Exercise 1 (4 Points). The Clayton Copula with parameter $\theta > 0$ is given by

$$C_\theta(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}.$$

Show that the Clayton copula is an archimedian copula.

We need to find a 2-monotone or convex φ so that $(u^{-\theta}+v^{-\theta}-1)^{-1/\theta} = \phi(\varphi^{-1}(u)+\varphi^{-1}(v))$. If we set the pseudo-inverse of the generator to $\varphi^{-1}(t) = t^{-\theta} - 1$ for $t \in (0,1]$ and t_0 for $t = 0$, then we get that $\varphi^{-1}(C_\theta(u,v)) = u^{-\theta} + v^{-\theta} - 2 = \varphi^{-1}(u) + \varphi^{-1}(v)$. That is, the Clayton copula is an archimedian copula with generator $\varphi(t) = (t+1)^{-1/\theta}$.

Let $U_1, U_2 \sim \mathrm{U}[0,1]$ independent random variables. Determine the function f_θ such that $(U_1, f_\theta(U_1,U_2)) \sim C_\theta$.

Exercise 3 (4 Points). Prove Proposition 3.4: Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be d-dimensional random vectors. Then

(i) $X \leq_{st} Y \quad \Longrightarrow \quad X \leq_{icx} Y, \quad X \leq_{uo} Y \quad \text{and} \quad X \geq_{lo} Y$

If $X \leq_{st} Y$, then

(iii) $X \leq_{sm} Y \quad \Longrightarrow \quad X \leq_c Y \quad \Longrightarrow \quad X \leq_{lo} Y \quad \text{and} \quad X \leq_{uo} Y$

First, $X \leq_c Y$ is defined by $X \leq_{uo} Y$ and $X \leq_{lo} Y$, so it is enough to that $X \leq_{sm} Y \implies X \leq_{lo} Y$ and $X \leq_{sm} \implies X \leq_{uo} Y$. To this end we follow Theorem 6.15. from [Rüs13]. If $X \leq_{sm} Y$, then for all φ with for all $\epsilon, \delta > 0$ and $x \in \mathbb{R}^n$ $\Delta_i^\varepsilon \Delta_j^\delta \varphi(x) \geq 0$ it holds that $E\varphi(X) \leq E\varphi(Y)$ *Here proof is missing.*

(iv) $X \leq_{sm} Y \quad \Longrightarrow \quad X \leq_{dcx} Y \implies \sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i$

See Remark 6.27.b in [Rüs13].

Exercise 5 (4 Points; Bonus). Show that the Markov product $A * B$ is a bivariate copula.

To this end we use proposition 1.10. To see that $A*B$ is a bivariate copula, we need to show that it is grounded, has uniform univariate marginals and is 2-increasing. Since A and B are grounded, $\int_0^1 \partial_2 A(0,t)\partial_1 B(t,v)dt = \int_0^1 \partial_2 A(0,t)\partial_1 B(t,v)dt = \int_0^1 \partial_2 A(u,t)\partial_1 B(t,0)dt = 0$, so $A*B$ is grounded. To see that $A*B$ has uniform univariate marginals, we calculate $A*B(u,1) = \int_0^1 \partial_2 A(u,t)\partial_1 B(t,1)dt = \int_0^1 \partial_2 A(u,t)dt = A(u,1) - A(u,0) = u$. An analogous calculation yields $A*B(1,v) = v$.

To see that $A*B$ is 2-increasing, according to equation (1.14) we need to see that for all $0 \le u, v \le 1$, $0 \le \varepsilon \le 1-u$ and $0 \le \delta \le 1-v$ we get $A*B(u+\varepsilon, v+\delta) - A*B(u, v+\delta) - A*B(u+\varepsilon, v) + A*B(u,v) \ge 0$. Employing yields

$$
\begin{aligned}
&A*B(u+\varepsilon, v+\delta) - A*B(u, v+\delta) - A*B(u+\varepsilon, v) + A*B(u,v) \\
=& \int_0^1 \big(\partial_2 A(u+\varepsilon,t)\partial_1 B(t,v+\delta) - \partial_2 A(u,t)\partial_1 B(t,v+\delta) \\
&- \partial_2 A(u+\varepsilon,t)\partial_1 B(t,v) + \partial_2 A(u,t)\partial_1 B(t,v)\big)dt \\
=& \int_0^1 \big[\partial_2\big(A(u+\varepsilon,t) - A(u,t)\big)\partial_1 B(t,v+\delta) - \partial_2\big(A(u+\varepsilon,t) - A(u,t)\big)\partial_1 B(t,v)\big]dt \\
=& \int_0^1 \big[\partial_2\big(A(u+\varepsilon,t) - A(u,t)\big)\partial_1\big(B(t,v+\delta) - B(t,v)\big)\big]dt\,.
\end{aligned}
$$

However, $\partial_2\big(A(u+\varepsilon,t) - A(u,t)\big) = \lim_{h\to 0}\frac{1}{h}\big(A(u+\varepsilon,t+h) - A(u,t+h) - A(u+\varepsilon,t) + A(u,t)\big) \ge 0$, because A is 2-increasing. Analogously $\partial_1\big(B(t,v+\delta) - B(t,v)\big) \ge 0$. Thus, the integrand is positive for every $0 \le t \le 1$ so that we get

$$\ge 0\,.$$

All together $A*B$ is a bivariate copula.

Moreover, show that for a bivariate copula C it holds that $\pi^2 * C = C * \Pi^2 = \Pi^2$.

The Markov product is defined by $\Pi * C = \int_0^1 \partial_2\Pi(u,t)\partial_1 C(t,v)dt = \int_0^1 u\partial_1 C(t,v)dt = u(C(1,v) - C(0,v)) = uv = \Pi$ *which however is not* $\Pi^2 * C$.

References

[Rüs13] RÜSCHENDORF, Ludger: Mathematical risk analysis. In: *Springer Ser. Oper. Res. Financ. Eng. Springer, Heidelberg* (2013)