# Substring Compression Variants

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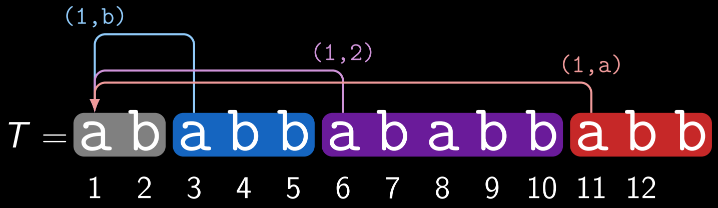



coding: (a,b)(1,b)(1,2)(1,a)

# setting

## text factorization

- input: text $T$ with length $n$
- output: factorization of $T$

examples of factorizations

- LZ77
- LZ78
- Lyndon factorization

goal: compute factorization in $\mathcal{O}(n)$ time

## substring compression

- index $T$ in a preprocessing step
- query: interval $[i..j] \subset [1..n]$
- output: factorization of $T[i..j]$

goal:

- query time linear to output size (output sensitive)
- index time linear in input size ($\mathcal{O}(n)$ time)

# why restricting index time?

trivial solution for substring compression:

- compute and store the factorizations of all $\Theta(n^2)$ substrings
- answer a query in $\mathcal{O}(1)$ via lookup
- however: index space is $\Omega(n^2)$ (hence time is also $\Omega(n^2)$)

# work on substring factorization

| factorization | construction time | query time | reference |
|---|---|---|---|
| LZ77 | $\mathcal{O}(n \lg n)$ | $\mathcal{O}(z \lg n \lg \lg n)$ | Cormode+'05 |
| LZ77 | $\mathcal{O}(n \lg n)$ | $\mathcal{O}(z \lg \lg n)$ | Keller+'14 |
| Lyndon | $\mathcal{O}(n \lg n)$ | $\mathcal{O}(z)$ | Babenko+'14 |
| Lyndon | $\mathcal{O}(n)$ | $\mathcal{O}(z)$ | Kociumaka'16 |
| LZ$X$ | $\mathcal{O}(n)$ | $\mathcal{O}(z)$ | this talk |

- $z$ : output size of respective factorization
- $X \in \{$ 78, Miller-Wegman (MW), Double (D) $\}$

References:


- Shibata, K.: "LZ78 Substring Compression with CDAWGs", SPIRE'24
- K.: "Substring Compression Variations and LZ78-Derivates", Information Systems'25
- K.: "Non-Overlapping LZ77 Factorization and LZ78 Substring Compression Queries with Suffix Trees", Algorithms'21

# factorizations in this talk

LZ78 derivations

- Lempel–Ziv 78 (LZ78) Ziv,Lempel'78
- Lempel–Ziv Double (LZD) Goto'15
- Lempel–Ziv-Miller–Wegman (LZMW) Miller+'85

why important?

- LZ78 widely used for image compression such as GIF or TIFF
- can be used as a grammar for more operations (unlike LZ77)

why the variants?

- number of LZ78 factors is lower bounded by $\Omega(\sqrt{n})$
- in contrast, the lower bound for LZD and LZMW is $\Omega(\lg n)$

# lower bound for LZ78

$$T = \mathtt{a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a} \cdots$$

1 2 3 4 5 6 7 8 9 10 11 12

coding:

since the length $|F_x|$ of the $x$-th factor is $x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\sqrt{n})$

# lower bound for LZ78

$$T = \texttt{a a a a a a a a a a a a} \cdots$$

1 2 3 4 5 6 7 8 9 10 11 12

coding: a

since the length $|F_x|$ of the $x$-th factor is $x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\sqrt{n})$

# lower bound for LZ78

(1,a)

$T = \texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\,\texttt{a}\cdots$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|

coding: a(1,a)

since the length $|F_x|$ of the $x$-th factor is $x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\sqrt{n})$

# lower bound for LZ78

(1,a)

(2,a)

$T = \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a} \cdots$

1 2 3 4 5 6 7 8 9 10 11 12

coding: a(1,a)(2,a)

since the length $|F_x|$ of the $x$-th factor is $x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\sqrt{n})$

# lower bound for LZ78

$(1,\mathtt{a})$ $(2,\mathtt{a})$ $(3,\mathtt{a})$

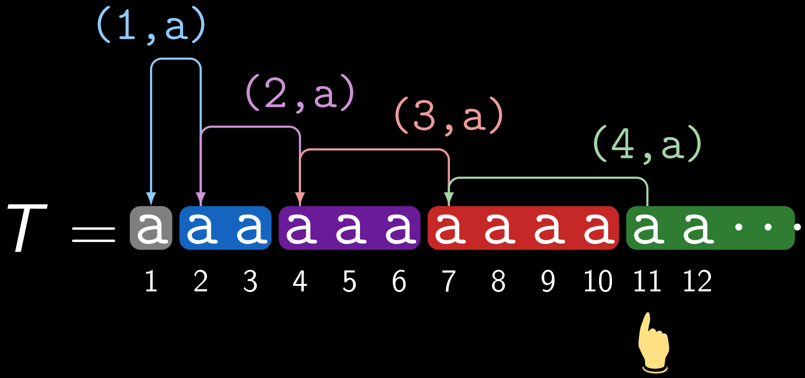
$T = \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a} \cdots$

1 2 3 4 5 6 7 8 9 10 11 12

coding: $\mathtt{a}(1,\mathtt{a})(2,\mathtt{a})(3,\mathtt{a})$

since the length $|F_x|$ of the $x$-th factor is $x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\sqrt{n})$

## lower bound for LZ78




coding: a(1,a)(2,a)(3,a)(4,a)

since the length $|F_x|$ of the $x$-th factor is $x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\sqrt{n})$

# lower bound for LZD

$$T = \texttt{a a a a a a a a a a a a} \cdots$$

1 2 3 4 5 6 7 8 9 10 11 12

coding:

since the length $|F_x|$ of the $x$-th factor is $2^x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZD

$$T = \texttt{a a a a a a a a a a a a} \cdots$$

1 2 3 4 5 6 7 8 9 10 11 12

coding: (a,a)

since the length $|F_x|$ of the $x$-th factor is $2^x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZD

$$T = \texttt{a a a a a a a a a a a a} \cdots$$

1 2 3 4 5 6 7 8 9 10 11 12

(1,1)

coding: (a,a)(1,1)

since the length $|F_x|$ of the $x$-th factor is $2^x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZD

(1,1)

(2,2)

$T =$ a a a a a a a a a a a a ···

1 2 3 4 5 6 7 8 9 10 11 12

coding: (a,a) (1,1) (2,2)

since the length $|F_x|$ of the $x$-th factor is $2^x$, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZMW

$T =$ a a a a a a a a a a a a $\cdots$

1 2 3 4 5 6 7 8 9 10 11 12

coding:

since the length $|F_x|$ of the $x$-th factor is the $(x-1)$st Fibonacci number,
$\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZMW

