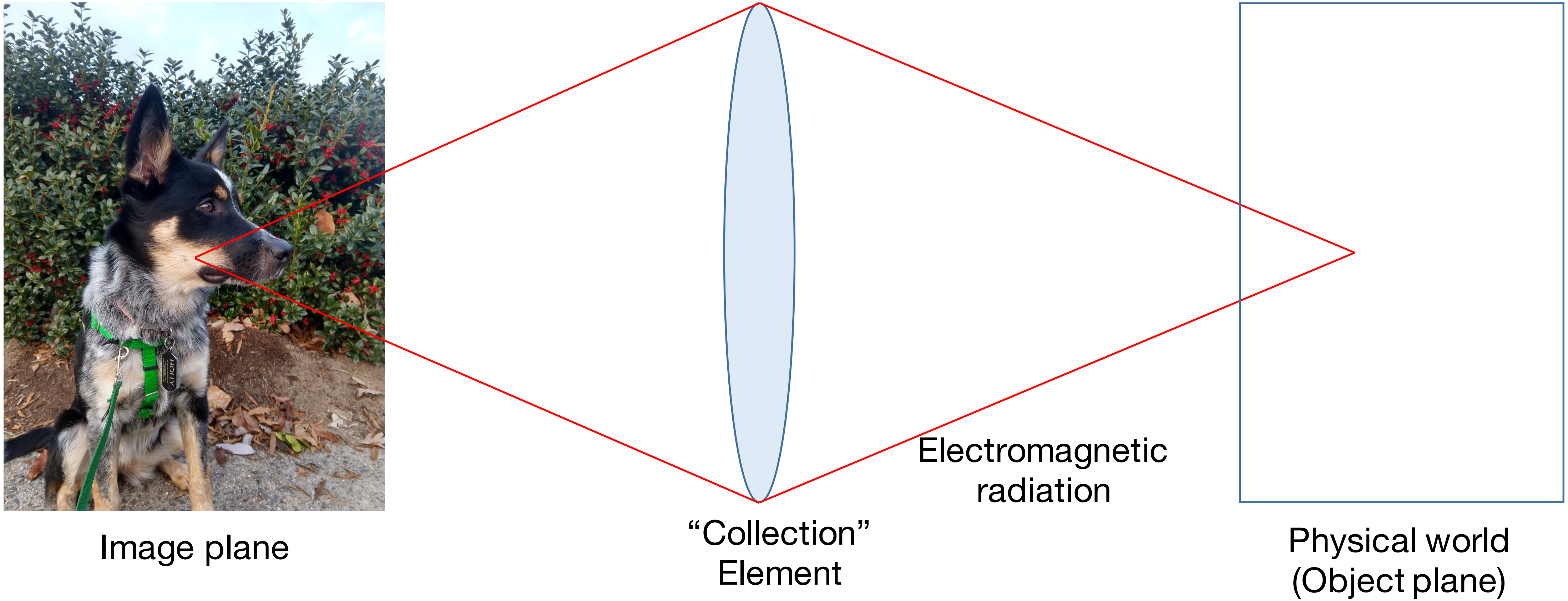
Machine Learning in Imaging

BME 590L
Roarke Horstmeyer

Lecture 2: Mathematical preliminaries for continuous functions

Last time: what is an image?

2. “Physical” Interpretation

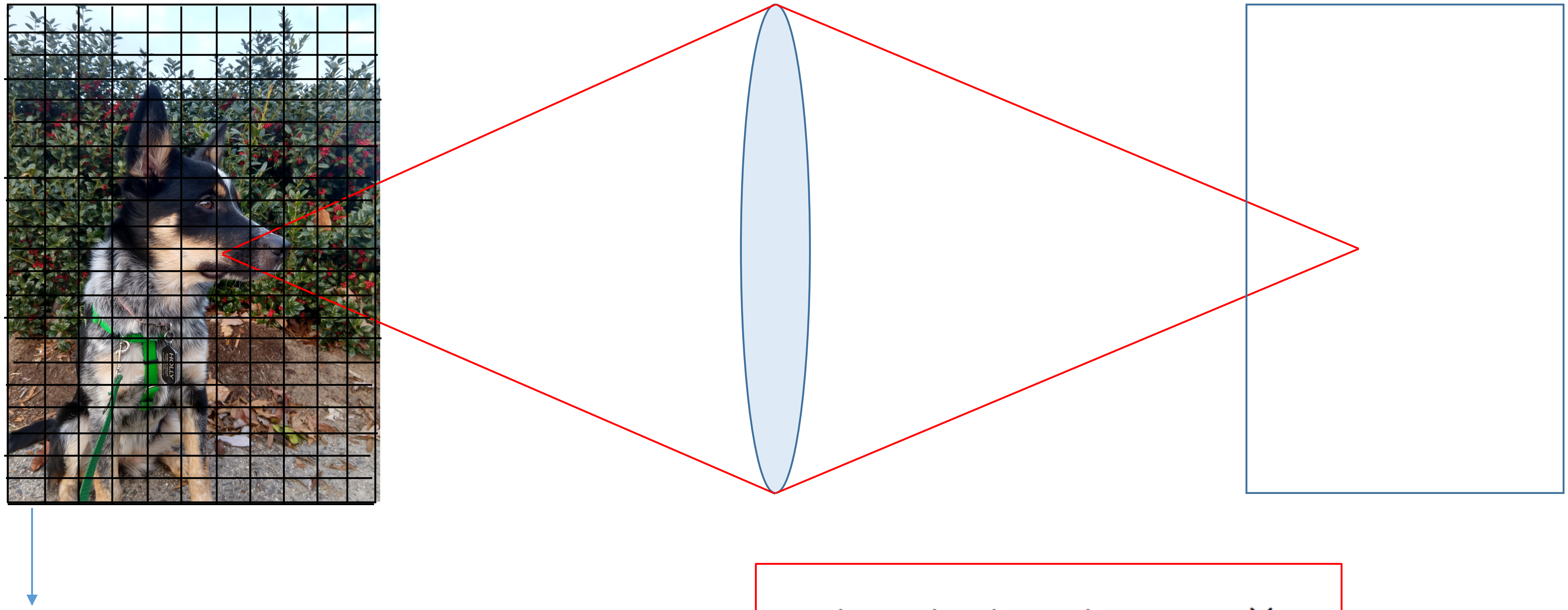


Continuous signal: $\boxed{I(x,y), (x,y) \in (-\infty, \infty)}$

Last time: what is an image?

3. "Digital" Interpretation

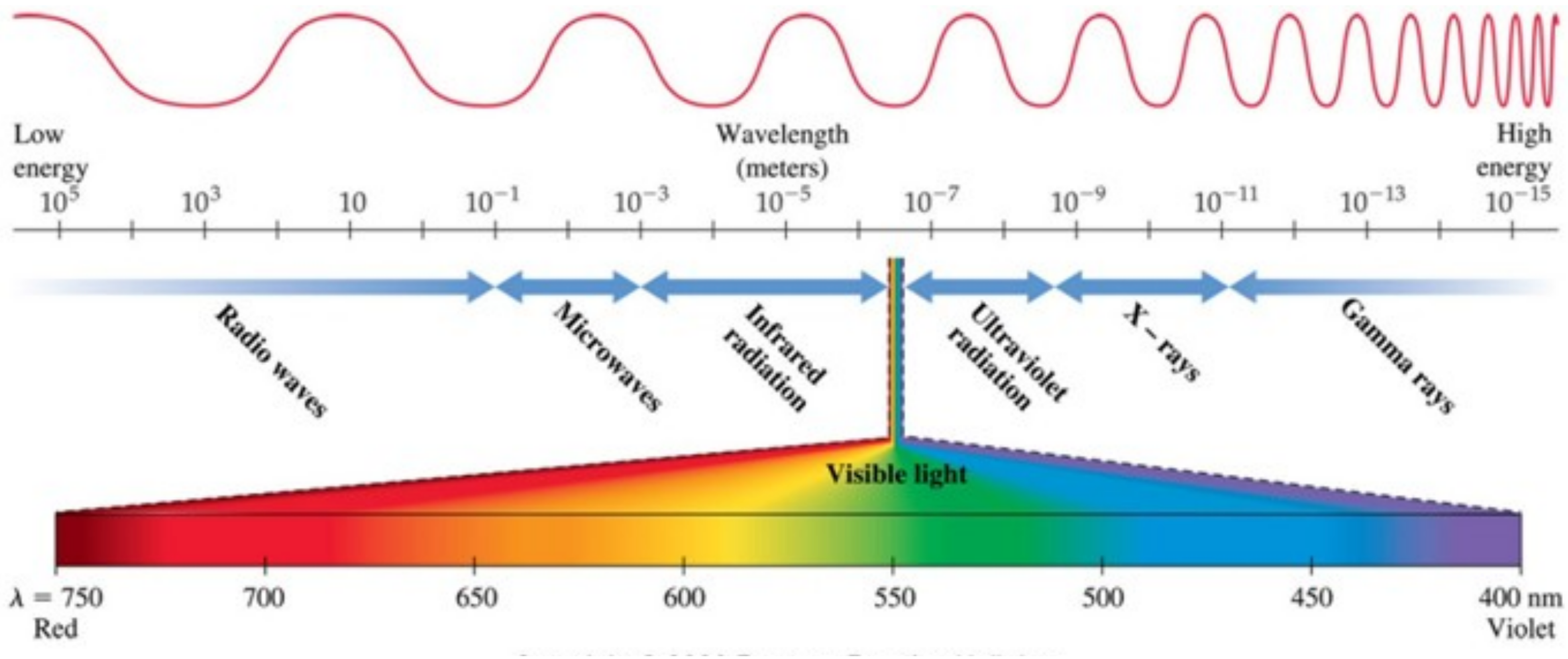
n x *m* array



Photons to electrons → Digitazation → *Discrete signal*

$$\boxed{I_s(x, y), (x, y) \in Z^{n \times m}}$$

Start at the beginning: Electromagnetic waves



Maxwell's equations

$$\nabla \times \vec{\mathcal{E}} = -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t}$$

$$\nabla \times \vec{\mathcal{H}} = \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t}$$

$$\nabla \cdot \epsilon \vec{\mathcal{E}} = 0$$

$$\nabla \cdot \mu \vec{\mathcal{H}} = 0.$$

→

Free-space propagation

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2}\frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$$

→

Scalar solution, 1 freq.

