A Robust Approach for Project Scheduling Problem

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1. Introduction

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3. Robust Scheduling with Bad Luck

4. Conclusions

Outline

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Problem Description

Eg: Minimal System Realization

Want to design a Low order Linear Time-Invariant(LTI) system

Denote by H_n the Hankel matrix with parameters $h_1, h_2, \cdots, h_{2n-1} \in \mathbb{R}$:

$$H_n = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdots & h_{n+1} \\ h_3 & h_4 & h_5 & \cdots & h_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n+1} & h_{n+2} & \cdots & h_{2n-1} \end{bmatrix}$$

Fact: Let $h_1, h_2, \cdots h_n$ be givien real numbers. Then there exists a minimal Linear Time-Invariant system of order r if and only if

$$r = \min_{h_{n+1}, \cdots, h_{2n-1} \in \mathbb{R}} \operatorname{rank}(H_n)$$

(Fazel)

Eg: Minimal System Realization(Continued)

The problem can be expressed as:

$$\begin{array}{|ll|}\hline \underset{H_n, s}{\text{minimize}} & \operatorname{rank} H_n \\ \text{subject to} & l_i \leq s_i \leq u_i, \quad i = 1, \cdots, n \\ & h_{n+1}, \cdots, h_{2n-1} \in \mathbb{R} \\ \hline \end{array}$$

where:

- $s_i = \sum_{k=1}^{i} h_k$ denote the terms in the step response.
- l_i lower bound of step response at the ith term.
- u_i upper bound of step response at the ith term.

(Fazel)

Outline

SDCMPCC formulation

Want to solve:

$$\min_X \{\text{rank}(X) \; : \; X \in \mathcal{C} \text{ and } X \in \mathbb{S}^n_+\}$$

Equivalently:

$$\begin{array}{ll} \min_{X,U} & n - < I, U > \\ \text{subject to} & X \in \mathcal{C} \\ & 0 \preceq U \preceq I \\ & 0 \preceq X \perp U \succeq 0 \end{array}$$

When X and U p.s.d, $X \perp U$ is equivalent to:

$$< X, U > = 0$$

Note that if X has the eigenvalue decomposition,

$$X = P^T \Sigma P$$

then we can choose

$$U = P_0 P_0^T$$

where P_0 is composed of columns in P corresponding to 0 eigenvalue of X.

Thus, it is obvious that $rank(X) = n - < I, U >$.

SDCMPCC formulation

We can apply the SDCMPCC formulation to the general case $X \in \mathbb{R}^{m \times n}$ by introducing an auxiliary variable Z:

$$Z = \begin{bmatrix} G & X^T \\ X & B \end{bmatrix} \succeq 0$$

For any X, can find matrix G and B such that $Z \succeq 0$ and $rank(Z) = rank(X)$

In the objective, we want to minimize the rank of Z.

Constraint Qualification of SDCMPCC Formulation

Common Constraint qualifications such as LICQ and Robinson CQ are violated for SDCMPCC.

Here we consider Local Calmness.

Definition

Suppose that $\bar{x}$ is a local optimal solution to the problem:

$$\underset{x \in X}{\text{minimize}} \;\; f(x) \;\; \text{subject to} \;\; x \in \mathcal{L} \text{ and } \; g(x) \in -\mathcal{K} \tag{1}$$

Problem(1) is said to be calm of order $\alpha > 0$ at $\bar{x}$ if there exists $M < \infty$ such that, for any sequence $\{z^q\}$ with $0 \neq z^q \to 0$ and any sequence $\{x_q\} \subset \mathcal{L}$ satisfying $x^q \to \bar{x}$ and $g(x^q) \in z^q - \mathcal{K}$, there holds

$$\frac{f(x^q) \; - \; f(\bar{x})}{||z^q||^{\alpha}} + M \; \geq \; 0 \tag{2}$$

Constraint Qualification of SDCMPCC Formulation

Huang et.al shows that local calmness or order 1 implies the existence of KKT multipliers:

Theorem

Let $\bar{x}$ be a local optimal solution to Problem(1) and (1) is calm of order 1 at $\bar{x}$. Then, there exists $\mu \in K^*$ such that the system:

$$0 \in \partial f(\bar{x}) + \mu(\nabla g(\bar{x})) + N_{\mathcal{L}}(\bar{x})$$

$$\mu(g(\bar{x})) = 0$$

is consistent. $N_{\mathcal{L}}(\bar{x})$ is the Clarke normal cone of $\mathcal{L}$ at $\bar{x}$.

Constraint Qualification of SDCMPCC Formulation

Proposition

Calmness of Order 1 holds at each local optimum $(\bar{X}, \bar{U})$ in the SDCMPCC Formulation.

In the proof, let (X^q, U^q) be a feasible solution to the perturbed SDCMPCC Formulation with perturbation parameter (z^q, r^q, h_1^q, h_2^q).

