# Belyi's theorem and *Dessins d'enfant*.

Koundinya Vajjha

September 12, 2018

- Give a few words about motivation, tell the story about Grothendieck's application to the CNRS.
- Read out the passage from page 77 of Lando and Zvonkin.

## Belyi's theorem

- A smooth alg. curve $X$ is *defined over a subfield* $K$ of $\mathbb{C}$ if it is isom. to the set of zeros, in some affine or projective space over $\mathbb{C}$, of a finite set of poly's in $K[X]$.
- An *algebraic number field* (or simply a *number field*) is a subfield $K$ of $\mathbb{C}$ which is a finite extension of $\mathbb{Q}$.
- In 1979, the Soviet mathematician Belyi gave a necessary and sufficient condition for a curve to be defined over $\overline{\mathbb{Q}}$.

Theorem 1 (Belyi). *Let $X$ be a comp. Riemann surface, that is, a smooth projective variety in $\mathbb{P}^N(\mathbb{C})$ for some $N$. $X$ is defined over $\overline{\mathbb{Q}}$ if and only if there is a non-constant meromorphic function $\beta : X \to \mathbb{P}^1$ which is ramified over at most three points.*

- $(X, \beta)$ is called a *Belyi pair*.
- The group $\mathrm{Aut}(\mathbb{P}^1)$ of automorphisms of $\mathbb{P}^1$ acts triply transitively on $\mathbb{P}^1$,and so by composing $\beta$ with a suitable automorphism we can (and generally will) assume that its critical values are contained in $\{0, 1, \infty\}$.
- Belyi functions allow us to construct *dessin d'enfants*!

# From Belyi pairs to Dessins.

Start with $X = \mathbb{P}^1$ and $\beta : \mathbb{P}^1 \to \mathbb{P}^1$. Where $\beta(z) = z^n$. This function is only ramified at 0 and $\infty$.

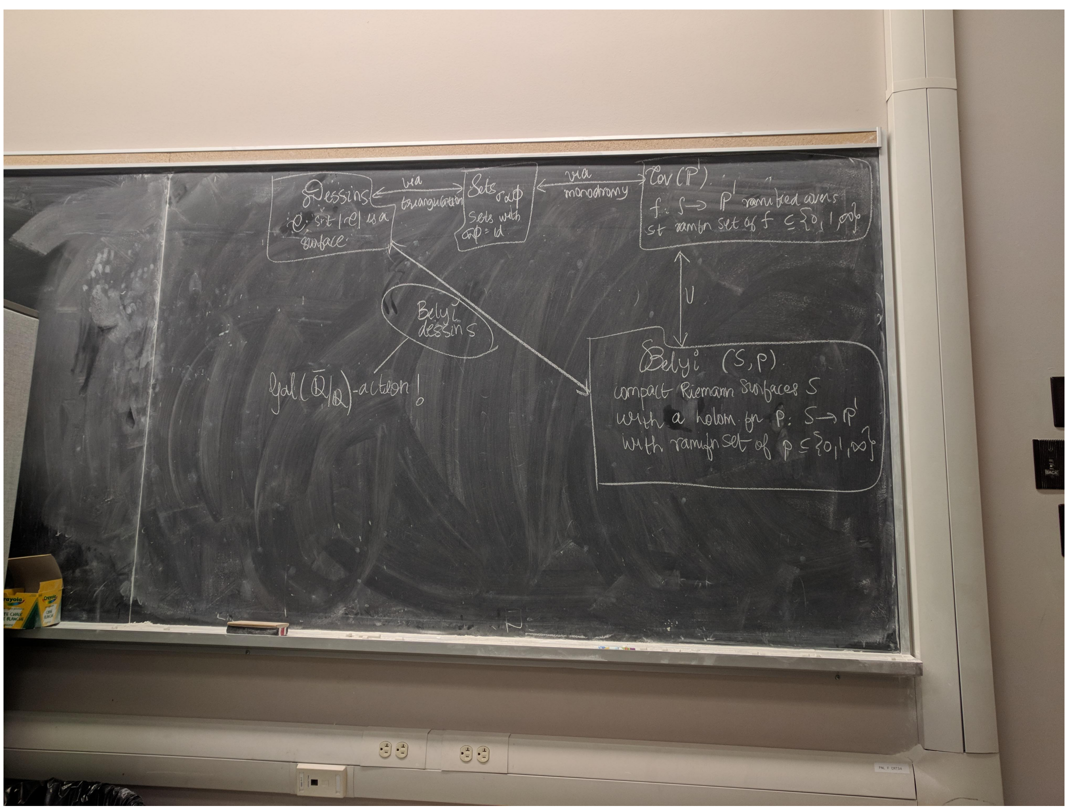
Given a Belyi pair $(X, \beta)$, we draw a "map" on $X$ as follows.

