Explicit Construction of Unique Tangent Polynomials

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1 Introduction

A simple question that I thought about recently was the following: when does a quartic polynomial have a line that is tangent to the curve at two distinct points. It is not hard to convince oneself that when such a line exists, it must be unique. What about higher order polynomials? When does an arbitrary polynomial have unique, lower order tangent polynomials? While these questions seem simple at first glance, for high degree polynomials they could have applications toward data fitting and model size reduction.

1.1 Statement of the problem

In this document we will consider polynomials of order n with real coefficients, which will be denoted $P_n(x)$. We know by the Fundamental Theorem of Algebra (FTA) that polynomials are defined uniquely (up to a multiplicative constant) by their roots counted with multiplicity. Obtaining a single point of tangency between $P_n(x)$ and $P_m(x)$ requires that $P_n(x) - P_m(x)$ has at least one root of multiplicity 2, so it requires two parameters to achieve. Therefore if $n > m$, the maximum number of points of tangency between $P_n(x)$ and $P_m(x)$ is $\lfloor \frac{n}{2} \rfloor$.

Counting parameters, if we want the *tangent polynomial* $P_m(x)$ to be uniquely defined by $P_n(x)$ and have the maximum number of possible tangency points, we need n even and $m = \frac{n}{2} - 1$. Then the problem of interest is to construct this unique "maximally tangent" polynomial $P_m(x)$ given $P_n(x)$, where n is even. Section 2 gives some easy examples of constructing maximally tangent polynomials, while Section 3 provides examples of using a more general method.

2 Polynomials of degree $2, 4, 6$

For an arbitrary parabola ($n = 2$), a unique maximally tangent polynomial of lesser degree will be a polynomial of degree 0. Hence we arrive at our first trivial result.

Theorem 2.0.1 (Degree 2)**.** Given a 2nd degree polynomial $P_2(x) = ax^2+bx+c$, there exists a unique 0th degree polynomial $P_0(x)$ that is tangent to $P_2(x)$ at a single point: $P_0(x) = c$.

Notice that Thm 2.0.1 is true for all 2nd degree polynomials. As we continue to go higher in degree, we will need to impose constraints on the polynomials $P_n(x)$ for which maximal tangency can be achieved.

For degree 4, we solve for $P_1(x)$ directly by forcing the tangency condition:

$$\begin{aligned} P_4(x) - P_1(x) &= x_3(x-x_1)^2(x-x_2)^2 \\ \implies P_1(x) &= P_4(x) - x_3(x-x_1)^2(x-x_2)^2 \end{aligned} \tag{1}$$

where x_1 and x_2 are the points of tangency.

Letting $P_4(x) = ax^4 + bx^3 + cx^2 + dx + e$ and requiring that $P_1(x)$ is degree 1, we expand the definition above and set the coefficients of the x^2, x^3 and x^4 terms to be 0. This allows us to solve for x_1, x_2, and x_3.

$$\begin{aligned} a - x_3 &= 0 \\ b + 2x_3(x_1+x_2) &= 0 \\ c - x_3(x_1^2 + 4x_1x_2 + x_2^2) &= 0 \\ \underset{\text{WLOG}}{\implies} &\begin{cases} x_3 = a \\ x_1 = \frac{-ab-\sqrt{3a^2b^2-8a^3c}}{4a^2} \\ x_2 = \frac{-ab+\sqrt{3a^2b^2-8a^3c}}{4a^2} \end{cases} \end{aligned} \tag{2}$$

Now, plugging x_1, x_2, and x_3 from (14) into (1) affords the desired expression for $P_1(x)$ after simplification. The tangency result is valid when the points of tangency x_1 and x_2 are real and distinct, which implies $3a^2b^2 - 8a^3c > 0$.

