An Interesting Iterative Map

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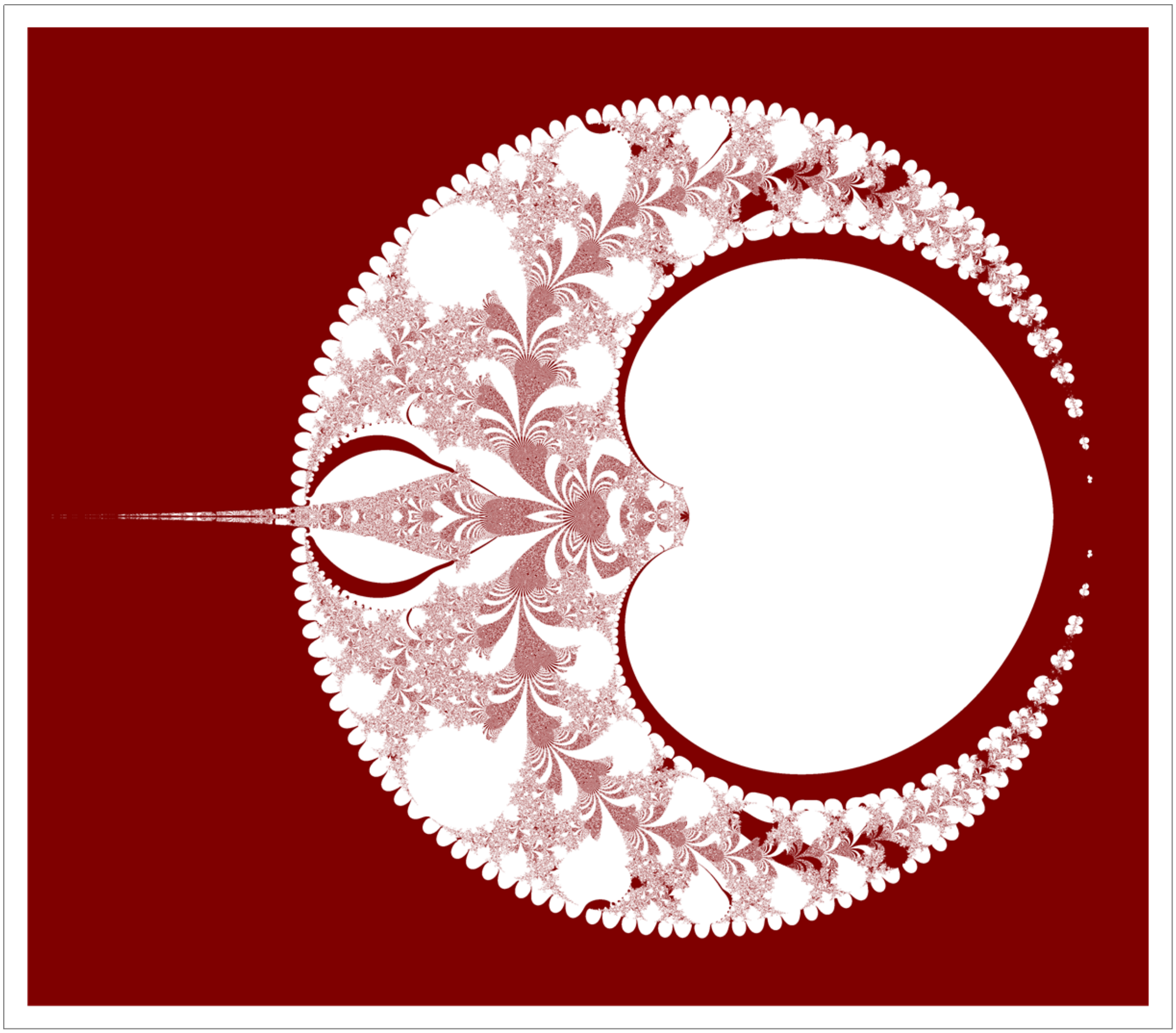

Figure 1: Result of running a crude algorithm to calculate $\{c \mid \Lambda_c^+(0) = 2\}$ shown in crimson (see Definition 2).

