A Conflict-Free Replicated JSON Datatype

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Abstract—Many applications model their data in a general-purpose storage format such as JSON. This data structure is modified by the application as a result of user input. Such modifications are well understood if performed sequentially on a single copy of the data, but if the data is replicated and modified concurrently on multiple devices, it is unclear what the semantics should be. In this paper we present an algorithm and formal semantics for a JSON data structure that automatically resolves concurrent modifications such that no updates are lost, and such that all replicas converge towards the same state. It supports arbitrarily nested list and map types, which can be modified by insertion, deletion and assignment. The algorithm performs all merging client-side and does not depend on ordering guarantees from the network, making it suitable for deployment on mobile devices with poor network connectivity, in peer-to-peer networks, and in messaging systems with end-to-end encryption.

Index Terms—CRDTs, Collaborative Editing, P2P, JSON, Optimistic Replication, Operational Semantics, Eventual Consistency.

1 Introduction

User's of mobile devices, such as smartphones, expect applications to continue working while the device is offline or has poor network connectivity, and to synchronize its state with the user's other devices when the network is available. Examples of such applications include calendars, address books, note-taking tools, to-do lists, and password managers. Similarly, collaborative work often requires several people to simultaneously edit the same text document, spreadsheet, presentation, graphic, and other kinds of document, with each person's edits reflected on the other collaborators' copies of the document with minimal delay.

What these applications have in common is that the application state needs to be replicated to several devices, each of which may modify the state locally. The traditional approach to concurrency control, serializability, would cause the application to become unusable at times of poor network connectivity [1]. If we require that applications work regardless of network availability, we must assume that users can make arbitrary modifications concurrently on different devices, and that any resulting conflicts must be resolved.

The simplest way to resolve conflicts is to discard some modifications when a conflict occurs, for example using a "last writer wins" policy. However, this approach is undesirable as it incurs data loss. An alternative is to let the user manually resolve the conflict, which is tedious and therefore should be avoided whenever possible.

Current applications solve this problem with a range of ad-hoc and application-specific mechanisms. In this paper we present a general-purpose datatype that provides the full expressiveness of the JSON data model, and supports concurrent modifications without loss of information. As we shall see later, our approach typically supports the automatic merging of concurrent modifications into a JSON data structure. We introduce a single, general mechansim (a multi-value register) into our model to record conflicting updates to leaf nodes in the JSON data structure. This mechanism then provides a consistent basis on which ap-

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plications can resolve any remaining conflicts through programmatic means, or via further user input. We expect that implementations of this datatype will drastically simplify the development of collaborative and state-synchronizing applications for mobile devices.

1.1 JSON data model

JSON is a popular general-purpose data encoding format, used in many databases and web services. It has similarities to XML, and we compare them in Section 3.2. The structure of a JSON document can optionally be constrained by a schema; however, for simplicity, this paper discusses only untyped JSON without an explicit schema.

A JSON document is a tree containing two types of branch node:

Map: A node whose children have no defined order, and where each child is labelled with a string *key*. A key uniquely identifies one of the children. We treat keys as immutable, but values as mutable, and key-value mappings can be added and removed from the map. A JSON map is also known as an *object*.

List: A node whose children have an order defined by the application. The list can be mutated by inserting or deleting list elements. A JSON list is also known as an *array*.

A child of a branch node can be either another branch node, or a leaf node. A leaf of the tree contains a primitive value (string, number, boolean, or null). We treat primitive values as immutable, but allow the value of a leaf node to be modified by treating it as a *register* that can be assigned a new value.

This model is sufficient to express the state of a wide range of applications. For example, a text document can be represented by a list of single-character strings; character-by-character edits are then expressed as insertions and deletions of list elements. In Section 3.1 we describe four more complex examples of using JSON to model application data.

1.2 Replication and conflict resolution

We consider systems in which a full copy of the JSON document is replicated on several devices. Those devices could be servers in datacenters, but we focus on mobile devices such as smartphones and laptops, which have intermittent network connectivity. We do not distinguish between devices owned by the same user and different users. Our model allows each device to optimistically modify its local replica of the document, and to asynchronously propagate those edits to other replicas.

Our only requirement of the network is that messages sent by one replica are eventually delivered to all other replicas, by retrying if delivery fails. We assume the network may arbitrarily delay, reorder and duplicate messages.

