Exercise 1 (4 Points). The Clayton Copula with parameter $\theta > 0$ is given by

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}\,.$$

Show that the Clayton copula is an archimedian copula.

We need to find a 2-monotone or convex φ so that $(u^{-\theta}+v^{-\theta}-1)^{-1/\theta} = \phi(\varphi^{-1}(u)+\varphi^{-1}(v))$. If we set the pseudo-inverse of the generator to $\varphi^{-1}(t) = t^{-\theta} - 1$ for $t \in (0, 1]$ and t_0 for $t = 0$, then we get that $\varphi^{-1}(C_\theta(u, v)) = u^{-\theta} + v^{-\theta} - 2 = \varphi^{-1}(u) + \varphi^{-1}(v)$. That is, the Clayton copula is an archimedian copula with generator $\varphi(t) = (t + 1)^{-1/\theta}$.

Exercise 3 (4 Points). Prove Proposition 3.4: Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ be d-dimensional random vectors. Then

(iv) $X \leq_{sm} Y \quad \Longrightarrow \quad X \leq_{dcx} Y \implies \sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i$

We follow remark 6.27.b in [Rüs13] to show that if $X \leq_{dcx} Y$ then $\sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i$. So let X, Y so that $X \leq_{dcx} Y$ and f convex. We want to show that $x \mapsto f\big(\sum_{i=1}^d x_i\big)$ is directionally convex, because then $\sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i$ as wanted. To this end let $x \in \mathbb{R}^d$, $\varepsilon, \delta > 0$ and $0 \leq i, j \leq d$. Because f is convex, for $a, b \in \mathbb{R}$ and $0 < \theta < 1$ we know that

$$\theta f(a) + (1-\theta) f(b) - f(\theta a + (1-\theta)b) \geq 0\,,$$

and also that

$$(1-\theta) f(a) + \theta f(b) - f((1-\theta)a + \theta b) \geq 0\,.$$

adding both inequalities and setting $a = \sum x_i$, $b = a + \varepsilon + \delta$ and $\theta = \frac{\varepsilon}{b-a}$ yields

$$f\Big(\sum x_i + \varepsilon + \delta\Big) - f\Big(\sum x_i + \varepsilon\Big) - f\Big(\sum x_i + \delta\Big) + f\Big(\sum x_i\Big) \geq 0\,.$$

By definition of Δ_j^ϵ this means

$$\Delta_j^\varepsilon \Delta_k^\delta f\Big(\sum x_i\Big) \geq 0\,,$$

so indeed $x \mapsto f(\sum x_i)$ is directionally convex. Now since $X \leq_{dcx} Y$ this means $Ef(\sum x_i) \leq Ef(\sum Y_k)$, so that $\sum x_i \leq_{cx} \sum Y_k$, since $f \in \mathcal{F}_{cx}$ was chosen arbitrarily.

Exercise 5 (4 Points; Bonus). Show that the Markov product $A * B$ is a bivariate copula.

To this end we use proposition 1.10. To see that $A*B$ is a bivariate copula, we need to show that it is grounded, has uniform univariate marginals and is 2-increasing. Since A and B are grounded, $\int_0^1 \partial_2 A(0,t)\partial_1 B(t,v)dt = \int_0^1 \partial_2 A(0,t)\partial_1 B(t,v)dt = \int_0^1 \partial_2 A(u,t)\partial_1 B(t,0)dt = 0$, so $A*B$ is grounded. To see that $A * B$ has uniform univariate marginals, we calculate $A * B(u,1) = \int_0^1 \partial_2 A(u,t)\partial_1 B(t,1)dt = \int_0^1 \partial_2 A(u,t)dt = A(u,1) - A(u,0) = u$. An analogous calculation yields $A * B(1,v) = v$.

To see that $A * B$ is 2-increasing, according to equation (1.14) we need to see that for all $0 \leq u, v \leq 1$, $0 \leq \varepsilon \leq 1-u$ and $0 \leq \delta \leq 1-v$ we get $A * B(u+\varepsilon, v+\delta) - A * B(u, v+\delta) - A * B(u+\varepsilon, v) + A * B(u,v) \geq 0$. Employing yields

$$
\begin{aligned}
&A * B(u+\varepsilon, v+\delta) - A * B(u, v+\delta) - A * B(u+\varepsilon, v) + A * B(u,v) \\
=&\int_0^1 \big(\partial_2 A(u+\varepsilon,t)\partial_1 B(t,v+\delta) - \partial_2 A(u,t)\partial_1 B(t,v+\delta) \\
&- \partial_2 A(u+\varepsilon,t)\partial_1 B(t,v) + \partial_2 A(u,t)\partial_1 B(t,v)\big)dt \\
=&\int_0^1 \big[\partial_2\big(A(u+\varepsilon,t) - A(u,t)\big)\partial_1 B(t,v+\delta) - \partial_2\big(A(u+\varepsilon,t) - A(u,t)\big)\partial_1 B(t,v)\big]dt \\
=&\int_0^1 \big[\partial_2\big(A(u+\varepsilon,t) - A(u,t)\big)\partial_1\big(B(t,v+\delta) - B(t,v)\big)\big]dt\,.
\end{aligned}
$$

However, $\partial_2\big(A(u+\varepsilon,t) - A(u,t)\big) = \lim_{h\to 0}\frac{1}{h}\big(A(u+\varepsilon,t+h) - A(u,t+h) - A(u+\varepsilon,t) + A(u,t)\big) \geq 0$, because A is 2-increasing. Analogously $\partial_1\big(B(t,v+\delta) - B(t,v)\big) \geq 0$. Thus, the integrand is positive for every $0 \leq t \leq 1$ so that we get

$$
\geq 0\,.
$$

All together $A * B$ is a bivariate copula.

References

[Rüs13] RÜSCHENDORF, Ludger: Mathematical risk analysis. In: *Springer Ser. Oper. Res. Financ. Eng. Springer, Heidelberg* (2013)