MATH 6740: Financial Mathematics and Simulation
Homework 2 solutions/presentation

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1. Q1

For each time interval Δt, the stock price's transition from $S(n)$ to $S(n+1)$ is based on a binary model, with $S(n+1) = uS(n)$ at probability p_u, and $S(n+1) = dS(n)$ at probability p_d. The coefficients u and d are:

$$\begin{cases} u = e^{\mu\Delta t + \sigma\sqrt{\Delta t}}, \\ d = e^{\mu\Delta t - \sigma\sqrt{\Delta t}}, \end{cases} \tag{1}$$

where μ and σ are the mean rate of return per unit time and variance per unit time respectively. Under the assumption that the market is arbitrage-free, the probability of up and down-market are:

$$\begin{aligned} q_u &= \frac{e^{r\Delta t} - e^{\mu\Delta t - \sigma\sqrt{\Delta t}}}{e^{\mu\Delta t + \sigma\sqrt{\Delta t}} - e^{\mu\Delta t - \sigma\sqrt{\Delta t}}} \\ &\approx \frac{r\Delta t + \frac{(r\Delta t)^2}{2} - \left[\mu\Delta t - \sigma\sqrt{\Delta t} + \frac{\left(\mu\Delta t - \sigma\sqrt{\Delta t}\right)^2}{2}\right]}{\left[\mu\Delta t + \sigma\sqrt{\Delta t} + \frac{\left(\mu\Delta t + \sigma\sqrt{\Delta t}\right)^2}{2}\right] - \left[\mu\Delta t - \sigma\sqrt{\Delta t} + \frac{\left(\mu\Delta t - \sigma\sqrt{\Delta t}\right)^2}{2}\right]} \\ &= \frac{(r-\mu)\sqrt{\Delta t} + \frac{r^2}{2}(\Delta t)^{\frac{3}{2}} + \sigma - \frac{\mu^2(\Delta t)^{\frac{3}{2}} + \sigma^2\sqrt{\Delta t} - 2\mu\sigma\Delta t}{2}}{2\sigma + 2\mu\sigma\Delta t} \\ &\approx \frac{1}{2} + \frac{r - \mu - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t}, \end{aligned} \tag{2a}$$

$$q_d = 1 - q_u = \frac{1}{2} - \frac{r - \mu - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t}. \tag{2b}$$

Since $X(n)$ is the number of heads in n coin tosses, we define a set of random variable $\{\xi_j\}$ to indicate if head shows up in the j^{th} coin toss. Therefore:

$$\xi_j = \begin{cases} 1 & w/\ p_u, \\ 0 & w/\ p_d, \end{cases} \tag{3}$$

with the mean and variance of ξ_j being:

$$E[\xi_j] = p_u, \tag{4a}$$

$$var(\xi_j) = E[\xi_j^2] - \left(E[\xi_j]\right)^2 = p_u(1 - p_u). \tag{4b}$$

$X(n)$ is defined as the sum of the first n elements in the set of i.i.d. variables $\{\xi_j\}$:

$$X(n) = \sum_{j=1}^{n} \xi_j, \tag{5}$$

$$E[X(n)] = \sum_{j=1}^{n} E[\xi_j] = np_u, \tag{6a}$$

$$var(X(n)) = \sum_{j=1}^{n} var(\xi_j) = np_u\left(1 - p_u\right). \tag{6b}$$

2. Q2

Define the random variable $Z(n)$ as:

$$Z(n) = \frac{2X(n) - n}{\sqrt{n}}. \tag{7}$$

Therefore:

$$E[Z(n)] = \frac{2}{\sqrt{n}} E[X(n)] - \sqrt{n} = \frac{2}{\sqrt{n}} np_u - \sqrt{n} = \sqrt{n}\left(2p_u - 1\right), \tag{8a}$$

$$var(Z(n)) = \frac{2}{\sqrt{n}} var(X(n)) = \frac{2}{\sqrt{n}} np_u\left(1 - p_u\right) = 2\sqrt{n} p_u\left(1 - p_u\right). \tag{8b}$$

