

MATH 6740: Financial Mathematics and Simulation

Homework 1 solutions/presentation

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1. Q1

The single-period market can be expressed by a 2×2 matrix \mathbf{M} and an initial price vector \mathbf{s} :

$$\mathbf{M} = \begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix}, \quad (1a)$$

$$\mathbf{s}_0 = \begin{bmatrix} s_0 & 1 \end{bmatrix}^T. \quad (1b)$$

To find an arbitrage for this market, we can attempt to construct a portfolio $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ that costs zero to set up:

$$c = \mathbf{s}_0^T \mathbf{x} = s_0 x_1 + x_2 = 0, \quad (2)$$

while making sure the future value of this portfolio $\mathbf{v} = \mathbf{M}\mathbf{x}$ is nonnegative at either market state and is not zero at both market states:

$$\mathbf{v} = \begin{bmatrix} v^u & v^d \end{bmatrix}^T, \text{ where } \begin{cases} v^u \geq 0, v^d \geq 0 \\ v^u + v^d > 0 \end{cases}. \quad (3)$$

According to the information given:

$$s^d = s^u < s_0 (1+r), \quad (4)$$

the future value of a portfolio \mathbf{x} would be:

$$\mathbf{v} = \mathbf{M}\mathbf{x} = \begin{bmatrix} s^u & 1+r \\ s^u & 1+r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s^u x_1 + (1+r)x_2 \\ s^u x_1 + (1+r)x_2 \end{bmatrix} = (s^u x_1 + (1+r)x_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (5)$$

Therefore, to satisfy the conditions set in Equations (2) and (3), we have:

$$\begin{cases} s_0 x_1 + x_2 = 0 \\ s^u x_1 + (1+r)x_2 > 0 \end{cases} \Rightarrow s^u x_1 - (1+r)s_0 x_1 > 0 \Rightarrow x_1 [s^u - (1+r)s_0] > 0. \quad (6)$$

Combined with the condition $s^u < s_0 (1+r)$ given in Equation (4), an arbitrage for this market is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -s_0 \end{bmatrix} x_1, \text{ where } x_1 < 0. \quad (7)$$

An explicit example is:

$$\mathbf{x} = \begin{bmatrix} -1 \\ s_0 \end{bmatrix}. \quad (8)$$

2. Q2

The incomplete market has been described with:

$$\mathbf{M} = \begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix}, \text{ where } 0 < r < s < 1, \quad (9)$$

$$\mathbf{s}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T.$$

To construct an arbitrage, we build a portfolio $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ with

$$\mathbf{x}^T \mathbf{s}_0 = x_1 + x_2 + x_3 = 0, \quad (10)$$

while making sure that the claim vector

$$\mathbf{v} = \mathbf{M}\mathbf{x} = \begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1 + (1+r)x_2 + (1+s)x_3] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (11)$$

is in the positive octant including its boundaries but excluding the origin. Combining the above 2 equations we have:

$$\mathbf{v} = (rx_2 + sx_3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad (12)$$

therefore as long as $x_2 \geq 0$, $x_3 \geq 0$, $x_2 + x_3 > 0$, we have a portfolio that takes advantage of the arbitrage. An explicit example would be:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad (13)$$

3. Q3

To “design” a 2×2 market with uniform risk-neutral probability, we assign value of $1/2$ to both q_1 and q_2 . With \mathbf{M} and \mathbf{s}_0 defined by Equation (1), we have:

$$\begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix} = (1+r) \begin{bmatrix} s_0 & 1 \end{bmatrix}, \quad (14)$$

\Downarrow

$$\begin{cases} s^u + s^d = 2(1+r)s_0 \\ 1+r = 1+r \end{cases}. \quad (15)$$

According to Equation (15), we can set up such a market $(\mathbf{M}, \mathbf{s}_0)$ with uniform risk-neutral probability with $r = .05$, $s_0 = 2$, $s^d = 1.5$, and $s^u = 2.7$:

$$\mathbf{M} = \begin{bmatrix} 2.7 & 1.05 \\ 1.5 & 1.05 \end{bmatrix}, \quad (16a)$$

$$\mathbf{s}_0 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T. \quad (16b)$$

4. Q4

To find in this market a risk-neutral probability vector $\mathbf{q} = [q_1 \ q_2 \ q_3]^T$, such that:

$$\mathbf{q}^T \mathbf{M} = (1+s) \mathbf{s}_0^T, \quad (17a)$$

$$\mathbf{q} > 0 \iff q_j > 0 \ \forall j. \quad (17b)$$

Equation (17a) in this case is therefore:

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \\ 1 & 1+r & 1+s \end{bmatrix} = (1+s) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

\Downarrow

$$(q_1 + q_2 + q_3) \begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix} = (1+s) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad (18)$$

\Downarrow

$$\begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix} = (1+s) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}. \quad (19)$$

The fact that $\sum_j q_j = 1$ is employed in the above derivation. Apparently Equation (18) is unsolvable as $\begin{bmatrix} 1 & 1+r & 1+s \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ are linearly independent given $0 < r < s$; Equation (19) further confirms this by futilely equating 1 and $1+r$ with $1+s$. Therefore, this incomplete market has no risk-neutral probability vector.