$A(\mathbf{r_1}) \cos(\mathbf{kr_1} - \omega t)$

Start at the beginning: optical fields and the black box

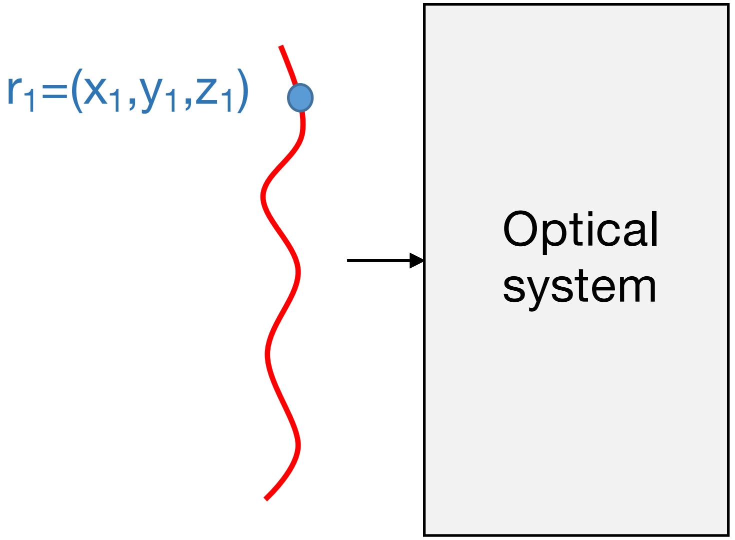
$$U(\mathbf{r}_1) = A(\mathbf{r}_1)\cos(\mathbf{k}\mathbf{r}_1 - \omega t)$$

(We will get into the details of optical fields in a few weeks)

The general idea:

1. We will treat light as a wave (an "optical field")

Start at the beginning: optical fields and the black box



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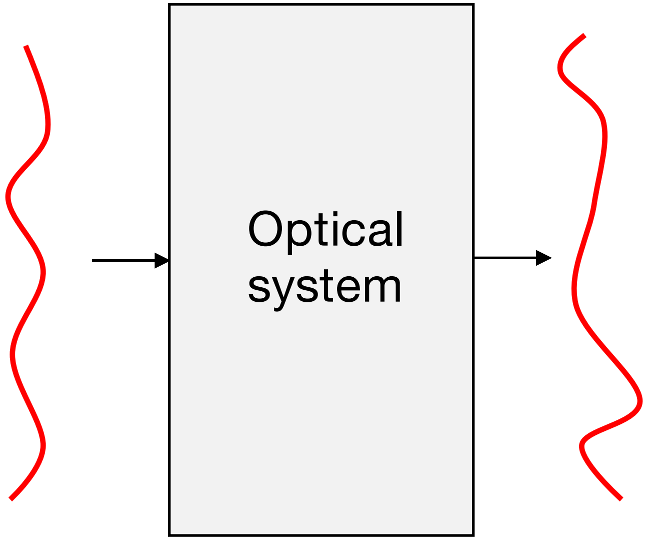
The general idea:

1. We will treat light as a wave (an "optical field")

2. It enters an optical system, which we treat as a black box

3. This black box has a number of useful properties

Start at the beginning: optical fields and the black box

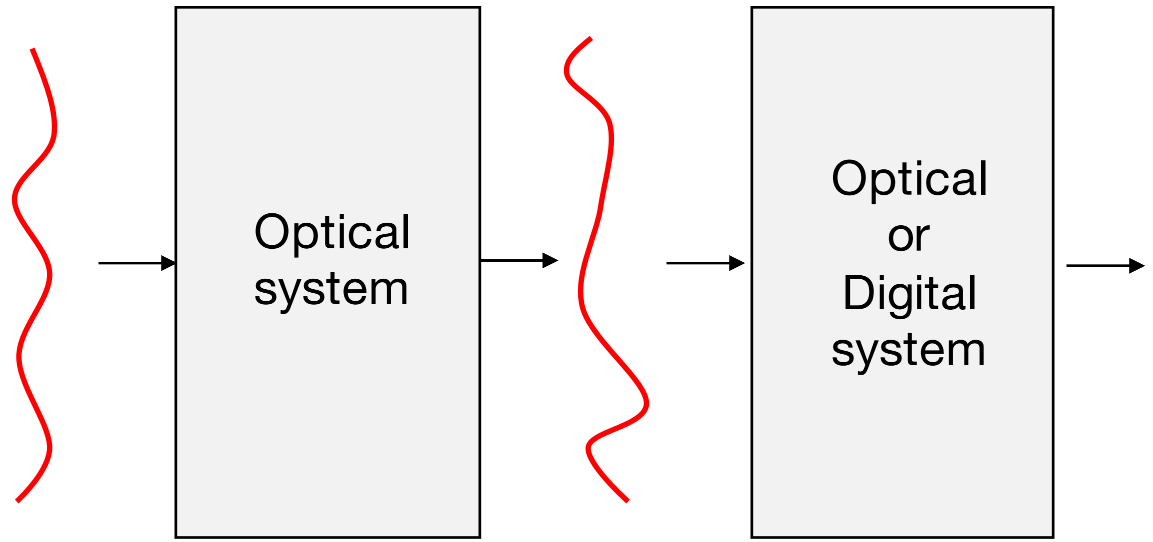


$A(\mathbf{r}_1)\cos(\mathbf{k}\mathbf{r}_1 - \omega t)$ $\quad$ $A(\mathbf{r}_2)\cos(\mathbf{k}\mathbf{r}_2 - \omega t)$

The general idea:

1. We will treat light as a wave (an "optical field")
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties
4. The black box outputs an optical field

Start at the beginning: optical fields and the black box



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The general idea:

1. We will treat light as a wave (an "optical field")
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties
4. The black box outputs an optical field, which then enters another optical system or a digital system
5. We can cascade these boxes…

Linear systems and the black box

Simplification #1: Let’s forget about light changing as a function of time. It does so way too fast, and way too slow:

$$A(\mathbf{r}) \cos(\mathbf{kr} - \omega t) \rightarrow A(\mathbf{r}) \cos(\mathbf{kr})$$

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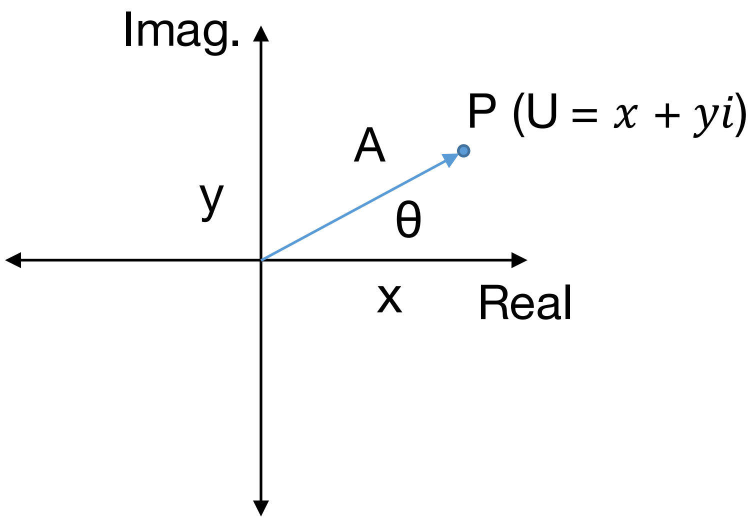
Simplification #2: We'll use complex numbers when required, it'll make our lives easier. This leads to the *complex* field, U(**r**):

$$A(\mathbf{r}) \cos(\mathbf{kr}) \leftrightarrow A(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}} = U(\mathbf{r})$$

Some things you need to recall about complex numbers

$U = x + iy$, $i = \sqrt{-1}$



$A = \sqrt{(x^2 + y^2)}$

$\theta = \mathrm{atan}(y/x)$

More useful representation:

$x = A \cos\theta$
$y = A \sin\theta$
$U = A (\cos\theta + i \sin\theta)$

$\boxed{U = A e^{i\theta}}$

A = Amplitude of field

θ = Phase of field

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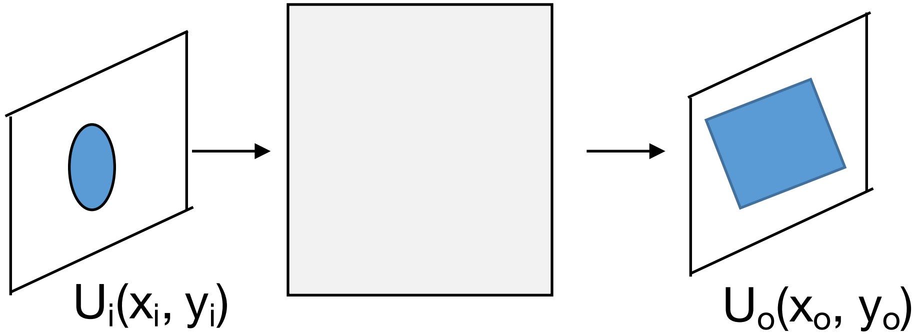
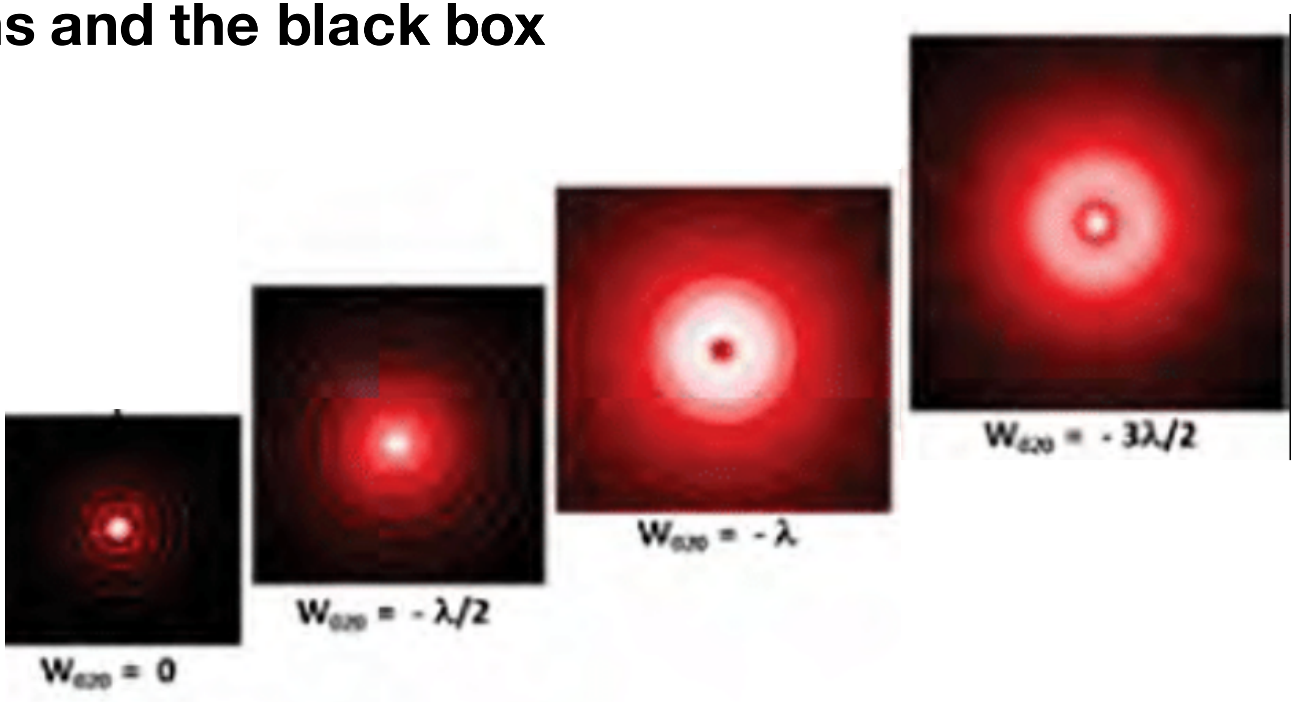
Simplification #3: Just consider mappings between planes across space. This is a critically important way of thinking for optics. Think "index card 1 to index card 2".

