MATH 6740: Financial Mathematics and Simulation
Homework 1 solutions/presentation

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1. Q1

The single-period market can be expressed by a 2×2 matrix $\mathbf{M}$ and an initial price vector $\mathbf{s}$:

$$\mathbf{M} = \begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix}, \tag{1a}$$

$$\mathbf{s}_0 = \begin{bmatrix} s_0 & 1 \end{bmatrix}^T. \tag{1b}$$

To find an arbitrage for this market, we can attempt to construct a portfolio $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ that costs zero to set up:

$$c = \mathbf{s}_0^T \mathbf{x} = s_0 x_1 + x_2 = 0, \tag{2}$$

while making sure the future value of this portfolio $\mathbf{v} = \mathbf{Mx}$ is nonnegative at either market state and is not zero at both market states:

$$\mathbf{v} = \begin{bmatrix} v^u & v^d \end{bmatrix}^T, \text{ where } \begin{cases} v^u \geq 0, \ v^d \geq 0 \\ v^u + v^d > 0 \end{cases}. \tag{3}$$

According to the information given:

$$s^d = s^u < s_0 (1+r), \tag{4}$$

the future value of a portfolio $\mathbf{x}$ would be:

$$\mathbf{v} = \mathbf{Mx} = \begin{bmatrix} s^u & 1+r \\ s^u & 1+r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s^u x_1 + (1+r) x_2 \\ s^u x_1 + (1+r) x_2 \end{bmatrix} = (s^u x_1 + (1+r) x_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5}$$

Therefore, to satisfy the conditions set in Equations (2) and (3), we have:

$$\begin{cases} s_0 x_1 + x_2 = 0 \\ s^u x_1 + (1+r) x_2 > 0 \end{cases} \quad \Rightarrow \quad s^u x_1 - (1+r) s_0 x_1 > 0 \quad \Rightarrow \quad x_1 \left[s^u - (1+r) s_0 \right] > 0. \tag{6}$$

Combined with the condition $s^u < s_0 (1+r)$ given in Equation (4), an arbitrage for this market is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -s_0 \end{bmatrix} x_1, \text{ where } x_1 < 0. \tag{7}$$

2. Q2

The incomplete market has been described with:

$$\mathbf{M} = \begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix}, \text{ where } 0 < r < s < 1, \qquad (8)$$
$$\mathbf{s}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T .$$

This market is arbitrage-free if and only if there exists a risk-neutral probability vector $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T$, such that:

$$\mathbf{q}^T \mathbf{M} = (1+s)\,\mathbf{s}_0^T, \qquad (9a)$$

$$\mathbf{q} > 0 \iff q_j > 0\; \forall j. \qquad (9b)$$

Equation (9a) in this case is therefore:

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix} = (1+s)\begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

$$\Downarrow$$

$$(q_1 + q_2 + q_3)\begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix} = (1+s)\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad (10)$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix} = (1+s)\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}. \qquad (11)$$

The fact that $\sum_j q_j = 1$ is employed in the above derivation. Apparently Equation (10) is unsolvable as $\begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ are linearly independent given $0 < r < s$; Equation (11) further confirms this by futilely equating 1 and $1 + r$ with $1 + s$. Therefore, this incomplete market has no arbitrage.

3. Q3

To "design" a 2×2 market with uniform risk-neutral probability, we assign value of $1/2$ to both q_1 and q_2. With $\mathbf{M}$ and $\mathbf{s}_0$ defined by Equation (1), we have:

$$\begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix} = (1+r)\begin{bmatrix} s_0 & 1 \end{bmatrix} \qquad (12)$$

$$\Downarrow$$

$$\begin{cases} s^u + s^d = 2\,(1+r)\,s_0 \\ 1 + r = 1 + r \end{cases} \qquad (13)$$

According to Equation (13), we can set up such a market $(\mathbf{M}, \mathbf{s}_0)$ with uniform risk-neutral probability with $r = .05$, $s_0 = 2$, $s^d = 1.5$, and $s^u = 2.7$:

$$\mathbf{M} = \begin{bmatrix} 2.7 & 1.05 \\ 1.5 & 1.05 \end{bmatrix}, \tag{14a}$$

$$\mathbf{s}_0 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T. \tag{14b}$$

4. Q4

No such risk-neutral probability vector $\mathbf{q}$ exists as shown with Equation (10).

