Contents

1 Preliminaries

1.1 Game Theory

1.1.1 Nash Equilibrium

Each player is choosing the best possible strategy given the strategies chosen by the other players.

$$s_i^* \in \arg\max_{s_i} U_i(s_1^*, \cdots, s_i^*, \cdots, s_n^*), \forall i = 1, \cdots n. \tag{1.1}$$

There is $\in$ instead of $=$ since this Nash Equilibrium does not need to be unique.

1.1.2 Reaction functions

A reaction function (or best reply or best response function) gives the best action for a player given the actions of the other players. The Nash Equilibrium is where all reaction functions intersect.

1.1.3 Symmetry

When the game is symmetric, i.e. all players are in the same conditions, have the same reaction function etcetera, then the players are anonymous (They are indistinguishable from each other except for name or index). Then all players choose the same strategy s^* in the Nash Equilibrium.

1.1.4 Sub-game perfect equilibrium

When players do not move simultaneously, i.e. the players move sequentially, we need to refine definition of Nash Equilibrium: the sub-game perfect equilibrium requires that the strategy profile under consideration is not only the equilibrium for the entire game but also for each sub-game.

1.1.5 Moves of nature

Many models in IO involve uncertainty. This is often modeled as a “move of nature” in a multistage game. The moment at which the uncertainty is resolved, is referred to as the move of nature. Such games can again be solved using backward induction *** Candidate equilibrium For ease of exposition, we will often use the concept of a candidate equilibrium. In many models, it is possible to make an educated guess as to what the equilibrium might be. We will refer to such an educated guess as a candidate equilibrium. A candidate equilibrium thus is a strategy profile that may be a (subgame perfect) Nash equilibrium, but for which we still have to check whether that really is the case. This approach for finding an equilibrium is often easier than deriving an equilibrium from scratch. Note however that if it turns out that the candidate equilibrium is a true equilibrium, we still have not established whether that equilibrium is unique. *** Mixed strategies Players are not restricted to always play some action a_i^* in equilibrium. A mixed strategy equilibrium has each player drawing its action from some probability distribution $F_i(a_i)$, defined on some domain A_i. Given the strategies played by all other players, each player i is indifferent between the actions among which it mixes. Hence, $\mathbb{E}\left[U_i(a_i)\right]$ is constant for all $a_i \in A_i$.

1.2 Models with differentiated products

1.2.1 Hotelling competition

With horizontal product differentiation different consumers prefer different products. With vertical product differentiation all consumers agree which product is preferable.

In Hotelling, consumers are normally distributed on line of unit length, the number of consumer is normalized to 1. Consumers either buy 1 unit of good, or none. Each consumer obtains gross utility v from consuming the product. There are 2 firms, one located at 0, one located at 1. Consumer located at x needs to travel distance x to buy from firm 0 and $1-x$ distance to buy from firm 1. Transportation costs are t per unit of distance. Marginal costs for both firms are constant and equal to c. From this information it can be concluded that in equilibrium both firms charge the same price since the game is symmetric. Suppose firm i charges price P_i. Then a consumer located at x buys from firm 0 iff

$$v - P_0 - tx > v - P_1 - t(1-x) \tag{1.2}$$

provided that $v - P_0 - tx > 0$. Assume entire market is covered: in equilibrium prices are s.t. everyone consumes. This implies that $v > 2t$.

Define a consumer z that is indifferent between buying from 0 or 1. Every consumer located at $x < z$ buys from 0, every consumer $x > z$ buys from 1. This consumer z is located at

$$P_0 + tz = P_1 + t(1-z) \tag{1.3}$$

$$\iff z = \frac{1}{2} + \frac{P_1 - P_0}{2t} \tag{1.4}$$

Total sales for firm 0 equal z and for firm 1 $1-z$. When they charge the same price $z = \frac{1}{2}$. Profits for firm 0 are:

$$\Pi_0 = (P_0 - c)z \tag{1.5}$$

$$= (P_0 - c)\left(\frac{1}{2}\frac{P_1 - P_0}{2t}\right) \tag{1.6}$$

Then we maximize wrt P_0 to get reaction function:

$$P_0 = R_0(P_1) = \frac{1}{2}(c + t + P_1) \tag{1.7}$$

And by symmetry:

$$P_1 = R_1(P_0) = \frac{1}{2}(c + t + P_0) \tag{1.8}$$

Thus, in equilibrium (by equation the two reaction functions):

$$P_0^* = P_1^* = c + t \tag{1.9}$$

$$\Pi_0 = \Pi_1 = \frac{1}{2}t \tag{1.10}$$

1.2.2 The circular city: Salop

Consumers are located uniformly on the edge of a circle with perimeter equal to 1. Consumers wish to buy 1 unit of good, have valuation v and transport cost t per unit of distance. Each firm is allowed to locate in 1 location. There is a fixed cost of entry f and marginal costs c. Hence, firm i has profit $(p_i - c)D_i - f$ if it enters (where D_i) is the demand it faces, and 0 otherwise.

In the first stage, the firms decide simultaneously whether to enter or not. Let n denote the number of entering firms. The firms are located equidistant from each other on the edge of the circle.

This can be solved using backward induction. Assume n firms have entered, and suppose all firms except firm 0 charge price p, then firm 0 charges price $R_0(p)$. In equilibrium, $R_0(p^*) = p^*$. Let firm 0 have location 0. The location of firm 0's right-hand neighbor is $\frac{1}{n}$ (firm 1). Consider consumers located between firm 0 and 1. If firm 0 charges price p_0, then the consumer z_{0-1} that is indifferent between firm 0 and 1 is:

$$p_0 + tz_{0-1} = p + t\left(\frac{1}{n} - z_{0-1}\right) \tag{1.11}$$

$$\iff z_{0-1} = \frac{1}{2n} + \frac{p - p_0}{2t} \tag{1.12}$$

The same holds for the left-hand side of firm 0, hence its profits are:

$$\Pi_0(p_0, p) = 2(p_0 - c)\left(\frac{1}{2n} + \frac{p - p_0}{2t}\right) \tag{1.13}$$

Maximize w.r.t. p_0 to find the reaction function:

$$R_0(p) = \frac{1}{2}\left(p + c + \frac{t}{n}\right) \tag{1.14}$$

Then we impose symmetry:

$$p^* = c + \frac{t}{n} \text{ with profits } \Pi^* = \frac{t}{n^2} \tag{1.15}$$

Now we move back to stage 1. Firms enter the market as long as profits are positive. Hence, the number of enterers is s.t.

$$\Pi^* - f = \frac{t}{n^2} - f = 0 \iff n^* = \sqrt{\frac{t}{f}} \tag{1.16}$$

$$\implies p^* = c + \sqrt{tf} \tag{1.17}$$

Consumer's average transportation cost is $\frac{t}{4n} = \frac{\sqrt{tf}}{4}$. When the entry cost tends to 0 the number of entering firms tends to infinity and the market price tends to marginal cost.

A social planner would choose n in order to minimize the sum of fixed cost and transportation costs: $\min_n \left(nf + \frac{t}{4n}\right) \implies n = \frac{1}{2}\sqrt{\frac{t}{f}} = \frac{1}{2}n^*$, hence the market generates too many firms. Similar results hold for quadratic transportation costs.

There are three natural extensions that would make the model more realistic: the introduction of a location choice, the possibility that firms do not enter simultaneously, and the possibility that a firm locates at several points in the product space.

1.2.3 Perloff and Salop

Suppose there are two firms, A and B, and a unit mass of consumers. Consumer j has a willingness to pay for the product of firm $i \in \{A, B\}$ that is given by $v + \epsilon_{ij}$, where ϵ_{ij} is the realization of a r.v. with cumulative distribution function F on some interval $[0, \bar{\epsilon}]$, and continuously differentiable density f. Assume v is high enough that all consumers buy in equilibrium. All draws are independent from each other, ϵ_{ij} can be interpreted as the match value between product i and consumer j. Assume all firms have constant marginal cost equal to c.

Firstly, suppose both firms charge same price p^*. This will happen in equilibrium since the game is symmetric. Then a consumer buys from A whenever $\epsilon_A > \epsilon_B$ and from B otherwise. To find the equilibrium we need to analyze what happens if firm A defects from a candidate equilibrium p^* by setting some different price. Without loss of generality assume A defects to some $p_A > p^*$. Denote $\Delta \equiv p_A - p^*$. Now, a consumer buys from firm A if $\epsilon_A - \Delta > \epsilon_B$. Assume for simplicity the ϵ are drawn from uniform distribution on $[0, 1]$. Then, the total sales of firm A from charging price $p_A = p^* + \Delta$ equal $q_A = \frac{1}{2}(1-\Delta)^2$. Hence the expected profits of firm A are

$$\pi_A = (p_A - c)q_A \tag{1.18}$$

Profit maximization requires:

$$\frac{\partial \pi_A}{\partial p_A} = (p_A - c)\frac{\partial q_A}{\partial p_A} + q_A = 0, \text{ with} \tag{1.19}$$

$$\frac{\partial q_A}{\partial p_A} = -(1 - \Delta) \tag{1.20}$$

Then if we take first order condition and impose symmetry, $\Delta = 0$ and $q_A = \frac{1}{2}$, we get:

$$(p_A - c)(-1) + \frac{1}{2} = 0 \tag{1.21}$$

$$\iff p^* = c + \frac{1}{2} \tag{1.22}$$

In the model description the assumption that v is high enough s.t. each consumer will buy in equilibrium was made. We thus need that a consumer with $(\epsilon_A, \epsilon_B) = (0, 0)$ would still be willing to buy. In this case, with $p^* = c + \frac{1}{2}$ this implies $v > c + \frac{1}{2}$

We can generalize the model for any distribution function F. Then:

$$q_A = \int_{\Delta}^{1} \left(\int_{0}^{\epsilon_A - \Delta} f(\epsilon_B)\, \mathrm{d}\epsilon_B \right) f(\epsilon_A)\, \mathrm{d}\epsilon_A \tag{1.23}$$

$$= \int_{\Delta}^{1} F(\epsilon_A - \Delta) f(\epsilon_A)\, \mathrm{d}\epsilon_A \tag{1.24}$$

Using Leibnitz' rule, this implies:

$$\frac{\partial q_A}{\partial p_A} = -\int_{\Delta}^{1} f(\epsilon_A - \Delta) f(\epsilon_A)\, \mathrm{d}\epsilon_A \tag{1.25}$$

Again we can find the equilibrium price by taking first-order condition (1.19) and using (1.25). In equilibrium $\Delta = 0$ and $q_A = \frac{1}{2}$. This yields:

$$p^* = c + \frac{1}{2\int_0^1 f^2(\epsilon)\,\mathrm{d}\epsilon} \tag{1.26}$$

Note that the Perloff-Salop model is similar in spirit to the Hotelling model, in the sense that each consumer has a certain valuation for each firm's product. The crucial difference is that in the Hotelling model, these valuations are perfectly negatively cor- related: a consumer that has a very high valuation for product A, say, necessarily has a very low valuation for product B, and vice-versa. In the Perloff-Salop model, these valuations are independent of each other. A consumer with a high valuation for product A may very well have a high valuation for product B as well. This also implies that the Perloff-Salop model can easily be generalized to more than 2 firms.

1.3 Exercises

1. Consider the following Cournot model. Two firms set quantities. Demand is given by $q = 1-p$. Marginal costs are either equal to 0 or 0.4, both with equal probability. Derive the Cournot equilibrium if

   (a) uncertainty is resolved before firms set their quantities.

