An application of the power mean $(p = -\frac{1}{2})$ toward bounding irrational 1D wave reflection coefficients in $\mathbb{Q}$

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The generalized mean of two positive real numbers a and b is defined as the following:

$$M_p(a,b) := \left(\frac{1}{2}(a^p + b^p)\right)^{\frac{1}{p}} \quad \text{for } p \in \mathbb{R}\backslash\{0\}. \tag{1}$$

Here, we will present a niche application of the generalized mean at $p = -\frac{1}{2}$ for bounding a specific class of functions in $\mathbb{Q}$.

1 Using $M_{-\frac{1}{2}}(a,b)$ to construct a specific function bound

$M_{-\frac{1}{2}}$ can be used to provide rational function bounds for the function composition $\frac{\sqrt{f}-\sqrt{g}}{\sqrt{f}+\sqrt{g}}$, which shows up in physics. Given $f : \mathbb{R} \to \mathbb{R}^+$, $g : \mathbb{R} \to \mathbb{R}^+$, and $c \in \mathbb{R}$, we can use $M_{-\frac{1}{2}}$ to obtain a bound for the function composition $Q(f,g)$:

$$a := \frac{1}{4}\Big(1 - \Big(\frac{g}{f}\Big)^{2c}\Big) \leq Q(f,g) := \frac{f^c - g^c}{f^c + g^c} = M_{-\frac{1}{2}}(a,b) \leq b := \frac{1}{4}\Big(\Big(\frac{f}{g}\Big)^{2c} - 1\Big) \tag{2}$$

Proof sketch of (2): Define $r := \left(\frac{g}{f}\right)^{2c}$. We first notice that r is always positive because range(f), range$(g) \subset \mathbb{R}^+$, and $r > 0$ implies $\text{sign}(1-r) = \text{sign}\big(\frac{1}{r}-1\big)$. Moreover, $1-r \leq \frac{1}{r}-1$ for all $r \in (0,\infty)$, which shows that $a \leq b$ for all f, g. As a result of the first of these observations, the equality $Q(f,g) = M_{-\frac{1}{2}}(a,b)$ can be algebraically verified by plugging a and b into (1) and simplifying for the cases where a and b have the same sign.

What makes the above bound interesting is that it changes the exponent from c to $2c$. This means that if $c = \frac{n}{2}$, $n \in \mathbb{Z}$, and range(f), range$(g) \subset \mathbb{Q}$, then (2) bounds $Q(f,g)$ in $\mathbb{Q}$.

2 Physics Example: Bounding an irrational reflection coefficient in $\mathbb{Q}$

When a wave travels from one medium to another, some component of the wave is reflected and another component is transmitted. Assuming energy is conserved, the relative amount of transmission versus reflection of the wave is directly dependent on the relative impedances of the two mediums. In the case of a transverse wave traveling along a string from a region of one tension (T_1) and mass density (μ_1) to a region of another tension (T_2) and mass density (μ_2), the impedance of the first region is $\sqrt{T_1\mu_1}$ and the impedance of the second region is $\sqrt{T_2\mu_2}$. Then the reflection and transmission coefficients (which correspond to the amount of the wave reflected versus transmitted) are the following:

$$\begin{aligned} R &= \frac{\sqrt{T_1\mu_1} - \sqrt{T_2\mu_2}}{\sqrt{T_1\mu_1} + \sqrt{T_2\mu_2}} \\ T &= \frac{2\sqrt{T_1\mu_1}}{\sqrt{T_1\mu_1} + \sqrt{T_2\mu_2}} \end{aligned} \tag{3}$$

We notice that the reflection coefficient R in **(3)** is exactly the same form as the function $Q(f, g)$ defined in the previous section. Applying the bound **(2)**, we obtain:

$$p = \frac{1}{4}\left(1 - \frac{T_2\mu_2}{T_1\mu_1}\right) \leq R \approx \mathrm{HM}(p, q) = \frac{T_1\mu_1 - T_2\mu_2}{2(T_1\mu_1 + T_2\mu_2)} \leq q = \frac{1}{4}\left(\frac{T_1\mu_1}{T_2\mu_2} - 1\right), \tag{4}$$

where $\mathrm{HM}(p, q)$ denotes the harmonic mean of the bounds p and q. What we notice here is that if $T_1\mu_1$ and $T_2\mu_2$ are rational, then **(4)** bounds R in $\mathbb{Q}$, which could potentially be nice for doing simple, back-of-the-envelope calculations. As a quick example before moving on, consider the case of impedance matching, where the impedances of the two mediums are almost identical, so nearly all of the wave is transmitted. Let $T_1\mu_1 = 999$ and $T_2\mu_2 = 1000$. Plugging these values into **(4)** affords:

$$\frac{-1}{3996} \leq \frac{\sqrt{999} - \sqrt{1000}}{\sqrt{999} + \sqrt{1000}} \approx \frac{-1}{3998} \leq \frac{-1}{4000}. \tag{5}$$

When R is small, as it is in this example, we can see that the bound performs quite well. The relative error between the true value $R \approx -2.50125078 \cdot 10^{-4}$ and the rational approximation ($\frac{-1}{3998} \approx -2.50125063 \cdot 10^{-4}$) is just $6 \cdot 10^{-8}$. In this example, obtaining a better rational approximation via Taylor expansion requires going up to the third order term, resulting in a much more complicated rational approximation (i.e., $R \approx \frac{-15984005}{63904047992}$).