Tentative solutions
Games, Competition and Markets
2024/25

Chapter 1

1. (a) It could turn out that marginal costs are 0, or that they are 0.4. In general, when they are c :

$$\pi_1 = (1 - q_1 - q_2 - c)\, q_1$$

so

$$\frac{\partial \pi_1}{\partial q_1} = 1 - 2q_1 - q_2 - c = 0.$$

Imposing symmetry:

$$q = \frac{1-c}{3}.$$

Hence, if it turns out marginal costs are zero, both firms set $q = 1/3$. If they are 0.4, firms set $q = .6/3 = .2$

(b) Expected profits now are:

$$\begin{aligned}\pi_1 &= \frac{1}{2}(1 - q_1 - q_2)\, q_1 + \frac{1}{2}(1 - q_1 - q_2 - 0.4)\, q_1 \\ &= (1 - q_1 - q_2 - 0.2)\, q_1\end{aligned}$$

so the equilibrium has $q = .8/3$ (which in fact, is just the average of the earlier options)

2. The indifferent consumer is now given by

$$v - tz^2 - p_0 = v - t(1-z)^2 - p_1$$

which yields

$$z = \frac{1}{2} + \frac{P_1 - P_0}{2t},$$

which is the exact same expression as with linear transport costs. The demand that each firm faces therefore does not change and hence the equilibrium will also be the same as in the lecture notes.

3. We proceed in the same fashion as in the lecture notes. Note however that for $n = 2$ one could also write profits of firm A as a function of prices p_A and p_B and then derive the best-reply function of firm A as a function of p_B.

We now have to take into account that not everyone will buy in equilibrium. In equilibrium, with equal prices, we would have sales of each firm being equal to

$$q = \frac{1}{2} - \frac{1}{2}p^2.$$

Suppose firm A sets a price slightly higher than the other firm. Sales then equal

$$q_A = \frac{1}{2}(1-\Delta)^2 - \frac{1}{2}p^2$$

Again

$$\pi_A = (p_A - c)q_A$$

so

$$\frac{\partial \pi_A}{\partial p_A} = p_A \frac{\partial q_A}{\partial p_A} + q_A = 0$$

Again

$$\frac{\partial q_A}{\partial p_A} = -(1-\Delta)$$

But equilibrium now has

$$q_A = \frac{1}{2} - \frac{1}{2}p^2.$$

So, imposing symmetry, we need

$$p(-1) + \left(\frac{1}{2} - \frac{1}{2}p^2\right) = 0$$

so

$$p^* = \sqrt{2} - 1$$

Chapter 2

1. Now equilibrium profits are $\frac{1}{n}(1-\lambda)$. At price $p \in [\underline{p}, 1]$, firm is expected profits are

$$E(\pi_i(p)) = \frac{1}{n}(1-\lambda)p + (1-F(p))^{n-1}\lambda p.$$

The lower bound still is $\underline{p} = \frac{1-\lambda}{1+\lambda}$ and equating expected profits with $\frac{1}{n}(1-\lambda)$.

$$F(p) = 1 - \left(\frac{(1-\lambda)(1-p)}{n\lambda p}\right)^{\frac{1}{n-1}}$$

2. (a) Consider the demand for firm 1. There are two cases to consider. Suppose that $p_2 \leq 2$. There are now 3 obvious choices for the price of firm 1. If it sets a price equal to 2, it only sells to half of all the uninformed consumers, so demand is $1 - \lambda$. If it sets a price equal to 3, it only sells to half of the uninformed high types, so demand is $(1 - \lambda)/2$. If it slightly undercuts firm 2, it will sell to all the informeds and half of all the uninformeds, so demand is $2\lambda + (1 - \lambda) = 1 + \lambda$. Profits in these three cases are $2(1 - \lambda)$, $3(1 - \lambda)/2$, and $p_2(1 + \lambda)$. Obviously, the first option always yields higher profits than the second option. The first option yields higher profits than the third option whenever

