A Robust Approach for Project Scheduling Problem

Xin Shen, Jubiao Yang
advised by: John E. Mitchell

Rensselaer Polytechnic Institute
Troy, NY 12180

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Outline

Introduction

Objective: to maximize the net present value of the project portfolio (sum of benefits and costs of portfolio projects discounted appropriately with hurdle rate).

Projects may have *dependencies*:

- nonsimultaneity (e.g. resource constraints on teams/equipments)
- single precedence (e.g. a project is decomposed into phases)
- alternative precedence (e.g. parallel-approach effort to overcome technical hurdles)

Introduction

Projects are subject to risks of bad luck (delay/failure/delay and failure).

Deterministic Approach: prepare for a certain bad-luck scenario beforehand (incl. scenario with no bad luck) and suggest a portfolio.

disproportionate depreciation of portfolio value can be caused by "chain reaction" of bad lucks, thanks to project dependencies.

Robust Approach: to have the largest portfolio value under the worst possible outcome scenario (resilience to bad luck).

Outline

Project dependencies

nonsimultaneity: if $i \nsim j$, then $\Delta_{ij} = \Delta_{ji} = -1$

alternative precedence: if $\{i_1, \cdots, i_N\} \vdash j$, then $\Delta_{i_1 j} = \cdots = \Delta_{i_N j} =$ a unique positive integer

single precedence: $i \succ j \quad \Leftrightarrow \quad \{i\} \vdash j$, thus a special case of $I \vdash j$ and can be treated the same way

e.g. for a project pool of $\{p_1, p_2, p_3, p_4, p_5\}$ with $p_1 \nsim p_2$, $p_2 \nsim p_3$, $p_3 \succ p_1$, $p_1 \succ p_4$, $\{p_2, p_5\} \vdash p_4$, $p_3 \succ p_4$, $\{p_1, p_3\} \vdash p_5$:

$$\mathbf{\Delta} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{array} \begin{array}{c} \begin{array}{ccccc} p_1 & p_2 & p_3 & p_4 & p_5 \end{array} \\ \begin{pmatrix} & -1 & & 1 & 1 \\ -1 & & -1 & 2 & \\ 1 & -1 & & 3 & 1 \\ & & & & \\ & & & 2 & \end{pmatrix} \end{array}$$

Model

Binary variable X_{jt}:

$$X_{jt} = \begin{cases} 1, & \text{if Project } j \text{ starts at the beginning of the } i^{th} \text{ month} \\ 0, & \text{otherwise} \end{cases}$$

User-controlled parameters q_j^δ and q_j^f:

$$q_j^\delta = \begin{cases} 1, & \text{if knew beforehand that Project } j \text{ would be delayed} \\ 0, & \text{otherwise} \end{cases}$$
$$q_j^f = \begin{cases} 1, & \text{if knew beforehand that Project } j \text{ would fail} \\ 0, & \text{otherwise} \end{cases}$$

Thus the adjusted durations and costs are:

$$\tilde{d}_j = d_j + q_j^\delta d_j^+, \ \tilde{c}_j = c_j + q_j^\delta c_j^+, \ \forall j \in J$$

- a project can start at most once:

$$\sum_{t=1}^{T} X_{jt} \leq 1 - q_i^f, \ \forall j \in J$$

- a project cannot start if it cannot complete by the deadline:

$$\sum_{t \geq T+1-\tilde{d}_j} X_{jt} = 0, \ \forall j \in J$$

- for $i \nsim j$, i cannot be started within d_j months after j started, vice versa:

$$\sum_{t-\tilde{d}_j+1 \leq t' \leq t+\tilde{d}_i-1} X_{jt'} + X_{it} \leq 1, \ \forall i \nsim j, \ \forall t \in \{1, \cdots, T\}$$

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- for $I \vdash j$, j cannot be started until at least one of the projects in I has been finished:

$$\sum_{i \in I} \sum_{t' \leq t - \tilde{d}_i} X_{it'} \geq X_{jt}, \ \forall I \vdash j, \ \forall t \in \{1, \cdots, T\}$$

- the objective function can be evaluated:

$$NPV_\gamma(S, T) = -\sum_{j,t} \gamma^t \cdot \tilde{c}_j \cdot X_{jt} + \sum_{j,t} \gamma^{t+\tilde{d}_j} \cdot b_j \cdot X_{jt}$$

