Mathematical Analysis and Techniques

- **Qualitative Analyses:**
- **(i) Existence and stability of equilibria.**
- **(ii) Existence and Non-existence of periodic solutions or solutions.**
- **(iii) Uniqueness of solutions/Boundedness**
- **(iv) Invariant region**
- **(v) Permanence of the model**
- **Methods of stability analysis: Techniques and Theories:**
- **- Linearisation**
- **- Routh- Herwitz method**
  - Bifurcations and manifolds

- **- Center Manifolds**
- **- Next generation operator method: Reproduction number**
- **- Fixed point theory**
- **Statistical: uncertainty and sensitivity analysis, optimization**

- (d) **Lyapunov functions and LaSalle's Invariance Principle**
- (e) **Dulac's criterion**
- (f) **Comparison Theorem**
- (g) **Statistical: uncertainty and sensitivity**
- **analysis; optimization etc.**

EQUILIBRIUM POINTS

- Steady states as t approaches infinity, ie, the behaviour of the model/epidemic in the long run
- **Rates of change = 0**
- System my have **none, one or more** equilibrium points in a virtually spacious pattern in state space
- Need to know if they **exist**
- **Write down condition for the existence**

TYPES OF EQUILIBRIUM POINTS

- **Disease-free equilibrium (DFE):- no disease and therefore no infectives: (S*,0,R*)**
- **The endemic equilibrium point (EEP): - the disease persists: (S*,I*,R*),ie, I* > 0**

STABILITY

- **Interest not just on existence but on their stability**
- **Properties of stability characterise how a system behave if its state is initiated close to a given equilibrium point**
- **System initiated with the state exactly equal to an equilibrium point will never move by definition**
- **If initiated close by the state may remain close by or it may move away**

Properties of matrices

- **Eigenvalues of matrices-**
- **- Use normal method** $Ax = \lambda x$
- **Metzler matrix or M- matrix**

  -A matrix with all its off-diagonal terms greater or equal to zero

  -M- matrices have positive or negative entries on the diagonal

- **If all of the diagonal entries are non-positive and if all the column sums are non-positive, the matrix is a compartmental matrix and cannot have eigenvalues with positive real part or purely imaginary eigenvalues**
- **For stability sometimes we do not need to determine the eigenvalues specifically**
- **We simply need the sign of the eigenvalues- theory of M-matrices is very useful in this respect**

- **Compound matrices $A^{[k]}$**

  - important to our purpose is its spectral property

Stability of equilibrium points

- **LINEARISATION**
  - **Find the Jacobian matrix**
  - **Find the eigenvalues, λ**
  - **If real part of λ < 0 = Asymptoticalstable**
  - **If real part of λ> 0 = Unstable**
  - **If real part of λ = 0 = Inconclusive**

EXAMPLE 2:

- **CONSIDER THE SYSTEM**

$$\frac{dS}{dt} = \mu N - \beta SI - \mu S$$

$$\frac{dI}{dt} = \beta SI - \gamma I - (\mu + \delta) I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

(a)Find the equilibrium points and classify them as DFE and EEP

(b) Find the condition for existence of EEP.

(c) Analyse the stability of the equilibrium points

(d) Write down the biological implication of your analyses

- **DFE = (N,0,0)**
- **EEP = ((μ+γ+ δ)/β, μ(βN-γ-μ–δ)/β(γ+μ+δ), γ(βN-γ-μ-δ)/β))**
- **R_0 = βN /(γ+μ+ δ)**
- **Endemic equilibrium point EEP exists when R_0 > 1.**
- **DFE is asymptotical stable if**

R_0 < 1 and unstable if R_0 > 1

EEP is asymptotical stable if R_0 > 1

And unstable if R_0 < 1

- **DFE = (N,0,0)**
- **EEP = $((\mu+\gamma)/\beta, \mu(\beta N-\gamma-\mu)/\beta(\gamma+\mu), \gamma(\beta N-\gamma-\mu)/\beta))$**
- **$R_0 = \beta N /\gamma+\mu$**
- **Endemic equilibrium point EEP exists when $R_0 > 1$.**
- **DFE is asymptotical stable if**

$R_0 < 1$ and unstable if $R_0 > 1$

EEP is asymptotical stable if $R_0 > 1$

And unstable if $R_0 < 1$

Reproduction Number (Rate, Ratio)

- Expected number of secondary individuals produced by an individual in its lifetime

- Number of newly infected cells produced by a single infected cell (in-host dynamics).

- number of individuals infected within a single infected individual's entire infectious lifetime (epidemiology)

This the basic reproduction number R_0 when there is no intervention

A brief history of R0

- Originally developed for demographics (1886)
- Independently studied for malaria (1911,1927)
- Now widely used for infectious disease (1975+)

- “One of the foremost and most valuable ideas that mathematical thinking has brought to epidemic theory” (Heesterbeek & Dietz, 1996).

A threshold criterion

- $R_0 = 1$ is a threshold below which the generation of secondary cases is insufficient to maintain the infection with the human community

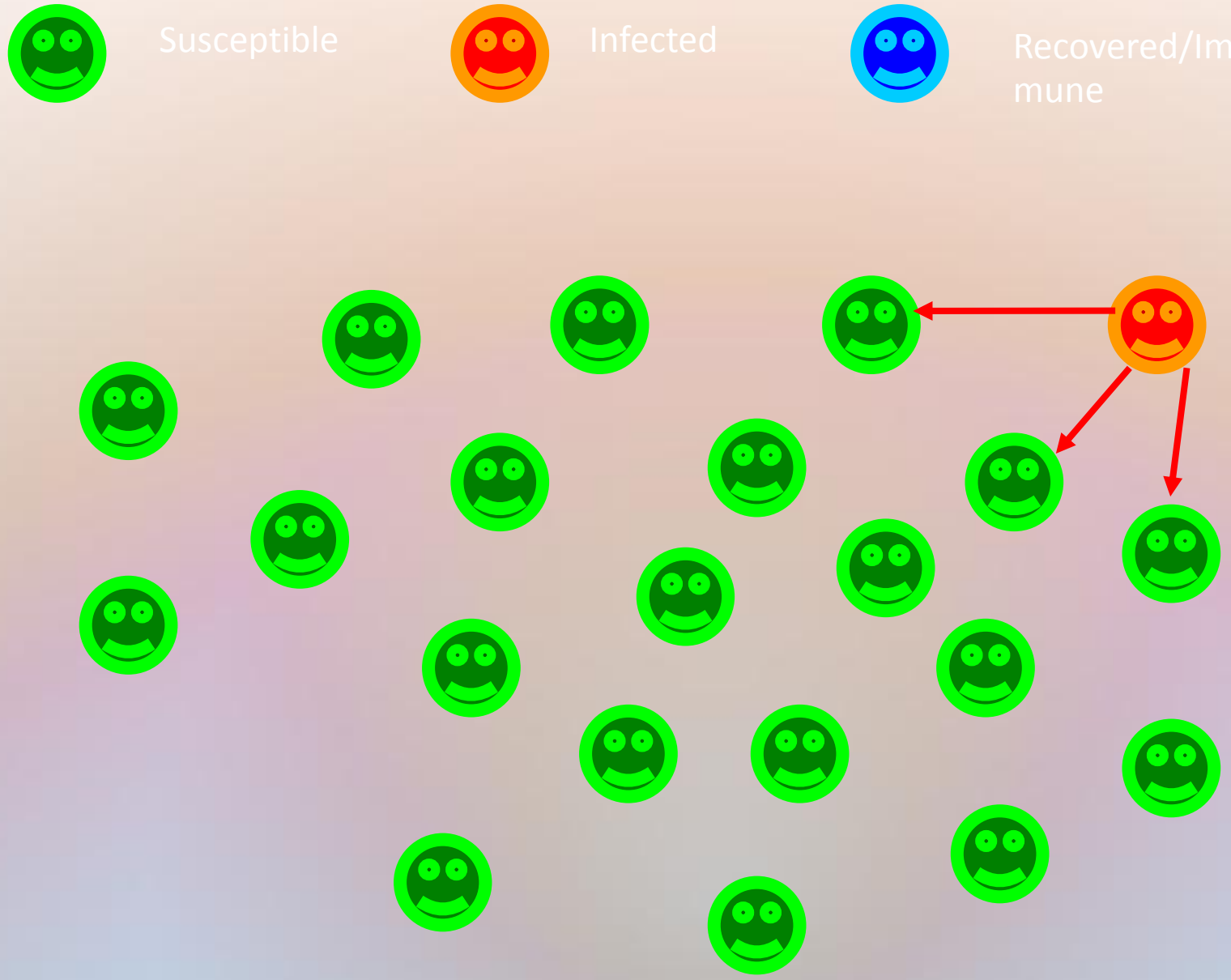
- If R0<1, each individual produces, on average, less than one new infected individual...

  ...and hence the disease dies out

- If R0>1, each individual produces more than one new infected individual...

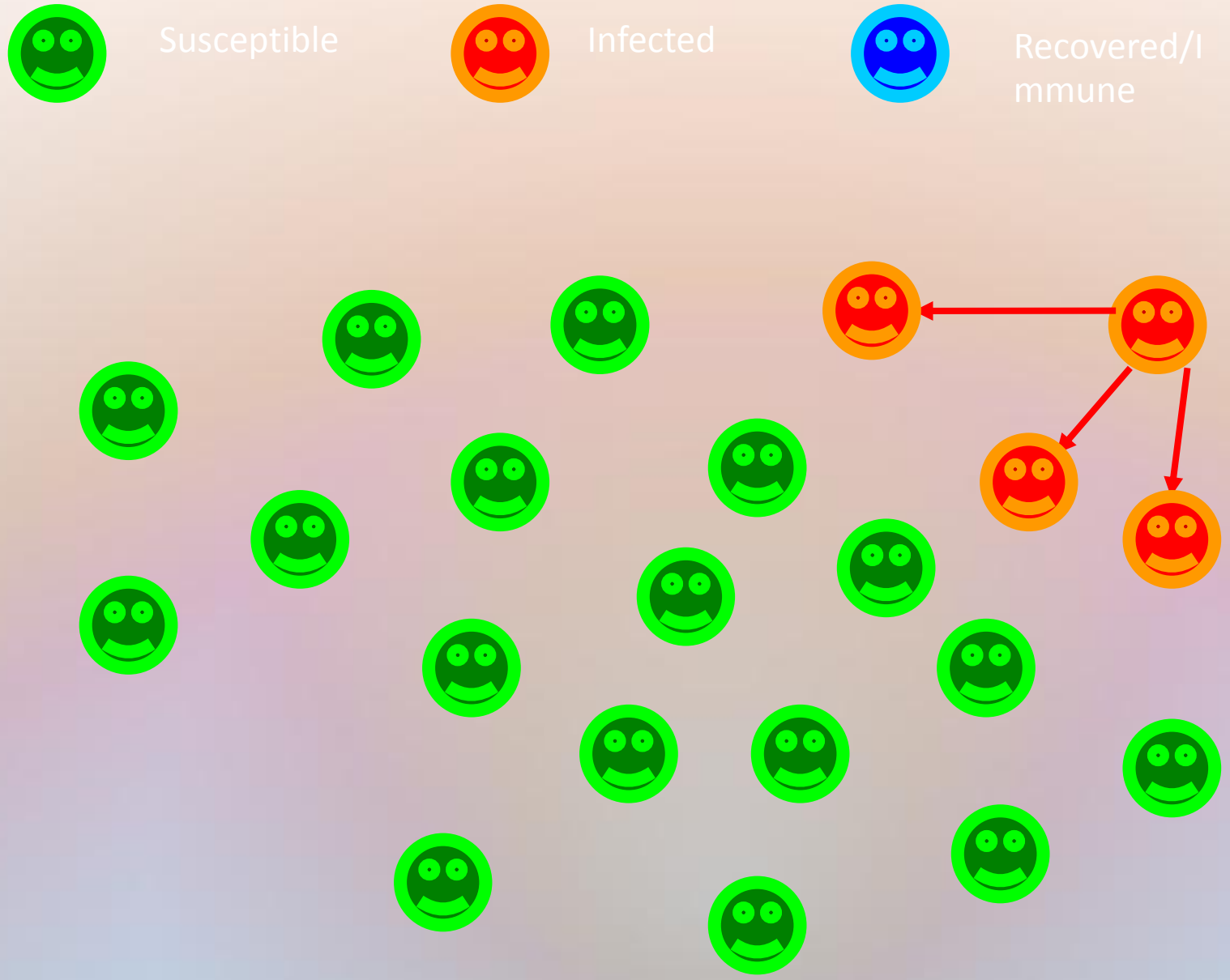
  ...and hence the disease is able to invade the susceptible population

