Nonstandard Sin and Cos Multiple Angle Formulae

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The canonical multiple angle formulae for sine and cosine involve summing over nonconstant powers of sine and cosine multiplied by binomial coefficients. However, it is also possible to obtain multiple angle formulas by summing over constant powers of cosine if a phase shift is introduced. Here, we give detailed proofs of two formulas which accomplish this.

$$-\frac{1}{4}\left[(-1)^n\binom{n}{\frac{n}{2}}+\binom{n}{\frac{n}{2}}\right]+\frac{2^{n-1}}{n}\sum_{k=1}^{n}\cos^n\left(x+\frac{2\pi k}{n}\right)=\cos(nx),\ \ \forall\, n\in\mathbb{N},\ x\in\mathbb{C} \tag{1}$$

$$\frac{1}{4}\left[(-1)^n\binom{n}{\frac{n}{2}}+\binom{n}{\frac{n}{2}}\right]-\frac{2^{n-1}}{n}\sum_{k=1}^{n}\cos^n\left(x+\frac{2\pi k+\frac{\pi}{2}}{n}\right)=\sin(nx),\ \ \forall\, n\in\mathbb{N},\ x\in\mathbb{C} \tag{2}$$

Proof. For the proof of **(1)**, we can start by taking the summation on the LHS of the equation, converting cosine to complex cosine, and then using the binomial theorem to raise the cosine term to the nth power:

$$\begin{aligned}S_n \equiv \frac{2^{n-1}}{n}\sum_{k=1}^{n}\cos^n\left(x+\frac{2\pi k}{n}\right) = \frac{2^{n-1}}{n}\sum_{k=1}^{n}\left(\frac{e^{i(x+\frac{2\pi k}{n})}+e^{-i(x+\frac{2\pi k}{n})}}{2}\right)^n = \\ \frac{1}{2n}\sum_{k=1}^{n}\sum_{j=0}^{n}\binom{n}{j}\left(e^{i(x+\frac{2\pi k}{n})}\right)^{n-j}\left(e^{-i(x+\frac{2\pi k}{n})}\right)^{j}\end{aligned} \tag{3}$$

After expanding the sum and multiplying $n-j$ and j into the exponents, we can simplify the exponents and pair terms with the same binomial coefficients. If n is odd, the binomial expansion has an even number of terms, so each term can be paired with the other term that has the same binomial coefficient. If n is even, the binomial expansion has an odd number of terms, so all but one term can be paired based on the binomial coefficients. As a result of this difference, we will treat these as separate cases:

Case 1: n odd.

$$S_n = \text{RHS of } \mathbf{(3)} = \frac{1}{2n}\sum_{k=1}^{n}\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\left(e^{i(n-2j)(x+\frac{2\pi k}{n})} + e^{-i(n-2j)(x+\frac{2\pi k}{n})}\right) \tag{4}$$

Expanding the exponents, we can eliminate a $2\pi ik$ term, as we know that $e^{2\pi ik} = 1$. We are now ready to convert back to real cosine:

$$\frac{1}{2n}\sum_{k=1}^{n}\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\left(e^{i((n-2j)x-\frac{4\pi jk}{n})} + e^{-i((n-2j)x-\frac{4\pi jk}{n})}\right) = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\cos\left((n-2j)x - \frac{4\pi jk}{n}\right) \tag{5}$$

We can use the angle difference formula $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$ to expand the cosine term in the sum. We also rearrange the order of the double summation, which leads to the following result:

$$S_n = \frac{1}{n}\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\left[\cos((n-2j)x)\sum_{k=1}^{n}\cos\left(\frac{4\pi jk}{n}\right) + \sin((n-2j)x)\sum_{k=1}^{n}\sin\left(\frac{4\pi jk}{n}\right)\right] \tag{6}$$

To evaluate the inner sums, we again break into cases.