$T = \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a}\ \texttt{a} \cdots$

1 2 3 4 5 6 7 8 9 10 11 12

coding: a

since the length $|F_x|$ of the $x$-th factor is the $(x-1)$st Fibonacci number, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZMW

$T = $ a a a a a a a a a a a a $\cdots$

1 2 3 4 5 6 7 8 9 10 11 12

coding: aa

since the length $|F_x|$ of the $x$-th factor is the $(x-1)$st Fibonacci number,
$\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZMW

2

$T = \texttt{a a a a a a a a a a a a} \cdots$

1 2 3 4 5 6 7 8 9 10 11 12

coding: aa2

since the length $|F_x|$ of the $x$-th factor is the $(x-1)$st Fibonacci number,
$\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZMW

$T = \texttt{a a a a a a a a a a a a} \cdots$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | a | a | a | a | a | a | a | a | a | a |

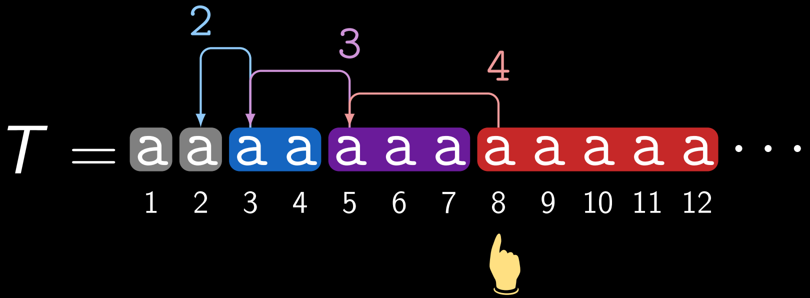
2

3

coding: aa23

since the length $|F_x|$ of the $x$-th factor is the $(x-1)$st Fibonacci number,
$\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

# lower bound for LZMW




coding: aa234

since the length $|F_x|$ of the $x$-th factor is the $(x-1)$st Fibonacci number, $\sum_{x=1}^{z} |F_x| = n \Leftrightarrow z \in \Theta(\lg n)$

## definition of LZ78

each factor represented as a pair

- index of a former factor (0 for the empty string)
- appended character

let $\mathrm{dst}_x$ denote the starting position of $F_x$ in $T$.

### Definition (LZ78)

A factorization $F_1 \cdots F_z$ of $T$ is LZ78 if

- $F_x = F_y c$, where
- $F_y$ is the longest factor among $F_0, F_1, \ldots, F_{x-1}$ being a prefix of $T[\mathrm{dst}_x..]$,
- $c = T[\mathrm{dst}_x + F_y]$,
- $F_0$ is the empty string

# LZ78

$T =$ a b a b b a b a b b a b b

| a | b | a | b | b | a | b | a | b | b | a | b | b |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

0 LZ trie

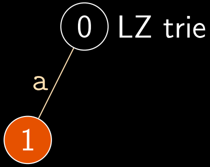
coding:

## LZ78

$T =$ a b a b b a b a b b a b b

1 2 3 4 5 6 7 8 9 10 11 12 13

coding: (0,a)

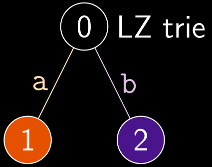


# LZ78

$T =$ a b a b b a b a b b a b b

1 2 3 4 5 6 7 8 9 10 11 12 13

coding: (0,a)(0,b)

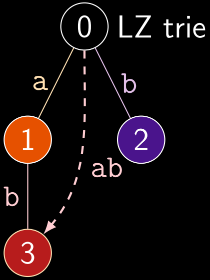


## LZ78

$T =$ a b a b b a b a b b a b b

1 2 3 4 5 6 7 8 9 10 11 12 13

coding: (0,a)(0,b)(1,b)

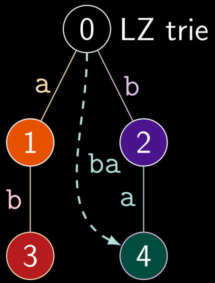


# LZ78

$T = \text{a b a b b a b a b b a b b}$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | a | b | b | a | b | a | b | b | a | b | b |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

coding: (0,a)(0,b)(1,b)(2,a)



# definition of LZD

each factor represented as a pair

- element is either a character or the index of a former factor
- greedily maximize the length by the first element first

let $\mathrm{dst}_x$ denote the starting position of $F_x$ in $T$.

## Definition (LZD)

A factorization $F_1 \cdots F_z$ of $T$ is LZD if

- $F_x = G_1 \cdot G_2$ with
- $G_1, G_2 \in \{F_1, \ldots, F_{x-1}\} \cup \Sigma$ such that
- $G_1$ and $G_2$ are respectively the longest possible prefixes of $T[\mathrm{dst}_x..]$ and of $T[\mathrm{dst}_x + |G_1|..]$.

## example for LZD

$T =$ a b a b b a b a b b a b b

1 2 3 4 5 6 7 8 9 10 11 12 13

coding:

# example for LZD

$T =$ a b a b b a b a b b a b b

1 2 3 4 5 6 7 8 9 10 11 12 13

coding: (a,b)

## example for LZD

(1,b)

| $T =$ | a | b | a | b | b | a | b | a | b | b | a | b | b |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

coding: (a,b)(1,b)

## example for LZD

(1,b)

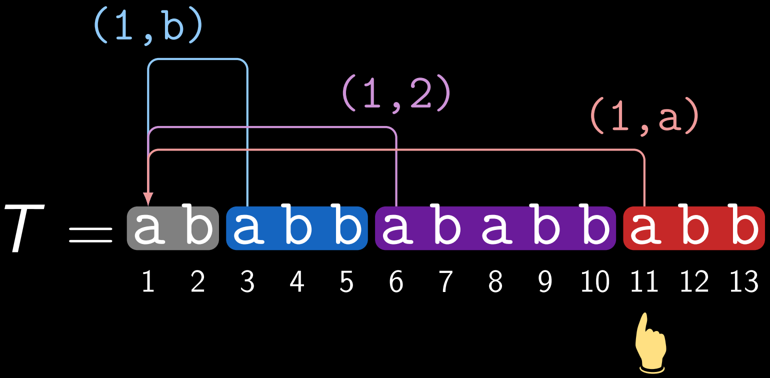
(1,2)

$T =$ a b a b b a b a b b a b b

| a | b | a | b | b | a | b | a | b | b | a | b | b |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

coding: (a,b)(1,b)(1,2)

# example for LZD




coding: (a,b)(1,b)(1,2)(1,a)

# definition of LZMW

- has like LZD two references
- however references need to be successive
- thus needs to store only one reference to a former factor index

**Definition (LZMW)**

A factorization $F_1 \cdots F_z$ of $T$ is LZMW if $F_x$ is the longest prefix of $T[\mathrm{dst}_x..]$ with $F_x \in \{F_{y-1}F_y : y \in [2..\mathrm{dst}_x - 1]\} \cup \Sigma$, for every $x \in [1..z]$.