$$U(\mathbf{r}) \rightarrow U(x,y)$$



Linear systems and the black box



Propagation of monochromatic light

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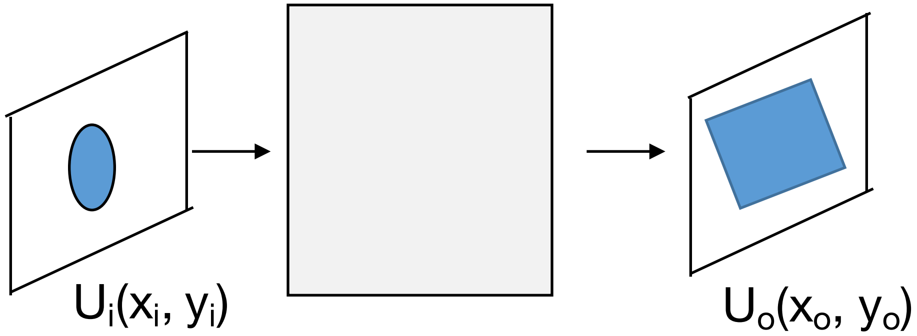
$U(\mathbf{r}) \rightarrow U(x,y)$



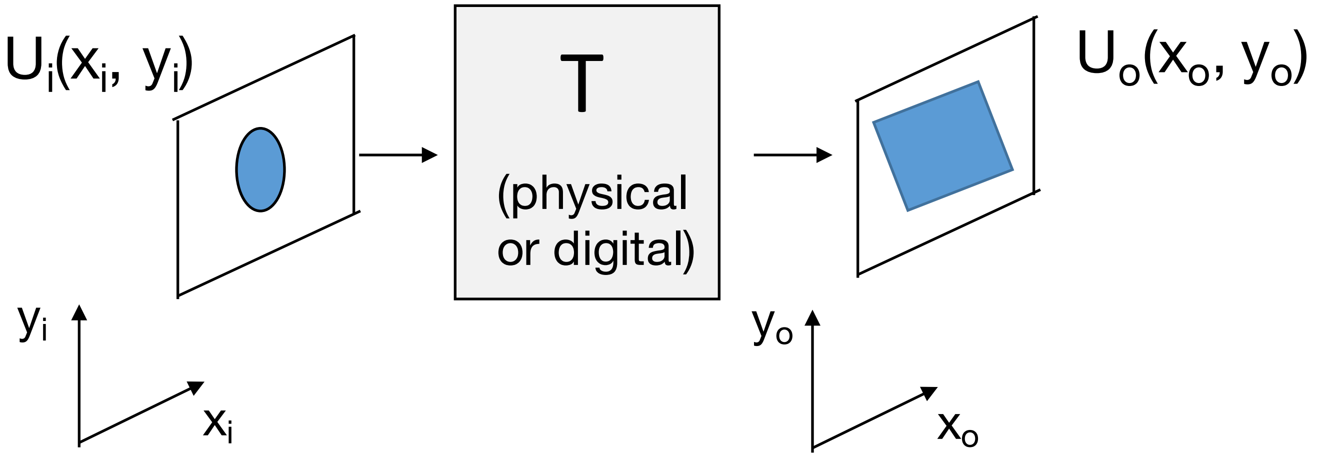
Linear systems and the black box

The "optical" black box system:

An optical black box system maps an input function $U_i(x_i,y_i)$ to an output function $U_o(x_o,y_o)$ via a transform T:

$$U_o(x_o,y_o) = T\,[\,U_i(x_i,y_i)\,]$$

Where T[ ] denotes the optical black box transformation



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Important properties of linear systems:

1. Homogeneity and additivity (superposition):

$$T\,[aU_1(x,y) + bU_2(x,y)] = aT\,[U_1(x,y)] + bT\,[U_2(x,y)]$$

2. Shift invariance: for shift distances d_x and d_y, we assume that,

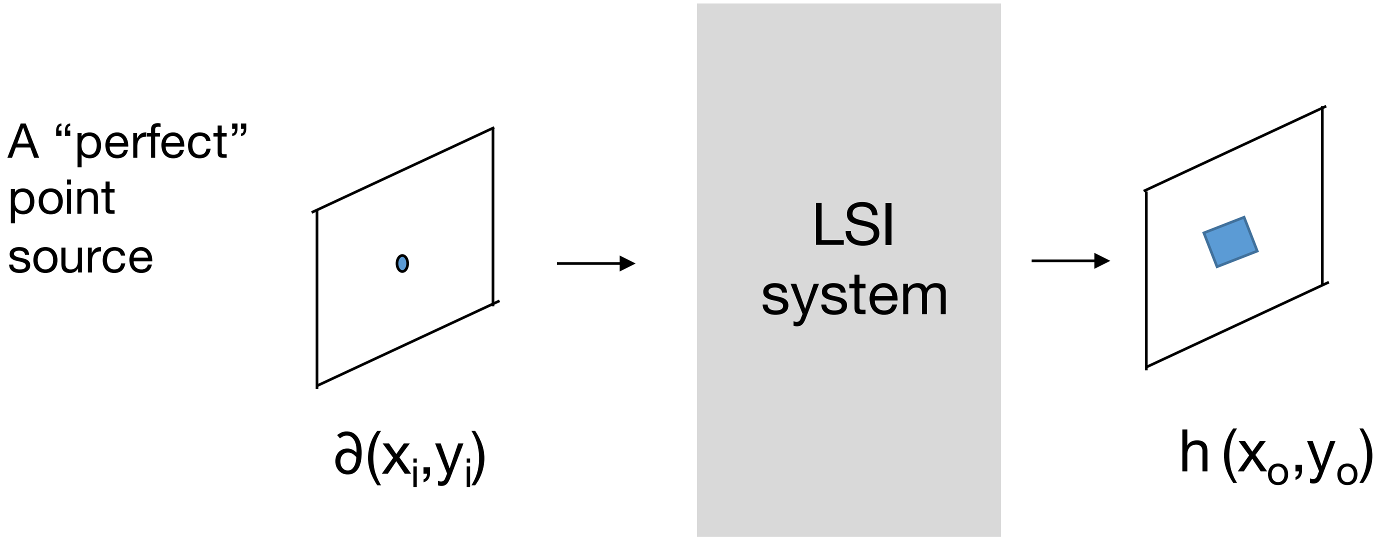
$$U_o(x_o - d_x, y_o - d_y) = T\,[U_i(x_i-d_x,\, y_i-d_y)]$$

Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

Input Dirac delta function into the black box:

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$



$$h(x_o,y_o) = T\,[\,\partial(x_i,y_i)\,]$$

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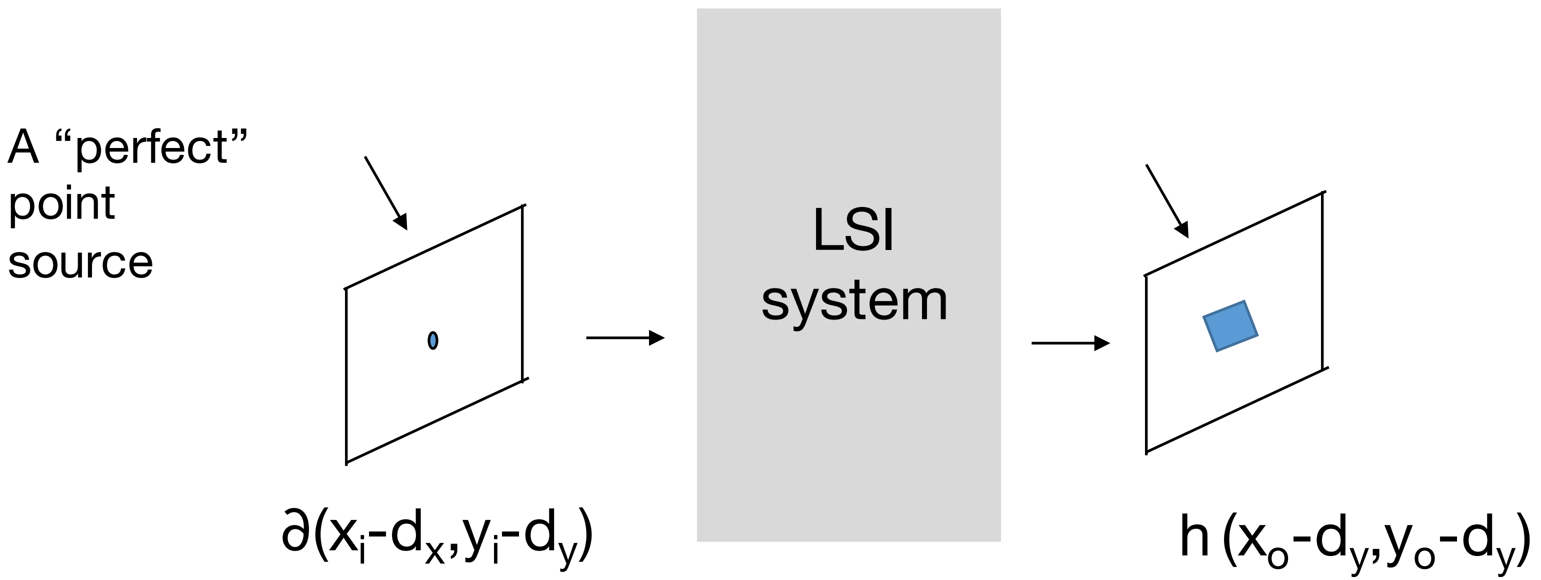
We know the system is shift invariant:



