Exercise 1 (Fréchet-Hoeffding bounds inequality; 4 Points)**.** Let C be a 2-dimensional copula. Prove that for every (u, v) in $[0, 1]^2$,

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v)\,.$$

Hint. Use the monotonicity in each component for the upper bound. For the lower bound use the inclusion–exclusion principle.

We follow the proof of Theorem 2.2.3 in [Nel06]. Let (u, v) be an arbitrary point in $[0, 1]^2$. By proposition 1.10.iii C is 2-increasing, so for $\varepsilon = 1 - u$ we get $C\big((u, v) + \varepsilon(1, 0)\big) = C(1, v) \geq C(u, v)$ and accordingly $C(u, 1) \geq C(u, v)$. By proposition 1.10.ii C has uniform univariate marginals, so $C(1, v) = v$ and $C(u, 1) = u$. All together $C(u, v) \leq \min(u, v)$. Also by proposition 1.10.i C is grounded, so $0 = C(0, v) \leq C\big((0, v) + u(1, 0)\big) = C(u, v)$, again because C is 2-increasing. Finally, because C is 2-increasing, the volume $V_C([u, 1] \times [v, 1]) = C(1, 1) - C(1, v) - C(u, 1) + C(u, v) = 1 - u - v + C(u, v) \geq 0$, *where however I don't understand how we can derive this from C being 2-increasing.* Solving for $C(u, v)$ yields $C(u, v) \geq u + v - 1$. All together we get $C(u, v) \geq \max(u + v - 1, 0)$.

Exercise 2 (Survival Copula; 4 Points)**.** For a pair (X, Y) of real-valued random variables with joint distribution function H, the joint survival function is given by $\overline{H}(x, y) = P[X > x, Y > y]$. The marginal distribution functions of X and Y are denoted by F and G, respectively. The corresponding survival functions are given by $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively. Let C be a copula for (X, Y), i.e, C is a copula such that $H(x, y) = C(F(x), G(y))$. Define the survival copula $\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ and show that

$$\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y))\,.$$

Moreover, prove that $\hat{C}$ is indeed a copula.

We follow the explanation in the beginning of section 2.6 of [Nel06]. We use the fact that $P(X > x, Y > y) = P(X > x) + P(Y > y) - P(\{X > x\} \cup \{Y > y\})$ so that

$$\overline{H}(x, y) = P(X > x) + P(Y > y) - 1 + P(X \leq x, Y \leq y)\,.$$

by definitions of $\overline{F}$ and $\overline{G}$ and the fact that $H(x, y) = C(F(x), G(y))$ we get

$$= \overline{F}(x) + \overline{G}(y) - 1 + C(F(x), G(y))\,.$$

Finally, again by definition of $\overline{F}$ and $\overline{G}$, we get

$$= \overline{F}(x) + \overline{G}(y) - 1 + C(1 - \overline{F}(x), 1 - \overline{G}(y))\,.$$

Now if we define the survival copula $\hat{C}$ as above, we arrive at $\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y))$. To see that $\hat{C}$ is a copula, we prove that it is grounded, has uniform univariate marginals and is 2-increasing. Because $\hat{C}(u, 0) = u - 1 + C(1 - u, 1) = 0$ and analogously $\hat{C}(0, v) = 0$, $\hat{C}$ is grounded. Also $\hat{C}(u, 1) = u + C(1 - u, 0) = u$ and accordingly for v, so that it has uniform univariate marginals. Finally we want to see whether $\hat{C}$ is 2-increasing. Also here because of the symmetry of $\hat{C}$ is suffices to consider only u. For all $u \in [0, 1]$ and $\varepsilon \in (0, 1-u]$ we get $\hat{C}((u+\varepsilon, v)) - \hat{C}((u, v)) = u+\varepsilon+v-1+C(1-u-\varepsilon, 1-v)-u-v+1-C(1-u, 1-v) = \varepsilon+C(1-u-\varepsilon, 1-v)-C(1-u, 1-v)$. *Here it is unclear, why this should be greater or equal than zero.*

Exercise 3 (Invariance properties; 4 Points)**.** Let X and Y be random variables with distribution function F_X and F_Y, respectively. Let F_{XY} be the joint distribution function of (X, Y) and C_{XY} be a copula for (X, Y), i.e, $F_{XY}(x, y) = C_{XY}(F_X(x), F_Y(y))$. If a and b are strictly increasing and continuous on $\operatorname{Ran}(X)$ and $\operatorname{Ran}(Y)$, respectively, then C_{XY} is also a copula for the random vector $(a(X), b(Y))$, i.e, the joint distribution function of $(a(X), b(Y))$ fulfills $F_{a(X)b(Y)}(x, y) = C_{XY}(F_{a(X)}(x), F_{b(Y)(Y)})$. Thus, C_{XY} is invariant under strictly increasing transformations of X and Y.