Transmission process (a)☑



One person acquires infection in the community and transmits to three neighbours (adapted from F. Nyabadza)

Transmission process (b)☑



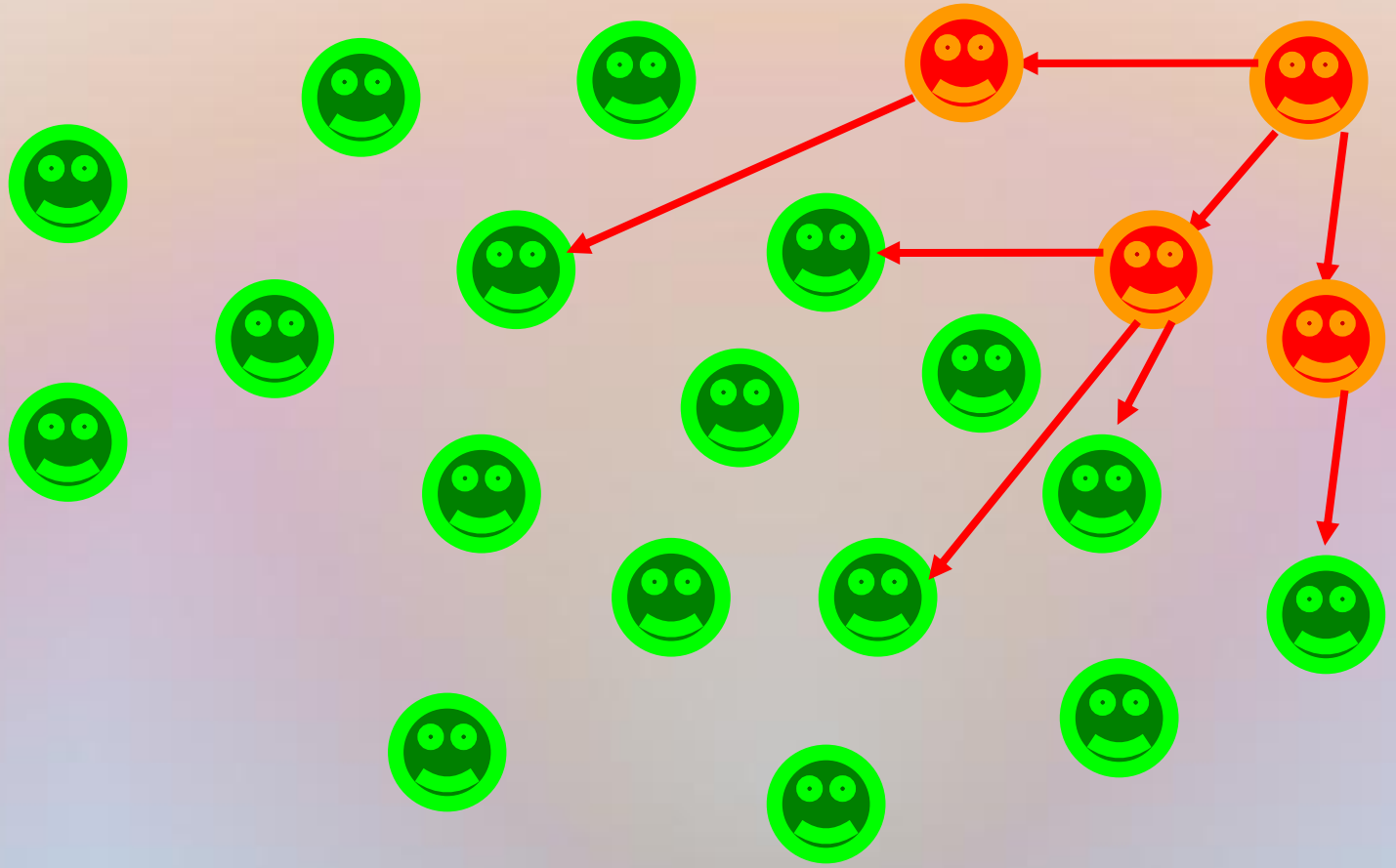
Three persons infected by one: R0 = 3☑

Transmission process (c)☑

Susceptible

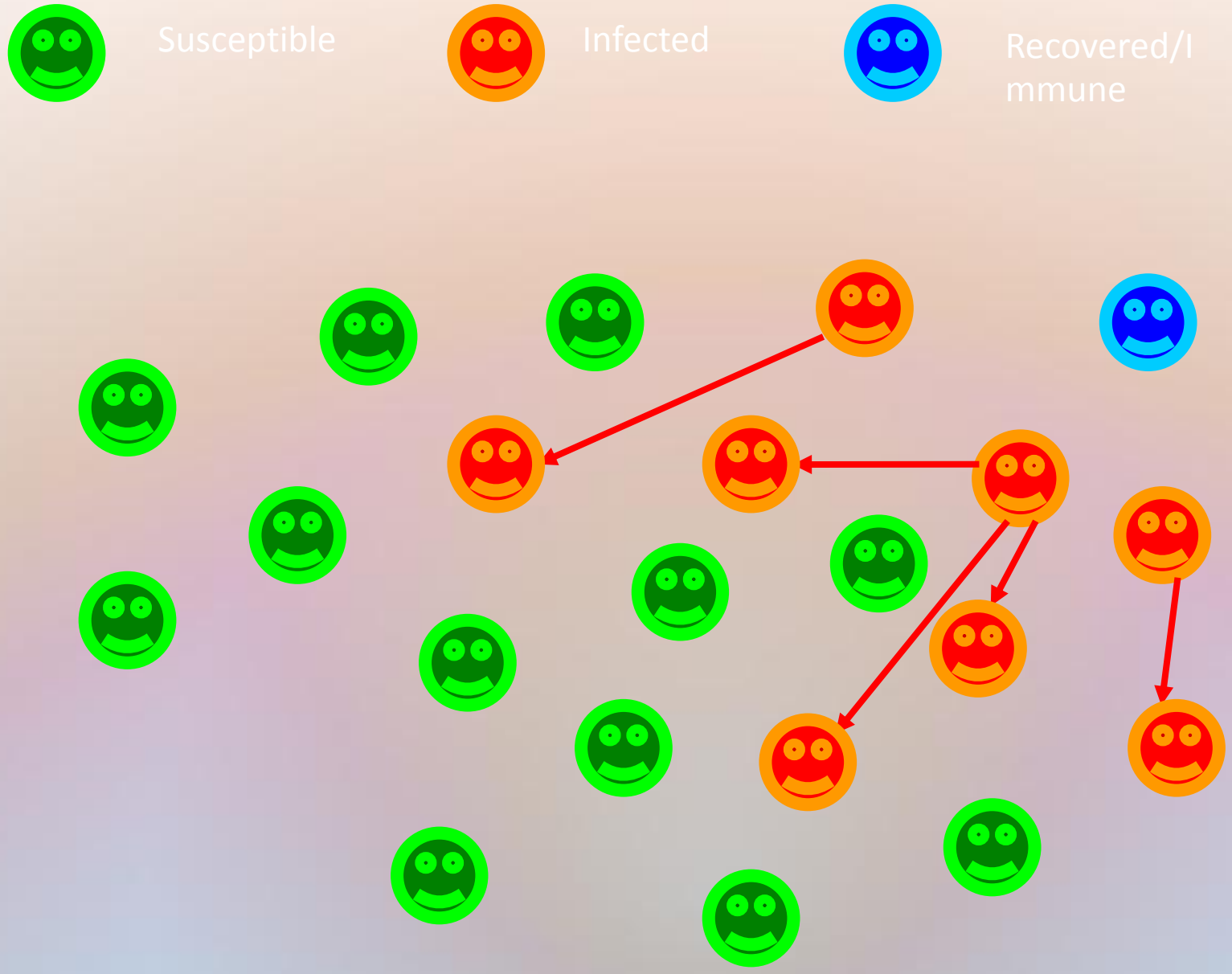
Infected

Recovered/Immune



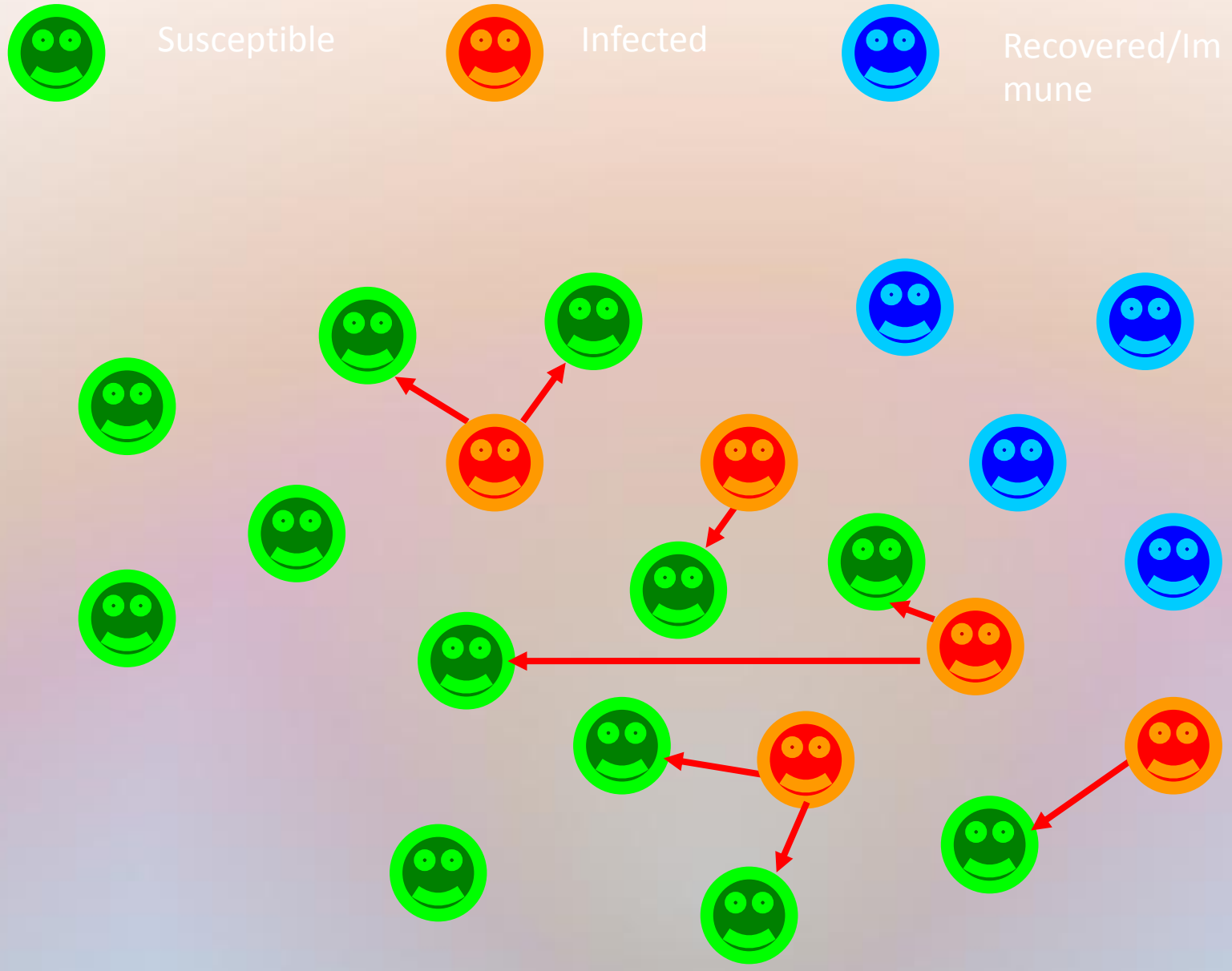
Transmission continues

Transmission process (d)☑



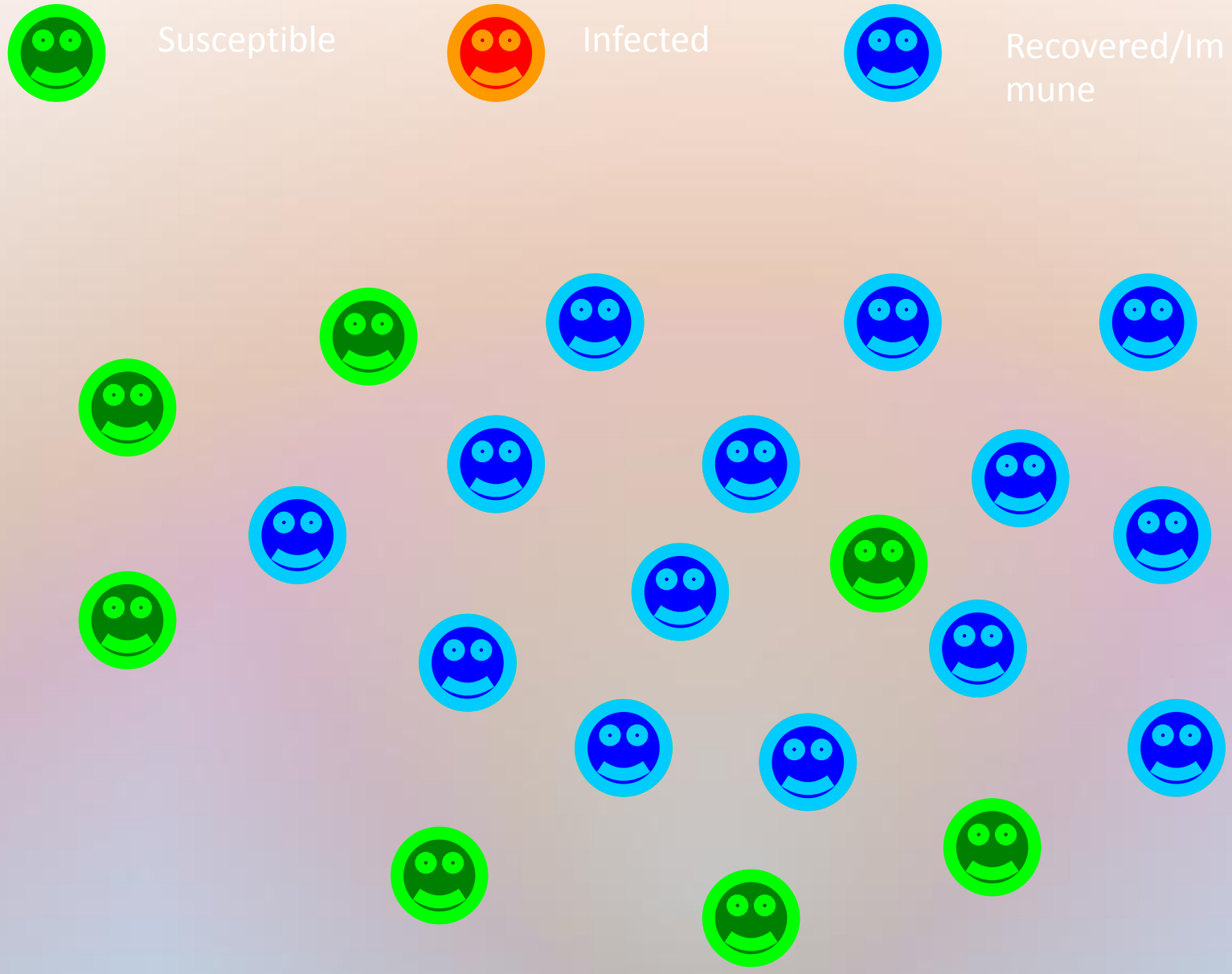
Transmission continues (Index case recovers)☑

Transmission process (e)☑



Transmission continues (more recoveries)☑

Transmission process (f)☑



The disease clears from the community

Basic reproductive number, R_0

Expected number of cases caused by a typical infectious individual in a susceptible population.

$R_0 \leq 1$

disease dies out

$R_0 > 1$

disease can invade

Outbreak dynamics

- probability of fade-out
- epidemic growth rate

Disease control

- threshold targets
- vaccination levels

What does R_0 tell you?

- Epidemic threshold

  NOTE: not every epidemic threshold parameter is R_0!
- Probability of successful invasion
- Initial rate of epidemic growth
- Prevalence at peak of epidemic
- Final size of epidemic (or the proportion of susceptibles remaining after a simple epidemic)
- Mean age of infection for endemic infection
- Critical vaccination threshold for eradication
- Threshold values for other control measures

- Reproduction number helps us to determine the effectiveness of control measures.

THE EFFECTIVE REPRODUCTIVE RATE R_e

