MATH 6740: Financial Mathematics and Simulation
Homework 5 solutions/presentation

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Solve Exercise Problem 4.9 in (Shreve, 2004, Chapter 4).

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1. Exercise 4.9 (Analytical solution to Black-Scholes equation)

1.1. (i)

Proof.

$$
\begin{aligned}
\text{LHS} \equiv Ke^{-r(T-t)}N'(d_-) &= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_-^2}{2}} \\
&= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(d_+-\sigma\sqrt{T-t}\right)^2}{2}} \\
&= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}e^{-\frac{\sigma^2(T-t)-d\sigma\sqrt{T-t}d_+}{2}} \\
&= N'(d_+)Ke^{-(T-t)\left(r+\frac{\sigma^2}{2}\right)+\sigma\sqrt{T-t}d_+} \\
&= xN'(d_+) = \text{RHS}, \qquad (1)
\end{aligned}
$$

since the stock price x can be related to d_+ as:

$$x = Ke^{-(T-t)\left(r+\frac{\sigma^2}{2}\right)+\sigma\sqrt{T-t}d_+}. \qquad (2)$$

□

1.2. (ii)

Proof.

$$
\begin{aligned}
c_x &= N(d_+) + xN'(d_+)\frac{\partial d_+}{\partial x} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial x} \\
&= N(d_+) + xN'(d_+)\frac{\partial d_+}{\partial x} - Ke^{-r(T-t)}N'(d_-)\frac{\partial\left(d_+-\sigma\sqrt{T-t}\right)}{\partial x} \\
&= N(d_+) + \frac{\partial d_+}{\partial x}\left(xN'(d_+) - Ke^{-r(T-t)}N'(d_-)\right) \\
&= N(d_+) + \frac{\partial d_+}{\partial x}\cdot 0 = N(d_+), \qquad (3)
\end{aligned}
$$

based on the conclusion from (i). □

1.3. (iii)

Proof. From the definition of d_- and d_+:

$$d_-(\tau, x) = d_+(\tau, x) - \sigma\sqrt{\tau} \quad \Rightarrow \quad \frac{\partial d_-}{\partial t} = \frac{\partial d_+}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}, \tag{4}$$

and again with the help from the conclusion of (i):

$$\begin{aligned} c_t &= xN'(d_+)\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}rN(d_-) - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial t} \\ &= -rKe^{-r(T-t)}N(d_-) + xN'(d_+)\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}N'(d_-)\left(\frac{\partial d_+}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right) \\ &= -rKe^{-r(T-t)}N(d_-) + \left(xN'(d_+) - Ke^{-r(T-t)}N'(d_-)\right)\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}N'(d_-)\frac{\sigma}{2\sqrt{T-t}} \\ &= -rKe^{-r(T-t)}N(d_-) + 0 \cdot \frac{\partial d_+}{\partial t} - \left(Ke^{-r(T-t)}N'(d_-)\right)\frac{\sigma}{2\sqrt{T-t}} \\ &= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+). \end{aligned} \tag{5}$$

□

1.4. (iv)

Proof. The gamma of the option price is:

$$c_{xx} = N'(d_+)\frac{\partial d_+}{\partial x} = N'(d_+)\frac{1}{\sigma\sqrt{T-t}}\frac{K}{x}\frac{1}{K} = \frac{N'(d_+)}{\sigma x\sqrt{T-t}}. \tag{6}$$

Therefore the residual of the Black-Scholes equation is:

$$\begin{aligned} \text{LHS} - \text{RHS} &= c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} - rc \\ &= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + rxN(d_+) \\ &\quad + \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) - rxN(d_+) + rKe^{-r(T-t)}N(d_-) \\ &= 0. \end{aligned} \tag{7}$$

□

1.5. (v)

Proof. To prove the terminal condition, two cases need to be considered:

1. $x > K$ and therefore $\ln(x/K) > 0$:

$$\lim_{t \nearrow T} d_+ = \lim_{\tau \searrow 0} d_+(\tau, x) = \lim_{\tau \searrow 0} \frac{\ln\frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = +\infty, \tag{8}$$

$$\lim_{t \nearrow T} d_- = \lim_{\tau \searrow 0} d_-(\tau, x) = \lim_{\tau \searrow 0} \left(d_+(\tau, x) - \sigma\sqrt{\tau}\right) = +\infty - 0 = +\infty. \tag{9}$$