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SDCMPCC formulation

Want to solve:

$$\min_X \{\mathrm{rank}(X) \ : \ X \in \mathcal{C} \text{ and } X \in \mathbb{S}^n_+\}$$

Equivalently:

$$\begin{array}{ll} \min_{X,U} & n - < I, U > \\ \text{subject to} & X \in \mathcal{C} \\ & 0 \preceq U \preceq I \\ & 0 \preceq X \perp U \succeq 0 \end{array}$$

When X and U p.s.d, $X \perp U$ is equivalent to:

$$< X, U > = 0$$

Note that if X has the eigenvalue decomposition,

$$X = P^T \Sigma P$$

then we can choose

$$U = P_0 P_0^T$$

where P_0 is composed of columns in P corresponding to 0 eigenvalue of X.

Thus, it is obvious that $rank(X) = n - < I, U >$.

SDCMPCC formulation

We can apply the SDCMPCC formulation to the general case $X \in \mathbb{R}^{m \times n}$ by introducing an auxiliary variable Z:

$$Z = \begin{bmatrix} G & X^T \\ X & B \end{bmatrix} \succeq 0$$

For any X, can find matrix G and B such that $Z \succeq 0$ and $rank(Z) = rank(X)$

In the objective, we want to minimize the rank of Z.

Constraint Qualification of SDCMPCC Formulation

Common Constraint qualifications such as LICQ and Robinson CQ are violated for SDCMPCC.

Here we consider Local Calmness.

Definition

Suppose that $\bar{x}$ is a local optimal solution to the problem:

$$\underset{x \in X}{\text{minimize}} \;\; f(x) \;\; \text{subject to} \;\; x \in \mathcal{L} \text{ and } \; g(x) \in -\mathcal{K} \tag{1}$$

Problem(1) is said to be calm of order $\alpha > 0$ at $\bar{x}$ if there exists $M < \infty$ such that, for any sequence $\{z^q\}$ with $0 \neq z^q \to 0$ and any sequence $\{x_q\} \subset \mathcal{L}$ satisfying $x^q \to \bar{x}$ and $g(x^q) \in z^q - \mathcal{K}$, there holds

$$\frac{f(x^q) \; - \; f(\bar{x})}{||z^q||^\alpha} + M \; \geq \; 0 \tag{2}$$

Constraint Qualification of SDCMPCC Formulation

Huang et.al shows that local calmness or order 1 implies the existence of KKT multipliers:

Theorem

Let $\bar{x}$ be a local optimal solution to Problem(1) and (1) is calm of order 1 at $\bar{x}$. Then, there exists $\mu \in K^*$ such that the system:

$$0 \in \partial f(\bar{x}) + \mu(\nabla g(\bar{x})) + N_{\mathcal{L}}(\bar{x})$$

$$\mu(g(\bar{x})) = 0$$

is consistent. $N_{\mathcal{L}}(\bar{x})$ is the Clarke normal cone of $\mathcal{L}$ at $\bar{x}$.

Constraint Qualification of SDCMPCC Formulation

Proposition

Calmness of Order 1 holds at each local optimum $(\bar{X}, \bar{U})$ in the SDCMPCC Formulation.

In the proof, let (X^q, U^q) be a feasible solution to the perturbed SDCMPCC Formulation with perturbation parameter (z^q, r^q, h_1^q, h_2^q).

Want to show the existence of $M < \infty$ that satisfies:

$$(n - < I, U^q >) - (n - < I, \bar{U} >) \geq -M||(z^q, r^q, h_1^q, h_2^q)||$$

for any $(X^q, U^q) \rightarrow (\bar{X}, \bar{U})$.

An upper bound of $n - < I, \bar{U} >$ is $rank(\bar{X})$.

Sketch of Proof

Want to get a lower bound for $n - < I, U^q >$. (X^q, U^q) is feasible to the perturbed problem:

$$\begin{aligned} \underset{X, U \in \mathbb{S}^n}{\text{minimize}} \quad & n - \ < I,\ U > \\ \text{subject to} \quad & X \ + \ z^q \in \tilde{\mathcal{C}} \cap \mathcal{S}^n_+ \\ & - < X,\ U > \leq \ r^q \\ & < X,\ U > \leq \ r^q \\ & I \ - \ U \ \succeq \ -h^q_1 I \\ & U \ \succeq \ -h^q_2 I \end{aligned}$$

The lower bound can be acquired by fixing $X \ = \ X^q$ in the perturbed problem.