- If there are any interventions we have the effective reproduction number R_e
- $R_e = R_0 x^*$
- x^* = the equilibrium fraction of susceptibles in the population
- Pathogen is already there
- Vaccinated/Treated population
- It fluctuates below and above unity in value as the incidence of infection

THE CONTACT NUMBER (σ)

- The average number of adequate contacts of a typical infective during the infectious period (hethcote,1987)
- adequate contact is one that is sufficient for transmission

THE REPLACEMENT NUMBER, R

- The average number of secondary infection produced by a typical infective during the epidemic
- The infections produced by those who were infected by the one individual who was introduced in the wholly susceptible population

THE THREE QUANTITIES: R_0,σ, R

- **All equal at the beginning of the spread of an infectious disease whose entire population is susceptible**
- **R0 is only defined at the time of invasion**
- **σ and R defined all times**
- **σ remains constant as the infection spread :**

σ = R used interchangeable

- **R always less than R0**

- **R always less than σ because after the invasion the susceptible fraction < 1 so that not all adequate contacts results in a new case**
- **$R \leq \sigma \leq R_0$**

HERD IMMUNITY

- A population is said to have herd immunity if a large enough fraction has been immunized to assure that the disease cannot be endemic
- Reduce R_0 to below 1
- Achieved by immunization

"Wellbee" says

BE WELL!

take

ORAL

POLIO

VACCINE

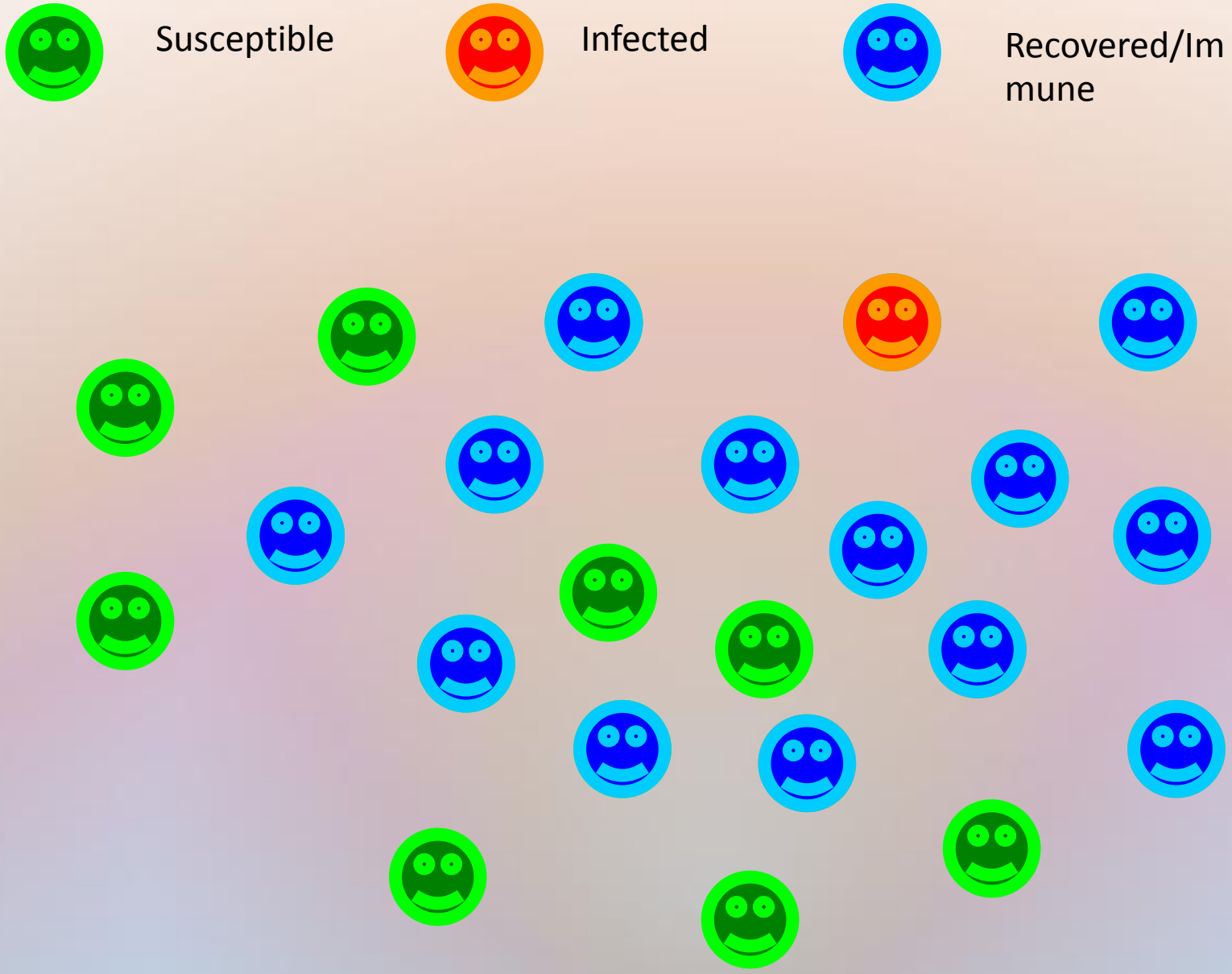
- tastes good
- works fast
- prevents polio

