Summations and Approximations and Recurrence Relations, Oh My!

Investigating the properties of convergent Dirichlet subseries

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Outline

Introduction and Motivation

- From real analysis: terms of the harmonic series approach 0 in the infinite limit, but the full series diverges.
- What about convergence of a sequence of subseries with harmonic series terms?

Example 1:

$$S = \quad 1 + \frac{1}{2} + \frac{1}{3}, \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \quad \frac{1}{3} + \frac{1}{4} + \frac{1}{5},$$
$$\ldots, \quad \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \to 0$$

- Because terms go to 0, fixed number of terms in each subseries $\implies$ sequence converges to 0.
- If we took more terms, could we get convergence to a finite, positive value?

Ground Rules: Indexing

- So far, decided sequence of partial sums includes *consecutive terms*.
- Need to decide the index where each successive partial sum (term of the sequence) will begin.

Example 2:

Index Method 1

$$S = 1, \quad \frac{1}{2}+\frac{1}{3}, \quad \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}, \quad \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}, \quad \ldots$$

Index Method 2

$$S = 1, \quad \frac{1}{2}+\frac{1}{3}, \quad \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}, \quad \frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}, \quad \ldots$$

Generalization to Dirichlet Series

The harmonic series is just a special case ($s = 1$) of the simplest type of Dirichlet series:

$$\sum_{j=1}^{\infty} \frac{1}{j^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \tag{1}$$

Because of its importance, Index Method 2 shown on the last slide will be formalized as the "Conservation of Terms" constraint.

Conservation of Terms (COT):

$$\sum_{k=1}^{\infty} \sum_{j=\lfloor g_s(k) \rceil}^{\lfloor g_s(k+1) \rceil - 1} \frac{1}{j^s} = \sum_{j=1}^{\infty} \frac{1}{j^s}$$

$g_s : \mathbb{R} \to \mathbb{R}$ such that $g_s(1) = 1$ and $g_s(k+1) \geq g_s(k) + 1$

Generalization to Dirichlet Series cont.

$$\sum_{j=1}^{\infty} \frac{1}{j^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots \tag{1}$$

This leads to the central question I will answer in this talk:

Question 1: *Following the COT constraint, how can we use consecutive terms of* (1) *to construct sequences of partial sums that converge to a nonzero value in the infinite limit?*

Generalization to Dirichlet Series cont.

Question 1: *Following the COT constraint, how can we use consecutive terms of* (1) *to construct sequences of partial sums that converge to a nonzero value in the infinite limit?*

For which $s \in \mathbb{R}$ is Question 1 interesting?

$$\lim_{k\to\infty} \sum_{j=g_s(k)}^{g_s(k+1)-1} \frac{1}{j^s} = \begin{cases} 0, & s \in (1,\infty) \\ \infty, & s \in (-\infty, 0) \\ \lim_{k\to\infty}(g_s(k+1) - g_s(k)), & s = 0 \\ \boxed{0 \text{ or } \lambda \in \mathbb{R}^+ \text{ or } \infty, \quad s \in (0,1]} \end{cases}$$

Solution for $s \in (0, 1)$

Question 1 asks us to find a $g_s(k)$ that satisfies the following:

$$\lim_{k \to \infty} \sum_{j=g_s(k)}^{g_s(k+1)-1} \frac{1}{j^s} = \lambda \in \mathbb{R}^+ \tag{2}$$

If we introduce the difference function $\Delta_s(k) = g_s(k+1) - g_s(k)$ to denote the number of terms in each partial sum, we can rewrite the COT constraint as a recurrence relation:

$$\begin{cases} g_s(1) = 1 \\ g_s(k+1) = g_s(k) + \Delta_s(k), \;\; \Delta_s(k) > 0 \;\forall\; k \in \mathbb{N}, s \in (0,1) \end{cases} \tag{3}$$

Solution for $s \in (0, 1)$ cont.

Need an ansatz for $\Delta_s(k)$ in order to solve the recurrence relation for $g_s(k)$. I claim that the following ansatz is correct:

$$\Delta_s(k) = b\,k^c, \quad b \in \mathbb{R}^+ \tag{4}$$

Plugging $\Delta_s(k)$ into the recurrence relation, we can solve the inhomogeneous recurrence using the RSolve method in Mathematica:

$$g_s(k) = 1 + b\zeta(-c, 1) - b\zeta(-c, k) \tag{5}$$

$$\zeta(s, q) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}, \quad \text{where } \mathrm{Re}(s) > 1 \text{ and } \mathrm{Re}(q) > 0$$

Solution for $s \in (0, 1)$ cont.

$$S_k = \sum_{j=\lfloor g_s(k) \rceil}^{\lfloor g_s(k+1) \rceil - 1} \frac{1}{j^s} = \sum_{j=\lfloor 1 + b\zeta(-c,1) - b\zeta(-c,k) \rceil}^{\lfloor b\zeta(-c,1) - b\zeta(-c,k) \rceil + \lfloor \Delta_s(k) \rceil} \frac{1}{j^s} \tag{6}$$

Proof sketch:
We desire to show that $\lim_{k \to \infty} S_k = \lambda \in \mathbb{R}^+$.

