# MATH 6740: Financial Mathematics and Simulation
# Homework 3 solutions/presentation

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Solve Exercise Problems 1.4, 1.6, and 1.7 in (Shreve, 2004, Chapter 1, p. 42-43).

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## 1. Exercise 1.4

### *1.1. Exercise 1.4 i)*

First follow (Shreve, 2004, Example 1.2.5, p. 11-12) to construct a uniformly distributed random variable $X$ taking values in $[0, 1]$ and defined on infinite coin-toss space $\Omega_\infty$ where $p_H = p_T = 0$, by defining

$$Y_n(\omega|_n) = \begin{cases} 1 & \text{if } \omega|_n = H, \\ 0 & \text{if } \omega|_n = T, \end{cases} \tag{1}$$

which is an indicator of whether the result of the $n^{\text{th}}$ toss is $\omega|_n = H$, where $\omega \in \Omega_\infty$, and

$$X = \sum_{n=1}^{\infty} \frac{Y_n}{2^n}. \tag{2}$$

(Shreve, 2004, Example 1.2.5, p. 11-12) concluded that $X \sim U(0, 1)$. Next (Shreve, 2004, Example 1.2.6, p. 12-13) showed that with the probability density function and cumulative distribution function of the standard normal distribution being:

$$\varphi(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}}, \tag{3}$$

$$N(x) = \int_{-\infty}^{x} \varphi(\xi) d\xi, \tag{4}$$

we can defined a new random variable $Z = N^{-1}(X)$, where $N^{-1}(\cdot)$ is the inverse function of $N(\cdot)$, which exists because $N(\cdot)$ is strictly increasing. For $\forall -\infty < a \le b < \infty$, we have

$$\begin{aligned} \mu_Z[a, b] &= P(\omega \in \Omega_\infty | a \le Z(\omega) \le b) \\ &= P(\omega \in \Omega_\infty | a \le N^{-1}(X(\omega)) \le b) \\ &= P(\omega \in \Omega_\infty | N(a) \le X(\omega) \le N(b)) \\ &= N(b) - N(a) = \int_a^b \varphi(\xi) d\xi. \end{aligned} \tag{5}$$

Therefore

$$Z = N^{-1}(X) = N^{-1}(\sum_{n=1}^{\infty} \frac{Y_n}{2^n}) \sim N(0,1). \tag{6}$$

*1.2. Exercise 1.4 ii)*

Define the indicator function $Y_n(\omega|_n)$ as before:

$$Y_n(\omega|_n) = \begin{cases} 1 & \text{if } \omega|_n = H, \\ 0 & \text{if } \omega|_n = T, \end{cases} \tag{7}$$

and a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ on $\Omega_\infty$, where each element only depends on the results of the first $n^{\text{th}}$ tosses, as:

$$X_n = \sum_{i=1}^{n} \frac{Y_i}{2^i}. \tag{8}$$

The binary number $\overline{Y_1 Y_2 \cdots Y_n} = \sum\limits_{i=1}^{n} 2^{n-i} Y_i$ is a random variable taking all integer values between 0 and $2^n - 1$ with equal probability $1/2^n$, therefore $X_n = \frac{1}{n}\overline{Y_1 Y_2 \cdots Y_n}$ is a random variable, with equal probability $1/2^n$, taking values in the set $\{0, \frac{1}{2^n}, \cdots, \frac{2^n-1}{2^n}\}$, which is an arithmetic series from 0 to $(2^n-1)/2^n$ with common distance $1/2^n$, therefore:

$$P(\omega \in \Omega | \frac{k-1}{2^n} \le X_n < \frac{k}{2^n}, k \in \mathbb{Z}, k \in [1, 2^n]) = \frac{1}{2^n}. \tag{9}$$

Now define a sequence of random variables $\{Z_n\}_{n=1}^{\infty}$, with each element $Z_n = N^{-1}(X_n)$, then for $\forall -\infty < a \le b < \infty$, there is:

$$\begin{aligned} \mu_{Z_n}[a,b] &= P(\omega \in \Omega_\infty | a \le Z_n(\omega) \le b) \\ &= P(\omega \in \Omega_\infty | a \le N^{-1}(X_n(\omega)) \le b) \\ &= P(\omega \in \Omega_\infty | N(a) \le X_n(\omega) \le N(b)) \\ &= P(\omega \in \Omega_\infty | \frac{\lceil N(a) \cdot 2^n \rceil}{2^n} \le X_n(\omega) < \frac{\lfloor N(b) \cdot 2^n \rfloor + 1}{2^n}) \\ &= \frac{\lfloor N(b) \cdot 2^n \rfloor - \lceil N(a) \cdot 2^n \rceil + 1}{2^n} \\ &= N(b) - N(a) + \frac{O(1)}{2^n}, \end{aligned} \tag{10}$$

therefore

$$\lim_{n \to \infty} \mu_{Z_n}[a,b] = \mu_Z[a,b], \tag{11}$$

and

$$\lim_{n \to \infty} Z_n(\omega) = Z(\omega), \quad \forall \omega \in \Omega_\infty, \tag{12}$$

where $Z_n(\omega)$ is defined, with the inverse of the standard normal distribution $N^{-1}(\cdot)$, as:

$$Z_n = N^{-1}(X_n) = N^{-1}(\sum_{i=1}^{n} \frac{Y_i}{2^i}). \tag{13}$$

## 2. Exercise 1.6

### 2.1. *Exercise 1.6 i)*

$$
\begin{aligned}
E[e^{uX}] &= \int_{-\infty}^{\infty} e^{ux} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{ux} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2-(2\mu+2u\sigma^2)x+\mu^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x-(\mu+u\sigma^2)]^2-2\mu u\sigma^2-u^2\sigma^4}{2\sigma^2}} dx \\
&= e^{u\mu+\frac{1}{2}u^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x-(\mu+u\sigma^2)]^2}{2\sigma^2}} dx \\
&= e^{u\mu+\frac{1}{2}u^2\sigma^2}. \qquad (14)
\end{aligned}
$$