Thus the corresponding terminal condition is:

$$\begin{aligned}\lim_{t \nearrow T} c(t,x) &= \lim_{\tau \searrow 0} \left(xN(d_+(\tau,x)) - Ke^{-r\tau} N(d_-(\tau,x)) \right) \\ &= x \cdot 1 - K \cdot 1 \cdot 1 = x - K. \end{aligned} \tag{10}$$

2. $0 < x < K$ and therefore $\ln(x/K) < 0$:

$$\lim_{t \nearrow T} d_+ = \lim_{\tau \searrow 0} d_+(\tau,x) = \lim_{\tau \searrow 0} \frac{\ln \frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = -\infty, \tag{11}$$

$$\lim_{t \nearrow T} d_- = \lim_{\tau \searrow 0} d_-(\tau,x) = \lim_{\tau \searrow 0} \left(d_+(\tau,x) - \sigma\sqrt{\tau} \right) = -\infty - 0 = -\infty. \tag{12}$$

The corresponding terminal condition is:

$$\begin{aligned}\lim_{t \nearrow T} c(t,x) &= \lim_{\tau \searrow 0} \left(xN(d_+(\tau,x)) - Ke^{-r\tau} N(d_-(\tau,x)) \right) \\ &= x \cdot 0 - K \cdot 1 \cdot 0 = 0. \end{aligned} \tag{13}$$

To summarize, the terminal condition is:

$$\lim_{t \nearrow T} c(t,x) = (x-K)^+, \quad x \in (0,K) \cup (K, +\infty). \tag{14}$$

□

1.6. (vi)

Proof.

$$\lim_{x \searrow 0} d_+ = \lim_{x \searrow 0} \frac{\ln \frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = -\infty, \tag{15}$$

$$\lim_{x \searrow 0} d_- = \lim_{x \searrow 0} \left(d_+(T-t,x) - \sigma\sqrt{T-t} \right) = -\infty. \tag{16}$$

Therefore one of the boundary condition is:

$$\lim_{x \searrow 0} c(t,x) = \lim_{x \searrow 0} \left[xN(d_+) - Ke^{-r(T-t)} N(d_-) \right] = 0 \cdot 0 - Ke^{-r(T-t)} \cdot 0 = 0. \tag{17}$$

□

1.7. (vii)

Proof.

$$\lim_{x \to +\infty} d_+ = \lim_{x \to +\infty} \frac{\ln \frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = +\infty, \tag{18}$$

$$\lim_{x \to +\infty} d_- = \lim_{x \to +\infty} \left(d_+(T-t,x) - \sigma\sqrt{T-t} \right) = +\infty. \tag{19}$$

Thus the other boundary condition is:

$$
\begin{aligned}
\lim_{x\to+\infty}\left[c(t,x)-x+Ke^{-r(T-t)}\right] &= \lim_{x\to+\infty}\left[xN(d_+)-Ke^{-r(T-t)}N(d_-)-x+Ke^{-r(T-t)}\right] \\
&= \lim_{x\to+\infty}\left[x\left(N(d_+)-1\right)\right]+Ke^{-r(T-t)}\lim_{x\to+\infty}\left[1-N(d_-)\right] \\
&= \lim_{x\to+\infty}\frac{N(d_+)-1}{x^{-1}}+Ke^{-r(T-t)}\cdot 0 \\
&= \lim_{x\to+\infty}\frac{N'(d_+)\frac{\partial d_+}{\partial x}}{-x^{-2}} = -\lim_{x\to+\infty}\frac{xe^{-\frac{d_+^2}{2}}}{\sigma\sqrt{2\pi\,(T-t)}} \\
&= -\frac{K}{\sigma\sqrt{2\pi\,(T-t)}}\lim_{d_+\to+\infty}\exp\left\{-\frac{1}{2}\left(d_+^2-2\sigma\sqrt{T-t}d_+ +(T-t)\left(2r+\sigma^2\right)\right)\right\} \\
&= -\frac{K}{\sigma\sqrt{2\pi\,(T-t)}}\lim_{d_+\to+\infty}\exp\left\{-\frac{1}{2}\left(d_+-\sigma\sqrt{T-t}\right)^2-r\,(T-t)\right\} \\
&= 0. \qquad (20)
\end{aligned}
$$