5. Q5

A generic 2×2 market $(\mathbf{M}, \mathbf{s}_0)$ is defined in Equation (1); that it is arbitrage-free implies that $s^d < (1 + r)\, s_0 < s^u$, and therefore makes it complete by effecting $\mathbf{M}$ to be full-rank. Thus there exists the inverse of $\mathbf{M}$:

$$\mathbf{M}^{-1} = \frac{1}{(s^u - s^d)(1 + r)} \begin{bmatrix} 1 + r & -(1 + r) \\ -s^d & s^u \end{bmatrix}. \tag{15}$$

To meet the contingency claims $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $\mathbf{v}^{II} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $\mathbf{v}^{III} = \begin{bmatrix} a & b \end{bmatrix}^T = a\mathbf{v}^I + b\mathbf{v}^{II}$, we need to design corresponding portfolios $\mathbf{x}^I$, $\mathbf{x}^{II}$, and $\mathbf{x}^{III}$ to hedge against those.

To hedge against $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$:

$$\begin{bmatrix} s^u & 1 + r \\ s^d & 1 + r \end{bmatrix} \begin{bmatrix} x_1^I \\ x_2^I \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\Downarrow$$

$$\begin{bmatrix} x_1^I \\ x_2^I \end{bmatrix} = \frac{1}{(s^u - s^d)(1 + r)} \begin{bmatrix} 1 + r & -(1 + r) \\ -s^d & s^u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s^u - s^d)(1 + r)} \begin{bmatrix} 1 + r \\ -s^d \end{bmatrix}. \tag{16}$$

The cost of this portfolio $\mathbf{x}^I$ is:

$$c_0^I = \mathbf{s}_0^T \mathbf{x}^I = \frac{(1 + r)\, s_0 - s^d}{(s^u - s^d)(1 + r)}. \tag{17}$$

Similarly, to hedge against $\mathbf{v}^{II} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$:

$$\begin{bmatrix} s^u & 1 + r \\ s^d & 1 + r \end{bmatrix} \begin{bmatrix} x_1^{II} \\ x_2^{II} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\Downarrow$$

$$\begin{bmatrix} x_1^{II} \\ x_2^{II} \end{bmatrix} = \frac{1}{(s^u - s^d)(1 + r)} \begin{bmatrix} 1 + r & -(1 + r) \\ -s^d & s^u \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s^u - s^d)(1 + r)} \begin{bmatrix} -(1 + r) \\ s^u \end{bmatrix}. \tag{18}$$

The cost of this portfolio $\mathbf{x}^{II}$ is:

$$c_0^{II} = \mathbf{s}_0^T \mathbf{x}^{II} = \frac{-(1 + r)\, s_0 + s^u}{(s^u - s^d)(1 + r)}. \tag{19}$$

To hedge against $\mathbf{v}^{III} = a\mathbf{v}^{I} + b\mathbf{v}^{II}$:

$$\mathbf{x}^{III} = \mathbf{M}^{-1}\mathbf{v}^{III} = \mathbf{M}^{-1}\left(a\mathbf{v}^{I} + b\mathbf{v}^{II}\right) = a\mathbf{x}^{I} + b\mathbf{x}^{II}, \tag{20}$$

and the cost of this portfolio is:

$$c_0^{III} = \mathbf{s}_0^T\mathbf{x}^{III} = \mathbf{s}_0^T\left(a\mathbf{x}^{I} + b\mathbf{x}^{II}\right) = ac_0^{I} + bc_0^{II} = \frac{(a-b)(1+r)\,s_0 + bs^u - as^d}{(s^u - s^d)(1+r)}. \tag{21}$$

6. Q6

Thanks to the fact that $\mathbf{M}$ is invertible, $\mathbf{q}^T\mathbf{M} = (1+r)\,\mathbf{s}_0^T$ yields $\mathbf{q}^T = (1+r)\,\mathbf{s}_0^T\mathbf{M}^{-1}$:

$$\begin{bmatrix} q_1 & q_2 \end{bmatrix} = \frac{1}{s^u - s^d}\begin{bmatrix} s_0 & 1 \end{bmatrix}\begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix} = \frac{1}{s^u - s^d}\begin{bmatrix} (1+r)\,s_0 - s^d \\ -(1+r)\,s_0 + s^u \end{bmatrix}^T. \tag{22}$$

Therefore the expected value of $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is:

$$E_{\mathbf{q}}\left[\mathbf{v}^I\right] = \mathbf{q}^T\mathbf{v}^I = \frac{(1+r)\,s_0 - s^d}{s^u - s^d}, \tag{23}$$

and the expected value of $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is:

$$E_{\mathbf{q}}\left[\mathbf{v}^{II}\right] = \mathbf{q}^T\mathbf{v}^{II} = \frac{-(1+r)\,s_0 + s^u}{s^u - s^d}. \tag{24}$$