Our algorithm works client-side and does not depend on any server to transform or process messages. This approach allows messages to be delivered via a peer-to-peer network as well as a secure messaging protocol with end-to-end encryption [2]. The details of the network implementation and cryptographic protocols are outside of the scope of this paper.

In Section 4 we define formal semantics describing how conflicts are resolved when a JSON document is concurrently modified on different devices. Our design is based on three simple principles:

- 1) All replicas of the data structure should automatically converge towards the same state.
- 2) No user input should be lost due to concurrent modifications.
- 3) If all sequential permutations of a set of updates lead to the same state, then concurrent execution of those updates also leads to the same state [3].

1.3 Our contributions

Our main contribution in this work is to define an algorithm and formal semantics for collaborative, concurrent editing of JSON data structures with automatic conflict resolution. Although similar algorithms have previously been defined for lists, maps and registers individually (see Section 2), to our knowledge this paper is the first to integrate all of these structures into an arbitrarily composable datatype that can be deployed on any network topology.

A key requirement of conflict resolution is that after any sequence of concurrent modifications, all replicas eventually converge towards the same state. In Section 4.4 and the appendix we prove a theorem to show that our algorithm satisfies this requirement.

Composing maps and lists into arbitrarily nested structures opens up subtle challenges that do not arise in flat structures, due to the possibility of concurrent edits at different levels of the tree. We illustrate some of those challenges by example in Section 3.1. Nested structures are an important requirement for many applications. Consequently, the long-term goal of our work is to simplify the development of applications that use optimistic replication by providing a general algorithm for conflict resolution whose details can largely be hidden inside an easy-to-use software library.

2 RELATED WORK

In this section we discuss existing approaches to optimistic replication, collaborative editing and conflict resolution.

2.1 Operational transformation

Algorithms based on *operational transformation* (OT) have long been used for collaborative editing applications [4], [5], [6], [7]. Most of them treat a document as a single ordered list (of characters, for example) and do not support nested tree structures that are required by many applications. Some algorithms generalize OT to editing XML documents [8], [9], [10], which provides nesting of ordered lists, but these algorithms do not support key-value maps as defined in this paper (see Section 3.2). The performance of OT algorithms degrades rapidly as the number of concurrent operations increases [11].

Most deployed OT collaboration systems, including Google Docs [12], Etherpad [13], Novell Vibe [14] and Apache Wave (formerly Google Wave [10]), rely on a single server to decide on a total ordering of operations [15], a design decision inherited from the Jupiter system [7]. This approach has the advantage of making the transformation functions simpler and less error-prone [16], but it does not meet our requirements, since we want to support peer-to-peer collaboration without requiring a single server.

Many secure messaging protocols, which we plan to use for encrypted collaboration, do not guarantee that different recipients will see messages in the same order [2]. Although it is possible to decide on a total ordering of operations by using an atomic broadcast protocol [17], which avoids relying on a single server, such protocols are equivalent to consensus [18], so they can only safely make progress if a majority of participants are online and reachable. We expect that in peer-to-peer systems of mobile devices it will frequently be the case that only a minority of participants are online at the same time, and so any algorithm requiring atomic broadcast would become unavailable. The strongest guarantee such a system can give is causal ordering [19].

The Google Realtime API [20] is to our knowledge the only implementation of OT that supports arbitrary nesting of lists and maps. Like Google Docs, it relies on a single server [15]. As a proprietary product, details of its algorithms have not been published.

2.2 CRDTs

Conflict-free replicated datatypes (CRDTs) are a family of data structures that support concurrent modification and that guarantee convergence of such concurrent updates. They work by attaching additional metadata to the data structure, and making modification operations commutative by construction. The JSON datatype described in this paper is a kind of CRDT.

CRDTs for registers, counters, maps and sets are widely known [21], [22], and have been implemented in various deployed systems such as Riak [23], [24]. For ordered lists, various algorithms have been proposed, including WOOT [25], RGA [26], Treedoc [27], Logoot [28] and LSEQ [29]. However, none of them support nesting: they assume that the elements of the CRDT map or list are primitive values, not another CRDT.

The problem of nesting one CRDT inside another (also known as *composition* or *embedding*) has only been studied more recently. Riak allows nesting of counters and registers inside maps, and of maps within other maps [23], [24]. Embedding counters inside maps raises questions of semantics, which have been studied by Baquero, Almeida and Lerche [30]. Almeida et al. [31] also define delta mutations for nested maps, and Baquero et al. [32] define a theoretical framework for composition of state-based CRDTs, based on lattices. None of this work integrates CRDTs for ordered lists, but the treatment of causality in these datatypes forms a basis for the semantics developed in this paper.

