A note on pseudo-binomial equations exhibiting D_n symmetry in $\mathbb{R}^2$

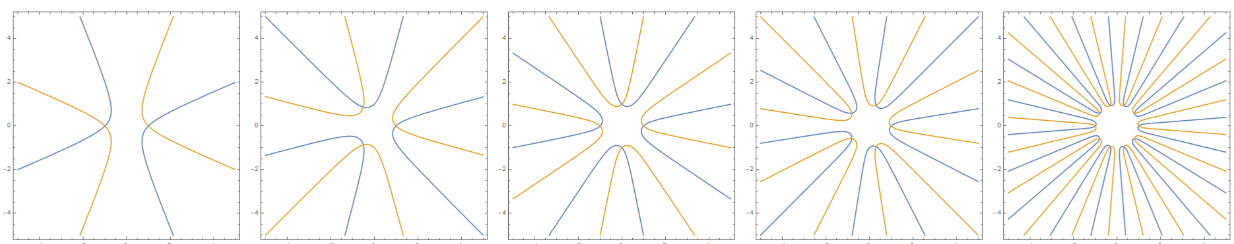
Kyle Fridberg

September 2020

Proposition 1. *Given $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = a(x^2 + y^2)^b$ where $a, b \in \mathbb{R}$, all curves given by the following two equations exhibit D_n symmetry about the origin in the xy-plane:*

$$f(x, y) = \sum_{k=0}^{n} i^{(k+1)k} \binom{n}{k} x^{n-k} y^k \tag{1}$$

$$f(x, y) = \sum_{k=0}^{n} i^{k(k-1)} \binom{n}{k} x^{n-k} y^k \tag{2}$$



(a) $n = 2$ (b) $n = 3$ (c) $n = 4$ (d) $n = 5$ (e) $n = 10$

Figure 1: Plots of (1) (blue) and (2) (yellow) with $f(x, y) = 1$ for different values of n. These curves represent a simple case of Proposition 1. Note the phase shift between the blue and yellow curves.

Proof. We first convert (1) and (2) to polar form by making the substitutions $x = r\cos\theta$ and $y = r\sin\theta$. Since the simplification steps are the identical for both equations, we will perform the simplifications together, denoting (1) and (2) as U_1 and U_2, respectively.

$$\begin{aligned} U_1 :\quad ar^{2b} &= \binom{n}{0}(r\cos\theta)^n - \binom{n}{1}(r\cos\theta)^{n-1}(r\sin\theta) - \binom{n}{2}(r\cos\theta)^{n-2}(r\sin\theta)^2 + ... \\ U_2 :\quad ar^{2b} &= \binom{n}{0}(r\cos\theta)^n + \binom{n}{1}(r\cos\theta)^{n-1}(r\sin\theta) - \binom{n}{2}(r\cos\theta)^{n-2}(r\sin\theta)^2 - ... \end{aligned} \tag{3}$$

After dividing both sides of the above equations by r^n, we can group the odd and even binomial terms, indexing over all integer values of k from 0 to n.

$$\begin{aligned} U_1: \quad ar^{2b-n} &= \sum_{k \text{ even}} (-1)^{\frac{k}{2}} \binom{n}{k} (\cos\theta)^{n-k} (\sin\theta)^k - \sum_{k \text{ odd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} (\cos\theta)^{n-k} (\sin\theta)^k \\ U_2: \quad ar^{2b-n} &= \sum_{k \text{ even}} (-1)^{\frac{k}{2}} \binom{n}{k} (\cos\theta)^{n-k} (\sin\theta)^k + \sum_{k \text{ odd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} (\cos\theta)^{n-k} (\sin\theta)^k \end{aligned} \tag{4}$$

Recognizing that the sums on the RHS of (4) are the canonical sine and cosine multiple angle formulas, we can greatly simplify the equations. To obtain the RHS as a single sine term, we apply the identity $\cos\theta \pm \sin\theta = \sqrt{2}\sin(\frac{\pi}{4} \pm \theta)$, which affords

$$\begin{aligned} U_1: \quad ar^{2b-n} &= \cos(n\theta) - \sin(n\theta) = \sqrt{2}\sin(\frac{\pi}{4} - n\theta), \\ U_2: \quad ar^{2b-n} &= \cos(n\theta) + \sin(n\theta) = \sqrt{2}\sin(\frac{\pi}{4} + n\theta). \end{aligned} \tag{5}$$

Remembering that $a, b \in \mathbb{R}$, we are free to define $c \equiv 2b - n$ and $d \equiv \frac{\sqrt{2}}{a}$ to make the equations as "clean" as possible:

$$\begin{aligned} U_1: \quad r^c &= d\sin(\frac{\pi}{4} - n\theta), \\ U_2: \quad r^c &= d\sin(\frac{\pi}{4} + n\theta). \end{aligned} \tag{6}$$

In the above equations (6), if $c = 1$ then we recognize that U_1 and U_2 each describe a phase-shifted rose curve, which has well-known D_n symmetry about the origin when n is a natural number. However, changing the power of r does not affect the D_n symmetry because $r = \sqrt{x^2 + y^2}$ is invariant under all rotations about the origin. Therefore, U_1 and U_2 each exhibit D_n symmetry in the xy-plane regardless of the values of c and d, which completes the proof of Proposition 1.

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