Exercise 1 (Fréchet-Hoeffding bounds inequality; 4 Points). Let C be a 2-dimensional copula. Prove that for every (u, v) in $[0,1]^2$,

$$\max(u+v-1,0) \leq C(u,v) \leq \min(u,v)\,.$$

Hint. Use the monotonicity in each component for the upper bound. For the lower bound use the inclusion–exclusion principle.

We follow the proof of Theorem 2.2.3 in [Nel06]. Let (u, v) be an arbitrary point in $[0,1]^2$. By proposition 1.10.iii C is 2-increasing, so for $\varepsilon = 1-u$ we get $C\big((u,v)+\varepsilon(1,0)\big) = C(1,v) \geq C(u,v)$ and accordingly $C(u,1) \geq C(u,v)$. By proposition 1.10.ii C has uniform univariate marginals, so $C(1,v) = v$ and $C(u,1) = u$. All together $C(u,v) \leq \min(u,v)$. Also by proposition 1.10.i C is grounded, so $0 = C(0,v) \leq C\big((0,v) + u(1,0)\big) = C(u,v)$, again because C is 2-increasing. Finally, because C is 2-increasing

$$0 \leq \Delta_u^{1-u}\Delta_v^{1-v}C(u,v)\,.$$

By definition of delta we get

$$\begin{aligned}&= \Delta_u^{1-u}\big(C(u,1)-C(u,v)\big)\\ &= C(1,1)-C(1,v)-C(u,1)+C(u,v).\end{aligned}$$

Because C has uniform univariate marginals, we arrive at

$$= 1-v-u+C(u,v).$$

Solving for $C(u,v)$ yields $C(u,v) \geq u+v-1$, so that all together we get $C(u,v) \geq \max(u+v-1,0)$.

Exercise 2 (Survival Copula; 4 Points). For a pair (X, Y) of real-valued random variables with joint distribution function H, the joint survival function is given by $\overline{H}(x, y) = P[X > x, Y > y]$. The marginal distribution functions of X and Y are denoted by F and G, respectively. The corresponding survival functions are given by $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively. Let C be a copula for (X, Y), i.e, C is a copula such that $H(x, y) = C(F(x), G(y))$. Define the survival copula $\hat{C}(u, v) = u+v-1+C(1-u, 1-v)$ and show that

$$\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y)) .$$

Moreover, prove that $\hat{C}$ is indeed a copula.

We follow the explanation in the beginning of section 2.6 of [Nel06]. We use the inclusion-exclusion principle, which tells us that $P(X > x, Y > y) = P(X > x) + P(Y > y) - P(\{X > x\} \cup \{Y > y\})$ so that

$$\overline{H}(x, y) = P(X > x) + P(Y > y) - 1 + P(X \leq x, Y \leq y) .$$

by definitions of $\overline{F}$ and $\overline{G}$ and the fact that $H(x, y) = C(F(x), G(y))$ we get

$$= \overline{F}(x) + \overline{G}(y) - 1 + C(F(x), G(y)) .$$

Finally, again by definition of $\overline{F}$ and $\overline{G}$, we get

$$= \overline{F}(x) + \overline{G}(y) - 1 + C(1 - \overline{F}(x), 1 - \overline{G}(y)) .$$

Now if we define the survival copula $\hat{C}$ as above, we arrive at $\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y))$. To see that $\hat{C}$ is a copula we use proposition 1.10 and prove that it is grounded, has uniform univariate marginals and is 2-increasing. Because $\hat{C}(u,0) = u - 1 + C(1-u,1) = 0$ and analogously $\hat{C}(0,v) = 0$, so $\hat{C}$ is grounded. Also $\hat{C}(u,1) = u + C(1-u,0) = u$ and accordingly for v, so that it has uniform univariate marginals. Finally we want to see whether $\hat{C}$ is 2-increasing. By definition of $\Delta_u^{\varepsilon_u}$, as already used in exercise 1, we see that

$$\begin{aligned}\Delta_u^{\varepsilon_u}\Delta_v^{\varepsilon_v}\hat{C}(u,v) &= \hat{C}(u+\varepsilon_u, v+\varepsilon_v) - \hat{C}(u+\varepsilon_u, v) \\ &\quad - \hat{C}(u, v+\varepsilon_v) + \hat{C}(u,v)\,.\end{aligned}$$

By definition of the survival copula we get

$$\begin{aligned}&= C(1-u-\varepsilon_u, 1-v-\varepsilon_v) - C(1-u-\varepsilon_u, 1-v) \\ &\quad - C(1-u, 1-v-\varepsilon_v) + C(1-u, 1-v)\,.\end{aligned}$$

Again by definition of Δ^ε we arrive at

$$= \Delta_u^{\varepsilon_u}\Delta_v^{\varepsilon_v} C(1-u-\varepsilon_u, 1-v-\varepsilon_v) \geq 0\,,$$

since C is a copula and thus 2-increasing.

