Convergence Values for a Family of Double Sums Involving Squares

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1 Results

$$\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} \sum_{k=1}^{j} \frac{1}{(2k)^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\psi(k+\frac{1}{2}) - \psi(k)}{(2k)^2} = -\frac{\pi^2}{24} \log(2) + \frac{5}{16}\zeta(3) \tag{1}$$

$$\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} \sum_{k=1}^{j} \frac{1}{(2k-1)^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\psi(k+\frac{1}{2}) - \psi(k)}{(2k-1)^2} = \frac{\pi^2}{8} \log(2) - \frac{7}{16}\zeta(3) \tag{2}$$

$$\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} \sum_{k=1}^{j} \frac{1}{(2k+1)^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\psi(k+\frac{1}{2}) - \psi(k)}{(2k+1)^2} = 2 - \frac{\pi^2}{8} + \frac{\pi^2}{8} \log(2) - \frac{21}{16}\zeta(3) \tag{3}$$

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} \sum_{k=1}^{j} \frac{1}{((2k-1)(2k)(2k+1))^2} &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{\psi(k+\frac{1}{2}) - \psi(k)}{((2k-1)(2k)(2k+1))^2} \\ &= \frac{5}{4} - \frac{\pi^2}{8} + \frac{\pi^2}{48} \log(2) - \frac{1}{8}\zeta(3)\end{aligned} \tag{4}$$

$$\begin{aligned}&\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} \sum_{k=1}^{j} \frac{144k^4 - 4k^3 - 43k^2 + \frac{21}{4}}{((2k-1)(2k)(2k+1))^2} \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left(\psi\left(k+\frac{1}{2}\right) - \psi(k)\right)\left(\frac{144k^4 - 4k^3 - 43k^2 + \frac{21}{4}}{((2k-1)(2k)(2k+1))^2}\right) = 1\end{aligned} \tag{5}$$

2 Derivation

Obtaining these results was made possible using the following Euler sum formula from Rao (1987) (restated in below form by Flajolet and Salvy 1998):

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^q} \sum_{j=1}^{n} \frac{(-1)^j}{j} = (\zeta(q) + \overline{\zeta}(q)) \log(2) - \frac{q+1}{2}\overline{\zeta}(q+1) + \frac{1}{2}\zeta(q+1) + \sum_{k=1}^{\frac{q}{2}-1} \zeta(2k)\overline{\zeta}(q+1-2k) \tag{6}$$

For the case $q = 2$, the above formula reduces to the following:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{j=1}^{n} \frac{(-1)^j}{j} = \frac{\pi^2}{4} \log(2) - \frac{5}{8}\zeta(3) \tag{7}$$

After pulling a factor of $\zeta(3)$ to the RHS, flipping the sign, and renaming the variables, we get

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \sum_{k=1}^{j-1} \frac{(-1)^{k+1}}{k} = -\frac{\pi^2}{4} \log(2) + \frac{13}{8}\zeta(3) \tag{8}$$

By grouping the terms with even and odd j from (8), we obtain the following expansion:

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \sum_{k=1}^{j-1} \frac{(-1)^{k+1}}{k} = \sum_{j=1}^{\infty} \frac{1}{(2j)^2} \sum_{k=0}^{2(j-1)} \frac{(-1)^k}{2j-k-1} - \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} \sum_{k=1}^{2(j-1)} \frac{(-1)^k}{2j-k-1} \tag{9}$$

While this result seems complicated, note the identity obtained by partial fraction decomposition:

$$\sum_{k=0}^{2(j-1)} \frac{(-1)^k}{2j-k-1} = 1 - \sum_{k=1}^{j-1} \frac{1}{2k(2k+1)} \tag{10}$$

Taking into account the $k = 0$ and $k = 1$ lower bounds in (9), we plug (10) into the RHS of (9). Then, we use the well-known convergence of the following sum

$$\sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} = 1 - \log(2)$$

to convert the inner summations from leading term summations to tail summations. Putting it all together and simplifying slightly we obtain:

$$\text{LHS of (8)} = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^3} - \log(2) \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} + \sum_{j=1}^{\infty} \left(\frac{1}{(2j)^2} - \frac{1}{(2j-1)^2} \right) \sum_{k=j}^{\infty} \frac{1}{2k(2k+1)} \tag{11}$$

Hence, after evaluating the first two sums on the RHS of (11) and reversing the order of the third sum, we have the following identity:

$$\sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} \sum_{j=1}^{k} \left(\frac{1}{(2j)^2} - \frac{1}{(2j-1)^2} \right) = \frac{3}{4}\zeta(3) - \frac{\pi^2}{6}\log(2) \tag{12}$$

Now, note that (1) is a known result. Moreover, it can be obtained by simply swapping the order of the double sum and evaluating the resulting expression in Mathematica. Subtracting (12) from (1), we obtain (2), our first main result.

The remainder of the results, (3)-(5), follow directly. In particular, subtracting (3) from (2) eliminates the double sum:

$$\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} \sum_{k=1}^{j} \left(\frac{1}{(2k-1)^2} - \frac{1}{(2k+1)^2} \right) = \sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} - \sum_{j=1}^{\infty} \frac{1}{2j(2j+1)^3} \tag{13}$$

Then, using Mathematica to calculate $\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)^3} = 3 - \frac{\pi^2}{8} - \log(2) - \frac{7}{8}\zeta(3)$, we plug in the convergence of (2) and rearrange to obtain (3).

Equation (4) is obtained just as easily. Using partial fraction decomposition,

$$\frac{1}{((2k-1)(2k)(2k+1))^2} = \frac{1}{(2k)^2} + \frac{1}{4}\left(\frac{1}{(2k-1)^2} + \frac{1}{(2k+1)^2} \right) + \frac{3}{4}\left(\frac{1}{2k+1} - \frac{1}{2k-1} \right). \tag{14}$$

The first terms are familiar, and will just give a contribution of $(1) + \frac{1}{4}((2)+(3))$. The second term gives a similar contribution to (13):

$$\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} \sum_{k=1}^{j} \left(\frac{1}{2k+1} - \frac{1}{2k-1} \right) = -\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)} + \sum_{j=1}^{\infty} \frac{1}{2j(2j+1)^2} \tag{15}$$

Again, we make use Mathematica to evaluate the second sum, $\sum_{j=1}^{\infty} \frac{1}{2j(2j+1)^2} = 2 - \frac{\pi^2}{8} - \log 2$. Summing all the individual contributions affords (4).

Finally, we can use (1)–(4) to obtain (5). We desire to satisfy

$$x\ (1) + y\ (2) + z\ (3) + w\ (4) = 1 \tag{16}$$

Using the convergence values of these sums, we obtain a unique solution to (16) over the rationals:

$$(x, y, z, w) = \left(\frac{79}{12}, \frac{13}{12}, \frac{4}{3}, -\frac{4}{3} \right) \tag{17}$$

By summing together the left hand sides of (1)–(4) multiplied by the corresponding coefficients in (17), we obtain (5).