$$2(1 - \lambda) > p_2(1 + \lambda),$$

thus if

$$p_2 < \frac{2(1 - \lambda)}{1 + \lambda}.$$

Now suppose that $2 < p_2 < 3$. Again, there are three obvious options for firm 1. If it sets a price equal to 2, it sells to all the informeds and half of the uninformeds, so demand is $2\lambda + (1 - \lambda) = 1 + \lambda$. If it sets a price equal to 3, it only sells to half of the uninformed high types, so demand is $(1 - \lambda)/2$. If it slightly undercuts firm 2, it will sell to all the informed high types, and to half of the uninformed high types, so demand is $\lambda + (1 - \lambda)/2 = (1 + \lambda)/2$. Profits in these three cases are $2(1 + \lambda)$, $3(1 - \lambda)/2$, and $p_2(1 + \lambda)/2$ respectively. Note that the third option yields higher profits than the first option if

$$p_2(1 + \lambda)/2 > 2(1 + \lambda),$$

i.e. if $p_2 > 4$. But firm 2 will never set such a price. Thus, the only relevant options in this interval are the first and the second. The first option yields higher profits if

$$2(1 + \lambda) > 3(1 - \lambda)/2,$$

i.e. if $\lambda > -1/7$, which is always satisfied. Hence, with $2 < p_2 < 3$,[1] the best reply for firm 1 is to always set $p_2 = 2$. This yields the following best reply function

$$p_1 = \begin{cases} 2 & \text{if } p_2 \leq \frac{2(1-\lambda)}{1+\lambda} \\ p_2 - \varepsilon & \text{otherwise} \end{cases}$$

[1]Note that if $p_2 = 3$, firm 1 will slightly undercut. But that implies that this will never be set, so for conciseness, we ignore that option in what follows.

Note that $2(1-\lambda)/(1+\lambda) \leq 2$ for all relevant λ. Hence, no firm will ever find it profitable to set a price higher than 2.

(b) With similar arguments as in the lecture notes, there is no equilibrium in pure strategies; firms have an incentive to undercut prices higher than $2(1-\lambda)/(1+\lambda)$, while for lower prices, the best reply is to set a price equal to 2. Hence we look for an equilibrium in mixed strategies on some interval $[\underline{p}, \bar{p}]$. Note that necessarily $\bar{p} = 2$: no firm will ever set a price higher than 2. If it does, it knows that it has the highest price for sure, but if that is true, it is better off setting $p = 2$. A firm that sets $p = 2$ will sell to half of all the uninformeds, and thus make profits $2(1-\lambda)$. A firm that sets price p will sell $2\lambda + (1-\lambda) = 1+\lambda$ if this price turns out to be the lowest. Otherwise it will sell $1-\lambda$, which is its share of all the uninformeds. Expected profits then equal

$$F(p)(1-\lambda)p + (1-F(p))(1+\lambda)p = p + p\lambda(1-2F(p)).$$

A mixed strategy necessarily has all prices yielding the same profits of $2(1-\lambda)$. This implies

$$F(p) = \frac{p - 2 + \lambda(p+2)}{2p\lambda}.$$

The lower bound of the interval can be found e.g. by plugging in $F(p) = 0$. This yields

$$\underline{p} = \frac{2(1-\lambda)}{1+\lambda}.$$

3. Consumer behavior does not change. But different types of consumers now have different $\hat{\varepsilon}$. Let's refer to the low search cost people as L and the high search cost people as H. From the analysis in the chapter $\hat{\varepsilon} = 1 - \sqrt{2s}$ so $\hat{\varepsilon}_L = 1 - 0.4\sqrt{2}$ and $\hat{\varepsilon}_H = 1 - 0.5\sqrt{2}$. A firm that defects now gets (following the slides)

$$\begin{aligned} q_A &= \lambda\left[\frac{1}{2}(1-\hat{\varepsilon}_H - \Delta)(1+\hat{\varepsilon}_H) + \frac{1}{2}\hat{\varepsilon}_H^2\right] \\ &+ (1-\lambda)\left[\frac{1}{2}(1-\hat{\varepsilon}_L - \Delta)(1+\hat{\varepsilon}_L) + \frac{1}{2}\hat{\varepsilon}_L^2\right]. \end{aligned}$$