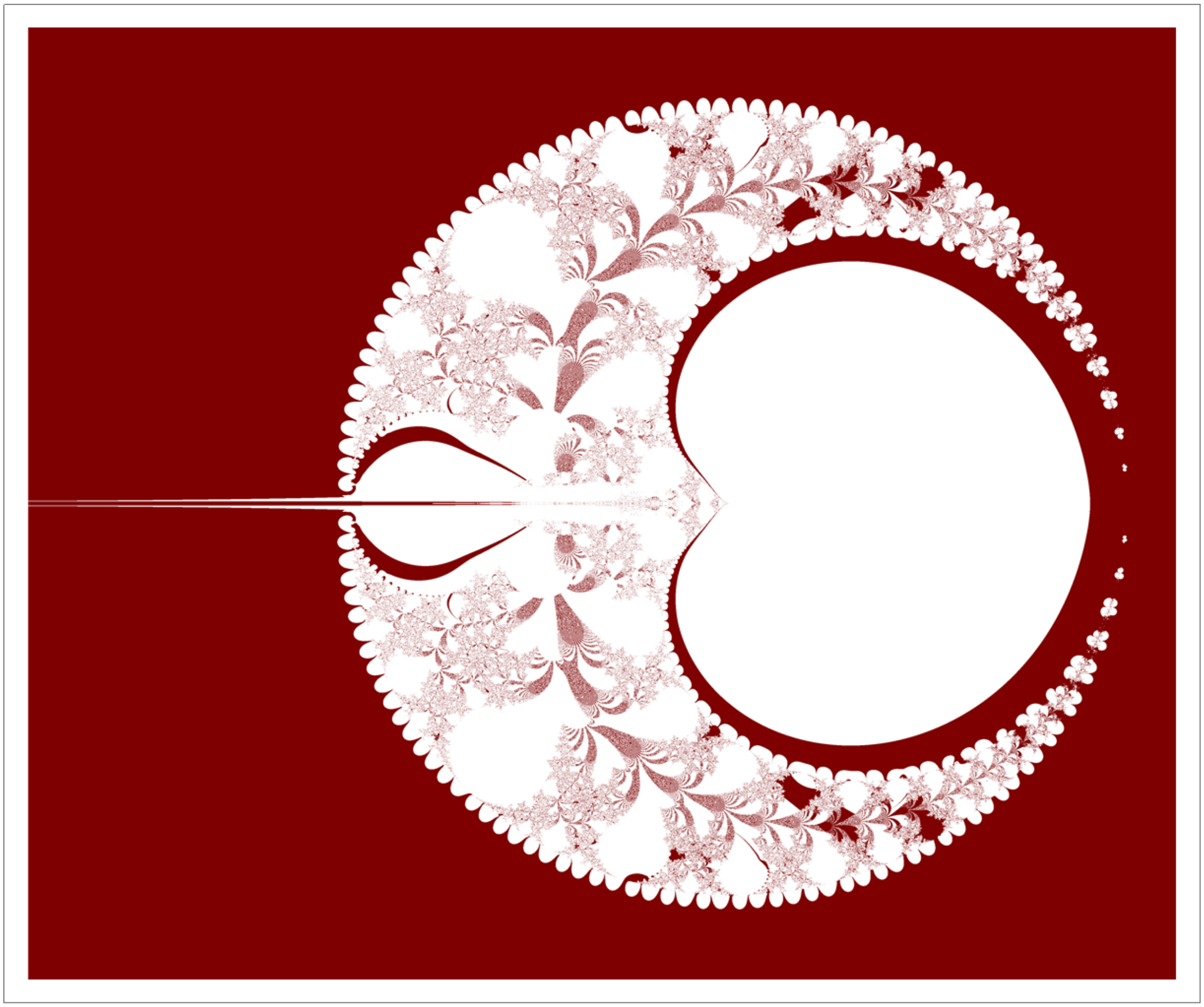

Figure 2: Result of running a crude algorithm to calculate $\{c \mid \Lambda_c^-(0) = 2\}$ shown in crimson (see Definition 2).

We are interested in the following iterative map:

Definition 1. $\forall\, z, c \in \mathbb{C}$

$$\lambda_c(z) := \left(\frac{1}{c \pm z}\right)^z$$

When it is necessary to resolve the sign $+/-$ in the denominator of the above quotient, $\lambda_c^+(z)$ and $\lambda_c^-(z)$ are used, respectively.

Lemma 1. $\forall\, n \in \mathbb{N}_0$, $\lambda_{\overline{c}}(n) = \overline{\lambda_c(n)}$. In words, under λ the orbit of the conjugate of c is equal to the conjugate of the orbit of c.

Proof. We seek an inductive proof of Lemma 1.

For $n = 0$: $\lambda_{\overline{c}}(0) = \overline{\lambda_c(0)} = 0$ by Definition 1.

Assume $\lambda_{\overline{c}}(k) = \overline{\lambda_c(k)}$ $(*)$.

For $n = k + 1$:

$$\lambda_{\overline{c}}(k+1) = \left(\frac{1}{\overline{c} \pm \lambda_{\overline{c}}(k)}\right)^{\lambda_{\overline{c}}(k)}$$

Using $(*)$ and that $\overline{a} + \overline{b} = \overline{a+b}$ $\ \forall\, a, b \in \mathbb{C}$,

$$\lambda_{\overline{c}}(k+1) = \left(\frac{1}{\overline{c \pm \lambda_c(k)}}\right)^{\overline{\lambda_c(k)}}$$

Now, we want to show the following equality:

$$\text{WTS:} \quad \left(\frac{1}{\overline{c \pm \lambda_c(k)}}\right)^{\overline{\lambda_c(k)}} = \overline{\left(\frac{1}{c \pm \lambda_c(k)}\right)^{\lambda_c(k)}}$$