□

Appendix A. Original Homework Questions (attached)

Exercise 4.9. For a European call expiring at time T with strike price K, the Black-Scholes-Merton price at time t, if the time-t stock price is x, is

$$c(t,x)=xN\big(d_+(T-t,x)\big)-Ke^{-r(T-t)}N\big(d_-(T-t,x)\big),$$

where

$$d_+(\tau,x)=\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K}+\left(r+\frac{1}{2}\sigma^2\right)\tau\right],$$
$$d_-(\tau,x)=d_+(\tau,x)-\sigma\sqrt{\tau},$$

and $N(y)$ is the cumulative standard normal distribution

$$N(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-\frac{z^2}{2}}dz=\frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty}e^{-\frac{z^2}{2}}dz.$$

The purpose of this exercise is to show that the function c satisfies the Black-Scholes-Merton partial differential equation

$$c_t(t,x)+rxc_x(t,x)+\frac{1}{2}\sigma^2x^2c_{xx}(t,x)=rc(t,x),\ 0\le t<T, x>0, \quad (4.10.3)$$

the *terminal condition*

$$\lim_{t\uparrow T}c(t,x)=(x-K)^+,\quad x>0, x\ne K, \qquad (4.10.4)$$

and the *boundary conditions*

$$\lim_{x\downarrow 0}c(t,x)=0,\quad \lim_{x\to\infty}\left[c(t,x)-\left(x-e^{-r(T-t)}K\right)\right]=0,\quad 0\le t<T. \ (4.10.5)$$

Equation (4.10.4) and the first part of (4.10.5) are usually written more simply but less precisely as

$$c(T,x)=(x-K)^+,\ x\ge 0$$

and

$$c(t,0)=0,\ 0\le t\le T.$$

For this exercise, we abbreviate $c(t,x)$ as simply c and $d_\pm(T-t,x)$ as simply $d_\pm$.

(i) Verify first the equation

$$Ke^{-r(T-t)}N'(d_-)=xN'(d_+). \qquad (4.10.6)$$

(ii) Show that $c_x=N(d_+)$. This is the *delta* of the option. (Be careful! Remember that d_+ is a function of x.)

(iii) Show that

$$c_t=-rKe^{-r(T-t)}N(d_-)-\frac{\sigma x}{2\sqrt{T-t}}N'(d_+).$$

This is the *theta* of the option.

(iv) Use the formulas above to show that c satisfies (4.10.3).

(v) Show that for $x>K$, $\lim_{t\uparrow T}d_\pm=\infty$, but for $0<x<K$, $\lim_{t\uparrow T}d_\pm=-\infty$. Use these equalities to derive the terminal condition (4.10.4).

(vi) Show that for $0\le t<T$, $\lim_{x\downarrow 0}d_\pm=-\infty$. Use this fact to verify the first part of boundary condition (4.10.5) as $x\downarrow 0$.

(vii) Show that for $0\le t<T$, $\lim_{x\to\infty}d_\pm=\infty$. Use this fact to verify the second part of boundary condition (4.10.5) as $x\to\infty$. In this verification, you will need to show that

$$\lim_{x\to\infty}\frac{N(d_+)-1}{x^{-1}}=0.$$

This is an indeterminate form $\frac{0}{0}$, and L'Hôpital's rule implies that this limit is

$$\lim_{x\to\infty}\frac{\frac{d}{dx}\left[N(d_+)-1\right]}{\frac{d}{dx}x^{-1}}.$$

Work out this expression and use the fact that

$$x=K\exp\left\{\sigma\sqrt{T-t}\,d_+-(T-t)\left(r+\frac{1}{2}\sigma^2\right)\right\}$$

to write this expression solely in terms of d_+ (i.e., without the appearance of any x except the x in the argument of $d_+(T-t,x)$). Then argue that the limit is zero as $d_+\to\infty$.

Shreve, S. E., 2004. Stochastic calculus for finance II: Continuous-time models. Vol. 11. Springer Science & Business Media.