Theorem 2.0.2 (Degree 4)**.** Given a 4th degree polynomial $P_4(x) = ax^4+bx^3+cx^2+dx+e$, iff $3a^2b^2 - 8a^3c > 0$ there exists a unique line $P_1(x)$ which is tangent to $P_4(x)$ at exactly 2 points:

$$P_1(x) = x\left(\frac{b(b^2-4ac)}{8a^2} + d\right) + \frac{-b^4 + 8ab^2c - 16a^2c^2}{64a^3} + e$$

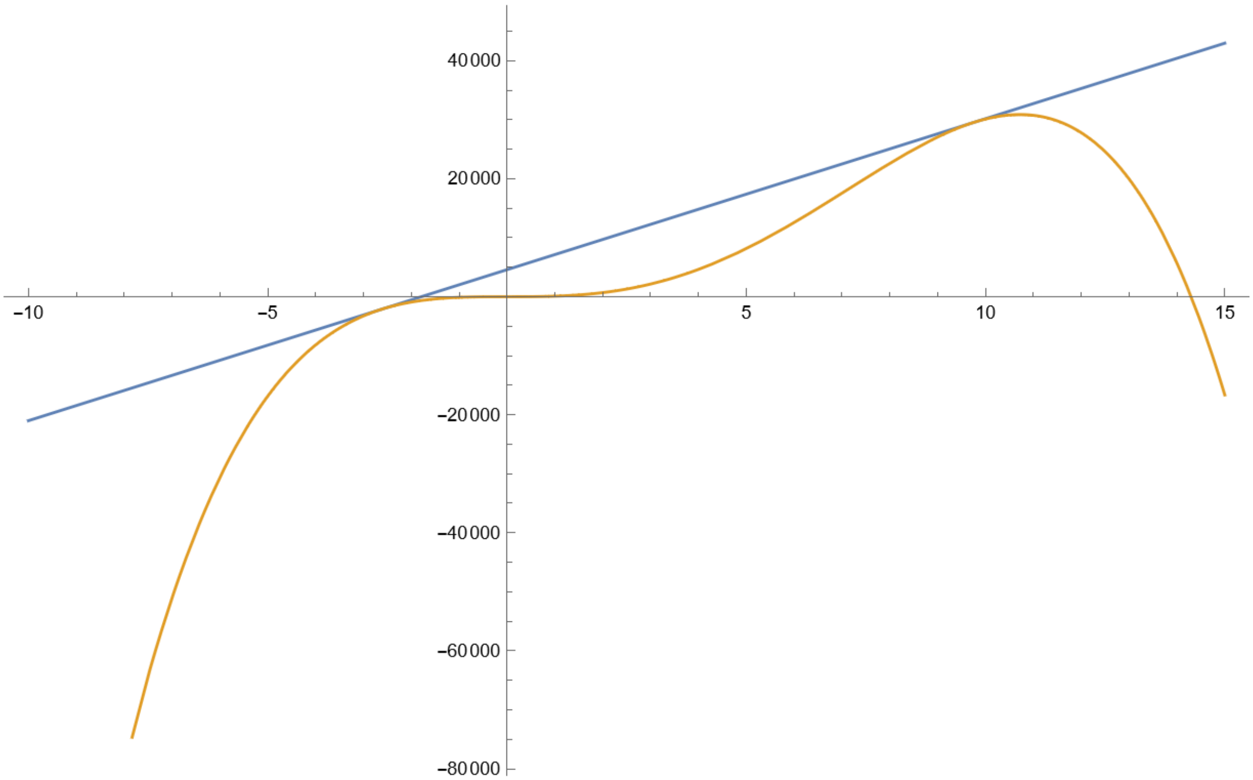

Figure 1: Example of $P_4(x) \implies P_1(x)$.

I computed the degree 6 case in the same way as degree 4; however, the calculations are more involved. Instead of writing the calculations all out, I'll show a better way to solve the problem in the next section, so here I will just state the degree 6 result.

Theorem 2.0.3 (Degree 6)**.** Given a 6th degree polynomial $P_6(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$, iff x_1, x_2, and x_3 are real then there exists a unique quadratic polynomial $P_2(x)$ which is tangent to $P_6(x)$ at exactly 3 points:

$$P_2(x) = x^2\left(\frac{-5b^4 + 24ab^2c - 16a^2c^2 - 32a^2bd}{64a^3} + e\right) + x\left(\frac{(b^2-4ac)(b^3-4abc+8a^2d)}{64a^4} + f\right) - \frac{(b^3-4abc+8a^2d)^2}{256a^5} + g$$