- *Factor out arithmetic mean from S_k. Since all terms of S_k approach the same value (0), substitute arithmetic mean for geometric mean in the infinite limit.
- Convert GM to factorial quotient, then apply Stirling's approximation, where error term approaches 0 in infinite limit.

Solution for $s \in (0, 1)$ cont.

Proof sketch cont.:
We desire to show that $\lim_{k\to\infty} S_k = \lambda \in \mathbb{R}^+$.

- Grouping terms and evaluating the infinite limit reveals that finite, nonzero convergence $\implies c = \frac{s}{1-s}$, so $\Delta_s(k) = b\,k^{\frac{s}{1-s}}$.
- Under this condition, we have the following convergence:

$$\lim_{k\to\infty} S_k = b^{1-s}\left(\frac{1}{1-s}\right)^s \in \mathbb{R}^+ \;\; \forall \;\; s \in (0,1) \tag{7}$$

- Hence we have a $g_s(k)$ that offers a solution to Question 1:

$$g_s(k) = 1 + b\,\zeta\left(\frac{-s}{1-s}, 1\right) - b\,\zeta\left(\frac{-s}{1-s}, k\right) \tag{8}$$

Solution for $s \in (0,1)$ cont.

Solution to Question 1 for $s \in (0,1)$:

$$S_k = \sum_{j=p}^{q} \frac{1}{j^s} \longrightarrow b^{1-s}\left(\frac{1}{1-s}\right)^s$$

$$p \equiv 1 + b\zeta\left(\frac{-s}{1-s}, 1\right) - b\zeta\left(\frac{-s}{1-s}, k\right)$$

$$q \equiv b\zeta\left(\frac{-s}{1-s}, 1\right) - b\zeta\left(\frac{-s}{1-s}, k+1\right)$$

Applications I: Approximating large Dirichlet series for $s \in (0,1)$

Example 3:

Suppose we want to calculate the value of the sum $P_{100} \equiv \sum_{j=1}^{10^{100}} \left(\frac{1}{j}\right)^{\frac{1}{3}}$.

Result: we have derived a convergent sequence of partial sums S_k such that $\sum_{k=1}^{\lceil k_{crit} \rceil} S_k \geq P_{100} \geq \sum_{k=1}^{\lfloor k_{crit} \rfloor} S_k$.

- Set the upper bounds of P_{100} and S_k equal. Numerically solve for k_{crit}.
- Multiply k_{crit} and the convergence value of S_k to obtain an approximation for P_{100}.

Applications I: Approximating large Dirichlet series for $s \in (0, 1)$ cont.

Example 3 cont.:
Let $b = 1$, $s = \frac{1}{3}$, and $q \equiv b\zeta(-\frac{s}{1-s}, 1) - b\zeta(-\frac{s}{1-s}, k+1)$:

$$\zeta\Big(-\frac{1}{2}, 1\Big) - \zeta\Big(-\frac{1}{2}, k+1\Big) = 10^{100} \implies k_{100} = 6.08\ldots \cdot 10^{66} \tag{9}$$

Convergence value of S_k: $\left(\frac{1}{1-s}\right)^s = \left(\frac{3}{2}\right)^{\frac{1}{3}}$.

$$P_{100} = \sum_{j=1}^{10^{100}} \left(\frac{1}{j}\right)^{\frac{1}{3}} \approx \left(\frac{3}{2}\right)^{\frac{1}{3}} \cdot k_{100} = 6.962383250419169 \cdot 10^{66}$$

Error between the true result and our approximation is $4.1 \cdot 10^{-13}$ %.

The standard integral method of approximating this sum gives an error of $1.9 \cdot 10^{-13}$ %.

Solution for $s = 1$

For $s = 1$, Question 1 asks us to split the harmonic series into a sequence of convergent subseries.

