MATH 6740: Financial Mathematics and Simulation
Homework 4 solutions/presentation

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Solve Exercise Problems 2.5 and 2.10 in (Shreve, 2004, Chapter 2), and Problems 3.1, 3.2, and 3.4 in (Shreve, 2004, Chapter 3).

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1. Exercise 2.5

The p.d.f. for random variable X is:

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy \\
&= \int_{-|x|}^{+\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-|x|}^{+\infty} (2|x|+y) \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} d(2|x|+y) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-|x|}^{+\infty} \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} d\frac{(2|x|+y)^2}{2} \\
&= \frac{1}{\sqrt{2\pi}} \left(\exp\{-\frac{|x|^2}{2}\right\} - \exp\{-\infty\}) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}, \qquad (1)
\end{aligned}$$

and the p.d.f. for random variable Y is:

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx \\
&= 2\int_{0}^{+\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{\max\{0,-y\}}^{+\infty} (2x+y) \exp\left\{-\frac{(2x+y)^2}{2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\max\{0,-y\}}^{+\infty} \exp\left\{-\frac{(2x+y)^2}{2}\right\} d\frac{(2x+y)^2}{2} \\
&= \frac{1}{\sqrt{2\pi}} \left(\exp\left\{-\frac{(2\max\{0,-y\}+y)^2}{2}\right\} - 0\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{|y|^2}{2}\right\} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}. \qquad (2)
\end{aligned}$$

Therefore both X and Y are standard normal random variables. The covariance of them is:

$$\begin{aligned}
cov(X,Y) &= E[(X-E[X])(Y-E[Y])] = E[XY] \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f_{X,Y}(x,y)dxdy \\
&= \int_{-\infty}^{+\infty} y\left(\int_{-\infty}^{+\infty} x\frac{2|x|+y}{\sqrt{2\pi}}\exp\left\{-\frac{(2|x|+y)^2}{2}\right\}dx\right)dy \\
&= \int_{-\infty}^{+\infty} y\cdot 0dy = 0, \qquad (3)
\end{aligned}$$

therefore X and Y are uncorrelated. However, since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

2. Exercise 2.10

$$\begin{aligned}
\int_A g(X)dP(X) &= \int_{-\infty}^{+\infty} g(x)1_{\omega\in A}f_X(x)dx \\
&= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\frac{yf_{X,Y}(x,y)}{f_X(x)}dy\right)1_{\omega\in A}f_X(x)dx \\
&= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} yf_{X,Y}(x,y)dy\right)1_{\omega\in A}dx \\
&= \int_{-\infty}^{+\infty} y\left(\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx\right)1_{\omega\in A}dy \\
&= \int_{-\infty}^{+\infty} y1_{\omega\in A}f_Y(y)dy = \int_A YdP(Y). \qquad (4)
\end{aligned}$$

3. Exercise 3.1

According to Definition 3.3.3(iii) in Shreve (2004), $W(u_2)-W(u_1)$ is independent of $\mathbb{F}(u_1)$; while according to Definition 3.3.3(i) in Shreve (2004), $\mathbb{F}(t)\subset\mathbb{F}(u_1)$. Therefore $W(u_2)-W(u_1)$ is independent of $\mathbb{F}(t)$.

4. Exercise 3.2

For $0\le s\le t$:

$$\begin{aligned}
E[W^2(t)-t|\mathbb{F}_s] &= E[(W(t)-W(s))^2+2W(t)W(s)-W^2(s)-t|\mathbb{F}(s)] \\
&= E[(W(t)-W(s))^2]+2W(s)E[W(t)-W(s)+W(s)|\mathbb{F}(s)]-W^2(s)-t \\
&= var(W(t)-W(s))+2W(s)\left(W(s)+E[W(t)-W(s)|\mathbb{F}(s)]\right)-W^2(s)-t \\
&= t-s+2W^2(s)-W^2(s)-t \\
&= W^2(s)-s, \qquad (5)
\end{aligned}$$

therefore $\{W^2(t)-t\}$ is a martingale.

5. Exercise 3.4

5.1. (i)

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \cdot \max_{0 \le k \le n-1} |W(t_{j+1}) - W(t_j)| \ge \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \,, \tag{6}$$

therefore the first variation of Brownian motion is:

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \ge \lim_{||\Pi|| \to 0} \frac{\sum\limits_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2}{\max\limits_{0 \le k \le n-1} |W(t_{j+1}) - W(t_j)|} = \frac{T}{0} = \infty. \tag{7}$$

5.2. (ii)

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \le \max_{0 \le k \le n-1} |W(t_{j+1}) - W(t_j)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2 \,, \tag{8}$$

therefore the cubic variation of Brownian motion is:

$$\begin{aligned} \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \le \lim_{||\Pi|| \to 0} \max_{0 \le k \le n-1} |W(t_{j+1}) - W(t_j)| \cdot \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2 \\ = 0 \cdot T = 0 \end{aligned} \tag{9}$$

Exercise 2.5. Let (X, Y) be a pair of random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{-\frac{(2|x|+y)^2}{2}\right\} & \text{if } y \geq -|x|, \\ 0 & \text{if } y < -|x|. \end{cases}$$

Show that X and Y are standard normal random variables and that they are uncorrelated but not independent.