So

$$\begin{aligned} \frac{\partial q_A}{\partial P_A} &= \lambda\left[-\frac{1}{2}(1+\hat{\varepsilon}_H\right] + (1-\lambda)\left[-\frac{1}{2}(1+\hat{\varepsilon}_L)\right] \\ &= -\frac{1}{2}(1+\hat{\varepsilon}_A) \end{aligned}$$

with

$$\begin{aligned}\hat{\varepsilon}_A &\equiv \lambda\hat{\varepsilon}_H + (1-\lambda)\varepsilon_L \\ &= 1-\sqrt{2}\cdot(0.5-0.1\lambda)\end{aligned}$$

the weighted average of the two. Equilibrium prices then equal, again following the notes,

$$\begin{aligned}p^* &= \frac{1}{1+\hat{\varepsilon}_A} \\ &= \frac{1}{2-\sqrt{2}\,(0.5-0.1\lambda)}\end{aligned}$$

4. This is based on Haan and Moraga-González (2011). The first part of the analysis is the same as that leading up to slide 37. However a fraction $a_1/\left(a_1+a_2\right)$ now visits firm 1 first, rather than 1/2. Hence, demand for firm 1 now equals

$$D_1 = \frac{a_1}{a_1+a_2}\left(1-\hat{\varepsilon}-\Delta\right) + \frac{a_2}{a_1+a_2}\left(1-\hat{\varepsilon}-\Delta\right)\hat{\varepsilon} + \frac{1}{2}\hat{\varepsilon}^2$$

with again $\Delta \equiv p_1 - p^*$, so

$$\frac{\partial D_1}{\partial p_1} = -\frac{a_1}{a_1+a_2} - \frac{a_2}{a_1+a_2}\hat{\varepsilon}.$$

Total profits are

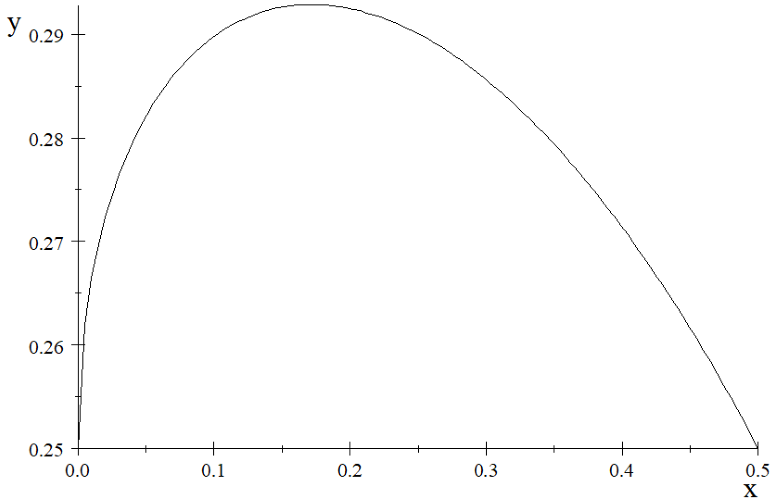
$$\pi_1 = p_1 \cdot D_1 - \frac{1}{4}a_1.$$

First-order conditions are

$$\begin{aligned}\frac{\partial \pi_1}{\partial p_1} &= D_1 + p_1\cdot\frac{\partial D_1}{\partial p_1} = 0 \\ \frac{\partial \pi_1}{\partial a_1} &= \frac{a_2}{\left(a_1+a_2\right)^2}\cdot\left(1-\hat{\varepsilon}-\Delta\right)\cdot p_1 - \frac{a_2}{\left(a_1+a_2\right)^2}\left(1-\hat{\varepsilon}-\Delta\right)\hat{\varepsilon}\cdot p_1 - \frac{1}{4} = 0\end{aligned}$$

We can now impose symmetry:

$$\begin{aligned}\frac{\partial \pi_1}{\partial p_1} &= \frac{1}{2} + p^*\cdot\left(-\frac{1}{2}-\frac{1}{2}\hat{\varepsilon}\right) = 0 \\ \frac{\partial \pi_1}{\partial a_1} &= \frac{1}{4a}\left(1-\hat{\varepsilon}\right)p^* - \frac{1}{4a}\left(1-\hat{\varepsilon}\right)\hat{\varepsilon}p^* - \frac{1}{4} = 0\end{aligned}$$