Let $X = 2^{1/3}(-5b^2 + 12ac)$ and $Y = 8(-5b^3 + 18abc - 27a^2d)$. Then

$$x_1 = \frac{1}{6a}\left(-b - \frac{X}{2^{2/3}(Y + \sqrt{X^3 + Y^2})^{1/3}} + \frac{(Y + \sqrt{X^3 + Y^2})^{1/3}}{2^{2/3}} \right)$$

$$x_2 = \frac{1}{6a}\left(-b + \frac{1 + i\sqrt{3}}{2} \frac{X}{2^{2/3}(Y + \sqrt{X^3 + Y^2})^{1/3}} - \frac{1 - i\sqrt{3}}{2} \frac{(Y + \sqrt{X^3 + Y^2})^{1/3}}{2^{2/3}} \right)$$

$$x_3 = \frac{1}{6a}\left(-b + \frac{1 - i\sqrt{3}}{2} \frac{X}{2^{2/3}(Y + \sqrt{X^3 + Y^2})^{1/3}} - \frac{1 + i\sqrt{3}}{2} \frac{(Y + \sqrt{X^3 + Y^2})^{1/3}}{2^{2/3}} \right)$$

These values are the roots of the polynomial $P_3(x) = 16a^3x^3 + 8a^2bx^2 + 2a(-b^2 + 4ac)x + b^3 - 4abc + 8a^2d$. To check that they are all real, it is sufficient to check that two of them are real.

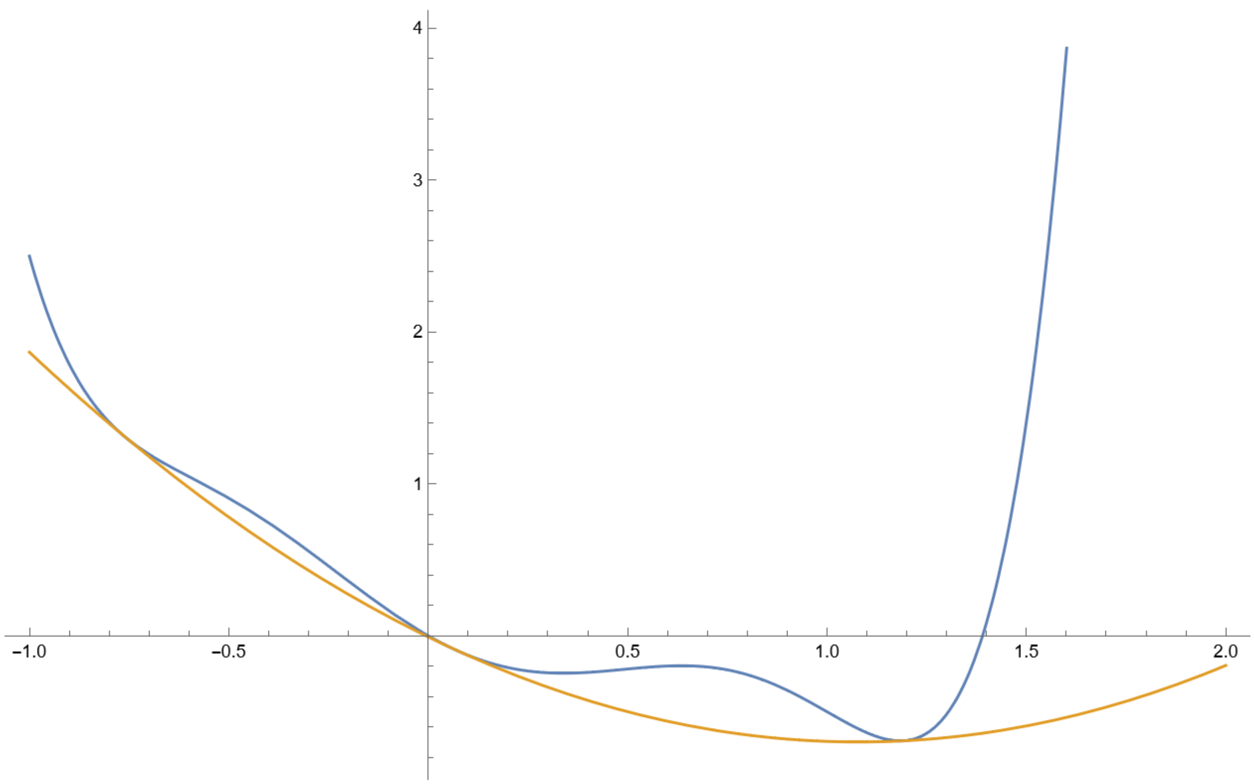

Figure 2: Example of $P_6(x) \implies P_2(x)$.

