The Results for Hysteresis in f & P_c

Equation: $$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left\{ f(S,p) + N_c h(S,p)\frac{\partial p}{\partial x} \right\} = 0 \quad \cdots (1)$$

$$P_c \in p^+(s) - p^-(s)\,\mathrm{sign}(\partial_t S) \quad \cdots (2)$$

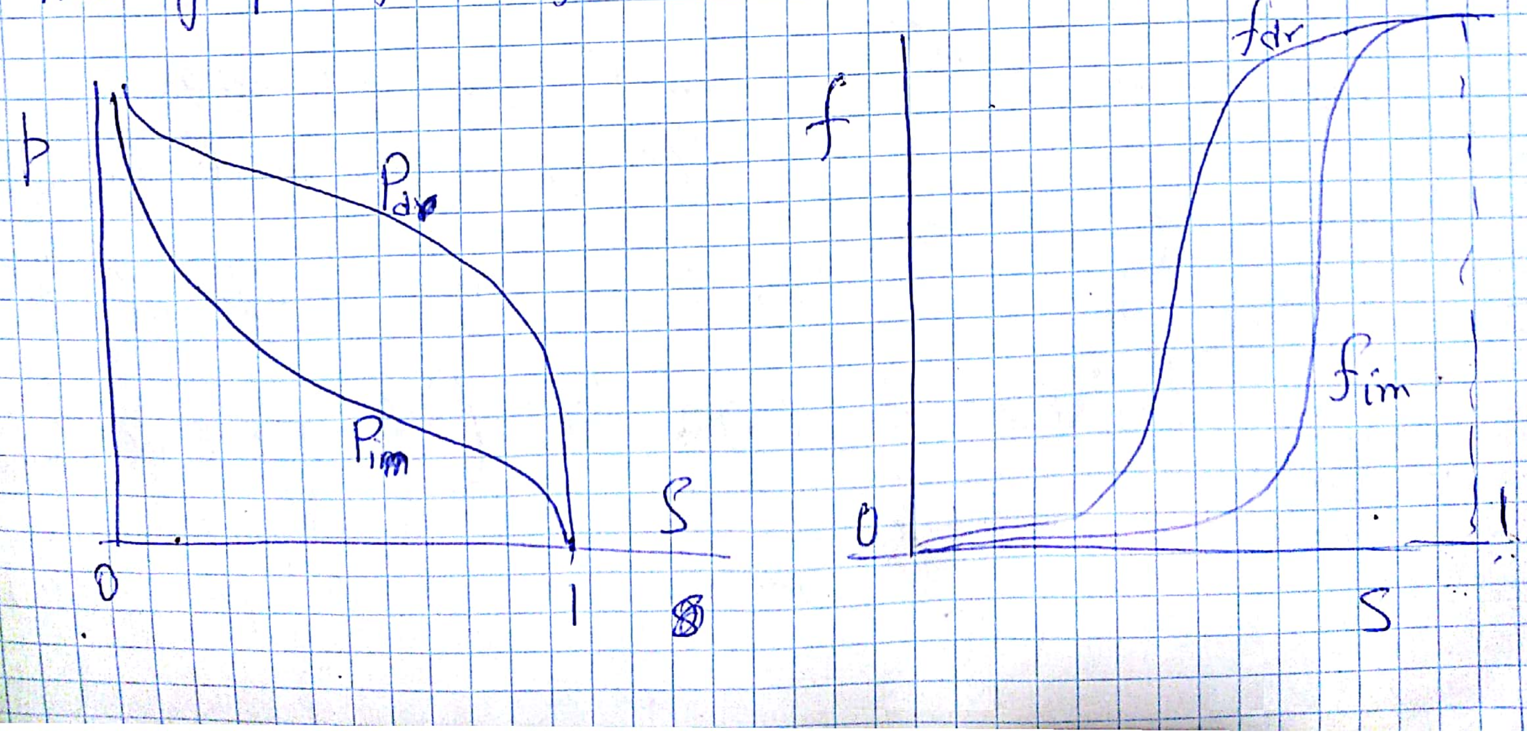
Model: Write (2) as

$$P_c \in P^+(s) - P^-(S)\,\mathrm{sign}(\partial_t S) - \varepsilon\,\partial_t S$$

which can be rewritten as;

$$\partial_t S = \begin{cases} \frac{1}{\varepsilon}(P_{im}(s) - p) & \text{if } p < P_{im}(s) \\ 0 & \text{if } p \in [P_{im}(S), P_{dr}(S)] \\ \frac{1}{\varepsilon}(P_{dr}(S) - p) & \text{if } p > P_{dr}(S) \end{cases}$$

