Multilevel Models with Varying Intercepts and Slopes

Nambari Short Course

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Overview

- Continue with radon data
  - Elaborate model to allow varying slopes
  - Interpretation of variance components
  - Inclusion and interpretation of individual- and group-level predictors
- Different representations of the multilevel model

Varying slopes in a multilevel model

Radon data

First recall the multilevel model with varying intercepts

Level 1: Distribution of household radon level within county j, regressing on where measurement was taken (X_{ij})

$$Y_{ij} \sim \mathcal{N}(\alpha_j + X_{ij}\beta, \sigma^2)$$

Level 2: Distribution of county-specific mean radon levels

$$\alpha_j \sim \mathcal{N}(\mu, \tau^2)$$

In this model, the slope on X_{ij} is considered constant across counties

Varying slopes in a multilevel model

Radon data

It is possible to allow both the intercept and slope to vary across counties

Level 1: Distribution of household radon level within county j, regressing on where measurement was taken (X_{ij})

$$Y_{ij} \sim \mathcal{N}(\alpha_j + X_{ij}\beta_j, \sigma^2)$$

Level 2: Distribution of county-specific mean radon levels

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \mu_\alpha \\ \mu_\beta \end{pmatrix}, \begin{pmatrix} \tau_\alpha^2 & \\ \rho\tau_\alpha\tau_\beta & \tau_\beta^2 \end{pmatrix}\right]$$

In this model, the slope on X_{ij} varies across counties

Varying slopes in a multilevel model

Radon data

A closer look at level 2:

Level 2: Distribution of county-specific mean radon levels

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \mu_\alpha \\ \mu_\beta \end{pmatrix}, \begin{pmatrix} \tau_\alpha^2 & \\ \rho\tau_\alpha\tau_\beta & \tau_\beta^2 \end{pmatrix}\right]$$

In this set up,

$$\begin{aligned} \tau_\alpha^2 &= \text{variance of intercepts} \\ \tau_\beta^2 &= \text{variance of slopes} \\ \rho &= \text{corr}(\alpha_j, \beta_j) \end{aligned}$$

Fitting the model using lmer

```
> ## Varying intercept & slopes w/ no group level predictors
> M3 <- lmer (y ~ x + (1 + x | county))

> display (M3)
lmer(formula = y ~ x + (1 + x | county))
            coef.est coef.se
(Intercept)  1.46     0.05
x           -0.68     0.09

Error terms:
 Groups   Name        Std.Dev. Corr
 county   (Intercept) 0.35
          x           0.34     -0.34
 Residual             0.75
---
number of obs: 919, groups: county, 85
AIC = 2180.3, DIC = 2153.9
deviance = 2161.1
```

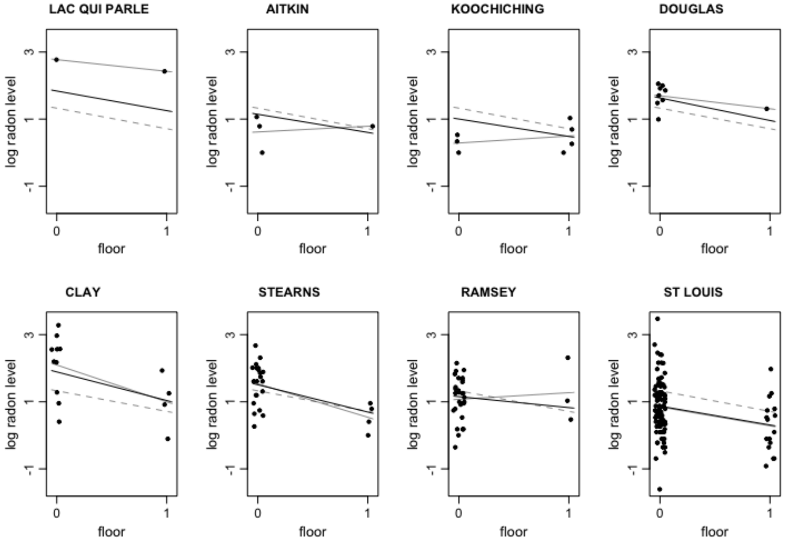
Fitting the model using lmer

```
> coef (M3)
$county
   (Intercept)          x
1    1.1445320 -0.5406088
2    0.9333792 -0.7708179
3    1.4716908 -0.6688827
4    1.5354379 -0.7525587
5    1.4270363 -0.6206673
6    1.4826565 -0.6877101
7    1.8193559 -0.4747484
[...]
81   1.8441153 -0.5360210
82   1.6002912 -0.7268221
83   1.6942411 -1.1511024
84   1.5991199 -0.7327255
85   1.3787929 -0.6531768
```

Fitting the model using lmer

```
> fixef (M3)
(Intercept)           x
  1.4627703  -0.6810982

> ranef (M3)
$county
    (Intercept)            x
1  -0.318238351  0.140489364
2  -0.529391140 -0.089719737
3   0.008920474  0.012215455
4   0.072667555 -0.071460543
5  -0.035734045  0.060430874
6   0.019886162 -0.006611892
7   0.356585564  0.206349823
[...]
81  0.381344973  0.145077244
82  0.137520843 -0.045723902
83  0.231470744 -0.470004198
84  0.136349565 -0.051627299
85 -0.083977464  0.027921421
```


