# *Dessins d'enfant* and the Galois Action

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This project is an outline of the arguments presented in the paper "*An elementary approach to dessins d'enfants and the Grothendieck-Teichmuller Group*" by Pierre Guilot.

The paper describes how dessin d'enfants arise naturally from classical theorems of different areas of mathematics, and how it leads us naturally into categorical equivalences between an appropriately defined category of dessins and many others. Indeed, it is due to these equivalences that it is possible to think of a dessin in many different contexts: as graphs embedded nicely on surfaces, finite sets with certain permutations, certain field extensions and some classes of algebraic curves (some defined over $\mathbb{C}$ and some over $\overline{\mathbb{Q}}$.

The term *dessins d'enfants* was coined by Grothendieck in his "*Esquisse d'un Programme*", in which a vast programme was laid out. In a nutshell, some of the categories mentioned above naturally carry an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the absolute Galois group of the rational field. This group therefore acts on the set of isomorphism classes of objects in any of the equivalent categories; in particular one can define an action of the absolute Galois group on graphs embedded on surfaces. In this situation however, the nature of the Galois action is really very mysterious - it is hoped that, by studying it, light may be shed on the structure of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It is the opportunity to bring some kind of basic, visual geometry to bear in the study of the absolute Galois group that makes *dessins d'enfants* – embedded graphs – so attractive.

A major open problem in the study of the theory of dessin d'enfants is to obtain "invariants" of the above Galois action: when can we say that two dessins are in the same Galois orbit? In this current outline, we define how a dessin is set up from a bipartite graph, how we can assemble a category of such objects and describe various categories equivalent to the category we define. This will lead us naturally to the Galois action and we will show that the action is faithful.

## 1 The category $\mathfrak{Dessins}$

Definition 1 (Bipartite Graphs). *We start with the definition of* bipartite graphs*, or* bigraphs *for short, where we assign colours black and white to the vertices, such that the edges only connect vertices of different colours. More formally, a bigraph consists of*

- *a set $B$, the elements of which we call the black vertices,*
- *a set $W$, the elements of which we call the white vertices,*
- *a set $D$, the elements of which we call the* darts*,*
- *two maps $\mathscr{B}\colon D \longrightarrow B$ and $\mathscr{W}\colon D \longrightarrow W$.*

To a bigraph $\mathscr{G}$ we may associate a topological space $|\mathscr{G}|$, called it's realization by attaching intervals along the edges and glueing according to the maps $\mathscr{B}$ and $\mathscr{W}$. Next we define cell complexes.

Definition 2 (Cell complexes). *Given a bigraph $\mathscr{G}$, a* loop *on $\mathscr{G}$ is a sequence of darts describing a cosed path on $\mathscr{G}$ alternating between black and white vertices. That is, it is a tupe $(d_1, d_2, \ldots, d_{2n}) \in D^{2n}$ such that $\mathscr{W}(d_{2i+1}) = \mathscr{W}(d_{2i+2})$ and $\mathscr{B}(d_{2i+2}) = \mathscr{B}(d_{2i+3})$, for $0 \leq i \leq n-1$, where $d_{2n+1}$ is to be understood as $d_1$. Loops on $\mathscr{G}$ form a set $L(\mathscr{G})$.*

*A* cell complex *consists now of*

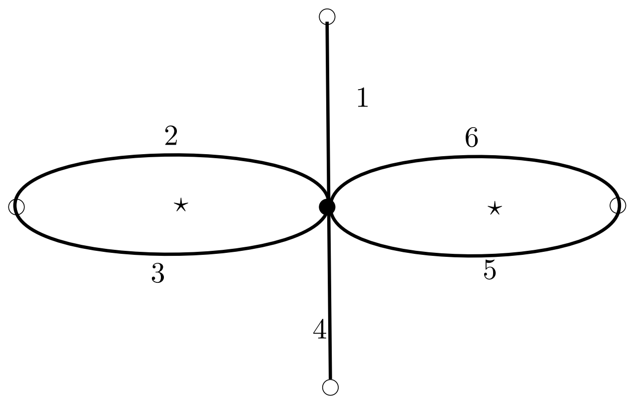
- *a bigraph $\mathscr{G}$,*
- *a set $F$, the elements of which we call the* faces*,*

