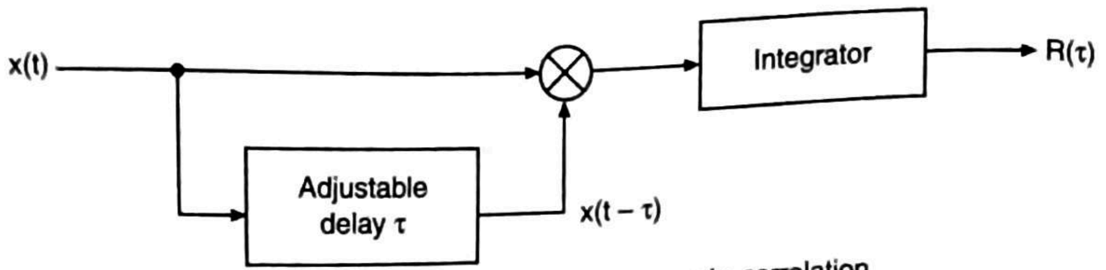


Fig. 2.97. Arrangement to measure auto-correlation

2.12.2 Analogy Between Autocorrelation and Convolution

The convolution of two continuous-time signals x(t) any y(t) may be defined as under :

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)d\tau \qquad ...(2.60)$$

Now, if we compare this equation with the definition of autocorrelation stated in equation (2.60), we find that the process of autocorrelation is very similar to that of convolution.

2.12.3 Interrelation Between Autocorrelation and ESD

For an energy signal x(t), the auto-correlation function and energy spectral density form a Fourier transform pair. This means that

$$R(\tau) \xleftrightarrow{F} \psi(f) \qquad ...(2.61)$$

Proof

For a complex valued signal x(t), the auto-correlation function is defined as under:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\cdot x^*(t-\tau)dt \qquad ...(2.62)$$

Let us substitute the value of $x^*(t-\tau)$ in the above equation. From the definition of inverse Fourier transform, we have

$$x(t-\tau) = \int_{-\infty}^{\infty} X(f)\cdot e^{j2\pi f(t-\tau)}df \qquad ...(2.63)$$

Taking complex conjugate of both sides, we have

$$x^*(t-\tau) = \left[\int_{-\infty}^{\infty} X(f)\cdot e^{j2\pi f(t-\tau)}df\right]^*$$

Therefore,
$$x^*(t-\tau) = \int_{-\infty}^{\infty} X^*(f)\cdot e^{-j2\pi f(t-\tau)}dt \qquad ...(2.64)$$

Substituting equation (2.64) into equation (2.62), we get

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\cdot\left[\int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi f(t-\tau)}df\right]dt$$

or $$R(\tau) = \int_{-\infty}^{\infty} X^*(f) \cdot e^{j2\pi f\tau} dt \left[\int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft}\, dt\right]$$

or $$R(\tau) = \int_{-\infty}^{\infty} X(f) \cdot X^*(f) \cdot e^{j2\pi f\tau} df = \int_{-\infty}^{\infty} |X(f)|^2\, e^{j2\pi f\tau}\, df$$

But, $$|X(f)|^2 = \psi(f)$$

Therefore, $$R(\tau) = \int_{-\infty}^{\infty} \psi(f) \cdot e^{j2\pi f\tau}\, df \quad ...(2.65)$$

This equation shows that $\psi(f)$ is the Fourier transform of $R(\tau)$.

Therefore $$F[R(\tau)] = \psi(f)$$

or $$R(\tau) \xleftrightarrow{F} \psi(f) \quad ...(2.66)$$

2.12.4 Auto-Correlation Function of Continuous-Time Power Signals

We can develop a formula for the auto-correlation function of power signals by following a procedure similar to that described for the energy signals.

The auto-correlation function of power signals is a measure of similarity between a periodic signals $x(t)$ and its delayed version $x(t-\tau)$. If the signal $x(t)$ is periodic with a period T_0, then the auto-correlation function over one complete period T_0 is defined as under:

Auto-correlation function of periodic signals

$$R(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot x(t-\tau)\, dt \quad ...(2.67)$$

Here, it may be noted that this expression is same as that for the auto-correlation function of energy signals, except for the inclusion of $\frac{1}{T_0}$ and change in the limits of integration.

For any period T, it is defined as under:

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \cdot x(t-\tau)\, dt \qquad \text{[For } x(t) \text{ to be real valued].}$$

For any period T with $x(t)$ complex, we have

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \cdot x^*(t-\tau) dt$$

For any period T with $x(t)$ complex and τ in negative direction, we have

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t+\tau) \cdot x^*(t)\, dt \quad ...(2.68)$$

Both $x(t)$ and $x^*(t-\tau)$ are advanced by τ to get $x(t+\tau)$ and $x^*(t)$ respectively.