3 A more general method (reparameterization)

As demonstrated in the previous section, it is straightforward to solve the problem (stated in section 1.1) for polynomials of degree $2, 4$, and 6 using only FTA and the quadratic and cubic formulae. However, for degree 8 polynomials the expressions yielded by this method are too messy for Mathematica to simplify. Moreover, for polynomials of degree greater than 8 the method fails entirely because there are no formulae for the roots of general quintic and higher polynomials.

Here, I will present a method that can be used to find the unique, maximally tangent polynomial of polynomials of degree $8, 10$, and conjectured: arbitrary even degree. The main takeaway of this document is the following conjecture:

Conjecture 3.0.1 (Maximally Tangent Polynomial Construction)**.**

Given a polynomial $P_{2n}(x)$ with real coefficients, the maximally tangent polynomial $P_{n-1}(x)$ (if it exists) can be explicitly solved for via reparameterization of the zeros of $P_{2n}(x)-P_{n-1}(x)$ followed by forward substitution.

In the last two subsections we present the reparameterization method via example for the cases of degree 8 and degree 10 polynomials.

3.1 Degree 8

Considering the degree 8 case, we define $P_3(x)$ analogously to how $P_1(x)$ was defined in (1). Taking $P_8(x) = ax^8+bx^7+cx^6+dx^5+ex^4+fx^3+gx^2+hx+i$, where i represents a constant coefficient, *not* the imaginary unit, we get a formula for $P_3(x)$.

$$\begin{aligned} P_3(x) &= P_8(x) - x_5((x-x_1)(x-x_2)(x-x_3)(x-x_4))^2 \\ &= P_8(x) - x_5((x^2-(x_1+x_4)x+x_1x_4)(x^2-(x_2+x_3)x+x_2x_3))^2 \end{aligned} \tag{3}$$

The expansion of the RHS motivates the following change of variables.

$$\begin{aligned} y &= x_1 + x_4 \\ z &= x_1x_4 \\ u &= x_2 + x_3 \\ v &= x_2x_3 \end{aligned} \tag{4}$$

This change of variables will help us to better see the underlying structure of the coefficients of $P_3(x)$. Plugging (4) into the RHS of (3) and expanding it out, we can iteratively reparameterize the coefficients to make them much simpler.

Coefficients of $P_3(x)$:

$$\begin{aligned}
&x^0 : i - x_5(vz)^2 \to i - x_5 q^2 \text{ where } q := vz\\
&x^1 : h + 2x_5 vz(vy + uz) \to h + 2x_5 qr \text{ where } r := vy + uz\\
&x^2 : g - x_5((vy + uz)^2 + 2vz(v + uy + z)) \to g - x_5(r^2 + 2qs) \text{ where } s := v + uy + z\\
&x^3 : f + 2x_5(vz(u + y) + (vy + uz)(v + uy + z)) \to f + 2x_5(qt + rs) \text{ where } t := u + y
\end{aligned} \tag{5}$$

With this change of variables, we will rewrite the system to set the higher order coefficients of $P_3(x)$ to be 0.

$$\begin{cases}
x^8 : a - x_5 = 0\\
x^7 : b + 2x_5 t = 0\\
x^6 : c - x_5(t^2 + 2s) = 0\\
x^5 : d + 2x_5(ts + r) = 0\\
x^4 : e - x_5(s^2 + 2(tr + q)) = 0
\end{cases} \tag{6}$$

We have now converted the problem into a system that can be solved for x_5, t, s, r, and q using forward substitution!

$$\begin{aligned}
x_5 &= a\\
t &= \frac{-b}{2a}\\
s &= \frac{-b^2 + 4ac}{8a^2}\\
r &= \frac{-b^3 + 4abc - 8a^2 d}{16a^3}\\
q &= \frac{-5b^4 + 24ab^2c - 16a^2c^2 - 32a^2bd + 64a^3e}{128a^4}
\end{aligned} \tag{7}$$