Case 1.1: $j \neq 0$

Consider the nth roots of unity denoted $z_k = e^{i\frac{2\pi k}{n}}$ for $k = 1, 2, ..., n$. Based on the inner sums in **(6)**, we desire to find the value of the following sum for $j \neq 0$:

$$\sum_{k=1}^{n} z_k^{2j} = \sum_{k=1}^{n} e^{i\frac{4\pi jk}{n}} = \sum_{k=0}^{n-1} e^{i\frac{4\pi jk}{n}} \tag{7}$$

(7) is just a finite geometric series, so we can use the fact that $S_n = \frac{a_1(1-r^n)}{1-r}$ for a geometric series with n terms and common ratio r. Here, it is important that the common ratio $r = e^{i\frac{2\pi(2j)}{n}}$ is not equal to 1. Given the constraint that $0 < 2j \leq n-1$, $2j$ is obviously not a multiple of n, so $e^{i\frac{2\pi(2j)}{n}} \neq 1$.

$$\sum_{k=0}^{n-1} e^{i\frac{4\pi jk}{n}} = \frac{1-e^{i4\pi j}}{1-e^{i\frac{4\pi j}{n}}} = 0 \implies \sum_{k=1}^{n}\cos\left(\frac{4\pi jk}{n}\right) = 0,\ \sum_{k=1}^{n}\sin\left(\frac{4\pi jk}{n}\right) = 0 \tag{8}$$

Plugging in the RHS results from **(8)**, we have shown that in the case $j \neq 0$, **(6)** evaluates to 0.

Case 1.2: $j = 0$

Plugging $j = 0$ into **(6)**, we obtain the result for odd n by showing that S_n exactly equals $\cos(nx)$:

$$S_n = \frac{1}{n}\binom{n}{0}\cos(nx)\sum_{k=1}^{n} 1 = \cos(nx) \tag{9}$$

Case 2: n even.

After solving the odd case, we can move onto the even case. Recall that when n is even, the binomial expansion on the RHS of **(3)** has an odd number of terms, so all but the middle term can be paired. Indeed, the middle term is a special case because $e^{i(x+\frac{2\pi k}{n})}$ and $e^{-i(x+\frac{2\pi k}{n})}$ are both raised to the power $\frac{n}{2}$, so their product simplifies to 1. From this, we obtain:

$$S_n = \text{RHS of } \mathbf{(3)} = \frac{1}{2n}\sum_{k=1}^{n}\sum_{j=0}^{\frac{n}{2}-1}\left[\binom{n}{j}\left(e^{i(n-2j)(x+\frac{2\pi k}{n})} + e^{-i(n-2j)(x+\frac{2\pi k}{n})}\right) + \binom{n}{\frac{n}{2}}\right] \tag{10}$$

We can then separate the sum to pull the constant term out of the expression:

$$S_n = \frac{1}{2}\binom{n}{\frac{n}{2}} + \frac{1}{2n}\sum_{k=1}^{n}\sum_{j=0}^{\frac{n}{2}-1}\binom{n}{j}\left(e^{i(n-2j)(x+\frac{2\pi k}{n})} + e^{-i(n-2j)(x+\frac{2\pi k}{n})}\right) \tag{11}$$

Now, we notice that the double summation term on the RHS of **(11)** is almost identical to the RHS of **(4)**, with the only difference being that j is summed up to $\frac{n}{2} - 1$ as opposed to $\frac{n-1}{2}$. However, by looking at Cases 1.1 and 1.2 we know that this upper bound on j is never used in the proof, as only $j = 0$ affords a nonzero value of the summation. The criterion of n being odd is also not used after the initial step. Therefore, following all the same logical steps as we used in Case 1 (equations **(5)** through **(9)**) affords:

$$S_n = \frac{1}{2}\binom{n}{\frac{n}{2}} + \cos(nx) \tag{12}$$

From Cases 1 and 2, we have shown: $S_n - \begin{cases} 0 \text{ for } n \text{ odd} \\ \frac{1}{2}\binom{n}{\frac{n}{2}} \text{ for } n \text{ even} \end{cases} = \cos(nx)$. Combining the results for odd and even n into one equation, we get

$$S_n - \frac{1}{4}\left[(-1)^n\binom{n}{\frac{n}{2}} + \binom{n}{\frac{n}{2}}\right] = \cos(nx), \ \ \forall\, n \in \mathbb{N}$$