Exercise 2.10. Let X and Y be random variables (on some unspecified probability space $(\Omega, \mathcal{F}, \mathbb{P})$), assume they have a joint density $f_{X,Y}(x,y)$, and assume $\mathbb{E}|Y| < \infty$. In particular, for every Borel subset C of $\mathbb{R}^2$, we have

$$\mathbb{P}\{(X,Y) \in C\} = \int_C f_{X,Y}(x,y)\, dx\, dy.$$

In elementary probability, one learns to compute $\mathbb{E}[Y|X = x]$, which is a *nonrandom* function of the *dummy variable* x, by the formula

$$\mathbb{E}[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy, \tag{2.6.1}$$

where $f_{Y|X}(y|x)$ is the *conditional density* defined by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

The denominator in this expression, $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,\eta) d\eta$, is the *marginal density* of X, and we must assume it is strictly positive for every x. We introduce the symbol $g(x)$ for the function $\mathbb{E}[Y|X = x]$ defined by (2.6.1); i.e.,

$$g(x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy = \int_{-\infty}^{\infty} \frac{y f_{X,Y}(x,y)}{f_X(x)}\, dy.$$

In measure-theoretic probability, conditional expectation is a *random variable* $\mathbb{E}[Y|X]$. This exercise is to show that when there is a joint density for (X, Y), this random variable can be obtained by substituting the random variable X in place of the dummy variable x in the function $g(x)$. In other words, this exercise is to show that

$$\mathbb{E}[Y|X] = g(X).$$

(We introduced the symbol $g(x)$ in order to avoid the mathematically confusing expression $E[Y|X = X]$.)

Since $g(X)$ is obviously $\sigma(X)$-measurable, to verify that $\mathbb{E}[Y|X] = g(X)$, we need only check that the partial-averaging property is satisfied. For every Borel-measurable function h mapping $\mathbb{R}$ to $\mathbb{R}$ and satisfying $\mathbb{E}|h(X)| < \infty$, we have

$$\mathbb{E}h(X) = \int_{-\infty}^{\infty} h(x) f_X(x)dx. \tag{2.6.2}$$

This is Theorem 1.5.2 in Chapter 1. Similarly, if h is a function of both x and y, then

$$\mathbb{E}h(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f_{X,Y}(x,y)dxdy \tag{2.6.3}$$

whenever (X,Y) has a joint density $f_{X,Y}(x,y)$. You may use both (2.6.2) and (2.6.3) in your solution to this problem.

Let A be a set in $\sigma(X)$. By the definition of $\sigma(X)$, there is a Borel subset B of $\mathbb{R}$ such that $A = \{\omega \in \Omega; X(\omega) \in B\}$ or, more simply, $A = \{X \in B\}$. Show the partial-averaging property

$$\int_A g(X)d\mathbb{P} = \int_A Y d\mathbb{P}.$$

Exercise 3.1. According to Definition 3.3.3(iii), for $0 \le t < u$, the Brownian motion increment $W(u) - W(t)$ is independent of the σ-algebra $\mathcal{F}(t)$. Use this property and property (i) of that definition to show that, for $0 \le t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

Exercise 3.2. Let $W(t)$, $t \ge 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \ge 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale. (Hint: For $0 \le s \le t$, write $W^2(t)$ as $(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s)$.)

Exercise 3.4 (Other variations of Brownian motion). Theorem 3.4.3 asserts that if T is a positive number and we choose a partition Π with points $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, then as the number n of partition points approaches infinity and the length of the longest subinterval $\|\Pi\|$ approaches zero, the sample quadratic variation

$$\sum_{j=0}^{n-1} \big(W(t_{j+1}) - W(t_j)\big)^2$$

approaches T for almost every path of the Brownian motion W. In Remark 3.4.5, we further showed that $\sum_{j=0}^{n-1} \big(W(t_{j+1}) - W(t_j)\big)(t_{j+1} - t_j)$ and $\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2$ have limit zero. We summarize these facts by the multiplication rules

$$dW(t)\,dW(t) = dt, \quad dW(t)\,dt = 0, \quad dt\,dt = 0. \tag{3.10.1}$$

(i) Show that as the number m of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample first variation

$$\sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|$$

approaches ∞ for almost every path of the Brownian motion W. (Hint:

$$\sum_{j=0}^{n-1} \big(W(t_{j+1}) - W(t_j)\big)^2$$

$$\leq \max_{0\leq k\leq n-1} \left|W(t_{k+1}) - W(t_k)\right| \cdot \sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|.)$$

(ii) Show that as the number n of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample cubic variation

$$\sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|^3$$

approaches zero for almost every path of the Brownian motion W.

Shreve, S. E., 2004. Stochastic calculus for finance II: Continuous-time models. Vol. 11. Springer Science & Business Media.