Burckhardt et al. [33] define *cloud types*, which are similar to CRDTs and can be composed. They define *cloud arrays*, which behave similarly to our map datatype, and *entities*, which are like unordered sets or relations; ordered lists are not defined in this framework.

Although CRDTs for registers, maps and ordered lists have existed for years in isolation, we are not aware of any prior work that allows them all to be composed into an arbitrarily nested CRDT with a JSON-like structure.

2.3 Other approaches

Many replicated data systems need to deal with the problem of concurrent, conflicting modifications, but the solutions are often ad-hoc. For example, in Dynamo [34], if several values are concurrently written to the same key, the database preserves all of these values, and leaves conflict resolution to application code – in other words, the only datatype it supports is a multi-value register. Naively chosen merge functions often exhibit anomalies such as deleted items reappearing [34]. We believe that conflict resolution is not a simple matter that can reasonably be left to application programmers.

Another frequently-used approach to conflict resolution is *last writer wins* (LWW), which arbitrarily chooses one among several concurrent writes as "winner" and discards the others. This approach is used in Apache Cassandra, and it is an option in many other systems including Riak and CouchDB. LWW does not meet our requirements, since we want no user input to be lost due to concurrent modifications.

Finally, systems such as Bayou [35] allow offline nodes to execute transactions tentatively, and confirm them when they are next online. This approach relies on all servers executing transactions in the same serial order, and deciding whether a transaction was successful depending on its preconditions. As discussed in Section 2.1, such serial ordering requirements are prohibitive in a peer-to-peer system of mobile devices. The possibility of tentative transactions later being rolled back also opens the risk of user input being lost.

3 Composing data structures

In this section we informally introduce our approach to collaborative editing of JSON data structures. A formal presentation of the algorithm follows in Section 4.

3.1 Concurrent editing examples

To illustrate some of the subtleties that arise when JSON documents are concurrently modified, we present some examples. In all examples we assume two replicas, labelled p (drawn on the left-hand side) and q (right-hand side). Local state for a replica is drawn in boxes and modifications to local state shown with labelled solid arrows; time runs down the page. Since replicas only mutate local state, we make communication of state changes between replicas explicit in our model. Network communication is depicted with dashed arrows.

Our first example is shown in Figure 1. In a document that maps "key" to a register with value "A", replica p sets the value of the register to "B", while replica q concurrently sets it to "C". As the replicas subsequently exchange edits via network communication, they detect the conflict. Since we do not want to simply discard one of the edits, and the strings "B" and "C" cannot be meaningfully merged, the system must preserve both concurrent updates. This datatype is known as a *multi-value register*: although a replica can only assign a single value to the register, reading the register may return a set of multiple values that were concurrently written. An implementation may keep metadata about the provenance of each value (who made the change, on which device, at what time) to assist the developer in automatically resolving such conflicts in an applicationspecific way, or deferring to users for manual resolution.

Another example is given in Figure 2. Here, one replica adds "red" to a map of colors, while concurrently another client first blanks out the entire map of colors, and then adds "green". As the replicas merge their edits, all changes must be preserved: "blue" must be absent from the final map, since it was removed by blanking out the map, while "red" and "green" must be present, since they were explicitly added.

The example in Figure 3 shows two replicas concurrently creating a new shopping list under the same map key "grocery", and adding items to the list. When the modifications are combined, the lists need to be merged. We preserve the ordering and adjacency of items inserted at each replica, so "ham" appears after "eggs", and "flour" appears after "milk" in the merged result. There is no information on which replica's items should appear first in the merged result, so the algorithm can make an arbitrary choice between "eggs, ham, milk, flour" and "milk, flour, eggs, ham", provided that all replicas end up with the same order.

Finally, Figure 4 shows a limitation of the principle of preserving all user input. In a to-do list application, one replica removes a to-do item from the list, while another replica concurrently marks the same item as done. As the changes are merged, the update of the map key "done" effectively causes the list item to be resurrected on replica p, leaving a to-do item without a title (since the title was deleted as part of deleting the list item). This behavior is consistent with the example in Figure 2, but it is perhaps surprising. In this case it may be more desirable to discard one of the concurrent updates, and thus preserve the implicit schema that a to-do item has both a "title" and a "done" field. We leave the analysis of developer expectations and the development of a schema language for future work.