   (b) uncertainty is resolved after firms set their quantities.

   Solution:

   (a) The game is symmetric, hence in equilibrium both firms set the same price.

   $$\pi_i = (1 - q_1 - q_2 - c)q_i \Longrightarrow \frac{\partial \pi_i}{\partial q_i} = 1 - q_j - 2q_i - c$$
   $$\overset{\text{FOC}}{\Longrightarrow} q_i = \frac{1 - q_j - c}{2}$$
   $$\text{Imposing symmetry: } q = \frac{1-c}{3}$$

   Then $c = 0$ gives $q = \frac{1}{3}$, $c = 0.4$ gives $q = \frac{1}{5}$.

   (b) The expected marginal cost is $\mathbb{E}[c] = 0.2$. We can fill this in the previous formula:

   $$q = \frac{1 - \mathbb{E}[c]}{3} = \frac{0.8}{3}$$

2. Consider a Hotelling model. Consumers are uniformly distributed on a line of unit length. Consumers either buy one unit of the good or none at all. Each consumer obtains gross utility v from consuming the product. We have two firms: one is located at 0, the other is located at 1. Marginal costs for both firms are constant and equal to c. However, transportation costs are constant: a consumer that has to travel a distance x incurs transport costs tx^2. Derive the equilibrium prices.

Solution: This game is symmetric hence in equilibrium both firms set the same price. Consider the indifferent consumer with location z, then:

$$\begin{aligned} v - P_0 - tz^2 &= v - P_1 - t(1-z)^2 \\ \iff P_0 + tz^2 &= P_1 + t(1-z)^2 \\ \implies z &= \frac{1}{2} + \frac{P_1 - P_0}{2t} \end{aligned}$$

Firm 0 its expected profits are:

$$\begin{aligned} \pi_0 &= (P_0 - c)z = (P_0 - c)\frac{1}{2} + \frac{P_1 - P_0}{2t} \\ \implies \frac{\partial \pi_0}{\partial P_0} &= \frac{p_1 - p_0 + t}{2t} - \frac{p_0 - c}{2t} \\ \overset{\text{FOC}}{\implies} P_0 &= \frac{1}{2}(P_1 + c + t) \end{aligned}$$

Imposing symmetry:

$$P = \frac{1}{2}(P + c + t) \iff P = c + t$$

3. Consider the Perloff-Salop model where consumers have a valuation that is uniformly distributed on $[0, 1]$. For simplicity, $c = 0$. Assume however that $v = 0$ such that not all consumers buy in equilibrium. Derive the equilibrium prices.

   Solution: In equilibrium, prices of both firms are equal to p. A consumer buys if $\max\{\epsilon_A, \epsilon_B\} > p$. This happens with probability:

$$\begin{aligned} \mathbb{P}\{\max\{\epsilon_A, \epsilon_B\} > p\} &= 1 - \mathbb{P}\{\max \epsilon_A, \epsilon_B \le p\} \\ &= 1 - \mathbb{P}\{\epsilon_A \le p\}\, \mathbb{P}\{\epsilon_B \le p\} \\ &= 1 - p^2 \end{aligned}$$

   Hence both firms face demand $q = \frac{1-p^2}{2}$.
   Suppose firm A sets the price slightly higher than firm B, $p_A = p + \Delta$. Then, demand is:

$$q_A = \frac{1}{2}(1 - \Delta)^2 - \frac{1}{2}p^2$$

   with profits:

$$\pi_A = (p_A - c)q_A$$

   FOC gives:

$$\begin{aligned} \pi_A = p_A \frac{\partial q_A}{\partial p_A} + q_A &= 0 \\ \text{with } \frac{\partial q_A}{\partial p_A} &= -(1 - \Delta) \\ \iff q_A &= (1 - \Delta)p_A \end{aligned}$$

   Imposing symmetry:

$$\begin{aligned} p &= \frac{1}{2} - \frac{p^2}{2} \\ p &= \sqrt{2} - 1 \end{aligned}$$

2 Consumer search

2.1 The standard Bertrand Model

Consider two firms that produce identical goods. There are no frictions, i.e the consumers buy from the firm that sets the lowest price. If they set the same price they both get half the market share. Assume firms can fulfill any demand they face. The demand function is $q = D(p)$, marginal costs constant equal c. Monopoly profits assumed to be strictly concave, bounded and the monopoly price is given by:

$$p_m = \arg\max_p (p-c)D(p) \tag{2.1}$$

Demand for firm i is written $D_i(p_i, p_j)$:

$$D_i(p_i,p_j) = \begin{cases} D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases} \tag{2.2}$$

The profits of firm i are then:

$$\Pi_i(p_i,p_j) = (p_i - c)D_i(p_i,p_j) \tag{2.3}$$

This profit function is discontinuous.

First consider the case when $p_j > p_m$. In such a case the best reply of firm i is to set $p_i = p_m$. Then it attracts all consumers, while charging the monopoly price.

Now suppose $c < p_j \leq p_m$. Firm i has three choices. Charging price $p_i > p_j$ it has zero profits. Charging price $p_i = p_j$, its sales are $\frac{1}{2}D(p_j)$. Given the concavity of the profit function and the fact that $p_j < p_m$, the best response of i is to set the price slightly below p_j, say $p_i = p_j - \epsilon$, with ϵ arbitrarily small.

Consider the case when $p_j < c$. Undercutting now yields negative profits. Hence, the best reply is to set any price $p_i > p)j$ with zero profit.

Finally, consider the case in which $p_j = c$. Undercutting again leads to negative profits. Both $p_i = p_j$ and $p_i > p_j$ yield 0 profits. Therefore the best reply is to set any $p_i \geq c$.

Summarizing:

$$R_i(p_j) = \begin{cases} \in (p_j, \infty) & \text{if } p_j < c \\ \in [c, \infty) & \text{if } p_j = c \\ p_j - \epsilon & \text{if } c < p_j \leq p_m \\ p_m & \text{if } p_j > p_m \end{cases} \tag{2.4}$$

From this we conclude there is unique Nash equilibrium where both firms charge price c, with zero profit for both firms.

This result is known as the *Bertrand paradox*. It qualifies as a paradox since it is hard to believe that two firms is already enough to restore competitive outcome in a market.

When only two firms are active in a market, one would expect them to make positive profits.

2.2 A model of sales

Consider a duopoly that competes in prices. Assume there is a mass of consumers that equals 1, where each consumer is willing to pay at most 1 for the product. Each consumer demands only one unit of product:

$$D(p) = \begin{cases} 1 & \text{if } p \leq 1 \\ 0 & \text{if } p > 1 \end{cases} \tag{2.5}$$

Assume 0 marginal costs, and that there are two types of consumers: A fraction of λ consumers is informed, which implies they can observe the prices of both firms. The remaining fraction of $1-\lambda$ is uninformed: they pick at random which firm to go to and buy the product if the price does not exceed 1.

The expected profits for firm i are given by:

$$\Pi_i(p_i, p_j) = \begin{cases} 0 & \text{if } p_i > 1 \\ \frac{1}{2}(1-\lambda)p_i & \text{if } 1 \geq p_i > p_j \\ \frac{1}{2}p_i & \text{if } p_i = p_j \leq 1 \\ \left[\lambda + \frac{1}{2}(1-\lambda)\right] p_i & \text{if } p_i < p_j \leq 1 \end{cases} \tag{2.6}$$

provided $p_i \leq 1$.

We can derive the best-response functions. First, note that whenever firm i would want to charge a higher price than j, the best option is to charge a price of 1, since it faces the same demand as any other price in $(p_j, 1]$, and maximizes its profits if it sets its price at 1. This implies the following. Suppose $p_j > 0$. The best choice for firm i is to either slightly undercut p_j which yields profits $\left[\lambda + \frac{1}{2}(1-\lambda)\right] p_j$ or to set $p_i = 1$, yields profits $\frac{1-\lambda}{2}$. Firm i strictly prefers the first strategy when:

$$\lambda p_j + \frac{1}{2}(1-\lambda) > \frac{1}{2}(1-\lambda) \tag{2.7}$$

$$\iff p_j > \frac{1-\lambda}{1+\lambda} \tag{2.8}$$

This yields the reaction functions:

$$R_i(p_j) = \begin{cases} 1 & \text{if } p_j \leq \frac{1-\lambda}{1+\lambda} \\ p_j - \epsilon & \text{if } p_j > \frac{1-\lambda}{1+\lambda} \end{cases} \tag{2.9}$$

There is no Nash equilibrium in pure strategies. Suppose firm 1 sets $p_1 = 1$. Firm 2 will slightly undercut. But now, firm 1 wants to undercut firm 2 again. This process continues until $p = \frac{1-\lambda}{1+\lambda}$. Then no firm has incentive to undercut the other, but each firm does have an incentive to defect by setting its price equal 1, where the whole process starts anew.

There is however, an equilibrium in mixed strategies. Suppose both firms draw their price from some continuous distribution function $F(p)$ on the support $\left[\underline{p}, \overline{p}\right]$. For this too be an equilibrium, we need that given that firm j uses the mixed strategy, any $p_i \in \left[\underline{p}, \overline{p}\right]$ must yield the same expected profit for firm i.

Note that the profits that firms make in the mixed strategy equilibrium can not be lower than $\frac{1-\lambda}{2}$, then a firm could just set price 1 and have more profits. This implies we can not have $\overline{p} > 1$. Consider the possibility that $\overline{p} < 1$. A firm charging price $\overline{p}$ is certain that it is charging the highest price on the market. Then it can do better by charging price 1 instead. Hence we must have $\overline{p} = 1$.

A firm charging $\overline{p} = 1$ makes profit $\frac{1-\lambda}{2}$, this implies that any $p \in \left[\underline{p}, \overline{p}\right]$ must yield those same profits. Consider a firm charging $\underline{p}$. With certainty, this firm charges the lowest price. Its profits then equal $\left[\lambda + \frac{1}{2}(1-\lambda)\right]\underline{p}$. This has to equal $\frac{1-\lambda}{2}$, we thus have that $\underline{p} = \frac{1-\lambda}{1+\lambda}$.

If firm i charges price p, the probability that firm j charges a lower price if $F(p)$. When setting some $p_i \in \left[\underline{p}, \overline{p}\right]$, expected profits are:

$$\mathbb{E}\left[\pi_i(p)\right] = F(p)\left(\frac{1}{2}(1-\lambda)\right)p + (1-F(p))\left(\lambda + \frac{1}{2}(1-\lambda)\right)p \tag{2.10}$$

$$= \frac{1}{2}(1-\lambda)p + (1-F(p))\lambda p \tag{2.11}$$

We thus need:

$$\frac{1}{2}(1-\lambda)p + (1-F(p))\lambda p = \frac{1-\lambda}{2}, \qquad \forall p \in \left[\frac{1-\lambda}{1+\lambda}, 1\right] \tag{2.12}$$

Solving for $F(p)$ yields:

$$F(p) = 1 - \frac{(1-\lambda)(1-p)}{2\lambda p} \tag{2.13}$$

The equilibrium profits of both firms in this equilibrium equal $\frac{1-\lambda}{2}$, which is strictly positive. Hence, there is no Bertrand paradox in this set-up.