Plugging (7) into (5) and simplifying just a bit, we get the equation for $P_3(x)$.

$$\begin{aligned}P_3(x) &= x^3\left(f + \frac{7b^5 - 40ab^3c + 48a^2b^2d - 64a^3cd + 16a^2b(3c^2 - 4ae)}{128a^4}\right)\\ &+ x^2\left(g - \frac{7b^6 - 60ab^4c + 64a^2b^3d - 256a^3bcd + 16a^2b^2(9c^2 - 4ae) + 64a^3(-c^3 + 2ad^2 + 4ace)}{512a^5}\right)\\ &+ x\left(h + \frac{(b^3 - 4abc + 8a^2d)(5b^4 - 24ab^2c + 32a^2bd + 16a^2(c^2 - 4ae))}{1024a^6}\right)\\ &+ i - \frac{(5b^4 - 24ab^2c + 32a^2bd + 16a^2(c^2 - 4ae))^2}{16384a^7}\end{aligned} \tag{8}$$

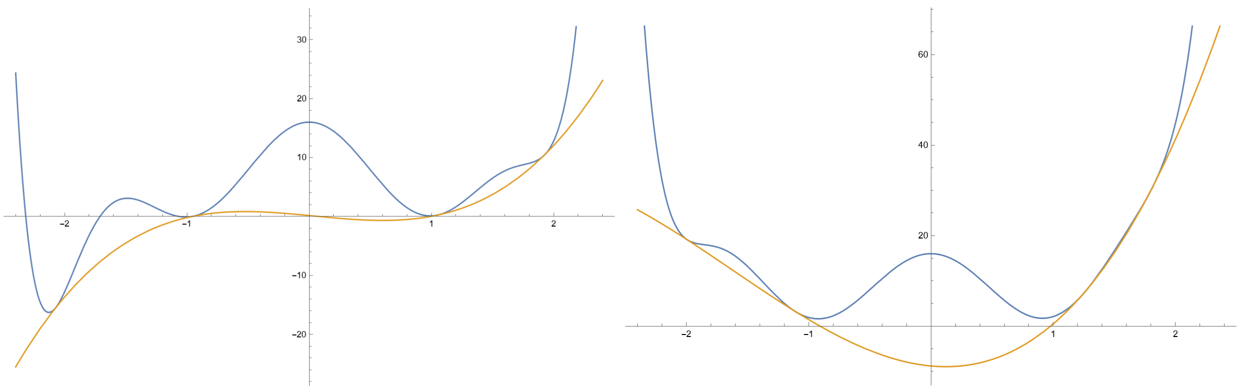

Figure 3: Two examples of $P_8(x) \implies P_3(x)$

A pertinent question to ask is when the equation for $P_3(x)$ (8) satisfies the maximal tangency condition with respect to $P_8(x)$. While we *can* develop a condition for the degree 8 case, the method does not generalize to degree 10 and higher. Nevertheless, it is included in the next section for completeness.

3.2 Maximal tangency condition for degree 8

By construction, $P_3(x)$ is tangent to $P_8(x)$ at 4 points in the real plane exactly when x_1, x_2, x_3, and x_4 are all real. Either we can calculate the roots x_i using the reparameterization variables or we can calculate the roots directly from $P_8(x) - P_3(x)$. First, let's try the latter. By construction,

$$\begin{aligned}
&\sqrt{x_5}(x-x_1)(x-x_2)(x-x_3)(x-x_4) = \sqrt{P_8(x)-P_3(x)} \\
&= \frac{1}{128a^{7/2}}\Big(-5b^4 + 8ab^3x + 8ab^2(3c-2ax^2) + 32a^2b(-d-cx+2ax^3) \\
&\qquad +16a^2(-c^2+4acx^2+4a(e+dx+2ax^4))\Big)
\end{aligned} \tag{9}$$

Setting the LHS of (9) equal to 0, we can use the quartic formula to obtain the values of the roots.