- E.g If a fraction p of say, μ N newborns is successfully immunized N is replaced by N(1-p) and therefore reduce R_0 by $R_0(1-p)$
- $R_0(1-p) < 1$ giving us $p > 1 - 1/R_0$

- **1978) for smallpox with $R_0 = 5$, 80% immunity provide herd immunity**
- **attainable since it requires a lower %age of the population to be immunized**
- **eliminated in 1972**
- **virus only maintained in labs**
- **eradication was possible after an intensive campaign for worldwide vaccination (Hethcote)**

- Last case was in Somalia
- for measles in us R_0 ranges from 5.4 to 6.3
- vaccine not always effective – does not reach everyone
- Therefore herd immunity against measles not been achieved

Transmission process (g)☑



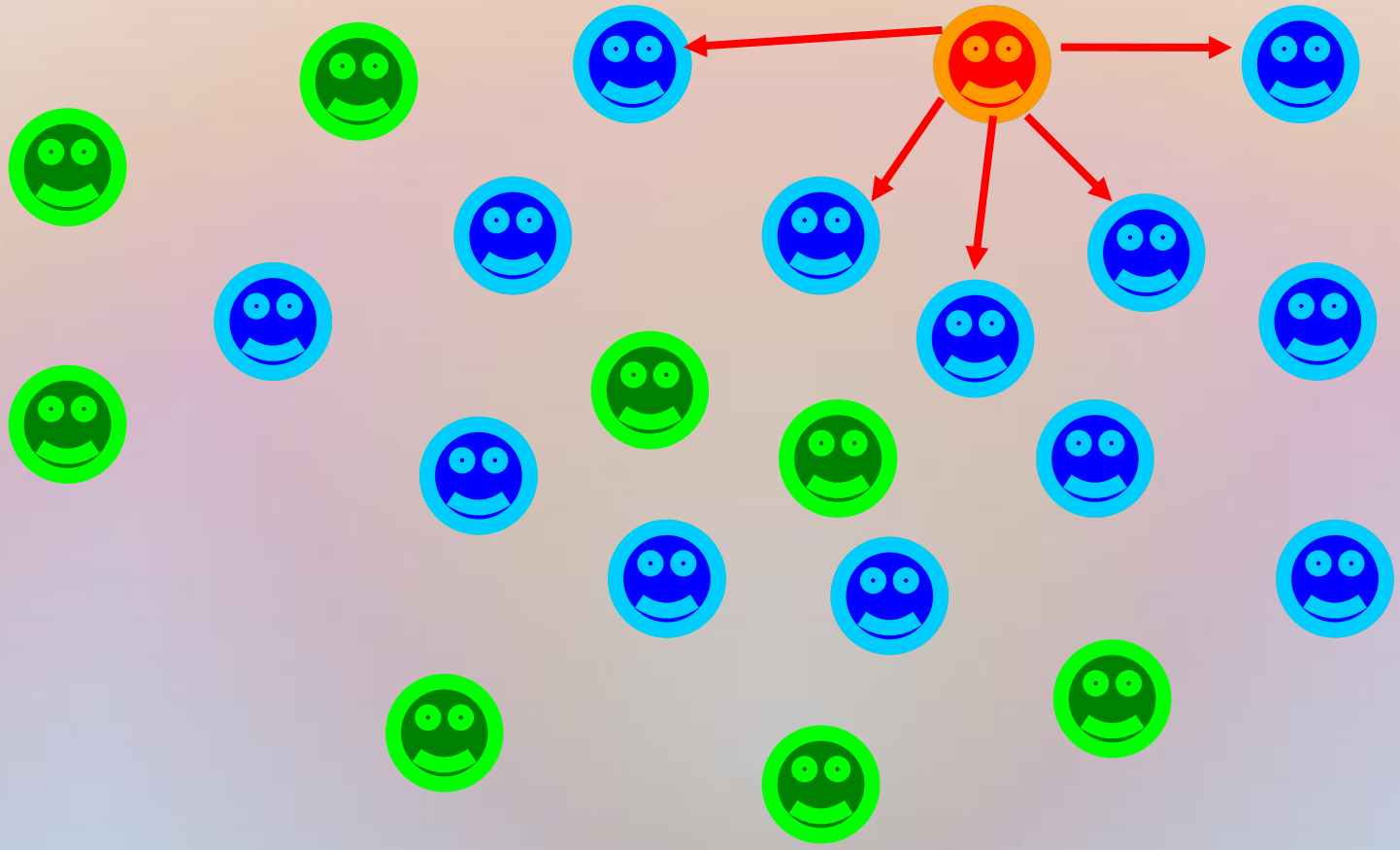
Suppose in infective sets in the community from elsewhere

Transmission process (h)☑

Susceptible

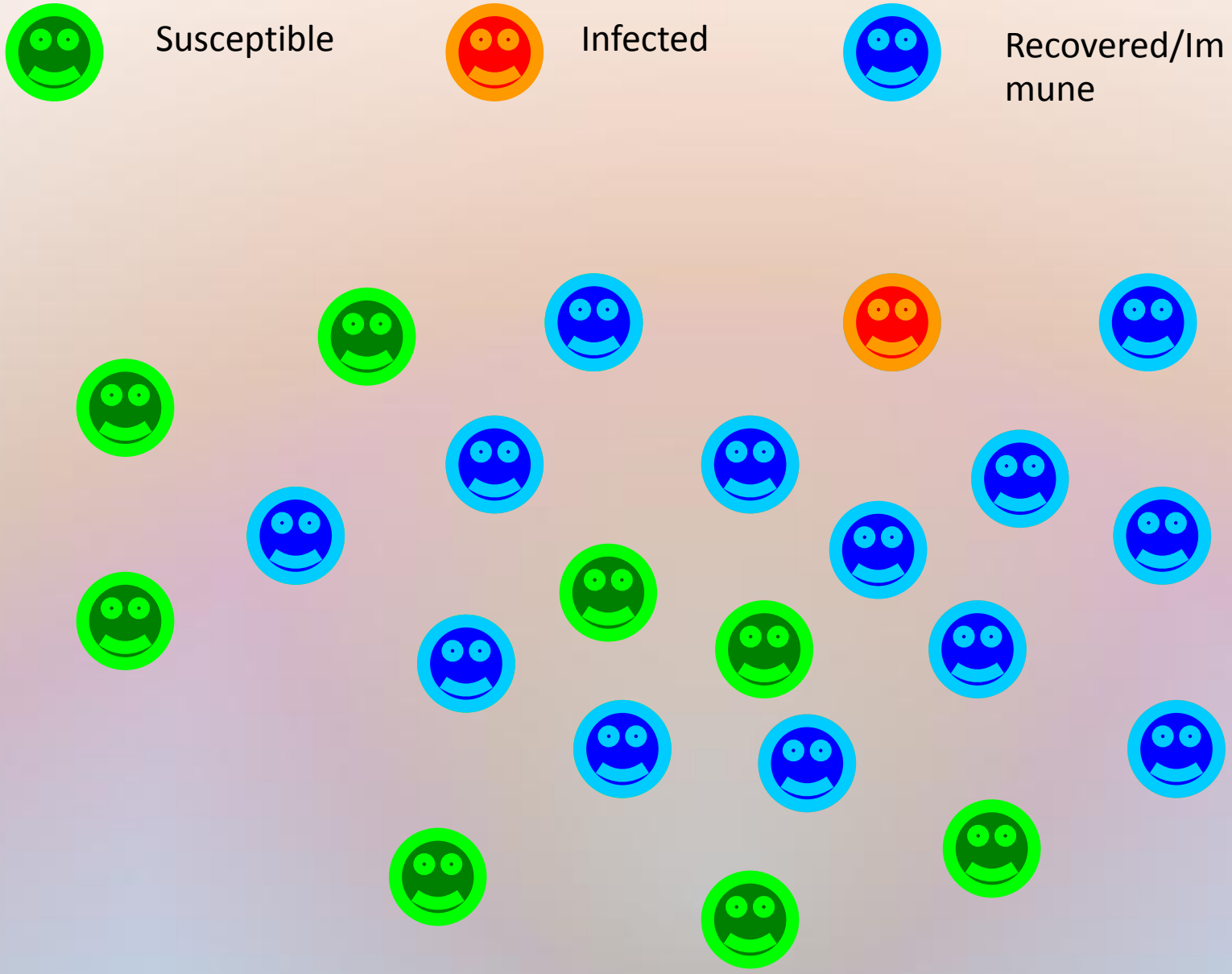
Infected

Recovered/Immune



No infection will occur due to herd immunity

Transmission process (g)☑



Suppose in infective sets in the community from elsewhere

The "gold standard" of R_0

- Takes into account the complete cycle
- Not restricted to ODEs
- But can get cumbersome for multiple states (eg schistosomiasis)

Recent uses: Scrapie in Cyprus (Gravenor *et al.*, 2004), Dengue in Brazil (Luz *et al.*, 2003).

DETERMINATION OF R_0

- Calculation either from **first principles** or **look at eigenvalues at disease free equilibrium**
- **Find the DFE**

- **Recall that the basic reproduction number, denoted by R_0, is the number of secondary infections caused by a single infective introduced into a population made up entirely of susceptible individuals ($S(0) \approx N$) over the course of the infection of this single infective.**

1. FROM FIRST PRINCIPLES - **Intuitive approach**

- R_0 = The transmission rate β x the average infectious period

EXAMPLE 2:

- **CONSIDER THE SYSTEM**
- $dS/dt = \mu N - \beta SI - \mu S$

  $dI/dt = \beta SI - (\gamma+\mu)I$

  $d R/dt = \gamma I - \mu R.$

(a) Find R_0 from the condition of the disease to start

(a)Find the equilibrium points and classify them as DFE and EEP

(b) Find the condition for existence of EEP.

(c) Analyse the stability of the equilibrium points

(d) Write down the biological implication of your analyses

- **DFE = (N,0,0)**
- **EEP = $((\mu+\gamma)/\beta, \mu(\beta N-\gamma-\mu)/\beta(\gamma+\mu), \gamma(\beta N-\gamma-\mu)/\beta))$**
- **$R_0 = \beta N /\gamma+\mu$**
- **Endemic equilibrium point EEP exists when $R_0 > 1$.**
- **DFE is asymptotical stable if**

$R_0 < 1$ and unstable if $R_0 > 1$

EEP is asymptotical stable if $R_0 > 1$

And unstable if $R_0 < 1$

R_0 = Per capita rate of infecting others ◅ Duration of infectiousness

… in a completely susceptible population.

Under frequency-dependent transmission:

Rate of infecting others = $\beta S/N$
= β in wholly susceptible population

Duration of infectiousness = 1/recovery rate
= $1/\gamma$

→ $R_0 = \beta / \gamma$

Example 1: Simple SIR model

Find R_0 for this system.

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Calculation of R_0

Use first principles or definition of R_0.