# example for LZMW

$$T = \mathtt{a\,b\,a\,b\,b\,a\,b\,a\,b\,b\,a\,b\,b}$$

1 2 3 4 5 6 7 8 9 10 11 12 13

coding:

## example for LZMW

| $T =$ | a | b | a | b | b | a | b | a | b | b | a | b | b |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

coding: a

# example for LZMW

$T =$ a b a b b a b a b b a b b

1 2 3 4 5 6 7 8 9 10 11 12 13

coding: ab

## example for LZMW

$$T = \text{a b a b b a b a b b a b b}$$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | a | b | b | a | b | a | b | b | a | b | b |

coding: ab2

## example for LZMW

| $T =$ | a | b | a | b | b | a | b | a | b | b | a | b | b |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

coding: ab23

# example for LZMW

2 3 4

$T =$ a b a b b a b a b b a b b

1 2 3 4 5 6 7 8 9 10 11 12 13

coding: ab234

# example for LZMW

2 3 4

$T =$ `a b a b b a b a b b a b b`

1 2 3 4 5 6 7 8 9 10 11 12 13

coding: `ab234b`

# LZD and LZMW computation

| time | space | reference |
|---|---|---|
| $\mathcal{O}(n \lg \sigma)$ | $\mathcal{O}(n)$ | Goto+'15 |
| $\Omega(n^{5/4})$ | $\mathcal{O}(z)$ | Goto+'15, Badkobeh+'17 |
| $\mathcal{O}(n + z \lg^2 n)$ expected | $\mathcal{O}(z)$ | Badkobeh+'17 |
| $\mathcal{O}(n)$ | $\mathcal{O}(n)$ | this talk |

where

- Goto+'15 only computes LZD
- $\sigma = n^{\mathcal{O}(1)}$ means that integer alphabets are supported

For LZ78: $\mathcal{O}(n)$ time and space achieved by Nakashima+'15

# our contributions

- for the whole text, we can compute LZD and LZMW in $\mathcal{O}(n)$ time and space
- compute the substring compression of LZD and LZMW with
  - $\mathcal{O}(n)$ index time for preprocessing
  - $\mathcal{O}(z)$ query time
- setting
  - $n$ : length of the input
  - integer alphabet
  - word RAM

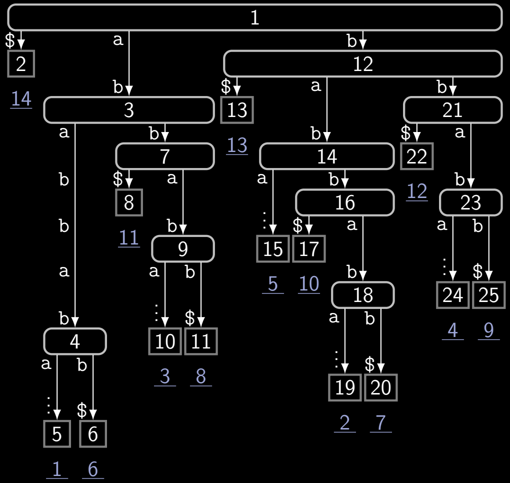
# tools

for computation, we leverage the following toolbox

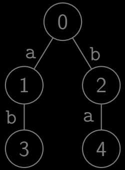
- suffix tree ST Weiner'73
  - linear-time construction of ST Farach-Colton'00
- weighted ancestor query data structure Gawrychowski'14
  - find an ancestor with string depth $d$ of any ST node and any $d$ in $\mathcal{O}(1)$ time
  - constructable in linear time Belazzougui'21
- lowest marked ancestor data structure Cole+'05
  - can mark any ST node in $\mathcal{O}(1)$ time
  - can find the lowest marked ancestor of any ST node in $\mathcal{O}(1)$ time

sum of needed space and time amounts to $\mathcal{O}(n)$ each

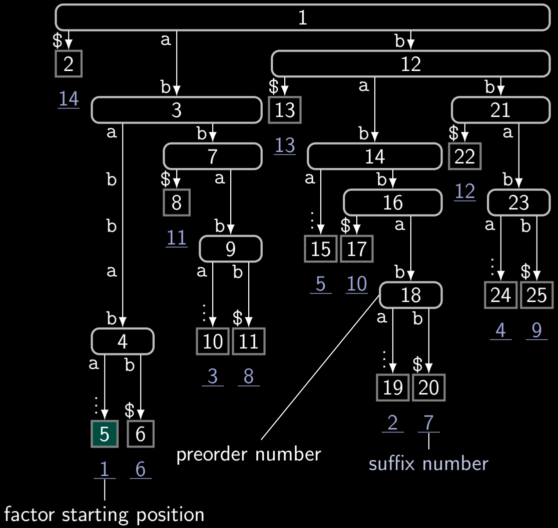
how used for LZ78 computation?