Sketch of Proof

By fixing $X = X^q$ we can get the following problem:

$$
\begin{array}{lll}
\underset{U \in \mathbb{S}^n}{\text{minimize}} & n - < I, U > & \\
\text{subject to} & - < X^q, U > \leq r^q, & y_1 \\
& < X^q, U > \leq r^q, & y_2 \\
& I - U \succeq -h_1^q I, & \Omega_1 \\
& U \succeq -h_2^q I, & \Omega_2
\end{array}
$$

where $y_1, y_2, \Omega_1, \Omega_2$ are the Lagrangian multipliers for the corresponding constraints.

U^q is feasible to the above problem.

Slater condion holds for the above problem. Can find a lower bound the objective by Strong Duality

Sketch of Proof

The dual problem is:

$$
\begin{aligned}
\underset{y_1, y_2 \in \mathbb{R},\, \Omega_1, \Omega_2 \in \mathbb{S}^n}{\text{maximize}} \quad & n + r^q\, y_1 + r^q\, y_2 - (1 + h_1^q) trace(\Omega_1) - h_2^q\, trace(\Omega_2) \\
\text{subject to} \quad & -y_1\, X^q + y_2\, X^q - \Omega_1 + \Omega_2 = -I \\
& y_1,\, y_2 \leq 0 \\
& \Omega_1,\, \Omega_2 \succeq 0
\end{aligned}
$$

By diagonalizing X^q we can get a tightened problem:

$$
\begin{aligned}
\underset{y_1, y_2 \in \mathbb{R},\, f, g \in \mathbb{R}^n}{\text{maximize}} \quad & n + r^q\, y_1 + r^q\, y_2 - (1 + h_1^q) \textstyle\sum_i f_i - h_2^q \sum_i g_i \\
\text{subject to} \quad & -y_1\, \lambda_i^q + y_2\, \lambda_i^q - f_i + g_i = -1, \forall i = 1 \cdots n \\
& y_1,\, y_2 \leq 0 \\
& f_i,\, g_i \geq 0, \forall i = 1 \cdots n
\end{aligned}
$$

Sketch of Proof

Since $X^q \to \bar{X}$, $\lambda_i^q \to \lambda_i$. Can get a lower bound for the objective of the dual problem, which is:

$$\text{rank}(X) - \frac{2r^q}{\tilde{\lambda}} - (n - rank(X))(h_1^q + (1 + h_1^q)\frac{2}{\tilde{\lambda}}||z^q||) - \frac{h_2^q}{\tilde{\lambda}}||\bar{X}||^*$$

where $\tilde{\lambda}$ is the smallest positive eigenvalue of $\bar{X}$. We can take

$$M = \frac{2}{\tilde{\lambda}} + \frac{1}{\tilde{\lambda}}||\bar{X}||^* + (n - rank(X))(1 + \frac{4}{\tilde{\lambda}})$$

KKT Condition of SDCMPCC Formulation

Given $\mathcal{C} = \{X \mid < A_i, X > \geq b_i, \forall i = 1 \cdots p\}$
The KKT condition is:

$$\begin{aligned} 0 &\preceq U && \perp -I + \mu X + Y \succeq 0 \\ 0 &\preceq X && \perp -\sum \lambda_i A_i + \mu U \succeq 0 \\ 0 &\preceq Y && \perp I - U \succeq 0 \\ 0 &\leq \lambda_i && \perp b_i - < A_i, X_i > \geq 0, \forall i = 1 \cdots, p \end{aligned} \tag{3}$$

Where λ, μ and Y are lagrangian multipliers corresponding to the constraints $A(X) = b$, $< X, U >= 0$ and $I - U \succeq 0$ respectively.

Any feasible pair (X, U) with U given by $P_0 P_0^T$ with columns of P_0 to be the eigenvectors in the null space of X ,is a KKT stationary point of the SDCMPCC Formulation.

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