- *a map $d\colon F \to L(\mathscr{G})$, called the boundary map.*

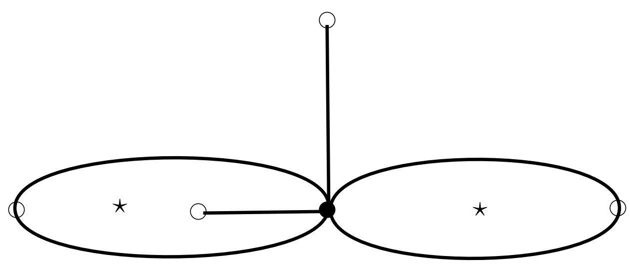
We can also talk about a topological realization $|\mathscr{C}|$ of a cell complex $\mathscr{C}$. This is done by attaching closed discs to the space $|\mathscr{G}|$ using the specified boundary maps. One can think of it as "filling in" the "holes" of $|\mathscr{G}|$ by closed discs and making appropriate identifications. One can make this precise by taking a copy of the closed disc $\mathbb{D}$ for each face $f$ and equipping the disjoint union of $|\mathscr{G}|$ and these faces with the quotient topology where we identify the arcs between points on the boundary of each disc with the vertices. This is made precise in the paper.

We shall often place a $*$ inside the faces, even when they are not labeled, to remind the reader to mentally fill in a disc. As an example, consider the following:

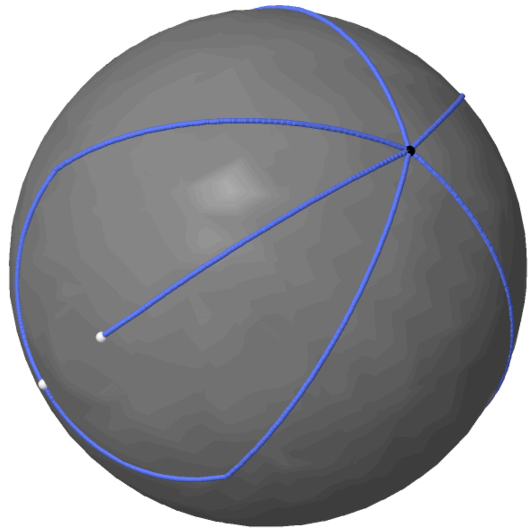
Example 1.1. *Look at the cell complex as follows, where we label the darts by the integers. There are two faces to this cell complex, which have boundary (2,3) and (5,6) respectively. We could also put a face on the "outside", which has boundary (1,1,2,3,4,4,5,6). The center of that face is placed "at infinity", so we make the identification of $|\mathscr{C}|$ with $S^2$.*




*Suppose now that we were to draw the following picture with a face "on the outside".*



*The topological realization of the above with a face "on the outside" is given as follows:*



Now we want to assemble these cell complexes into a category, and so we need a correct notion of a morphism of these. To define a morphism of cell complexes, we first resort to *triangulations*.

Definition 3 (Triangulation). *Given a cell complex $\mathscr{C}$, we may triangulate the faces of $|\mathscr{C}|$ by adding a point in the interior of each face (the point $*$) and connecting it to each vertex on the boundary. So if a face $f$ has boundary $(d_1, \ldots, d_{2n})$ we identify $2n$ subspaces of $|\mathscr{C}|$ each homeomorphic to a triangle. We denote them $t_i^f$ for $1 \leq i \leq 2n$. We write $T$ for the set of all triangles in the complex. There is a map $\mathscr{D}\colon T \to D$ which associates $t_i^f$ with $\mathscr{D}(t_i^f) = d_i$, there is also a map $\mathscr{F}\colon T \to F$ with $\mathscr{F}(t_i^f) = f$.*

Each $t \in T$ has vertices which we may call $\bullet$, $\circ$ and $*$ unambiguously. Its sides will be called $\bullet - \circ$, $* - \bullet$ and $* - \circ$. Each $t$ also has a neighbouring triangle obtained by reflecting in the $* - \bullet$ side; call it $a(t)$. Likewise, we may reflect in the $* - \circ$ side and obtain a neighbouring triangle, which we call $c(t)$. In other words, $T$ comes equipped with two permutations $a$ and $c$, of order two and having no fixed points.