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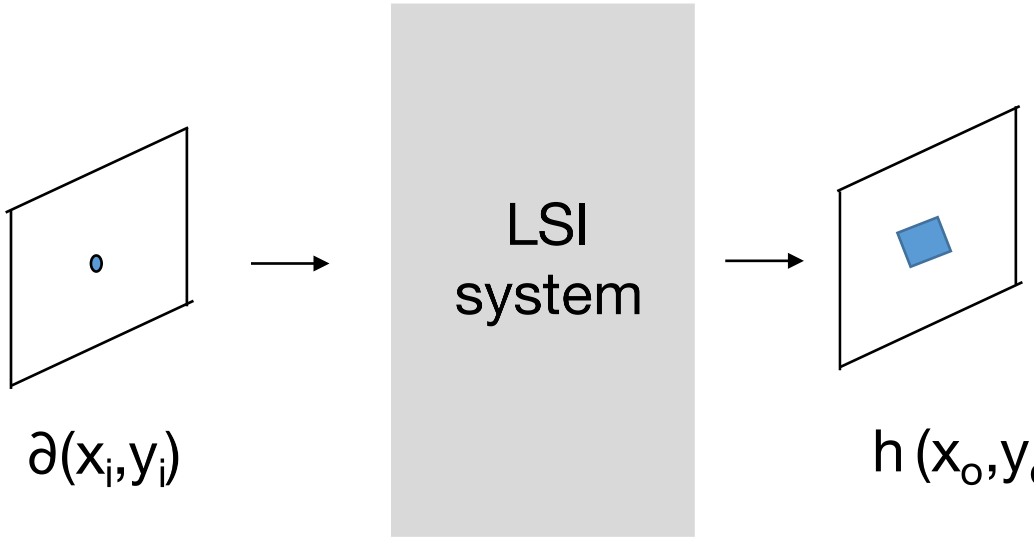
A “perfect” point source

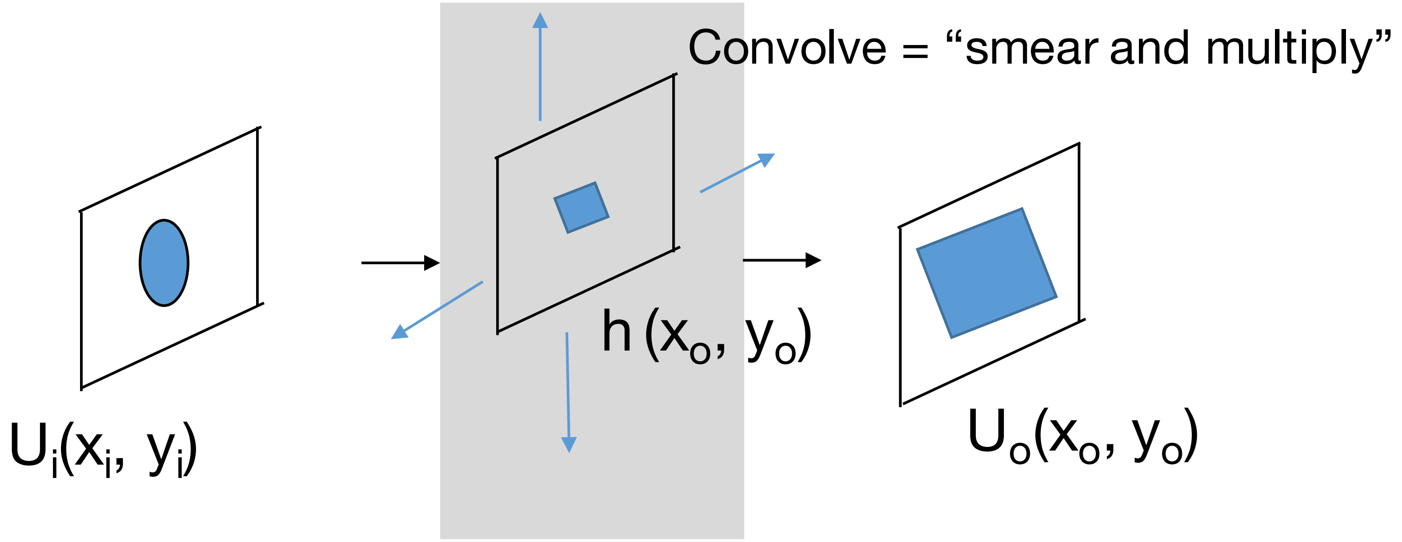


$h(x_o,y_o)$ is the system’s point-spread function

Point-spread function $h(x_o,y_o) = T\,[\,\partial(x_i,y_i)\,]$

Black box transforms as a convolution

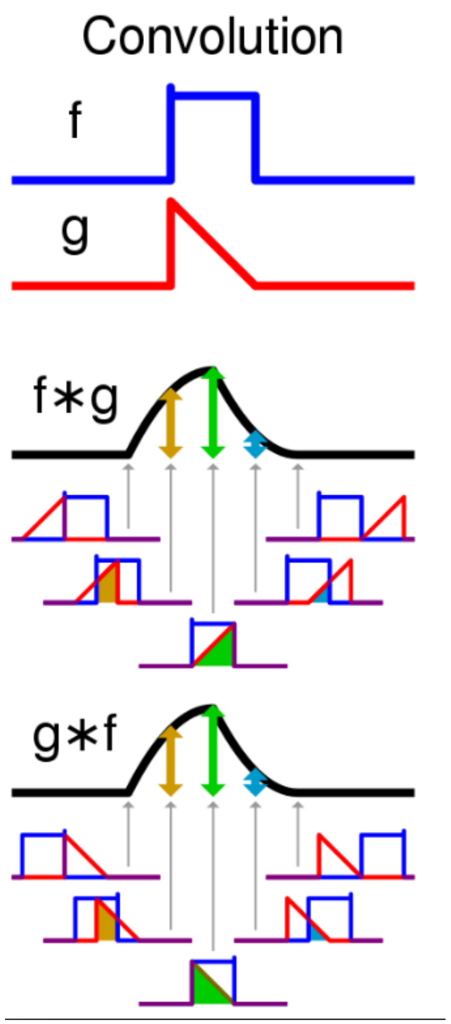
Knowing the point-spread function, it is direct to model any output of the black box, given an input:



$$U_o(x_o, y_o) = \iint_{-\infty}^{\infty} U_i(x_i, y_i) h(x_o - x_i, y_o - y_i) dx_i dy_i$$

Output of linear system is a convolution of the input with its point-spread function

1D convolution example



Steps to perform a convolution:

1. Flip one signal (the second one = the PSF)

2. Position PSF right before overlap

With incremental steps:

3. Step PSF over to position x_o

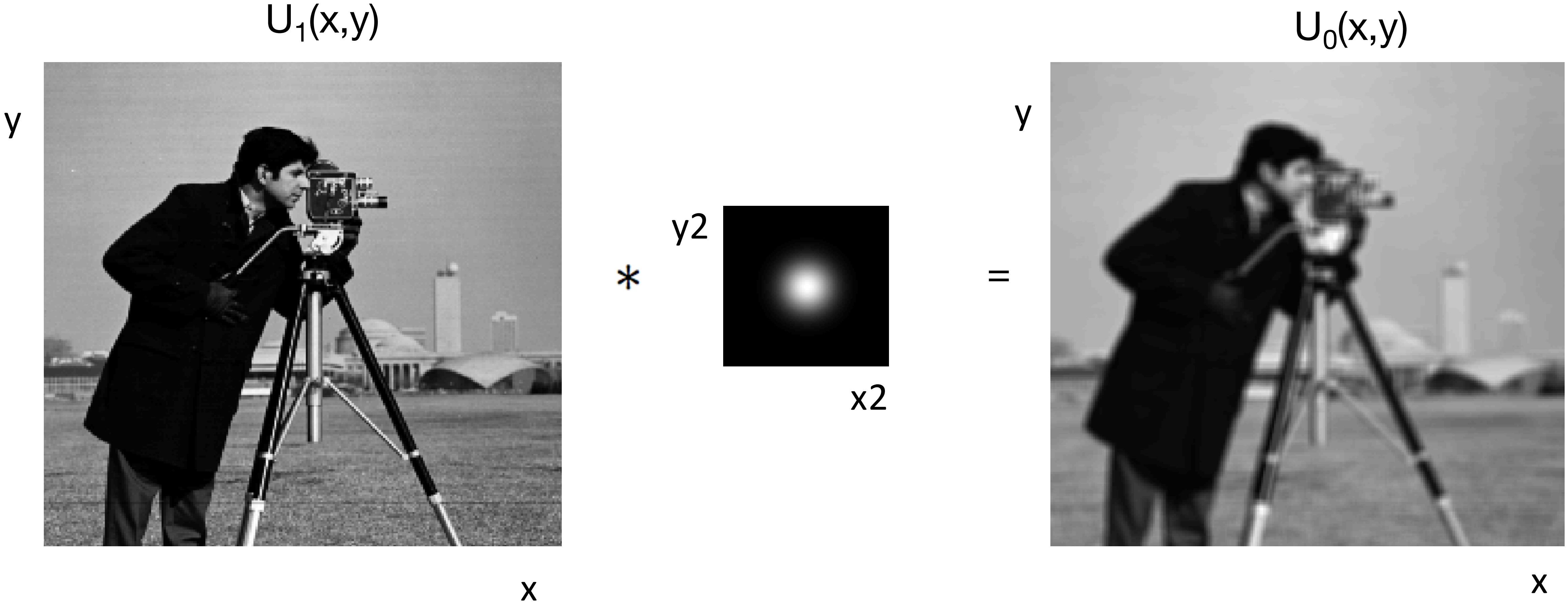
4. Compute *area* of overlap of two functions

5. Convolution value at x_o= area of overlap

6. Repeat 3-5 until signals do not overlap

2D convolution example

- Direct extension of 1D concept to 2D functions
- Note – it is effectively the same with discrete functions = matrices