We follow the proof of theorem 2.4.3 in [Nel06]. Because a and b are strictly increasing, we can apply a^{-1} inside of P. That is, $F_{a(X)}(x) = P(a(X) \leq x) = P(X \leq a^{-1}(x)) = F_X(a^{-1}(x))$ and accordingly $F_{b(Y)}(y) = F_Y(b^{-1}(y))$ so that for every $x, y \in \overline{\mathbb{R}}$ by Sklar's theorem we get

$$
\begin{aligned}
C_{a(X)b(Y)}(F_{a(X)}(x), F_{b(Y)}(y)) &= P(a(X) \leq x, b(Y) \leq y)\,, \\
&= P(X \leq a^{-1}(x), Y \leq b^{-1}(y))\,,
\end{aligned}
$$

with the same trick as above. Using Sklar's theorem again, we can rewrite this to

$$
\begin{aligned}
&= C_{XY}(F_X(a^{-1}(x)), F_Y(b^{-1}(y))\,, \\
&= C_{XY}(F_{a(X)}(x), F_{b(Y)}(y))\,,
\end{aligned}
$$

by the relation we found out in the beginning. Now X and Y are continuous. That means that $\operatorname{Ran} F_{a(X)} = F_{b(Y)} = [0, 1]$ so we get the equality indeed for every $(x, y) \in [0, 1]^2$ and together with $F_{a(X)b(Y)}(x, y) = C_{a(X)b(Y)}(F_{a(X)}(x), F_{b(Y)}(y))$ we get $F_{a(X)b(Y)}(x, y) = C_{XY}(F_X(x), F_Y(y))$.

Exercise 4 (Distributional transform; 4 Points)**.** Let X be a real-valued random variable with distribution function F, and let $V \sim \mathrm{U}(0, 1)$ be an independent random variable uniformly distributed on $(0, 1)$. The *generalized distribution function* of X is defined by

$$
F(x, \lambda) = P(X < x) + \lambda P(X = x) \quad \text{or equivalently} \quad F(x, \lambda) = F(x-) + \lambda\big(F(x) - F(x-)\big)\,,
$$

and the *generalized distribution transform* U of X is given by $U = F(X, V)$. Show that $U \sim \mathrm{U}(0, 1)$.

Hint: Define $p = P\big(X < F^{-1}(\alpha)\big)$ *and* $q = P\big(X = F^{-1}(\alpha)\big)$ *and show the equality*

$$\{U \leq \alpha\} = \{X < F^{-1}(\alpha)\} \cup \{X = F^{-1}(\alpha), p + Vq \leq \alpha\}\,.$$

You can use without proof that $F(F^{-1}(\alpha)) \leq \alpha$ *and if* F *is continuous at* $F^{-1}(\alpha)$ *that it holds that* $F(F^{-1}(\alpha)) = \alpha$.

We follow the proof of proposition 2.1 in [Rü09]. By definition of F we have $U = F(X, V) \leq \alpha$ if and only if $(X, V) \in \{(x, \lambda)\colon P(X < x) + \lambda P(X = x) \leq \alpha\}$. Now consider the case that $0 = q = P(X = F^{-1}(\alpha))$. Then $P(\{U \leq \alpha\}) = P(\{P(X < x) \leq \alpha\}) = P(\{X < F^{-1}(\alpha)\}) = P(X \leq F^{-1}(\alpha)) = \alpha$ since F is continuous at α. Now if $q > 0$, then $(X, V) \in \{(x, \lambda)\colon P(X < x) + \lambda P(X = x) \leq \alpha\}$ is equivalent to $(X, V) \in \{X < F^{-1}(\alpha)\} \cup \{X = F^{-1}(\alpha), p + Vq \leq \alpha\}$, *however I don't really see how this works.* Employing we again get

$$P(U \leq \alpha) = P(F(X, V) \leq \alpha)\,.$$

Solving $q + V\beta \leq \alpha$ for V we get

$$= q + \beta P(V \leq (\alpha - q)/\beta)\,.$$

However I don't quite get how this works. Since $V \sim \mathrm{U}(0, 1)$ we get

$$= q + \beta\frac{\alpha - q}{\beta} = \alpha\,.$$

References

[Nel06] Nelsen, Roger B.: *An introduction to copulas.* Springer, 2006

[Rü09] Rüschendorf, Ludger: On the distributional transform, Sklar's theorem, and the empirical copula process. In: *Journal of Statistical Planning and Inference* 139 (2009), Nr. 11, 3921-3927. https://www.sciencedirect.com/science/article/pii/S037837580900158X