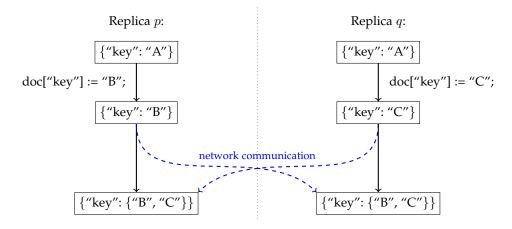


Fig. 1. Concurrent assignment to the register at doc["key"] by replicas p and q.

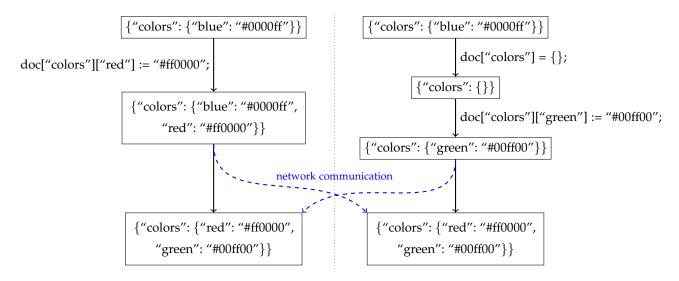


Fig. 2. Modifying the contents of a nested map while concurrently the entire map is overwritten.

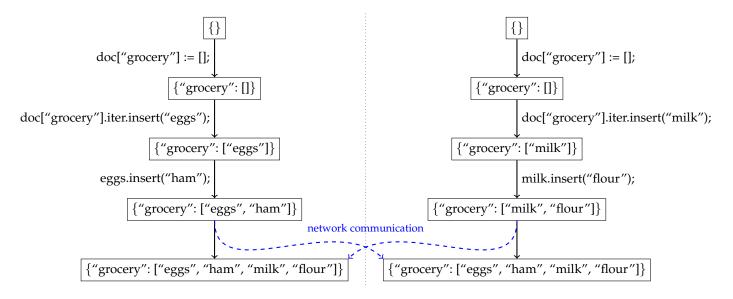


Fig. 3. Two replicas concurrently create ordered lists under the same map key.

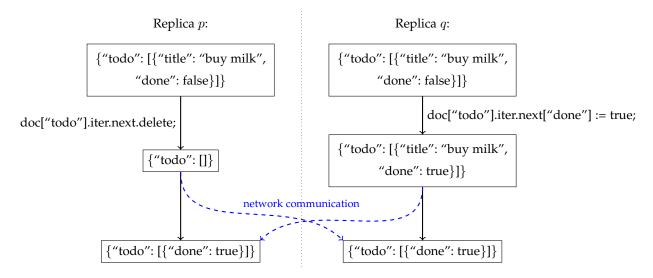


Fig. 4. One replica removes a list element, while another concurrently updates its contents.

3.2 JSON versus XML

The most common alternative to JSON is XML, and collaborative editing of XML documents has been previously studied [8], [9], [10]. Besides the superficial syntactical differences, the tree structure of XML and JSON appears quite similar. However, there is an important difference that we should highlight.

JSON has two collection constructs that can be arbitrarily nested: maps for unordered key-value pairs, and lists for ordered sequences. In XML, the children of an element form an ordered sequence, while the attributes of an element are unordered key-value pairs. However, XML does not allow nested elements inside attributes – the value of an attribute can only be a primitive datatype. Thus, XML supports maps within lists, but not lists within maps. In this regard, XML is less expressive than JSON: the example in Figure 3 cannot occur in XML.

Some applications may attach map-like semantics to the children of an XML document, for example by interpreting the child element name as key. However, this key-value structure is not part of XML itself, and would not be enforced by existing collaborative editing algorithms for XML. If multiple children with the same key are concurrently created, existing algorithms would create duplicate children with the same key rather than merging them like in Figure 3.

3.3 Document editing API

To define the semantics for collaboratively editable data structures, we first define a simple command language that is executed locally at any of the replicas, and which allows that replica's local copy of the document to be queried and modified. Performing read-only queries has no side-effects, but modifying the document has the effect of producing *operations* describing the mutation. Those operations are immediately applied to the local copy of the document, and also enqueued for asynchronous broadcasting to other replicas.

