Partitioning Divergent Dirichlet Series into Sequences of Convergent Subseries

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1 Introduction and Motivation

We know from real analysis that although the terms of the harmonic series approach 0 in the infinite limit, the full series diverges.

But what if instead of looking at the full series, we looked at the convergence of a sequence of subseries? As an arbitrary example, consider the sequence of subseries where each subseries has 3 terms:

$$S = \ 1 + \frac{1}{2} + \frac{1}{3}, \ \ \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \ \ \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \ \ \ldots, \ \ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}, \ \ \ldots \tag{1}$$

We see that the above sequence of partial harmonic sums converges to 0 in the infinite limit. But if we took more terms, could we get convergence to a finite, positive value? Clearly, if we take any fixed number of terms, like the 3 that we took above, the sequence will converge to 0. But what if we compensate for the decreasing size of each term in the partial sum by increasing the number of terms in each successive partial sum?

2 Ground Rules

So far, we have decided that the sequence of partial sums will include only *consecutive terms*. But we still need to decide the index where each successive partial sum (term of the sequence) will begin. There are two obvious ways we could choose to do this, illustrated below using examples where the number of terms in each successive partial sum doubles.

Index Method 1:

Each partial sum S_n starts with the nth term of the harmonic series.

$$S = 1, \quad \frac{1}{2}+\frac{1}{3}, \quad \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}, \quad \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}, \quad \ldots \tag{2}$$

Index Method 2:

Each partial sum S_n starts with the kth term of the harmonic series, where k is one more than the index of the term where the previous partial sum left off.

$$S = 1, \quad \frac{1}{2}+\frac{1}{3}, \quad \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}, \quad \frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}, \quad \ldots \tag{3}$$

While I have worked with both methods in this project so far, in this presentation we will focus primarily on Index Method 2, as it affords more interesting and useful results.

The method matters! It is worth noting that the sequence (3) converges to a value less than 1 while the sequence (2) diverges.

3 Generalization to Dirichlet Series

Harmonic series is just a special case ($s = 1$) of the simplest type of Dirichlet series (which provides the classical definition for the Riemann zeta function when $\mathrm{Re}(s) > 1$):

$$\sum_{j=1}^{\infty} \frac{1}{j^s} \tag{4}$$

Now, we ask the same question about the above series that we did about the Harmonic series.

Question 1: *Can we construct a sequence of partial sums of consecutive terms of* (21) *that converges to a finite, positive value in the infinite limit?*

Because of its importance, Index Method 2 shown on the last slide will be formalized as the "Conservation of Terms" constraint:

Conservation of Terms (COT):

$$\boxed{\begin{aligned} \sum_{k=1}^{\infty} \sum_{j=g_s(k)}^{g_s(k+1)-1} \frac{1}{j^s} = \sum_{j=1}^{\infty} \frac{1}{j^s} \\ g_s : \mathbb{R} \to \mathbb{R} \text{ such that } g_s(1) = 1 \text{ and } g_s(k+1) \geq g_s(k) + 1 \end{aligned}}$$

Together with the fact that terms of the partial sums are consecutive, the COT constraint is what endows this work with a practical application, which I will discuss toward the end of the talk.

4 Generalization to Dirichlet Series cont.

Question 1: *Can we construct a sequence of partial sums of consecutive terms of* (21) *that converges to a finite, positive value in the infinite limit?*

For which $s \in \mathbb{R}$ is Question 1 interesting?

Case 1: For $s \in (1, \infty)$, the series from $j = 1$ to ∞ is convergent. As a result, we have the following limit for all $g_s(k)$:

$$\lim_{k\to\infty} \sum_{j=g_s(k)}^{g_s(k+1)-1} \frac{1}{j^s} = 0$$

Case 2: For $s \in (-\infty, 0)$, the series from $j = 1$ to ∞ is divergent and we have the following limit for all $g_s(k)$:

$$\lim_{k\to\infty} \sum_{j=g_s(k)}^{g_s(k+1)-1} \frac{1}{j^s} = \infty$$

Case 3: For $s \in [0, 1]$, we have to be more careful. While the full series from $j = 1$ to ∞ is divergent, we can get different results from the limit below depending on the $g_s(k)$ that we choose:

$$\lim_{k\to\infty} \sum_{j=g_s(k)}^{g_s(k+1)-1} \frac{1}{j^s} = \begin{cases} 0, s \neq 0 \\ c > 0 \\ \infty \end{cases}$$

As a result, Question 1 is only interesting for Case 3. Answering it for this case will compose the bulk of the presentation.