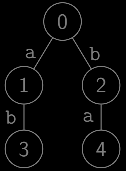


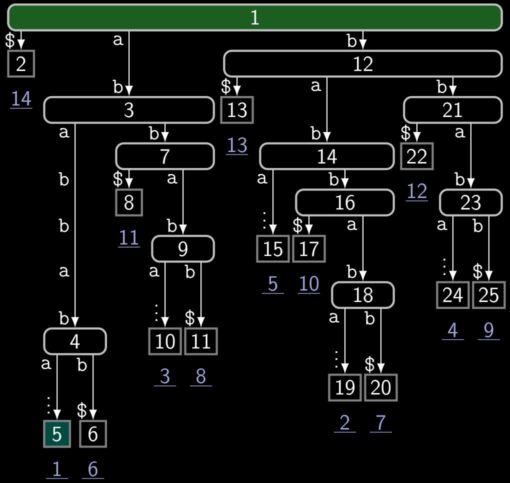

suffix tree of $T\$ =$ ababbababbabb$



suffix tree of $T\$ =$ ababbababbabb$

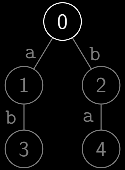


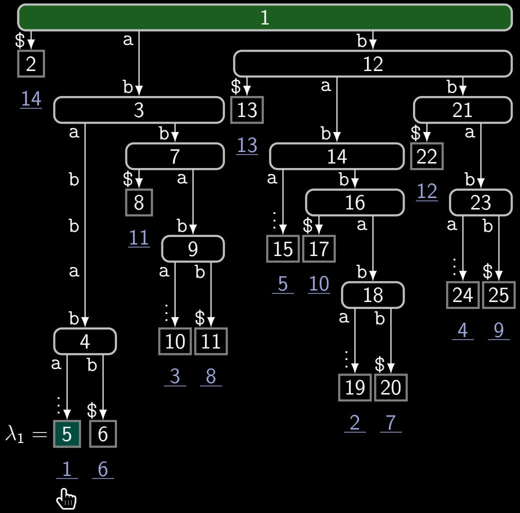





suffix tree of $T\$ =$ ababbababbabb\$

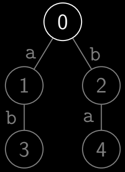
- ST root represents empty factor

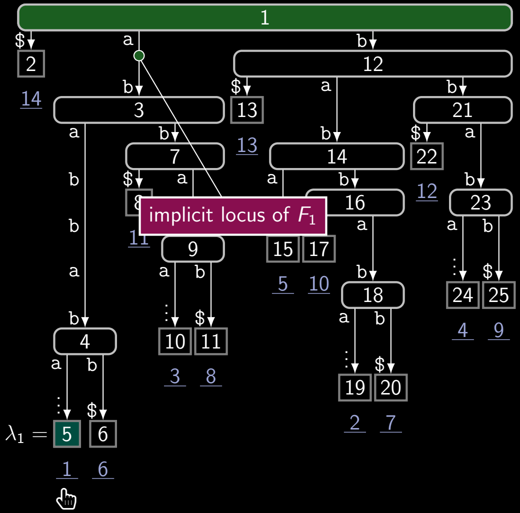





suffix tree of $T\$ =$ ababbabababbabb\$

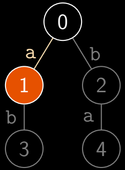
- ST root represents empty factor
- find suffix number = factor starting position

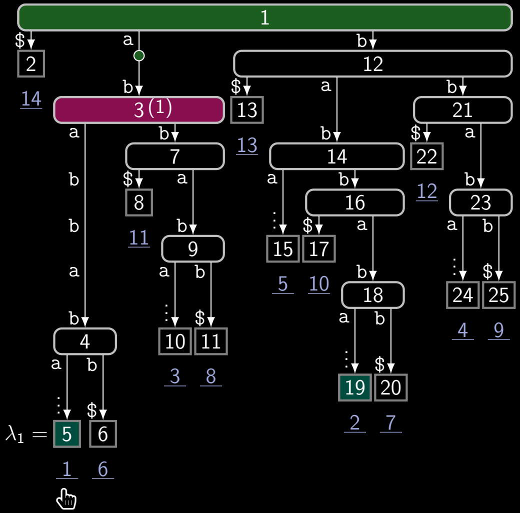





suffix tree of $T\$ =$ ababbababbabb$

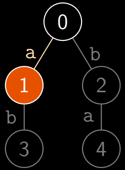
- ST root represents empty factor
- find suffix number = factor starting position
- create child of lowest *marked* ancestor

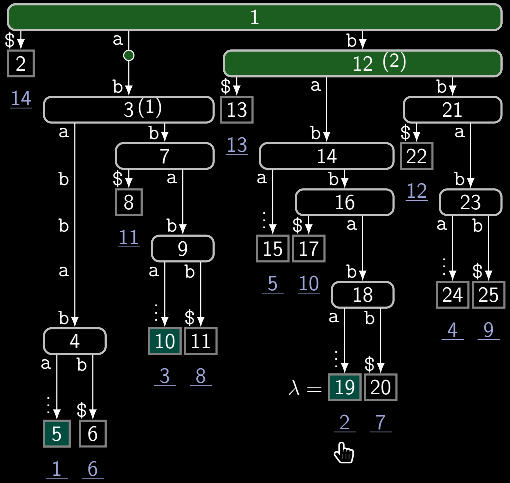




suffix tree of $T\$ =$ ababbababbabb$

- ST root represents empty factor
- find suffix number = factor starting position
- create child of lowest *marked* ancestor
- explicit nodes witness factors



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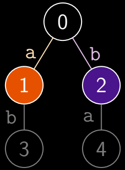

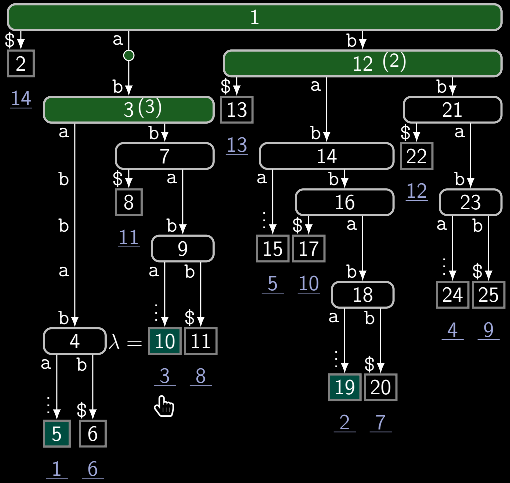

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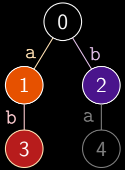

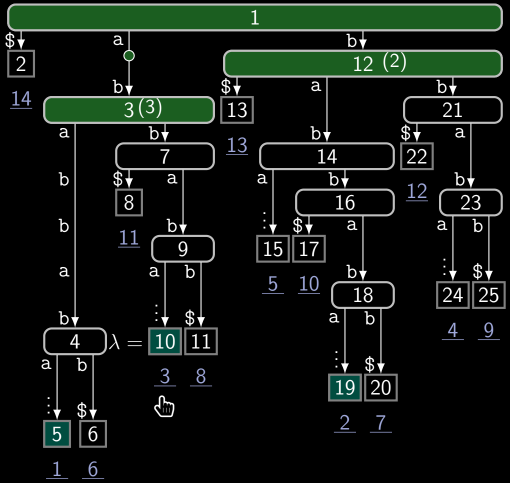

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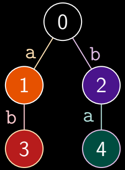


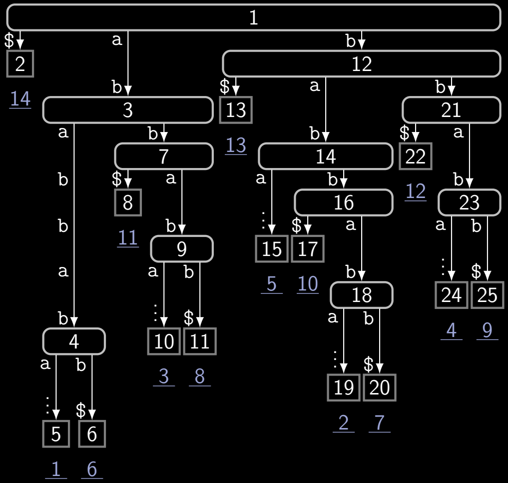
## time complexity

for processing $F_x$

- take leaf $\lambda$ corresponding to the starting position $\mathrm{dst}_x$ of $F_x$
- compute the lowest marked ancestor $v$ of $\lambda$
- given $\ell$ is the string length of $v$, the length of $F_x$ is $\ell$
- if $v$ refers to an implicit node, use the stored length instead of $\ell$

each step takes $\mathcal{O}(1)$ time, so we have $\mathcal{O}(z)$ total time, where $z$ is the number of processed factors