2.12.5 Interrelation Between Autocorrelation and PSD (MDU, Rohtak, Sem. Exam., 2008-09)

For a power signal $x(t)$, the auto-correlation function and the power spectral density (PSD) form a Fourier transform pair. This means that

$$R(\tau) \xleftrightarrow{F} S(f) \quad ...(2.69)$$

Equation (2.69) states that the power spectral density S(f) is the Fourier transform of the auto-correlation function R(τ). This can be written in an expanded form as under:

$$S(f) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j2\pi f\tau}\, d\tau \quad ...(2.70)$$

The other meaning of equation (2.69) is that the auto-correlation function R(τ) is the inverse Fourier transform of power spectral density S(f). This can be expressed in an expanded form as under :

$$R(\tau) = \int_{-\infty}^{\infty} S(f) \cdot e^{j2\pi f t}\, df \quad ...(2.71)$$

Equations (2.70) and (2.71) are known as **Einstein-Wiener-Khintchine relations.**

EXAMPLE 2.40. Determine the autocorrelation of the continuous time signal given by $x(t) = A\,\text{rect}\left(\frac{t}{2}\right)$.

Solution: The given signal is represented graphically as shown in figure 2.98.

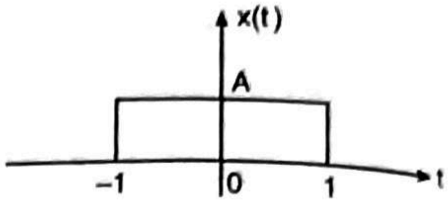


Fig. 2.98.

It is expressed mathematically as :

$$x(t) = \begin{cases} A & \text{for } -1 \le t \le +1 \\ 0 & \text{otherwise} \end{cases}$$

We know that the autocorrelation function of continuous-time energy signals is given by

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x(t-\tau)\, dt$$

Now, let us consider figures 2.99 (b), (c) and (d) which show the shifted signal x(t – τ) and hence the possibility of overlap of x(t) and x(t – τ). This shows that if τ is less than – 2, then there will not be any overlap and if τ > 2, then, also, there will not be any overlap of x(t) and x(t – τ). And, if there is no overlap, then the product x(t)·x(t – τ) = 0 and the autocorrelation also is zero.

The actual overlapping as shown in figure 2.99 (c) takes place for t = – 1 to t = (τ + 1). Hence, the autocorrelation function is given by

$$R(\tau) = \underbrace{\int_{-\infty}^{-1} x(t)x(t-\tau)\, dt}_{\text{No overlap}} + \int_{-1}^{(\tau+1)} x(t)\cdot x(t-\tau)\, dt + \underbrace{\int_{\tau+1}^{\infty} x(t)\cdot x(t-\tau)\, dt}_{\text{No overlap}}$$

$$= \int_{-\infty}^{-1} 0\, dt + \int_{-1}^{\tau+1} A\cdot A\, dt + \int_{\tau+1}^{\infty} 0\, dt$$

$$R(\tau) = 0 + A^2[t]_{-1}^{\tau+1} + 0 = A^2[(\tau+1) - (-1)] = A^2[\tau+2]$$

Therefore,

$$R(\tau) = A^2\,[\tau+2] \quad \text{for} \quad -2 \le \tau \le 0 \quad ...(i)$$

Now, let us consider figure 2.99 (d) which shows that the overlapping will take place if 0 ≤ τ ≤ 2. The actual overlapping as shown in figure 2.99 (d) takes place for t = 1 to t = (τ – 1). Hence, the autocorrelation function is given by

$$R(\tau) = \underbrace{\int_{-\infty}^{\tau-1} x(t)x(t-\tau)\, dt}_{\text{No overlap}} + \underbrace{\int_{\tau-1}^{1} x(t)\cdot x(t-\tau)\, dt}_{\text{Overlap}} + \underbrace{\int_{1}^{\infty} x(t)\, x(t-\tau)\, dt}_{\text{No overlap}}$$

EXAMPLE 2.48. **Obtain spectral density, autocorelation and signal energy when:**

$$v(t) = A \text{ sinc } [4 W (t + t_d)]$$

(WBUT, Kolkata, Sem. Exam., 2007-08)

Solution: We know that

$$A \sin c\ (2Wt) \xleftrightarrow{F} \frac{A}{2W} \text{rect}\left[\frac{f}{2W}\right]$$

Therefore, $A \sin c\ (4Wt) \xleftrightarrow{F} \frac{A}{4W} \text{rect}\left[\frac{f}{4W}\right]$