5 Solution for $s \in (0,1)$

For now, I would like to focus on $s \in (0,1)$ (i.e. just the interior of the range $s \in [0,1]$).

Under the COT constraint, Question 1 asks us to find a $g_s(k)$ that satisfies the following:

$$\lim_{k\to\infty} \sum_{j=g_s(k)}^{g_s(k+1)-1} \frac{1}{j^s} = c \in \mathbb{R}^+ \tag{5}$$

Calculating $g_s(k)$ is a little tricky, so we will use a guess-and-check approach.

Recall the COT constraint:

$$\sum_{k=1}^{\infty} \sum_{j=g_s(k)}^{g_s(k+1)-1} \frac{1}{j^s} = \sum_{j=1}^{\infty} \frac{1}{j^s}$$
$$g_s : \mathbb{R} \to \mathbb{R} \text{ such that } g_s(1) = 1 \text{ and } g_s(k+1) \geq g_s(k) + 1$$

If we introduce the difference function $\Delta_s(k) = g_s(k+1) - g_s(k)$ to denote the number of terms in each partial sum, we can rewrite the COT constraint as a recurrence relation:

$$\begin{cases} g_s(1) = 1 \\ g_s(k+1) = g_s(k) + \Delta_s(k), \;\; \Delta_s(k) > 0 \;\; \forall \;\; k \in \mathbb{N}, s \in (0,1) \end{cases} \tag{6}$$

We now need an ansatz for $\Delta_s(k)$ in order to solve the recurrence relation for $g_s(k)$. I claim that the following ansatz is correct:

$$\Delta_s(k) = b\,k^c, \quad b \in \mathbb{R}^+, \;\; c = ? \text{ (expect } s \text{ dependence)} \tag{7}$$

To show that this is correct, we can solve the recurrence relation and show that the sum (5) converges to a finite, nonzero value in the infinite limit. Plugging **(7)** into **(6)**, we can solve the inhomogeneous recurrence using the RSolve method in Mathematica:

$$g_s(k) = 1 + b\zeta(-c,1) - b\zeta(-c,k) \tag{8}$$

With $g_s(k)$ in hand, we can now find the convergence value of (5). For clarity, let's assign p and q to represent the upper and lower bounds of the sum, respectively.

$$\begin{aligned} p \equiv \lfloor g_s(k) \rceil &= \lfloor 1 + b\zeta(-c,1) - b\zeta(-c,k) \rceil \\ q \equiv \lfloor g_s(k) - 1 \rceil + \lfloor \Delta_s(k) \rceil &= \lfloor b\zeta(-c,1) - b\zeta(-c,k) \rceil + \lfloor \Delta_s(k) \rceil \end{aligned} \tag{9}$$

We take the nearest integer function $\lfloor \cdot \rceil$ of the bounds so that the sum makes sense. Note that the choice to round $\Delta_s(k)$ as a separate term ensures that the sum from $j = p$ to q will contain $q - p + 1 = \lfloor \Delta_s(k) \rceil$ terms. Then, applying this to (5), we can factor out the arithmetic mean:

$$S_k = \sum_{j=p}^{q} \frac{1}{j^s} = \lfloor \Delta_s(k) \rceil \underbrace{\left[\frac{1}{\lfloor \Delta_s(k) \rceil} \sum_{j=p}^{q} \frac{1}{j^s} \right]}_{\text{arithmetic mean}} \tag{10}$$

We know by the AM-GM inequality that the arithmetic mean of a set of numbers is equal to the geometric mean of that set exactly when all the numbers in the set are equal. If we take the infinite limit ($k \to \infty$) of **(10)**, all terms of the sum converge to 0 (need uniform convergence?), so we can substitute the arithmetic mean for the geometric mean.

$$S_\infty = \lim_{k\to\infty} \lfloor \Delta_s(k) \rceil \underbrace{\left[\frac{1}{\lfloor \Delta_s(k) \rceil} \sum_{j=p}^{q} \frac{1}{j^s} \right]}_{\text{arithmetic mean}} = \lim_{k\to\infty} \lfloor \Delta_s(k) \rceil \underbrace{\left[\prod_{j=p}^{q} \frac{1}{j^s} \right]^{\frac{1}{\lfloor \Delta_s(k) \rceil}}}_{\text{geometric mean}} \tag{11}$$