Question: how many infectives are caused by a single infective introduced into a wholly susceptible population

$dI/dt \approx (\beta N - \gamma)I$ since the total population is wholly susceptible

- **If $(\beta N - \gamma)I > 0$ then I(t) increases**

$\forall \Rightarrow \beta N/\gamma - 1 > 0$

- **This infective individual makes βN contacts per unit time producing new infections with a mean infectious period of 1/γ. Therefore, the basic reproduction number is**

$\forall \therefore R_0 = \beta N/\gamma$

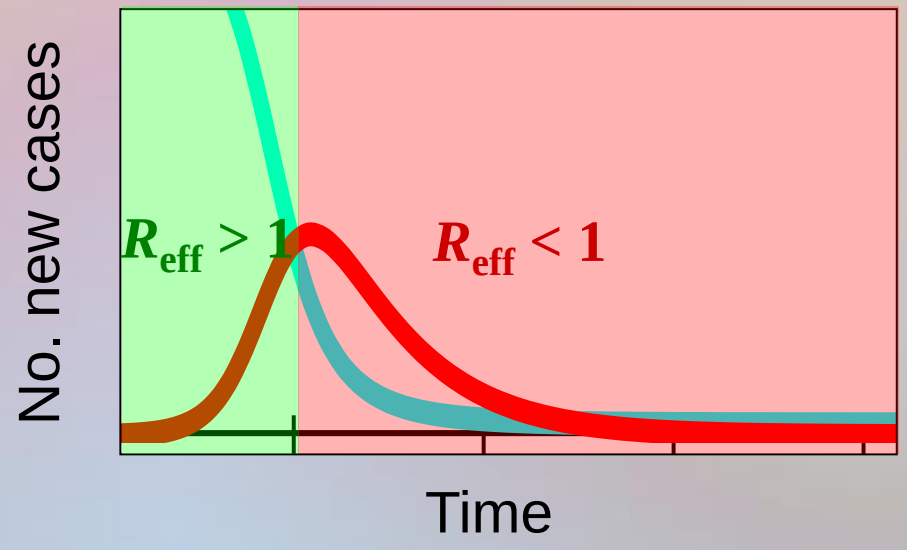
Effective reproductive number

Expected number of cases caused by a typical infectious individual in a population that is not wholly susceptible.

$$R_{effective} = R_0 \times S/N$$

Endemic disease: At equilibrium $R_{eff} = 1$, so that $S^*/N = 1/R_0$

Epidemic disease: R_{eff} changes as epidemic progresses, as susceptible pool is depleted.



Note: Sometimes "effective reproductive number" is used to describe transmission in the presence of disease control measures.

This is also called $R_{control}$.

$R_{effective}$ and herd immunity

$$R_{effective} = R_0 \checkmark S/N$$

If a sufficiently high proportion of the population is immune, then $R_{effective}$ will be below 1 and the disease cannot circulate.

The remaining susceptibles are protected by **herd immunity**.

The critical proportion of the population that needs to be immune is determined by a simple calculation:

- For $R_{eff} < 1$, we need $S/N < 1/R_0$
- Therefore we need a proportion $1-1/R_0$ to be immune.

2. Existence of the infectious individuals for the epidemic to start:

a) This implies that at a lower level dI/dt > 0 with S approx = N

b) The existence of the endemic equilibrium point, ie, I* > 0

Stability analysis of the disease free equilibria

- R_0 = (dominant eigenvalue) x (infectious period) + 1

EXAMPLE 3

- **Consider for the model**
- **$dS/dt = \mu N - \beta SI - \mu S$**

  $dI/dt = \beta SI - (\alpha+\gamma+\mu)I$

  $dR/dt = \gamma I - \mu R$.

(a) Draw a compartmentalised model

(b) Write down the assumptions of the model.

(c) Find the equilibrium points and classify them and write down the condition for their existence

(d) Find R_0 and analyse the steady states

Existence of the endemic equilibrium

- Find the conditions of existence and this gives a threshold value which corresponds to R_0 since the disease persists when $R_0>1$.

EXAMPLE 4

- **Consider for the model**
- **$dS/dt = \mu N - \beta SI - \mu S$**

  $dI/dt = ßSI - (\alpha+\gamma+\mu)I$

  $dR/dt = \gamma I - \mu R$.

(a) Find the equilibrium points and classify them and write down the condition for their existence

(b) Hence find R_0

EXERCISE 5

- **Consider for the model**
- **$dS/dt = \mu N - \beta SI - \mu S$**

  $dE/dt = ßSI - (\varepsilon+\mu)E$

  $dI/dt = \varepsilon E - (\alpha+\gamma+\mu)I$

  $dR/dt = \gamma I - \mu R$.

(a) Draw a compartmentalised model

(b) Write down the assumptions of the model.

(c) Find the equilibrium points and classify them and write down the condition for their existence

(d) Find R_0 and analyse the steady states

- In cases of diseases with varying infectiousness periods, the basic reproduction number can be calculated as the sum of the reproduction number for each transition time into the disease. An example of this is the HIV/AIDS model

- The classes which contribute to transmission are I_1 and I_2 *and therefore* R_0 *will be a sum of* R_0 *from* I_1 *and* R_0 *from* I_2.
- *The calculation becomes complicated and therefore another method has to be used*

EXAMPLE 2:

- **CONSIDER THE SYSTEM**
- **$dS/dt = \mu N - \beta SI - \mu S$**

  $dI/dt = \beta SI - (\gamma+\mu)I$

  $d R/dt = \gamma I - \mu R.$

(a)Find the equilibrium points and classify them as DFE and EEP

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(c) Analyse the stability of the equilibrium points

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- **DFE = (N,0,0)**
- **EEP = $((\mu+\gamma)/\beta, \mu(\beta N-\gamma-\mu)/\beta(\gamma+\mu), \gamma(\beta N-\gamma-\mu)/\beta)$**
- **$R_0 = \beta N /\gamma+\mu$**
- **Endemic equilibrium point EEP exists when $R_0 > 1$.**
- **DFE is asymptotical stable if**

$R_0 < 1$ and unstable if $R_0 > 1$

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And unstable if $R_0 < 1$

Stability analysis of the disease free equilibria

- R_0 = (dominant eigenvalue) x (infectious period) + 1
- Jacobian method

The Jacobian method

- Stability of the disease-free equilibrium is determined by the largest eigenvalue of the Jacobian
- This is a threshold condition that can derive an $R_{0,J}$
- May not produce a biologically meaningful value

Recent use: malaria vaccination in Africa (Smith, 2007).

EXERCISE 9

- **Consider for the model**
- **$dS/dt = \mu N - \beta SI - \mu S$**

  $dE/dt = ßSI - (\varepsilon+\mu)E$

  $dI/dt = \varepsilon E - (\alpha+\gamma+\mu)I$

  $dR/dt = \gamma I - \mu R$.

(a) Draw a compartmentalised model

(b) Write down the assumptions of the model.

(c) Find the equilibrium points and classify them and write down the condition for their existence

(d) Find R_0 and analyse the steady states

Eg: Malaria model

- $dS_h/dt = \Pi_h - \beta_m S_h I_m - \mu S_h + \alpha I_h$
- $dI_h/dt = \beta_m S_h I_m - (\gamma + \alpha + \mu) I_h$

$dR_h/dt = \gamma I_h - \mu R_h$

$dS_m/dt = \Pi_m - \beta_h S I_m - \mu S_m$

$dI_m/dt = \beta_h S_m I_h - \mu I_m$

We next use the next generation method by P. van den Driessche, J. Watmough (2002) to find the reproduction number.

The next-generation method

- A general method for deriving R_0 when the population is divided into discrete, disjoint cases
- Can be used for models with underlying age structure or spatial structure.

Procedure

- For the model:
- Decide which states are infected?
- Which states are uninfected?
- Find disease-free equilibrium point

Calculating the next generation R_0

Let

- F_i be the rate of appearance of new infections in compartment *i*
- V_i be the transfer of individuals out of compartment *i* by all other means
- x_0 be the disease-free equilibrium
- R_0 is the largest eigenvalue of

$$\left[\frac{\partial F_i(x_0)}{\partial x_j}\right] \cdot \left[\frac{\partial V_i(x_0)}{\partial x_j}\right]^{-1}$$

- INTERPRETATION OF FV^{-1}
- - consider the fate of an infected individual introduced into compartment k of a disease free population.
- - the (i,j) entry is the average length of time this individual spends in compartment j during its lifetime if population is near DFE and barring re-infection.

- - the (i,j) entry of F is the rate at which infected individuals in compartment j produce new infections in compartment i

∀ ∴ the (i,j) entry of FV^{-1} is the expected number of new infections into compartment k.

- FV^{-1} is the next generation matrix of the model

- $R_0 = \rho\ FV^{-1}$ where FV^{-1} is the spectral radius of the matrix FV^{-1} .

$\forall \Rightarrow R_0$ = DOMINANT EIGENVALUE OF FV^{-1}

EXERCISE

- **Consider for the model**
- **$dS/dt = \mu N - \beta SI - \mu S$**

  $dE/dt = ßSI - (\varepsilon+\mu)E$

  $dI/dt = \varepsilon E - (\alpha+\gamma+\mu)I$

  $dR/dt = \gamma I - \mu R$.

(a) Draw a compartmentalised model

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R_0 in disease control shocker!

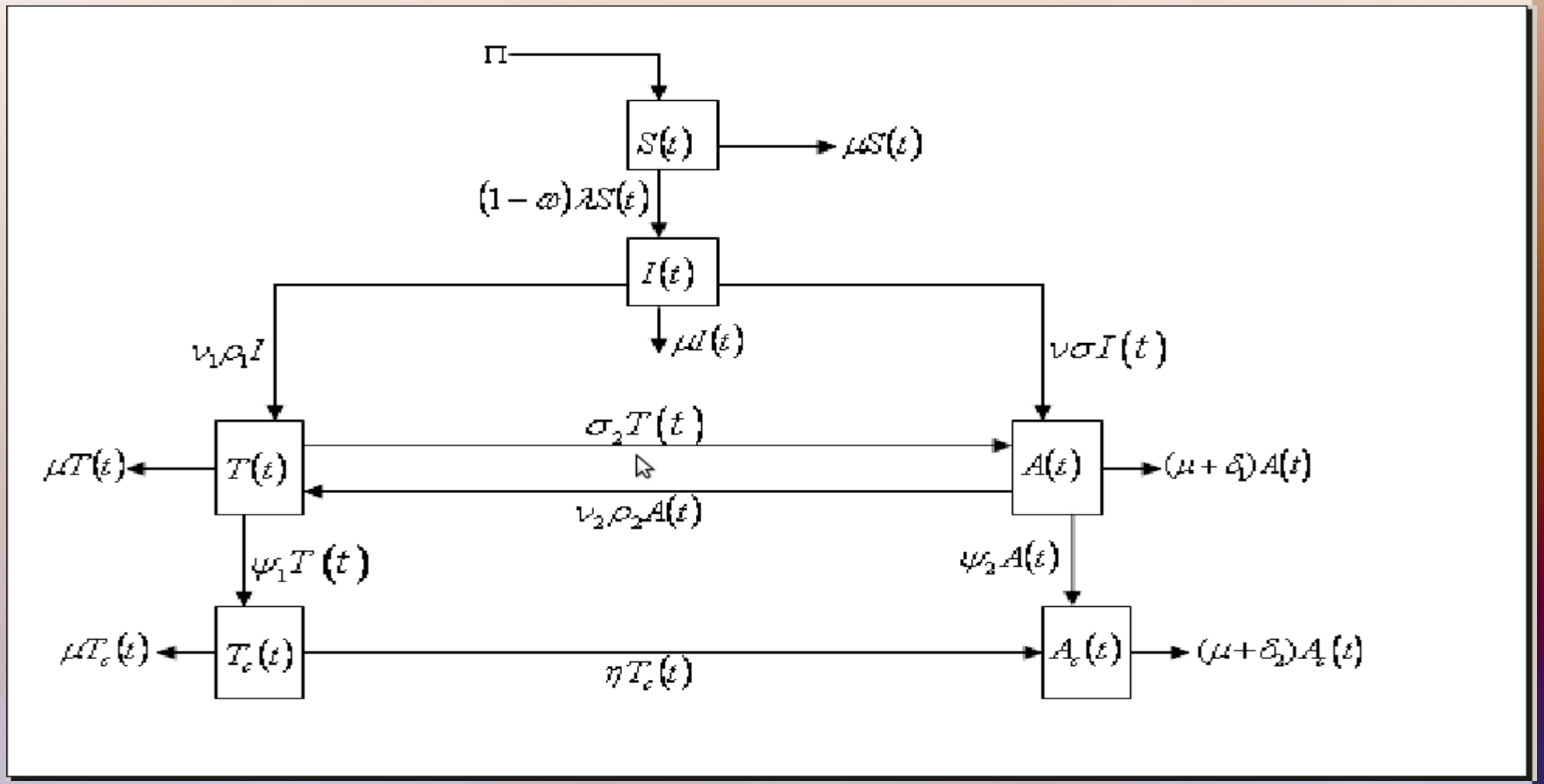
Malaria: Low R_0 used to justify possibility of elimination from an island in the Gulf of Guinea (Hagmann *et al.*, 2003)