Now, let us use the time shifting property of Fourier transform to write

$$A \text{ sinc } [4W(t + t_d)] \xleftrightarrow{F} e^{-j\omega t_d} \frac{A}{4W} \text{rect}\left[\frac{f}{4W}\right]$$

Hence,
$$V(f) = e^{-j\omega t_d} \frac{A}{4W} \text{rect}\left[\frac{f}{4W}\right] \quad ...(i)$$

(ii) Next, we find ESD $\psi(f)$.

$$\psi(f) = |V(f)|^2 = V(f) \cdot V^*(f)$$

But
$$V^*(f) = e^{j\omega t_d} \frac{A}{4W} \text{rect}\left[\frac{f}{4W}\right] \quad ...(ii)$$

Therefore,
$$\psi(f) = \frac{A^2}{(4W)^2} \text{rect}\left[\frac{f}{4W}\right] \quad \textbf{Ans.}$$

(iii) Then, let us find the auto correlation $R(\tau)$.

We have
$$R(\tau) = f^{-1}[\psi(f)] = f^{-1} \frac{A^2}{(4W)^2} \text{rect}\left(\frac{f}{4W}\right)$$

Hence,
$$R(\tau) = \frac{A^2}{(4W)^2} \text{sinc}\,(4W\tau) \quad \textbf{Ans.}$$

(iv) Lastly, let us determine the signal energy.

$$E = R(0) = \frac{A^2}{(4W)^2} \text{sinc}\,(4W\tau) = \frac{A^2}{(4W)^2} \quad \textbf{Ans.}$$

2.13 CROSS CORRELATION OF CONTINUOUS-TIME (CT) SIGNALS

In the previous section, we have discussed about the auto-correlation which is a measure of similarity between a signal and its shifted version. Now, let us discuss the other type of correlation which is cross correlation. Cross correlation measures the similarity between two completely different signals say x(t) and y(t).

2.13.1 Cross-Correlation of Continuous-time (CT) Energy Signals

Let $x_1(t)$ and $x_2(t)$ denote a pair of real valued energy signals. Then, the cross-correlation function of this pair of signals is defined as under :

$$R_{12}(t) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t - \tau)\, dt \quad ...(2.72)$$

If the two signals $x_1(t)$ and $x_2(t)$ are somewhat similar then the value of the cross-correlation function $R_{12}(\tau)$ will be finite over some range of τ. Thus, corss-correlation gives the measure of similarity or coherence, between them.

Equation (2.72) defines one possible value of the cross-correlation function for a specified value of the delay variable τ. We can define the second cross-correlation function of the energy signals $x_1(t)$ and $x_2(t)$ as under:

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t) \cdot x_1(t-\tau)\, dt \quad ...(2.73)$$

If the signals $x_1(t)$ and $x_2(t)$ are complex valued signals of finite energy then the cross-correlation is defined as under:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2^*(t-\tau)\, dt \quad ...(2.74)$$

It is important to note the value or $R_{12}(\tau)$ will be finite if the two signals $x_1(t)$ and $x_2(t)$ have some similarity between them. If $R_{12}(\tau) = 0$, then the two signals will not have any similarity between them.

2.13.2 Analogy Between Cross-Correlation and Convolution

Convolution between two continuous-time (CT) signals $x_1(t)$ and $x_2(t)$ may be defined as under :

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau)\, d\tau$$

If we compare this equation with the definition of cross corelation stated in equation (2.72), then we can realize the similarity between cross-correlation and convolution.

2.13.3 Cross-Correlation of Continuous-time (CT) Power Signals

Cross-correlation may be defined for two different power signals $x_1(t)$ and $x_2(t)$. If $x_1(t)$ and $x_2(t)$ represent two different power signals, then cross-correlation between them is defined as under :

$$R_{12}(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} x_1(t) \cdot x_2^*(t-\tau)\, dt \quad ...(2.75)$$

Similarly, we can define a second cross-correlation function as under:

$$R_{21}(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} x_2(t) \cdot x_1^*(t-\tau)\, dt \quad ...(2.76)$$

If the two periodic signals $x_1(t)$ and $x_2(t)$ have the same time period T_0, then the cross-correlation is defined as under:

$$R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_1(t) \cdot x_2^*(t-\tau)\, dt \quad ...(2.77)$$

and the second cross-correlation for the two periodic signals is defined as under:

$$R_{21}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_2(t) \cdot x_1^*(t-\tau)\, dt \quad ...(2.78)$$

Important Point: Note that the equations for the auto-correlation function and cross-correlation function are identical except for the fact that it has been defined for two different power signals $x_1(t)$ and $x_2(t)$ whereas the auto-correlation is defined for the same signal $x(t)$.