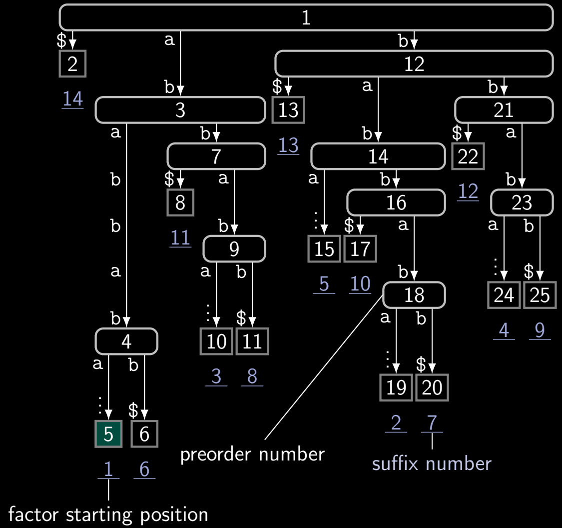
# . . . and how about LZD?



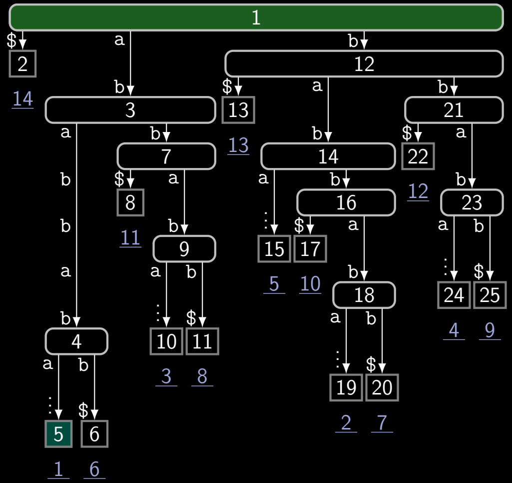

suffix tree of $T\$ = \mathtt{ababbababbabb\$}$

$T = \mathtt{ababbababbabb}$

suffix tree of $T\$ = \texttt{ababbababbabb\$}$



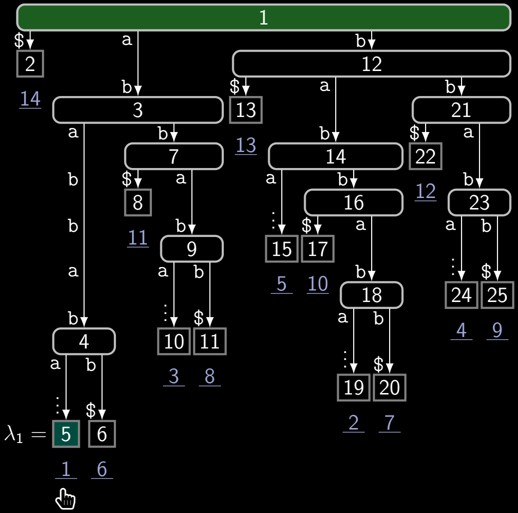

$T = \texttt{ababbababbabb}$




suffix tree of $T\$$ = ababbababbabb$

- ST root represents empty factor

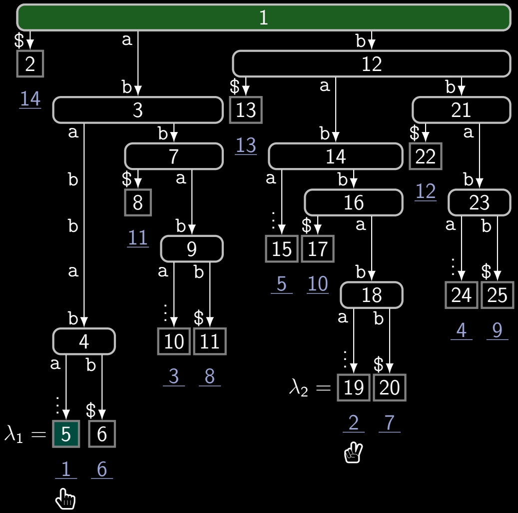
$T$ = ababbababbabb




suffix tree of $T\$ = \mathtt{ababbababbabb\$}$

- ST root represents empty factor
- compute pair $F_1 = (e_L, e_R)$ of first factor
- suffix number of $\lambda_1$ is $\mathrm{dst}_1 = 1$
- lowest marked ancestor of $\lambda_1$ is ST root, so $e_L = T[1] = \mathtt{a}$

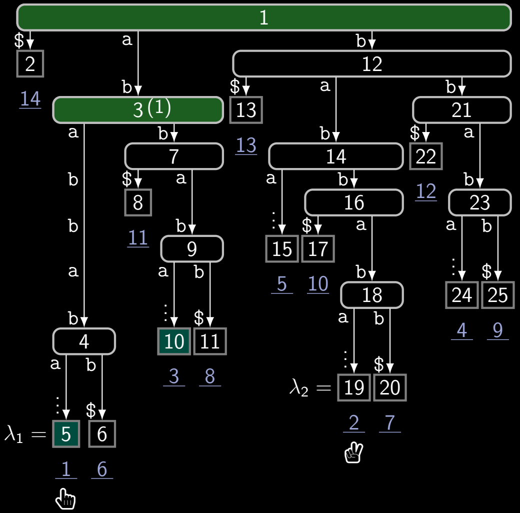
$T = \mathtt{ababbababbabb}$



suffix tree of $T\$ = \mathtt{ababbababbabb\$}$

- ST root represents empty factor
- compute pair $F_1 = (e_L, e_R)$ of first factor
- suffix number of $\lambda_1$ is $\mathrm{dst}_1 = 1$
- lowest marked ancestor of $\lambda_1$ is ST root, so $e_L = T[1] = \mathtt{a}$
- $\lambda_2$ is leaf with suffix number 2