West Nile Virus: R_0 used to evaluate public health policies of mosquito vs bird control, concluding that bird control would actually enhance transmission (Wonham *et al.*, 2004).

Limitations

- Models used “in the field” are usually simple, deterministic and non-structured
- R_0 typically quantified after epidemic has run its course
- R_0 values usually used to justify severe or costly control measures, rather than affecting public health measures directly.

Example 1


Π
S(t)
μS(t)
(1 − ω)λS(t)
I(t)
μI(t)
ν1σ1I
νσI(t)
σ2T(t)
μT(t)
T(t)
A(t)
(μ + δ)A(t)
ν2σ2A(t)
ψ1T(t)
ψ2A(t)
μTc(t)
Tc(t)
Ac(t)
(μ+δ2)Ac(t)
ηTc(t)

Derivation of Ro from the model flow chart

- **Reproduction number is a sum of the contributions of the sexual active population at the disease-free equilibrium**
- **The class can have one path or many paths indicating how infections are introduced in the class.**

- **The contribution to the class will be a product of the contribution into each class leading to the required infectious class or sum of the different paths bringing the infection into the infectious class.**
- **All the paths are connected to the susceptible and the primary infective class**
- **Eg SIR model and SEIR model**

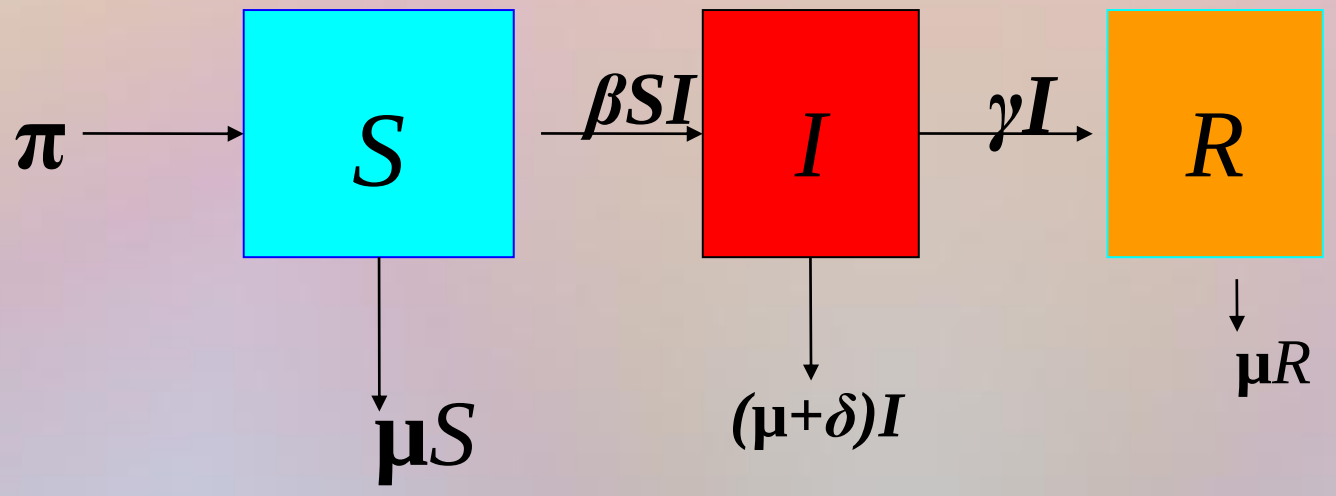
- **SIR Model equations are**

$$\frac{dS}{dt} = \pi - \beta SI - \mu S$$

$$\frac{dI}{dt} = \beta SI - (\mu + \delta) I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

The flow chart of SIR model



- Entry into I is by $\boldsymbol{\beta S}$
- $S^* = \boldsymbol{\pi}/\mu$
- Proportion out of I $= \boldsymbol{(\mu+\delta+\gamma)}$

- $\boldsymbol{R_0} = \underline{\boldsymbol{\beta S^*}}$
- $\boldsymbol{(\mu+\delta+\gamma)}$
- $= \underline{\boldsymbol{\beta\pi/\mu}}$
- $\boldsymbol{(\mu+\delta+\gamma)}$
- $= \underline{\boldsymbol{\beta\pi}}$
- $\mu\,(\mu+\boldsymbol{\delta+\gamma})$

SEIR Model

- **Modification of the assumptions can now lead to a model where infected individuals recover with temporary immunity**
- **The immunity wanes and the individual becomes susceptible again and therefore go back to the susceptibles class S**

EXERCISE

- **Consider for the model**
- **$dS/dt = \mu N - \beta SI - \mu S$**

  $dE/dt = ßSI - (\varepsilon+\mu)E$

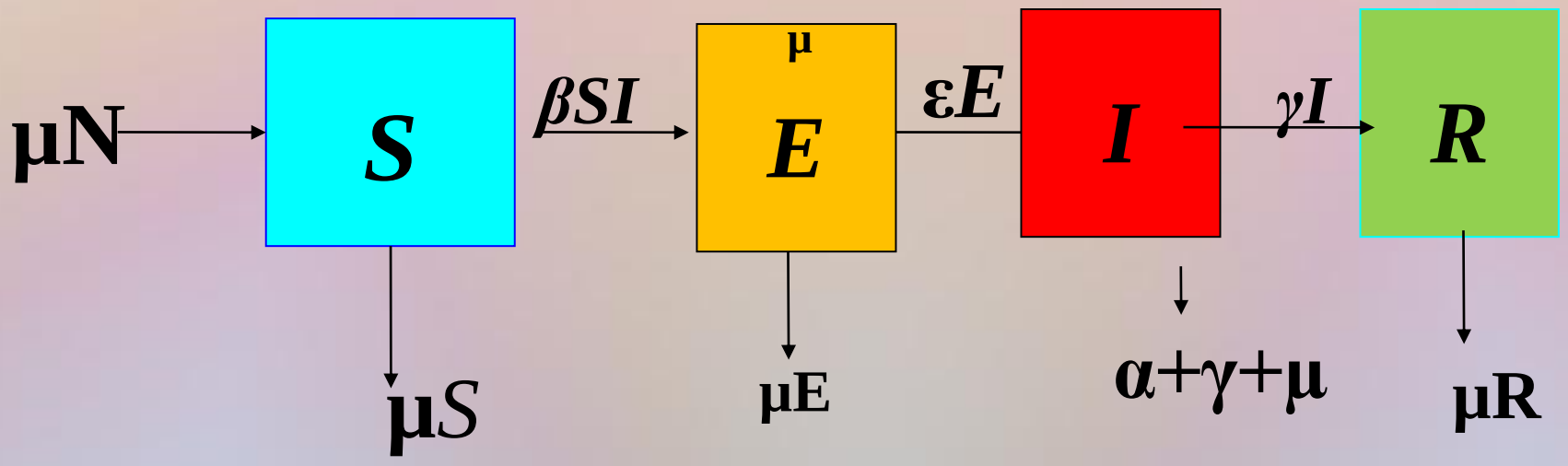
  $dI/dt = \varepsilon E - (\alpha+\gamma+\mu)I$

  $dR/dt = \gamma I - \mu R$.