Can use same recurrence relation to solve for $g_1(k)$, but need a new difference function:

$$\Delta_1(k) = b\,c^k, \quad b \in \mathbb{R}^+, \;\; c \in (1, \infty) \tag{10}$$

Plugging $\Delta_1(k)$ into the recurrence relation, we again solve it using the RSolve method in Mathematica:

$$g_1(k) = 1 + \frac{b\,c\,(c^{k-1} - 1)}{c - 1} \tag{11}$$

Solution for $s = 1$ cont.

Plugging in $g_1(k)$, Mathematica easily evaluates the following limit:

$$\lim_{k\to\infty} \sum_{j=g_1(k)}^{g_1(k+1)-1} \frac{1}{j} = \log c\,, \quad b > 0 \text{ and } c > 1 \tag{12}$$

Convergence not dependent on b. Why?

$$\lim_{k\to\infty} \underbrace{\sum_{j=g_1(k)}^{g_1(k)+\Delta_1(k)-1} \frac{1}{j}}_{\text{Indexing Method 2}} = \lim_{k\to\infty} \underbrace{\sum_{j=k}^{k+\Delta_1(g_1^{-1}(k))-1} \frac{1}{j}}_{\text{Indexing Method 1}} = \lim_{k\to\infty} \sum_{j=k}^{ck+c(b-1)} \frac{1}{j} = \log c \tag{13}$$

Solution for $s = 1$ cont.

We have now answered Question 1 for $s = 1$:

$$\boxed{\begin{aligned} S_k = \sum_{j=p}^{q} \frac{1}{j} \quad &\longrightarrow \quad \log c \\[2em] p \equiv 1 + &\frac{b(c^k - c)}{c-1} \\ q \equiv &\frac{b(c^{k+1} - c)}{c-1} \end{aligned}}$$

Application II: Approximating large harmonic sums

There are already good approximations (to an arbitrary accuracy) for harmonic sums:

$$\mathrm{H}(n) = \sum_{j=1}^{n} \frac{1}{j} = \log n + \gamma + \mathcal{O}\left(\frac{1}{n}\right) \tag{14}$$

We can obtain a different type of approximation using our harmonic partial sum results. For $M \geq N$:

$$\boxed{\sum_{j=1}^{M} \frac{1}{j} \approx \sum_{j=1}^{N} \frac{1}{j} + \log\left(\frac{2M+1}{2N+1}\right)}$$

Applications II: Approximating large harmonic sums cont.

How good is the harmonic series approximation?

$$\sum_{j=1}^{10^{100}} \frac{1}{j} \approx \sum_{j=1}^{10^{50}} \frac{1}{j} + \log\left(\frac{2 \cdot 10^{100} + 1}{2 \cdot 10^{50} + 1}\right) = 230.8357249643061$$

(error less than default Mathematica calculation precision)

$$\sum_{j=1}^{10^{5}} \frac{1}{j} \approx \sum_{j=1}^{10^{4}} \frac{1}{j} + \log\left(\frac{2 \cdot 10^{5} + 1}{2 \cdot 10^{4} + 1}\right) = 12.090146130275887$$

($3.4 \cdot 10^{-9}$% error)

$$\approx \log(10^5) + \gamma = 12.090141129871762$$

($4.1 \cdot 10^{-5}$% error)

Additional partial sums of this form

Example 4

$$S_k = \underbrace{\sqrt{\sqrt{\frac{2}{1}} - 1}}_{k=1},\ \underbrace{\sqrt{\sqrt{\frac{3}{2}} - 1} + \sqrt{\sqrt{\frac{4}{3}} - 1}}_{k=2},$$

$$\underbrace{\sqrt{\sqrt{\frac{5}{4}} - 1} + \sqrt{\sqrt{\frac{6}{5}} - 1} + \sqrt{\sqrt{\frac{7}{6}} - 1}}_{k=3},$$

$$\dots,\ \sum_{j=\frac{1}{2}k(k-1)+1}^{\frac{1}{2}k(k+1)} \sqrt{\sqrt{\frac{j+1}{j}} - 1} \longrightarrow 1$$

Additional partial sums of this form cont.