The graphs for f and P_c are



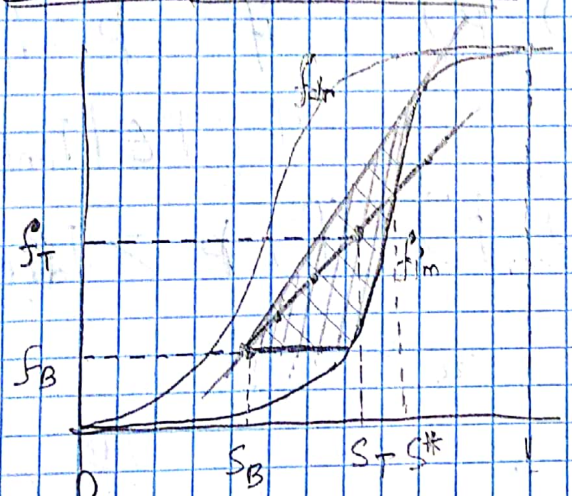
Travelling Waves: with $\eta = ct - x$,

$$(4)\begin{cases} S' = \Psi_\varepsilon(S,p) = \dfrac{1}{C\varepsilon}\begin{cases} P_{im}(S) - u & \text{if } u < P_{im}(S) \\ 0 & \text{if } u \in [P_{im}, P_{dr}] \\ P_{dr}(S) - u & \text{if } u > P_{dr}(S) \end{cases} \\ u' = \dfrac{1}{N_c h(S,p)}\left\{ f(S,p) - f_B - c(S - S_B) \right\} \end{cases}$$

with $\quad c = \dfrac{f_T - f_B}{S_T - S_B}$,

We divide the domains;

Case 1: $f_T > f_B$

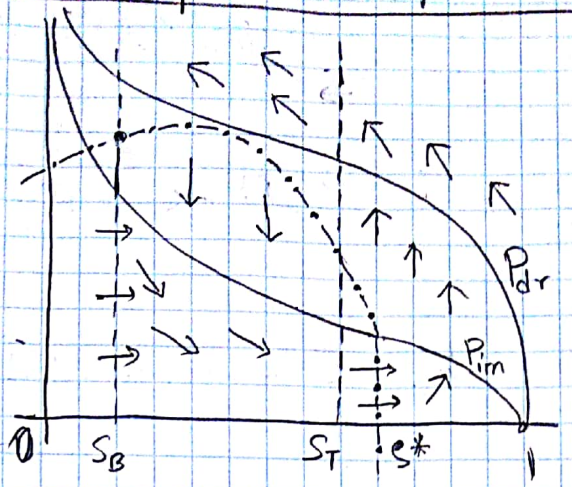


~~lets assume that~~
~~(S_B, f_B) is in the shaded area.~~

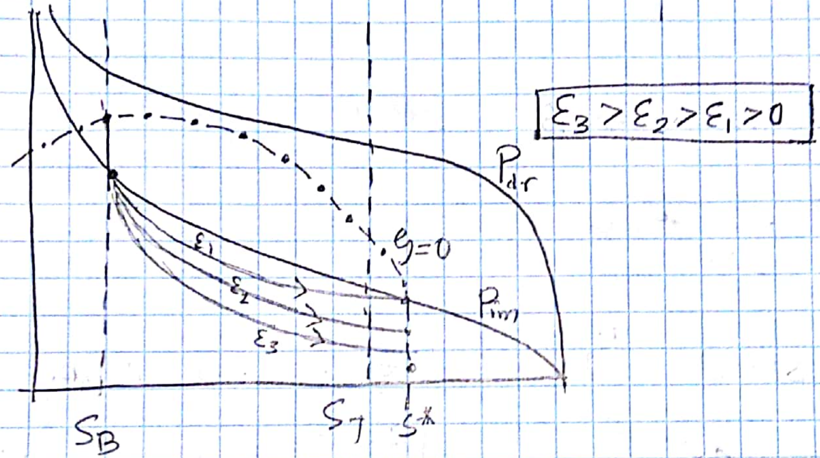
For every point in the line
$$f(S,p) = f_B + c(S - S_B).$$
And here the form of
$f(S,p)$ is assumed to be

$$f(S,p) = \begin{cases} f_{dr}(S) & \text{if } p > P_{dr}(S) \\ f^+(S) - f^-(S)\left(\dfrac{P^+(S) - p}{P^-(S)}\right) & \text{if } p \in [P_{im}, P_{dr}] \\ f_{im}(S) & \text{if } p < P_{im}(S) \end{cases}$$

So the phase space is;



The Orbits will be



The orbits are emitted from $(S_B, P_{im}(S_B))$.

