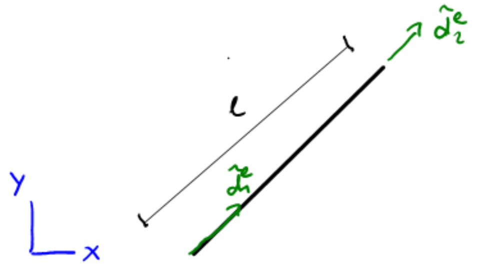
Übung 5: Stabelemente

5.1 a) Elementsteifigkeitsmatrix aufstellen



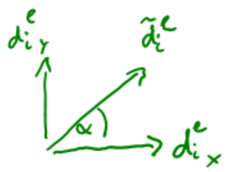
$$\delta\Pi_e^{int,h} := \int_0^l \delta\tilde{\varepsilon}^h EA \tilde{\varepsilon}^h \, d\tilde{x}$$

$$\tilde{\varepsilon}^h = \underline{\underline{B}}^e \underline{\tilde{d}}^e \qquad \underline{\tilde{d}}^e = \begin{bmatrix} \tilde{d}_1^e \\ \tilde{d}_2^e \end{bmatrix}$$

$$\delta\tilde{\varepsilon}^h = \underline{\underline{B}}^e \delta\underline{\tilde{d}}^e$$

$$\Rightarrow \quad \delta\Pi_e^{int,h} = (\delta\underline{\tilde{d}}^e)^T \underbrace{\int_0^l (\underline{\underline{B}}^e)^T EA \, \underline{\underline{B}}^e \, d\tilde{x}}_{:= \underline{\underline{\tilde{k}}}^e} \; \underline{\tilde{d}}^e$$

Knotenverschiebung $\tilde{d}_i^e$ in x- und y-Richtung



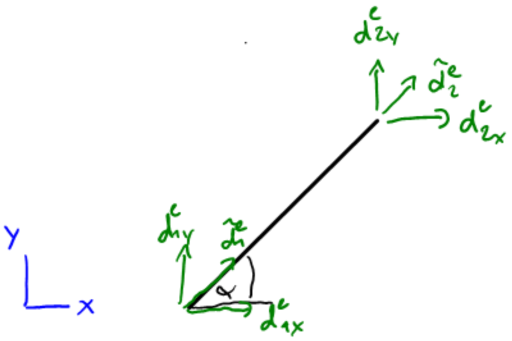
$$d_{ix}^e = \tilde{d}_i^e \cos\alpha \qquad \underline{d}_i^e := \begin{bmatrix} d_{ix}^e \\ d_{iy}^e \end{bmatrix} \quad \underline{n} := \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$d_{iy}^e = \tilde{d}_i^e \sin\alpha$$

$$\underline{n}^T \,|\, \underline{d}_i^e = \tilde{d}_i^e \, \underline{n} \qquad (\underline{n}^T \underline{n} = 1)$$

$$\underline{n}^T \underline{d}_i^e = \tilde{d}_i^e$$

gesamtes Element:



$$\begin{bmatrix} \tilde{d}_1^e \\ \tilde{d}_2^e \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{n}^T & \underline{0} \\ \underline{0} & \underline{n}^T \end{bmatrix}}_{\underline{\underline{T}}^e\text{: Transformationsmatrix}} \begin{bmatrix} \underline{d}_1^e \\ \underline{d}_2^e \end{bmatrix}$$

$$\underline{\tilde{d}}^e = \underline{\underline{T}}^e \underline{d}^e$$

$$(\delta\underline{\tilde{d}}^e)^T = (\delta\underline{d}^e)^T (\underline{\underline{T}}^e)^T$$

$$\Rightarrow \quad \delta\Pi_e^{int,h} = (\delta\underline{d}^e)^T \underbrace{(\underline{\underline{T}}^e)^T \, \underline{\underline{\tilde{k}}}^e \, \underline{\underline{T}}^e}_{\underline{\underline{k}}^e} \, \underline{d}^e$$

Matrix $\underline{\underline{\tilde{k}}}^e$

$$\left.\begin{aligned} \underline{\underline{\tilde{k}}}^e &= \int_0^l \underline{\underline{B}}^{eT} EA \, \underline{\underline{B}}^e \, d\tilde{x} \\ \underline{\underline{B}}^e &= \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \quad (\text{const in } \tilde{x}) \end{aligned}\right\} \quad \underline{\underline{\tilde{k}}}^e = EA \, \underline{\underline{B}}^{eT} \underline{\underline{B}}^e \underbrace{\int_0^l d\tilde{x}}_{l}$$

$$\underline{\underline{B}}^{eT}\underline{\underline{B}}^{e} = \frac{1}{\ell^2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \underline{\underline{\tilde{k}}}^{e} = \frac{EA}{\ell^{\cancel{2}}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cancel{\ell}$$

Transformationsmatrix $\underline{\underline{T}}^e$

$c := \cos\alpha$

$s := \sin\alpha$

$$\underline{n} = \begin{bmatrix} c \\ s \end{bmatrix} \quad \Rightarrow \quad \underline{\underline{T}}^e = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

