

**Exercise 1** (4 Points). The Clayton Copula with parameter  $\theta > 0$  is given by

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}.$$

Show that the Clayton copula is an archimedian copula.

We need to find a 2-monotone or convex  $\varphi$  so that  $(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} = \phi(\varphi^{-1}(u) + \varphi^{-1}(v))$ . If we set the pseudo-inverse of the generator to  $\varphi^{-1}(t) = t^{-\theta} - 1$  for  $t \in (0, 1]$  and  $t_0$  for  $t = 0$ , then we get that  $\varphi^{-1}(C_\theta(u, v)) = u^{-\theta} + v^{-\theta} - 2 = \varphi^{-1}(u) + \varphi^{-1}(v)$ . That is, the Clayton copula is an archimedian copula with generator  $\varphi(t) = (t + 1)^{-1/\theta}$ .

**Exercise 3** (4 Points). Prove Proposition 3.4: Let  $X = (X_1, \dots, X_d)$  and  $Y = (Y_1, \dots, Y_d)$  be  $d$ -dimensional random vectors. Then

$$(iv) \quad X \leq_{sm} Y \implies X \leq_{dcx} Y \implies \sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i$$

See Remark 6.27.b in [Rüs13].

**Exercise 5** (4 Points; Bonus). Show that the Markov product  $A * B$  is a bivariate copula.

To this end we use proposition 1.10. To see that  $A * B$  is a bivariate copula, we need to show that it is grounded, has uniform univariate marginals and is 2-increasing. Since  $A$  and  $B$  are grounded,  $\int_0^1 \partial_2 A(0, t) \partial_1 B(t, v) dt = \int_0^1 \partial_2 A(0, t) \partial_1 B(t, v) dt = \int_0^1 \partial_2 A(u, t) \partial_1 B(t, 0) dt = 0$ , so  $A * B$  is grounded. To see that  $A * B$  has uniform univariate marginals, we calculate  $A * B(u, 1) = \int_0^1 \partial_2 A(u, t) \partial_1 B(t, 1) dt = \int_0^1 \partial_2 A(u, t) dt = A(u, 1) - A(u, 0) = u$ . An analogous calculation yields  $A * B(1, v) = v$ .

To see that  $A * B$  is 2-increasing, according to equation (1.14) we need to see that for all  $0 \leq u, v \leq 1$ ,  $0 \leq \varepsilon \leq 1 - u$  and  $0 \leq \delta \leq 1 - v$  we get  $A * B(u + \varepsilon, v + \delta) - A * B(u, v + \delta) - A * B(u + \varepsilon, v) + A * B(u, v) \geq 0$ . Employing yields

$$\begin{aligned} & A * B(u + \varepsilon, v + \delta) - A * B(u, v + \delta) - A * B(u + \varepsilon, v) + A * B(u, v) \\ &= \int_0^1 (\partial_2 A(u + \varepsilon, t) \partial_1 B(t, v + \delta) - \partial_2 A(u, t) \partial_1 B(t, v + \delta) \\ &\quad - \partial_2 A(u + \varepsilon, t) \partial_1 B(t, v) + \partial_2 A(u, t) \partial_1 B(t, v)) dt \\ &= \int_0^1 [\partial_2 (A(u + \varepsilon, t) - A(u, t)) \partial_1 B(t, v + \delta) - \partial_2 (A(u + \varepsilon, t) - A(u, t)) \partial_1 B(t, v)] dt \\ &= \int_0^1 [\partial_2 (A(u + \varepsilon, t) - A(u, t)) \partial_1 (B(t, v + \delta) - B(t, v))] dt. \end{aligned}$$

However,  $\partial_2 (A(u + \varepsilon, t) - A(u, t)) = \lim_{h \rightarrow 0} \frac{1}{h} (A(u + \varepsilon, t + h) - A(u, t + h) - A(u + \varepsilon, t) + A(u, t)) \geq 0$ , because  $A$  is 2-increasing. Analogously  $\partial_1 (B(t, v + \delta) - B(t, v)) \geq 0$ . Thus, the integrand is positive for every  $0 \leq t \leq 1$  so that we get

$$\geq 0.$$

All together  $A * B$  is a bivariate copula.

## References

- [Rüs13] RÜSCHENDORF, Ludger: Mathematical risk analysis. In: *Springer Ser. Oper. Res. Financ. Eng. Springer, Heidelberg* (2013)