2.3 The Diamond paradox

2.3.1 Model

Consider a slight change of the basic model with homogeneous products and equal marginal costs. We assume that consumers have to incur search costs to find out which price each firm charges. These search costs are s, where s can be arbitrarily small. Visiting the first shop is free, but to visit any additional shop, a consumer has to incur search costs s. Suppose again that there is a unit mass of consumers that all have a willingness to pay for one unit of the product that equals v. Firms have constant marginal costs $c < v$. It is crucial that consumers cannot observe prices before they visit a shop. For simplicity, we assume that there are only two firms.

The timing of the game is as follows. First, firms set prices that are not observable. Second, consumers visit the shop of their first choice, and observe the price that is set by that shop. Third, consumers either buy from this shop, decide to visit the other shop, or decide not to buy at all. The equilibrium now also depends on what consumers expect. That makes it hard to derive reaction functions in the usual manner; these reaction functions now also depend on the behavior of consumers. We therefore choose a different manner of analysis. We first postulate a candidate equilibrium. Then, we derive whether any firm (or the consumers, for that matter), has an incentive to defect from that equilibrium.

A natural candidate equilibrium seems to be the one that is the equilibrium in the standard Bertrand model, in which both firms charge a price c, consumers split evenly in the choice of their first shop, and consumers also decide to buy at the shop they first visit. It is easy to see, however, that this cannot be an equilibrium. Suppose that firm 1 deviates and increases its price to $c + \frac{s}{2}$. First, such a defection cannot influence the number of consumers that visit this shop, as consumers can only observe the price after they have visited. Hence, all consumers that visit this shop are now unpleasantly surprised. Yet, they still have no reason to choose a different strategy in the sub-game that follows. Going to the other shop implies a cost-saving of $\frac{s}{2}$ due to a lower price (note that we consider a deviation from the Nash equilibrium by firm 1, which implies that firm 2 is still charging price c), but an increase in search cost by s. Hence, consumers will not continue search and the deviation is profitable.

But the same argument holds for any symmetric candidate equilibrium that has $p < v$. For a price $p \geq v$, consumers also will not switch for a small price increase, but a firm is not willing to defect such a manner, as by definition it would decrease its profits.

Thus, the unique equilibrium has both firms charging the monopoly price, even with infinitesimally small search costs. This is known as the *Diamond paradox*. Also note that this result does not hinge on the number of firms. For any $n \geq 2$, the analysis applies and the unique equilibrium has prices equal to the monopoly price.

2.3.2 What if the first visit is not free?

In this model, the assumption that the first visit is free is somewhat odd. Suppose that a consumer would also have to pay s to visit the first firm. Following the logic above, the unique Nash equilibrium would have all firms charging the monopoly price $pm = v$. But now suppose that consumers also have to pay s to visit the first firm. This will not change the logic of the analysis; it would still be an equilibrium to have $p = v$. Once the consumer has visited the first firm, the search costs s are already sunk so she could either buy at that firm and pay 1, or go home without the product and still having incurred search costs s.

But a rational consumer will foresee this. Knowing that all firms will charge price v, and knowing that she has to incur search costs s to visit one, she knows that it is not a good idea to enter this market, since doing so so will leave her with a negative surplus of $-s$. Firms would like to commit to charge a lower price of $v - s$, but cannot do so; once a consumer enters their shop they have an incentive to defect to charging a price v anyhow. Hence, the market will break down and cease to exist.

We can get around this by assuming that each consumer has an individual downward sloping demand function $D(p)$. At a monopoly price of p_m, each consumer then demands more than one unit and still has some consumer surplus S. Firms do not have an incentive to defect to a higher price: Again consumers will not switch for a small price increase, but a firm is not willing to defect such a manner, as by definition it would decrease its profits. As long as its consumer surplus S this is bigger than the search cost s, the market would still exist, as the consumer would still be willing to pay the first visit.

2.3.3 Implications

This result, that even infinitessimally small search costs are enough to switch from an equilibrium with marginal cost pricing to one with monopoly pricing, is known as the *Diamond paradox*. It shows that even the slightest perturbation of the original model is already enough to end up in the other extreme outcome.

Another surprising outcome of the model is that, although we have introduced search costs, people do not actually end up searching in the equilibrium of this model: in equilibrium, they also end up buying from the first firm they encounter.

To find a way out of the Diamond paradox, we can do two things. First, we can assume that not all consumers have positive search costs. If some consumers can observe all prices, firms may still have an incentive to defect from the monopoly price. Second, we may assume that products are differentiated. We will discuss these two approaches in what follows. But before we can do so, we first have to delve somewhat deeper in optimal consumer search.

2.4 Optimal search

Suppose firms sell homogeneous products and prices at each firm are drawn from a cumulative distribution $F(p)$, with a different draw for each firm. There are n firms, and the consumer has unit demand. A consumer incurs search costs s per firm it visits.

We focus on *sequential search*: after each firm a consumer visits, he can decide whether to continue search. With *sequential search* the consumer has to decide beforehand how many firms to visit. With *perfect recall* a consumer can decide at some point in the process that he wants to go back to a firm he visited before, he can do so for free.

Consider $n = 2$. Our consumer has visited 1 firm, denote as firm 1. At that firm, price p_1 is observer. Suppose firm 2 is visited. If a price $p_2 > p_1$ is observed, the consumer returns to firm 1 and buys there. If $p_2 < p_1$ the consumer buys from firm 2 and his utility would increase by $p_1 - p_2$. Hence, the benefit from visiting the second firm is given by:

$$b(p_1) = \int_0^{p_1} (p_1 - p_2)\, \mathrm{d}F(p_2) \tag{2.14}$$

The expected cost of visiting firm 2 is s. Hence the consumer is willing to visit the last firm whenever $p_1 > \hat{p}$, where $\hat{p}$ implicitly defined by:

$$b(\hat{p}) = s \tag{2.15}$$

For existence of $\hat{p}$, note that:

$$\begin{aligned} b(\overline{p}) &= \int_0^{\overline{p}} (\overline{p} - p_2)\, \mathrm{d}F(p) & (2.16) \\ &= \overline{p} - \mathbb{E}\left[p\right], \text{ and } b(0) = 0 & (2.17) \end{aligned}$$

$b(p)$ is strictly increasing in p. These observations imply that $\hat{p}$ always exists and is unique, provided that $s < \overline{p} - \mathbb{E}\left[p\right]$.

Now suppose $n = 3$. At the first firm again, p_1 is observed. Denote $B_1(p)$ as the expected net profit of doing one more search when the current best price is p, and there is just one firm left to search. Hence, $B_1(p) \equiv b(p) - s$. By construction $B_1(\hat{p}) = 0$. Denote $B_2(p)$ as the expected net benefit of doing another search when the current best price is p and there are two firms left to search.

We can use backward induction to evaluate $B_2(\hat{p})$. Suppose that our consumer is at the first of three shops, observer $p_1 = \hat{p}$ and considers whether it is worthwhile to to also visit shop 2. First suppose $p_2 < \hat{p}$. In that case the best price after visiting is p_2. As $B_1(p_2) \leq B_1(\hat{p}) = 0$ it will not be worthwhile to visit firm 3. Now suppose $p_2 \geq \hat{p}$. In that case the best price is still $p_1 = \hat{p}$. It will also not be worthwhile to visit firm 3 as by construction $B_1(\hat{p}) = 0$. In other words $B_2(p) = 0$, so the consumer will use the same reservation value regardless of whether there are 1 or 2 firms left to search. By induction this also holds for $3, 4, \cdots$ firms left to search. Hence, *the reservation price a consumer uses is independent of the number of firms she has left to search.* This implies to derive the optimal search rule, it is not even necessary for the consumer to know how many firms there are left to search.

2.5 Search with homogeneous products

Suppose there are n firms and a unit mass of consumers. Customers have unit demand and a willingness to pay of 1. A fraction λ of consumers has zero reach costs. The remaining fraction $1 - \lambda$ has reach costs $s \in (0, 1)$. Again assume that the first visit is free.

An equilibrium in pure strategies does not exist. Firms either have incentive to charge a slightly lower price to capture all the shoppers or a much higher price to make a substantial profit on the non-shoppers that happen to visit.

Given the reservation price $\hat{p}$ that non-shoppers will use we can derive the price distribution $F(p)$ that firms will use. Given the $F(p)$ that the firms will use we can derive the reservation price $\hat{p}$ that consumers will use.

Suppose that one firm charges a price $p > \hat{p}$. Then both non-shoppers and shoppers will not buy from this firm, hence this can not be part of an equilibrium. This implies in equilibrium all firms set $p \leq \hat{p}$ which implies there will be no search in equilibrium, all non-shoppers buy from the first firm they encounter. *The equilibrium thus necessarily has all shoppers buying from the cheapest firm, and all non-shoppers buying from the first firm they encounter.*

2.6 Search with differentiated products

Suppose there are n firms. Their costs are zero. There is a unit mass of consumers that all have search costs s. If a consumer buys from i, a utility is obtained:

$$U_j^i(p_i, \epsilon_i) = v + \epsilon_i - p_i, \tag{2.18}$$

where v is large enough such that s.t. a consumer always buys in equilibrium. We assume ϵ_i is the realization of a r.v. with CDF F and continuously differentiable density f. Assume F is log-concave. ϵ_{ij} can be interpreted as the match value between consumer j and product i. To find out this value, j first has to visit firm i. By assumption, firm i never finds out ϵ_{ij}.

Assume that all firms charge some price p^*. Suppose our consumer visits firm i and finds match value ϵ_{ij}. Buying from i yields utility $v + \epsilon_{ij} - p^*$. A visit to some other firm k will give utility $v + \epsilon_{kj} - p^*$. This is higher than utility of buying from i if $\epsilon_{kj} > \epsilon_{ij}$. Then the expected benefit from searching once more is then:

$$b(\epsilon_{ij}) = \int_{\epsilon_{ij}}^{\infty} (\epsilon - \epsilon_{ij}) \, \mathrm{d}F(\epsilon) \tag{2.19}$$

The consumer searches until a match value that is at least equal to $\hat{\epsilon}$, where $b(\hat{\epsilon}) = s$. Given that s is low enough, such a $\hat{\epsilon}$ always exists and is unique.

2.6.1 2 firms, uniform distribution

If all firms charge p^* and the consumer finds ϵ_i at the first firm, the expected benefit from visiting the other firm is:

$$b(\epsilon_i) = \frac{(1 - \epsilon_i)^2}{2} \tag{2.20}$$

Since visiting one more firm costs s, you will do so whenever $b(\epsilon_i) > s$, hence if $\epsilon_i > \hat{\epsilon}$, with:

$$\hat{\epsilon} = 1 - \sqrt{2s} \tag{2.21}$$

When $s = 0$, consumers will continue searching for any value of ϵ they find at the first firm. The higher s, the lower $\hat{\epsilon}$, hence the earlier they are going to settle for the product they find at the first firm they visit.

Suppose that a consumer visits firm i first and finds price p. If this consumer buys from i, utility $v + \epsilon_i - p$ is obtained. A visit to the other firm will give utility $v + \epsilon_j - p^*$.

Hence, j is preferred over i whenever $\epsilon_j > \epsilon_i + \Delta$, with $\Delta \equiv p^* - p$. If prices were equal the consumer would buy from i whenever $\epsilon_i > \hat{\epsilon}$. But if firm i now charges a price that is Δ lower, buying from i gives an additional utility of Δ relative to buying from the other firm. Hence, this consumer now buys from i whenever $\epsilon_i + \Delta > \hat{\epsilon}$.