$$\begin{aligned}
W &= -35b^4 + 160ab^2c - 80a^2c^2 - 240a^2bd + 384a^3e \\
V &= -245b^6 + 1680ab^4c - 3408a^2b^2c^2 + 2432a^3c^3 - 1008a^2b^3d \\
&\quad + 576a^3bcd + 3456a^4d^2 + 4032a^3b^2e - 9216a^4c \\
X &= \frac{1}{3}\Big(\frac{2^{4/3}W}{(V+\sqrt{-4W^3+V^2})^{1/3}} + 2^{2/3}(V+\sqrt{-4W^3+V^2})^{1/3} \Big) \\
Y &= \frac{7b^2-16ac}{3} \\
Z &= 2(-7b^3+24abc-32a^2d) \\
x_1 &= \frac{1}{8a}\Big(-b - \sqrt{Y+X} - \sqrt{2Y - X - \frac{Z}{\sqrt{Y+X}}} \Big) \\
x_2 &= \frac{1}{8a}\Big(-b - \sqrt{Y+X} + \sqrt{2Y - X - \frac{Z}{\sqrt{Y+X}}} \Big) \\
x_3 &= \frac{1}{8a}\Big(-b + \sqrt{Y+X} - \sqrt{2Y - X + \frac{Z}{\sqrt{Y+X}}} \Big) \\
x_4 &= \frac{1}{8a}\Big(-b + \sqrt{Y+X} + \sqrt{2Y - X + \frac{Z}{\sqrt{Y+X}}} \Big)
\end{aligned} \tag{10}$$

Checking that three of these roots are real is sufficient for showing that the tangency condition holds. However, these expressions are rather messy. We can obtain a better check using the parameterized system.

$$\frac{-b}{2a} = u + y \tag{11}$$
$$\frac{-b^2 + 4ac}{8a^2} = v + uy + z \tag{12}$$
$$\frac{-b^3 + 4abc - 8a^2d}{16a^3} = vy + uz \tag{13}$$
$$\frac{-5b^4 + 24ab^2c - 16a^2c^2 - 32a^2bd + 64a^3e}{128a^4} = vz \tag{14}$$

Solving by forward substitution, we get

$$\begin{aligned}
&(11) \implies y = \frac{-b}{2a} - u \\
&(12) \implies v = \frac{-b^2 + 4ac}{8a^2} + u\left(\frac{b}{2a} + u\right) - z \\
&(13) \implies z = \frac{1}{\frac{b}{2a} + 2u}\left(\frac{-b^3 + 4abc - 8a^2d}{16a^3} - \left(\frac{b}{2a} + u\right)\left(\frac{b^2 - 4ac}{8a^2} - u\left(\frac{b}{2a} + u\right)\right)\right) \\
&(14) \implies -(b + 4au)^2(-5b^4 + 24ab^2c - 16a^2c^2 - 32a^2bd + 64a^3e) \\
&\quad + 4(-b^3 + ab^2u + 4ab(c + 2au^2) + 4a^2(-d + cu + 2au^3)) \\
&\quad \cdot (b^3 - 2ab^2u + 4ab(-c + 2au^2) + 8a^2(d + cu + 2au^3)) = 0
\end{aligned} \tag{15}$$

Finally, solving (15) for u in Mathematica affords 6 possible values of u. That u has 6 possible values arises from the symmetry of the 4 roots: $u \in \{x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4\}$.

$$
\begin{aligned}
W &= 35b^4 - 160ab^2c + 240a^2bd + 16a^2(5c^2 - 24ae)\\
V &= -245b^6 + 1680ab^4c - 48a^2b^2(71c^2 + 21bd) + 64a^3(38c^3 + 9bcd + 63b^2e) + 1152a^4(3d^2 - 8ce)\\
X &= (V + \sqrt{4W^3 + V^2})^{1/3}\\
Y &= 7b^2 - 16ac\\
u_1 &= \frac{1}{4a}\left(-b - \sqrt{\frac{1}{3}\left(Y - \frac{2^{4/3}W}{X} + 2^{2/3}X\right)}\right)\\
u_2 &= \frac{1}{4a}\left(-b + \sqrt{\frac{1}{3}\left(Y - \frac{2^{4/3}W}{X} + 2^{2/3}X\right)}\right)\\
u_3 &= \frac{1}{4a}\left(-b - \sqrt{\frac{1}{3}\left(Y + \frac{1+i\sqrt{3}}{2}\left(\frac{2^{4/3}W}{X}\right) - \frac{1-i\sqrt{3}}{2}(2^{2/3}X)\right)}\right)\\
u_4 &= \frac{1}{4a}\left(-b + \sqrt{\frac{1}{3}\left(Y + \frac{1+i\sqrt{3}}{2}\left(\frac{2^{4/3}W}{X}\right) - \frac{1-i\sqrt{3}}{2}(2^{2/3}X)\right)}\right)\\
u_5 &= \frac{1}{4a}\left(-b - \sqrt{\frac{1}{3}\left(Y + \frac{1-i\sqrt{3}}{2}\left(\frac{2^{4/3}W}{X}\right) - \frac{1+i\sqrt{3}}{2}(2^{2/3}X)\right)}\right)\\
u_6 &= \frac{1}{4a}\left(-b + \sqrt{\frac{1}{3}\left(Y + \frac{1-i\sqrt{3}}{2}\left(\frac{2^{4/3}W}{X}\right) - \frac{1+i\sqrt{3}}{2}(2^{2/3}X)\right)}\right)
\end{aligned} \tag{16}
$$