Hence

$$
\begin{aligned}
p^* &= \frac{1}{1+\hat{\varepsilon}} \\
a^* &= (1-\hat{\varepsilon})^2 p^*
\end{aligned}
$$

Consumer behavior does not change, so we still have $\hat{\varepsilon} = 1 - \sqrt{2s}$. Note that prices are still the same as in the regular model. This is intuitive: in equilibrium each firm will choose the same level of advertising, so in equilibrium consumers will again visit firms randomly with equal probability. Equilibrium profits are given by

$$
\begin{aligned}
\pi^* &= \frac{1}{2}p^* - \frac{1}{4}a^* = \frac{1}{2(1+\hat{\varepsilon})} - \frac{1}{4}(1-\hat{\varepsilon})^2 \cdot \frac{1}{1+\hat{\varepsilon}} - \left(\frac{1}{4}\frac{1+2\varepsilon-\varepsilon^2}{1+\varepsilon}\right) \\
&= \frac{1}{4}\frac{1+2\hat{\varepsilon}-\hat{\varepsilon}^2}{1+\hat{\varepsilon}} = \frac{1}{4}\frac{1+2\left(1-\sqrt{2s}\right)-\left(1-\sqrt{2s}\right)^2}{2-\sqrt{2s}}
\end{aligned}
$$

$$
\frac{1}{4}\frac{1+2\left(1-\sqrt{2s}\right)-\left(1-\sqrt{2s}\right)^2}{2-\sqrt{2s}}
$$

We need that $s \in (0, 1/2)$. Profits are then plotted in the figure above.

That is, they're increasing up to $3 - 2\sqrt{2} \approx 0.17157$ and decreasing afterwards.

Chapter 3

1. (a) The number of consumers that are informed about firm 1 now equal

$$\phi_1 = \Phi_1 + \frac{1}{2}(1 - \Phi_1) = \frac{1}{2}(1 + \Phi_1)$$

Demand is

$$D_1 = \phi_1 \left[(1 - \phi_2) + \phi_2 \left(\frac{1}{2} + \frac{P_2 - P_1}{2t} \right) \right]$$

So

$$\pi_1 = (P_1 - c) D_1 - a (\Phi_1)^2 / 2$$

$$\frac{\partial \pi_1}{\partial P_1} = \phi_1 \left[1 - \phi_2 + \phi_2 \left(\frac{1}{2} + \frac{P_2 - 2P_1}{2t} \right) \right] = 0$$

$$P_1 = \frac{P_2 + t}{2} + \frac{1 - \phi_2}{\phi_2}$$

$$a\Phi_1 = \frac{1}{2} P_1 \left[1 - \frac{1}{2}(1 + \Phi_1) + \frac{1}{2}(1 + \Phi_2) \left(\frac{1}{2} + \frac{P_2 - 2P_1}{2t} \right) \right]$$

(b) Probably: the same intuition still holds. The spillovers in advertisements (in the sense that now ads effectively will reach more people due to word of mouth) will be internalized (i.e., taken into account when setting the advertisement intensity) both by the social planner and in the decentralized equilibrium. Firms will still mix in equilibrium and an ad with the highest price will still capture all the consumer surplus and therefore still be put out at the socially optimal rate. Because of the mixed strategy equilibrium ads with other prices generate the same private benefits as the highest-priced ad and hence will also be put out at the socially optimal rate.

2. The logic is still the same, in that a fraction $\Phi_1 (1 - \Phi_2)$ will definitely buy from firm 1, $(1 - \Phi_1) \Phi_2$ definitely buy from 2, while $\Phi_1 \Phi_2$ are selective.