Following our paper [van Duijn & Mitra, 2018]:

Proposition 1.1: For $\varepsilon>0$ small enough

$$(S_\varepsilon, p_\varepsilon) \longrightarrow (s^*, P_{im}(s^*)) \text{ as } \eta \to \infty$$

Proof: We analyze the nature of the point $(s^*, P_{im}(s^*))$. If approached from $p<P_{im}(s)$ the eigenvalues are

$$\boxed{\lambda^{\pm} = \frac{P_{im}'(s^*)}{2c\varepsilon}\left[1 \pm \sqrt{1 - \frac{4c\varepsilon \frac{\partial g}{\partial s}}{(P_{im}'(s^*))^2}}\right]}$$

Here $\frac{\partial g}{\partial s} = \frac{1}{N_c h(s^*)}\left[f'(s^*) - c\right] > 0$

So $\lambda^{\pm} < 0$; and the eigen directions are;

$$\frac{du}{ds} = \frac{P_{d\sigma}'(s^*)}{2}\left[1 \pm \sqrt{1 - \frac{4c\varepsilon}{(P_{d\sigma}')}\frac{\partial g}{\partial s}}\right].$$

So if $(s_\varepsilon, p_\varepsilon)$ is close enough to $(s^*, P_{im}(s^*))$ then it is captured by $(s^*, P_{im}(s^*))$.

Finally, to show that it gets close enough we show that for a fixed $s_\varepsilon = s$, $p_\varepsilon \to P_{im}(s)$ for $\varepsilon \to 0$ and $s \in [s_B, s^*]$.

we get, $\frac{dp^\varepsilon}{ds} = \frac{c\varepsilon g}{P_{im}^\varepsilon - p^\varepsilon}$ from (4).

so defining $v^\varepsilon = P_{im}(s^\varepsilon) - p^\varepsilon$

$$P_{im}' v^\varepsilon - \frac{d}{ds}\left(\frac{|v_\varepsilon|^2}{2}\right) = c\varepsilon g$$

$$\frac{d}{ds}\left(\frac{|v_\varepsilon|^2}{2}\right) = P_{im}' v^\varepsilon - c\varepsilon g \leq -c\varepsilon g$$

or $$|v_\varepsilon|^2 \leq -2c\varepsilon \int_{s_B}^{s} \frac{1}{N_c h_i(s)}\left[f_i - f_B - c(s - s_B)\right]$$

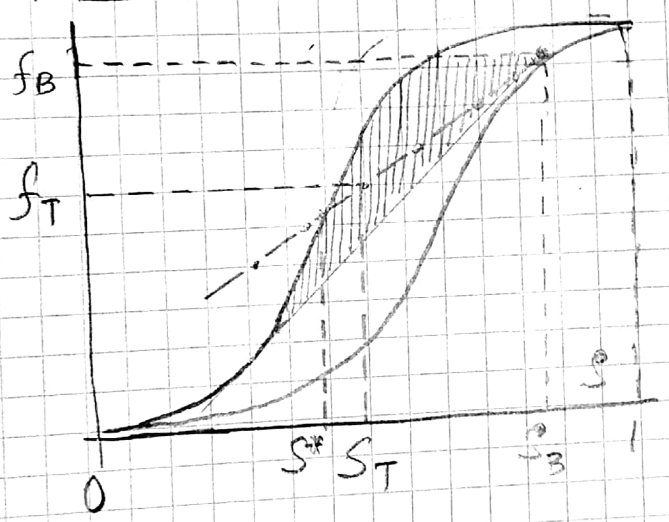
So if $\varepsilon \to 0$ then $|v_\varepsilon(s)|^2 \to 0$.

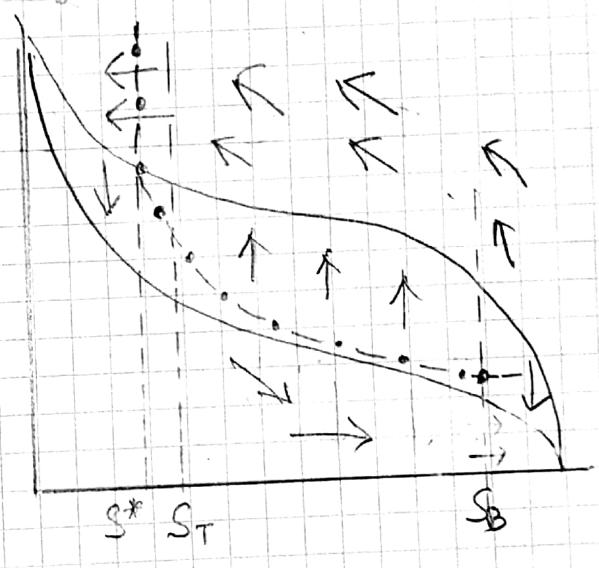
So this proves that the limiting orbit is,

$$(s^\varepsilon, p^\varepsilon) \to (s, P_{im}(s)) \text{ for } \varepsilon \to 0$$

And $(s, P_{im}(s)) \to (s^*, P_{im}(s^*))$.

Case II : $f_B > f_T$





So if (S_T, f_T) is in the shaded region then the travelling wave exists.