The syntax of the command language is given in Figure 5. It is not a full programming language, but rather

```
CMD
             let x = EXPR
                                   x \in VAR
             EXPR := v
                                   v \in VAL
             EXPR.insert(v)
                                   v \in VAL
             EXPR.delete
             yield
             CMD; CMD
EXPR
             doc
                                   x \in VAR
             EXPR[key]
                                    key \in String
             EXPR.iter
             EXPR.next
             EXPR.keys
             EXPR.values
 VAR
                                   x \in \text{VarString}
        ::=
 VAL
                                   n \in \text{Number}
        ::=
             n
                                    str \in String
             true | false | null
             {} | []
```

Fig. 5. Syntax of command language for querying and modifying a document.

```
doc := {};
let list = doc["shopping"].iter;
list.insert("eggs");
let eggs = list.next;
eggs.insert("milk");
list.insert("cheese");

// Final state:
{"shopping": ["cheese", "eggs", "milk"]}
eggs.values // evaluates to {"eggs"}
eggs.next.values // evaluates to {"milk"}
```

Fig. 6. Example of programmatically constructing a JSON document.

an API through which the document state is queried and modified. We assume that the program accepts user input and issues a (possibly infinite) sequence of commands to the API. We model only the semantics of those commands, and do not assume anything about the program in which the command language is embedded. The API differs slightly from the JSON libraries found in many programming languages, in order to allow us to define consistent merge semantics.

We first explain the language informally, before giving its formal semantics. The expression construct EXPR is used to construct a *cursor* which identifies a position in the document. An expression starts with either the special token doc, identifying the root of the JSON document tree, or a variable x that was previously defined in a let command. The expression defines, left to right, the path the cursor takes as it navigates through the tree towards the leaves: the subscript operator [key] selects a key within a map, iter starts iterating over an ordered list and next moves to the next element of an ordered list.

The expression construct EXPR can also query the state of the document: keys returns the set of keys in the map at the current cursor, and values returns the contents of the multi-value register at the current cursor. (values is not defined if the cursor refers to a map or list.)

A command CMD either sets the value of a local variable (let), performs network communication (yield), or modifies the document. A document can be modified by assigning the value of a register (using the assignment operator :=), by inserting an element into a list (insert), or by deleting an element from a list or a map (delete). The preceding expression EXPR defines the cursor that identifies the part of the document being modified.

Figure 6 shows an example sequence of commands that constructs a new document representing a shopping list. First doc is set to {}, the empty map literal. The second line navigates to the key "shopping" and performs iter, which treats the value at that key as a list and selects the head of the list. Since the key does not exist, it is implicitly created and set to the empty list. Finally, three items are inserted into the list. The insert command adds a new list element after the current cursor position, or at the head if the cursor is at the head of the list. The variable list refers to the head, so cheese is inserted before eggs, but the variable eggs refers to the list element "eggs", so milk is inserted after eggs.

A few features of this language deliberately differ from most mainstream programming languages: keys in maps are implicitly created when they are first accessed, so there is no need for a command to put a new key-value pair into a map; lists can only be navigated by iteration (next) but not by index; and the language has literals for creating empty maps and lists, but not for non-empty collections. As we shall see later, these features are helpful for achieving desirable semantics in the presence of concurrent modifications.

4 FORMAL SEMANTICS

The state of replica p is described by A_p , a finite partial function. The evaluation rules of the command language

```
\begin{split} A_p &= \{ \; \mathsf{mapT}(\mathsf{doc}) \mapsto \{ \; \mathsf{listT}(\text{"shopping"}) \mapsto \{ \\ & \mathsf{next}(\mathsf{head}) \mapsto id_3, \\ & \mathsf{regT}(id_3) \mapsto \{ \; id_3 \mapsto \text{"cheese"}) \, \}, \\ & \mathsf{next}(id_3) \mapsto id_1, \; \mathsf{pres}(id_3) \mapsto \{ id_3 \}, \\ & \mathsf{regT}(id_1) \mapsto \{ \; id_1 \mapsto \text{"eggs"}) \, \}, \\ & \mathsf{next}(id_1) \mapsto id_2, \; \mathsf{pres}(id_1) \mapsto \{ id_1 \}, \\ & \mathsf{regT}(id_2) \mapsto \{ \; id_2 \mapsto \text{"milk"}) \, \}, \\ & \mathsf{next}(id_2) \mapsto \mathsf{tail}, \; \mathsf{pres}(id_2) \mapsto \{ id_2 \} \\ & \; \}, \; \mathsf{pres}(\text{"shopping"}) \mapsto \{ id_1, id_2, id_3 \} \; \}, \\ & \mathsf{list} \mapsto \mathsf{cursor}(\langle \mathsf{mapT}(\mathsf{doc}), \mathsf{listT}(\text{"shopping"}) \rangle, \; \mathsf{head}), \\ & \mathsf{eggs} \mapsto \mathsf{cursor}(\langle \mathsf{mapT}(\mathsf{doc}), \mathsf{listT}(\text{"shopping"}) \rangle, \; id_1) \\ \} \end{split}
```

