Übung 2: Variationsrechnung

2.1) Stab unter Eigengewicht

a) Elastisches Potential des Stabes

$$\Pi = \int_B \left(\frac{1}{2} \sigma(\varepsilon) \varepsilon - \rho(-g) u \right) dv$$

$$\sigma(\varepsilon) = E\varepsilon \quad , \quad \varepsilon = \frac{du}{dx} = u' \quad , \quad \int_B dv = A \int_0^l dx$$

$$\Rightarrow \Pi = A \int_0^l \left(\frac{1}{2} E (u')^2 + \rho g u \right) dx \Rightarrow \min_u$$

Variation nach $u := u(x)$

$$\delta_u \Pi = \frac{\partial \Pi}{\partial u'} \delta u' + \frac{\partial \Pi}{\partial u} \delta u = A \int_0^l (E u' \delta u' + \rho g \delta u) dx = 0$$

partielle Integration

$$\boxed{NR: \underbrace{\int_0^l (u' \delta u)' dx}_{u' \delta u|_{x=l} - \cancel{u' \delta u|_{x=0}}} = \int_0^l u' \delta u' dx + \int_0^l u'' \delta u \, dx}$$

0, $\delta u|_{x=0} = 0$ (Dirichlet RB / „essentielle" RB)

$$\Rightarrow \delta_u \Pi = EAu'|_{x=l} \delta u - \int_0^l EAu'' \delta u \, dx + \int_0^l A\rho g \delta u \, dx = 0$$

$$= \underbrace{EAu'|_{x=l}}_{=0} \delta u - \int_0^l (\underbrace{EAu'' - A\rho g}_{=0}) \delta u = 0$$

Neumann RB / natürliche RB $\qquad$ Euler-Lagrange-Gleichung

DG: $\quad Eu'' = \rho g$

RB: $\quad EAu'(x=l) = 0$

$\quad\quad\; u(x=0) = 0$

b) Lösung der DG

$$\left.\begin{array}{l} u' = \frac{\rho g}{E} x + C_1 \\ u'(x=l) = 0 \end{array}\right\} C_1 = -\frac{\rho g}{E} l \Rightarrow u' = \frac{\rho g}{E}(x - l)$$

$$\left.\begin{array}{l} u = \frac{\rho g}{2E} x^2 - \frac{\rho g}{E} l x + C_2 \\ u(x=0) = 0 \end{array}\right\} C_2 = 0 \Rightarrow u = \frac{\rho g}{E}\left(\frac{1}{2} x^2 - x l\right)$$

2.2) Balken unter Streckenlast

a) Variation nach $w := w(x)$

$$\delta_w \Pi = \int_0^\ell \left(EI w'' \delta w'' \overset{*}{-} q_0 \delta w\right) dx = 0$$, essentielle RB hier: $w|_{x=0} = 0$, $w|_{x=\ell} = 0$

partielle Integration

$$\boxed{NR1: \int_0^\ell w'' \delta w'' \, dx = \underbrace{\int_0^\ell (w'' \delta w')' dx}_{w''\delta w'|_{x=\ell} - w''\delta w'|_{x=0}} - \int_0^\ell w''' \delta w' \, dx \quad *}$$

* Hinweis: Vorzeichen in der alten Aufgabenstellung falsch

partielle Integration

$$\boxed{NR2: -\int_0^\ell w''' \delta w' \, dx = \underbrace{-\int_0^\ell (w''' \delta w)' dx}_{-w'''\cancel{\delta w}|_{x=0} + w'''\cancel{\delta w}|_{x=\ell}} + \int_0^\ell w^{IV} \delta w \, dx \quad **}$$

$\delta w|_{x=0} = 0$ $\qquad$ $\delta w|_{x=\ell} = 0$

$$\Rightarrow \delta_w \Pi = EI w''|_{x=\ell} \delta w' - EI w''|_{x=0} \delta w' + \int_0^\ell (EI w^{IV} - q_0) \delta w \, dx = 0$$

DG: $EI w^{IV} = q_0$

RB:

$$\left.\begin{array}{l} EI w''(x=\ell) = 0 \quad (:= -M) \\ EI w''(x=0) = 0 \end{array}\right\} \text{natürliche RB}$$

$$\left.\begin{array}{l} w(x=0) = 0 \\ w(x=\ell) = 0 \end{array}\right\} \text{essentielle RB}$$

b) Lösung der DG

$$EI w'' = -\frac{1}{2} q_0 x^2 + c_1 x + \cancel{c_2}^{\,0} \quad \overset{\text{Auswerten der RB}}{\Rightarrow} \quad c_2 = 0, \; c_1 = \frac{1}{2} q_0 \ell$$

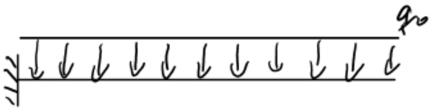
$$EI w' = -\frac{1}{6} q_0 x^3 + \frac{1}{4} q_0 \ell x^2 + c_3$$

$$EI w = -\frac{1}{24} q_0 x^4 + \frac{1}{12} q_0 \ell x^3 + c_3 x + \cancel{c_4}^{\,0}$$

$$c_3 = \frac{1}{\ell}\left(\frac{1}{24} q_0 \ell^4 - \frac{1}{12} q_0 \ell^4\right) = -\frac{1}{24} q_0 \ell^3$$

$$EI w = -\frac{1}{24} q_0 x^4 + \frac{1}{12} q_0 \ell x^3 - \frac{1}{24} q_0 \ell^3 x$$

c) Kragarm



$w|_{x=0} = 0$

$w'|_{x=0} = 0$

$\delta w'|_{x=0} = 0$

$* \quad w''\delta w'|_{x=l} - \cancel{w''\delta w'|_{x=0}}$

$\delta w|_{x=0} = 0$

$** \quad -w'''\delta w|_{x=l} + \cancel{w'''\delta w|_{x=0}}$

natürliche RB: $EIw''(x=l) = 0 \quad (:= -M)$

$EIw'''(x=l) = 0 \quad (:= -Q)$