   In equilibrium, firm 1 again sells to half of all the informeds. That now implies

$$q_1 = \frac{1}{2} \left(1 - (1 - \Phi)^2 \right) = \frac{1}{2} \Phi (2 - \Phi).$$

But now consider a defection both in Φ as well as in price. Demand then equals

$$q_1 = \Phi_1 \left[(1 - \Phi_2) + \frac{1}{2} \Phi_2 \left(1 - (p_1 - p^*) \right)^2 \right]$$

Profits:

$$\pi_1 = (p_1 - c)\, q_1 - \frac{1}{2} a \Phi_1^2$$

FOCs:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial p_1} &= (p_1 - c) \frac{\partial q_1}{\partial p_1} + q_1 = 0 \\
\frac{\partial \pi_1}{\partial \Phi_1} &= \left[(1 - \Phi_2) + \frac{1}{2} \Phi_2 \left(1 - (p_1 - p^*) \right)^2 \right] (p_1 - c) - a \Phi_1
\end{aligned}$$

Now

$$\frac{\partial q_1}{\partial p_1} = -\Phi_1 \Phi_2 \left(1 - (p_1 - p^*) \right)$$

Symmetry:

$$\begin{aligned}
-\,(p - c)\, \Phi^2 + q_1 &= 0 \\
(1 - \Phi + \frac{1}{2} \Phi)(p - c) - a \Phi &= 0
\end{aligned}$$

so

$$\begin{aligned}
p - c - \frac{2 - \Phi}{2 \Phi} &= 0 \\
(1 - \frac{1}{2} \Phi)(p - c) - a \Phi &= 0
\end{aligned}$$

This implies

$$\begin{aligned}
p &= c + \frac{2 - \Phi}{2 \Phi} \\
4 - 4\Phi + \Phi^2 (1 - 4a) &= 0 \implies \Phi = \frac{4 - 8\sqrt{a}}{2 - 8a}.
\end{aligned}$$

Calculating the Φ that maximizes social welfare is now rather complicated: it involves the expected value of the highest of two match values.

Chapter 4

1. (a) With quality $q_2 = 36$, willingness to pay for the low types is $\sqrt{36} = 6$, that of the high types $\theta\sqrt{36} = 6\theta$. The monopolist can choose to either set a price such that only the high types buy, or a price such that everyone buys. Doing the first entails setting price 6θ and making profits $(6\theta - 2)/2 = 3\theta - 1$. Doing the second entails setting price 6 and making profits $6 - 2 = 4$. The latter yields higher profits iff $\theta < 5/3$. With quality $q_1 = 9$, willingness to pay for the low types is $\sqrt{9} = 3$, that of the high types $\theta\sqrt{9} = 3\theta$. The monopolist can choose to either set a price such that only the high types buy, or a price such that everyone buys. Doing the first entails setting price 3θ and making profits $(3\theta - 2)/2 = \frac{3}{2}\theta - 1$. Doing the second entails setting price 3 and making profits $3 - 2 = 1$. The latter yields higher profits iff $\theta < 4/3$.

   (b) Using the standard result from the lecture notes, if it would offer both qualities, it would set

$$
\begin{aligned}
p_1 &= \sqrt{9} = 3 \\
p_2 &= \theta\left(\sqrt{36} - \sqrt{9}\right) + 3 = 3\theta + 3
\end{aligned}
$$

   so profits would equal

$$
\frac{6 + 3\theta}{2} - 2 = \frac{3}{2}\theta + 1.
$$

   This is never higher than the profits from only selling the low quality. Only selling the high quality yields $\max\{3\theta - 1, 4\}$ But for every θ, this is higher than what we get from selling both qualities. Hence, it is never profitable to sell both qualities.

   (c) Denote the quality of the low quality product as x^2. When offering both products, the monopolist sets prices

$$
\begin{aligned}
p_1 &= x \\
p_2 &= x + \theta(6 - x),
\end{aligned}
$$

   so profits are

$$
\frac{2x + \theta(6 - x)}{2} - 2 = \frac{1}{2}x(2 - \theta) + 3\theta - 2.
$$

For $\theta < 2$, this is increasing in x, which suggests the profit-maximizing thing would be to set $x = 6$ (so quality also equals 36). Note that this would boil down to offering only the high quality product and selling it to both, as $p_1 = p_2 = 6$. For $\theta > 2$, this is decreasing in x, so it would be profit-maximizing to set $x = 0$. Note that this boils down to only selling the high-quality product to the high types.