Since the geometric mean is a product, we can rewrite it in the following way:

$$S_\infty = \lim_{k\to\infty} \lfloor \Delta_s(k) \rceil \left[\frac{1}{p} \cdot \frac{1}{p+1} \cdots \frac{1}{q-1} \cdot \frac{1}{q} \right]^{\frac{s}{\lfloor \Delta_s(k) \rceil}} = \lim_{k\to\infty} \lfloor \Delta_s(k) \rceil \left[\frac{(p-1)!}{q!} \right]^{\frac{s}{\lfloor \Delta_s(k) \rceil}} \tag{12}$$

We now use Stirling's approximation: $n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \big(1 + \mathcal{O}(\frac{1}{n})\big)$. It can be readily verified that $\lim_{k\to\infty} p \to \infty$ and $\lim_{k\to\infty} q \to \infty$ for all $s \in (0,1)$. As a result, the $\mathcal{O}(\frac{1}{n})$ term of Stirling's approximation goes to 0, so it can be omitted from the calculation. Plugging in $p - 1 = q - \lfloor \Delta_s(k) \rceil$ and applying Stirling's approximation:

$$S_\infty = \lim_{k\to\infty} \lfloor \Delta_s(k) \rceil \left[\frac{\sqrt{2\pi}(q - \lfloor \Delta_s(k) \rceil)^{q - \lfloor \Delta_s(k) \rceil + \frac{1}{2}} e^{-(q - \lfloor \Delta_s(k) \rceil)}}{\sqrt{2\pi}(q)^{q+\frac{1}{2}} e^{-q}} \right]^{\frac{s}{\lfloor \Delta_s(k) \rceil}} \tag{13}$$

Cancelling out the $\sqrt{2\pi}$ and e^{-q} terms from the numerator and denominator, we can group terms with like exponents and simplify:

$$S_\infty = \lim_{k\to\infty} \lfloor \Delta_s(k) \rceil \left[\left(1 - \frac{\lfloor \Delta_s(k) \rceil}{q}\right)^{q} \cdot \left(1 - \frac{\lfloor \Delta_s(k) \rceil}{q}\right)^{\frac{1}{2}} \cdot (q - \lfloor \Delta_s(k) \rceil)^{-\lfloor \Delta_s(k) \rceil} \cdot e^{\lfloor \Delta_s(k) \rceil} \right]^{\frac{s}{\lfloor \Delta_s(k) \rceil}} \tag{14}$$

In the limit as $k \to \infty$, we have that $q \to \infty$, $\Delta_s(k) \to \infty$, and $\frac{\Delta_s(k)}{q} \to 0$. Resultantly, since $\lim_{k\to\infty} \frac{\lfloor \Delta_s(k) \rceil}{\Delta_s(k)} = 1$, we can eliminate the nearest integer function from all terms in the above expression. We now turn our attention to the $(q - \Delta_s(k))^{-\Delta_s(k)}$ term in **(14)**.

The Hurwitz Zeta function has a well-known relationship with the Bernoulli polynomials: $\zeta(-c, k) = \frac{-B_{c+1}(k)}{c+1}$. Instead of using the full formula for the Bernoulli polynomials generalized to non-integer values, it is sufficient to use that the leading term of $B_n(x)$ is $B_0 x^n = x^n$. From this, we obtain the following:

$$q - \Delta_s(k) = b\,\zeta(-c, 1) - b\,\zeta(-c, k) = \frac{b\,k^{c+1}}{c+1} - \mathcal{O}(k^c) \tag{15}$$

Plugging (15) into (14) along with $\Delta_s(k) = b\,k^c$, we are finally ready to evaluate the limit.

$$\begin{aligned} S_\infty &= \lim_{k\to\infty} b\,k^c \left[\cancelto{e^{-b\,k^c}}{\left(1 - \frac{b\,k^c}{q}\right)^q} \cdot \cancelto{1}{\left(1 - \frac{b\,k^c}{q}\right)^{\frac{1}{2}}} \cdot \left(\frac{b\,k^{c+1}}{c+1} - \mathcal{O}(k^c)\right)^{-b\,k^c} \cdot e^{b\,k^c} \right]^{\frac{s}{b\,k^c}} \\ &= \lim_{k\to\infty} b\,k^c \left[\frac{c+1}{b\,k^{c+1} - \mathcal{O}(k^c)}\right]^s = \lim_{k\to\infty} \frac{b\,k^c (c+1)^s}{b^s\,k^{(c+1)s} - \mathcal{O}\big(k^{(c+1)s-1}\big)} \\ &= \lim_{k\to\infty} \frac{b\,(c+1)^s}{b^s\,k^{(c+1)s-c} - \mathcal{O}\big(k^{(c+1)s-c-1}\big)} \end{aligned} \tag{16}$$