The next step is to find the demand of firm i if it charges p rather than p^*. There are 3 ways in which a consumer ends up buying from firm i.

First, with probability $\frac{1}{2}$ firm i is visited first, and with this consumer buys right away if $\epsilon_i > \hat{\epsilon} - \Delta$, which happens with probability $1 - \hat{\epsilon} + \Delta$. Hence the probability this occurs is:

$$\frac{1}{2}(1 - \hat{\epsilon} + \Delta) \tag{2.22}$$

Second, with probability $\frac{1}{2}$ the consumer visits firm j first. With probability $\hat{\epsilon}$ a match value that is too low is found. Then a match that is high enough at firm i is found with probability $1 - \hat{\epsilon} + \Delta$. Hence the joint probability of this is:

$$\frac{1}{2}\hat{\epsilon}(1 - \hat{\epsilon} + \Delta) \tag{2.23}$$

Third, it could be the case that she visits both firms, but finds a match value at both firms that is too low. Then, she will just go back to the firm that with hindsight offers the best deal. The consumer buys from i in this way when all of the following conditions hold:

$$\epsilon_i < \hat{\epsilon} - \Delta \tag{2.24}$$
$$\epsilon_j < \hat{\epsilon} \tag{2.25}$$
$$\epsilon_i > \epsilon_j - \Delta \tag{2.26}$$

This happens with probability:

$$\frac{1}{2}(\epsilon^2 - \Delta^2) \tag{2.27}$$

Then we can sum 2.22, 2.23 and 2.27 to obtain the probability that a firm buys from i:

$$D_i(p^*, \Delta) = \frac{1}{2}(1 - \hat{\epsilon} + \Delta) + \frac{1}{2}\hat{\epsilon}(1 - \hat{\epsilon} + \Delta) + \frac{1}{2}(\epsilon^2 - \Delta^2) \tag{2.28}$$
$$= \frac{1}{2} + \frac{1}{2}\Delta(1 + \hat{\epsilon} - \Delta) \tag{2.29}$$

Note if both firms charge the same price, hence $\Delta = 0$, $D_i = \frac{1}{2}$.

Using the definition of Δ:

$$D_i(p^*, p) = \frac{1}{2} + \frac{1}{2}(p^* - p)(1 + \hat{\epsilon} - p^* + p) \tag{2.30}$$

Profits of firm i equal $\pi_i = pD_i$, FOC gives:

$$\frac{\partial D_i}{\partial p} + D_i = 0 \tag{2.31}$$

To solve for equilibrium, impose symmetry, hence $D_i = \frac{1}{2}$. Moreover:

$$\frac{\partial D_i}{\partial p} = p^* - p - \frac{1}{2}(1 + \hat{\epsilon}) \tag{2.32}$$

$$= -\frac{1}{2}(1 + \hat{\epsilon}), \text{ with symmetry} \tag{2.33}$$

Hence, the equilibrium price is:

$$p^* = \frac{1}{1 + \hat{\epsilon}} \tag{2.34}$$

$$= \frac{1}{2 - \sqrt{2s}}, \text{ using 2.21} \tag{2.35}$$

Hence when the search costs are zero, the price equals $\frac{1}{2}$, and the equilibrium prices increase with the search costs. With higher search costs, a consumer that visits a firm is less likely to walk away, hence firms have more market power and hence charge higher prices.

2.6.2 Solution of the general model

Suppose all other firms charge p^* and firm 1 defects to some $p \neq p^*$. Consumers expect all firms to charge p^*. Suppose a consumer visits firm i first and finds price p. If this consumer buys from i, utility $v + \epsilon_i - p_i$. A visit to some other firm j gives $v + \epsilon_j - p^*$. Hence, j is preferred over i if $\epsilon_j > \epsilon_i - \Delta$, with $\Delta \equiv p_i - p^*$. If the consumer buys from j rather than i the benefit is $\epsilon_j - (\epsilon_i - \Delta)$. Hence, the expected benefit of searching one more time is:

$$b(\epsilon_i; \Delta) = \int_{\epsilon_i - \Delta}^{\infty} (\epsilon - (\epsilon_i - \Delta)) \, \mathrm{d}F(\epsilon) \tag{2.36}$$

The consumer buys from i whenever $\epsilon_i > \hat{\epsilon} + \Delta$.

All consumers visit firms in a random order, so a share $\frac{1}{n}$ visits firm 1 first. They buy there with probability $1 - F(\hat{\epsilon} + \Delta)$. Another share $\frac{1}{n}$ "plans" to visit 1 second (if they don't like the product enough they find at the first firm they visit). These consumers buy at their first firm with probability $1 - F(\hat{\epsilon})$. Hence, the probability that such a consumer buys from firm 1 is $F(\hat{\epsilon})(1 - F(\hat{\epsilon} + \Delta))$. This pattern continues.

Some customers visit all n firms, but find a match value below $\hat{\epsilon}$ every time. Such a consumer returns to the firm that, with hindsight, offered the highest match value. The total share of consumers that buy from 1 this way is:

$$R_1(p_1, p^*) = \int_{-\infty}^{\hat{\epsilon} + \Delta} F(\epsilon - \Delta)^{N-1} \, \mathrm{d}F(\epsilon). \tag{2.37}$$

Such a consumer has $\epsilon_1 < \hat{\epsilon} + \Delta$ and all other match values smaller than $\epsilon - \Delta$.

Combining terms, total demand for firm 1 when charging p_1 while others charge p^* is:

$$D_1(p_1, p^*) = \frac{1}{n} \sum_{i=1}^{n} F(\hat{\epsilon})^{i-1}(1 - F(\hat{\epsilon} + \Delta)) + R_1(p_1, p^*) \tag{2.38}$$

$$= \frac{1}{n} \left[\frac{1 - F(\hat{\epsilon})^n}{1 - F(\epsilon)} \right] (1 - F(\hat{\epsilon} + \Delta)) + \int_{-\infty}^{\hat{\epsilon}+\Delta} F(\epsilon - \Delta)^{n-1} \,\mathrm{d}F(\epsilon). \tag{2.39}$$

Profits of firm 1 are given by $\pi_1(p_1, p^*) = p_1 D_1(p_1, p^*)$. Take the FOC and set $\frac{\partial \pi_1(p_1,p^*)}{\partial p_1} = 0$. Equilibrium requires that this is satisfied for $p_1 = p^*$. Note:

$$\frac{\partial \pi_1(p_1, p^*)}{\partial p_1} = D_1(p_1, p^*) + p_1 \frac{\partial D_1(p_1, p^*)}{\partial p_1} = 0, \tag{2.40}$$

and:

$$\frac{\partial D_1(p_1, p^*)}{\partial p_1} = -\frac{1}{n} \left[\frac{1 - F(\hat{\epsilon})^n}{1 - F(\hat{\epsilon})} \right] f(\hat{\epsilon} + \Delta) + F(\hat{\epsilon})^{n-1} f(\hat{\epsilon} + \Delta) \tag{2.41}$$

$$- (n-1) \int_{-\infty}^{\hat{\epsilon}+\Delta} F(\epsilon - \Delta)^{n-2} f^2(\epsilon) \,\mathrm{d}\epsilon. \tag{2.42}$$

Imposing symmetry, and using integration by parts:

$$\frac{\partial D_1(p^*, p^*)}{\partial p_1} = -\frac{1}{n} \left[\frac{1 - F(\hat{\epsilon})^n}{1 - F(\hat{\epsilon})} \right] f(\hat{\epsilon}) \tag{2.43}$$

$$+ F(\hat{\epsilon})^{n-1} f(\hat{\epsilon}) - (n-1) \int_{-\infty}^{\hat{\epsilon}} F(\epsilon)^{n-2} f^2(\epsilon) \,\mathrm{d}\epsilon \tag{2.44}$$

$$= -\frac{1}{n} \left[\frac{1 - F(\hat{\epsilon})^n}{1 - F(\hat{\epsilon})} \right] + \int_{-\infty}^{\hat{\epsilon}} f'(\epsilon) F(\epsilon)^{n-1} \,\mathrm{d}\epsilon. \tag{2.45}$$

Firms are symmetric and in equilibrium all consumer buy. This implies that $D_1(p^*, p^*) = \frac{1}{n}$. Using 2.40, again imposing symmetry:

$$p^* = -\frac{D_1(p^*, p^*)}{\partial D_1(p^*, p^*)/\partial p_1} = \left(\frac{1 - F(\hat{\epsilon})^n}{1 - F(\hat{\epsilon})} f(\hat{\epsilon}) - n \int_{-\infty}^{\hat{\epsilon}} f'(\epsilon) F(\epsilon)^{n-1} \,\mathrm{d}\epsilon \right)^{-1}. \tag{2.46}$$

Assume now that F is $\mathrm{Unif}(0, 1)$, then:

$$D_1(p_1, p^*) = \frac{1}{n} \left[\frac{1 - \hat{\epsilon}^n}{1 - \hat{\epsilon}} \right] (1 - \hat{\epsilon} - \Delta) + \int_{-\infty}^{\hat{\epsilon}+\Delta} (\epsilon - \Delta)^{n-1} \,\mathrm{d}\epsilon, \tag{2.47}$$

so:

$$\frac{\partial D_1(p^*, p^*)}{\partial p_1} = -\frac{1}{n} \left[\frac{1 - \hat{\epsilon}^n}{1 - \hat{\epsilon}} \right]. \tag{2.48}$$

Under this assumption the derivative of the number of returning consumers with respect to p_1, that is, $\frac{\partial R_1(p_1,p^*)}{\partial p_1}$, is equal to zero. The equilibrium price now equals:

$$p^* = \frac{1 - \hat{\epsilon}}{1 - \hat{\epsilon}^n}. \tag{2.49}$$

from 2.19:

$$b(\hat{\epsilon}) = \int_{\hat{\epsilon}}^{1} (\epsilon - \hat{\epsilon})\,\mathrm{d}\epsilon = \frac{1}{2}(1-\hat{\epsilon})^2, \tag{2.50}$$

hence:

$$\hat{\epsilon} = 1 - \sqrt{2s} \tag{2.51}$$

$$\implies p^* = \frac{\sqrt{2s}}{1-(1-\sqrt{2s})^n}. \tag{2.52}$$

Suppose the number of firms increases (this does not affect $\hat{\epsilon}$). Hence:

$$\frac{\partial p^*}{\partial n} = \frac{\partial}{\partial n}\left(\frac{1-\hat{\epsilon}}{1-\hat{\epsilon}^n}\right) = \hat{\epsilon}^n \ln[\hat{\epsilon}]\,\frac{1-\hat{\epsilon}}{(1-\hat{\epsilon}^n)^2} < 0, \tag{2.53}$$

as $\ln[\hat{\epsilon}] < 0$. Hence, *having more firms leads to lower prices.*

Suppose the search costs increase. Note $\hat{\epsilon}$ is defined by $b(\hat{\epsilon}) = s$. $b(\hat{\epsilon})$ is decreasing in ϵ, so we have that an increase in search costs leads to a lower $\hat{\epsilon}$ (If search costs are higher, a consumer is willing to settle for a lower match value). We have:

$$\frac{\partial p^*}{\partial \hat{\epsilon}} = \frac{\partial}{\partial \hat{\epsilon}}\left(\frac{1-\hat{\epsilon}}{1-\hat{\epsilon}^n}\right) = \frac{(1-\hat{\epsilon})n\hat{\epsilon}^{n-1} - (1-\hat{\epsilon}^n)}{(1-\hat{\epsilon}^n)^2} = \frac{\hat{\epsilon}^{n-1}(n(1-\hat{\epsilon})+\hat{\epsilon}) - 1}{(1-\hat{\epsilon}^n)^2} < 0 \tag{2.54}$$

Hence, *equilibrium prices are increasing in search costs*; as search costs increase, firms have more market power over the consumers that do visit them, which implies that they can set higher prices.