which is the equation **(1)** that we wanted to prove. ■

Proof. The proof of **(2)** is very similar to the proof of **(1)**, but we will include it here nevertheless. Again, we start by taking the summation on the LHS of the equation, converting to complex cosine, and using the binomial theorem to raise the cosine term to the nth power:

$$T_n \equiv \frac{-2^{n-1}}{n}\sum_{k=1}^{n}\cos^n\left(x+\frac{2\pi k+\frac{\pi}{2}}{n}\right)=\frac{-2^{n-1}}{n}\sum_{k=1}^{n}\left(\frac{e^{i(x+\frac{2\pi k}{n}+\frac{\pi}{2n})}+e^{-i(x+\frac{2\pi k}{n}+\frac{\pi}{2n})}}{2}\right)^n = \frac{-1}{2n}\sum_{k=1}^{n}\sum_{j=0}^{n}\binom{n}{j}\left(e^{i(x+\frac{2\pi k}{n}+\frac{\pi}{2n})}\right)^{n-j}\left(e^{-i(x+\frac{2\pi k}{n}+\frac{\pi}{2n})}\right)^{j} \tag{13}$$

After expanding the sum and multiplying the power terms $n-j$ and j into the exponents, we can simplify the exponents and pair terms with the same binomial coefficients. If n is odd, the binomial expansion has an even number of terms, so each term can be paired with the other term that has the same binomial coefficient. If n is even, the binomial expansion has an odd number of terms, so all but one term can be paired based on the binomial coefficients. As a result of this difference, we will again treat these as separate cases:

Case 1: n odd.

$$T_n = \text{RHS of } \textbf{(13)} = \frac{-1}{2n}\sum_{k=1}^{n}\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\left(e^{i(n-2j)(x+\frac{2\pi k}{n}+\frac{\pi}{2n})}+e^{-i(n-2j)(x+\frac{2\pi k}{n}+\frac{\pi}{2n})}\right) \tag{14}$$

Expanding the exponent terms above, we can separate $i(2\pi k+\frac{\pi}{2})$ from the first exponent and $-i(2\pi k+\frac{\pi}{2})$ from the second exponent, as we know that $e^{i(2\pi k+\frac{\pi}{2})}=i$ and $e^{-i(2\pi k+\frac{\pi}{2})}=-i$. The resulting multiplication by i allows us to convert back to real sine:

$$\frac{-1}{2n}\sum_{k=1}^{n}\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\left(ie^{i((n-2j)x-2j(\frac{2\pi k}{n}+\frac{\pi}{2n}))}-ie^{-i((n-2j)x-2j(\frac{2\pi k}{n}+\frac{\pi}{2n}))}\right) = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\sin\left((n-2j)x-2j\Big(\frac{2\pi k}{n}+\frac{\pi}{2n}\Big)\right) \tag{15}$$

We can use the angle difference formula $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$ to expand the sine term in the sum. We then rearrange the order of the double summation, which leads to the following result:

$$T_n=\frac{1}{n}\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\left[\sin((n-2j)x)\sum_{k=1}^{n}\cos\left(\frac{4\pi jk}{n}+\frac{\pi j}{n}\right)-\cos((n-2j)x)\sum_{k=1}^{n}\sin\left(\frac{4\pi jk}{n}+\frac{\pi j}{n}\right)\right] \tag{16}$$

To evaluate the inner sums, we again break into cases.

Case 1.1: $j \neq 0$

Consider the nth roots of unity denoted $z_k = e^{i\frac{2\pi k}{n}}$ for $k = 1, 2, ..., n$. Based on the inner sums in **(16)**, we desire to find the value of the following sum for $j \neq 0$:

$$\sum_{k=1}^{n} e^{\frac{\pi j}{n}} z_k^{2j} = e^{\frac{\pi j}{n}} \sum_{k=1}^{n} e^{i\frac{4\pi jk}{n}} = e^{\frac{\pi j}{n}} \sum_{k=0}^{n-1} e^{i\frac{4\pi jk}{n}} \tag{17}$$