Example 5

$$S_k = \underbrace{\left(\frac{1}{1^3} - \frac{1}{3^3}\right)^{\frac{1}{4}}}_{k=1},\ \underbrace{\left(\frac{1}{2^3} - \frac{1}{4^3}\right)^{\frac{1}{4}} + \left(\frac{1}{3^3} - \frac{1}{5^3}\right)^{\frac{1}{4}}}_{k=2},$$

$$\underbrace{\left(\frac{1}{4^3} - \frac{1}{6^3}\right)^{\frac{1}{4}} + \left(\frac{1}{5^3} - \frac{1}{7^3}\right)^{\frac{1}{4}} + \left(\frac{1}{6^3} - \frac{1}{8^3}\right)^{\frac{1}{4}} + \left(\frac{1}{7^3} - \frac{1}{9^3}\right)^{\frac{1}{4}}}_{k=3},$$

$$\dots,\ \sum_{j=2^{k-1}+1}^{2^k} \left(\frac{1}{(j-1)^3} - \frac{1}{(j+1)^3}\right)^{\frac{1}{4}} \longrightarrow 6^{\frac{1}{4}} \log 2$$

Future Work

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- Generalize Question 1 for non-Dirichlet series.
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- Generalize Question 1 for non-Dirichlet series.
  - e.g. Consider the summand $\frac{1}{\log j}$. Does there exist a $\Delta(k)$ that affords nonzero convergence?
- Which value(s) of b lead to fastest convergence of sequence of Dirichlet partial sums?

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- Extend results to $s \in \mathbb{C}$
- Generalize Question 1 for non-Dirichlet series.
  - e.g. Consider the summand $\frac{1}{\log j}$. Does there exist a $\Delta(k)$ that affords nonzero convergence?
- Which value(s) of b lead to fastest convergence of sequence of Dirichlet partial sums?

Thank you for listening!

Applications II: Approximating large harmonic sums

There are already very good approximations (to an arbitrary accuracy) for harmonic sums:

$$\begin{aligned} \mathrm{H}(n) = \sum_{j=1}^{n} \frac{1}{j} &= \psi^{(0)}(n+1) + \gamma \\ &= \log n + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right) \end{aligned} \tag{15}$$

We can obtain a different type of approximation using our harmonic partial sum results. Let M be the upper bound of the larger sum and let N be the upper bound of the smaller sum. k_M and k_N are the unique solutions to $q = M$ and $q = N$, respectively.

$$\sum_{j=1}^{M} \frac{1}{j} \approx \sum_{j=1}^{N} \frac{1}{j} + (k_M - k_N) \log c = \sum_{j=1}^{N} \frac{1}{j} + \log\left(\frac{cb + (c-1)M}{cb + (c-1)N}\right) \tag{16}$$

Applications II: Approximating large harmonic sums cont.

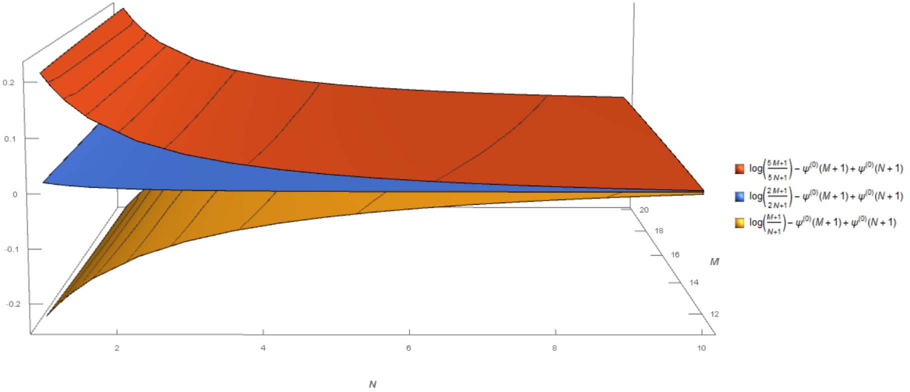
Rewrite the harmonic numbers in terms of the digamma function, set the error equal to 0, solve for b, and then find the value of b in the limit where $M, N \to \infty$, $M > N$:

$$\lim_{N\to\infty}\left(\lim_{M\to\infty} b\right) = \frac{c-1}{2c} \tag{17}$$

Since this value of b minimizes the error of the approximation in the infinite limit, let's plug it back in to obtain the optimal approximation:

$$\boxed{\sum_{j=1}^{M}\frac{1}{j} \approx \sum_{j=1}^{N}\frac{1}{j} + \log\left(\frac{2M+1}{2N+1}\right)}$$

Applications II: Approximating large harmonic sums cont.



Additional partial sums of this form

For clarity, let's rename some of the bounds we've been using:

$$
\begin{aligned}
p_s(k) &\equiv \left\lfloor 1 + b\,\zeta\!\left(\frac{-s}{1-s}, 1\right) - b\,\zeta\!\left(\frac{-s}{1-s}, k\right) \right\rfloor \\
q_s(k) &\equiv \left\lceil b\,\zeta\!\left(\frac{-s}{1-s}, 1\right) - b\,\zeta\!\left(\frac{-s}{1-s}, k\right) \right\rceil + \left\lceil b\,k^{\frac{s}{1-s}} \right\rceil \\
p_1(k) &\equiv \left\lfloor 1 + \frac{b\,c\,(c^{k-1}-1)}{c-1} \right\rfloor \\
q_1(k) &\equiv \left\lceil \frac{b\,c\,(c^{k-1}-1)}{c-1} \right\rceil + \left\lceil b\,c^k \right\rceil
\end{aligned}
\tag{18}
$$

Additional partial sums of this form cont.