2.7 Exercises

1. Consider Varian's model of sales as described in Section 2.3. Suppose the number of firms is $n > 2$. Derive the mixed strategy equilibrium.

   Solution: Now the profits in equilibrium are $\frac{1}{n}(1-\lambda)$. With probability $(1 - F(p))^{n-1}$, firm i charging price p sets the lowest price. Then expected profits are:

   $$\mathbb{E}[\pi_i(p)] = \frac{1}{n}(1-\lambda)p + (1-F(p))^{n-1}\lambda p.$$

   The lower bound is still $\underline{p} = \frac{1-\lambda}{1+\lambda}$, then equate expected profits with equilibrium profits:

   $$F(p) = 1 - \left(\frac{(1-\lambda)(1-p)}{n\lambda p}\right)^{\frac{1}{n-1}}.$$

2. Consider the following variation of Varian (1980)'s model of sales. A duopoly competes in prices. Costs of production are zero. There is a mass 1 of consumers that is willing to pay at most 2, and a mass 1 of consumers that is willing to pay at most 3. For both groups there is a fraction λ that is informed and a fraction $1-\lambda$ that is uninformed.

(a) Derive the best-reply function of each firm.
(b) Derive the Nash equilibrium in prices.

Solution:

(a) Setting $p = 2$ is always more profitable than setting price $p > 2$ for either firm, regardless of the choice of the other firm. Since even if $p_2 = 0$ (i.e. all informed customers go to firm 2) then still $p_1 = 2$ gives profit $\pi_1 = 2(1 - \lambda)$ while $p_1 = 3$ gives profits $\pi_1 = \frac{3(1-\lambda)}{2}$, which is clearly smaller. This difference increases even further if $p_2 \geq 2$.
Suppose that $p_2 = 2$. Then there are 3 choices for firm 1:
   - $p_1 = 2 \Longrightarrow q_1 = 1 \Longrightarrow \pi_1 = 2$
   - $p_1 = p_2 - \epsilon$ with ϵ arbitrarily small. This gives demand $q_1 = 2\lambda + (1-\lambda) = 1 + \lambda \Longrightarrow \pi_1 = 2(1 + \lambda)$

   It is clear that the third choice is the best option.
Suppose $p_2 < 2$. Then there are 2 choices for firm 1:
   - $p_1 = 2 \Longrightarrow q_1 = 1 - \lambda \Longrightarrow \pi_1 = 2(1 - \lambda)$
   - $p_1 = p_2 - \epsilon$ with ϵ arbitrarily small. This gives demand $q_1 = 2\lambda + (1-\lambda) = 1 + \lambda \Longrightarrow \pi_1 = p_2(1 + \lambda)$

   The first option is optimal if $p_2(1+\lambda) \leq 2(1-\lambda) \Longleftrightarrow p_2 \leq \frac{2(1-\lambda)}{1+\lambda}$. This yields the best-reply function:

$$p_1 = \begin{cases} 2 & \text{if } p_2 \leq \frac{2(1-\lambda)}{1+\lambda} \\ p_2 - \epsilon & \text{otherwise} \end{cases}$$

(b) There is no equilibrium in pure strategies, we need to look for a mixed-strategy equilibrium on the interval $[\underline{p}, \overline{p}]$, where we now from the previous part that $\overline{p} = 2$.
If a firm sets $p = 2$ its price is the highest for sure. This gives profits $2(1 - \lambda)$. A firm that sets price p will sell $1 + \lambda$ if this price is the lowest. Otherwise, it will only sell $1 - \lambda$. Then the expected profits are:

$$F(p)(1 - \lambda)p + (1 - F(p))(1 + \lambda)p = p + p\lambda(1 - 2F(p))$$

   For a mixed strategy equilibrium, this expected profit needs to equal the profit of $p = 2$. Hence:

$$F(p) = \frac{p - 2 + \lambda(p + 2)}{2p\lambda}$$

   We can find lower bound $\underline{p}$ by solving $F(p) = 0$. Hence, $\underline{p} = \frac{2(1-\lambda)}{1+\lambda}$.

3. Consider a model of search with differentiated products as analyzed in this chapter. There are two firms that each have zero costs. Match values are uniformly distributed on $[0, 1]$. A share λ of consumers have search costs $s = 0.25$. The other $1 - \lambda$ have search costs $s = 0.16$. Derive the equilibrium in prices.

Solution: Suppose both firms charge price p. Let's refer to the low search cost consumers with L and to the high search cost consumers with H. We have $\hat{\epsilon} =$

$1-\sqrt{2s}$, so $\hat{\epsilon}_L = 1-0.4\sqrt{2}$ and $\hat{\epsilon}_H = 1-0.5\sqrt{2}$. Suppose firm A defects to price $p_A = p-\Delta$. Let $\hat{\epsilon}_A = \lambda\hat{\epsilon}_H + (1-\lambda)\hat{\epsilon}_L$. Then (As per the result in ch. 2.6.1):

$$\begin{aligned} q_A &= \frac{1}{2} + \frac{1}{2}\Delta(1+\hat{\epsilon}_A - \Delta) \\ &= \frac{1}{2}(p-p_A)(1+\hat{\epsilon}_A - p + p_A) \\ &= \frac{1}{2} + \frac{1}{2}\left(p(1+\hat{\epsilon}_A - p) + p_A(2p-1-\hat{\epsilon}_A - p_A)\right). \end{aligned}$$

Profits are $\pi_A = p_A q_A$, hence the FOC is:

$$\frac{\partial q_A}{\partial p_A} + q_A = 0$$

We know:

$$\begin{aligned} \frac{\partial q_A}{\partial p_A} &= p - p_A - \frac{1}{2}(1+\hat{\epsilon}_A) \\ \text{(with symmetry)} &= -\frac{1}{2}(1+\hat{\epsilon}_A). \end{aligned}$$

With symmetry, $q_A = \frac{1}{2}$. Then fill in into the FOC:

$$\begin{aligned} -\frac{1}{2}(1+\hat{\epsilon}_A)p^* + \frac{1}{2} &= 0 \\ \implies p^* &= \frac{1}{1+\hat{\epsilon}_A}. \end{aligned}$$

$\hat{\epsilon}_A = \lambda\hat{\epsilon}_H + (1-\lambda)\hat{\epsilon}_L = 1-\sqrt{2}(0.4+0.1\lambda)$. So the final answer is:

$$p^* = \frac{1}{2-\sqrt{2}(0.4+0.1\lambda)}$$

4. In the model of search with differentiated products described in these notes, we assumed that consumers pick a firm at random to visit first. Let us now assume that that is no longer the case, and can advertise to attract consumers. More precisely, if firm 1 puts out a_1 ads and firm 2 puts out a_2, then the probability that firm 1 will be visited first is given by $a_1/(a_1+a_2)$. The cost of each ad are $1/4$, Suppose that match values are uniformly distributed on $[0,1]$, and that firms set prices and advertising levels simultaneously. Derive equilibrium profits as a function of search costs s.

   Solution: We now have demand for firm 1:

   $$D_1 = \frac{a_1}{a_1+a_2}(1-\hat{\epsilon}-\Delta) + \frac{a_2}{a_1+a_2}(1-\hat{\epsilon}-\Delta)\hat{\epsilon} + \frac{1}{2}\hat{\epsilon}^2.$$

   Fill in $\Delta = p_1 - p^*$ and calculate the derivative:

   $$\frac{\partial D_1}{\partial p_1} = -\frac{a_1}{a_1+a_2} - \frac{a_2}{a_1+a_2}\hat{\epsilon}.$$

Total profits are equal to $\pi_1 = p_1 D_1 - \frac{1}{4}a_1$. Then calculate the FOC's wrt. to p_1 and a_1:

$$\frac{\partial \pi_1}{\partial p_1} = D_1 + p_1 \frac{\partial D_1}{\partial p_1},$$
$$\frac{\partial \pi_1}{\partial a_1} = \frac{a_2}{(a_1+a_2)^2}(1-\hat{\epsilon}-\Delta)p_1 - \frac{a_2}{(a_1+a_2)^2}(1-\hat{\epsilon}-\Delta)\hat{\epsilon}p_1 - \frac{1}{4} = 0.$$

Then impose symmetry:

$$\frac{\partial \pi_1}{\partial p_1} = \frac{1}{2} + p^* \left(-\frac{1}{2} - \frac{1}{2}\hat{\epsilon}\right) = 0,$$
$$\frac{\partial \pi_1}{\partial a_1} = \frac{1}{4a}(1-\hat{\epsilon})p^* - \frac{1}{4a}(1-\hat{\epsilon})\hat{\epsilon}p^* - \frac{1}{4} = 0.$$

Hence:

$$p^* = \frac{1}{1+\hat{\epsilon}},$$
$$a^* = (1-\hat{\epsilon})^2 p^*.$$

Then fill this into the profit function (with $\hat{\epsilon} = 1 - \sqrt{2s}$):

$$\pi^* = \frac{1}{2}p^* - \frac{1}{4}a^* = \frac{1}{4}\frac{1 + 2(1-\sqrt{2}s) - (1-\sqrt{2s})^2}{2-\sqrt{2s}}.$$

3 Advertising

3.1 The Butters Model

3.1.1 Setup

Firms produce a homogeneous product at unit cost c. There is free entry of firms and there are N consumers that can only learn a firm's existence and price by receiving an ad from that firm. Each firm decides how many ads to send, and which price to put in those ads. Ads are randomly distributed among customers at a cost of k per ad. Consumers have unit demand and are willing to pay at most R. We assume that $R > c + k$. We are interested in the equilibrium in terms of prices and advertising levels, and whether the market provides too much or too little advertising from a welfare perspective.

3.1.2 Analysis

Suppose that A ads are sent. There will be three kinds of consumers. Some customers receive no ads (uninformed). Some consumers only receive ads from 1 firm (captive) and some consumers receive ads from both firms (selective). These customers buy from the lowest price.

Let Φ denote the probability that a consumer receives at least 1 ad. The probability that a consumer is uninformed equals $1-\Phi = \left(1-\frac{1}{N}\right)^A$. If N is large enough, this is approximately $e^{\frac{A}{N}}$. Thus, if a proportion of Φ consumers are to receive at least 1 ad. Then the number of ads that has to be sent equals:

$$A(\Phi) = N \ln\left[\frac{1}{1-\Phi}\right]. \tag{3.1}$$

A pure strategy equilibrium in prices does not exist. If all other firms charge some price $p \in (k+c, R]$, then this firm has incentive to slightly undercut. If all other firms charge $p = k+c$ however, then this firm has incentive to charge price $p = R$ and sell to its captive consumers. The equilibrium has firms mixing on $[k+c, R]$. For any price below $k+c$ a firm would be better off not sending ads.