Comparing (16) with (10), we see that (16) is much easier to evaluate. Moreover, it is again sufficient to check that 3 of the u_i above are real in order to show that the 4-point tangency condition holds.

3.3 Degree 10

Finally, we move on to the degree 10 case. Here, our original method from Section 2 could not be used because solving for x_i requires finding the roots of an arbitrary quintic, which is impossible. However, our reparameterization method still works because it avoids solving for the roots. Moreover, the parameterization here is nearly identical to the degree 8 case, allowing us to only perform a couple additional calculations.

Define $P_{10}(x) = ax^{10} + bx^9 + cx^8 + dx^7 + ex^6 + fx^5 + gx^4 + hx^3 + ix^2 + jx + k$. Let us apply the same change of variables we applied previously.

$$\begin{aligned} y &= x_1 + x_4 \\ z &= x_1 x_4 \\ u &= x_2 + x_3 \\ v &= x_2 x_3 \end{aligned} \tag{17}$$

This change of variables will help us to better see the underlying structure of the coefficients of $P_4(x)$. Plugging (17) into $P_4(x) = P_{10}(x) - x_6((x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5))^2$ and expanding it out, we can iteratively reparameterize the coefficients of $P_4(x)$ to make them much simpler. Additionally, we know from the highest order term that $x_6 = a$, so we will apply this result everywhere below.

Coefficients of $P_4(x)$:

$$\begin{aligned}
&x^0 : k - a(vzx_5)^2 \to k - ap^2 \text{ where } p := vzx_5 \\
&x^1 : j + 2ap(vyx_5 + vz + uzx_5) \to j + 2apq \text{ where } q := vyx_5 + vz + uzx_5 \\
&x^2 : i - a(q^2 + 2p(vx_5 + vy + uyx_5 + uz + zx_5)) \to i - a(q^2 + 2pr),\ r := vx_5 + vy + uyx_5 + uz + zx_5 \\
&x^3 : h + 2a(p(v + ux_5 + uy + yx_5 + z) + qr) \to h + 2a(ps + qr) \text{ where } s := v + ux_5 + uy + yx_5 + z \\
&x^4 : g - a(r^2 + 2(p(u + x_5 + y) + qs)) \to g - a(r^2 + 2(pt + qs)) \text{ where } t := u + x_5 + y
\end{aligned} \tag{18}$$

With this change of variables, we will rewrite the system to set the higher order coefficients to be 0.

$$\begin{cases} x^{10} : a - x_6 = 0 \\ x^9 : b + 2x_6 t = 0 \\ x^8 : c - x_6(t^2 + 2s) = 0 \\ x^7 : d + 2x_6(ts + r) = 0 \\ x^6 : e - x_6(s^2 + 2(tr + q)) = 0 \\ x^5 : f + 2x_6(rs + tq + p) = 0 \end{cases} \tag{19}$$

Since $x_6 = a$, notice that with this parameterization the coefficients of x_0 through x_3 are identical to the coefficients we had in the degree 8 case. Saving time by using the q, r, s, t we solved for in the degree 8 case, we only need to solve the equation for the coefficient of x^5:

$$p = \frac{-7b^5 + 40ab^3c - 48a^2bc^2 - 48a^2b^2d + 64a^3cd + 64a^3be - 128a^4f}{256a^5} \tag{20}$$