Want to show the existence of $M < \infty$ that satisfies:

$$(n - < I, U^q >) - (n - < I, \bar{U} >) \geq -M||(z^q, r^q, h_1^q, h_2^q)||$$

for any $(X^q, U^q) \to (\bar{X}, \bar{U})$.

An upper bound of $n - < I, \bar{U} >$ is $rank(\bar{X})$.

Sketch of Proof

Want to get a lower bound for $n- < I, U^q >$. (X^q, U^q) is feasible to the perturbed problem:

$$
\begin{aligned}
\underset{X,U \in \mathbb{S}^n}{\text{minimize}} \quad & n - \ < I, \ U > \\
\text{subject to} \quad & X \ + \ z^q \in \tilde{\mathcal{C}} \cap \mathcal{S}_+^n \\
& - < X, \ U > \leq \ r^q \\
& < X, \ U > \leq \ r^q \\
& I \ - \ U \ \succeq \ -h_1^q \, I \\
& U \ \succeq \ -h_2^q \, I
\end{aligned}
$$

The lower bound can be acquired by fixing $X \ = \ X^q$ in the perturbed problem.

Sketch of Proof

By fixing $X = X^q$ we can get the following problem:

$$
\begin{aligned}
\underset{U \in \mathbb{S}^n}{\text{minimize}} \quad & n - < I, U > \\
\text{subject to} \quad & - < X^q, U > \leq r^q, \qquad y_1 \\
& < X^q, U > \leq r^q, \qquad y_2 \\
& I - U \succeq -h_1^q I, \qquad \Omega_1 \\
& U \succeq -h_2^q I, \qquad \Omega_2
\end{aligned}
$$

where $y_1, y_2, \Omega_1, \Omega_2$ are the Lagrangian multipliers for the corresponding constraints.

U^q is feasible to the above problem.

Slater condion holds for the above problem. Can find a lower bound the objective by Strong Duality

Sketch of Proof

The dual problem is:

$$
\begin{aligned}
\underset{y_1, y_2 \in \mathbb{R},\, \Omega_1, \Omega_2 \in \mathbb{S}^n}{\text{maximize}} \quad & n + r^q y_1 + r^q y_2 - (1 + h_1^q)\, trace(\Omega_1) - h_2^q\, trace(\Omega_2) \\
\text{subject to} \quad & -y_1 X^q + y_2 X^q - \Omega_1 + \Omega_2 = -I \\
& y_1, y_2 \leq 0 \\
& \Omega_1, \Omega_2 \succeq 0
\end{aligned}
$$

By diagonalizing X^q we can get a tightened problem:

$$
\begin{aligned}
\underset{y_1, y_2 \in \mathbb{R},\, f, g \in \mathbb{R}^n}{\text{maximize}} \quad & n + r^q y_1 + r^q y_2 - (1 + h_1^q) \textstyle\sum_i f_i - h_2^q \sum_i g_i \\
\text{subject to} \quad & -y_1 \lambda_i^q + y_2 \lambda_i^q - f_i + g_i = -1, \forall i = 1 \cdots n \\
& y_1, y_2 \leq 0 \\
& f_i, g_i \geq 0, \forall i = 1 \cdots n
\end{aligned}
$$

Sketch of Proof

Since $X^q \to \bar{X}$, $\lambda_i^q \to \lambda_i$. Can get a lower bound for the objective of the dual problem, which is:

$$\text{rank}(X) - \frac{2r^q}{\tilde{\lambda}} - (n - rank(X))(h_1^q + (1 + h_1^q)\frac{2}{\tilde{\lambda}}||z^q||) - \frac{h_2^q}{\tilde{\lambda}}||\bar{X}||^*$$

where $\tilde{\lambda}$ is the smallest positive eigenvalue of $\bar{X}$. We can take

$$M = \frac{2}{\tilde{\lambda}} + \frac{1}{\tilde{\lambda}}||\bar{X}||^* + (n - rank(X))(1 + \frac{4}{\tilde{\lambda}})$$

KKT Condition of SDCMPCC Formulation

Given $\mathcal{C} = \{X \mid < A_i, X > \geq b_i, \forall i = 1 \cdots p\}$
The KKT condition is:

$$\begin{aligned} 0 \preceq U \quad & \perp -I + \mu X + Y \succeq 0 \\ 0 \preceq X \quad & \perp -\sum \lambda_i A_i + \mu U \succeq 0 \\ 0 \preceq Y \quad & \perp I - U \succeq 0 \\ 0 \leq \lambda_i \quad & \perp b_i - < A_i, X_i > \geq 0, \forall i = 1 \cdots, p \end{aligned} \tag{3}$$

Where λ, μ and Y are lagrangian multipliers corresponding to the constraints $A(X) = b$, $< X, U >= 0$ and $I - U \succeq 0$ respectively.

Any feasible pair (X, U) with U given by $P_0 P_0^T$ with columns of P_0 to be the eigenvectors in the null space of X ,is a KKT stationary point of the SDCMPCC Formulation.

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