There is free entry of firms, implying that equilibrium profits are zero. Denote by $x(P)$ the probability that an ad with price P will be accepted by the customer receiving it. Then $x(P)$ is the probability that a consumer does not receive an ad with a lower price. This is decreasing in P. Equilibrium requires that for each $P \in [k+c, R]$, we have:

$$(P-c)x(P) - k = 0. \tag{3.2}$$

This implies that $x(c+k) = 1$ and $x(R) = \frac{k}{R-c}$. In equilibrium we also need that a consumer will accept price R exactly equals the probability that he does not receive any other ad, so $x(R) = 1-\Phi$. Equilibrium thus requires:

$$\Phi^* = \frac{1-k}{R-c}. \tag{3.3}$$

3.1.3 Social optimum

Now consider the amount of advertising that a social planner would choose. Prices are just transfers between consumers and firms, so they will not affect total welfare. When one additional consumer that was initially uninformed learns about the existence of some firm, the social benefit is $R-c$. Thus, social benefits are $N\Phi(R-c)$. However, there is a cost of reaching these consumers, which is given by (3.1). The social planner thus maximizes:

$$\max_{\Phi}\left\{\Phi N(R-c) - kN\ln\left[\frac{1}{1-\Phi}\right]\right\}. \tag{3.4}$$

Taking FOC:

$$N(R-c) - \frac{kN}{1-\Phi} = 0. \tag{3.5}$$

Solving for Φ yields:

$$\Phi^S = \frac{1-k}{R-c}. \tag{3.6}$$

This implies that $\Phi^S = \Phi^*$. Thus, the market provides the socially optimal level of advertising.

This can be understood as follows. Consider the private benefit to a firm of sending an ad at the price R. This benefit equals $(R - c)$ times the probability that the consumer receives no other ad. But this is also the social benefit of sending an ad, since the ad increases social surplus by $R - c$ but only if no other ads are received by the consumer. Thus, the highest-priced firm appropriates all consumer surplus and steals no business from rivals. Therefore, it advertises at the socially optimal rate. Now consider an ad at a price lower than R. The private benefits to a firm of sending such an ad are equal to those of sending an ad at price $P = R$, as expected profits are always zero. This implies that social benefits also equal private benefits at any price below R.

3.2 Informative advertising, differentiated products

Consumers are uniformly distributed on a line of unit length. Each has unit demand and is willing to pay at most R, but also faces transportation costs t per unit of distance. Two firms are located at 0 and 1. Ads are sent randomly. The cost of reaching a fraction of Φ_i is denoted by $A(\Phi_i)$. Suppose that $A(\Phi_i) = \frac{a(\Phi_i)^2}{2}$, with $a > \frac{t}{2}$.

There are three kinds of customers. Suppose that firms 1 and 2 inform fractions Φ_1 and Φ_2 of consumers. A fraction $(1 - \Phi_1)(1 - \Phi_2)$ is uninformed. A fraction $\Phi_1(1 - \Phi_2)$ is captive on firm 1 (and similarly for firm 2). A fraction $\Phi_1\Phi_2$ is selective. Suppose that the market is always covered, in the sense that a customer who has received at least 1 ad will always buy. Also assume that the number of selective consumers is sufficiently large that firms are willing to compete for them. This is true if advertising is not too costly. Firm 1's demand function is:

$$D_1(P_1, P_2, \Phi_1, \Phi_2) = \Phi_1 \left[(1 - \Phi_2) + \Phi_2 \frac{(P_2 - P_1 + t)}{2t}\right]. \tag{3.7}$$

Consider a game in which firms simultaneously choose price and advertising levels. Profits of firm 1 equal:

$$\begin{aligned} \pi_1 &= (P_1 - c)D_1 - A(\Phi_1) && (3.8)\\ &= (P_1 - c)\Phi_1 \left[(1 - \Phi_2) + \Phi_2 \frac{(P_2 - P_1 + t)}{2t}\right] - \frac{a(\Phi_1)^2}{2}. && (3.9)\end{aligned}$$

Taking first order condition wrt. P_1:

$$\frac{\partial \pi_1}{\partial P_1} = \Phi_1 \left[(1 - \Phi_1) + \Phi_2 \frac{P_2 - 2P_1 + c + t}{2t}\right] = 0. \tag{3.10}$$

This gives the reaction fuction:

$$P_1 = \frac{P_2 + t + c}{2} + \frac{1 - \Phi_2}{\Phi_2} t. \tag{3.11}$$

Now taking first order condition wrt. to Φ_1:

$$a\Phi_1 = (P_1 - c)\left[(1 - \Phi_2) + \Phi_2 \frac{P_2 - P_1 + t}{2t}\right]. \tag{3.12}$$

We can impose symmetry and solve (3.11) and (3.12) simultaneously. From (3.11):

$$p^* = c + t\frac{2 - \Phi^*}{\Phi^*}. \tag{3.13}$$

From (3.12):

$$a\Phi = t\frac{2-\Phi}{\Phi}\left((1-\Phi) + \frac{\Phi}{2}\right) \tag{3.14}$$

$$\iff a\Phi^2 = \left(1 - \frac{1}{2}\Phi\right)^2 \tag{3.15}$$

$$\implies \sqrt{a}\Phi = \left(1 - \frac{1}{2}\Phi\right). \tag{3.16}$$

Hence:

$$\Phi^* = \frac{2}{1 + \sqrt{2a/t}}, \tag{3.17}$$

$$\text{and } P^* = c + \sqrt{2at}. \tag{3.18}$$

Equilibrium profits are:

$$\Pi^* = \frac{2a}{\left(1 + \sqrt{2a/t}\right)^2}. \tag{3.19}$$

The equilibrium price is higher than in the case without advertising. The equilibrium advertising level is higher when advertising is less costly, and when products are more differentiated (that is, when t is higher). Most surprisingly, equilibrium profits are increasing in the cost of advertising. When a increases, this implies an increase in costs, but will also lower the equilibrium level of advertising. In this model, the net effect is positive.

From a welfare point of view, prices are just transfers from consumers to firms. Consumers that are informed about the existence of both firms will face transportation costs $\frac{t}{4}$ on average. Those that only know the existence of one firm face transportation costs $\frac{t}{2}$ on average. Hence, social welfare is given by:

$$\mathrm{SW}(\Phi) = \Phi^2(v - c - t/4) + 2\Phi(1-\Phi)(v - c - t/2) - a\Phi^2. \tag{3.20}$$

Maximizing wrt. to Φ yields:

$$\Phi^S = \frac{2(v-c) - t}{2(v-c) + 2a - 3t/2}. \tag{3.21}$$

This implies that the market equilibrium can either have too much or too little advertising, depending on the exact parameters of the model.

3.3 Exercises

1. (a) Consider the Grossman/Shapiro model as described in the lecture notes, but with $t = 1$ and $c = 0$. However, different from the model in the notes, half of

consumers that are not informed by the ads of firm 1, still learn about product 1 through word-of-mouth. Something similar holds for firm 2. Write down the profit functions for this case, and derive the system of equations that pin down equilibrium prices and equilibrium numbers of informed consumers (hence do not explicitly solve for those values; that is too much hassle. But do write equilibrium prices in terms of equilibrium fractions of informed consumers and vice versa)

(b) Suppose we would do the same in a Butters model (that is, assume that half of consumers that are not informed by the ads of firm 1, still learn about product 1 through word-of-mouth). Argue whether the result that the amount of ads are socially optimal would still hold (hence do not derive, only give an intuitive argument).

Solution:

(a) The number of consumers that is now informed about firm 1 is:

$$\phi_i = \Phi_1 + \frac{1}{2}(1 - \Phi_1) = \frac{1}{2}(1 + \Phi_1).$$

Then demand is:

$$D_1 = \phi_1 \left[(1 - \phi_2) + \phi_2 \left(\frac{1}{2} + \frac{P_2 - P_1}{2t} \right) \right]$$

Hence, the profit is:

$$\begin{aligned}
\pi_1 &= (P_1 - c)D_1 - \frac{a(\Phi_1)^2}{2} \\
\Longrightarrow \frac{\partial p_1}{\partial P_1} &= \phi_1 \left[(1 - \phi_2) + \phi_2 \left(\frac{1}{2} + \frac{P_2 - 2P_1}{2t} \right) \right] = 0 \\
\Longrightarrow P_1 &= \frac{P_2 + t}{2} + \frac{1 - \phi_2}{\phi_2} \\
\Longrightarrow a\Phi_1 &= \frac{1}{2} P_1 \left[1 - \frac{1}{2}(1 + \Phi_1) + \frac{1}{2}(1 + \Phi_2) \left(\frac{1}{2} + \frac{P_2 - 2P_1}{2t} \right) \right].
\end{aligned}$$

(b) Probably: the same intuition still holds. The spillovers in advertisements (in the sense that now ads effectively will reach more people due to word of mouth) will be internalized (i.e., taken into account when setting the advertisement intensity) both by the social planner and in the decentralized equilibrium. Firms will still mix in equilibrium and an ad with the highest price will still capture all the consumer surplus and therefore still be put out at the socially optimal rate. Because of the mixed strategy equilibrium ads with other prices generate the same private benefits as the highest-priced ad and hence will also be put out at the socially optimal rate.

2. Solve the Grossman-Shapiro model in the case that preferences of consumers are given by the Perloff-Salop model described in the first chapter, rather than the Hotelling model. (Do not solve for the welfare optimum: that becomes too complicated.)

Solution: In equilibrium (equal prices and ads), firm 1 sells to half the informed consumers:

$$q_1 = \frac{1}{2}(1-(1-\Phi)^2) = \frac{1}{2}\Phi(2-\Phi).$$

Consider when firm 1 defects in both price and advertising:

$$q_1 = \Phi_1\left[(1-\Phi_2) + \frac{1}{2}\Phi_2(1-(p_1-p))^2\right].$$

Then profits are:

$$\pi_1 = (p_1-c)q_1 - \frac{1}{2}a\Phi_1^2.$$

Take FOC:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial p_1} &= (p_1-c)\frac{\partial q_1}{\partial p_1} + q_1 = 0 \\
\frac{\partial \pi_1}{\partial \Phi_1} &= \left[(1-\Phi_2) + \frac{1}{2}\Phi_2(1-(p_1-p))^2\right](p_1-c) - a\Phi_1,
\end{aligned}$$

where:

$$\frac{\partial q_1}{\partial p_1} = -\Phi_1\Phi_2(1-(p_1-p)).$$

Then we impose symmetry:

$$\begin{aligned}
-(p-c)\Phi^2 + q_1 &= 0 \\
\left(1-\Phi+\frac{1}{2}\Phi\right)(p-c) - a\Phi &= 0 \\
&\Longrightarrow \\
p-c-\frac{2-\Phi}{2\Phi} &= 0 \\
\left(1-\frac{1}{2}\Phi\right)(p-c) - a\Phi &= 0 \\
&\Longrightarrow \\
&\quad p = c + \frac{2-\Phi}{2\Phi} \\
&\quad \Phi = \frac{4-8\sqrt{a}}{2-8a}
\end{aligned}$$

4 Menu Pricing

4.1 Menu pricing

A monopolist wants to price discriminate, but there is no exogenous signal of each customer's demand function. In other words: the monopolist is unable to divide the customers into groups purely on the basis of some exogenous variable such as age. Some consumers have a higher elasticity of demand than others, but the monopolist is not able to tell which is which. The monopolist can offer a menu of bundles where the consumers can choose from.

