MATH 6740: Financial Mathematics and Simulation
Homework 1 solutions/presentation

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1. Q1

The single-period market can be expressed by a 2×2 matrix $\mathbf{M}$ and an initial price vector $\mathbf{s}$:

$$\mathbf{M} = \begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix}, \tag{1a}$$

$$\mathbf{s}_0 = \begin{bmatrix} s_0 & 1 \end{bmatrix}^T. \tag{1b}$$

To find an arbitrage for this market, we can attempt to construct a portfolio $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ that costs zero to set up:

$$c = \mathbf{s}_0^T \mathbf{x} = s_0 x_1 + x_2 = 0, \tag{2}$$

while making sure the future value of this portfolio $\mathbf{v} = \mathbf{M}\mathbf{x}$ is nonnegative at either market state and is not zero at both market states:

$$\mathbf{v} = \begin{bmatrix} v^u & v^d \end{bmatrix}^T, \text{ where } \begin{cases} v^u \geq 0, \ v^d \geq 0 \\ v^u + v^d > 0 \end{cases}. \tag{3}$$

According to the information given:

$$s^d = s^u < s_0 (1+r), \tag{4}$$

the future value of a portfolio $\mathbf{x}$ would be:

$$\mathbf{v} = \mathbf{M}\mathbf{x} = \begin{bmatrix} s^u & 1+r \\ s^u & 1+r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s^u x_1 + (1+r) x_2 \\ s^u x_1 + (1+r) x_2 \end{bmatrix} = (s^u x_1 + (1+r) x_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5}$$

Therefore, to satisfy the conditions set in Equations (2) and (3), we have:

$$\begin{cases} s_0 x_1 + x_2 = 0 \\ s^u x_1 + (1+r) x_2 > 0 \end{cases} \quad \Rightarrow \quad s^u x_1 - (1+r) s_0 x_1 > 0 \quad \Rightarrow \quad x_1 \left[s^u - (1+r) s_0 \right] > 0. \tag{6}$$

Combined with the condition $s^u < s_0 (1+r)$ given in Equation (4), an arbitrage for this market is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -s_0 \end{bmatrix} x_1, \text{ where } x_1 < 0. \tag{7}$$

An explicit example is:

$$\mathbf{x} = \begin{bmatrix} -1 \\ s_0 \end{bmatrix}. \tag{8}$$

2. Q2

The incomplete market has been described with:

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix}, \text{ where } 0 < r < s < 1, \\ \mathbf{s}_0 &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T. \end{aligned} \tag{9}$$

To construct an arbitrage, we build a portfolio $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ with

$$\mathbf{x}^T \mathbf{s}_0 = x_1 + x_2 + x_3 = 0, \tag{10}$$

while making sure that the claim vector

$$\mathbf{v} = \mathbf{M}\mathbf{x} = \begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1 + (1+r)\,x_2 + (1+s)\,x_3] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{11}$$

is in the positive octant including its boundaries but excluding the origin. Combining the above 2 equations we have:

$$\mathbf{v} = (r x_2 + s x_3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{12}$$

therefore as long as $x_2 \geq 0,\ x_3 \geq 0,\ x_2 + x_3 > 0$, we have a portfolio that takes advantage of the arbitrage. An explicit example would be:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \tag{13}$$

3. Q3

To "design" a 2×2 market with uniform risk-neutral probability, we assign value of $1/2$ to both q_1 and q_2. With $\mathbf{M}$ and $\mathbf{s}_0$ defined by Equation (1), we have:

$$\begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix} = (1+r) \begin{bmatrix} s_0 & 1 \end{bmatrix}, \tag{14}$$

$$\Downarrow$$

$$\begin{cases} s^u + s^d = 2\,(1+r)\, s_0 \\ 1 + r = 1 + r \end{cases}. \tag{15}$$

According to Equation (15), we can set up such a market $(\mathbf{M}, \mathbf{s}_0)$ with uniform risk-neutral probability with $r = .05$, $s_0 = 2$, $s^d = 1.5$, and $s^u = 2.7$:

$$\mathbf{M} = \begin{bmatrix} 2.7 & 1.05 \\ 1.5 & 1.05 \end{bmatrix}, \tag{16a}$$

$$\mathbf{s}_0 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T. \tag{16b}$$

4. Q4

To find in this market a risk-neutral probability vector $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T$, such that:

$$\mathbf{q}^T \mathbf{M} = (1+s)\, \mathbf{s}_0^T, \tag{17a}$$

$$\mathbf{q} > 0 \iff q_j > 0 \;\forall j. \tag{17b}$$

Equation (17a) in this case is therefore:

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix} = (1+s) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

$$\Downarrow$$

$$(q_1 + q_2 + q_3) \begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix} = (1+s) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \tag{18}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix} = (1+s) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}. \tag{19}$$

The fact that $\sum_j q_j = 1$ is employed in the above derivation. Apparently Equation (18) is unsolvable as $\begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ are linearly independent given $0 < r < s$; Equation (19) further confirms this by futilely equating 1 and $1 + r$ with $1 + s$. Therefore, this incomplete market has no risk-neutral probability vector.