Fig. 7. Internal state A_p of replica p after the execution of the commands in Figure 6.

inspect and modify this local state A_p , and they are independent of the state A_q of any other replica q. The only communication between replicas occurs in the evaluation of the yield command, which we discuss later. For now, we concentrate on the execution of commands at a single replica p.

An illustrative example of the replica state A_p is given in Figure 7, which corresponds to the shopping list example of Figure 6. For each local variable defined with a let command, A_p maps the variable name to a cursor. In addition, A_p maps mapT(doc) to a nested partial function representing the contents of the document. mapT denotes that the document doc is of type map. The only map entry is the key "shopping" of type listT. The list is represented in a manner resembling a linked list, with each list element assigned a unique identifier (id_1, id_2, id_3) , and special head and tail atoms denoting the beginning and end of the list, respectively.

4.1 Expression evaluation

Figure 8 gives the rules for evaluating EXPR expressions in the command language, which are evaluated in the context of the local replica state A_p . The EXEC rule formalizes the assumption that commands are executed sequentially. The LET rule allows the program to define a local variable, which is added to the local state, and the corresponding VAR rule allows the program to retrieve the value of a previously defined variable.

The rules in Figure 8 show how an expression is evaluated to a cursor, which unambiguously identifies a particular position in a JSON document by describing a path from the root of the document tree to some branch or leaf node. A cursor consists only of immutable keys and identifiers, so it can be sent over the network to another replica, where it can be used to locate the same position in the document.

For example,

```
cursor(\langle mapT(doc), listT("shopping") \rangle, id_1)
```

is a cursor representing the list element "eggs" in Figure 6. It can be interpreted as a path through the structure in

$$\operatorname{Exec} \frac{\operatorname{cmd}_1 : \operatorname{CMD} \quad A_p, \operatorname{cmd}_1 \Rightarrow A'_p}{A_p, \operatorname{cmd}_1 ; \operatorname{cmd}_2 ; \ldots \rangle} = \operatorname{A'_p, \operatorname{cmd}_2 ; \ldots \rangle} = \operatorname{A'_p,$$

Fig. 8. Rules for evaluating expressions.

Figure 7, read from left to right: starting from the doc map at the root, it traverses through the map entry "shopping" of type listT, and finishes with the list element with identifier id_1 .

In general, $\operatorname{cursor}(\langle k_1,\ldots,k_{n-1}\rangle,\,k_n)$ consists of a (possibly empty) vector of keys $\langle k_1,\ldots,k_{n-1}\rangle$, and a final key k_n which is always present. k_n can be thought of as the final element of the vector, with the distinction that it is not tagged with a datatype, whereas the elements of the vector are tagged with the datatype of the branch node, either mapT or listT.

The DOC rule in Figure 8 defines the simplest cursor cursor($\langle \rangle$, doc), referencing the root of the document using the special atom doc. The GET rule navigates a cursor to a particular key within a map. For

example, the expression doc["shopping"] evaluates to $cursor(\langle mapT(doc) \rangle$, "shopping") by applying the DOC and GET rules. Note that the expression doc[...] implicitly asserts that doc is of type mapT, and this assertion is encoded in the cursor.

The ITER rule shifts the cursor into a list and positions it at the head of the list. This rule applies even if the list is empty or nonexistent in A_p . The four rules $\mathrm{NEXT}_{1,2,3,4}$ handle iteration through a linked list by setting the final key in the cursor to the identifier of the next list element. The NEXT rules apply only if the list exists in A_p , and NEXT_4 recursively descends the local state according to the vector of keys in the cursor. $\mathrm{NEXT}_{2,3}$ are conditional on an entry $\mathrm{pres}(k')$ in A_p , which encodes the *presence* of an element in the list: if the presence set is empty, that means the list

element was deleted, and so NEXT₃ skips over the element. If the presence set is nonempty, NEXT₂ applies.