Useful properties of the convolution

1. Commutativity $$U(x) * h(x) = h(x) * U(x)$$

$\Rightarrow$ You can choose which signal to "flip"

2. Associativity $$U(x) * [V(x) * W(x)] = [U(x) * V(x)] * W(x)]$$

$\Rightarrow$ Can change order $\rightarrow$ sometimes one order is easier than another

3. Distributivity $$U(x) * [h_1(x) * h_2(x)] = U(x) * h_1(x) + U(x) * h_2(x)$$

Signals in space and spatial frequency

- What we have so far:
  - Continuous & (possibly) complex function for images across space
  - Black-box linear transformation from one domain to the next via convolution

- Analogy:
  - Time-varying voltage/current going through a circuit
  - Audio signal passing through a filter

Complex function of time -> frequency

Fourier Transforms

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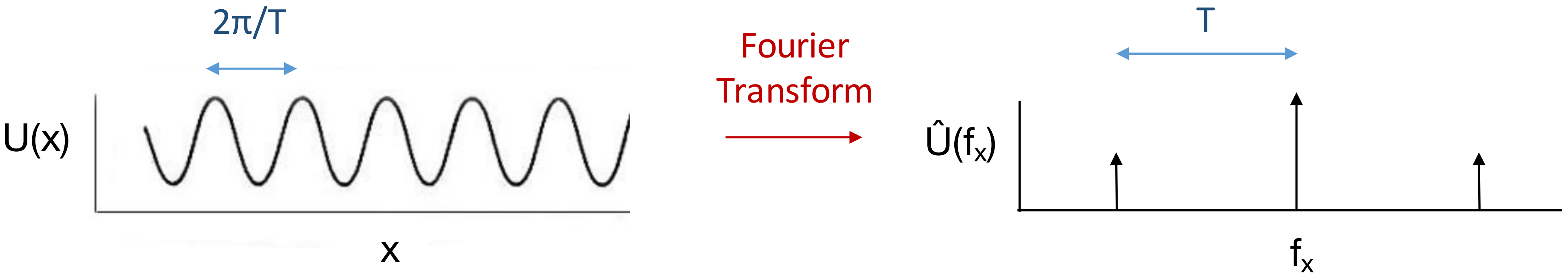
  Complex function of time -> frequency

- Here, we have 2D (complex) function across space (x,y) -> *spatial* frequency (f_x, f_y)



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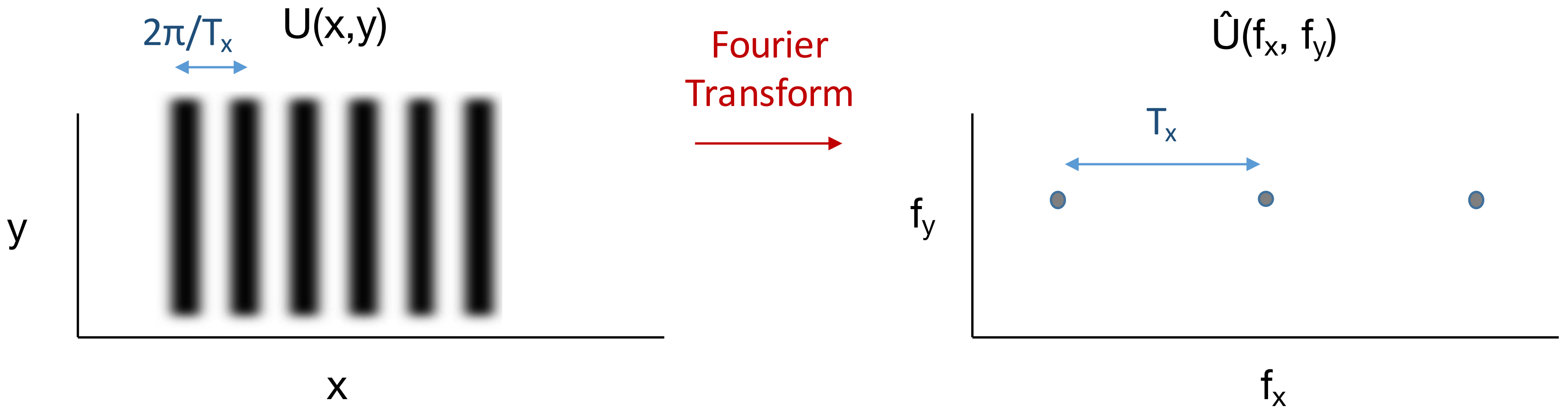

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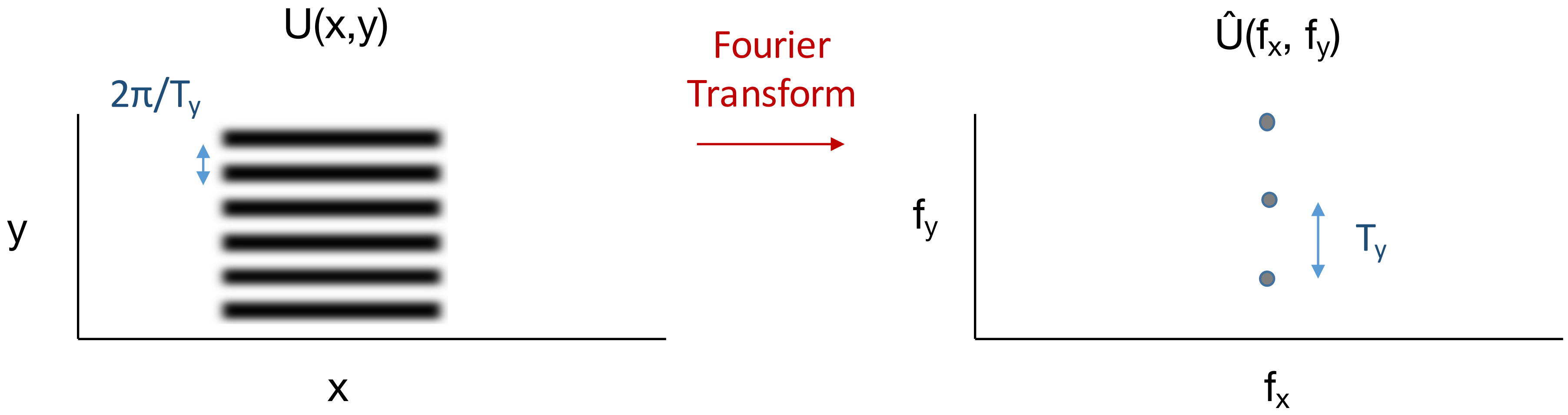


Continuous Fourier transforms – for 2D images

Decomposition of a signal into elementary functions of form, $\exp\left(-2\pi i(f_x x + f_y y)\right)$:

$$\mathcal{F}\{U(x,y)\} = \hat{U}(f_x, f_y) = \iint_{-\infty}^{\infty} U(x,y) \exp\left(-2\pi i(f_x x + f_y y)\right) dx\, dy$$

U is absolutely integrable & no infinite discontinuities. The inverse Fourier transform is,

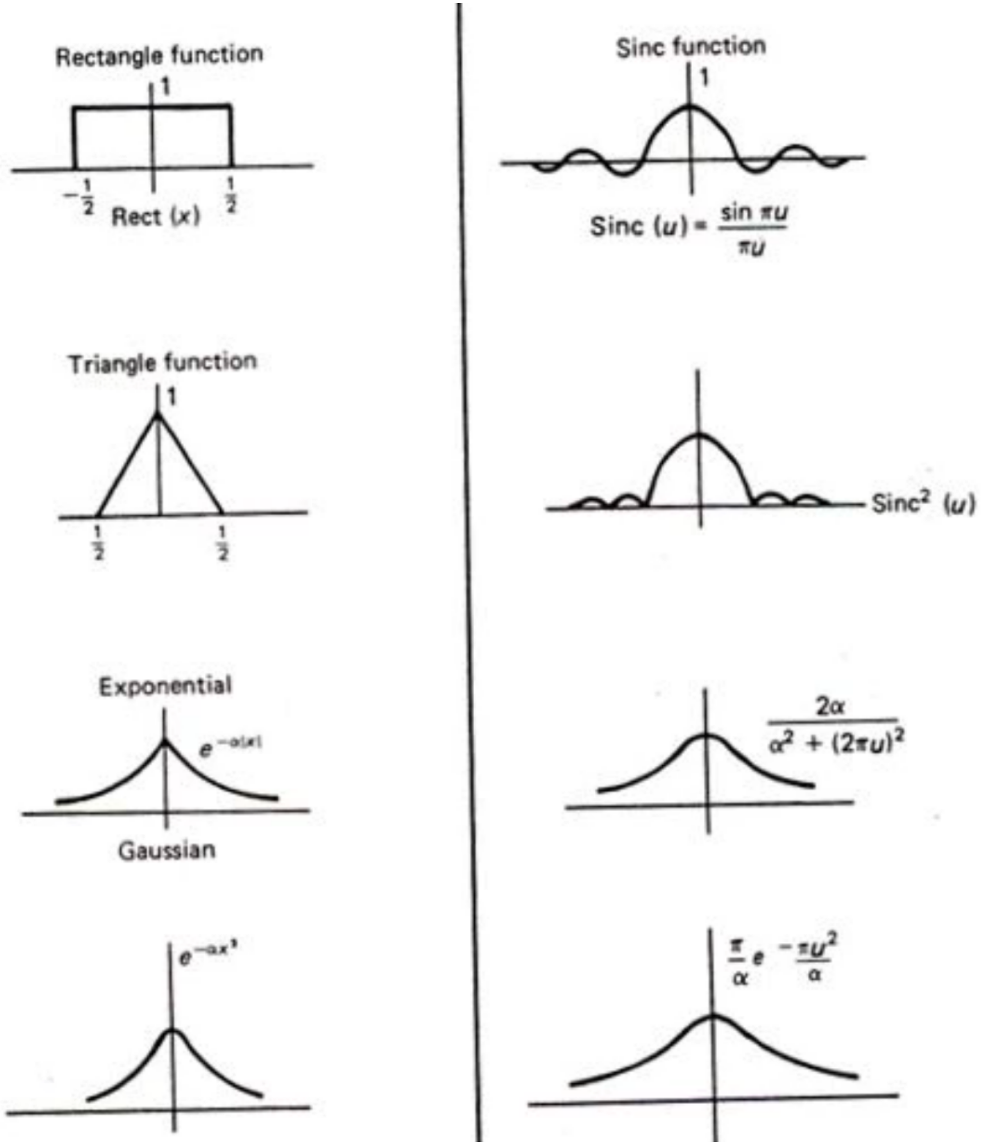
$$\mathcal{F}^{-1}\{\hat{U}(f_x, f_y)\} = U(x,y) = \iint_{-\infty}^{\infty} \hat{U}(f_x, f_y) \exp\left(2\pi i(f_x x + f_y y)\right) df_x\, df_y$$