Define: $c \pm \lambda_c(k) = e^{\ln(r) + i\theta}$ and $\lambda_c(k) = x + iy$ for $r, \theta, x, y \in \mathbb{R}$.

$$\text{LHS:} \quad \left(\frac{1}{e^{\ln(r) - i\theta}}\right)^{x - iy} = e^{-x\ln(r) + y\theta + i(y\ln(r) + x\theta)}$$

$$\overline{\text{RHS:}} \quad \left(\frac{1}{e^{\ln(r) + i\theta}}\right)^{x + iy} = e^{-x\ln(r) + y\theta - i(y\ln(r) + x\theta)}$$

Then LHS $=$ RHS $= e^{-x\ln(r) + y\theta + i(y\ln(r) + x\theta)}$.

Hence, if Lemma 1 is true for $n = k$, then it is true for $n = k + 1$. Since Lemma 1 is true for $n = 0$, it is true for all $n \in \mathbb{N}_0$.

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Definition 2.

$$\Lambda_c(z) := \# \text{ of limit points of } \lambda_c(z) \text{ as it is iterated to infinity.}$$

When it is necessary to resolve the sign $+/-$ in the denominator of the expression for $\lambda_c(z)$, $\Lambda_c^+(z)$ and $\Lambda_c^-(z)$ are used, respectively.

Lemma 2. By Lemma 1, $\Lambda_c(0)$ is symmetric over the real axis.

Conjectures about the orbit of 0 under λ:

1) The set $S = \{c \mid \Lambda_c(0) \neq 2\}$ is bounded.

2) The set $T = \{c \mid \Lambda_c(0) = 1\}$ is simply connected.

3) $\text{Range}\big(\Lambda_c(0)\big) = \mathbb{N} \cup \{\infty\}$

4) It is possible to obtain series expansions for $\lambda_c(0)$ at $c = \infty$. Moreover, repeated functional iteration changes only the tail of these expansions, hence the leading terms are valid as $\lambda_c(0)$ is iterated to infinity (see expansions (1) – (4) computed using Mathematica). These series imply $\Lambda_c(0) = 2$ when $|c|$ is sufficiently large.

$$\lim_{n\to\infty}\lambda_c^+(2n+1) = 1-\frac{\log(c)}{c}+\frac{1}{2c^2}\Big(2\log(c)+\log(c)^2-2\log(c)^3\Big)$$
$$+\frac{1}{6c^3}\Big(-6-6\log(c)-18\log(c)^2+11\log(c)^3+9\log(c)^4-9\log(c)^5\Big)$$
$$+\frac{1}{24c^4}\Big(48+48\log(c)+96\log(c)^2+36\log(c)^3-251\log(c)^4+80\log(c)^5+96\log(c)^6-64\log(c)^7\Big)$$
$$+\mathcal{O}\Big(\frac{1}{c^5}\Big) \tag{1}$$

$$\lim_{n\to\infty}\lambda_c^+(2n) = \frac{1}{c}+\frac{1}{c^2}\Big(-1+\log(c)^2\Big)+\frac{1}{2c^3}\Big(2+4\log(c)-4\log(c)^2-\log(c)^3+3\log(c)^4\Big)$$
$$+\frac{1}{6c^4}\Big(-6-27\log(c)+6\log(c)^2+45\log(c)^3-26\log(c)^4-12\log(c)^5+16\log(c)^6\Big)+\mathcal{O}\Big(\frac{1}{c^5}\Big) \tag{2}$$

$$\lim_{n\to\infty}\lambda_c^-(2n+1) = 1-\frac{\log(c)}{c}+\frac{1}{2c^2}\Big(-2\log(c)+\log(c)^2-2\log(c)^3\Big)$$
$$+\frac{1}{6c^3}\Big(6-6\log(c)+18\log(c)^2-13\log(c)^3+9\log(c)^4-9\log(c)^5\Big)$$
$$+\frac{1}{24c^4}\Big(48-48\log(c)+240\log(c)^2-180\log(c)^3+253\log(c)^4-136\log(c)^5+96\log(c)^6-64\log(c)^7\Big)$$
$$+\mathcal{O}\Big(\frac{1}{c^5}\Big) \tag{3}$$

$$\lim_{n\to\infty}\lambda_c^-(2n) = \frac{1}{c}+\frac{1}{c^2}\Big(1+\log(c)^2\Big)+\frac{1}{2c^3}\Big(2-4\log(c)+4\log(c)^2-\log(c)^3+3\log(c)^4\Big)$$
$$+\frac{1}{6c^4}\Big(6-39\log(c)+30\log(c)^2-45\log(c)^3+28\log(c)^4-12\log(c)^5+16\log(c)^6\Big)+\mathcal{O}\Big(\frac{1}{c^5}\Big) \tag{4}$$