A Simple Formula for Isogonal Conjugate Points via Blaschke Products

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1 Motivating Theorems

Theorem 1.1. *p and q are isogonal conjugates with respect to triangle ABC if and only if there exists an inscribed ellipse in ABC with p and q as its foci.*

Theorem 1.2 (Theorem 2.9 in Finding Ellipses (by Daepp, Gorkin, Shaffer, and Voss))**.** *Let B be a Blaschke product of degree 3 with zeros 0, a_1, and a_2. For $\lambda \in \mathbb{T}$, let z_1, z_2, and z_3 denote the three distinct solutions to $B(z) = \lambda$. Then the lines joining z_j and z_k, for $j \neq k$, are tangent to the ellipse given by*

$$|w - a_1| + |w - a_2| = |1 - \overline{a_1} a_2|$$

An easy consequence of theorem 1.1 and 1.2 is the next corollary.

Corollary 1.2.1 (See Corollary 4.4 in Finding Ellipses)**.** *Let E be an ellipse with foci a_1 and a_2. If E can be circumscribed by a triangle inscribed in $\mathbb{T}$, then E is the Blaschke 3-ellipse with foci at a_1 and a_2 with major axis of length $|1 - \overline{a_1} a_2|$. Moreover, by Thm 1.1, a_1 and a_2 are isogonal conjugates.*

Notation:
Now that the connection between Blaschke products and isogonal conjugates is clearly stated, we can introduce some tools and notation for exploring this connection:

Let a, b, c be the three vertices of a nondegenerate triangle inscribed in the unit circle ($\mathbb{T}$). Then let $p, q \in \mathbb{D}$ (the unit disk) be the two foci of an ellipse inscribed in triangle abc, where only p is known. We desire to leverage the connection of Blaschke products and ellipses and isogonal conjugates in order to find a formula for q (the isogonal conjugate of p).

Lemma 1.3 (See Lemma 3.4 in Finding Ellipses)**.** *For a degree-3 Blaschke product, there exists* $\lambda \in \mathbb{T}$ *such that* $B(a) = \lambda$*,* $B(b) = \lambda$*, and* $B(c) = \lambda$*.*

Lemma 1.4 (See Theorem 15.30 and Lemma 4.2 in Finding Ellipses)**.** *Using the degree-3 Blaschke product* $B(z)$ *with* λ *defined by Lemma 1.3 and zeros at* 0*,* p*, and* q*, we can define the following rational function which has nice properties:*

$$F(z) := \frac{B(z)}{z(B(z) - \lambda)} = \frac{m_1}{z-a} + \frac{m_2}{z-b} + \frac{m_3}{z-c}$$

These properties include:

$$m_1 + m_2 + m_3 = 1 \tag{1}$$

$$\frac{1}{m_j} = 1 + \frac{1-|p|^2}{|z_j - p|^2} + \frac{1-|q|^2}{|z_j - q|^2} \quad \textit{where } z_1 = a, z_2 = b, z_3 = c \tag{2}$$

$$\begin{aligned} \frac{m_1}{p-a} + \frac{m_2}{p-b} + \frac{m_3}{p-c} &= 0 \\ \frac{m_1}{q-a} + \frac{m_2}{q-b} + \frac{m_3}{q-c} &= 0 \end{aligned} \tag{3}$$

2 Solving for q using lemma 1.3

I initially tried approaching this problem using lemma 1.4 because it seemed to be the most promising approach. However, approaching the problem in this way was computationally intractable for the general case (i.e. I was only able to solve it in Mathematica for simple cases where the argument of p was the same as the argument of a, b, or c). Instead, the simplest approach of using lemma 1.3 was actually the best approach, which I will show below.