The KEYS_{1,2,3} rules allow the application to inspect the set of keys that are defined in a map. This set is determined by examining the local state, and excluding any keys for which the presence set is empty (indicating that the key has been deleted).

Finally, the VAL_{1,2,3} rules allow the application to read the contents of a register at a particular cursor position, using a similar recursive rule structure as the NEXT rules. A register is expressed using the regT type annotation in the local state, and the VAL rules only apply if the register identified by the cursor exists in A_p . Although a replica can only assign a single value to a register, a register can nevertheless contain multiple values if multiple replicas concurrently assign values to it.

4.2 Generating operations

When commands mutate the state of the document, they generate *operations* that describe the mutation. In our semantics, a command never directly modifies the local replica state A_p , but only generates an operation. That operation is then immediately applied to A_p so that it takes effect locally, and the operation is also asynchronously broadcast to the other replicas. After an operation is received a from remote replica, it is applied locally when its causal dependencies are satisfied, as detailed below.

4.2.1 Lamport timestamps

Every operation in our model is given a unique identifier, which is used in the local state and in cursors. For example, in Figure 7, $id_{1,2,3}$ are used to identify list elements and also the values of registers. Those identifiers $id_{1,2,3}$ are in fact the identifiers of the operations that inserted the list elements or performed the register assignments.

In order to generate globally unique operation identifiers without requiring synchronous coordination between replicas we use Lamport timestamps [36]. A Lamport timestamp is a pair (c,p) where $p \in \text{ReplicaID}$ is the unique identifier of the replica on which the edit is made (for example, a hash of its public key), and $c \in \mathbb{N}$ is a counter that is stored at each replica and incremented for every operation. Since each replica generates a strictly monotonically increasing sequence of counter values c, the pair (c,p) is unique.

If a replica receives an operation with a counter value c that is greater than the locally stored counter value, the local counter is increased to the value of the incoming counter. This ensures that if operation o_1 causally happened before o_2 (that is, the replica that generated o_2 had received and processed o_1 before o_2 was generated), then o_2 must have a greater counter value than o_1 . Only concurrent operations can have equal counter values.

We can thus define a total ordering < for Lamport timestamps:

$$(c_1, p_1) < (c_2, p_2)$$
 iff $(c_1 < c_2) \lor (c_1 = c_2 \land p_1 < p_2)$.

If one operation happened before another, this ordering is consistent with causality (the earlier operation has a lower timestamp). If two operations are concurrent, their order according to < is arbitrary but deterministic. This ordering

property is important for our definition of the semantics of ordered lists.

4.2.2 Operation structure

An operation is a tuple of the form

```
op(id: \mathbb{N} \times \text{ReplicaID}, \\ deps: \mathcal{P}(\mathbb{N} \times \text{ReplicaID}), \\ cur: \text{cursor}(\langle k_1, \dots, k_{n-1} \rangle, \, k_n), \\ mut: \text{insert}(v) \mid \text{delete} \mid \text{assign}(v) \quad v: \text{VAL} \\ )
```

where id is the Lamport timestamp that uniquely identifies the operation, cur is the cursor describing the position in the document being modified, and mut is the mutation that was requested at the specified position.

deps is the set of causal dependencies of the operation, given as a set of Lamport timestamps. The rules below define deps to be the set of all operation IDs that had already been applied to the document at the time when the operation was generated. In a real implementation, this set would become impracticably large, so a compact representation of causal history would be used instead – for example, version vectors [37], state vectors [4], or dotted version vectors [38]. However, to avoid ambiguity in our semantics we give the dependencies as a simple set of operation IDs.

The purpose of the causal dependencies *deps* is to impose a partial ordering on operations: an operation can only be applied after all operations that "happened before" it have been applied. In particular, this means that the sequence of operations generated at one particular replica will be applied in the same order at every other replica. Operations that are concurrent, i.e. where there is no causal dependency, can be applied in any order.

4.2.3 Semantics of generating operations

The evaluation rules for commands are given in Figure 9. The MAKE-ASSIGN, MAKE-INSERT and MAKE-DELETE rules define how these respective commands mutate the document: all three delegate to the MAKE-OP rule to generate and apply the operation. MAKE-OP generates a new Lamport timestamp by choosing a counter value that is 1 greater than any existing counter in $A_p(\mathsf{ops})$, the set of all operation IDs that have been applied to replica p.