- Take the segment $[0, 1] \subset \mathbb{P}^1$. Colour the point 0 black and the point 1 white and let $*$ be the point at infinity.

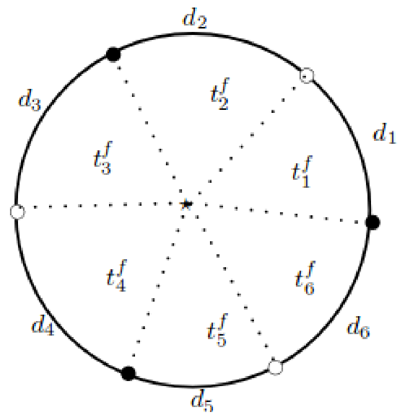
- Let $H = \beta^{-1}([0,1]) \subset X$. Since outside $\{0,1,\infty\}$, $\beta$ is a finite covering, $H$ will be a disjoint union of segments.
- The black vertices of $H$ are the preimages of 0, the white ones are preimages of 1, their valencies being equal to multiplicities of the corresponding critical points. This is because locally, the covering looks like $z \mapsto z^k$, that is, a star.
- Set $F = \beta^{-1}(\infty)$. Because $\mathbb{P}^1 \setminus [0,1]$ is an open subset in $\mathbb{P}^1$ homeomorphic to the disc. So $\beta^{-1}(\mathbb{P}^1 \setminus [0,1])$ is a disjoint union of open discs each containing one element of $F$. Faces of $H$ contain preimages of infinity, one preimage in each, and multiplicity of pole is the valency of the face, where the valency $\deg(f)$ of a face $f$ is the number of edges incident to this face. This is because on each such disc $\beta$ looks like $z \mapsto z^m$ for some $m$. Draw a triangle $T$ in the Riemann sphere, with vertices $0, 1$ and $\infty$, and colour the edges red, blue and green say. Now look at the inverse image $T$fi of $T$. This is a triangulation of $X$ and we colour its edges according to their image in $T$. Let the 0-1 edge be red. At a pole of $\beta$, $k$ blue and $k$ green edges emerge, where $k$ is the order of the pole. So $2k$ triangles of $T$ meet at that point, and they combine to form a face of $\beta^{-1}(T)$ with $2k$ edges. (The faces of $\beta^{-1}(T)$ have an even number of edges since it's a bipartite graph).

## From Dessins to Belyi Pairs.

In this part I will be very vague, because it takes a lot of time to set up the ideas precisely. However, I do want to draw the following zig-zag of equivalences.



- $\mathfrak{Dessins}$ For each dessin, make a triangulation.




Assemble these into a category, $\mathfrak{Dessins}$ of objects "bipartite graphs whose realization is a surface" and morphisms the "triangulation preserving" ones.

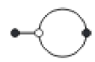
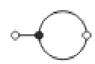
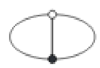
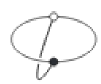
- $\mathfrak{Sets}$ Once we do this we immediately get a set of triangles and three permutations on them whose product is the identity. That is, an element of $\mathfrak{Set}_{\sigma,\alpha,\phi}$. It is also possible to go the other way to land in $\mathfrak{Dessins}$.