3. Q3

According to the Central Limit Theorem, as $n \to \infty$ (therefore $\Delta t = t/n \to 0$):

$$\frac{X(n) - np_u}{\sqrt{np_u\left(1 - p_u\right)}} \to N\left(0, 1\right), \tag{9}$$

$$\frac{X(n) - n\left(\frac{1}{2} + \frac{r-\mu-\frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t}\right)}{\sqrt{n\left(\frac{1}{4} - \frac{1}{4}\frac{\left(r-\mu-\frac{\sigma^2}{2}\right)^2}{\sigma^2}\Delta t\right)}} \to N\left(0, 1\right),$$

$$\frac{2X(n) - n - n\frac{r-\mu-\frac{\sigma^2}{2}}{\sigma}\sqrt{\Delta t}}{\sqrt{n}} \to N\left(0, 1\right),$$

$$\frac{2X(n) - n}{\sqrt{n}} - \frac{r - \mu - \frac{\sigma^2}{2}}{\sigma}\sqrt{n\Delta t} \to N\left(0, 1\right),$$

$$\frac{2X(n) - n}{\sqrt{n}} \to N\left(\frac{r - \mu - \frac{\sigma^2}{2}}{\sigma}\sqrt{t}, 1\right). \tag{10}$$

3.1. (b)

$$
\begin{aligned}
S(t) &= S(0)u^{\#H}d^{\#T} = S(0)e^{\left(\mu\Delta t+\sigma\sqrt{\Delta t}\right)\#H}e^{\left(\mu\Delta t-\sigma\sqrt{\Delta t}\right)\#T} \\
&= S(0)e^{\mu\Delta t(\#H+\#T)}e^{\sigma\sqrt{\Delta t}(\#H-\#T)} \\
&= S(0)e^{\mu\Delta tn}e^{\sigma\sqrt{\Delta t}Z(n)\sqrt{n}} \\
&= S(0)e^{\mu t+Z(n)\sigma\sqrt{t}},
\end{aligned}
\tag{11}
$$

therefore:

$$lnS(t) = lnS(0) + \mu t + Z(n)\cdot\sigma\sqrt{t}. \tag{12}$$

From the conclusion from Equation (10):

$$Z(t) \sim N\left(\frac{r-\mu-\frac{\sigma^2}{2}}{\sigma}\sqrt{t}, 1\right),$$

$$Z(t)\cdot\sigma\sqrt{t} \sim N\left(\left(r-\mu-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right),$$

$$lnS(t) \sim N\left(lnS(0)+\mu t+\left(r-\mu-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right),$$

$$lnS(t) \sim N\left(lnS(0)+\left(r-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right). \tag{13}$$

The expectation of $S(t)$ is:

$$E[S(t)] = S(0)\cdot e^{rt}, \tag{14}$$

which can be derived as the following or calculated according to log-normal distribution properties.

$$\overset{=\ln S(t)}{x} \sim N\left(\ln S(0)+\left(r-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$$

$$
\begin{aligned}
E[S(t)] &= E[e^{x}] \\
&= \int_{-\infty}^{+\infty} e^{x}\frac{1}{\sqrt{2\pi t}\,\sigma}e^{-\frac{[x-\ln S(0)-(r-\frac{\sigma^2}{2})t]^2}{2\sigma^2 t}}dx \\
&= \frac{1}{\sqrt{2\pi t}\,\sigma}\int_{-\infty}^{+\infty} e^{-\frac{x^2-2\ln S(0)x-(2r-\sigma^2)tx+[\ln S(0)+(r-\frac{\sigma^2}{2})t]^2-2\sigma^2 tx}{2\sigma^2 t}}dx \\
&= \frac{1}{\sqrt{2\pi t}\,\sigma}\int_{-\infty}^{+\infty} e^{-\frac{x^2-2\ln S(0)x-(2r+\sigma^2)tx+[\ln S(0)+(r+\frac{\sigma^2}{2})t]^2-2\sigma^2 t[\ln S(0)+rt]}{2\sigma^2 t}}dx \\
&= e^{\ln S(0)+rt}\underbrace{\frac{1}{\sqrt{2\pi t}\,\sigma}\int_{-\infty}^{+\infty} e^{-\frac{[x-\ln S(0)-(r+\frac{\sigma^2}{2})t]^2}{2\sigma^2 t}}dx}_{1}.
\end{aligned}
$$

$$\Delta\ \left[\ln S(0)+rt\pm\frac{\sigma^2}{2}t\right]^2$$

$$= 2\frac{\sigma^2}{2}t\cdot[\ln S(0)+rt]$$

4. Q4

$$X_n = \begin{cases} +1, & p_u = 1/2, \\ -1, & p_d = 1/2, \end{cases} \tag{15}$$

$$S_n = S_{n-1} + X_n. \tag{16}$$

To show that the stochastic process $\{S_n\}$ is a Martingale:

$$\begin{aligned} E_p[S_{n+1} \mid \mathbb{F}_n] &= E_p[S_n + X_{n+1} \mid \mathbb{F}_n] \\ &= (S_n \mid \mathbb{F}_n) + E_p[X(n+1) \mid \mathbb{F}_n] \\ &= s_n + \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1)\right) \\ &= s_n, \end{aligned} \tag{17}$$

where $\mathbb{F}_n$ is the information known at Step n, and s_n is the observed value of S_n, hence $s_n = (S_n \mid \mathbb{F}_n)$. Equation (17) proved that $\{S_n\}$ is a Martingale by definition.

5. Q5

$$\begin{aligned} E[S_n] &= E[S_0 + \sum_{j=1}^{n} X_j] \\ &= E[S_0] + \sum_{j=1}^{n} E[X_j] \\ &= 0. \end{aligned} \tag{18}$$

$$\begin{aligned} var(S_n) &= E[S_n^2] - (E[S_n])^2 \\ &= E[\left(S_0 + \sum_{j=1}^{n} X_j\right)^2] - (E[S_n])^2 \\ &= E[\sum_{j=1}^{n} X_j^2] - 0 \\ &= \sum_{j=1}^{n} E[X_j^2] \\ &= n. \end{aligned} \tag{19}$$

6. Q6

Define a martingale:

$$S_n = e^{\sigma M_n} (\cosh \sigma)^{-n}, \tag{20}$$

where M_n is symmetric random walk starting at $M_0 = 0$. The corresponding stopped martingale is also a martingale:

$$S_{n\wedge\tau_m} = e^{\sigma M_{n\wedge\tau_m}} (\cosh\sigma)^{-n\wedge\tau_m}, \tag{21}$$

where τ_m is the stopping time when the random walk first arrives and stops at $M_n = m$. Therefore

$$E[S_{n\wedge\tau_m} \mid \mathbb{F}_0] = S_{0\wedge\tau_m} = S_0 = 1. \tag{22}$$

Taking the limit of the above equation when $n \to \infty$:

$$\begin{aligned}
\lim_{n\to\infty} E[S_{n\wedge\tau_m}] &= \lim_{n\to\infty} E[e^{\sigma M_{n\wedge\tau_m}} (\cosh\sigma)^{-n\wedge\tau_m}] \\
&= E[\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_m}} (\cosh\sigma)^{-n\wedge\tau_m}] \\
&= E[e^{\sigma m} (\cosh\sigma)^{-\tau_m}] = 1,
\end{aligned} \tag{23}$$

where the second equality is based on Lebesgue's Dominated Convergence Theorem, while the third equality is based on the fact that $P(\tau_m < \infty) = 1$. Since only τ_m is stochastic in the above equation:

$$E[(\cosh\sigma)^{-\tau_m}] = e^{-\sigma m}. \tag{24}$$

Define $\alpha = (\cosh\sigma)^{-1} = 2/\,(e^{\sigma} + e^{-\sigma})$, and choose $\sigma > 0$ out of symmetry:

$$e^{-\sigma} = \frac{1 - \sqrt{1-\alpha^2}}{\alpha} \qquad \Rightarrow \qquad -\sigma = \ln\frac{1 - \sqrt{1-\alpha^2}}{\alpha}, \tag{25}$$

therefore:

$$E[\alpha^{\tau_m}] = e^{-\sigma m} = \left(\frac{1 - \sqrt{1-\alpha^2}}{\alpha}\right)^m. \tag{26}$$

Since the preceding derivations implied that $m > 0$, for any non-zero stopping criterion level m, because the random walk is symmetric in this case, the moment generating function becomes:

$$E[\alpha^{\tau_m}] = \left(\frac{1 - \sqrt{1-\alpha^2}}{\alpha}\right)^{|m|}. \tag{27}$$