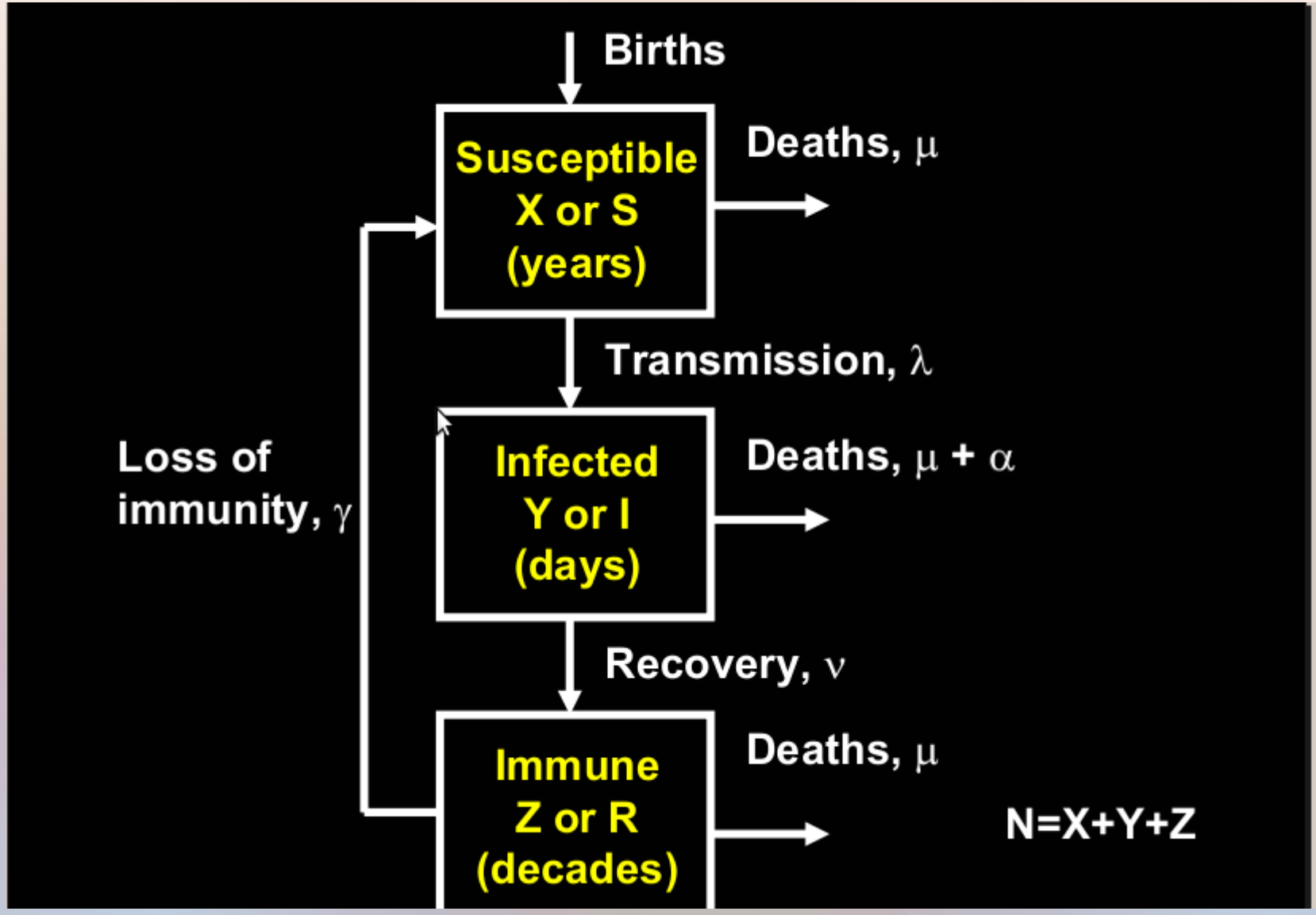
(a) Draw a flowchart

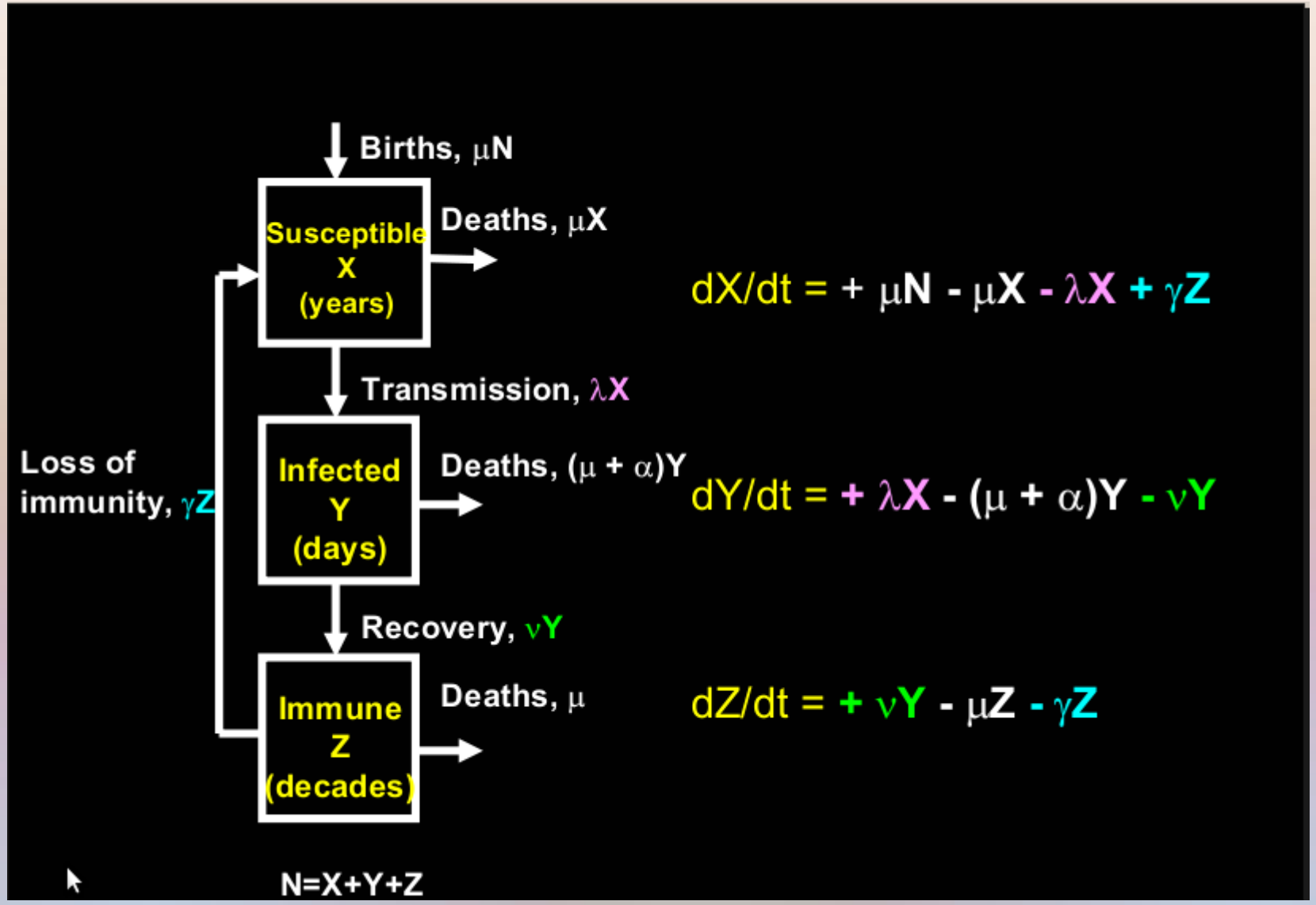
(b) Find R_0 from the flowchart

The flow chart of SEIR model



- Entry into I is by βS
- $S^* = N$
- ***Proportion out of S into E*** $= \beta S$
- Proportion out of E $= \varepsilon + \mu$
- ***Proportion out of E into I*** $= \varepsilon$
- Proportion out of I $= \varepsilon + \gamma + \mu$
- $R_0 = \dfrac{\beta S^*}{\varepsilon + \mu} \; X \; \dfrac{\varepsilon}{\alpha + \gamma + \mu}$
-
- $= \beta \varepsilon N$


Births
Susceptible
X or S
(years)
Deaths, μ
Transmission, λ
Infected
Y or I
(days)
Deaths, μ + α
Loss of
immunity, γ
Recovery, ν
Immune
Z or R
(decades)
Deaths, μ
N=X+Y+Z


Births, μN
Susceptible
X
(years)
Deaths, μX
dX/dt = + μN - μX - λX + γZ
Transmission, λX
Loss of
immunity, γZ
Infected
Y
(days)
Deaths, (μ + α)Y
dY/dt = + λX - (μ + α)Y - νY
Recovery, νY
Immune
Z
(decades)
Deaths, μ
dZ/dt = + νY - μZ - γZ
N=X+Y+Z

EXERCISE 5

- **Consider for the following HIV/AIDS model**
- **$dS/dt = \mu N - \beta SI1 - \mu S$**

  $dI1/dt = ßSI1 - (\varepsilon+\mu)I1$

  $dI2/dt = \varepsilon I1 - (\alpha+\gamma+\mu)I2$

  $dA/dt = \gamma I2 - \mu A.$

(a) Draw a compartmentalised model

(b) Write down the assumptions of the model.

(c) Find the equilibrium points and classify them and write down the condition for their existence

(d) Find R_0 and analyse the steady states

- In cases of diseases with varying infectiousness periods, the basic reproduction number can be calculated as the sum of the reproduction number for each transition time into the disease. An example of this is the HIV/AIDS model

- The classes which contribute to transmission are I_1 and I_2 *and therefore* R_0 *will be a sum of* R_0 *from* I_1 *and* R_0 *from* I_2.
- *The calculation becomes complicated and therefore another method has to be used*

EXAMPLE 1:

- **CONSIDER THE SYSTEM**
- **$dS/dt = \mu N - \beta SI - \mu S$**

$dI/dt = \beta SI - (\gamma+\mu)I$

$d R/dt = \gamma I - \mu R.$

(a)Find the equilibrium points and classify them as DFE and EEP

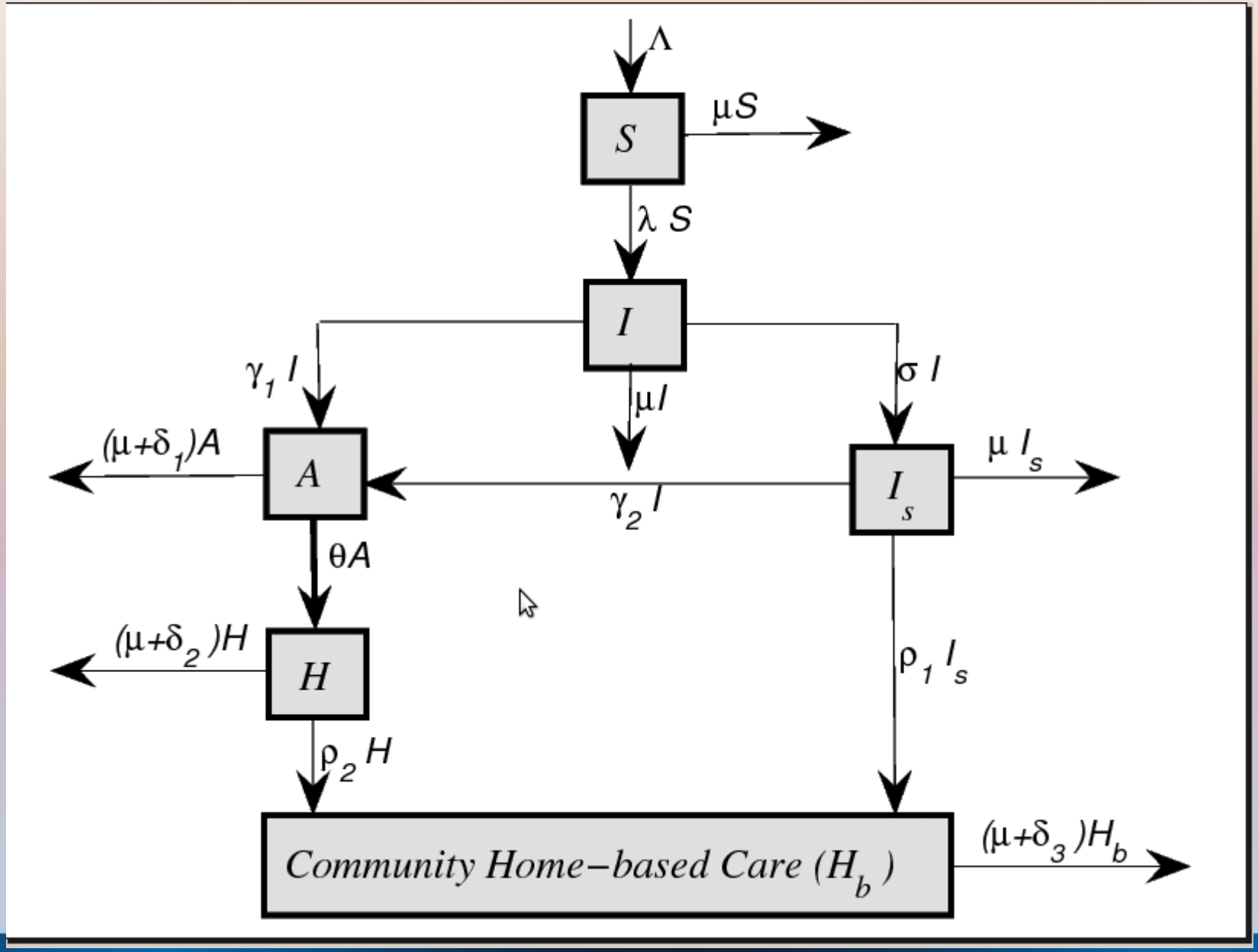
(b) Find the R_0 and find the condition for existence of EEP.

