# MATH 6740: Financial Mathematics and Simulation
# Homework 4 solutions/presentation

Jubiao "Jack" Yang[a]

[a]*Rensselaer Polytechnic Institute, Troy, NY 12180*

Solve Exercise Problems 2.5 and 2.10 in (Shreve, 2004, Chapter 2), and Problems 3.1, 3.2, and 3.4 in (Shreve, 2004, Chapter 3).

## 1. Exercise 2.5

The p.d.f. for random variable $X$ is:

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy \\
&= \int_{-|x|}^{+\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-|x|}^{+\infty} (2|x|+y) \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} d(2|x|+y) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-|x|}^{+\infty} \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} d\frac{(2|x|+y)^2}{2} \\
&= \frac{1}{\sqrt{2\pi}} \left(\exp\left\{-\frac{|x|^2}{2}\right\} - \exp\{-\infty\}\right) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\},
\end{aligned}
\tag{1}
$$

and the p.d.f. for random variable $Y$ is:

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx \\
&= 2\int_0^{+\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{\max\{0,-y\}}^{+\infty} (2x+y) \exp\left\{-\frac{(2x+y)^2}{2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\max\{0,-y\}}^{+\infty} \exp\left\{-\frac{(2x+y)^2}{2}\right\} d\frac{(2x+y)^2}{2} \\
&= \frac{1}{\sqrt{2\pi}} \left(\exp\left\{-\frac{(2\max\{0,-y\}+y)^2}{2}\right\} - 0\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{|y|^2}{2}\right\} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}.
\end{aligned}
\tag{2}
$$

Therefore both $X$ and $Y$ are standard normal random variables. The covariance of them is:

$$
\begin{aligned}
cov(X,Y) &= E[(X - E[X])(Y - E[Y])] = E[XY] \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{X,Y}(x,y) dx dy \\
&= \int_{-\infty}^{+\infty} y \left( \int_{-\infty}^{+\infty} x \frac{2|x| + y}{\sqrt{2\pi}} \exp\left\{ -\frac{(2|x| + y)^2}{2} \right\} dx \right) dy \\
&= \int_{-\infty}^{+\infty} y \cdot 0 dy = 0,
\end{aligned}
\tag{3}
$$

therefore $X$ and $Y$ are uncorrelated. However, since $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$, $X$ and $Y$ are not independent.

## 2. Exercise 2.10

$$
\begin{aligned}
\int_A g(X) dP(X) &= \int_{-\infty}^{+\infty} g(x) 1_{\omega \in A} f_X(x) dx \\
&= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \frac{y f_{X,Y}(x,y)}{f_X(x)} dy \right) 1_{\omega \in A} f_X(x) dx \\
&= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} y f_{X,Y}(x,y) dy \right) 1_{\omega \in A} dx \\
&= \int_{-\infty}^{+\infty} y \left( \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx \right) 1_{\omega \in A} dy \\
&= \int_{-\infty}^{+\infty} y 1_{\omega \in A} f_Y(y) dy = \int_A Y dP(Y).
\end{aligned}
\tag{4}
$$

## 3. Exercise 3.1

According to Definition 3.3.3(iii) in Shreve (2004), $W(u_2) - W(u_1)$ is independent of $\mathbb{F}(u_1)$; while according to Definition 3.3.3(i) in Shreve (2004), $\mathbb{F}(t) \subset \mathbb{F}(u_1)$. Therefore $W(u_2) - W(u_1)$ is independent of $\mathbb{F}(t)$.

## 4. Exercise 3.2

For $0 \leq s \leq t$:

$$
\begin{aligned}
E[W^2(t) - t | \mathbb{F}_s] &= E[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t | \mathbb{F}(s)] \\
&= E[(W(t) - W(s))^2] + 2W(s) E[W(t) - W(s) + W(s) | \mathbb{F}(s)] - W^2(s) - t \\
&= var(W(t) - W(s)) + 2W(s)(W(s) + E[W(t) - W(s) | \mathbb{F}(s)]) - W^2(s) - t \\
&= t - s + 2W^2(s) - W^2(s) - t \\
&= W^2(s) - s,
\end{aligned}
\tag{5}
$$

therefore $\{W^2(t) - t\}$ is a martingale.

## 5. Exercise 3.4

*5.1. (i)*

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \cdot \max_{0 \le k \le n-1} |W(t_{j+1}) - W(t_j)| \ge \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2, \qquad (6)$$

therefore the first variation of Brownian motion is:

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \ge \lim_{||\Pi|| \to 0} \frac{\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2}{\max_{0 \le k \le n-1} |W(t_{j+1}) - W(t_j)|} = \frac{T}{0} = \infty. \qquad (7)$$

*5.2. (ii)*

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \le \max_{0 \le k \le n-1} |W(t_{j+1}) - W(t_j)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2, \qquad (8)$$

therefore the cubic variation of Brownian motion is:

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \le \lim_{||\Pi|| \to 0} \max_{0 \le k \le n-1} |W(t_{j+1}) - W(t_j)| \cdot \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2$$

$$= 0 \cdot T = 0 \qquad (9)$$

**Exercise 2.5.** Let $(X, Y)$ be a pair of random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} & \text{if } y \geq -|x|, \\ 0 & \text{if } y < -|x|. \end{cases}$$

Show that $X$ and $Y$ are standard normal random variables and that they are uncorrelated but not independent.