Taking derivatives on both sides, and using Lebesgue's Dominated Convergence Theorem to move the derivative inside the expectation:

$$\begin{aligned}
\frac{d}{d\alpha} E[\alpha^{\tau_m}] &= E[\frac{d}{d\alpha}\alpha^{\tau_m}] \;=\; E[\tau_m \alpha^{\tau_m - 1}] \\
&= \frac{d}{d\alpha}\left(\frac{1 - \sqrt{1-\alpha^2}}{\alpha}\right)^{|m|} \\
&= \frac{|m|\left(1 - \sqrt{1-\alpha^2}\right)^{|m|-1} \alpha^{-|m|+1}}{\sqrt{1-\alpha^2}} - |m|\frac{\left(1 - \sqrt{1-\alpha^2}\right)^{|m|}}{\alpha^{|m|+1}} \\
&= |m|\frac{\left(1 - \sqrt{1-\alpha^2}\right)^{|m|}}{\alpha^{|m|+1}\sqrt{1-\alpha^2}},
\end{aligned} \tag{28a}$$

$$
\begin{aligned}
\frac{d^2}{d\alpha^2}E[\alpha^{\tau_m}] &= E[\frac{d^2}{d\alpha^2}\alpha^{\tau_m}] \;=\; E[\left(\tau_m^2-\tau_m\right)\alpha^{\tau_m-2}] \\
&= |m|\frac{d}{d\alpha}\frac{\left(1-\sqrt{1-\alpha^2}\right)^{|m|}}{\alpha^{|m|+1}\sqrt{1-\alpha^2}} \\
&= |m|\frac{\left(1-\sqrt{1-\alpha^2}\right)^{|m|}\left[-|m|\alpha^2+|m|+\sqrt{1-\alpha^2}\left(2\alpha^2-1\right)\right]}{\alpha^{|m|+2}\left(1-\alpha^2\right)^2}, \qquad (28b)
\end{aligned}
$$

therefore:

$$
E[\tau_m] = \lim_{\alpha\to 1^-} E[\tau_m\alpha^{\tau_m-1}] = +\infty, \qquad (29a)
$$

$$
\begin{aligned}
var(\tau_m) &= E[\tau_m^2]-(E[\tau_m])^2 \;=\; \lim_{\alpha\to 1^-}\left[E[\left(\tau_m^2-\tau_m\right)\alpha^{\tau_m-2}]+E[\tau_m\alpha^{\tau_m-1}]-\left(E[\tau_m\alpha^{\tau_m-1}]\right)^2\right] \\
&= \lim_{\alpha\to 1^-}\frac{\begin{pmatrix}\left(1-\sqrt{1-\alpha^2}\right)^{|m|}\left[-m^2\alpha^2+m^2+|m|\sqrt{1-\alpha^2}\left(2\alpha^2-1\right)\right]\alpha^{|m|} \\ +|m|\alpha^{|m|+1}\left(1-\alpha^2\right)^{3/2}\left(1-\sqrt{1-\alpha^2}\right)^{|m|}-m^2\left(1-\alpha^2\right)\left(1-\sqrt{1-\alpha^2}\right)^{2|m|}\end{pmatrix}}{\alpha^{2|m|+2}\left(1-\alpha^2\right)^2} \\
&= +\infty. \qquad (29b)
\end{aligned}
$$

Appendix A. Original Homework Questions (attached)

Chjan Lim, Professor Mathematical Sciences

For Q1 – 3, Stock price S(t) is given by an exponential expression involving μ the mean rate of return per unit time and σ (variance per unit time), namely

$S(t) = S(n\Delta t) = S(0)exp(\mu t + \sigma\sqrt{t})$. **This lognormal model will be discretized using** $t = n(\Delta t)$ **in Q 1 – 3.**

1. **Calculate (a) mean and variance of X(n), where X(n) equals the number of Heads minus the number of Tails in n coin tosses.**
2. **Calculate mean and variance of (2X(n) – n) /sqrt(n)**
3. **Use the Central Limit thm to show that (2X(n) – n) /sqrt(n) converges as Δt tends to zero, to a Normal distribution with mean = sqrt(t) (μ + σ^2/2 – r) / σ and variance equal to 1.**

Hint: what are the limits of the mean and var in Q2 as the time period Δt goes to 0

3 b) Verify that S(t) is lognormally distributed with mean = log S(0) + (r – σ^2/2) t and var = (σ^2) t.

For Q4 and more, S(n) is NOT stock price but given by definition below.

4. **Consider the symmetric random walk:**

S(n) = S(n-1) + X(n) with S(0) = 0 and X(n) = -1 or + 1 with equal probability.

Show that the stochastic process {S(n)} is a Martingale

5. **Calculate E[S(n)] and var (S(n))**
6. **Harder problems on First Exit Times: The first time t at which the random walker above hits the level S(t) = 10 say is a random variable, so we can ask for its mean and variance. We will briefly outline without some of the proofs the methods for calculating such Optional Stopping times, this and next week.**