From (16) we find that the limit converges to a finite, nonzero value if the leading power of k in the denominator is 0. Setting this constraint, we obtain c in terms of s:

$$\text{finite, nonzero convergence} \implies (c+1)s - c = 0 \implies c = \frac{s}{1-s} \tag{17}$$

Then if we desire finite, nonzero convergence, we must plug the value $c = \frac{s}{1-s}$ into the RHS of (16):

$$S_\infty = \lim_{k\to\infty} \frac{b\left(\frac{1}{1-s}\right)^s}{b^s - \mathcal{O}(\frac{1}{k})} = \boxed{b^{1-s}\left(\frac{1}{1-s}\right)^s} \tag{18}$$

Since the convergence value $b^{1-s}\left(\frac{1}{1-s}\right)^s$ is finite and nonzero for all $s \in (0,1)$, $b \in \mathbb{R}^+$, this proves that the ansatz $\Delta_s(k) = b\,k^{\frac{s}{1-s}}$ was a correct guess (up to the first order term) for solving the recursion. Hence, plugging in $c = \frac{s}{1-s}$, we have finally obtained $g_s(k)$:

$$g_s(k) = 1 + b\zeta\left(\frac{-s}{1-s}, 1\right) - b\zeta\left(\frac{-s}{1-s}, k\right) \tag{19}$$

Using this, we have answered Question 1 for $s \in (0,1)$:

$$\boxed{\begin{gathered} S_k = \sum_{j=p}^{q} \frac{1}{j^s} \quad \longrightarrow \quad b^{1-s}\left(\frac{1}{1-s}\right)^s \\[1em] p \equiv 1 + b\zeta\left(\frac{-s}{1-s}, 1\right) - b\zeta\left(\frac{-s}{1-s}, k\right) \\ q \equiv b\zeta\left(\frac{-s}{1-s}, 1\right) - b\zeta\left(\frac{-s}{1-s}, k+1\right) \end{gathered}}$$

6 Applications Part I: Approximating large Dirichlet sums for $s \in (0,1)$

An immediate application that follows from the convergence results is the ability to approximate very large sums to high accuracy.

Example 6.1:

Mathematica is able to numerically evaluate the following sum:

$$\sum_{j=1}^{10^{100}} \left(\frac{1}{j}\right)^{\frac{1}{3}} = 6.962383250419197 \cdot 10^{66}$$

However, applying $q \equiv b\zeta(-\frac{s}{1-s}, 1) - b\zeta(-\frac{s}{1-s}, k+1)$, and taking $b = 1$, $s = \frac{1}{3}$, we can instead solve the following equation and take the largest solution:

$$\zeta\Big(-\frac{1}{2}, 1\Big) - \zeta\Big(-\frac{1}{2}, k+1\Big) = 10^{100} \implies k_{max} = 6.082201995573401 \cdot 10^{66} \tag{20}$$

Then the sum will be approximately equal to k_{max} multiplied by the convergence value: $\left(\frac{1}{1-s}\right)^s = \left(\frac{3}{2}\right)^{\frac{1}{3}}$.

$$\sum_{j=1}^{10^{100}} \left(\frac{1}{j}\right)^{\frac{1}{3}} \approx \left(\frac{3}{2}\right)^{\frac{1}{3}} \cdot k_{max} = 6.962383250419169 \cdot 10^{66}$$

The error between the true result and our approximation is $4.1 \cdot 10^{-13}$ %. We can quickly use this technique to compute very large sums that Mathematica cannot handle, with the knowledge that the error decreases as the upper bound of the sum increases.

The next example shows an easy way to further improve the accuracy of the approximation.