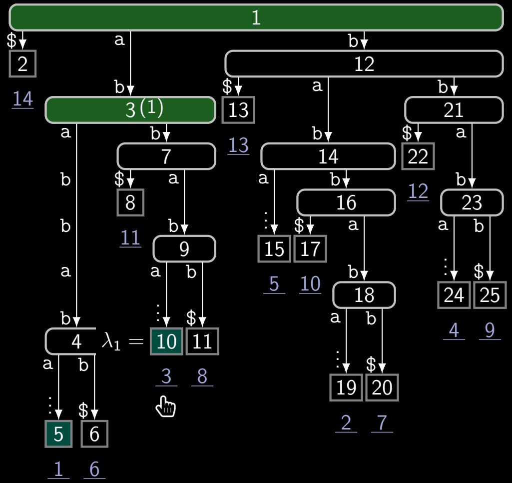
$T = \mathtt{ababbababbabb}$




$T$ = ab|abbababbabb

suffix tree of $T\$$ = ababbababbabb\$

- ST root represents empty factor
- compute pair $F_1 = (e_{\mathrm{L}}, e_{\mathrm{R}})$ of first factor
- suffix number of $\lambda_1$ is $\mathrm{dst}_1 = 1$
- lowest marked ancestor of $\lambda_1$ is ST root, so $e_{\mathrm{L}} = T[1] = \mathtt{a}$
- $\lambda_2$ is leaf with suffix number 2
- lowest marked ancestor of $\lambda_2$ is ST root, so $e_{\mathrm{R}} = T[2] = \mathtt{b}$
- mark ancestor of $\lambda_1$ with string depth 2 with 1

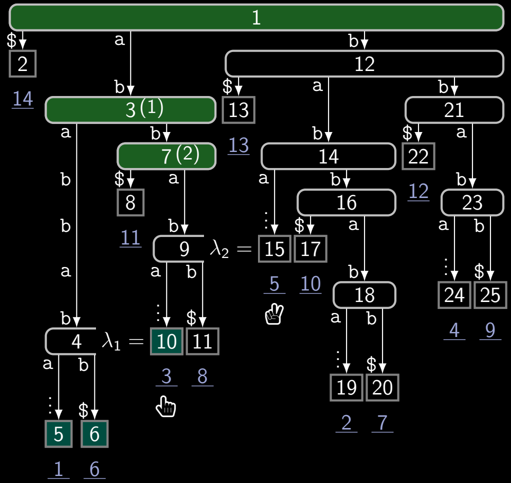



suffix tree of $T\$ = \mathtt{ababbababbabb\$}$

process $F_2$

- suffix number of $\lambda_1$ is $\mathrm{dst}_2 = 3$
- lowest marked ancestor of $\lambda_1$ is 3, so $e_L = 1$ (mark of 3)

$T = \mathtt{ab}|\mathtt{abbababbabb}$

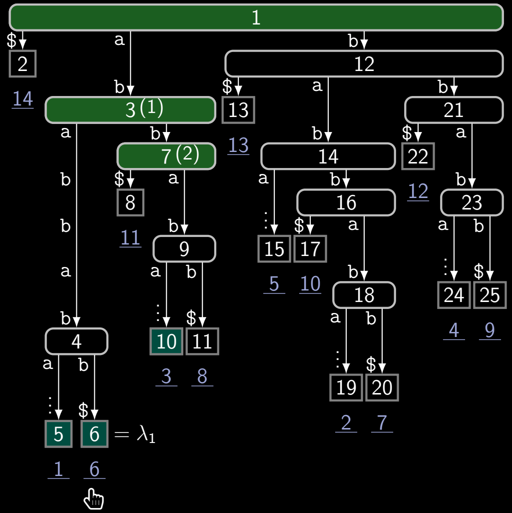


suffix tree of $T\$ = \mathtt{ababbababbbabb\$}$

process $F_2$

- suffix number of $\lambda_1$ is $\mathrm{dst}_2 = 3$
- lowest marked ancestor of $\lambda_1$ is 3, so $e_L = 1$ (mark of 3)
- like before, $e_R = T[2] = \mathtt{b}$
- mark ancestor of $\lambda_1$ with string depth $|F_2| = 3$ with 2

$T = \mathtt{ab}|\mathtt{abb}|\mathtt{ababbabb}$

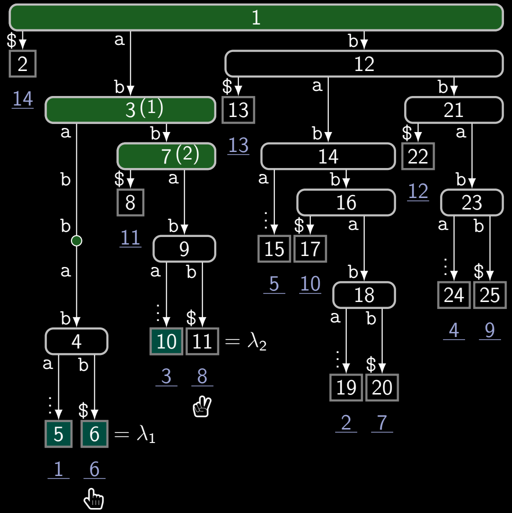



suffix tree of $T\$ = \text{ababbababbbabb\$}$

process $F_3$

- suffix number of $\lambda_1$ is $\text{dst}_3 = 6$
- lowest marked ancestor of $\lambda_1$ is 3, so $e_L = 1$ (mark of 3)

$T = \text{ab}|\text{abb}|\text{ababbabb}$



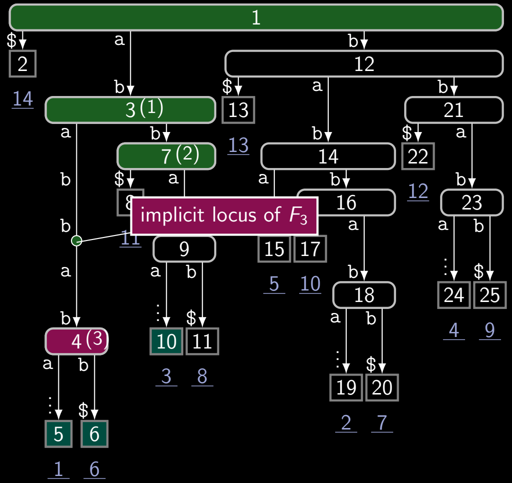

suffix tree of $T\$ = \texttt{ababbababbabb\$}$

process $F_3$

- suffix number of $\lambda_1$ is $\mathrm{dst}_3 = 6$
- lowest marked ancestor of $\lambda_1$ is 3, so $e_L = 1$ (mark of 3)
- lowest marked ancestor of $\lambda_2$ is 7, so $e_L = 2$ (mark of 2)
- however: ancestor of $\lambda_1$ with string depth $|F_3| = 5$ does not exist!

