Übung 1: Tensorrechnung und Kontinuumsmechanik

1.1) Tensormultiplikation

a) $\underline{I} : (\underline{A} \otimes \underline{b}) = \delta_{ij} A_{kl} b_m \, \overbrace{(\underline{e}_i \otimes \underline{e}_j) : (\underline{e}_k}^{\delta_{ik}} \otimes \underline{e}_l \otimes \underline{e}_m)$, $\quad \underbrace{\underline{e}_j) : (\underline{e}_k \otimes \underline{e}_l}_{\delta_{jl}}$

$$= \delta_{ij} \cancel{\delta_{ik}} \cancel{\delta_{jl}} A_{\underset{i\,j}{\cancel{kl}}} b_m \underline{e}_m = \delta_{ij} A_{ij} b_m \underline{e}_m$$

schneller: $\underline{I} : (\underline{A} \otimes \underline{b}) = \delta_{ij} A_{ij} b_k \underline{e}_k$ $\quad (i,\ j,\ k)$

b) $(\underline{I} \otimes \underline{I})^{\overset{23}{T}} : \underline{B} = \cancel{\delta_{ik}} \cancel{\delta_{jl}} B_{\underset{i\,j}{\cancel{kl}}} \underline{e}_i \otimes \underline{e}_j = B_{ij} \underline{e}_i \otimes \underline{e}_j = \underline{B}$ $\quad (i,\ j,\ l,\ k)$

Identitätstensor 4. Ordnung: $\underline{\underline{\mathbb{I}}} := (\underline{I} \otimes \underline{I})^{\overset{23}{T}}$ mit $\underline{\underline{\mathbb{I}}} : \underline{B} = \underline{B}$

c) $\underline{B} : (\underline{A} \otimes \underline{A}^T) : \mathbb{L} : (\underline{B} \otimes \underline{B}^T) : \underline{A} = B_{ij} A_{ij} A_{lk} \mathbb{L}_{klmn} B_{mn} B_{po} A_{op}$ $\quad (i,\ j,\ k,\ l,\ m,\ n,\ o,\ p)$

d) $\underline{I}^{*} (\underline{A} \; \underline{B}) = \cancel{\delta_{ij}} A_{\underset{i}{\cancel{j}}k} B_{kl} \underline{e}_i \otimes \underline{e}_l = A_{ik} B_{kl} = \underline{A} \; \underline{B}$ $\quad (i,\ j,\ k,\ l)$

e) $(\underline{I} \otimes \underline{I}) : \underline{B} = \delta_{ij} \cancel{\delta_{kl}} B_{\underset{l}{\cancel{k}}l} \underline{e}_i \otimes \underline{e}_j = \underline{I} (\mathrm{tr}\, \underline{B})$ $\quad (i,\ j,\ l,\ k)$

↳ $(\underline{I} \otimes \underline{I})$ ist kein Identitätstensor!

f) $\underline{A}^T : \mathbb{A} : \underline{B} = A_{ji} \mathbb{A}_{ijkl} B_{kl}$ $\quad (i,\ j,\ k,\ l)$

1.2) Differentialoperationen

a) $\frac{1}{2}(\operatorname{grad}\underline{v} + \operatorname{grad}^T\underline{v}) : \underline{A}$ mit $\underline{A}^T = \underline{A}$ symmetrisch

$$\operatorname{grad}\underline{v} = v_{i,j}\,\underline{e}_i \otimes \underline{e}_j\,, \quad \operatorname{grad}^T\underline{v} = v_{j,i}\,\underline{e}_i \otimes \underline{e}_j\,, \quad A_{ij} = A_{ji}$$

$$\frac{1}{2}(\operatorname{grad}\underline{v} + \operatorname{grad}^T\underline{v}) : \underline{A} = \frac{1}{2}(v_{i,j} + v_{j,i})\,A_{ij} = \frac{1}{2}(v_{i,j}A_{ij} + v_{j,i}A_{ji}) = v_{i,j}A_{ij}$$
$$= \operatorname{grad}\underline{v} : \underline{A}$$