Using 1.3, we obtain 3 equations in terms of the 3 unknowns λ, q, and $\overline{q}$

$$\begin{aligned} a\left(\frac{a-p}{1-a\overline{p}}\right)\left(\frac{a-q}{1-a\overline{q}}\right) &= \lambda \\ b\left(\frac{b-p}{1-b\overline{p}}\right)\left(\frac{b-q}{1-b\overline{q}}\right) &= \lambda \\ c\left(\frac{c-p}{1-c\overline{p}}\right)\left(\frac{c-q}{1-c\overline{q}}\right) &= \lambda \end{aligned} \tag{4}$$

We then eliminate λ to get two equations in two unknowns (q and $\overline{q}$), and we cross multiply to clear denominators.

$$\begin{aligned} a(a-p)(a-q)(1-b\overline{p})(1-b\overline{q}) &= b(b-p)(b-q)(1-a\overline{p})(1-a\overline{q}) \\ b(b-p)(b-q)(1-c\overline{p})(1-c\overline{q}) &= c(c-p)(c-q)(1-b\overline{p})(1-b\overline{q}) \end{aligned} \tag{5}$$

We can now solve (5) for q and $\overline{q}$ in Mathematica and simplify the result to obtain a beautiful formula for q.

$$q = \frac{\overline{p}(ab+bc+ca-abc\overline{p}) - (a+b+c-p)}{\overline{p}p-1} \tag{6}$$

As an initial check, we can see that (6) is symmetric in the triangle vertices a, b, and c. To verify that this formula gives the correct results, I checked it against a different formula that I derived geometrically.

3 Geometric computation of the isogonal conjugate to check equation 6

Rather than constructing the isogonal conjugate using reflections, I made use of the following theorem.

Theorem 3.1. *p and q are isogonal conjugates with respect to triangle abc if and only if they have a common pedal circle with center at the midpoint of segment pq.*

Theorem 3.1 offers a straightforward way to construct the isogonal conjugate (q) of a point p inside triangle abc.

To summarize, we start with point p inside triangle abc. Then, we construct the pedal triangle of p and find the circumcircle of that pedal triangle. Then, construct a vector $\vec{v}$ from p to the center of the circumcircle (C). Because C is the midpoint of pq by Thm 3.1, we have that $q = p + 2\vec{v}$.

I am happy to write up more details of this computation if anyone wants to see it. However, since it is straightforward but computationally messy, I'll leave it out for now. Since I did the computation using vectors in $\mathbb{R}^2$, the result for $q = (q_1, q_2)$ is in terms of $p = (p_1, p_2)$, $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$. It is not a nice-looking formula:
Finally, we can plug in values and check that (6) gives results that are consistent with the geometrically derived formula above.