Definition 4 (Morphism of cell complexes). *A* morphism *between $\mathscr{C} = (\mathscr{G}, F, d)$ and $\mathscr{C}' = (\mathscr{G}', F', d')$. We define this to be given by a morphism $\mathscr{G} \to \mathscr{G}'$ (thus including a map $\Delta\colon D \to D'$) and a map $\Theta\colon T \to T'$ which*

1. *verifies that for each triangle $t$, one has $\mathscr{D}'(\Theta(t)) = \Delta(\mathscr{D}(t))$,*
2. *is compatible with the permutations $a$ and $c$, that is $\Theta(a(t)) = a(\Theta(t))$ and $\Theta(c(t)) = c(\Theta(t))$.*

*It is immediate that morphisms induce continuous maps between the topological realizations. These continuous maps restrict to homeomorphisms between the triangles.*

We now have a topological space $|\mathscr{C}|$ associated to every cell complex $\mathscr{C}$. So it is now natural to ask: *under what conditions on $\mathscr{C}$ is $|\mathscr{C}|$ a surface?* Answering this question, we have:

Proposition 5. *Let $\mathscr{C}$ be a complex. Then $|\mathscr{C}|$ is a topological surface if and only if the following conditions are met:*

1. *each vertex has positive degree,*
2. *each dart is on the boundary of precisely two faces, counting multiplicities,*
3. *all the connectivity graphs (which is a graph $\mathscr{C}_b$ for a vertex $b \in B$ with vertices $\mathscr{B}^{-1}(b)$, the set of darts whose black vertex is $b$, as its set of vertices and there is an edge between two darts whenever they appear consecutively in some face $f$ in $\mathscr{C}$) are connected.*

*Necessary and sufficient conditions for $|\mathscr{C}|$ to be a surface-with-boundary are obtained by replacing (2) with the condition that each dart is on the boundary of either one or two faces, counting multiplicities.*

Note that if we assume that the darts in $\mathscr{C}$ are on the boundary of no more than two faces, then it follows that each vertex in $\mathscr{C}_b$ is connected to at most two others. Thus when $\mathscr{C}_b$ is connected, it is either a straight path or a circle.

We now quickly collect a series of intermediate results:

Proposition 6. *For a cell complex $\mathscr{C}$, we have the following:*

1. *The space $|\mathscr{C}|$ is connected if and only if the realization $|\mathscr{G}|$ is.*
2. *The space $|\mathscr{C}|$ is compact if and only if the complex is finite ( i.e., the sets $B, W, D$ and $F$ are finite.*
3. *If $|\mathscr{C}|$ is a compact, connected surface, then $|\mathscr{C}|$ is orientable if and only if it is possible to assign a colour to each triangle, black or white in such a way that two triangles having a side in common are never of the same colour.*
4. *A morphism $\mathscr{C} \to \mathscr{C}'$ where $\mathscr{C}$ and $\mathscr{C}'$ are oriented compact, connected surfaces is called orientation preserving if it sends black triangles to black triangles and white triangles to white triangles.*
5. *If $|\mathscr{C}|$ is a compact, connected, oriented surface without boundary, then there is a bijection between the black trianges and the darts.*

A cell complex $|\mathscr{C}|$ which is a surface is called a *dessin d'enfant* or simply a *dessin*. We write $\mathfrak{Dessins}$ for the category whose objects are compact, oriented dessins without boundary, and whose morphisms are the orientation-preserving maps of cell complexes. We write $\mathfrak{UDessins}$ for the category whose objects are compact dessins without boundary and whose morphisms are just morphisms of cell complexes.