Example 6.2:

Suppose we want to approximate the value of S_{20}, given that we know the sum of S_9.

$$S_{20} = \sum_{j=1}^{10^{20}} \left(\frac{1}{j}\right)^{\frac{e}{\pi}} = \ ?, \quad S_9 = \sum_{j=1}^{10^9} \left(\frac{1}{j}\right)^{\frac{e}{\pi}} = 114.25770004704884$$

To begin, we solve two equations involving q (taking $b = 1$, $s = \frac{e}{\pi}$):

$$\zeta\Big(\frac{e}{e-\pi}, 1\Big) - \zeta\Big(\frac{e}{e-\pi}, k+1\Big) = 10^{20} \implies k_{20} = 648.3944757727119$$
$$\zeta\Big(\frac{e}{e-\pi}, 1\Big) - \zeta\Big(\frac{e}{e-\pi}, k+1\Big) = 10^{9} \implies k_{9} = 20.891619830301128$$

Now, we can subtract the two values of k to add the tail approximation onto S_9:

$$S_{20} \approx S_9 + (k_{20} - k_9)\left(\frac{1}{1-s}\right)^s = 3669.038413650589$$

In this case, we can calculate that $S_{20} = 3669.10695865657$, thus the error in our approximation is 0.0019%. If we had approximated the entire sum S_{20}, the error would have been 0.11%.

7 Solutions for the boundary values $s = 0, 1$

$$\sum_{j=1}^{\infty} \frac{1}{j^s} \tag{21}$$

Question 1: *Can we construct a sequence of partial sums of consecutive terms of* (21) *that converges to a finite, positive value in the infinite limit?*

Until this point, we have been focusing entirely on $s \in (0, 1)$, but the boundary values $s = 0, 1$ are also worth discussing.

$s = 0$ is the trivial case where every term of the Dirichlet series is 1. In the sequence of partial sums, there are $\Delta_0(k)$ terms total (by definition), so the value of the kth partial sum is $\Delta_0(k)$. Hence, if $\lim_{k\to\infty} \Delta_0(k) = c \in \mathbb{Z}^+$, the sequence of partial sums converges to c. However, if $\lim_{k\to\infty} \Delta_0(k) \to \infty$, the sequence diverges.

The $s = 1$ case is much more interesting, as it represents the harmonic sum.

Recall the recurrence relation we used to find $g_s(k)$ for $s \in (0, 1)$. We will use this same recurrence relation to solve for $g_1(k)$, but we will need a new difference function.

$$\begin{cases} g_1(1) = 1 \\ g_1(k+1) = g_1(k) + \Delta_1(k), \;\; \Delta_1(k) > 0 \;\; \forall \;\; k \in \mathbb{N} \end{cases} \tag{22}$$

I now claim that the following choice of $\Delta_1(k)$ (the difference function $g_1(k+1) - g_1(k)$) satisfies the desired properties:

$$\Delta_1(k) = b\,c^k, \quad b \in \mathbb{R}^+, \;\; c \in (1, \infty) \tag{23}$$

To show that this is correct, we can solve the recurrence relation and show that the partial sum converges to a finite, nonzero value in the infinite limit. Plugging (23) into (22), we can solve the recurrence using the RSolve method in Mathematica:

$$g_1(k) = 1 + \frac{b\,c\,(c^{k-1} - 1)}{c - 1} \tag{24}$$

Then, defining the lower and upper bounds as p and q, respectively, we have:

$$\begin{aligned} p \equiv \lfloor g_1(k) \rceil &= \left\lfloor 1 + \frac{b\,c\,(c^{k-1} - 1)}{c - 1} \right\rceil \\ q \equiv \lfloor g_1(k) - 1 \rceil + \lfloor \Delta_1(k) \rceil &= \left\lfloor \frac{b\,c\,(c^{k-1} - 1)}{c - 1} \right\rceil + \lfloor \Delta_1(k) \rceil \end{aligned} \tag{25}$$

Unlike in the case $s \in (0,1)$, Mathematica easily evaluates the following limit once the nearest integer function is removed from the bounds:

$$\lim_{k\to\infty} \sum_{j=p}^{q} \frac{1}{j} = \log c\,, \quad b > 0 \text{ and } c > 1 \tag{26}$$

What we notice here is that the convergence value of $\log c$ is independent of the constant b that appears in $\Delta_1(k)$. To see why this occurs, let's first recall Indexing Method 1 that we introduced at the start. Assuming $g_1(k)$ is invertible on the domain $k \in [R, \infty)$, where $R \in \mathbb{Z}^+$, the relationship between Indexing Method 2 and Indexing Method 1 is the following:

$$\lim_{k\to\infty} \sum_{j=g_1(k)}^{g_1(k)+\Delta_1(k)-1} \frac{1}{j} = \lim_{k\to\infty} \sum_{j=k}^{k+\Delta_1(g_1^{-1}(k))-1} \frac{1}{j} \tag{27}$$

Now, since $g_1(k)$ is an exponential function with an elementary inverse, we can do some algebra to obtain $\Delta_1(g^{-1}(1,k)) = 1 - k + c(k + b - 1)$. Then, plugging $g^{-1}(1,k)$ into the above sum, we have:

$$\lim_{k\to\infty} \sum_{j=k}^{k+\Delta_1(g_1^{-1}(k))-1} \frac{1}{j} = \lim_{k\to\infty} \sum_{j=k}^{ck+c(b-1)} \frac{1}{j} = \lim_{k\to\infty} \sum_{j=k}^{ck} \frac{1}{j} = \log c \tag{28}$$

This inversion shows us why the convergence result is independent of b.