b) $\operatorname{div}(\underline{v}\ \underline{T}) = (v_i T_{ij})_{,j} = v_{i,j}T_{ij} + v_i T_{ij,j} = \operatorname{grad}\underline{v} : \underline{T} + \underline{v}\ \operatorname{div}\underline{T}$

c) $$\frac{\partial(\operatorname{tr}\underline{A})}{\partial\underline{A}} = \frac{\partial(\underline{A}:\underline{I})}{\partial\underline{A}} = \frac{\partial A_{ij}\delta_{ij}}{\partial A_{kl}}\,\underline{e}_k \otimes \underline{e}_l = \Big(\underbrace{\frac{\partial A_{ij}}{\partial A_{kl}}}_{\delta_{ik}\delta_{jl}}\delta_{ij} + \overbrace{\cancel{A_{ij}\frac{\partial \delta_{ij}}{\partial A_{kl}}}}^{0}\Big)\,\underline{e}_k \otimes \underline{e}_l$$
$$= \cancel{\delta_{ik}}\cancel{\delta_{jl}}\underbrace{\cancel{\delta_{ij}}}_{kl}\,\underline{e}_k \otimes \underline{e}_l = \delta_{kl}\,\underline{e}_k \otimes \underline{e}_l = \underline{I}$$

d) $$\underbrace{\frac{\partial\underline{A}}{\partial\underline{A}}}_{=\underline{\underline{I}}} : \underline{B} = \underbrace{\frac{\partial A_{ij}}{\partial A_{kl}}}_{\delta_{ik}\delta_{jl}} B_{kl}\,\underline{e}_i \otimes \underline{e}_j = \cancel{\delta_{ik}}\cancel{\delta_{jl}}\,B_{\underset{i\,j}{\cancel{kl}}}\,\underline{e}_i \otimes \underline{e}_j = \underline{B}$$

e) $$\operatorname{tr}(\operatorname{grad}(\underline{v}\ \underline{T})) = \operatorname{grad}(\underline{v}\ \underline{T}) : \underline{I} = (v_i T_{ij})_{,\underset{j}{\cancel{k}}}\,\cancel{\delta_{jk}} = (v_i T_{ij})_{,j} = \operatorname{div}(\underline{v}\ \underline{T})$$

f) $$\frac{\partial(\underline{A}\ \underline{B})}{\partial\underline{A}} = \frac{\partial A_{ij}B_{jk}}{\partial A_{lm}}\,\underline{e}_i \otimes \underline{e}_k \otimes \underline{e}_l \otimes \underline{e}_m = \delta_{il}\cancel{\delta_{jm}}\,B_{\underset{m}{\cancel{j}}k}\,\underline{e}_i \otimes \underline{e}_k \otimes \underline{e}_l \otimes \underline{e}_m$$
$$+ A_{ij}\underbrace{\cancel{\frac{\partial B_{jk}}{\partial A_{lm}}}}_{0,\ \underline{B}\ \text{konst. in}\ \underline{A}}\,\underline{e}_i \otimes \underline{e}_k \otimes \underline{e}_l \otimes \underline{e}_m$$

$$= (\underline{I} \otimes \underline{B}^T)^{\overset{23}{T}} \quad \text{oder}$$
$$= (\underline{B} \otimes \underline{I}^T)^{\overset{14}{T}} = (\underline{B} \otimes \underline{I})^{\overset{14}{T}}$$

Hinweis zur Notation:
Um einheitlich mit der Vorlesung zu bleiben wird die einfache Überschiebung ($z.B.\ \underline{A}\,\underline{B} = A_{ij}B_{jk}\,\underline{e}_i \otimes \underline{e}_k$) ohne Punkt gekennzeichnet, während das „Skalarprodukt" von Tensoren beliebiger Ordnung mit Punkt gekennzeichnet wird
($z.B.\ \underline{A}\cdot\underline{B} = \underline{A}:\underline{B} = A_{ij}B_{ij}$ oder $\underline{a}\cdot\underline{b} = \underline{a}\,\underline{b} = a_i b_i$)