4.1.1 The Model

Assume that the consumers have a surplus of consuming quality q of a good that equals $U(\theta, q)$, where θ is a parameter that differs per consumer. The cost of providing quality q is $c(q)$. We make the following assumptions:

1. $U(\theta, 0) = 0$
2. $\frac{\partial U}{\partial q} > 0$
3. $\frac{\partial^2 U}{\partial q^2} < 0$
4. $c' \geq 0$
5. $c'' \geq 0$

We assume that there are two types of consumers, with taste parameters θ_1, θ_2, with $\theta_1 < \theta_2$. The proportion of type-1 consumers is λ, hence $(1 - \lambda)$ type 2. The costs of providing a product with quality q is given by $c(q)$, with $c' > 0$ and $c'' \geq 0$.

Suppose the monopolist offers two packages. One is denoted (p_1, q_1), the other (p_2, q_2). We assume the following single-crossing condition is satisfied:

$$U(\theta_2, q_2) - U(\theta_2, q_1) \geq U(\theta_1, q_2) - U(\theta_1, q_1) \tag{4.1}$$

A high type is always willing to pay more for any quality upgrade than a low type is.

The monopolist maximizes its profits:

$$\max_{p_1, q_1, p_2, q_2} \lambda(p_1 - c(q_1)) + (1 - \lambda)(p_2 - c(q_1)) \tag{4.2}$$

$$\text{s.t.} \begin{cases} U(\theta_1, q_1) - p_1 \geq U(\theta_1, q_2) - p_2 & \text{(IC-1)} \\ U(\theta_2, q_2) - p_2 \geq U(\theta_2, q_1) - p_1 & \text{(IC-2)} \\ U(\theta_1, q_1) - p_1 \geq 0 & \text{(IR-1)} \\ U(\theta_2, q_2) - p_2 \geq 0 & \text{(IR-2)} \end{cases} \tag{4.3}$$

(IC-1), (IC-2) require that the net utility of buying the bundle designed for you is at least as high as buying the other bundle. (IR-1), (IR-2) require that you get positive utility from buying the bundle designed for you.

4.1.2 Benchmark: complete information

Consider the case in which the monopolist knows the types of the consumers it faces. Hence, it does not need to worry about a type 1 choosing bundle 2 and vice-versa, simply because a type 1 consumer only has the possibility to choose a type 1 bundle, and a type 2 only can choose bundle 2. This means that the incentive constraints do not have to be satisfied.

The monopolist now designs bundle i to maximize:

$$\max_{p_i, q_i} p_i - c(q_i) \tag{4.4}$$

$$\text{s.t. } U(\theta_i, q_i) - p_i \geq 0. \tag{4.5}$$

In the profit-maximizing solution the constraint is binding. Otherwise the monopolist can increase p_i to get more profit. Then the problem becomes:

$$\max_{q_i} U(\theta_i, q_i) - c(q_i), \tag{4.6}$$

which implies that the optimal solution is given by:

$$\frac{\partial U(\theta_i, q_i)}{\partial q_i} = \frac{\mathrm{d}c(q_i)}{\mathrm{d}q_i}, \tag{4.7}$$

$$p_i = U(\theta_i, q_i). \tag{4.8}$$

The consumer is indifferent between buying the bundle and not buying, in this case.

For a given bundle q_i, the monopolist always maximizes its profits by charging an amount that equals a consumer's willingness to pay for this bundle, so $p_i = U_i(\theta_i, q_i)$. Then the profit-maximizing bundle to offer to type i consumers is the one where marginal cost equals marginal utility.

4.1.3 Social optimum

Now consider social welfare. As always, social welfare equals the firm's profits plus the consumer surplus. For a consumer of type i, this implies:

$$\mathrm{SW}_i = (p - c(q_i)) + (U(\theta_i, q_i) - p) = U(\theta_i, q_i) - c(q_i). \tag{4.9}$$

This is exactly (4.6). Hence, the monopoly will maximize social welfare.

4.1.4 Solving the model with incomplete information

We now solve (4.3) with incomplete information. We do not have the assumption that the monopolist can observe the type of a consumer, and can offer that consumer only one package. Now we assume that that the monopolist needs to offer both types of packages to the consumers, and hope that each consumer will buy the correct one.

Note that we have:

$$U(\theta_2, q_2) - p_2 \geq U(\theta_2, q_1) - p > U(\theta_1, q_1) - p_1. \tag{4.10}$$

The first inequality is simply (IC-2), the second inequality then follows from $\theta_2 > \theta_1$. Suppose that (IR-1) is not binding. Then, (4.10) implies that (IR-2) is also not binding. Then, the monopolist could just increase p_1 and p_2 and increase its profits. Hence, at the profit-maximizing solution, we must have (IR-1) binding. By (4.10) we then have that (IR-2) is strictly satisfied. This means that (IR-2) can be removed from the list of restrictions, since it is implied by (IR-1).

Suppose that (IC-2) is not binding. We then have:

$$U(\theta_2, q_2) - p_2 > U(\theta_2, q_1) - p_1 > U(\theta_1, q_1) - p_1 = 0. \tag{4.11}$$

This implies that increasing p_2 by some small value ϵ does not violate (IR-2). Hence this cannot be profit-maximizing. At the profit-maximizing solution, (IC-2) is binding.

Add (IC-1) and (IC-2) to find:

$$U(\theta_2, q_2) - U(\theta_2, q_1) \geq U(\theta_1, q_2) - U(\theta_1, q_1). \tag{4.12}$$

This is exactly the single-crossing solution, provided that $q_2 \geq q_1$. Hence, at the profit-maximizing solution, $q_2 \geq q_1$. The fact that (IC-2) is binding implies:

$$p_2 - p_1 = U(\theta_2, q_2) - U(\theta_2, q_1) > U(\theta_1, q_2) - U(\theta_1, q_1), \tag{4.13}$$

using $\theta_2 > \theta_1$, $q_2 \geq q_1$ and the single crossing condition. This implies:

$$U(\theta_1, q_2) - p_2 < U(\theta_1, q_1) - p_1. \tag{4.14}$$

Hence, if (IC-2) is binding, (IC-1) is strictly satisfied. This means we can delete (IC-1) from the list of restrictions. We can write the monopolist's problem as:

$$\max_{q_1, q_2} \lambda(p_1 - c(q_1)) + (1 - \lambda)(p_2 - c(q_2)) \tag{4.15}$$

$$\text{s.t.} \quad \begin{cases} p_2 = U(\theta_2, q_2) - U(\theta_2, q_1) + p_1 \\ p_1 = U(\theta_1, q_1) \end{cases} \tag{4.16}$$

or

$$\max_{q_1, q_2} \Pi(q_1, q_2) = \lambda(U(\theta_1, q_1) - c(q_1)) \tag{4.17}$$

$$+ (1 - \lambda)(U(\theta_2, q_2) - U(\theta_2, q_1) + U(\theta_1, q_1) - c(q_2)). \tag{4.18}$$

FOC:

$$\frac{\partial \Pi}{\partial q_1} = \frac{\partial U}{\partial q_1}(\theta_1, q_1) - (1 - \lambda)\frac{\partial U}{\partial q_1}(\theta_2, q_1) - \lambda\frac{\partial c}{\partial q_1}(q_1) = 0, \tag{4.19}$$

$$\frac{\partial \Pi}{\partial q_2} = (1 - \lambda)\left(\frac{\partial U}{\partial q_2}(\theta_2, q_2) - \frac{\partial c}{\partial q_2}(q_2)\right) = 0 \tag{4.20}$$

$$\Longrightarrow \frac{\partial U}{\partial q_1}(\theta_1, q_1) = \lambda\frac{\partial c}{\partial q_1}(q_1) + (1 - \lambda)\frac{\partial U}{\partial q_1}(\theta_2, q_1), \tag{4.21}$$

$$\frac{\partial U}{\partial q_2}(\theta_2, q_2) = \frac{\partial c}{\partial q_2}(q_2). \tag{4.22}$$

Note that $q_2 > q_1$. Hence, as $\frac{\partial^2 U}{\partial q^2}(\theta_2, q) < 0$, we have $\frac{\partial U}{\partial \theta_{2}, q_1} > \frac{\partial U}{\partial q_2}(\theta_2, q_2) = \frac{\partial c}{\partial q_2}(q_2) \geq \frac{\partial c}{\partial q_1}(q_1)$. This implies that $\lambda\frac{\partial c}{\partial q_1}(q_1) + (1 - \lambda)\frac{\partial U}{\partial q_1}(\theta_2, q_1) > \frac{\partial c}{\partial q_1}(q_1)$, so the optimal solution has $\frac{\partial U}{\partial q_1}(\theta_1, q_1) > \frac{\partial c}{\partial q_1}(q_1)$ (marginal cost is smaller than marginal utility).

There are several things to note:

1. Note that (4.22) implies that the type 2 consumes exactly the amount that he would also choose to consume would he face a price c. Hence, the type 2 consumes the amount that is socially optimal. This is a general result in these types of models: there is *no distortion at the top.*

2. Also note from that the type 2 gets a strictly positive surplus: with $U(\theta_2, q_2) > p_2$, a type 2 pays less for package 2 than he would be willing to pay. The difference is type 2's informational rent: if the monopolist would know his type, he would not get any surplus. Hence, the surplus is due to type 2 having private information regarding his type.

3. From (IR-1) binding, we have that a type 1 pays exactly the amount he is willing to pay for q_1 units of the product. Hence, the lowest type does not obtain any utility from consuming the good: there is no surplus at the bottom.

4. Under the assumption that, in equilibrium, both types are served (which was made throughout the entire analysis), we have that $\frac{\partial U}{\partial q_1}(\theta_1, q_1) > \frac{\partial c}{\partial q_1}(q_1)$. With $\frac{\partial^2 U}{\partial q_1^2}(\theta_1, q_1) < 0$, this implies that type 1 consumes less than his socially optimal amount. In fact, from Lemma 3, we have that the package (p_1, q_1) is designed such that a type 2 is exactly indifferent between consuming (p_1, q_1) and (p_2, q_2). This is also a general result: in a model with more types, we always have that a type is indifferent between consuming his own package and the package designed for the next-highest type. In a sense, this also holds for the lowest type: he is indifferent between consuming the package designed for him, and consuming nothing.

5. Note that the rhs of (4.21) is decreasing in λ, again conditional on both types being served. This implies that as λ, the fraction of types 1 in the population, increases, the package designed for type 1 moves in the direction of his socially optimal package. Also, as θ_2 and θ_1 move closer together, the same is true.

6. Finally, note that throughout the analysis we assume that the monopolist finds it profitable to offer two different packages to the two different types of consumers. We solved the model under that restriction. However, that may not always be the case. Depending on the parameters, the monopolist may find it more profitable to only offer one package. For example, if there are relative few low types, then a monopolist may find it more profitable to only offer a package aimed at the high type, and not selling anything to the low types. In that case, it can charge a higher price to the high types, but has to give up any profits it was making on the low types.

If we would implement the complete information equilibrium in the incomplete information case, the incentive compatibility constraint of the high type would not be satisfied: he would prefer consuming bundle q_1 and obtain utility $U(\theta_2, q_1) - p_1 > 0$ rather than consuming bundle q_2 and obtain utility $U(\theta_2, q_2) - p_2 = 0$.