$T = \texttt{ab}|\texttt{abb}|\texttt{ababb}|\texttt{abb}$

suffix tree of $T\$ = \mathtt{ababbababbabb\$}$




maintaining reference for $F_3$

- locus of $F_3$ can be witnesses by node 4
- let node 4 store length of $F_3$; mark node 4

$T = \mathtt{ab|abb|ababb|abb}$

# time complexity

basically doubling the time for LZ78
for processing $F_x$

- take leaf $\lambda_1$ corresponding to the starting position $\mathrm{dst}_x$ of $F_x$
- compute the lowest marked ancestor $v_1$ of $\lambda_1$
- given $\ell_1$ is the string length of $v_1$, take leaf $\lambda_2$ having suffix number $\mathrm{dst}_x + \ell_1$
- compute the lowest marked ancestor $v_2$ of $\lambda_2$
- length of $F_x$ is $\ell_1 + \ell_2$, where $\ell_2$ is the string length of $v_2$
- if $v_1$ (or $v_2$) refers to an implicit node, use the stored length instead of $\ell_1$ (or $\ell_2$)

each step takes $\mathcal{O}(1)$ time, so we have $\mathcal{O}(z)$ total time, where $z$ is the number of processed factors

# LZMW

LZMW computation works similarly

- mark the locus of $F_{x-1}F_x$ instead of $F_x$
- need only one lowest marked ancestor query ($v_2$ not needed)

# summary

- can compute LZ$X$ in $\mathcal{O}(n)$ time, in the computational model
  - $n$ : length of the input
  - alphabet can be integer
  - word RAM

  $X \in \{$ 78, Miller-Wegman (MW), Double (D) $\}$

for substring compression:

- $\mathcal{O}(n)$ index time
- $\mathcal{O}(z)$ query time, where $z$ is the number of factors to output

Open question: Substring compression in compressed space possible?

# substring compression in compressed space

- up till now, all substring compression solutions for LZ77, Lyndon factorization, etc., need $\mathcal{O}(n)$ space
- can we improve space by sacrificing time?

## Answer

For LZ$X$: Yes by a reduction to the stabbing-max problem

# reduction to stabbing-max problem

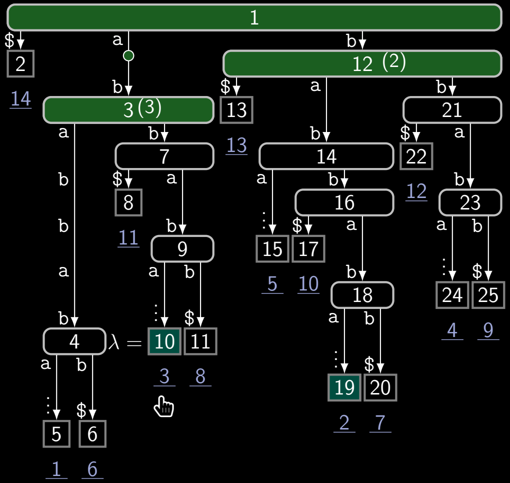
**Definition (stabbing-max problem)**

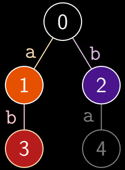
- input: dynamic set of $m$ weighted intervals $\mathcal{S} = \{(\mathcal{I}_i, w_i)\}_i$ with $\mathcal{I}_i \subset [1..n]$ and weight $w_i \in [1..n]$
- supports two operations:
  - $\mathrm{query}(k)$: return the heaviest interval containing $k$, i.e., $\mathrm{argmax}_i\{w_i \mid k \in \mathcal{I}_i\}$
  - $\mathrm{add}(\mathcal{I}, w)$: add $(\mathcal{I}, w)$ to $\mathcal{S}$

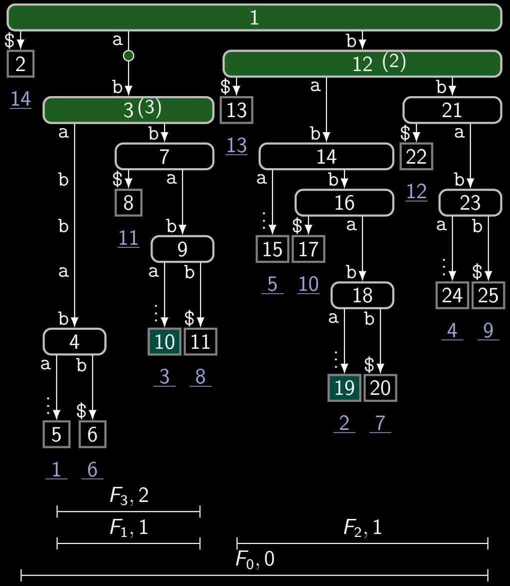
An implementation is due to Tarjan'79:

- query and add in $\mathcal{O}(\lg m)$ time
- $\mathcal{O}(m)$ words

LZ78 factorization having $F_3$ computed

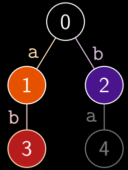




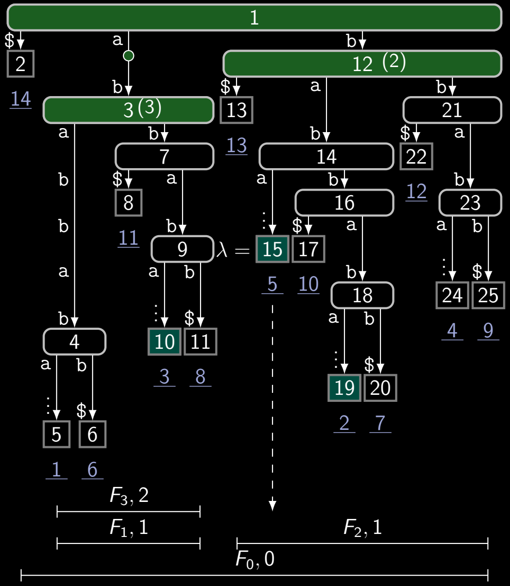
LZ78 factorization having $F_3$ computed