The summation in **(17)** represents a finite geometric series, so we can use the fact that $S_n = \frac{a_1(1-r^n)}{1-r}$ for a geometric series with n terms and common ratio r. Here, it is important that the common ratio $r = e^{i\frac{2\pi(2j)}{n}}$ is not equal to 1. Given the constraint that $0 < 2j \leq n-1$, $2j$ is obviously not a multiple of n, so $e^{i\frac{2\pi(2j)}{n}} \neq 1$.

$$e^{\frac{\pi j}{n}} \sum_{k=0}^{n-1} e^{i\frac{4\pi jk}{n}} = e^{\frac{\pi j}{n}} \left(\frac{1 - e^{i4\pi j}}{1 - e^{i\frac{4\pi j}{n}}} \right) = 0 \implies \sum_{k=1}^{n} \cos\left(\frac{4\pi jk}{n} + \frac{\pi j}{n}\right) = 0, \ \sum_{k=1}^{n} \sin\left(\frac{4\pi jk}{n} + \frac{\pi j}{n}\right) = 0 \tag{18}$$

Plugging in the RHS results from **(18)**, we have shown that in the case $j \neq 0$, **(16)** evaluates to 0.

Case 1.2: $j = 0$

Plugging $j = 0$ into **(16)**, we obtain the result for odd n by showing that T_n exactly equals $\sin(nx)$:

$$T_n = \frac{1}{n} \binom{n}{0} \sin(nx) \sum_{k=1}^{n} 1 = \sin(nx) \tag{19}$$

Case 2: n even.

After solving the odd case, we can move onto the even case. Recall that when n is even, the binomial expansion on the RHS of **(13)** has an odd number of terms, so all but the middle term can be paired. Indeed, the middle term is a special case because $e^{i(x+\frac{2\pi k}{n}+\frac{\pi}{2n})}$ and $e^{-i(x+\frac{2\pi k}{n}+\frac{\pi}{2n})}$ are both raised to the power $\frac{n}{2}$, so their product simplifies to 1. From this, we obtain:

$$T_n = \text{RHS of } \mathbf{(13)} = \frac{-1}{2n} \sum_{k=1}^{n} \sum_{j=0}^{\frac{n}{2}-1} \left[\binom{n}{j} \left(e^{i(n-2j)(x+\frac{2\pi k}{n}+\frac{\pi}{2n})} + e^{-i(n-2j)(x+\frac{2\pi k}{n}+\frac{\pi}{2n})} \right) + \binom{n}{\frac{n}{2}} \right] \tag{20}$$

We can then separate the sum to pull the constant term out of the expression:

$$T_n = -\frac{1}{2} \binom{n}{\frac{n}{2}} - \frac{1}{2n} \sum_{k=1}^{n} \sum_{j=0}^{\frac{n}{2}-1} \binom{n}{j} \left(e^{i(n-2j)(x+\frac{2\pi k}{n}+\frac{\pi}{2n})} + e^{-i(n-2j)(x+\frac{2\pi k}{n}+\frac{\pi}{2n})} \right) \tag{21}$$

Now, we notice that the double summation term on the RHS of **(21)** is almost identical to the RHS of **(14)**, with the only difference being that j is summed up to $\frac{n}{2}-1$ as opposed to $\frac{n-1}{2}$. However, by looking at Cases 1.1 and 1.2 we know that this upper bound on j is never used in the proof, as only $j=0$ affords a nonzero value of the summation. The criterion of n being odd is also not used after the initial step. Therefore, following all the same logical steps as we used in Case 1 (equations **(15)** through **(19)**) affords:

$$T_n = -\frac{1}{2}\binom{n}{\frac{n}{2}} + \sin(nx) \tag{22}$$

From Cases 1 and 2, we have shown: $T_n + \begin{cases} 0 \text{ for } n \text{ odd} \\ \frac{1}{2}\binom{n}{\frac{n}{2}} \text{ for } n \text{ even} \end{cases} = \sin(nx)$. Combining the results for odd and even n into one equation, we get

$$T_n + \frac{1}{4}\left[(-1)^n\binom{n}{\frac{n}{2}} + \binom{n}{\frac{n}{2}}\right] = \sin(nx), \;\; \forall\, n \in \mathbb{N}$$

which is the equation **(2)** that we wanted to prove.

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