In equilibrium, the monopolist decreases the price for bundle 2 and decrease q_1. Both actions make it less attractive for a type 2 to choose the bundle of a type 1. In the new optimum, the low type pays exactly what he is willing to pay, whereas the high type is indifferent between consuming his own bundle or that of a low type, i.e. $\frac{\partial U}{\partial q_2}(\theta_2, q_2) - P_2 =$

$\frac{\partial U}{\partial q_1}(\theta_2, q_1) - P_1$. *What the company is trying to do is prevent the passengers who can pay the second-class fare from traveling third-class; it hits the poor, not because it wants to hurt them, but to frighten the rich.*

4.2 Damaged Goods

Suppose that a monopolist offers a good with quality q_2. The monopolist can also "damage" the good it produces. By doing so it produces a quality $q_1 < q_2$. Marginal costs for the damaged goods are $c_1 > c_2$. The higher cost is because of the damaging.

Assume there are two types of consumers, with valuations $\theta_i V(q)$, where $\theta_2 > \theta_1$. Also, assume a proportion λ of the population has type 1. We make two additional assumptions:

1. When the monopolist only sells the high-end product, it would find it profitable to serve only the type 2 consumers.
2. When the monopolist would only sell the damaged good, it would sell find it most profitable to sell that to both types of consumers.

Consider assumption 1. When it chooses to serve only the high types, the profit-maximizing price would be $p_2 = \theta_2 V(q_2)$. Profits then equal:

$$\Pi_H = (1-\lambda)(\theta_2 V(q_2) - c_2). \tag{4.23}$$

Suppose the monopolist would serve both consumers. Then it would set $p_2 = \theta_1 V(q_2)$, profits then equal:

$$\Pi = \theta_1 V(q_2) - c_2. \tag{4.24}$$

The assumption that only the high types are served boils down to:

$$(1-\lambda)(\theta_2 V(q_2) - c_2) > \theta_1 V(q_2) - c_2 \tag{4.25}$$
$$\implies (\theta_1 - (1-\lambda)\theta_2)V(q_2) < \lambda c_2. \tag{4.26}$$

Now consider the case that the monopolist only produces the damaged good. When only serving high types, the profit-maximizing price would be $p_1 = \theta_2 V(q_1)$. Profits then equal:

$$\Pi_D = (1-\lambda)(\theta_2 V(q_1)). \tag{4.27}$$

Suppose it would serve both consumers, the price is then $p_1 = \theta_1 V(q_1)$, with profits:

$$\Pi = \theta_1 V(q_1) - c_1. \tag{4.28}$$

The assumption that he would serve both types thus boils down to assuming that:

$$\theta_1 V(q_1) - c_1 > (1-\lambda)(\theta_2 V(q_1)) \tag{4.29}$$

We are now able to tackle the problem of the monopolist that produces both goods. It then has to set p_1 and p_2 as to maximize:

$$\Pi_{HD} = \lambda(p_1 - c_1) + (1 - \lambda)(p_2 - c_2). \tag{4.30}$$

Note that the difference between this problem and the one discussed earlier, is that the qualities q_1 and q_2 are exogenously given. With the same arguments as before, we can show that in equilibrium the individual rationality constraint of type 1, and the incentive compatibility constraint of type 2 have to bind:

$$p_1 = \theta_1 V(q_1) \tag{4.31}$$
$$p_2 = \theta_2(V(q_2) - V(q_1)) + p_1. \tag{4.32}$$

The monopolist's profits are thus given by:

$$\Pi_{HD} = \Pi_H + (\theta_1 V(q_1) - c_1) - (1 - \lambda)(\theta_2 V(q_1) - c_1). \tag{4.33}$$

Using (4.29) this implies that $\Pi_{HD} > \Pi_H$. Hence, offering both the high-quality and the damaged goods strictly increases the monopolist's profits. The types 2 will also be strictly better off with the introduction of the damaged good: in the case without damaged goods, they were kept at zero surplus. Now, they earn a strictly positive surplus, for the usual reasons. In this simplified specification, the types 1 are indifferent between whether the damaged good is introduced: in both cases, their net surplus is zero. In Deneckere and McAfee (1996), the low types are strictly better off. This is due to the fact that they assume that each individual has a downward sloping demand curve for both qualities. In such a case, by setting a single price for each quality, the monopolist is never able to capture the entire consumer surplus. Therefore, with two products, the types 1 are left with some consumer surplus as well. Note that this result would disappear if the monopolist would simply sell packages of the high-quality and the damaged good.

4.3 Exercises

1. There are two types of consumers. High types have a willingness to pay $\theta\sqrt{q}$ for a product of quality q, with $\theta > 1$, low types have willingness to pay $\sqrt{q}$. Both types occur in equal numbers. A monopolist is able to produce two qualities $q_1 = 9$ and $q_2 = 36$. Production costs are 2 per unit, regardless of the quality produced.

   (a) Suppose the monopolist can only produce the high quality product. Determine the price that it will charge and the profits it will make (note: this will depend on θ). Do the same in case the monopolist can only produce the low quality product.

   (b) Suppose the monopolist can produce both qualities. Determine the profit-maximizing prices, and resulting profits. For what values of θ will the monopolist choose to produce both qualities?

   (c) Now suppose that the monopolist can choose the quality of the low quality product, whereas the quality of the high-quality product is still fixed at 36. Determine what low quality he is going to set. Explain.

   Solution:

(a) With quality $q_2 = 36$, low type has willingness to pay $\sqrt{36} = 6$. For the high type 6θ. The monopolist can choose to set price s.t. only the high type buys or everyone buys. Doing the first, set price to 6θ, then profits are $3\theta - 1$. Doing the second leads to profits of 4. This is higher if $\theta < 5/3$.

With quality $q_1 = 9$, willingness to pay for low type is 3, for high type 3θ. The same analysis as before yields that the monopolist sets price 3 if $\theta < 4/3$ and price 3θ else.

(b) Using the result from the lecture notes, the prices are:

$$p_1 = \sqrt{9}$$
$$p_2 = \theta(\sqrt{36} - \sqrt{9}) + 3 = 3\theta + 3.$$

This leads to profits:

$$\frac{6 + 3\theta}{2} - 2 = \frac{3}{2}\theta + 1$$

This is never higher than only selling the low quality. Only selling the high quality leads to profits $\max\{3\theta - 1, 4\}$, which is always higher than when selling both qualities.

(c) Let x^2 be the quality of the low quality product. When offering both products, the monopolist sets prices:

$$p_1 = x$$
$$p_2 = x + \theta(6 - x),$$

which leads to profits:

$$\frac{2x + \theta(6 - x)}{2} - 2 = \frac{1}{2}x(2 - \theta) + 3\theta - 2.$$

For $\theta < 2$ this is increasing in x, which means that the monopolist will set $x = 6$ (so quality is $6^2 = 36$). This means that the monopolist only sells the high quality product to both types. For $\theta > 2$, this is decreasing in x so the monopolist will set $x = 0$. This means it only sells high quality to high type.

2. A monopolist sells cars. She faces two types of consumers. The utility a consumer obtains from the car is solely determined by the expected amount of kilometers that the car is able to drive. Denote this expected kilometrage as $\mathbb{E}[k]$, denoted in units of 100.000 kilometer. Type 1 consumers have a utility of owning the car that equals $3\sqrt{\mathbb{E}[k]}$. Type 2 consumers have a utility of owning the car that equals $4\sqrt{\mathbb{E}[k]}$. The fraction of type 1 consumers in the population is 1/2. The fraction of type 2 consumers is also 1/2. Initially, the monopolist produces cars that run 100,000 kilometers for sure, so $\mathbb{E}[k] = 1$. The costs of producing such a car are 1. But the monopolist realizes that she is also able to produce a car that is able to run for 100,000 kilometers with probability 1/2, and will not run at all (so 0 kilometers) with probability 1/2 as well. Consumers can perfectly observe this. The costs of producing such a low-quality car are denoted by c_L.

(a) Determine for which values of c_L it is profitable for the monopolist to also supply the low-quality car, given that she can charge prices for both types of cars in a profit-maximizing manner.

(b) Now suppose that $c_L = 1$ and the monopolist can choose the expected kilometrage of the cheap car to be any value below 100,000. Which value will he choose? Again, of course, we assume that after he has chosen this value, he can charge prices for both types of cars in a profit-maximizing manner.

Solution:

(a) Suppose the monopolist sells both cars. Then the maximization problem is:

$$\max \frac{1}{2}(P_H - 1) + \frac{1}{2}(P_L - c_L)$$
$$\text{s.t.} \begin{cases} 3\sqrt{1/2} - P_L \geq 0 & \text{(IR-1)} \\ 4\sqrt{1} - P_H \geq 0 & \text{(IR-2)} \\ 3\sqrt{1/2} - P_L \geq 3\sqrt{1} - P_H & \text{(IC-1)} \\ 4\sqrt{1} - P_H \geq 4\sqrt{1/2} - P_L & \text{(IC-2)} \end{cases}$$

From the same argument as the notes we have that (IR-1) and (IC-2) are binding, hence:

$$\begin{aligned} P_L &= \sqrt{1/2} \\ 4 - P_H &= 3\sqrt{1/2} - P_L \\ &\implies \\ P_L &= 3\sqrt{1/2} \\ P_H &= 4 - \sqrt{1/2}. \end{aligned}$$

Then, profits are:

$$\begin{aligned} \Pi &= \frac{1}{2}\left(4 - \sqrt{1/2} - 1\right) + \frac{1}{2}\left(3\sqrt{1/2} - c_L\right) \\ &= \frac{1}{2}\sqrt{2} + \frac{3}{2} - \frac{1}{2}c_L. \end{aligned}$$

Suppose the monopolist sells only high quality car. Only selling to high types implies $P = 4$, so $\Pi = \frac{1}{2}(4 - 1) = 3/2$. Selling to both gives $P = 3$ so $\Pi = 3 - 1 = 2$, hence in this case the monopolist would sell to both. Also supplying low quality car is profitable if

$$\begin{aligned} \frac{1}{2}\sqrt{2} + \frac{3}{2} - \frac{1}{2}c_L &> 2 \\ &\implies c_L < \sqrt{2} - 1. \end{aligned}$$

(b) Let γ be the quality of the low-quality car. Suppose the monopolist offers both types of cars. Then the maximization problem is:

$$\max \frac{1}{2}(P_H - 1) + \frac{1}{2}(P_L - 1)$$
$$\text{s.t.} \begin{cases} 3\sqrt{\gamma} - P_L \geq 0 & \text{(IR-1)} \\ 4\sqrt{1} - P_H \geq 0 & \text{(IR-2)} \\ 3\sqrt{\gamma} - P_L \geq 3\sqrt{1} - P_H & \text{(IC-1)} \\ 4\sqrt{1} - P_H \geq 4\sqrt{\gamma} - P_L & \text{(IC-2)} \end{cases}$$

(IR-1) and (IC-2) binding implies:

$$\begin{aligned} P_L &= 3\sqrt{\gamma} \\ 4 - P_H &= 4\sqrt{\gamma} - P_L \\ &\Longrightarrow \\ P_L &= 3\sqrt{\gamma} \\ P_H &= 4 - \sqrt{\gamma} \end{aligned}$$

Then, the profits are:

$$\begin{aligned} \Pi &= \frac{1}{2}\left(4 - \sqrt{\gamma} - 1\right) + \frac{1}{2}\left(3\sqrt{\gamma} - 1\right) \\ &= \sqrt{\gamma} + 1. \end{aligned}$$

This is strictly increasing in γ and thus $\gamma = 1$. This implies the monopolist only sells the high quality car.