We have now answered Question 1 for $s = 1$:

$$\boxed{\begin{aligned} S_k = \sum_{j=p}^{q} \frac{1}{j} \quad &\longrightarrow \quad \log c \\ p \equiv 1 + \frac{b(c^k - c)}{c-1} \\ q \equiv \frac{b(c^{k+1} - c)}{c-1} \end{aligned}}$$

8 Applications Part II: Approximating large harmonic sums and increasing speed of convergence

There are already very good approximations (to an arbitrary accuracy) for harmonic sums:

$$\begin{aligned} \mathrm{H}(n) &= \sum_{j=1}^{n} \frac{1}{j} = \psi^{(0)}(n+1) + \gamma \\ &= \log n + \gamma + \frac{1}{2n} + \mathcal{O}\Big(\frac{1}{n^2}\Big) \end{aligned} \tag{29}$$

We can obtain a different type of approximation by using our harmonic partial sum results. Because the sequence of partial sums from the previous section have consecutive terms and follow the COT constraint, the following rough approximation holds:

$$\begin{gathered} \text{Let } k_M \text{ be the solution to } \frac{b(c^{k+1}-c)}{c-1} = M \text{ , then:} \\ \sum_{j=1}^{M} \frac{1}{j} \approx k_M \log(c) = \log\Big(1 + \frac{M}{bc}(c-1)\Big) \end{gathered} \tag{30}$$

However, this approximation is rather useless, as it depends on the arbitrary values b and c which are not in the original sum. A much better approximation is where we use the partial sum convergence value to approximate just the tail of the sum, where the error for each term $(S_k - \log(c))$ is smaller. In this case, let M be the upper bound of the larger sum and let N be the upper bound of the smaller sum. The tail sum runs from $j = N+1$ to M.

$$\sum_{j=1}^{M} \frac{1}{j} \approx \sum_{j=1}^{N} \frac{1}{j} + \log c(k_M - k_N) = \sum_{j=1}^{N} \frac{1}{j} + \log\left(\frac{cb + (c-1)M}{cb + (c-1)N}\right) \tag{31}$$

Now, we want to find the value of b that minimizes the error between the true value of the harmonic sum and the approximation. To do this, we rewrite the harmonic numbers in terms of the digamma function, set the error equal to 0, solve for b, and then find the value of b in the limit where $M, N \to \infty$, $M > N$:

$$\begin{aligned} \psi^{(0)}(N+1) - \psi^{(0)}(M+1) + \log\left(\frac{cb + (c-1)M}{cb + (c-1)N}\right) &= 0 \\ b &= \frac{(c-1)\big(Me^{\psi^{(0)}(N+1)} - Ne^{\psi^{(0)}(M+1)}\big)}{c\big(e^{\psi^{(0)}(M+1)} - e^{\psi^{(0)}(N+1)}\big)} \\ \lim_{M,N\to\infty} b = \lim_{N\to\infty}\left(\lim_{M\to\infty} \frac{(c-1)\big(Me^{\psi^{(0)}(N+1)} - Ne^{\psi^{(0)}(M+1)}\big)}{c\big(e^{\psi^{(0)}(M+1)} - e^{\psi^{(0)}(N+1)}\big)}\right) &= \frac{c-1}{2c} \end{aligned} \tag{32}$$

Since we found the value of b that minimizes the error of the approximation in the infinite limit, let's plug it back into the approximation. Plugging in this value of b also causes c to cancel out, making the approximation independent of b and c, as desired:

$$\boxed{\sum_{j=1}^{M} \frac{1}{j} \approx \sum_{j=1}^{N} \frac{1}{j} + \log\left(\frac{2M+1}{2N+1}\right)}$$