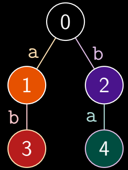
- represent each factor as an interval

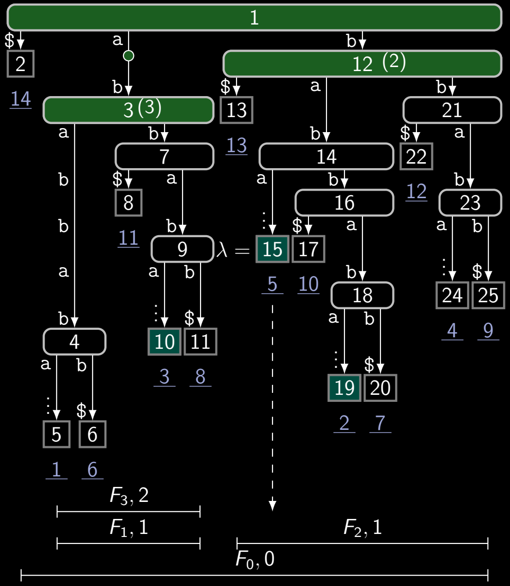


LZ78 factorization having $F_3$ computed

- represent each factor as an interval
- reference is highest stabbed interval



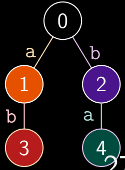


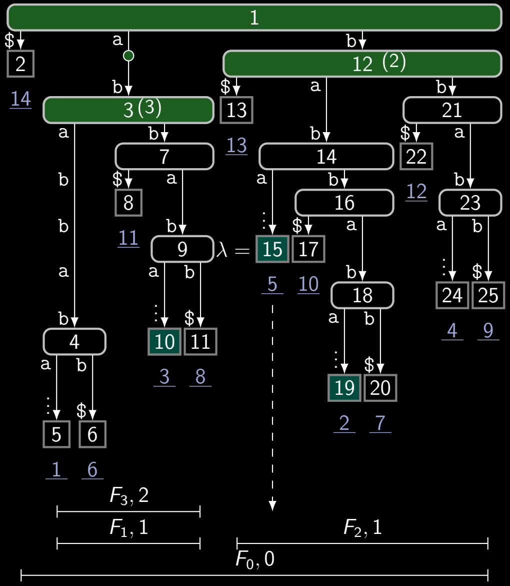

LZ78 factorization having $F_3$ computed

- represent each factor as an interval
- reference is highest stabbed interval

need two helper arrays $\mathrm{LCP}[1..n]$ and $\mathrm{ISA}[1..n]$

- $\mathrm{ISA}[i]$: rank of leaf with suffix number $i$ ($= i$'s SA-position)






LZ78 factorization having $F_3$ computed

- represent each factor as an interval
- reference is highest stabbed interval

need two helper arrays LCP[1..$n$] and ISA[1..$n$]

- ISA[$i$]: rank of leaf with suffix number $i$ ($= i$'s SA-position)
- LCP[$i$]: string depth of lowest common ancestor of $i$-th leaf with its preceding leaf

## the reduction

for a substring $Y$ of $T$, let $\mathrm{range}(Y) = [i..j]$ be the maximum SA-interval of $Y$, i.e., $Y$ is a prefix of $T[k..]$ if and only if $\mathrm{ISA}[k] \in [i..j]$.

- represent computed factors by $\mathcal{S}$
- for LZ78: A factor $F_x$ is represented by
  - $\mathcal{I}_x = \mathrm{range}(Y)$
  - weight $w_x = |F_x|$
- find the next factor starting at $T[i..]$ with $\mathrm{query}(\mathrm{ISA}[i])$:
- if $\mathrm{query}(\mathrm{ISA}[i]) = (\mathcal{I}_x, w_x)$, then $F_x$ is the longest already computed factor being a prefix of $T[i..]$

### conclusion

can compute LZ78 with

- stabbing-max data structure
- access to $T[i]$ and $\mathrm{ISA}[j]$
- $\mathrm{range}(Y)$ for a substring $Y$

# Implementation: r-index

Define

- $\mathrm{PSV}(x,d) = \max(\{0\} \cup \{y \in [1..x-1] \mid \mathsf{LCP}[y] < d\})$ and
- $\mathrm{NSV}(x,d) = \min(\{n\} \cup \{y \in [x..n-1] \mid \mathsf{LCP}[y] < d\})$.

Then $\mathrm{range}(Y) = [\mathrm{PSV}(\mathsf{ISA}[i],|Y|), \mathrm{NSV}(\mathsf{ISA}[i],|Y|)-1]$ for $Y$ being a prefix of $T[i..]$.

r-index Gagie'20

- $T[i]$ and $\mathsf{ISA}[j]$ in $\mathcal{O}(\lg \frac{n}{r})$ time
- PSV and NSV in $\mathcal{O}(\frac{\lg n}{\lg\lg n} + \lg \frac{n}{r})$ time.

## Theorem

- *$\mathcal{O}(r \lg \frac{n}{r})$ words of space*
- *LZ78 in $\mathcal{O}(z\left(\lg\lg \frac{r}{\lg n} + \lg \frac{n}{r} + \lg z\right))$ time with $\mathcal{O}(z)$ extra space*

# Implementation: CDAWG

CDAWG Blumer'85

- $\mathcal{O}(e)$ space, where $e$ is the number of edges in the CDAWG
- $T[i]$ and ISA$[j]$ in $\mathcal{O}(\lg n)$ time Belazzougui'17
- range in $\mathcal{O}(\lg n)$ time by centroid-path decomposition Shibata,K'25

**Theorem**

- *$\mathcal{O}(e)$ space*
- *LZ78 in $\mathcal{O}(z \lg n)$ time with $\mathcal{O}(z)$ extra space*

# Implementation: $\delta$-index

$\delta$-index Kempa, Kociumaka'23

- $\mathcal{O}(\delta \lg \frac{n \lg \sigma}{\delta \lg n})$ space, where $\delta = \max\{\frac{d_k}{k} \mid k \in [1..n]\}$, and $d_k$ is the number of distinct $k$-length substrings in $T$
- $T[i]$ and $\mathrm{ISA}[j]$ in $\mathcal{O}(\log^{4+\varepsilon} n)$ time, where $\varepsilon > 0$ is a given constant.
- longest common prefix between two suffixes in $\mathcal{O}(\lg n)$ time.
- compute $\mathrm{PSV}(x, d)$ and $\mathrm{NSV}(x, d)$ by binary search and LCE queries. (time dwarfed by access to $T[i]$)

**Theorem**

- $\mathcal{O}(\delta \lg \frac{n \lg \sigma}{\delta \lg n})$ *space*
- *LZ78 in* $\mathcal{O}(z \lg^{4+\varepsilon} n)$ *time with* $\mathcal{O}(z)$ *extra space*

# final recap

| data structure | space in words | query time |
|---|---|---|
| suffix tree | $\mathcal{O}(n)$ | $\mathcal{O}(z)$ |
| CDAWG | $\mathcal{O}(e)$ | $\mathcal{O}(z \lg n)$ |
| $r$-index | $\mathcal{O}(r \lg \frac{n}{r})$ | $\mathcal{O}(z \left(\lg\lg \frac{r}{\lg n} + \lg \frac{n}{r} + \lg z\right))$ |
| $\delta$-index | $\mathcal{O}(\delta \lg \frac{n \lg \sigma}{\delta \lg n})$ | $\mathcal{O}(z \lg^{4+\varepsilon} n)$ |

where

- $\delta$: substring complexity
- $e$: # CDAWG edges
- $r$: # runs in the BWT
- $\delta \leq r \leq e \leq n$
- $z$: LZX factors of query substring

Are there other compression methods having whose substring compression can be computed in compressed space?

Thank you for listening. Any questions are welcome!

# open problems

- LZ77-based substring compression use geometric data structures, which are heavyweight. Is there some other approach?
- allowing non-greedy choices for LZD/LZMW, the variant computing the fewest factors is NP-hard? (For LZ78, the flexible parsing is optimal)
- Lyndon factorization can be computed with $\mathcal{O}(1)$ space and $\mathcal{O}(n)$ time, so is there likely a trade-off for substring computation?
- "LZSE: an LZ-style compressor supporting $\mathcal{O}(\log n)$ time random access" (arxiv'25) seems to be computable with the presented tools