Trigonometric Sum Conjectures

Kyle Fridberg

13 April 2021

Making use of symbolic computation in Mathematica, we observed the closed form convergence of the following trigonometric sums for small n ($n \leq 15$). Then, guessing the patterns, we wrote down the formulas below and numerically verified them from $n = 1$ to 55. The formulas are surprising becuase they seem to suggest that sequences of period 11 (see sequences 5 and 6) arise in the sums (2) and (4).

Are these formulas correct? If not, at what values of n do they break down?

Conjecture 1. For all $n \in \mathbb{N}$:

$$\sum_{k=1}^{\infty} \frac{\big(\sin(k)\cos(k)\big)^{2n-1}}{(2k-1)(2k+1)} = \frac{1}{2^{4n-2}} \sum_{j=1}^{n} (-1)^j \binom{2n-1}{n-j} \log\left|\tan\left(\frac{2j-1}{2}\right)\right| \sin(2j-1) \tag{1}$$

$$\sum_{k=1}^{\infty} \frac{\big(\sin(k)\cos(k)\big)^{2n}}{(2k-1)(2k+1)} = \frac{\pi}{2^{4n+1}} \sum_{j=1}^{n} \overline{\boldsymbol{e}}_j \binom{2n}{n-j} \sin(2j) \tag{2}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\big(\sin(k)\cos(k)\big)^{2n-1}}{\big((2k-1)(2k+1)\big)^2} = \frac{1}{2^{4n-1}} \sum_{j=1}^{n} (-1)^{j+1} \binom{2n-1}{n-j} \Bigg(&\log\left|\tan\left(\frac{2j-1}{2}\right)\right| \sin(2j-1) \\ &+ \operatorname{Im}\Big(\operatorname{Li}_2\big(e^{i(2j-1)}\big) - \operatorname{Li}_2\big(-e^{i(2j-1)}\big)\Big) \cos(2j-1)\Bigg)\end{aligned} \tag{3}$$

$$\sum_{k=1}^{\infty} \frac{\big(\sin(k)\cos(k)\big)^{2n}}{\big((2k-1)(2k+1)\big)^2} = \frac{\pi}{2^{4n+3}} \left[\frac{\pi}{2}\binom{2n}{n} + \sum_{j=1}^{n} \overline{\boldsymbol{e}}_j \binom{2n}{n-j} \Big((4j - \pi\, \overline{\boldsymbol{v}}_j)\cos(2j) - 2\sin(2j)\Big)\right] \tag{4}$$

Where $\overline{\boldsymbol{e}}$ and $\overline{\boldsymbol{v}}$ are the following special sequences not found in the OEIS:

$$\overline{\boldsymbol{e}} := (\boldsymbol{e}, \boldsymbol{e}, \boldsymbol{e}, ...) = (1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, 1, 1, -1, -1, -1, ...)$$

$$\boldsymbol{e} := (1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1) \tag{5}$$

$$\overline{\boldsymbol{v}} := (\boldsymbol{v_1}, 14 + \boldsymbol{v_1}, 2 \cdot 14 + \boldsymbol{v_1}, 3 \cdot 14 + \boldsymbol{v_1}, ...) = (1, 3, 3, 5, 7, 7, 9, 11, 11, 13, 15, 15, 17, 17, 19, 21, 21, 23, 25, 25, ...)$$

$$\boldsymbol{v_1} := (1, 3, 3, 5, 7, 7, 9, 11, 11, 13, 15) \tag{6}$$

It is worth noting that $\overline{\boldsymbol{e}}_j$ has a simple closed-form expression:

$$\overline{\boldsymbol{e}}_j = \text{sign}\Bigg(\tan\left(\frac{2\pi j}{11} - \frac{\pi}{44}\right)\Bigg) = (-1)^{\left\lceil \frac{4j}{11} \right\rceil - 1} \tag{7}$$

Using $\overline{\boldsymbol{e}}_j$, we can also find a closed form expression for $\overline{\boldsymbol{v}}_j$:

$$\overline{\boldsymbol{v}}_j = 2\Bigg(j + 1 - 2\left\lceil \frac{2j}{11} \right\rceil\Bigg) + (-1)^{\left\lceil \frac{4j}{11} \right\rceil - 1} \tag{8}$$