## 2 Categories equivalent to $\mathfrak{Dessins}$

### 2.1 The categories $\mathfrak{Sets}_{a,b,c}$ and $\mathfrak{Sets}_{\phi,\alpha,\sigma}$

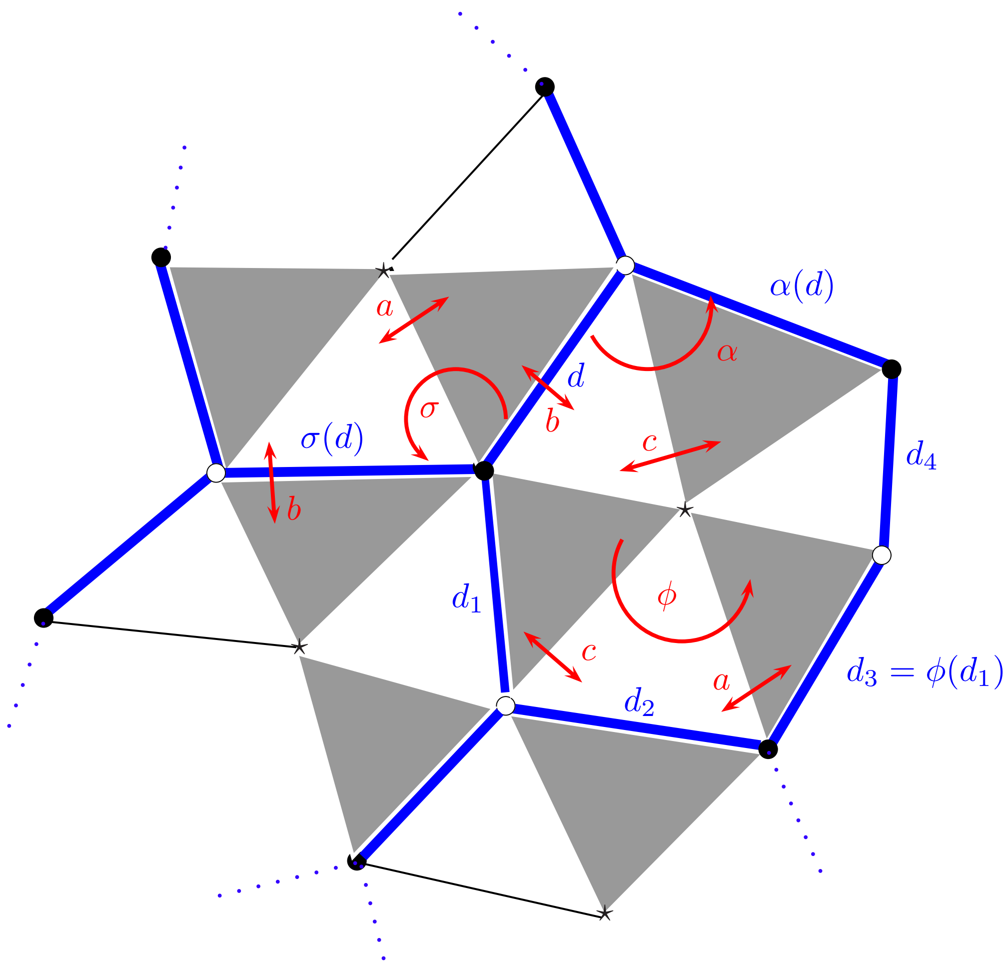
Definition 7 (Permutations *a, b, c*). *Let $\mathscr{C}$ be a dessin. Shortly after the definition of triangulation (Definition 3) we remarked about how the triangulation of a cell complex naturally gives rise to two permutations $a$ and $c$. Now any triangle $t \in T$ determines a dart $d = \mathscr{D}(t)$ and $d$ belongs to one or two triangles (exactly two when $|\mathscr{C}|$ has no boundary. We may thus define another permutation $b$ of $T$ by requiring*

$$b(t) = \begin{cases} t \text{ if no other triangle has } d \text{ as a side}\,, \\ t' \text{ if } t' \text{ has } d \text{ as a side and } t' \neq t\,. \end{cases}$$