- $\mathfrak{Cov}(\mathbb{P}^1)$ Covers with ramification set included in $\{0, 1, \infty\}$ correspond to finite (unramified) covers of the space with these points excluded. Also, finite unramified coverings correspond to fibers of a basepoint with the monodromy action. With the base point $* = \frac{1}{2}$ (say), one has $\pi_1(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}, *) = \langle \sigma, \alpha \rangle$, the free group on the two distinguished generators $\sigma$ and $\alpha$; these are respectively the homotopy classes of the loops $t \mapsto \frac{1}{2}e^{2i\pi t}$ and $t \mapsto 1 - \frac{1}{2}e^{2i\pi t}$. The category of finite, right $\pi_1(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}, *)$-sets is precisely the category $\mathfrak{Sets}_{\sigma,\alpha,\phi}$ already mentioned.

- $\mathfrak{Belyi}$ Up until now, we have not used the complex structure on $\mathbb{P}^1$. Note that when $p \colon S \to R$ is a ramified cover, and $R$ is equipped with a complex structure, there is a unique complex structure on $S$ such that $p$ is holomorphic. So, given a compact Riemann surface $S$, and a meromorphic function $p$ on $S$, such that the ramification set of $p$ is contained in $\{0, 1, \infty\}$, we get a dessin and conversely. The pair $(S, p)$ is called a Belyi pair.

Even though this is very vague and quick, the point I want you to take home is that it is possible to go the other way, from a dessin to a Belyi pair.

Examples of Belyi pairs and Belyi dessins:

| Dessin | $\hat{X}$ | Equation for the cover |
|---|---|---|
|  | $\mathbb{P}^1(\mathbb{C})$ | $\beta_1(x) = x^3$ |
|  | $\mathbb{P}^1(\mathbb{C})$ | $\beta_2(x) = 1 - \beta_1(x) = 1 - x^3$ |
|  | $\mathbb{P}^1(\mathbb{C})$ | $\beta_3(x) = \frac{(4-x)(1+2x)^2}{27x}$ |
|  | $\mathbb{P}^1(\mathbb{C})$ | $\beta_4(x) = 1 - \beta_3(x) = \frac{4(x-1)^3}{27x}$ |
|  | $\mathbb{P}^1(\mathbb{C})$ | $\beta_5(x) = \frac{x^3+3x^2}{4}$ |
|  | $\mathbb{P}^1(\mathbb{C})$ | $\beta_6(x) = \frac{x^3}{x^3-1}$ |
|  | $y^2 = x^3 + 1$ | $\beta_7(x,y) = \frac{1}{2}(1+y)$ |

Definition 2 (Pure Dessin/Map).

- *A pure dessin (or a map) is a dessin in which every white vertex has valency* 2.
- *In such cases, we simply omit drawing the white vertex, implicitly assuming their existence at the "midpoint" of every edge. A Belyi function corresponding to a map is called a pure Belyi function.*

# Applications: A bound of Davenport-Stothers-Zannier

- Belyi functions can also be used in other applications.
- This is one of the most spectacular applications of Belyi functions.
- Let $P$ and $Q$ be two complex polynomials, what is the smallest possible degree of the polynomial $R = P^3 - Q^2$? The claim was that
  1. $deg(R) \geq \frac{1}{2}(deg(P)) + 1$
  2. This bound is sharp, i.e., is attained infinitely often.
- $\deg(P^3) \neq \deg(Q^2) \Rightarrow \deg(R) \geq \deg(P^3) = 3 \cdot \deg(P)$. So we may suppose that
  1. $P$ and $Q$ are monic.
  2. $\deg P = 2m$, $\deg Q = 3m$ for $m \in \mathbb{N}$: so leading terms cancel. Sharpness now says that $deg(R) = m + 1$ for infinitely many values of $m$.

- Sharpness took 16 years to prove!

Theorem 3. *The bound is sharp. In fact, it is attained for every value of m.*

*Proof.* Set $R(x) = P^3(x) - Q^2(x)$ and consider the function

$$f(x) = \frac{P^3(x)}{R(x)} \quad \text{so that} \quad f(x) - 1 = \frac{Q^2(x)}{R(x)}$$

