

MATH 6740: Financial Mathematics and Simulation

Homework 3 solutions/presentation

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Solve Exercise Problems 1.4, 1.6, and 1.7 in (Shreve, 2004, Chapter 1, p. 42-43).

1. Exercise 1.4

1.1. Exercise 1.4 i)

First follow (Shreve, 2004, Example 1.2.5, p. 11-12) to construct a uniformly distributed random variable X taking values in $[0, 1]$ and defined on infinite coin-toss space Ω_∞ where $p_H = p_T = 0$, by defining

$$Y_n(\omega|_n) = \begin{cases} 1 & \text{if } \omega|_n = H, \\ 0 & \text{if } \omega|_n = T, \end{cases} \quad (1)$$

which is an indicator of whether the result of the n^{th} toss is $\omega|_n = H$, where $\omega \in \Omega_\infty$, and

$$X = \sum_{n=1}^{\infty} \frac{Y_n}{2^n}. \quad (2)$$

(Shreve, 2004, Example 1.2.5, p. 11-12) concluded that $X \sim U(0, 1)$. Next (Shreve, 2004, Example 1.2.6, p. 12-13) showed that with the probability density function and cumulative distribution function of the standard normal distribution being:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad (3)$$

$$N(x) = \int_{-\infty}^x \varphi(\xi) d\xi, \quad (4)$$

we can defined a new random variable $Z = N^{-1}(X)$, where $N^{-1}(\cdot)$ is the inverse function of $N(\cdot)$, which exists because $N(\cdot)$ is strictly increasing. For $\forall -\infty < a \leq b < \infty$, we have

$$\begin{aligned} \mu_Z[a, b] &= P(\omega \in \Omega_\infty; a \leq Z(\omega) \leq b) \\ &= P(\omega \in \Omega_\infty; a \leq N^{-1}(X(\omega)) \leq b) \\ &= P(\omega \in \Omega_\infty; N(a) \leq X(\omega) \leq N(b)) \\ &= N(b) - N(a) = \int_a^b \varphi(\xi) d\xi. \end{aligned} \quad (5)$$

Therefore

$$Z = N^{-1}(X) = N^{-1}\left(\sum_{n=1}^{\infty} \frac{Y_n}{2^n}\right) \sim N(0, 1). \quad (6)$$

1.2. Exercise 1.4 ii)

Define the indicator function $Y_n(\omega|_n)$ as before:

$$Y_n(\omega|_n) = \begin{cases} 1 & \text{if } \omega|_n = H, \\ 0 & \text{if } \omega|_n = T, \end{cases} \quad (7)$$

and a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ on Ω_{∞} , where each element only depends on the results of the first n^{th} tosses, as:

$$X_n = \sum_{i=1}^n \frac{Y_i}{2^i}. \quad (8)$$

The binary number $\overline{Y_1 Y_2 \cdots Y_n} = \sum_{i=1}^n 2^{n-i} Y_i$ is a random variable taking all integer values between 0 and $2^n - 1$ with equal probability $1/2^n$, therefore $X_n = \overline{Y_1 Y_2 \cdots Y_n} / 2^n$ is a random variable, with equal probability $1/2^n$, taking values in the set $\{0, \frac{1}{2^n}, \dots, \frac{2^n-1}{2^n}\}$, which is an arithmetic series from 0 to $(2^n - 1)/2^n$ with common distance $1/2^n$, thus:

$$P(\omega \in \Omega; \frac{k-1}{2^n} \leq X_n < \frac{k}{2^n}, k \in \mathbb{Z}, k \in [1, 2^n]) = \frac{1}{2^n}, \quad (9)$$

therefore X_n 's cumulative distribution function is càdlàg. Now define a sequence of random variables $\{Z_n\}_{n=1}^{\infty}$, with each element $Z_n = N^{-1}(X_n)$, then for $\forall -\infty < a \leq b < \infty$, there is:

$$\begin{aligned} \mu_{Z_n}[a, b] &= P(\omega \in \Omega_{\infty}; a \leq Z_n(\omega) \leq b) \\ &= P(\omega \in \Omega_{\infty}; a \leq N^{-1}(X_n(\omega)) \leq b) \\ &= P(\omega \in \Omega_{\infty}; N(a) \leq X_n(\omega) \leq N(b)) \\ &= P(\omega \in \Omega_{\infty}; \frac{\lfloor N(a) \cdot 2^n \rfloor}{2^n} \leq X_n(\omega) < \frac{\lfloor N(b) \cdot 2^n \rfloor + 1}{2^n}) \\ &= \frac{\lfloor N(b) \cdot 2^n \rfloor - \lfloor N(a) \cdot 2^n \rfloor + 1}{2^n} \\ &= N(b) - N(a) + \frac{O(1)}{2^n}, \end{aligned} \quad (10)$$

therefore

$$\lim_{n \rightarrow \infty} \mu_{Z_n}[a, b] = \mu_Z[a, b], \quad (11)$$

and

$$\lim_{n \rightarrow \infty} Z_n(\omega) = Z(\omega), \quad \forall \omega \in \Omega_{\infty}, \quad (12)$$

where $Z_n(\omega)$ is defined, with the inverse of the standard normal distribution $N^{-1}(\cdot)$, as:

$$Z_n = N^{-1}(X_n) = N^{-1}\left(\sum_{i=1}^n \frac{Y_i}{2^i}\right). \quad (13)$$