### 2.2. *Exercise 1.6 ii)*

For the convex function $\varphi(x) = e^{ux}$:

$$E[\varphi(X)] = E[e^{uX}] = e^{u\mu+\frac{1}{2}u^2\sigma^2} \geq e^{u\mu} = \varphi(E[X]), \qquad (15)$$

which satisfies Jensen's inequality.

## 3. Exercise 1.7

### 3.1. *Exercise 1.7 i)*

$$
\begin{aligned}
f(x) &= \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{2n\pi}} \sum_{i=0}^{\infty} \frac{1}{i!} \left(-\frac{x^2}{2n}\right)^i \\
&= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \left(\frac{1}{n^{1/2}} - \frac{x^2}{2n^{3/2}} + \frac{x^4}{8n^{5/2}} + f(x)O(\frac{1}{n^{7/2}})\right) \\
&= 0. \qquad (16)
\end{aligned}
$$

### 3.2. *Exercise 1.7 ii)*

Since $f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}}$ is essentially the probability density function of a random variable with normal distribution $X \sim N(0, n)$, therefore $\int_{-\infty}^{\infty} f_n(x)dx = 1$, thus

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = 1. \qquad (17)$$

*3.3. Exercise 1.7 iii)*

The Monotone Convergence Theorem requires that

$$0 \leq f_1 \leq f_2 \leq \cdots \quad \text{almost everywhere.} \tag{18}$$

In this instance, the range of $x$ where $f_n(x) > f_{n+1}(x)$ can be determined with, since $f_n(x) > 0$:

$$\begin{aligned}
\frac{f_n(x)}{f_{n+1}(x)} = \frac{\sqrt{n+1} \cdot e^{-\frac{x^2}{2n}}}{\sqrt{n} \cdot e^{-\frac{x^2}{2(n+1)}}} > 1 \quad \Rightarrow \quad e^{\frac{x^2}{2n} - \frac{x^2}{2(n+1)}} &< \sqrt{\frac{n+1}{n}} \\
e^{\frac{x^2}{2n(n+1)}} &< \sqrt{\frac{n+1}{n}} \\
\frac{x^2}{2n(n+1)} &< \frac{1}{2} \ln \frac{n+1}{n} \\
-n(n+1) \ln \frac{n+1}{n} < \quad x \quad &< n(n+1) \ln \frac{n+1}{n},
\end{aligned} \tag{19}$$

which is a non-trivial range. Therefore the fact that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) dx \neq \int_{-\infty}^{\infty} f(x) dx$$

does not violate the Monotone Convergence Theorem.

Appendix A. Original Homework Questions (attached)

**Exercise 1.4.** (i) Construct a standard normal random variable $Z$ on the probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P})$ of Example 1.1.4 under the assumption that the probability for head is $p = \frac{1}{2}$. (Hint: Consider Examples 1.2.5 and 1.2.6.)

(ii) Define a sequence of random variables $\{Z_n\}_{n=1}^\infty$ on $\Omega_\infty$ such that

$$\lim_{n\to\infty} Z_n(\omega) = Z(\omega) \text{ for every } \omega \in \Omega_\infty$$

and, for each $n$, $Z_n$ depends only on the first $n$ coin tosses. (This gives us a procedure for approximating a standard normal random variable by random variables generated by a finite number of coin tosses, a useful algorithm for Monte Carlo simulation.)

**Exercise 1.6.** Let $u$ be a fixed number in $\mathbb{R}$, and define the convex function $\varphi(x) = e^{ux}$ for all $x \in \mathbb{R}$. Let $X$ be a normal random variable with mean $\mu = \mathbb{E}X$ and standard deviation $\sigma = \left[\mathbb{E}(X-\mu)^2\right]^{\frac{1}{2}}$, i.e., with density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

(i) Verify that

$$\mathbb{E}e^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}.$$

(ii) Verify that Jensen's inequality holds (as it must):

$$\mathbb{E}\varphi(X) \geq \varphi(\mathbb{E}X).$$

**Exercise 1.7.** For each positive integer $n$, define $f_n$ to be the normal density with mean zero and variance $n$, i.e.,

$$f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}}.$$

(i) What is the function $f(x) = \lim_{n\to\infty} f_n(x)$?
(ii) What is $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx$?
(iii) Note that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx \neq \int_{-\infty}^{\infty} f(x)\,dx.$$

Explain why this does not violate the Monotone Convergence Theorem, Theorem 1.4.5.

Shreve, S. E., 2004. Stochastic calculus for finance II: Continuous-time models. Vol. 11. Springer Science & Business Media.