(c) Analyse the stability of the equilibrium points

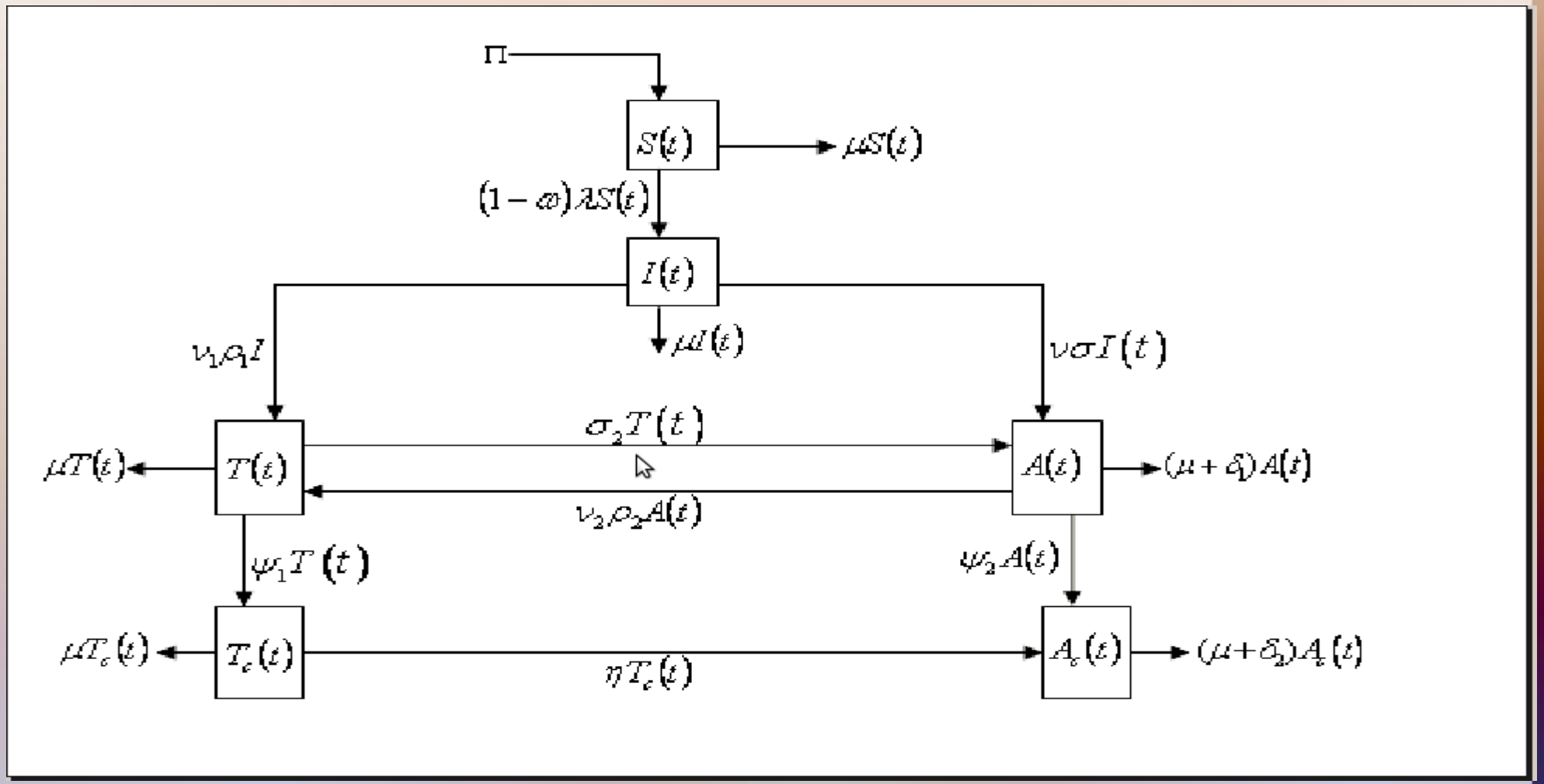
(d) Write down the biological implication of your analyses

2. Find Reproduction numbers for the following models from the flow charts, eg 2 and 3

Example 2


Λ
S
μS
λ S
I
γ1 I
μI
σ I
(μ+δ1)A
A
μ Is
Is
γ2 I
θA
(μ+δ2)H
H
ρ1 Is
ρ2 H
Community Home–based Care (Hb)
(μ+δ3)Hb

Example 3


Π
S(t)
μS(t)
(1 − ω)λS(t)
I(t)
μI(t)
ν1σ1I
νσI(t)
σ2T(t)
μT(t)
T(t)
A(t)
(μ + δ)A(t)
ν2σ2A(t)
ψ1T(t)
ψ2A(t)
μTc(t)
Tc(t)
Ac(t)
(μ+δ2)Ac(t)
ηTc(t)

Modelling an epidemic with intervention

- Interventions to at least reduce the disease if possible eradicate it
- Effective reproduction number R_e
- Help to reduce R_e to below 1.
- Types of intervention:
- - Vaccination –for immunity and treatment
- -Education on behavioural change
- - Treatment with Drugs
- - Condoms

- -rate of acquisition of new sexual partners for STIs
- Concepts to consider:
- - drug efficacy
- - drug compliance
- - vaccines

Example1

- $dS/dt = \mu N - \beta SI - \mu S$

  $dE/dt = ßSI - (\varepsilon+\mu)E$

  $dI/dt = \varepsilon E - (\alpha+\gamma+\mu)I$

  $dR/dt = \gamma I - \mu R.$

- **For the model treating the exposed and the infectious leads to a class of Treated individuals (T)**

- $dS/dt = \mu N - \beta SI - \mu S$

$dE/dt = ßSI - (\varepsilon+\mu+\lambda)E$

$dI/dt = \varepsilon E - (\alpha+\gamma+\mu+\tau)I$

$dT/dt = \lambda E + \tau I - (\eta+\mu)T$

$dR/dt = \gamma I + \eta T - \mu R.$

Where λ,τ and η are rates of treatments

Proportions treated

- $dS/dt = \mu N - \beta SI - \mu S$

$dE/dt = ßSI + \sigma pT - (\varepsilon+\mu+\lambda)E$

$dI/dt = \varepsilon E - (\alpha+\gamma+\mu+\tau)I$

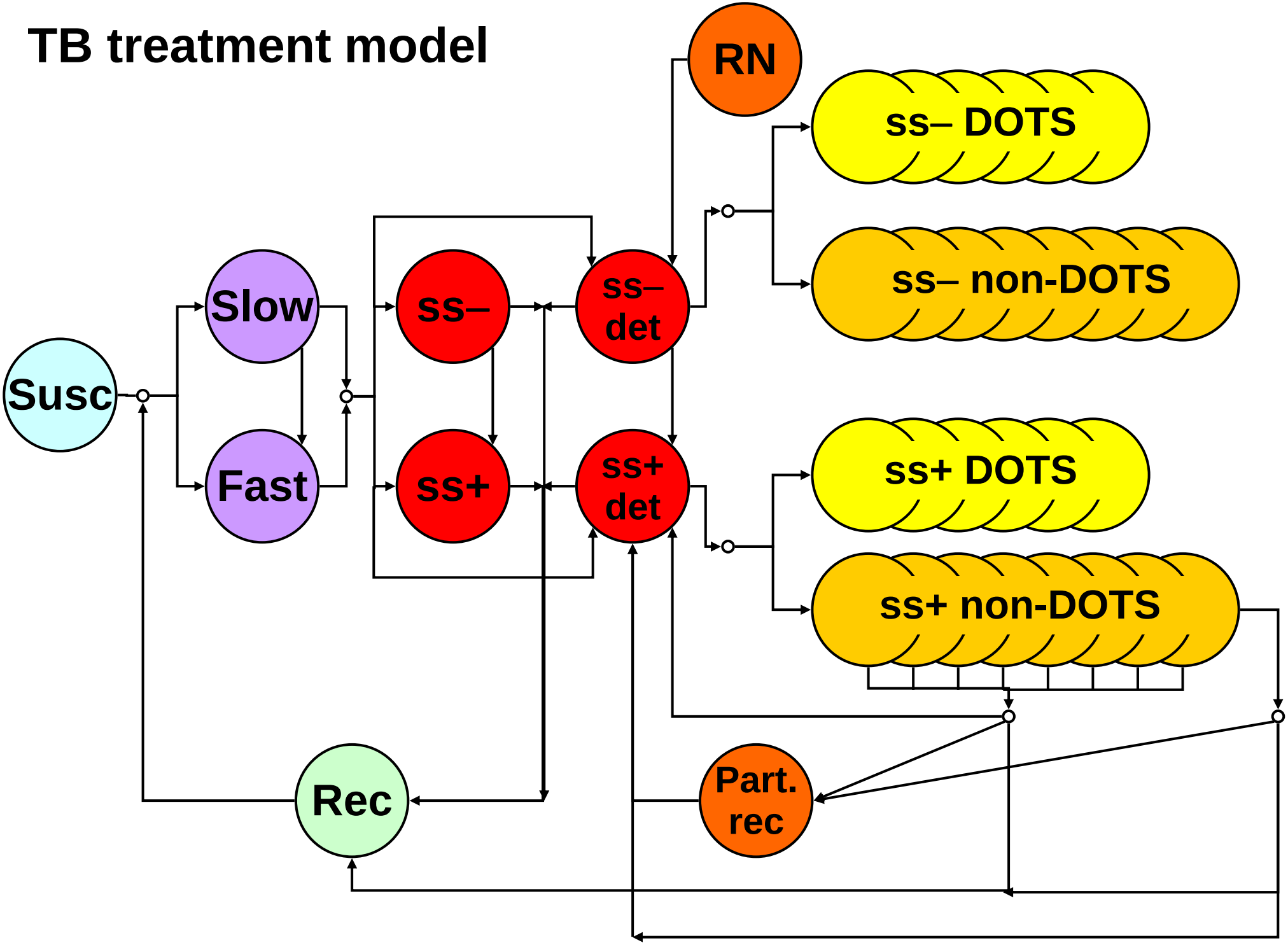
$dT/dt = \lambda E + \tau I - (\sigma(1-p)+\mu)T$

$dR/dt = \gamma I + \sigma T - \mu R.$

σ is the rate of recovery due to treatment.

λ is the rate of treatment for the exposed and the infectious individuals

TB treatment model



TB model with treatment

- $dS/dt = \pi - \lambda_T S - \mu S$

$dL/dt = p\lambda_T S + \rho T_t - (\lambda_R + \mu + \alpha)L$

$dT/dt = (1-p)\lambda_T S + \lambda_R L + \alpha L - (\delta_T + \mu + \tau)T$

$dT_t/dt = \tau T - (\rho + \mu)T_t$

$\lambda_T = \beta_T(T + \eta_T T_t)/N, \quad \lambda_T = \beta\eta_R T/N,$

$N = S + L + T + T_t.$

- Find the equilibrium points
- Find the reproduction number, R_0 of the basic model without treatment
- Find the reproduction number, R_e, of the basic model with treatment
- Compare the reproduction numbers by analysing them.
- Find the endemic equilibrium points for both models in terms of the reproduction numbers.
- **Write the EEP in terms of the Reproduction number where possible**