Additional Details:

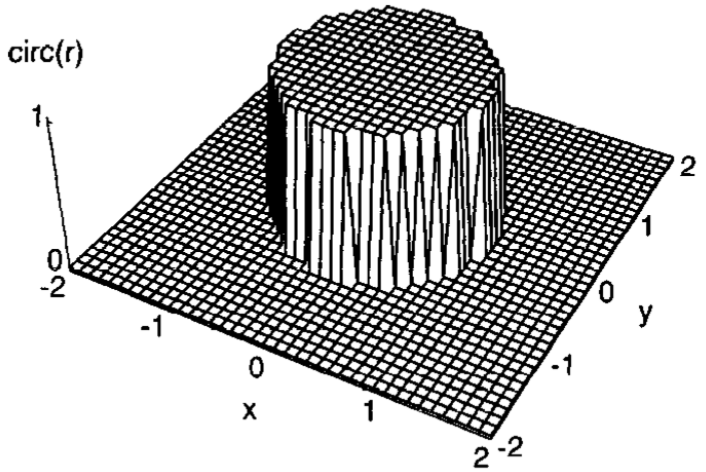
- Goodman Chapter 2.1
- Mathworld/Wikipedia, Fourier Transform

A few examples of Fourier transform pairs, 1D

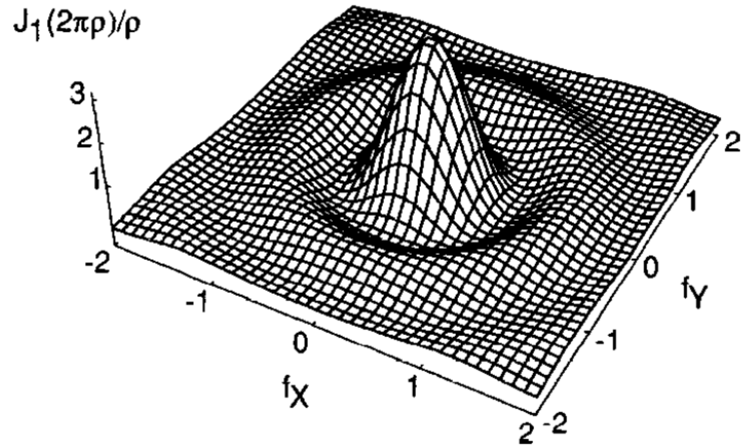
$U(x)$ $\hat{U}(f_x)$



Examples of Fourier transform pairs, 2D



(a)



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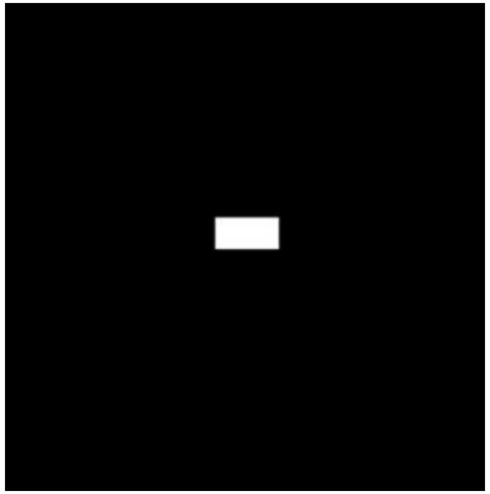
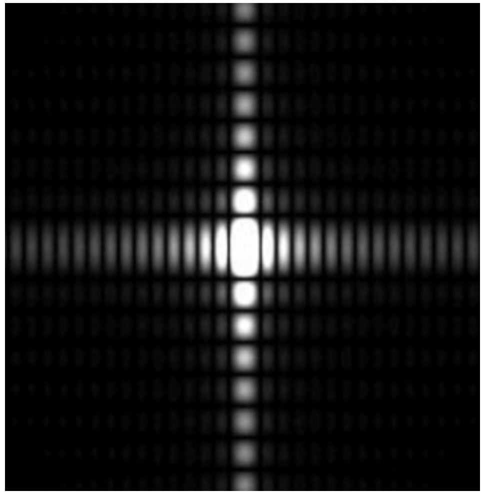
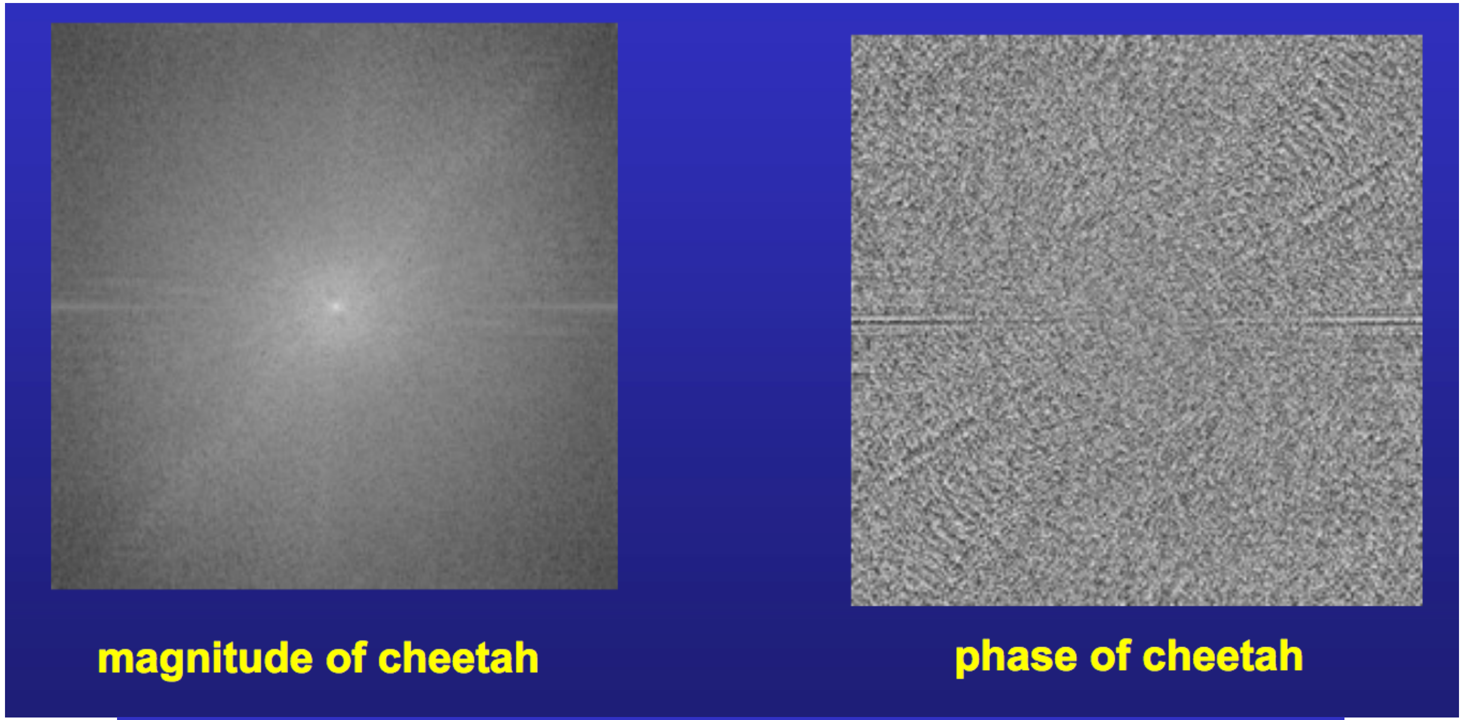




$U_1(x,y)$ $U_2(x,y)$

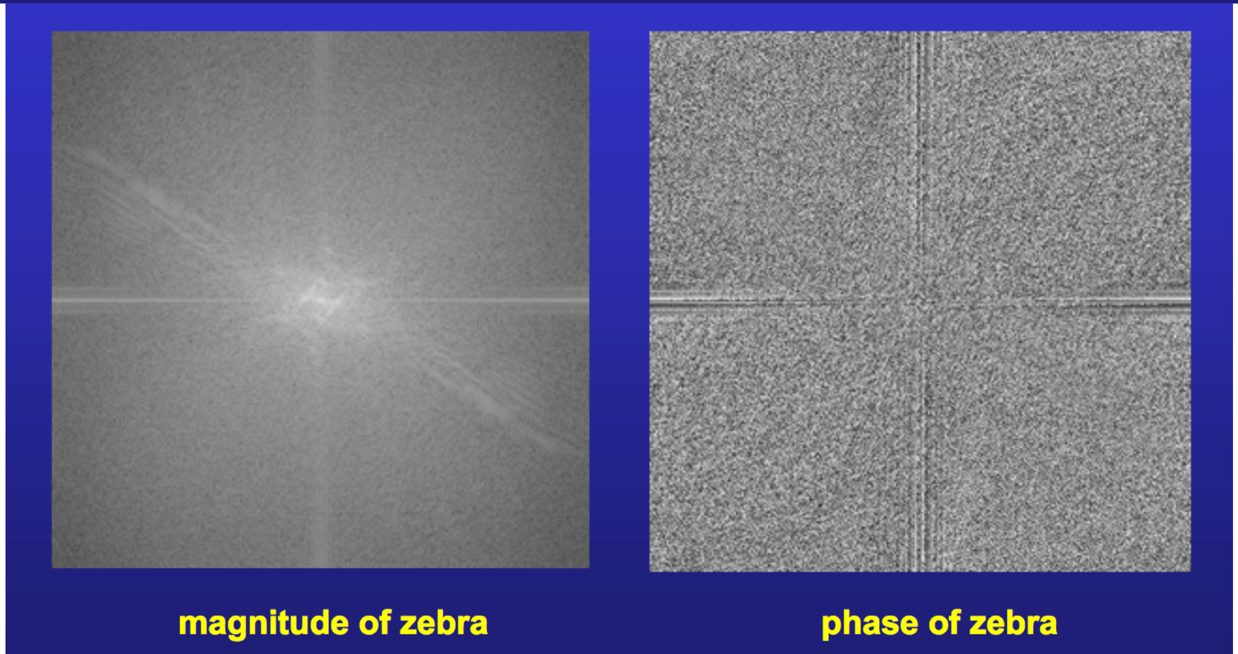
$\hat{U}_1(f_x, f_y)$



magnitude of cheetah

phase of cheetah

$\hat{U}_2(f_x, f_y)$



magnitude of zebra

phase of zebra

Important properties of the Fourier transform

- Linearity
- Scaling
- Shift
- Parseval's Theorem (energy conservation)
- Fourier integral theorem