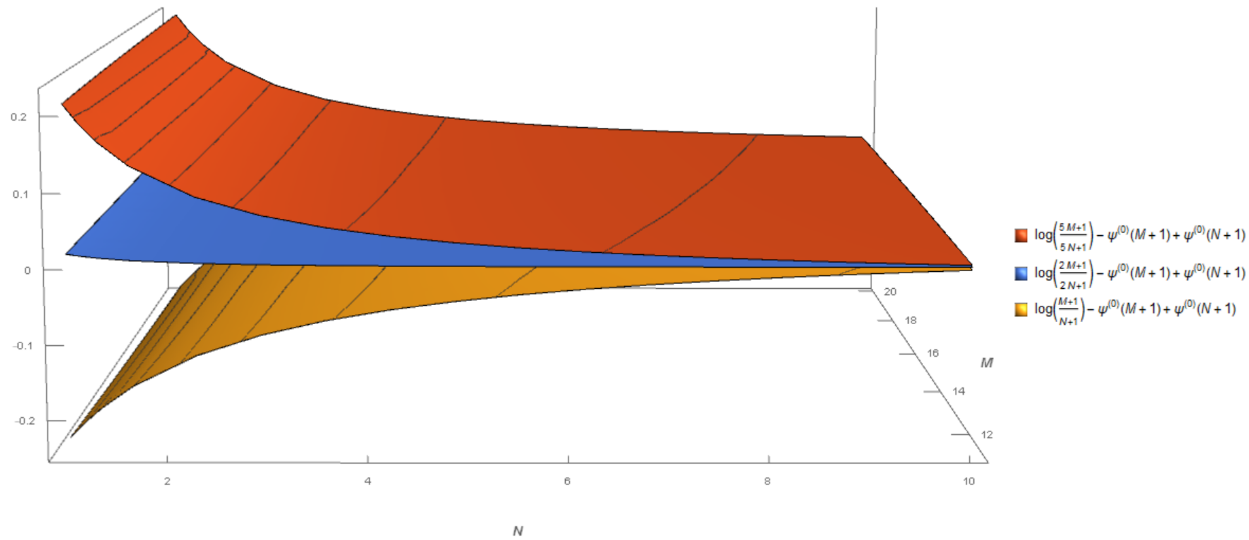

Figure 1: Visually we can see that the error of the above approximation (written using the digamma function) converges to 0 fastest compared with other choices of the quotient inside the log function.

How good is the above approximation?

$$\begin{gathered} \sum_{j=1}^{10^{100}} \frac{1}{j} = 230.8357249643061 \\ \sum_{j=1}^{10^{50}} \frac{1}{j} + \log\left(\frac{2 \cdot 10^{100} + 1}{2 \cdot 10^{50} + 1}\right) = 230.8357249643061 \\ \text{(error less than calculation precision)} \end{gathered} \tag{33}$$

$$\begin{gathered} \sum_{j=1}^{10^{5}} \frac{1}{j} = 12.090146129863427 \\ \sum_{j=1}^{10^{4}} \frac{1}{j} + \log\left(\frac{2 \cdot 10^{5} + 1}{2 \cdot 10^{4} + 1}\right) = 12.090146130275887 \;\; (3.4 \cdot 10^{-9}\% \text{ error}) \\ \log(10^5) + \gamma = 12.090141129871762 \;\; (4.1 \cdot 10^{-5}\% \text{ error}) \end{gathered} \tag{34}$$

Conjecture 8.0.1. The choice $b = \frac{c-1}{2c}$, which minimizes approximation error in the infinite limit, is also the choice of b that maximizes the speed of convergence of the sequence of partial sums.

If this conjecture is correct, fastest convergence of the sequence of partial sums is obtained by the following bounds:

$$\boxed{\begin{aligned} S_k = \sum_{j=p}^{q} \frac{1}{j} \quad &\longrightarrow \quad \log c \\ p \equiv \frac{1}{2}(c^{k-1}+1) \\ q \equiv \frac{1}{2}(c^{k}-1) \end{aligned}}$$

9 Additional partial sums of this form

For clarity, let's rename some of the bounds we've been using:

$$\begin{aligned}
p_s(k) &\equiv \left\lfloor 1 + b\,\zeta\Big(\frac{-s}{1-s}, 1\Big) - b\,\zeta\Big(\frac{-s}{1-s}, k\Big) \right\rceil \\
q_s(k) &\equiv \left\lfloor b\,\zeta\Big(\frac{-s}{1-s}, 1\Big) - b\,\zeta\Big(\frac{-s}{1-s}, k\Big) \right\rceil + \left\lfloor b\, k^{\frac{s}{1-s}} \right\rceil \\
p_1(k) &\equiv \left\lfloor 1 + \frac{b\, c\,(c^{k-1}-1)}{c-1} \right\rceil \\
q_1(k) &\equiv \left\lfloor \frac{b\, c\,(c^{k-1}-1)}{c-1} \right\rceil + \left\lfloor b\, c^k \right\rceil
\end{aligned} \tag{35}$$