1.3) Elastizitätstensor und Voigt-Notation

a) $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon} = (\lambda \underline{I} \otimes \underline{I} + 2\mu \underline{\underline{I}}) : \underline{\varepsilon}$

NR: $\lambda(\underline{I} \otimes \underline{I}) : \underline{\varepsilon} = \lambda (\operatorname{tr} \underline{\varepsilon}) \underline{I}$ (vgl. A 1.1 e)

$2\mu \underline{\underline{I}} : \underline{\varepsilon} = 2\mu \underline{\varepsilon}$ (vgl. A 1.1 b)

$= \lambda (\operatorname{tr} \underline{\varepsilon}) \underline{I} + 2\mu \underline{\varepsilon}$

b) Matrixnotation

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

(Definitionssache ...)

$$\underline{\underline{\sigma}}^{9\times 1} = [\sigma_{11} \ \sigma_{22} \ \sigma_{33} \ \sigma_{23} \ \sigma_{13} \ \sigma_{12} \ \sigma_{32} \ \sigma_{31} \ \sigma_{21}]^T$$

$$\underline{\underline{I}}^{9\times 1} = [1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

$$[1 \quad 1 \quad 1 \quad \underline{\underline{0}}^{6\times 1}]$$

$$(\underline{\underline{I}} \otimes \underline{\underline{I}})^{9\times 9} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \underline{\underline{0}}^{6\times 1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \underline{\underline{0}}^{3\times 6} \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & \\ & \underline{\underline{0}}^{6\times 3} & & \underline{\underline{0}}^{6\times 6} \end{bmatrix}$$

$$\underline{\underline{I}}^{9\times 9} = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & \underline{\underline{0}} & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & \underline{\underline{0}} & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

$$\underline{\underline{\mathbb{C}}}^{9\times 9} = \begin{bmatrix} \lambda & \lambda & \lambda & \\ \lambda & \lambda & \lambda & \underline{\underline{0}}^{3\times 6} \\ \lambda & \lambda & \lambda & \\ & \underline{\underline{0}}^{6\times 3} & & \underline{\underline{0}}^{6\times 6} \end{bmatrix} + \begin{bmatrix} 2\mu & & & & & & & & \\ & 2\mu & & & & & & \underline{\underline{0}} & \\ & & 2\mu & & & & & & \\ & & & 2\mu & & & & & \\ & & & & 2\mu & & & & \\ & & & & & 2\mu & & & \\ & & \underline{\underline{0}} & & & & 2\mu & & \\ & & & & & & & 2\mu & \\ & & & & & & & & 2\mu \end{bmatrix}$$

$$\underline{\underline{\varepsilon}}^{9\times 1} = \left[\varepsilon_{11} \; \varepsilon_{22} \; \varepsilon_{33} \; \varepsilon_{23} \; \varepsilon_{13} \; \varepsilon_{12} \; \varepsilon_{32} \; \varepsilon_{31} \; \varepsilon_{21}\right]^T$$

$$\underline{\underline{\sigma}}^{9\times 1} = \underline{\underline{\mathbb{C}}}^{9\times 9} \underline{\underline{\varepsilon}}^{9\times 1}$$

Annahmen für Voigt-Notation:

-> Symmetrie von $\underline{\sigma}$: $\sigma_{ij} = \sigma_{ji}$

-> Symmetrie von $\underline{\varepsilon}$: $\varepsilon_{ij} = \varepsilon_{ji}$

-> „kleine" Symmetrie von $\underline{\mathbb{C}}$: $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk}$

(Anmerkung: aus „großer" Symmetrie $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$ folgt $\underline{\underline{\mathbb{C}}}^v = (\underline{\underline{\mathbb{C}}}^v)^T$.
Jedoch keine notwendige Bedingung für Voigt-Notation)