Having done this, we plug all the variables back into (18) and simplify to obtain the final result. To save space here, I will just show the formula in terms of the variables obtained by solving (19).

$$P_4(x) = x^4(g-a(r^2+2(pt+qs)))+x^3(h+2a(ps+qr))+x^2(i-a(q^2+2pr))+x(j+2apq)+k-ap^2 \tag{21}$$

When is $P_4(x)$ tangent to $P_{10}(x)$ at 5 points? Since we cannot directly solve for the solutions x_1, x_2, x_3, x_4, and x_5 to $P_{10}(x) - P_4(x) = 0$ in Mathematica, this question requires further investigation.

In the future, it is desirable to state the reparameterization method more generally and then use it to prove Conjecture 3.0.1.

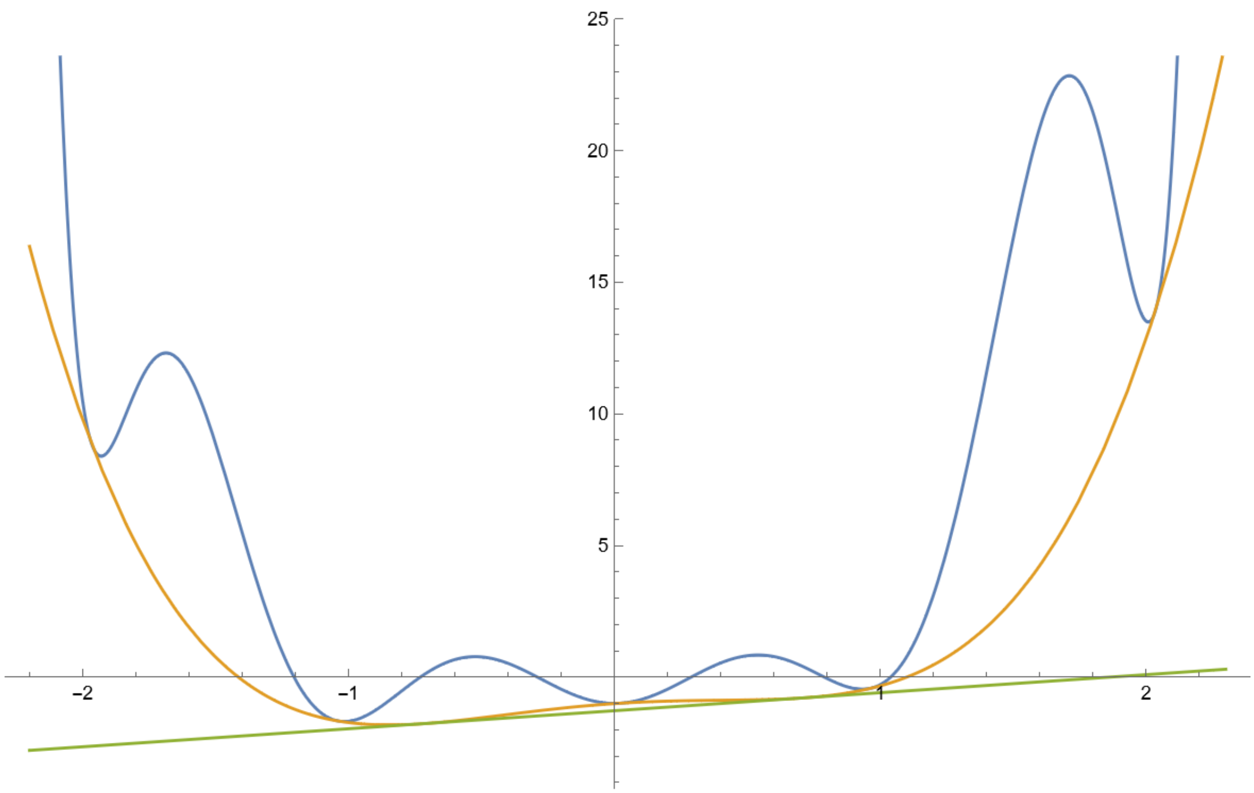

Figure 4: Example of $P_{10}(x) \implies P_4(x) \implies P_1(x)$.