5. Q5

A generic 2×2 market $(\mathbf{M}, \mathbf{s}_0)$ is defined in Equation (1); that it is arbitrage-free implies that $s^d < (1+r)s_0 < s^u$, and therefore makes it complete by effecting \mathbf{M} to be full-rank. Thus there exists the inverse of \mathbf{M} :

$$\mathbf{M}^{-1} = \frac{1}{(s^u - s^d)(1+r)} \begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix}. \quad (20)$$

To meet the contingency claims $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $\mathbf{v}^{II} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $\mathbf{v}^{III} = \begin{bmatrix} a & b \end{bmatrix}^T = a\mathbf{v}^I + b\mathbf{v}^{II}$, we need to design corresponding portfolios \mathbf{x}^I , \mathbf{x}^{II} , and \mathbf{x}^{III} to hedge against those.

To hedge against $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$:

$$\begin{aligned} \begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix} \begin{bmatrix} x_1^I \\ x_2^I \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \Downarrow \\ \begin{bmatrix} x_1^I \\ x_2^I \end{bmatrix} &= \frac{1}{(s^u - s^d)(1+r)} \begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s^u - s^d)(1+r)} \begin{bmatrix} 1+r \\ -s^d \end{bmatrix}. \end{aligned} \quad (21)$$

The cost of this portfolio \mathbf{x}^I is:

$$c_0^I = \mathbf{s}_0^T \mathbf{x}^I = \frac{(1+r)s_0 - s^d}{(s^u - s^d)(1+r)}. \quad (22)$$

Similarly, to hedge against $\mathbf{v}^{II} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$:

$$\begin{aligned} \begin{bmatrix} s^u & 1+r \\ s^d & 1+r \end{bmatrix} \begin{bmatrix} x_1^{II} \\ x_2^{II} \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \Downarrow \\ \begin{bmatrix} x_1^{II} \\ x_2^{II} \end{bmatrix} &= \frac{1}{(s^u - s^d)(1+r)} \begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s^u - s^d)(1+r)} \begin{bmatrix} -(1+r) \\ s^u \end{bmatrix}. \end{aligned} \quad (23)$$

The cost of this portfolio \mathbf{x}^{II} is:

$$c_0^{II} = \mathbf{s}_0^T \mathbf{x}^{II} = \frac{-(1+r)s_0 + s^u}{(s^u - s^d)(1+r)}. \quad (24)$$

To hedge against $\mathbf{v}^{III} = a\mathbf{v}^I + b\mathbf{v}^{II}$:

$$\mathbf{x}^{III} = \mathbf{M}^{-1} \mathbf{v}^{III} = \mathbf{M}^{-1} (a\mathbf{v}^I + b\mathbf{v}^{II}) = a\mathbf{x}^I + b\mathbf{x}^{II}, \quad (25)$$

and the cost of this portfolio is:

$$c_0^{III} = \mathbf{s}_0^T \mathbf{x}^{III} = \mathbf{s}_0^T (a\mathbf{x}^I + b\mathbf{x}^{II}) = ac_0^I + bc_0^{II} = \frac{(a-b)(1+r)s_0 + bs^u - as^d}{(s^u - s^d)(1+r)}. \quad (26)$$

6. Q6

Thanks to the fact that \mathbf{M} is invertible, $\mathbf{q}^T \mathbf{M} = (1+r) \mathbf{s}_0^T$ yields $\mathbf{q}^T = (1+r) \mathbf{s}_0^T \mathbf{M}^{-1}$:

$$\begin{bmatrix} q_1 & q_2 \end{bmatrix} = \frac{1}{s^u - s^d} \begin{bmatrix} s_0 & 1 \end{bmatrix} \begin{bmatrix} 1+r & -(1+r) \\ -s^d & s^u \end{bmatrix} = \frac{1}{s^u - s^d} \begin{bmatrix} (1+r) s_0 - s^d \\ -(1+r) s_0 + s^u \end{bmatrix}^T. \quad (27)$$

Therefore the expected value of $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is:

$$E_{\mathbf{q}}[\mathbf{v}^I] = \mathbf{q}^T \mathbf{v}^I = \frac{(1+r) s_0 - s^d}{s^u - s^d}, \quad (28)$$

and the expected value of $\mathbf{v}^I = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is:

$$E_{\mathbf{q}}[\mathbf{v}^{II}] = \mathbf{q}^T \mathbf{v}^{II} = \frac{-(1+r) s_0 + s^u}{s^u - s^d}. \quad (29)$$