CS412 Machine Learning Spring 2024 : Homework 2, Question 1

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1 MLE Estimation for the parameter α

$$\hat{\alpha}_{MLE} = \arg\max_{\alpha} L(\alpha)$$

$$L(\alpha) = \prod_{i=1}^{n} \alpha(1-x_i)^{\alpha-1}$$

$$\begin{aligned}
\ln L(\alpha) &= \ln\left(\prod_{i=1}^{n} \alpha(1-x_i)^{\alpha-1}\right) \\
&= \sum_{i=1}^{n} \ln\left(\alpha(1-x_i)^{\alpha-1}\right) \\
&= \sum_{i=1}^{n} \left(\ln(\alpha) + (\alpha-1)\ln(1-x_i)\right) \\
&= n\ln(\alpha) + (\alpha-1)\sum_{i=1}^{n} \ln(1-x_i)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\alpha}\ln L(\alpha) &= \frac{d}{d\alpha}\left(n\ln(\alpha) + (\alpha-1)\sum_{i=1}^{n} \ln(1-x_i)\right) \\
&= \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(1-x_i)
\end{aligned}$$

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \ln(1-x_i) = 0$$

$$\Rightarrow \hat{\alpha}_{MLE} = \frac{n}{-\sum_{i=1}^{n} \ln(1-x_i)}$$

2 MAP estimation for the parameter α

$$\hat{\alpha}_{MAP} = \arg\max_{\alpha} p(\alpha) \cdot L(\alpha)$$

$$p(\alpha) = \lambda \alpha^{\lambda-1} e^{-\lambda\alpha}$$

$$L(\alpha) = \prod_{i=1}^{n} \alpha(1-x_i)^{\alpha-1}$$

$$p(\alpha|X) \propto p(\alpha) \cdot L(\alpha)$$

$$\Rightarrow \ln p(\alpha|X) \propto \sum_{i=1}^{n} \ln\left(\alpha(1-x_i)^{\alpha-1}\right) + \ln\left(\lambda\alpha^{\lambda-1}e^{-\lambda\alpha}\right)$$

$$\Rightarrow \ln p(\alpha|X) = n\ln(\alpha) + (\alpha-1)\sum_{i=1}^{n}\ln(1-x_i) + (\lambda-1)\ln(\alpha) - \lambda\alpha + C$$

$$\frac{d}{d\alpha}\ln p(\alpha|X) = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-x_i) + \frac{\lambda-1}{\alpha} - \lambda = 0$$

$$\frac{d}{d\alpha}\ln p(\alpha|X) = \frac{n+\lambda-1}{\alpha} - \lambda = -\sum_{i=1}^{n}\ln(1-x_i)$$

$$\Rightarrow \hat{\alpha}_{MAP} = \frac{n+\lambda-1}{\lambda + \sum_{i=1}^{n}\ln(1-x_i)}$$