LAC QUI PARLE
AITKIN
KOOCHICHING
DOUGLAS
CLAY
STEARNS
RAMSEY
ST LOUIS
log radon level
floor
-1
1
3
0
1

Including group-level predictor

Recall that U_j is uranium level in county j. Including group-level predictors entails expanding the level 2 model as follows:

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \gamma_0^\alpha + \gamma_1^\alpha U_j \\ \gamma_0^\beta + \gamma_1^\beta U_j \end{pmatrix}, \begin{pmatrix} \tau_\alpha^2 & \\ \rho\tau_\alpha\tau_\beta & \tau_\beta^2 \end{pmatrix}\right]$$

$$\begin{aligned} \gamma_1^\alpha &= \text{uranium effect on intercept} \\ \gamma_1^\beta &= \text{uranium effect on slope} \end{aligned}$$

Fitting the model in lmer

```
> ## Including group level predictors
> M4 <- lmer (y ~ x + u.full + x:u.full + (1 + x | county))

> display (M4)
lmer(formula = y ~ x + u.full + x:u.full + (1 + x | county))
            coef.est coef.se
(Intercept)  1.47     0.04
x           -0.67     0.08
u.full       0.81     0.09
x:u.full    -0.42     0.23

Error terms:
 Groups   Name        Std.Dev. Corr
 county   (Intercept) 0.12
          x           0.31     0.41
 Residual             0.75
---
number of obs: 919, groups: county, 85
AIC = 2142.6, DIC = 2101.9
deviance = 2114.2
```

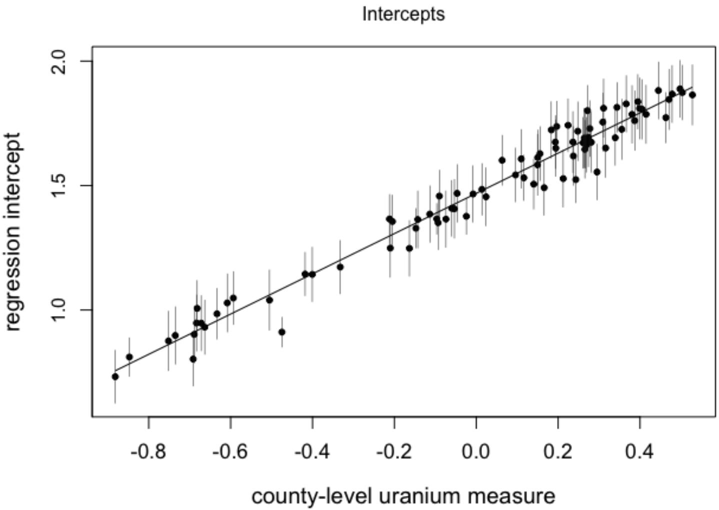
Fitting the model in lmer

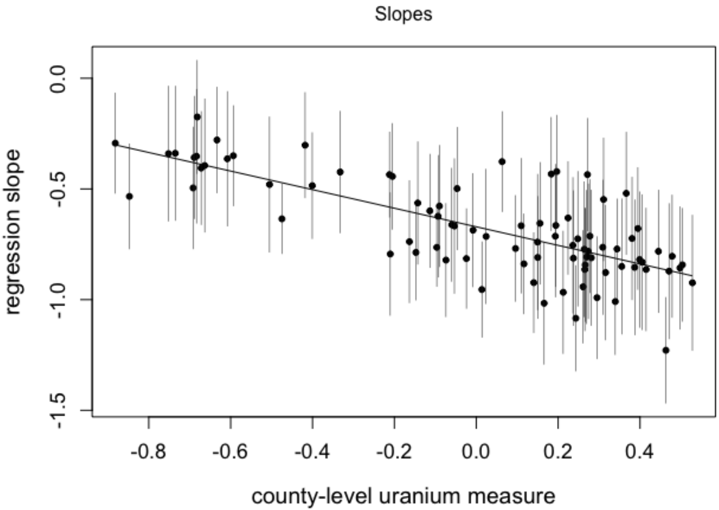
```
> coef (M4)
$county
   (Intercept)          x    u.full   x:u.full
1     1.458668 -0.6468782 0.8081345 -0.4195149
2     1.495642 -0.8889935 0.8081345 -0.4195149
3     1.476934 -0.6465966 0.8081345 -0.4195149
4     1.527552 -0.5989640 0.8081345 -0.4195149
5     1.479225 -0.6232842 0.8081345 -0.4195149
6     1.448104 -0.6916993 0.8081345 -0.4195149
7     1.581426 -0.3212303 0.8081345 -0.4195149
8     1.504498 -0.5967115 0.8081345 -0.4195149
```

Fitting the model in lmer

```
> fixef (M4)
(Intercept)           x      u.full    x:u.full
  1.4685942  -0.6709437   0.8081345  -0.4195149

> ranef (M4)
$county
    (Intercept)            x
1  -0.009926311  0.024065524
2   0.027047629 -0.218049771
3   0.008339473  0.024347096
4   0.058957543  0.071979706
5   0.010630259  0.047659481
6  -0.020490584 -0.020755637
7   0.112831703  0.349713432
8   0.035903697  0.074232254
```


Intercepts
regression intercept
2.0
1.5
1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
county-level uranium measure


Slopes
regression slope
0.0
-0.5
-1.0
-1.5
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
county-level uranium measure