*Each of the permutations a, b and c have order two and have no fixed points.*

Definition 8 (Permutations $\phi, \alpha, \sigma$). *Now consider the permutations $\sigma = ab$, $\alpha = bc$ and $\phi = ca$. Each preserves the subset of $T$ comprised by the black triangles, so we may see $\sigma$, $\alpha$ and $\phi$ as permutation of $D$. It is immediate that they satisfy $\sigma\alpha\phi = 1$, the identity permutation. (Here if $\sigma$ and $\tau$ are permutations on a set $X$, then $\sigma\tau$ is defined to be the permutation $x \mapsto \tau(\sigma(x))$)*

The following picture shows how these permutations act:




These permutations are relevant to our dicussion because of the following theorem:

Theorem 9. *Consider the category $\mathfrak{Sets}_{a,b,c}$ whose objects are the finite sets $T$ equipped with three distinguished permutations $a$, $b$, $c$,each of order two and having no fixed points, and whose arrows are the equivariant maps. Then the assigment $\mathscr{C} \to T$ (the set of triangles) extends to an equivalence of categories between $\mathfrak{UDessins}$ and $\mathfrak{Sets}_{a,b,c}$.*

*Likewise, consider the category $\mathfrak{Sets}_{\sigma,\alpha,\phi}$ whose objects are the finite sets $D$ equipped with three distinguished permutations $\sigma$, $\alpha$, $\phi$ satisfying $\sigma\alpha\phi = 1$, and whose arrows are the equivariant maps. Then the assigment $\mathscr{C} \to D$ extends to an equivalence of categories between $\mathfrak{Dessins}$ and $\mathfrak{Sets}_{\sigma,\alpha,\phi}$.*

It may be of interest to note that for a dessin $\mathscr{C}$ the group generated by the permutations $a$, $b$, $c$ is called the *full cartographic group* and the group generated by $\sigma$, $\tau$, $\alpha$ is called the *cartographic group* or *monodromy group* of $\mathscr{C}$.

# 3 The category $\mathfrak{Cov}(\mathbb{P}^1)$.

From now on, we shall be interested only in compact, oriented dessins without boundary. Let $S$ and $R$ be topological surfaces. A map $p\colon S \to R$ is called a *ramified cover* if there exists for each $s \in S$ a couple of charts, centered around $s$ and $p(s)$ respectively, in which the map $p$ becomes $z \mapsto z^e$ for some integer $e \geq 1$ called the *ramification index at s*.

Note that the set of all $s \in S$ whose ramification index is > 1 is discrete in $S$, so closed and hence finite as $S$ is compact. Its image in $R$ under $p$ is called the *ramification set* and written $R_r$. It follows that the restriction

$$p\colon S \smallsetminus f^{-1}(R_r) \longrightarrow R \smallsetminus R_r$$

is a finite covering in the traditional sense. Now, it is a classical result that one can go the other way around: namely, start with a compact topological surface $R$, let $R_r$ denote a finite subset of $R$, and let $p\colon\ U \longrightarrow R \smallsetminus R_r$ denote a finite covering map; then one can construct a compact surface $S$ together with a ramified cover $\bar{p}\colon\ S \to R$ such that $U$ identifies with $\bar{p}^{-1}(R \smallsetminus R_r)$ and $p$ identifies with the restriction of $\bar{p}$. The ramification set of $\bar{p}$ is then contained in $R_r$.

So given the above result, once we fix the ramification set to be a subset of a given finite set $R_r$, ramified covers are in one-one correspondence with covering maps. To make this more precise, let us consider two ramified covers $p\colon\ S \to R$ and $p'\colon\ S' \to R$ both having a ramification set contained in $R_r$, and let us define a morphism between them to be a continuous map $h\colon\ S \to S'$ such that $p' \circ h = p$. Morphisms, of covering maps above $R \smallsetminus R_r$ are defined similarly. We may state:

Theorem 1. *The category of finite coverings of $R \smallsetminus R_r$ is equivalent to the category of ramified covers of $R$ with ramification set included in $R_r$.*

From classical algebraic topology, we have the following result:

Theorem 2. *Assume that $R$ is connected, and pick a base point $* \in R \smallsetminus R_r$. The category of coverings of $R \smallsetminus R_r$ is equivalent to the category of right $\pi_1(R \smallsetminus R_r, *)$-sets. The functor giving the equivalence sends $p\colon\ U \to R \smallsetminus R_r$ to the fibre $p^{-1}(*)$ with the monodromy action.*

We shall now specialize to $R = \mathbb{P}^1 = S^2$ and $R_r = \{0, 1, \infty\}$. With the base point $* = \frac{1}{2}$ (say), one has $\pi_1(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}, *) = \langle \sigma, \alpha \rangle$, the free group on the two distinguished generators $\sigma$ and $\alpha$; these are respectively the homotopy classes of the loops $t \mapsto \frac{1}{2}e^{2i\pi t}$ and $t \mapsto 1 - \frac{1}{2}e^{2i\pi t}$. The category of finite, right $\pi_1(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}, *)$-sets is precisely the category $\mathfrak{Sets}_{\sigma,\alpha,\phi}$ already mentioned.

The following result combines theorem 9 from the previous section, theorem 1 above, as well as theorem 2:

Theorem 3. *The category $\mathfrak{Dessins}$ of oriented, compact dessins without boundary is equivalent to the category $\mathfrak{Cov}(\mathbb{P}^1)$ of ramified covers of $\mathbb{P}^1$ having ramification set included in $\{0, 1, \infty\}$.*

We do not need to go through the permutations to get to the above equivalence of categories. Here is a result with more geometric intuition:

Proposition 10. *Let $\mathscr{C}$ correspond to $p\colon\ S \to \mathbb{P}^1$ in the above equivalence of categories. Then $|\mathscr{C}| \cong S$, under a homeomorphism taking $|\mathscr{G}|$ to the inverse image $p^{-1}([0, 1])$.*

## 4 The category $\mathfrak{Belyi}$.

When $p\colon\ S \to R$ is a ramified cover, and $R$ is equipped with a complex structure, there is a unique complex structure on $S$ such that $p$ is complex analytic which is a classical result of Riemann surfaces. Any morphism between $S$ and $S'$, over $R$, is then itself complex analytic. Conversely if $S$ and $R$ both have complex structures, an analytic map $S \to R$ is a ramified cover as soon as it is not constant on any connected component of $S$.

We may state yet another equivalence of categories. Recall that an analytic map $S \to \mathbb{P}^1$ is called a meromorphic function on $S$.

Theorem 4. *The category $\mathfrak{Dessins}$ is equivalent to the category $\mathfrak{Belyi}$ of compact Riemann surfaces with a meromorphic function whose ramification set is contained in $\{0, 1, \infty\}$.*

Example 4.1. *Let us illustrate the results up to now with dessins on the sphere, so let $\mathscr{C}$ be such that $|\mathscr{C}|$ is homeomorphic to $S^2$. By the above, $\mathscr{C}$ corresponds to a Riemann surface $S$ equipped with a Belyi map $p\colon\ S \to \mathbb{P}^1$.*

*By proposition 10, $S$ is itself topologically a sphere. The uniformization theorem states that there is a complex isomorphism $\vartheta\colon\ \mathbb{P}^1 \longrightarrow S$, so we may as well replace $S$ with $\mathbb{P}^1$ equipped with $F = p \circ \vartheta$. Then $(\mathbb{P}^1, F)$ is a Belyi pair isomorphic to $(S, p)$.*

*Now $F\colon\ \mathbb{P}^1 \to \mathbb{P}^1$, which is complex analytic and not constant, must be given by a rational fraction, as is classical.* The bigraph $\mathscr{G}$ can be realized as the inverse image $F^{-1}([0, 1])$ where $F\colon\ \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is a rational fraction.