**Exercise 2.10.** Let $X$ and $Y$ be random variables (on some unspecified probability space $(\Omega, \mathcal{F}, \mathbb{P})$), assume they have a joint density $f_{X,Y}(x,y)$, and assume $\mathbb{E}|Y| < \infty$. In particular, for every Borel subset $C$ of $\mathbb{R}^2$, we have

$$\mathbb{P}\{(X, Y) \in C\} = \int_C f_{X,Y}(x, y)\, dx\, dy.$$

In elementary probability, one learns to compute $\mathbb{E}[Y|X = x]$, which is a *nonrandom* function of the *dummy variable* $x$, by the formula

$$\mathbb{E}[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy, \qquad (2.6.1)$$

where $f_{Y|X}(y|x)$ is the *conditional density* defined by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

The denominator in this expression, $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, \eta) d\eta$, is the *marginal density* of $X$, and we must assume it is strictly positive for every $x$. We introduce the symbol $g(x)$ for the function $\mathbb{E}[Y|X = x]$ defined by (2.6.1); i.e.,

$$g(x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy = \int_{-\infty}^{\infty} \frac{y f_{X,Y}(x, y)}{f_X(x)} dy.$$

In measure-theoretic probability, conditional expectation is a *random variable* $\mathbb{E}[Y|X]$. This exercise is to show that when there is a joint density for $(X, Y)$, this random variable can be obtained by substituting the random variable $X$ in place of the dummy variable $x$ in the function $g(x)$. In other words, this exercise is to show that

$$\mathbb{E}[Y|X] = g(X).$$

(We introduced the symbol $g(x)$ in order to avoid the mathematically confusing expression $E[Y|X = X]$.)

Since $g(X)$ is obviously $\sigma(X)$-measurable, to verify that $\mathbb{E}[Y|X] = g(X)$, we need only check that the partial-averaging property is satisfied. For every Borel-measurable function $h$ mapping $\mathbb{R}$ to $\mathbb{R}$ and satisfying $\mathbb{E}|h(X)| < \infty$, we have

$$\mathbb{E}h(X) = \int_{-\infty}^{\infty} h(x)f_X(x)dx. \qquad (2.6.2)$$

This is Theorem 1.5.2 in Chapter 1. Similarly, if $h$ is a function of both $x$ and $y$, then

$$\mathbb{E}h(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f_{X,Y}(x,y)dxdy \qquad (2.6.3)$$

whenever $(X,Y)$ has a joint density $f_{X,Y}(x,y)$. You may use both (2.6.2) and (2.6.3) in your solution to this problem.

Let $A$ be a set in $\sigma(X)$. By the definition of $\sigma(X)$, there is a Borel subset $B$ of $\mathbb{R}$ such that $A = \{\omega \in \Omega; X(\omega) \in B\}$ or, more simply, $A = \{X \in B\}$. Show the partial-averaging property

$$\int_A g(X)d\mathbb{P} = \int_A Y\,d\mathbb{P}.$$

**Exercise 3.1.** According to Definition 3.3.3(iii), for $0 \le t < u$, the Brownian motion increment $W(u) - W(t)$ is independent of the $\sigma$-algebra $\mathcal{F}(t)$. Use this property and property (i) of that definition to show that, for $0 \le t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

**Exercise 3.2.** Let $W(t)$, $t \ge 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \ge 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale. (Hint: For $0 \le s \le t$, write $W^2(t)$ as $(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s)$.)

**Exercise 3.4 (Other variations of Brownian motion).** Theorem 3.4.3 asserts that if $T$ is a positive number and we choose a partition $\Pi$ with points $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, then as the number $n$ of partition points approaches infinity and the length of the longest subinterval $\|\Pi\|$ approaches zero, the sample quadratic variation

$$\sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2$$

approaches $T$ for almost every path of the Brownian motion $W$. In Remark 3.4.5, we further showed that $\sum_{j=0}^{n-1}\left(W(t_{j+1}) - W(t_j)\right)(t_{j+1} - t_j)$ and $\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2$ have limit zero. We summarize these facts by the multiplication rules

$$dW(t)\,dW(t) = dt, \quad dW(t)\,dt = 0, \quad dt\,dt = 0. \qquad (3.10.1)$$

(i) Show that as the number $m$ of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample first variation

$$\sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|$$

approaches $\infty$ for almost every path of the Brownian motion $W$. (Hint:

$$\sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2$$

$$\leq \max_{0 \leq k \leq n-1} \left|W(t_{k+1}) - W(t_k)\right| \cdot \sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|.)$$

(ii) Show that as the number $n$ of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample cubic variation

$$\sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|^3$$

approaches zero for almost every path of the Brownian motion $W$.

Shreve, S. E., 2004. Stochastic calculus for finance II: Continuous-time models. Vol. 11. Springer Science & Business Media.