1. Design a flow network $G = (V, E, s, t, c)$ with $|V| \leq 4$ and $c : V \times V \to 0, 1$ such that max–flow function for G is not unique. On the flow network you design, show at least two different max–flow functions and state the value of the max–flow.

2. How would you use the output of the Floyd-Warshall algorithm to detect a negative cycle?

3. Suppose that we are given a flow network G and max flow f on G. We know that all the capacities are integers in G, and all flow values are also integers. Suppose that the capacity of an edge (u,v) is increased by 1.

   (a) How can you decide if the increase in the capacity affects the max flow value? Find an efficient algorithm to decide this.

   (b) If f is not max-flow any more, what is the new max-flow value?

   (c) How do you find the new max-flow function in an efficient way?

4. (a) **Graph and Johnson's Algorithm Problem**

Consider the following directed graph $G = (V, E)$, where:

- $V = \{1, 2, 3, 4, 5\}$
- $E = \{(1, 2), (2, 5), (3, 2), (3, 4), (4, 1), (4, 5), (5, 1), (5, 3)\}$

Suppose that we are given the following weight function $w : E \to \mathbb{R}$ for the edges:

$$\begin{aligned} w(1, 2) &= -5 \\ w(2, 5) &= 8 \\ w(3, 2) &= 1 \\ w(3, 4) &= -4 \\ w(4, 1) &= 7 \\ w(4, 5) &= 6 \\ w(5, 1) &= 0 \\ w(5, 3) &= 2 \end{aligned}$$

For all pairs shortest path problem, if we would like to apply Dijkstra's algorithm, we have to reweight the edges, as suggested by Johnson's algorithm.

Johnson's algorithm is based on first finding some suitable weights for the vertices of the graph. Let us use x_1, x_2, x_3, x_4, x_5 as the weight of the nodes 1, 2, 3, 4, 5 respectively. After we find a suitable weight for each node, we will define a new weight function $\tilde{w} : E \to \mathbb{R}$ for the edges, such that all edges will have non-negative weights when we use

$$\tilde{w}(i, j) = w(i, j) - x_i + x_j \quad \forall (i, j) \in E$$

For each edge (i, j) in E, write down the constraint on the difference of x_i and x_j based on the way $\tilde{w}$ is defined in Equation (1) above.

(b) Draw the graph G that would solve this system of difference constraints if applied the Bellman - Ford algorithm.