*Let us take this opportunity to explain the terminology* dessins d'enfants *(children's drawings), and stress again some remarkable features. By drawing a simple picture, we may as in example 1.1 give enough information to describe a cell complex $\mathscr{C}$.*

*Very often it is evident that $|\mathscr{C}|$ is a sphere, as we have seen in this example. What the theory predicts is that we can find a rational fraction $F$ such that the drawing may be recovered as $F^{-1}([0,1])$. This works with pretty much any planar, connected drawing that you can think of, and gives these drawings a rigidified shape.*

*To be more precise, the fraction $F$ is unique up to an isomorphism of $\mathbb{P}^1$, that is, up to precomposing with a Moebius transformation. This allows for rotation and stretching, but still some features will remain unchanged. For example the darts around a given vertex will all have the same angle $\frac{2\pi}{e}$ between them, since $F$ looks like $z \mapsto z^e$ in conformal charts.*

## 5 The category $\mathfrak{Etale}(\mathbb{C}(x))$

If $R$ is a compact connected Riemann surface, all the meromorphic functions on S assemble into a field $\mathscr{M}(R)$. If $S$ is not assumed to be connected, then $\mathscr{M}(S)$ breaks up as a direct sum of fields, one for each connected component of $S$. This is called the *étale algebra* of $S$. An *étale algebra over a field $K$* is an étale algebra which is also a $K$-algebra which is finite dimensional over $K$. We have now:

Theorem 5. *Fix a compact, connected Riemann surface $R$. The category of compact Riemann surfaces $S$ with a ramified cover $S \to R$ is equivalent to the opposite category of étale algebras over $\mathscr{M}(R)$. The equivalence is given by $S \mapsto \mathscr{M}(S)$, and the degree of $S \to R$ is equal to the dimension of $\mathscr{M}(S)$ as a vector space over $\mathscr{M}(R)$. Where degree of a ramified cover is the degree of the underlying cover after removing the ramification set.*

Now in order to get an equivalence of $\mathfrak{Dessins}$ with $\mathfrak{Etale}(\mathbb{C}(x))$, we set $R = \mathbb{P}^1$. But we need to understand how "ramification" translates into a statement about étale algebras. Note that we have $\mathscr{M}(\mathbb{P}^1) = \mathbb{C}(x)$ where $x$ is the identity on $\mathbb{P}^1$.

Definition 11 (Ramification in Étale algebras). *We give the definition in steps:*

1. *If $k$ is a field, consider a finite Galois extension $L$ of $k(x)$. We shall say that $L/k(x)$ is* not ramified at 0 *when it embeds into the extension $k((x))/k(x)$ where $k((x))$ is the field of formal power series in $x$.*

2. *For any $s \in k$, construct $L_s = L \otimes_{k(x)} k(x)$ where we see $k(x)$ as an algebra over $k(x)$* via *the map $k(x) \to k(x)$ which sends $x$ to $x + s$. When $L_s/k(x)$ is not ramified at $0$, we say that $L/k(x)$* is not ramified at *$s$.*

3. *Finally, using the map $k(x) \to k(x)$ which sends $x$ to $x^{-1}$, we get an extension $L_\infty/k(x)$, proceeding as above. When the latter is not ramified at $0$, we say that $L/k(x)$ is* not ramified at $\infty$.

If the above conditions are not satisfied, we will say that $L$ is ramified at $s$. It is a theorem that the definitions of topological ramification and algebraic ramification agree.

Lemma 5.1. *Let $p\colon S \to \mathbb{P}^1$ be a ramified cover, with $S$ connected, and assume that the corresponding extension $\mathscr{M}(S)/\mathbb{C}(x)$ is Galois. Then for any $s \in \mathbb{P}^1$, the ramification set $\mathbb{P}^1_r$ contains $s$ if and only if $\mathscr{M}(S)/\mathbb{C}(x)$ ramifies at $s$ in the algebraic sense.*

*In particular, the ramification set in contained in $\{0, 1, \infty\}$ if and only if the extension $\mathscr{M}(S)/\mathbb{C}(x)$ does not ramify at $s$ whenever $s \notin \{0, 1, \infty\}$.*

Finally, an étale algebra over $k(x)$ will be said not to ramify at $s$ when it is a direct sum of field extensions, none of which ramifies at $s$. This clearly corresponds to the topological situation when $k = \mathbb{C}$, and we have established the following.