Exercise 3 (Invariance properties; 4 Points)**.** Let X and Y be random variables with distribution function F_X and F_Y, respectively. Let F_{XY} be the joint distribution function of (X, Y) and C_{XY} be a copula for (X, Y), i.e, $F_{XY}(x, y) = C_{XY}(F_X(x), F_Y(y))$. If a and b are strictly increasing and continuous on $\mathrm{Ran}(X)$ and $\mathrm{Ran}(Y)$, respectively, then C_{XY} is also a copula for the random vector $(a(X), b(Y))$, i.e, the joint distribution function of $(a(X), b(Y))$ fulfills $F_{a(X)b(Y)}(x, y) = C_{XY}(F_{a(X)}(x), F_{b(Y)}(y))$. Thus, C_{XY} is invariant under strictly increasing transformations of X and Y.

We follow the proof of theorem 2.4.3 in [Nel06]. Because a and b are strictly increasing, we can apply a^{-1} inside of P. That is we have $F_{a(X)}(x) = P(a(X) \leq x) = P(X \leq a^{-1}(x)) = F_X(a^{-1}(x))$ and accordingly $F_{b(Y)}(y) = F_Y(b^{-1}(y))$ so that for every $x, y \in \overline{\mathbb{R}}$ by Sklar's theorem we get

$$\begin{aligned} C_{a(X)b(Y)}(F_{a(X)}(x), F_{b(Y)}(y)) &= P(a(X) \leq x, b(Y) \leq y)\,, \\ &= P(X \leq a^{-1}(x), Y \leq b^{-1}(y))\,, \end{aligned}$$

with the same trick as above. Using Sklar's theorem again, we can rewrite this to

$$\begin{aligned} &= C_{XY}(F_X(a^{-1}(x)), F_Y(b^{-1}(y))\,, \\ &= C_{XY}(F_{a(X)}(x), F_{b(Y)}(y))\,, \end{aligned}$$

yet again by the relation we found out in the beginning. Now X and Y are continuous. That means that $\mathrm{Ran}\, F_{a(X)} = F_{b(Y)} = [0, 1]$ so we get the equality indeed for every $(x, y) \in [0, 1]^2$ and together with $F_{a(X)b(Y)}(x, y) = C_{a(X)b(Y)}(F_{a(X)}(x), F_{b(Y)}(y))$ we get $F_{a(X)b(Y)}(x, y) = C_{XY}(F_X(x), F_Y(y))$.

Exercise 4 (Distributional transform; 4 Points). Let X be a real-valued random variable with distribution function F, and let $V \sim \mathrm{U}(0,1)$ be an independent random variable uniformly distributed on $(0,1)$. The *generalized distribution function* of X is defined by

$$F(x,\lambda) = P(X < x) + \lambda P(X = x)\,,$$

or equivalently

$$F(x,\lambda) = F(x-) + \lambda\big(F(x) - F(x-)\big)\,,$$

and the *generalized distribution transform* U of X is given by $U = F(X,V)$. Show that $U \sim \mathrm{U}(0,1)$.

Hint: Define $p = P\big(X < F^{-1}(\alpha)\big)$ and $q = P\big(X = F^{-1}(\alpha)\big)$ and show the equality

$$\{U \leq \alpha\} = \{X < F^{-1}(\alpha)\} \cup \{X = F^{-1}(\alpha), p + Vq \leq \alpha\}\,.$$

You can use without proof that $F(F^{-1}(\alpha)) \leq \alpha$ and if F is continuous at $F^{-1}(\alpha)$ that it holds that $F(F^{-1}(\alpha)) = \alpha$.

We follow the proof of proposition 2.1 in [Rü09]. By definition of F we have $U = F(X,V) \leq \alpha$ iff $(X,V) \in \{(x,\lambda)\colon P(X < x) + \lambda P(X = x) \leq \alpha\}$. Now consider the case that $0 = q = P(X = F^{-1}(\alpha))$. Then

$$\begin{aligned} P(\{U \leq \alpha\}) &= P(\{P(X < x) \leq \alpha\}) = P(\{X < F^{-1}(\alpha)\}) \\ &= P(X \leq F^{-1}(\alpha)) = F(F^{-1}(\alpha)) = \alpha\,, \end{aligned}$$

since F is continuous at α. Now if $q > 0$, then

$$(X,V) \in \{(x,\lambda)\colon P(X < x) + \lambda P(X = x) \leq \alpha\}$$

is equivalent to

$$(X,V) \in \{X < F^{-1}(\alpha)\} \cup \{X = F^{-1}(\alpha), p + Vq \leq \alpha\},$$

however also here I don't really see how this works. Employing we again get

$$P(U \leq \alpha) = P(F(X,V) \leq \alpha)\,.$$

Solving $q + V\beta \leq \alpha$ for V we get

$$= q + \beta P(V \leq (\alpha - q)/\beta)\,.$$

However I don't quite get how this works. Since $V \sim \mathrm{U}(0,1)$ we get

$$= q + \beta \frac{\alpha - q}{\beta} = \alpha\,.$$

So that in both cases $P(U \leq \alpha) = \alpha$ and thus $U \sim \mathrm{U}(0,1)$.

References

[Nel06] Nelsen, Roger B.: *An introduction to copulas.* Springer, 2006

[Rü09] Rüschendorf, Ludger: On the distributional transform, Sklar's theorem, and the empirical copula process. In: *Journal of Statistical Planning and Inference* 139 (2009), Nr. 11, 3921-3927. https://www.sciencedirect.com/science/article/pii/S037837580900158X