2. (a) Suppose the monopolist would sell both cars. His problem is then to

$$
\begin{aligned}
&\max \frac{1}{2}\left(P_H - 1\right) + \frac{1}{2}\left(P_L - c_L\right) \\
&\text{s.t} \\
&\begin{cases}
3\sqrt{1/2} - P_L \geq 0 & \text{(IR-1)} \\
4\sqrt{1} - P_H \geq 0 & \text{(IR-2)} \\
3\sqrt{1/2} - P_L \geq 3\sqrt{1} - P_H & \text{(IC-1)} \\
4\sqrt{1} - P_H \geq 4\sqrt{1/2} - P_L & \text{(IC-2)}
\end{cases}
\end{aligned}
$$

With the usual arguments, we have that the optimal solution has (IR-1) and (IC-2) binding. Hence

$$
\begin{aligned}
P_L &= 3\sqrt{1/2} \\
4 - P_H &= 3\sqrt{1/2} - P_L
\end{aligned}
$$

or

$$
\begin{aligned}
p_L &= 3\sqrt{1/2} \\
p_H &= 4 - \sqrt{1/2}.
\end{aligned}
$$

Profits then equal

$$
\begin{aligned}
\Pi &= \frac{1}{2}\left(4 - \sqrt{1/2} - 1\right) + \frac{1}{2}\left(3\sqrt{1/2} - c_L\right) \\
&= \frac{1}{2}\sqrt{2} + \frac{3}{2} - \frac{1}{2}c_L
\end{aligned}
$$

Suppose you only sell the high quality car. There are two options: only sell to the high types or sell to both. Only selling to the high types would imply $P = 4$,so $\Pi = \frac{1}{2} \cdot (4 - 1) = \frac{3}{2}$. Selling to both would imply $P = 3$ and $\Pi = 1 \cdot (3 - 1) = 2$. Hence the preferred option would then be to sell to both types. Also supplying the low quality car would be profitable if

$$
\frac{1}{2}\sqrt{2} + \frac{3}{2} - \frac{1}{2}c_L > 2, \tag{1}
$$

which implies

$$c_L < \sqrt{2} - 1 = 0.41421 \tag{2}$$

(b) Denote the quality (i.e. the expected number of kilometres) of the low-quality car as γ. Suppose that the monopolist would choose to supply both types of car. Its problem then is to

$$\max \frac{1}{2}(P_H - 1) + \frac{1}{2}(P_L - 1) \tag{3}$$

$$\text{s.t} \tag{4}$$

$$\begin{cases} 3\sqrt{\gamma} - P_L \geq 0 & \text{(IR-1)} \\ 4\sqrt{1} - P_H \geq 0 & \text{(IR-2)} \\ 3\sqrt{\gamma} - P_L \geq 3\sqrt{1} - P_H & \text{(IC-1)} \\ 4\sqrt{1} - P_H \geq 4\sqrt{\gamma} - P_L & \text{(IC-2)} \end{cases} \tag{5}$$

With the usual arguments, we have that the optimal solution has (IR-1) and (IC-2) binding. Hence

$$P_L = 3\sqrt{\gamma} \tag{6}$$

$$4 - P_H = 4\sqrt{\gamma} - P_L \tag{7}$$

or

$$p_L = 3\sqrt{\gamma} \tag{8}$$

$$p_H = 4 - \sqrt{\gamma}. \tag{9}$$

Profits then equal

$$\Pi = \frac{1}{2}(4 - \sqrt{\gamma} - 1) + \frac{1}{2}(3\sqrt{\gamma} - 1) \tag{10}$$

$$= \sqrt{\gamma} + 1 \tag{11}$$

which is strictly increasing in γ and thus $\gamma = 1$. Hence, the monopolist is best off by only offering a car of quality 1.

References

Haan, M. A. and Moraga-González, J. L. (2011). Advertising for attention in a consumer search model. *Economic Journal*, pages 552–579.