2. Exercise 1.6

2.1. Exercise 1.6 i)

$$\begin{aligned}
E[e^{uX}] &= \int_{-\infty}^{\infty} e^{ux} f(x) dx \\
&= \int_{-\infty}^{\infty} e^{ux} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2 - (2\mu + 2u\sigma^2)x + \mu^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x - (\mu + u\sigma^2)]^2 - 2\mu u\sigma^2 - u^2\sigma^4}{2\sigma^2}} dx \\
&= e^{u\mu + \frac{1}{2}u^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x - (\mu + u\sigma^2)]^2}{2\sigma^2}} dx \\
&= e^{u\mu + \frac{1}{2}u^2\sigma^2}.
\end{aligned} \tag{14}$$

2.2. Exercise 1.6 ii)

For the convex function $\varphi(x) = e^{ux}$:

$$E[\varphi(X)] = E[e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2} \geq e^{u\mu} = \varphi(E[X]), \tag{15}$$

which satisfies *Jensen's inequality*.

3. Exercise 1.7

3.1. Exercise 1.7 i)

$$\begin{aligned}
f(x) &= \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}} \\
&= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n\pi}} \cdot \lim_{n \rightarrow \infty} e^{-\frac{x^2}{2n}} \\
&= 0.
\end{aligned} \tag{16}$$

3.2. Exercise 1.7 ii)

Since $f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}}$ is essentially the probability density function of a random variable with normal distribution $X \sim N(0, n)$, therefore $\int_{-\infty}^{\infty} f_n(x) dx = 1$, thus

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = 1. \tag{17}$$

3.3. Exercise 1.7 iii)

The *Monotone Convergence Theorem* requires that

$$0 \leq f_1 \leq f_2 \leq \dots \quad \text{almost everywhere.} \quad (18)$$

In this instance, the range of x where $f_n(x) > f_{n+1}(x)$ can be determined with, since $f_n(x) > 0$:

$$\begin{aligned} \frac{f_n(x)}{f_{n+1}(x)} &= \frac{\sqrt{n+1} \cdot e^{-\frac{x^2}{2n}}}{\sqrt{n} \cdot e^{-\frac{x^2}{2(n+1)}}} > 1 \quad \Rightarrow \quad e^{\frac{x^2}{2n} - \frac{x^2}{2(n+1)}} < \sqrt{\frac{n+1}{n}} \\ e^{\frac{x^2}{2n(n+1)}} &< \sqrt{\frac{n+1}{n}} \\ \frac{x^2}{2n(n+1)} &< \frac{1}{2} \ln \frac{n+1}{n} \\ -n(n+1) \ln \frac{n+1}{n} &< x < n(n+1) \ln \frac{n+1}{n}, \end{aligned} \quad (19)$$

which is a non-trivial range. Therefore the fact that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx \neq \int_{-\infty}^{\infty} f(x) dx$$

does not violate the *Monotone Convergence Theorem*.

Appendix A. Original Homework Questions (attached)

- Exercise 1.4.** (i) Construct a standard normal random variable Z on the probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P})$ of Example 1.1.4 under the assumption that the probability for head is $p = \frac{1}{2}$. (Hint: Consider Examples 1.2.5 and 1.2.6.)
- (ii) Define a sequence of random variables $\{Z_n\}_{n=1}^\infty$ on Ω_∞ such that

$$\lim_{n \rightarrow \infty} Z_n(\omega) = Z(\omega) \text{ for every } \omega \in \Omega_\infty$$

and, for each n , Z_n depends only on the first n coin tosses. (This gives us a procedure for approximating a standard normal random variable by random variables generated by a finite number of coin tosses, a useful algorithm for Monte Carlo simulation.)

Exercise 1.6. Let u be a fixed number in \mathbb{R} , and define the convex function $\varphi(x) = e^{ux}$ for all $x \in \mathbb{R}$. Let X be a normal random variable with mean $\mu = \mathbb{E}X$ and standard deviation $\sigma = [\mathbb{E}(X - \mu)^2]^{\frac{1}{2}}$, i.e., with density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

- (i) Verify that

$$\mathbb{E}e^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}.$$

- (ii) Verify that Jensen's inequality holds (as it must):

$$\mathbb{E}\varphi(X) \geq \varphi(\mathbb{E}X).$$

Exercise 1.7. For each positive integer n , define f_n to be the normal density with mean zero and variance n , i.e.,

$$f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}}.$$

- (i) What is the function $f(x) = \lim_{n \rightarrow \infty} f_n(x)$?
- (ii) What is $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx$?
- (iii) Note that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx \neq \int_{-\infty}^{\infty} f(x) dx.$$

Explain why this does not violate the Monotone Convergence Theorem, Theorem 1.4.5.