Exercise 1 (4 Points). The Clayton Copula with parameter $\theta > 0$ is given by

$$C_\theta(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}.$$

Show that the Clayton copula is an archimedian copula.

We need to find a 2-monotone or convex φ so that $(u^{-\theta}+v^{-\theta}-1)^{-1/\theta} = \phi(\varphi^{-1}(u)+\varphi^{-1}(v))$. If we set the pseudo-inverse of the generator to $\varphi^{-1}(t) = t^{-\theta} - 1$ for $t \in (0,1]$ and t_0 for $t = 0$, then we get that $\varphi^{-1}(C_\theta(u,v)) = u^{-\theta} + v^{-\theta} - 2 = \varphi^{-1}(u) + \varphi^{-1}(v)$. That is, the Clayton copula is an archimedian copula with generator $\varphi(t) = (t+1)^{-1/\theta}$.

Let $U_1, U_2 \sim \mathrm{U}[0,1]$ independent random variables. Determine the function f_θ such that $(U_1, f_\theta(U_1,U_2)) \sim C_\theta$.

Exercise 3 (4 Points). Prove Proposition 3.4: Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be d-dimensional random vectors. Then

(i) $X \leq_{st} Y \quad \Longrightarrow \quad X \leq_{icx} Y, \quad X \leq_{uo} Y \quad \text{and} \quad X \geq_{lo} Y$

If $X \leq_{st} Y$, then

(iii) $X \leq_{sm} Y \quad \Longrightarrow \quad X \leq_c Y \quad \Longrightarrow \quad X \leq_{lo} Y \quad \text{and} \quad X \leq_{uo} Y$

Since we don't have the script at hand, we use the definition 6.1 from [Rüs13]. First, $X \leq_c Y$ is defined by $X \leq_{uo} Y$ and $X \leq_{lo} Y$, so it is enough to that $X \leq_{sm} Y \implies X \leq_{lo} Y$ and $X \leq_{sm} \implies X \leq_{uo} Y$. To this end we follow Theorem 6.15. from [Rüs13]. If $X \leq_{sm} Y$, then for all φ with for all $\epsilon, \delta > 0$ and $x \in \mathbb{R}^n$ $\Delta_i^\varepsilon \Delta_j^\delta \varphi(x) \geq 0$ it holds that $E\varphi(X) \leq E\varphi(Y)$ *Here proof is missing.*

(iv) $X \leq_{sm} Y \quad \Longrightarrow \quad X \leq_{dcx} Y \implies \sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i$

See Remark 6.27.b in [Rüs13].

Exercise 5 (4 Points; Bonus)**.** Show that the Markov product $A \times B$ is a bivariate copula. Moreover, show that for a bivariate copula C it holds that $\pi^2 * C = C * \Pi^2 = \Pi^2$.

The Markov product is defined by $\Pi * C = \int_0^1 \partial_2 \Pi(u,t) \partial_1 C(t,v) dt = \int_0^1 u \partial_1 C(t,v) dt = u(C(1,v) - C(0,v)) = uv$.

References

[Rüs13] Rüschendorf, Ludger: Mathematical risk analysis. In: *Springer Ser. Oper. Res. Financ. Eng. Springer, Heidelberg* (2013)