We can use binomial expansions to prove the convergence of these following two partial sums for $s \in (0,1)$, where $\alpha, \beta, \lambda, \mu \in \mathbb{R}$ and $\alpha + \beta \neq 1$. It is worth noting that the special case $\beta = 0$ leads to some interesting pseudo-telescoping sums.

$$\lim_{k\to\infty} \sum_{j=p_s(k)}^{q_s(k)} \Big(j^{\beta}\, \left|(j+\lambda)^{\alpha} - (j+\mu)^{\alpha}\right| \Big)^{\frac{s}{1-(\alpha+\beta)}} = b^{1-s} \Big(\frac{1}{1-s}\Big)^{s} \cdot\ \left|\alpha\,(\lambda-\mu)\right|^{\frac{s}{1-(\alpha+\beta)}} \tag{36}$$

$$\lim_{k\to\infty} \sum_{j=p_1(k)}^{q_1(k)} \Big(j^{\beta}\, \left|(j+\lambda)^{\alpha} - (j+\mu)^{\alpha}\right| \Big)^{\frac{1}{1-(\alpha+\beta)}} = \log c\ \cdot\ \left|\alpha\,(\lambda-\mu)\right|^{\frac{1}{1-(\alpha+\beta)}} \tag{37}$$

Example 9.1

Consider the sequence of partial sums given by (36) for the parameters $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\lambda = 1$, $\mu = 0$, $s = \frac{1}{2}$, and $b = 1$:

$$\begin{aligned}
S_k = \sum_{j=p_{\frac{1}{2}}(k)}^{q_{\frac{1}{2}}(k)} \sqrt{\sqrt{\frac{j+1}{j}} - 1} \;=\; &\underbrace{\sqrt{\sqrt{\frac{2}{1}}-1}}_{k=1},\ \underbrace{\sqrt{\sqrt{\frac{3}{2}}-1} + \sqrt{\sqrt{\frac{4}{3}}-1}}_{k=2}, \\
&\underbrace{\sqrt{\sqrt{\frac{5}{4}}-1} + \sqrt{\sqrt{\frac{6}{5}}-1} + \sqrt{\sqrt{\frac{7}{6}}-1}}_{k=3},\ \dots,\ \underbrace{\Big(\qquad 1 \qquad\Big)}_{k\to\infty}
\end{aligned} \tag{38}$$

The sequence of partial sums of the form (36) and (37) can be used to generate approximations of the full summation in the same way as we discussed in sections 6 and 8, respectively.

Example 9.2

By considering the following limit, where the integral evaluates to an ordinary hypergeometric function,

$$\lim_{k\to\infty}\sum_{j=p_s(k)}^{q_s(k)}\left(\frac{1}{(j-1)^2}-\frac{1}{(j+1)^2}\right)^{\frac{1}{8n}}=\lim_{k\to\infty}\int_{p_s(k)}^{q_s(k)}\left(\frac{1}{(x-1)^2}-\frac{1}{(x+1)^2}\right)^{\frac{1}{8n}}dx \tag{39}$$

we can apply equation 36 to obtain

$$\lim_{k\to\infty}k^9\left({}_2F_1\Big(\Big(\frac{2}{4}\Big)^2,\Big(\frac{3}{4}\Big)^2,\Big(\frac{5}{4}\Big)^2,ak^{16}+bk^{11}\Big)-{}_2F_1\Big(\Big(\frac{2}{4}\Big)^2,\Big(\frac{3}{4}\Big)^2,\Big(\frac{5}{4}\Big)^2,ak^{16}\Big)\right)=\frac{9b}{20(-a)^{\frac{5}{4}}}. \tag{40}$$

Note: this hypergeometric function can be defined as follows:

$$
\begin{aligned}
{}_2F_1\Big(\Big(\frac{2}{4}\Big)^2,\Big(\frac{3}{4}\Big)^2,\Big(\frac{5}{4}\Big)^2,z\Big) &:= \sum_{n=0}^{\infty}c_n z^n\\
c_n &:= \prod_{k=0}^{n-1}\frac{(2^2+16k)(3^2+16k)}{(4^2+16k)(5^2+16k)}.
\end{aligned}
$$