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IsogonalConjugateP[p1_, p2_, a1_, a2_, b1_, b2_, c1_, c2_] :=
  {((b2 (c1 - p1) + c2 p1 - c1 p2 + b1 (-c2 + p2))
       (a2^2 (b1 + c1 - p1) + b1 c1 p1 - b2 c2 p1 + b2 c1 p2 + b1 c2 p2 - a2 (c2 (b1 - p1) + b2 (c1 - p1) + (b1 + c1) p2)) +
     a1^2 (b1 (-c1 c2 + b2 (c1 - p1) + p1 p2) + (c2 - p2) (b2^2 + c1 p1 + c2 p2 - b2 (c2 + p2))) +
     a1 (b2^2 c1 c2 + 2 b2 c2 (-b2 + c2) p1 - c1 c2 p1^2 - 2 b2 c1 c2 p2 + (b2^2 + c1^2 - c2^2) p1 p2 + c1 c2 p2^2 + b1^2 (c1 c2 - p1 p2) -
        b1 b2 (c1^2 + c2^2 - p1^2 - 2 c2 p2 + p2^2) +
        a2 (b1^2 (-c1 + p1) + b2^2 (-c1 + p1) - p1 (c1^2 + c2^2 - c1 p1) + 2 b2 (c1 - p1) p2 + 2 c2 p1 p2 - c1 p2^2 +
           b1 (c1^2 + c2^2 - p1^2 - 2 c2 p2 + p2^2)))) /
   (p1 (-b2^2 c2 + b1 c2 (-b1 + p1) + b2 (c1^2 + c2^2 - c1 p1)) + c1 (b1^2 + b2^2 - b1 c1) p2 - b1 c2^2 p2 + (-b2 c1 + b1 c2) p2^2 +
     a1^2 (b2 (c1 - p1) + c2 p1 - c1 p2 + b1 (-c2 + p2)) + a2^2 (b2 (c1 - p1) + c2 p1 - c1 p2 + b1 (-c2 + p2)) +
     a2 (b1^2 (-c1 + p1) + b2^2 (-c1 + p1) - p1 (c1^2 + c2^2 - c1 p1) + c1 p2^2 + b1 (c1^2 + c2^2 - p1^2 - p2^2)) +
     a1 (-c2 p1^2 + b1^2 (c2 - p2) + b2^2 (c2 - p2) + c1^2 p2 + c2 (c2 - p2) p2 + b2 (-c1^2 - c2^2 + p1^2 + p2^2))),
  (a2^2 (-((b1 - c1) (b1 c1 + b2 c2 - (b1 + c1) p1 + p1^2)) + (b1 b2 - c1 c2 + (-b2 + c2) p1) p2) +
     ((-b2 c1 - b1 c2 + a1 (b2 + c2)) (a1 - p1) + (-((a1 - b1) (a1 - c1)) + b2 c2) p2) (b2 (c1 - p1) + c2 p1 - c1 p2 + b1 (-c2 + p2)) +
     a2 (-c1 c2 (b2^2 + p1^2) + (b2^2 + c1^2 - c2^2) p1 p2 + c1 c2 p2^2 - b1^2 (c1 (c2 - 2 p2) + p1 p2) +
        a1 (c2 (b1 - p1)^2 + b2^2 (c2 - p2) + (-b1^2 + c1^2 + c2^2 + 2 b1 p1 - 2 c1 p1) p2 - c2 p2^2 - b2 ((c1 - p1)^2 + (c2 - p2) (c2 + p2))) +
        b1 (2 c1 (c2 p1 - c1 p2) + b2 ((c1 - p1)^2 + (c2 - p2) (c2 + p2))))) /
   (p1 (-b2^2 c2 + b1 c2 (-b1 + p1) + b2 (c1^2 + c2^2 - c1 p1)) + c1 (b1^2 + b2^2 - b1 c1) p2 - b1 c2^2 p2 + (-b2 c1 + b1 c2) p2^2 +
     a1^2 (b2 (c1 - p1) + c2 p1 - c1 p2 + b1 (-c2 + p2)) + a2^2 (b2 (c1 - p1) + c2 p1 - c1 p2 + b1 (-c2 + p2)) +
     a2 (b1^2 (-c1 + p1) + b2^2 (-c1 + p1) - p1 (c1^2 + c2^2 - c1 p1) + c1 p2^2 + b1 (c1^2 + c2^2 - p1^2 - p2^2)) +
     a1 (-c2 p1^2 + b1^2 (c2 - p2) + b2^2 (c2 - p2) + c1^2 p2 + c2 (c2 - p2) p2 + b2 (-c1^2 - c2^2 + p1^2 + p2^2)))};
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The two formulas were consistent in all cases I tried except for when at least one of a, b, and c does not lie on the unit circle or they are all collinear.

It makes sense that (6) requires $a, b, c \in \mathbb{T}$, as this is a condition imposed by Theorem 1.2.

4 Further remarks

Here, we have successfully used the degree-3 Blaschke product and ellipse relationship to obtain a simple formula (6) for the isogonal conjugate of point p inside a triangle abc where each of the vertices a, b and c lie on the unit circle in the complex plane. This formula is much simpler than the standard formula for the isogonal conjugate, which involves trilinear coordinates.

Every triangle in $\mathbb{C}$ may be transformed via translation and scaling to a triangle inscribed in $\mathbb{T}$. As such, equation 6 may be readily applied to find the isogonal conjugate q of a point p with respect to any nondegenerate triangle.