5. Q5

A generic 2×2 market $(\mathbf{M}, \mathbf{s}_0)$ is defined in Equation (1); that it is arbitrage-free implies that $s^d < (1+r)\, s_0 < s^u$, and therefore makes it complete by effecting $\mathbf{M}$ to be full-rank. Thus there exists the inverse of $\mathbf{M}$:

$$\mathbf{M}^{-1} = \frac{1}{\left(s^u - s^d\right)(1+r)} \begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix}. \tag{20}$$

To meet the contingency claims $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $\mathbf{v}^{II} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $\mathbf{v}^{III} = \begin{bmatrix} a & b \end{bmatrix}^T = a\mathbf{v}^I + b\mathbf{v}^{II}$, we need to design corresponding portfolios $\mathbf{x}^I$, $\mathbf{x}^{II}$, and $\mathbf{x}^{III}$ to hedge against those.

To hedge against $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$:

$$\begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix} \begin{bmatrix} x_1^I \\ x_2^I \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\Downarrow$$

$$\begin{bmatrix} x_1^I \\ x_2^I \end{bmatrix} = \frac{1}{\left(s^u - s^d\right)(1+r)} \begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\left(s^u - s^d\right)(1+r)} \begin{bmatrix} 1+r \\ -s^d \end{bmatrix}. \tag{21}$$

The cost of this portfolio $\mathbf{x}^I$ is:

$$c_0^I = \mathbf{s}_0^T \mathbf{x}^I = \frac{(1+r)\, s_0 - s^d}{\left(s^u - s^d\right)(1+r)}. \tag{22}$$

Similarly, to hedge against $\mathbf{v}^{II} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$:

$$\begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix} \begin{bmatrix} x_1^{II} \\ x_2^{II} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\Downarrow$$

$$\begin{bmatrix} x_1^{II} \\ x_2^{II} \end{bmatrix} = \frac{1}{\left(s^u - s^d\right)(1+r)} \begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\left(s^u - s^d\right)(1+r)} \begin{bmatrix} -(1+r) \\ s^u \end{bmatrix}. \tag{23}$$

The cost of this portfolio $\mathbf{x}^{II}$ is:

$$c_0^{II} = \mathbf{s}_0^T \mathbf{x}^{II} = \frac{-(1+r)\, s_0 + s^u}{\left(s^u - s^d\right)(1+r)}. \tag{24}$$

To hedge against $\mathbf{v}^{III} = a\mathbf{v}^I + b\mathbf{v}^{II}$:

$$\mathbf{x}^{III} = \mathbf{M}^{-1}\mathbf{v}^{III} = \mathbf{M}^{-1}\left(a\mathbf{v}^I + b\mathbf{v}^{II}\right) = a\mathbf{x}^I + b\mathbf{x}^{II}, \tag{25}$$

and the cost of this portfolio is:

$$c_0^{III} = \mathbf{s}_0^T \mathbf{x}^{III} = \mathbf{s}_0^T \left(a\mathbf{x}^I + b\mathbf{x}^{II}\right) = ac_0^I + bc_0^{II} = \frac{(a-b)(1+r)\, s_0 + bs^u - as^d}{\left(s^u - s^d\right)(1+r)}. \tag{26}$$

6. Q6

Thanks to the fact that $\mathbf{M}$ is invertible, $\mathbf{q}^T\mathbf{M} = (1+r)\,\mathbf{s}_0^T$ yields $\mathbf{q}^T = (1+r)\,\mathbf{s}_0^T\mathbf{M}^{-1}$:

$$\begin{bmatrix} q_1 & q_2 \end{bmatrix} = \frac{1}{s^u - s^d}\begin{bmatrix} s_0 & 1 \end{bmatrix}\begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix} = \frac{1}{s^u - s^d}\begin{bmatrix} (1+r)\,s_0 - s^d \\ -(1+r)\,s_0 + s^u \end{bmatrix}^T. \tag{27}$$

Therefore the expected value of $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is:

$$E_{\mathbf{q}}\left[\mathbf{v}^I\right] = \mathbf{q}^T\mathbf{v}^I = \frac{(1+r)\,s_0 - s^d}{s^u - s^d}, \tag{28}$$

and the expected value of $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is:

$$E_{\mathbf{q}}\left[\mathbf{v}^{II}\right] = \mathbf{q}^T\mathbf{v}^{II} = \frac{-(1+r)\,s_0 + s^u}{s^u - s^d}. \tag{29}$$

Appendix A. Original Homework Questions (attached)

Math of Finance 6740 Spring 2016

Instructor: Chjan Lim, Professor Mathematical Sciences

Worksheet 1 for Presentation this Friday 02/19

1. Let S_0 be the t = 0 price of a risky asset and $S^d = S^u < S_0\,(1+r)$ where r > 0 is the riskless interest rate in a 2 by 2 single period market in payoff format for one riskless and one risky asset. Construct explicitly an arbitrage for this market.

2. Consider the 3 by 3 incomplete market M = $\begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix}$ where 0 < r < s < 1, and the t = 0 prices of the three assets are 1, 1, 1. Does this market have an arbitrage. Construct one explicitly if there is one.

3. Give an explicit example of a 2 by 2 market (M, S_0) with unifom risk neutral probability (that is equal for up and down market states).

4. For the incomplete 3 by 3 market in Q2, find a risk neutral probability vector q = $\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ or show there is none.

5. For the generic 2 by 2 market (M, S_0) that is arbitrage free (AF), calculate the prices $C_0(V)$ of the claims $V = \binom{1}{0}$ and $V = \binom{0}{1}$ and then price the generic claim $V = \binom{a}{b}$ where a and b are any real numbers.

6. Calculate the expected values $E_q\,[V]$ of the claims $V = \binom{1}{0}$ and $V = \binom{0}{1}$ under the risk neutral probability q = $\binom{q_1}{q_2}$ for the AF generic 2 by 2 market in Q5.