Additional Details:

- Goodman Chapter 2.1
- Mathworld/Wikipedia, Fourier Transform

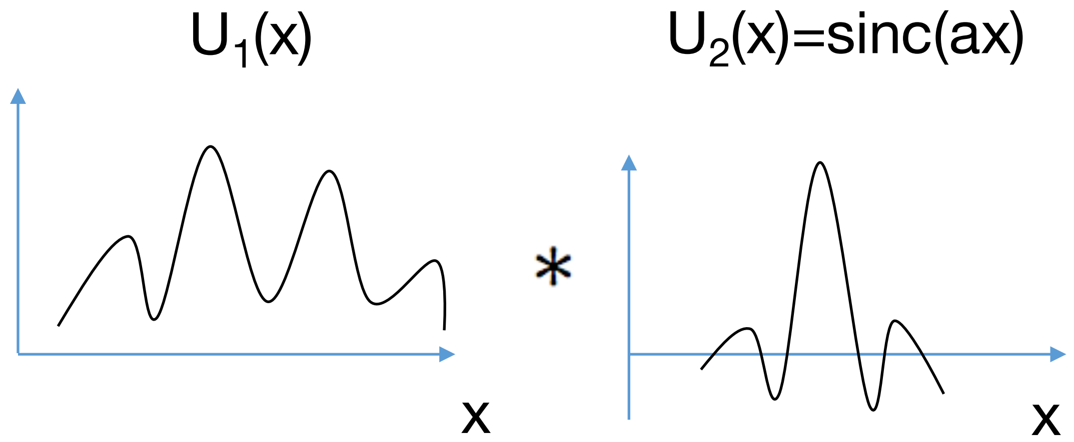
Convolution - Fourier Transform relationship: Convolution Theorem

Convolution theorem. If $\mathcal{F}\{g(x, y)) = G(f_X, f_Y)$ and $\mathcal{F}\{h(x, y)\} = H(f_X, f_Y)$, then

$$\mathcal{F}\left\{\iint_{-\infty}^{\infty} g(\xi, \eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta\right\} = G(f_X, f_Y)\, H(f_X, f_Y).$$

"The convolution of two functions in space can be performed by a multiplication in the Fourier domain (spatial frequency domain)"

Example of convolution theorem, 1D



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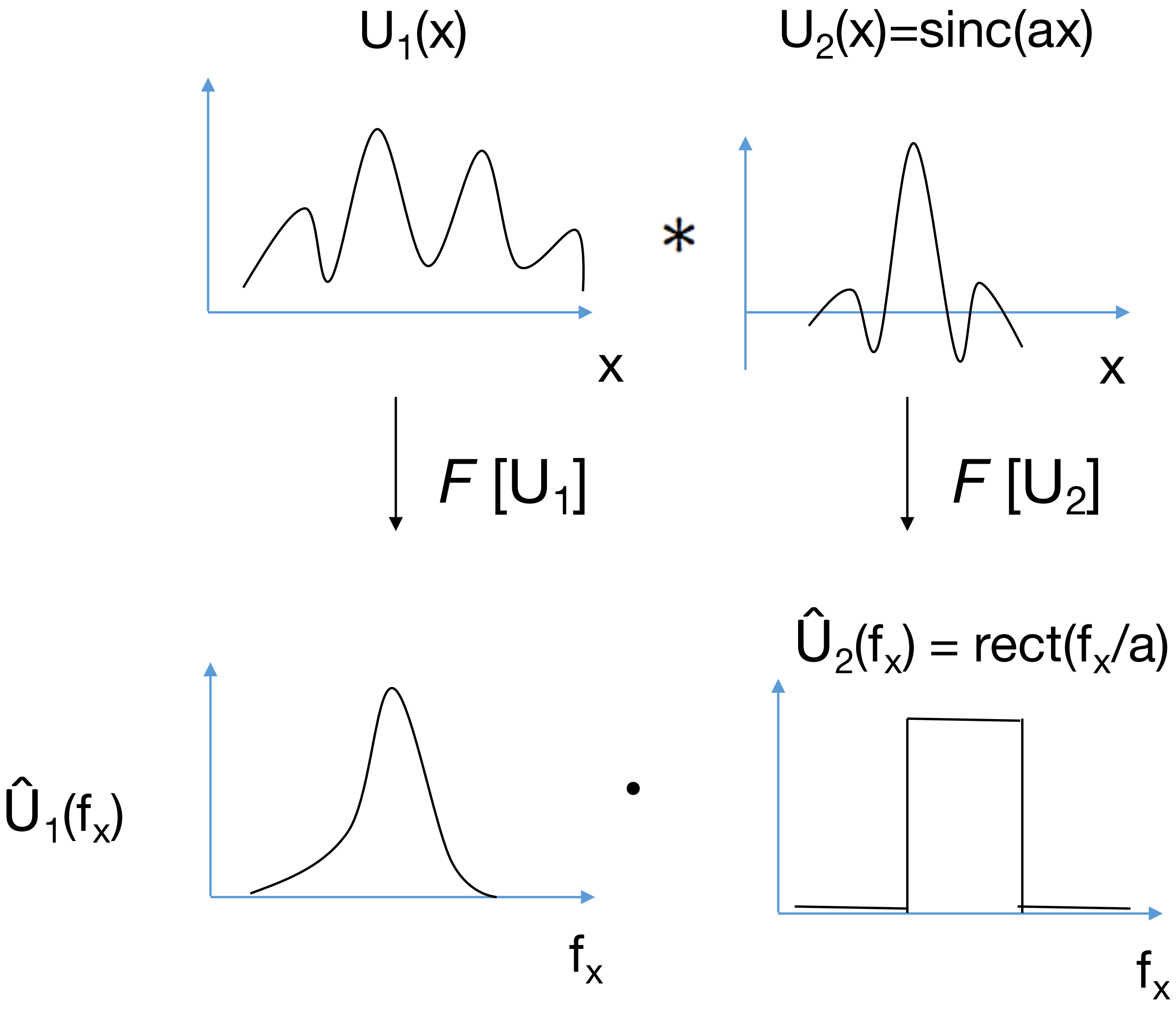

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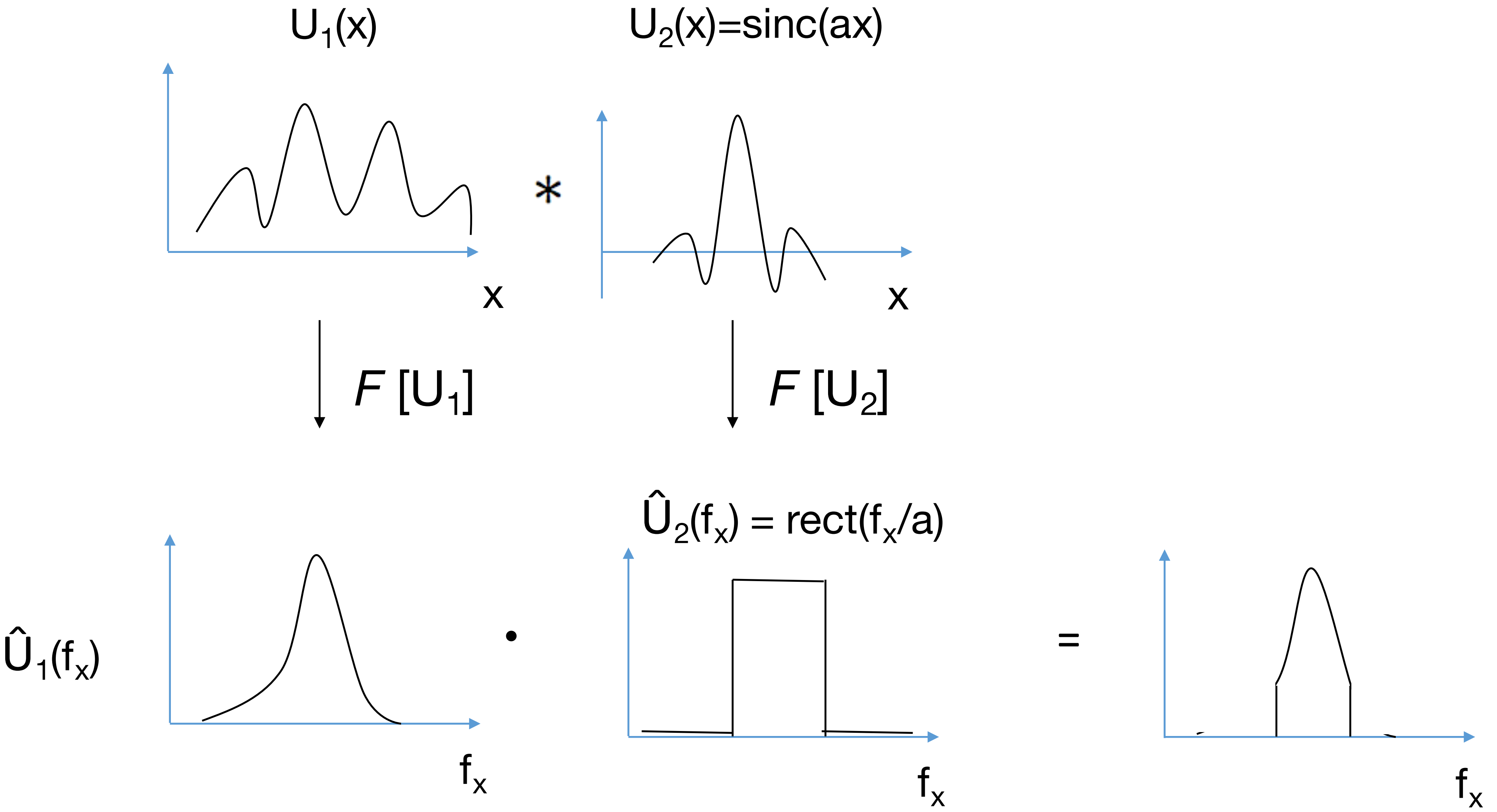