I used binomial expansions to prove the convergence of these following two partial sums for $s \in (0,1)$, where $\alpha, \beta, \lambda, \mu \in \mathbb{R}$ and $\alpha + \beta \neq 1$.

$$\lim_{k\to\infty} \sum_{j=p_s(k)}^{q_s(k)} \left(j^{\beta} \left|(j+\lambda)^{\alpha} - (j+\mu)^{\alpha}\right|\right)^{\frac{s}{1-(\alpha+\beta)}} = b^{1-s}\left(\frac{1}{1-s}\right)^{s} \cdot \left|\alpha\left(\lambda-\mu\right)\right|^{\frac{s}{1-(\alpha+\beta)}} \tag{19}$$

$$\lim_{k\to\infty} \sum_{j=p_1(k)}^{q_1(k)} \left(j^{\beta} \left|(j+\lambda)^{\alpha} - (j+\mu)^{\alpha}\right|\right)^{\frac{1}{1-(\alpha+\beta)}} = \log c \cdot \left|\alpha\left(\lambda-\mu\right)\right|^{\frac{1}{1-(\alpha+\beta)}} \tag{20}$$

Additional partial sums of this form cont.

Example 4

Consider the sequence of partial sums given by (19) for the parameters $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\lambda = 1$, $\mu = 0$, $s = \frac{1}{2}$, and $b = 1$:

$$S_k = \sum_{j=p_{\frac{1}{2}}(k)}^{q_{\frac{1}{2}}(k)} \sqrt{\sqrt{\frac{j+1}{j}} - 1} = \underbrace{\sqrt{\sqrt{\frac{2}{1}} - 1}}_{k=1}, \underbrace{\sqrt{\sqrt{\frac{3}{2}} - 1} + \sqrt{\sqrt{\frac{4}{3}} - 1}}_{k=2}, \underbrace{\sqrt{\sqrt{\frac{5}{4}} - 1} + \sqrt{\sqrt{\frac{6}{5}} - 1} + \sqrt{\sqrt{\frac{7}{6}} - 1}}_{k=3}, \ldots, \underbrace{\left(\quad 1 \quad \right)}_{k\to\infty} \tag{21}$$

Convergence Plot

The Hurwitz zeta function has no known inverse. That said, it has a relationship with the Bernoulli polynomials for some values of its argument:

$$\zeta(-\alpha, k) = \frac{-B_{\alpha+1}(k)}{\alpha+1}, \ \ \alpha \in \mathbb{N}$$

Hence, for $s = \frac{m}{m+1}$, $m \in \mathbb{N}$, we can write the indexing function $g_s(k)$ in terms of the Bernoulli polynomials. Then, we can invert the Bernoulli polynomials on the domain $k \in [u, \infty)$ where $u \in \mathbb{R}^+$ in order to convert Indexing Method 2 into Indexing Method 1.

Convergence Plot cont.

Example 6: $s = \frac{3}{4}$

$$g_{\frac{3}{4}}(k) = \frac{1}{2}(k(k-1))^2 + 1$$

$$\text{On } k \in [1, \infty), \quad g_{\frac{3}{4}}^{-1}(k) = \frac{1}{2}\left(1 + \sqrt{1 + 8\sqrt{k-1}}\right)$$

Then, since we have an expression for $g_s^{-1}(k)$, we can convert Indexing Method 2 to Indexing Method 1:

$$\lim_{k\to\infty} \underbrace{\sum_{j=g_1(k)}^{g_1(k)+\Delta_1(k)-1} \frac{1}{j^{\frac{3}{4}}}}_{\text{Indexing Method 2}} = \lim_{k\to\infty} \underbrace{\sum_{j=k}^{k+\Delta_1(g_1^{-1}(k))-1} \frac{1}{j^{\frac{3}{4}}}}_{\text{Indexing Method 1}} \tag{22}$$

Convergence Plot cont.

$$\lim_{k\to\infty} \sum_{j=k}^{k+\left(\frac{1}{2}\left(1+\sqrt{1+8\sqrt{k-1}}\right)\right)^3-1} \frac{1}{j^{\frac{3}{4}}} = 2\sqrt{2} \tag{23}$$