Elementsteifigkeitsmatrix $\underline{\underline{k}}^e = \underline{\underline{T}}^{eT}\underline{\underline{\tilde{k}}}^e\underline{\underline{T}}^e$

$$\underline{\underline{\tilde{k}}}^e\underline{\underline{T}}^e = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} = \frac{EA}{\ell}\begin{bmatrix} c & s & -c & s \\ -c & -s & c & s \end{bmatrix}$$

$$\underline{\underline{T}}^{eT}\underline{\underline{\tilde{k}}}^e\underline{\underline{T}}^e = \frac{EA}{\ell}\begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}\begin{bmatrix} c & s & -c & s \\ -c & -s & c & s \end{bmatrix} = \frac{EA}{\ell}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Spezialfälle auswerten

$$\alpha \in \{90°, 270°\}: \quad c = 0, \; s = 1 \quad \Rightarrow \underline{\underline{k}}^e = \frac{EA}{\ell}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$\alpha \in \{0°, 180°\}: \quad c = 1, \; s = 0 \quad \Rightarrow \underline{\underline{k}}^e = \frac{EA}{\ell}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

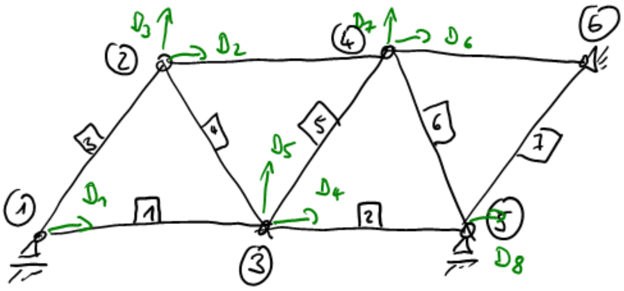
c) Lösungsvektor $\underline{\underline{D}}$

$$\begin{bmatrix} k_{33}^{1} + k_{33}^{3} & k_{34}^{1} \\ k_{43}^{1} & k_{44}^{1} + k_{44}^{2} \end{bmatrix} \begin{bmatrix} D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\underline{\underline{D}} = \underline{\underline{K}}^{-1} \underline{\underline{F}}$$

$$\underline{\underline{D}} = \begin{bmatrix} 0{,}185 \\ -0{,}330 \end{bmatrix} \cdot 10^{-3} \text{ m}$$

zu 5.26) Assemblierung (Matlab)



connectivity:

element / elementnode	1	2	3	[...]
1	1	3	1	[...]
2	3	5	2	

assembleid

node / nodal dof	1	2	3	[...]	5
$d_x^{(i)}$	D_1	D_2	D_4	[...]	D_8
$d_y^{(i)}$	0	D_3	D_5		0
	I		3 I		3

Update der globalen Matrix

1

$$\begin{array}{cc|l} & \overbrace{1 \quad\quad 0}^{I} \quad\quad \overbrace{4 \quad\quad 5}^{3} & \\ \hline \underline{\underline{k}}^{[1]} = & \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} & \begin{matrix} \left.\begin{matrix} 1 \\ 0 \end{matrix}\right\} I \\ \left.\begin{matrix} 4 \\ 5 \end{matrix}\right\} 3 \end{matrix} \end{array}$$

I3 = [1 3 4]

↳ update mit Assemble3.m

— $\underline{\underline{K}}(1,1) = \underline{\underline{K}}(1,1) + \underline{\underline{k}}^{[1]}(1,1)$

~ $\underline{\underline{K}}(5,4) = \underline{\underline{K}}(5,4) + \underline{\underline{k}}^{[1]}(3,4)$

[...]

2

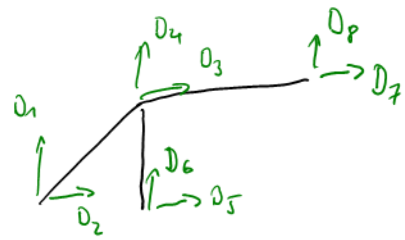
$$\begin{array}{cc|l} & 4 \quad\quad 5 \quad\quad 8 \quad\quad 0 & \\ \hline \underline{\underline{k}}^{[2]} = & \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} & \begin{matrix} 4 \\ 5 \\ 8 \\ 0 \end{matrix} \end{array}$$

I3 = [1 2 3] => Assemble3.m

— $\underline{\underline{K}}(4,8) = \underline{\underline{K}}(4,8) + \underline{\underline{k}}^{[2]}(3,1)$

[...]

5.2f) Weglassen der Dirichlet-Randbedingungen



- → Lineares Gleichungssystem lässt sich nicht lösen
- → Mehr Unbekannte als unabh. Gleichungen
- → Mathematisch: $\underline{\underline{K}}$ nicht invertierbar ($\det \underline{\underline{K}} = 0$)
- → Mechanisch: Freikörperbewegungen treten auf (dynamische Rechnung wäre erforderlich)

$$\begin{bmatrix} k^1_{11} & k^1_{12} & k^1_{13} & k^1_{14} & 0 & 0 & 0 & 0 \\ k^1_{21} & k^1_{22} & k^1_{23} & k^1_{24} & 0 & 0 & 0 & 0 \\ k^1_{31} & k^1_{32} & k^1_{33}+k^3_{33} & k^1_{34} & 0 & 0 & k^3_{37} & 0 \\ k^1_{41} & k^1_{42} & k^1_{43} & k^1_{44}+k^2_{44} & 0 & k^1_{46} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k^2_{64} & 0 & k^2_{66} & 0 & 0 \\ 0 & 0 & k^3_{73} & 0 & 0 & 0 & k^3_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_1 \\ F_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$