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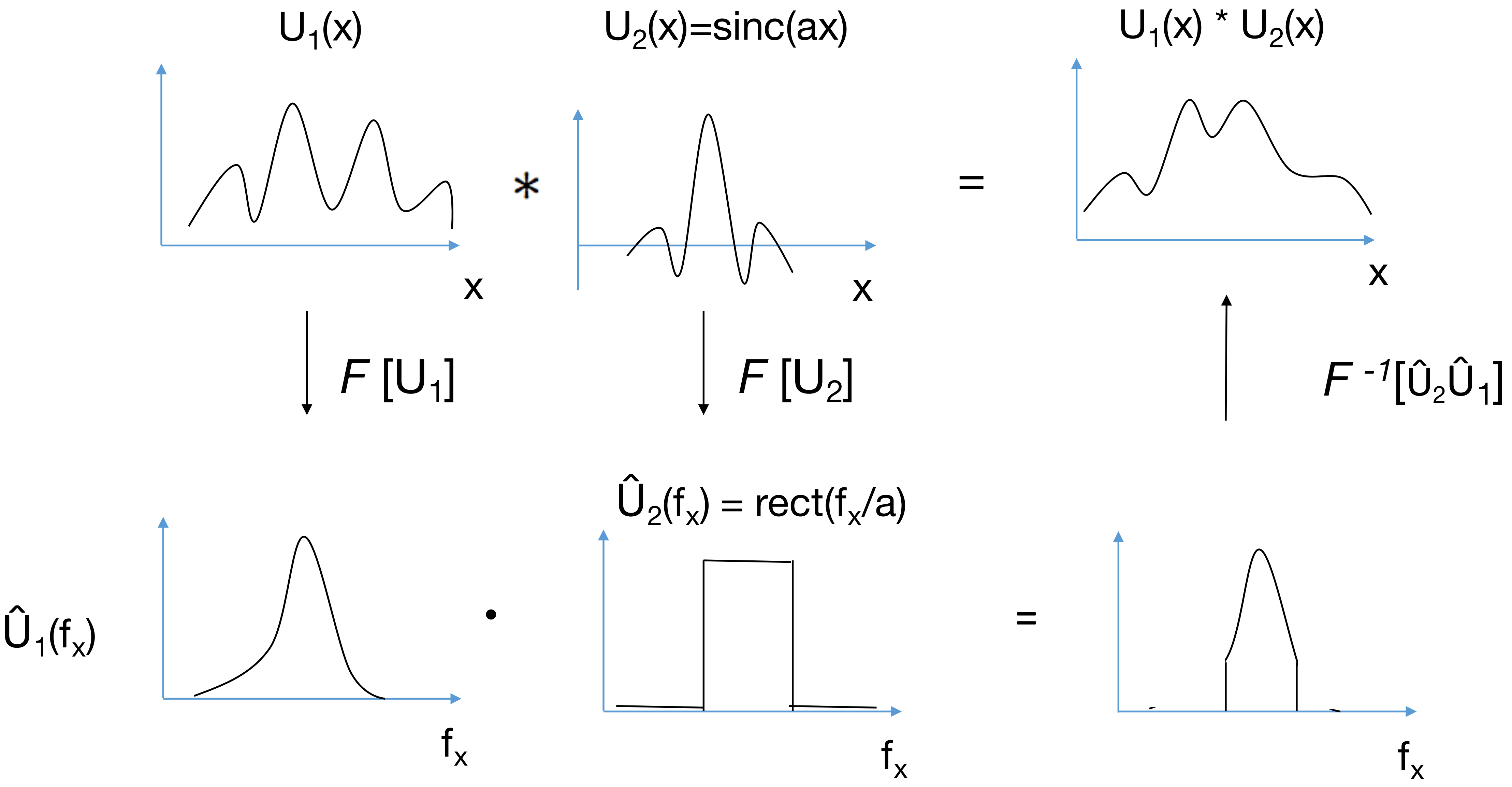
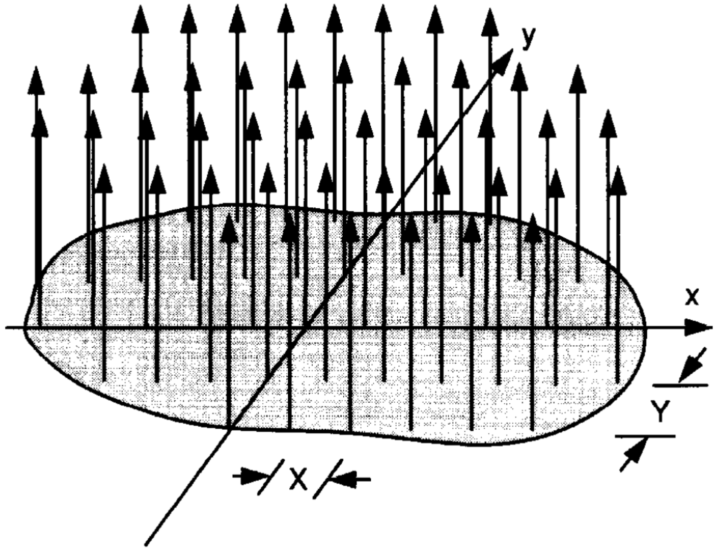


The Sampling Theorem – from Goodman Section 2.4.1

$$U_s(x,y) = \text{comb}(x/X)\text{comb}(y/Y)U(x,y)$$



Signal sampling occurs with:

- CMOS (pixel) sensors, PMTs, SPADs
- A-to-D after antennas
- A-to-D after acoustic transducers

Sampling interval width X and Y

The Sampling Theorem – from Goodman Section 2.4.1

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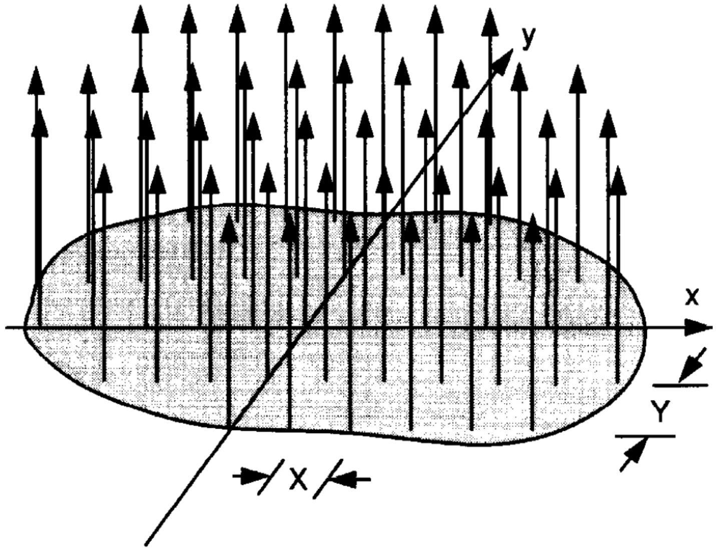

Sampling interval width X and Y

$$\hat{U}_s(f_x,f_y) = \mathcal{F}\left[\text{comb}(x/X)\text{comb}(y/Y)\right] * \hat{U}(f_x,f_y)$$

The Sampling Theorem – from Goodman Section 2.4.1

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$$\mathcal{F}\left[\mathrm{comb}(x/X)\mathrm{comb}(y/Y)\right] = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$

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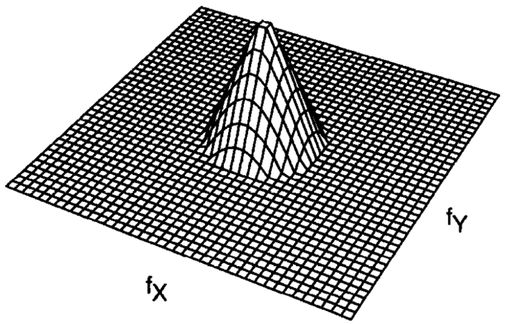
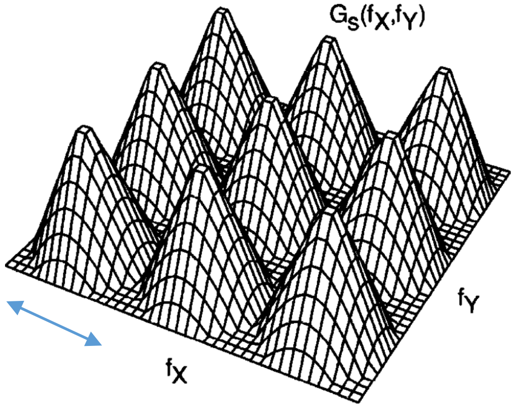


(a)



(b)

Signal extends from ($-B_x$, $-B_y$) to (B_x, B_y) in Fourier domain

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

Bandwidth (B_x, B_y) of signal

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F\,[\bullet]$

$$h(x, y) = 4B_xB_y\mathrm{sinc}(2B_xx)\mathrm{sinc}(2B_yy)$$

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$$\xrightarrow{F[\bullet]} \quad h(x, y) = 4B_xB_y\mathrm{sinc}(2B_xx)\mathrm{sinc}(2B_yy)$$

$$h(x, y) * (U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y)) = U(x, y)$$

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$$\xrightarrow{F[\bullet]} \quad h(x, y) = 4B_xB_y\mathrm{sinc}(2B_xx)\mathrm{sinc}(2B_yy)$$

$$h(x, y) * (U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y)) = U(x, y)$$

$$U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y) = XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX, mY)\delta(x - nX, y - mY)$$

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F[\bullet]$

$$h(x, y) = 4B_x B_y \mathrm{sinc}(2B_x x)\mathrm{sinc}(2B_y y)$$

$$h(x, y) * (U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y)) = U(x, y)$$

$$U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y) = XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX, mY)\delta(x - nX, y - mY)$$

$$U(x, y) = 4B_x B_y XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX, mY)\mathrm{sinc}\left[2B_x(x - nX)\right]\mathrm{sinc}\left[2B_y(y - mY)\right]$$

The Sampling Theorem

When sampled appropriately, a discrete signal can *exactly* reproduce a continuous signal:

$$\underline{U(x,y)} = 4B_x B_y XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \underline{U(nX, mY)} \operatorname{sinc}\left[2B_x(x - nX)\right] \operatorname{sinc}\left[2B_y(y - mY)\right]$$

Continuous signal:

- EM field
- Sound wave
- MR signal

Discretized signal:

- Detected EM field
- Sampled sound wave
- Sampled MR signal

What does the Sampling Theorem mean for us?