Theorem 6. *The category $\mathfrak{Dessins}$ is equivalent to the category $\mathfrak{Etale}(\mathbb{C}(x))^{op}$ of finite, étale algebras over $\mathbb{C}(x)$ that are not ramified outside of $\{0, 1, \infty\}$, in the algebraic sense.*

In fact, more is true.

Theorem 7. *The category $\mathfrak{Dessins}$ is anti-equivalent to the category $\mathfrak{Etale}(\overline{\mathbb{Q}}(x))$ of finite, étale extensions of $\overline{\mathbb{Q}}(x)$ that are not ramified outside of $\{0, 1, \infty\}$, in the algebraic sense.*

# 6 The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

In this section we finally indicate how the absolute Galois group of the rationals acts on the set of isomorphism classes of dessins. The action is most natural on the objects of the category $\mathfrak{Etale}(\overline{\mathbb{Q}}(x))$ and understanding it in $\mathfrak{Dessins}$ is difficult because of the zig-zag of equivalences we need to go through before we land up in $\mathfrak{Dessins}$. But in some cases, such as in Genus 0, we can study the action.

Let $\lambda\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ be an element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We extend it to a map $\overline{\mathbb{Q}}(x) \to \overline{\mathbb{Q}}(x)$ which fixes $x$, and use the same letter $\lambda$ to denote it. In this situation the tensor product operation

$$- \otimes_\lambda \overline{\mathbb{Q}}(x)$$

defines a functor from $\mathfrak{Etale}(\overline{\mathbb{Q}}(x))$ to itself. In more details, if $L/\overline{\mathbb{Q}}(x)$ is an étale algebra, one considers

$${}^\lambda L = L \otimes_\lambda \overline{\mathbb{Q}}(x)\,.$$

The notation suggests that we see $\overline{\mathbb{Q}}(x)$ as a module over itself *via* the map $\lambda$. We turn ${}^\lambda L$ into an algebra over $\overline{\mathbb{Q}}(x)$ using the map $t \mapsto 1 \otimes t$.

To describe this in more concrete terms, as well as verify that ${}^\lambda L$ is an tale algebra over $\overline{\mathbb{Q}}(x)$ whenever $L$ is, it is enough to consider field extensions, since the operation clearly commutes with direct sums. So if $L \cong \overline{\mathbb{Q}}(x)[y]/(P)$ is a field extension of $\overline{\mathbb{Q}}(x)$, with $P \in \overline{\mathbb{Q}}(x)[y]$ an irreducible polynomial, then ${}^\lambda L \cong \overline{\mathbb{Q}}(x)[y]/({}^\lambda P)$, where ${}^\lambda P$ is what you get when the (extented) map $\lambda$ is applied to the coefficients of $P$. Clearly ${}^\lambda P$ is again irreducible (if it could be factored as a product, the same could be said of $P$ by applying $\lambda^{-1}$). Therefore ${}^\lambda L$ is again a field extension of $\overline{\mathbb{Q}}(x)$, and coming back to the general case, we do conclude that ${}^\lambda L$ is an étale algebra whenever $L$ is. What is more, the ramification condition satisfied by the objects of $\mathfrak{Etale}(\overline{\mathbb{Q}}(x))$ is obviously preserved.

Let $\mu \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Note that $y \otimes s \otimes t \mapsto y \otimes \mu(s)t$ yields an isomorphism

$${}^\mu\left({}^\lambda L\right) = L \otimes_\lambda \overline{\mathbb{Q}}(x) \otimes_\mu \overline{\mathbb{Q}}(x) \longrightarrow L \otimes_{\mu\lambda} \overline{\mathbb{Q}}(x) = {}^{\mu\lambda} L\,.$$

As a result, the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts (on the left) on the set of isomorphism classes of objects in $\mathfrak{Etale}(\overline{\mathbb{Q}}(x))$, or in any category equivalent to it. We state this separately in $\mathfrak{Dessins}$.

Theorem 8. *The absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts on the set of isomorphism classes of compact, oriented dessins without boundaries.*