Aufgabe 1 (4 Points).

Definition 1 (T-Forward-Measure). Let $(B_t)_{t\leq T}$, $B_t = e^{\int_0^t r_s ds}$ be the bank account/numeraire in a financial market. If $\mathbb{Q}$ is a risk-neutral measure, then the forward measure $\mathbb{Q}^T$ on $\mathscr{F}_T$ is defined via the Radon Nikodym density process Z with respect to $\mathbb{Q}$, given by

$$Z_t = \frac{P_t(T)}{P_0(T)B_t}\,.$$

Aufgabe 1 (4 Points). Let $F_t(T,S)$ be the simple forward rate for $[T,S]$ prevailing at t which is given by

$$F_t(T,S) = \frac{1}{S-T}\left(\frac{P_t(T)}{P_t(S)} - 1\right), \quad t \in [0,T]\,.$$

Show that $(F_t(T,S))_{t\in[0,T]}$ is a martingale with respect to some forward measure Q^U; that is

$$F_t(T,S) = E_{Q^U}\left[F_T(T,S)|\mathscr{F}_t\right] \quad \text{für alle } t \in [0,T]\,.$$

What is U?

Hint. Use the identity

$$P_t(T) = B_t E_Q\left[\frac{1}{B_T}\middle|\mathscr{F}_t\right], \quad t \in [0,T]\,.$$

Solution. this is Exercise 1 from sheet 13 form Probability Theory 2. We claim that $U = S$. According to Definition 1, the forward measure $\mathbb{Q}^S$ with respect to $\mathbb{Q}$ is given by

$$Z = \frac{1}{P_0(S)}\frac{P(S)}{B}\,. \tag{1}$$

We use the definition of $F_T(T,S)$ to get

$$E_{Q^S}[F_T(T,S)|\mathscr{F}_t] = \frac{1}{S-T}\left(E_{Q^S}\left[\frac{1}{P_T(S)}\middle|\mathscr{F}_t\right]-1\right).$$

By measure change with Z from equation (1) we get

$$= \frac{1}{S-T}\left(\frac{1}{Z_t}E_Q\left[\frac{1}{P_T(S)}\middle|\mathscr{F}_t\right]-1\right).$$

Employing the definition of Z in equation (1) we get

$$= \frac{1}{S-T}\left(\frac{B_t}{P_t(S)}E_Q\left[\frac{1}{B_T}\middle|\mathscr{F}_t\right]-1\right).$$

Finally, we use the hint to arrive at

$$= \frac{1}{S-T}\left(\frac{P_t(T)}{P_t(S)-1}\right) = F_t(T,S).$$

Literatur

[BM06] Brigo, Damiano ; Mercurio, Fabio: *Interest rate models-theory and practice: with smile, inflation and credit.* Bd. 2. Springer, 2006