=> Zeile 4–6 und 7–9 liefern die selbe Aussage => streiche Zeile 7–9

$$\underline{\underline{\sigma}}^v = \left[\sigma_{11} \; \sigma_{22} \; \sigma_{33} \; \sigma_{23} \; \sigma_{13} \; \sigma_{12}\right]^T$$

$$\underline{\underline{\mathbb{C}}}^v = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & \underline{\underline{0}}^{3\times 3} & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & 0 & 0 \\ & \underline{\underline{0}}^{3\times 3} & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix}$$

Faktor 2 „herübergezogen" auf $\underline{\underline{\varepsilon}}^v$

1.4) Analytische Lösung eines Randwertproblems

a) Impulsbilanz

statisches Problem

$$-\operatorname{div} \underline{\sigma} = \cancel{\underline{b}} - \cancel{\rho \ddot{\underline{x}}}$$

keine Volumenkräfte (Erdbeschleunigung)

$$\underline{u} = \begin{bmatrix} \alpha x_1 \\ \beta x_2 \\ \gamma x_3 \end{bmatrix} \Rightarrow \operatorname{grad} \underline{u} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

$$\underline{\varepsilon} = \frac{1}{2}\left(\operatorname{grad} \underline{u} + \operatorname{grad}^T \underline{u}\right) := \operatorname{sym}(\operatorname{grad} \underline{u})$$

$$\Rightarrow \underline{\varepsilon} \overset{\text{hier}}{=} \operatorname{grad} \underline{u}$$

$$\operatorname{tr} \underline{\varepsilon} = \alpha + \beta + \gamma$$

$$\Rightarrow \underline{\sigma} = \lambda \begin{bmatrix} \alpha+\beta+\gamma & 0 & 0 \\ 0 & \alpha+\beta+\gamma & 0 \\ 0 & 0 & \alpha+\beta+\gamma \end{bmatrix} + 2\mu \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

$\Rightarrow -\operatorname{div} \underline{\sigma} = \underline{0}$ ✓ , da $\underline{\sigma}$ konstant in $\underline{x}$

Randbedingungen:

$$x_3 = \frac{t}{2}: \quad \underline{\sigma}\, \underline{e}_3 = \underline{0} \Rightarrow \lambda(\alpha+\beta+\gamma) + 2\mu\gamma = 0 \qquad (1)$$

$$x_2 = \frac{b}{2}: \quad \underline{\sigma}\, \underline{e}_2 = \underline{0} \Rightarrow \lambda(\alpha+\beta+\gamma) + 2\mu\beta = 0 \qquad (2)$$

$$x_1 = l: \quad \underline{\sigma}\, \underline{e}_1 = \begin{bmatrix} -q \\ 0 \\ 0 \end{bmatrix} \Rightarrow \lambda(\alpha+\beta+\gamma) + 2\mu\alpha = -q \qquad (3)$$

$$(1)-(2): \quad 2\mu\gamma - 2\mu\beta = 0 \Rightarrow \gamma = \beta \qquad (4)$$

$$(1)-(3): \quad 2\mu\gamma - 2\mu\alpha = -q \Rightarrow \alpha = \gamma - \frac{q}{2\mu} \qquad (5)$$

(4) und (5) in (2):

$$\lambda\left(\gamma - \frac{q}{2\mu} + 2\gamma\right) + 2\mu\gamma = 0 \Rightarrow \gamma(3\lambda + 2\mu) = \frac{q\lambda}{2\mu} \Leftrightarrow \gamma = \beta = \frac{q\lambda}{2\mu(3\lambda+2\mu)}$$

in (5): $\alpha = \frac{q\lambda}{2\mu(3\lambda+2\mu)} - \frac{q}{2\mu} = -\frac{q(2\lambda+2\mu)}{2\mu(3\lambda+2\mu)} = -\frac{q(\lambda+\mu)}{\mu(3\lambda+2\mu)}$

b) Verschiebung im Punkt A

$\lambda = 80{,}769$ GPa , $\mu = 121{,}154$ GPa $\Rightarrow \alpha = -0{,}00238$

$\Rightarrow \underline{u}(l,0,0) = -0{,}238\,\text{mm}\,\underline{e}_1$