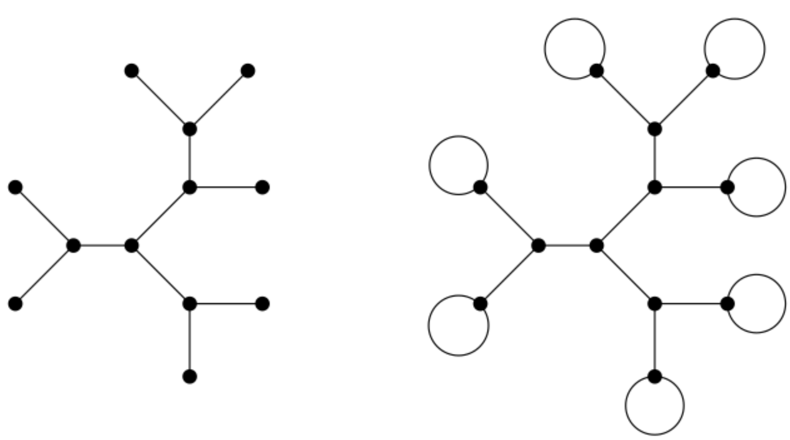
We ask ourselves: *what if $f$ was a (pure) Belyi map?* The combinatorial conditions on the corresponding Belyi dessin are as follows:

- The numerator of $f$ is $P^3$, $\deg(P) = 2m$ which means that the dessin has $2m$ black vertices, all of them degree 3.
- The numerator of $f - 1$ is $Q^2$, $\deg(Q) = 3m$ and so because we assumed $f$ was a pure Belyi function, this means that the dessin has $3m$ edges.
- For the number of faces $F$, Euler's formula now gives us:

  $$2m - 3n + F = 2 \Rightarrow F = m + 2$$

  Recall that the centers of the faces are the $f$-preimages of $\infty$. One of the faces, the "outer" one, has its center at infinity; the other centers are the roots of $R$; in order to have $\deg R = m + 1$ we must ensure that all the faces except the outer one are of degree 1.

In order to prove the statement, we should construct a planar map with $3m$*edges*, $2n$ vertices of degree 3 and all its $m + 1$ "finite" faces having degree 1. *Does there exist, for every m, a plane map with* $3m$ *edges, with* $2m$ *vertices of degree* 3*, and with all the faces except the outer one having degree* 1*?*



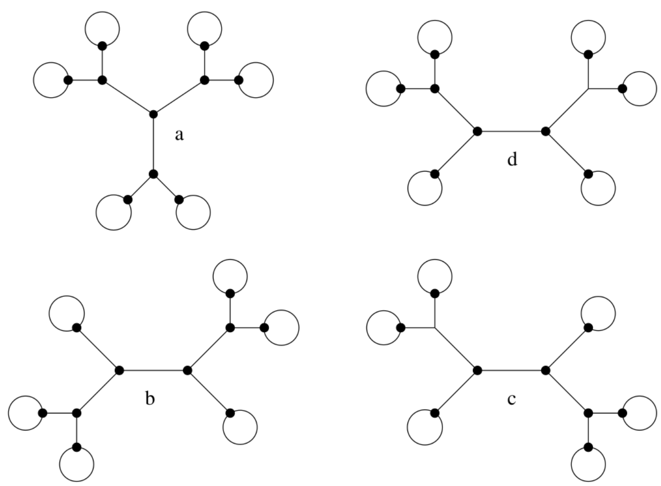
The answer is yes! The image above shows two stages for $m = 6$: first stage is to draw a tree with all the internal vertices degree 3 and the second stage is to attach a loop to each leaf.

To make this formal, for a natural number $m$, we plot $m-1$ vertices, and to these, we add $2(m-1)+1 = 2m-1$ edges, so that the end result is a tree with internal vertices all of degree 3. We now end up with $m-1+2 = m+1$ leaves, where we add a loop. Thus we add $m+1$ new edges to complete the construction. So in total, we have $(2m-1)+(m+1) = 3m$ edges and $2m$ vertices of degree 3.

Now that we have shown the existence, the Belyi function corresponding to this map will now have the required properties. Thus we have shown that the bound is sharp.

□

Note however, for example, that for $m = 5$, there might be other ways in which the map could have been constructed:




This clearly shows the orbits $\{a\}, \{b, c\}, \{d\}$ so that the corresponding field of moduli for $\{a\}$ and $\{d\}$ is $\mathbb{Q}$ and for the other orbit it is quadratic.