MAKE-OP constructs an op() tuple of the form described above, and delegates to the APPLY-LOCAL rule to process the operation. APPLY-LOCAL does three things: it evaluates the operation to produce a modified local state A_p' , it adds the operation to the queue of generated operations $A_p(\text{queue})$, and it adds the operation ID to the set of processed operations $A_p(\text{ops})$.

The yield command, inspired by Burckhardt et al. [33], performs network communication: sending and receiving operations to and from other replicas, and applying operations from remote replicas. The rules APPLY-REMOTE, SEND, RECV and YIELD define the semantics of yield. Since any of these rules can be used to evaluate yield, their effect is nondeterministic, which models the asynchronicity of the network between replicas: a message sent by one replica

$$\begin{aligned} \operatorname{Make-Assign} & \frac{A_p, \exp r \implies cur \quad val : \operatorname{VAL} \quad A_p, \operatorname{makeOp}(cur, \operatorname{assign}(val)) \implies A'_p}{A_p, \exp r := val \implies A'_p} \\ & \frac{A_p, \exp r \implies cur \quad val : \operatorname{VAL} \quad A_p, \operatorname{makeOp}(cur, \operatorname{insert}(val)) \implies A'_p}{A_p, \exp r.\operatorname{insert}(val) \implies A'_p} \\ & \operatorname{Make-Insert} & \frac{A_p, \exp r \implies cur \quad val : \operatorname{VAL} \quad A_p, \operatorname{makeOp}(cur, \operatorname{insert}(val)) \implies A'_p}{A_p, \exp r.\operatorname{insert}(val) \implies A'_p} \\ & \operatorname{Make-Delete} & \frac{A_p, \exp r \implies cur \quad A_p, \operatorname{makeOp}(cur, \operatorname{delete}) \implies A'_p}{A_p, \operatorname{paphy}(\operatorname{op}(\operatorname{ctr} + 1, p), A_p(\operatorname{ops}), \operatorname{cur}, \operatorname{mut})) \implies A'_p} \\ & \operatorname{Make-Op} & \frac{ctr = \operatorname{max}(\{0\} \cup \{c_i \mid (c_i, p_i) \in A_p(\operatorname{ops})\} \quad A_p, \operatorname{apphy}(\operatorname{op}(\operatorname{ctr} + 1, p), A_p(\operatorname{ops}), \operatorname{cur}, \operatorname{mut})) \implies A'_p}{A_p, \operatorname{makeOp}(\operatorname{cur}, \operatorname{mut}) \implies A'_p} \\ & \operatorname{Apply-Local} & \frac{A_p, \operatorname{op} \implies A'_p}{A_p, \operatorname{apphy}(\operatorname{op}) \implies A'_p(\operatorname{queue}) \cup \{op\}, \operatorname{ops} \mapsto A'_p(\operatorname{ops}) \cup \{op.id\}\}} \\ & \operatorname{Apply-Remote} & \frac{op \in A_p(\operatorname{recv}) \quad op.id \notin A_p(\operatorname{ops}) \quad op.deps \subseteq A_p(\operatorname{ops}) \quad A_p, \operatorname{op} \implies A'_p}{A_p, \operatorname{yield} \implies A'_p(\operatorname{ops} \mapsto A'_p(\operatorname{ops}) \cup \{op.id\}\}} \\ & \operatorname{SEND} & \frac{q : \operatorname{ReplicaID}}{A_p, \operatorname{yield} \implies A_p(\operatorname{send}) \cup A_p(\operatorname{queue})} \\ & \operatorname{Recv} & \frac{q : \operatorname{ReplicaID}}{A_p, \operatorname{yield} \implies A'_p(\operatorname{recv} \mapsto A_p(\operatorname{recv}) \cup A_q(\operatorname{send}))} \\ & \operatorname{Yield} & \frac{A_p, \operatorname{yield} \implies A'_p \quad A'_p, \operatorname{yield} \implies A''_p}{A_p, \operatorname{yield} \implies A''_p} \end{aligned}$$

Fig. 9. Rules for generating, sending, and receiving operations.

arrives at another replica at some arbitrarily later point in time, and there is no mesage ordering guarantee in the network.

The SEND rule takes any operations that were placed in $A_p(\text{queue})$ by APPLY-LOCAL and adds them to a send buffer $A_p(\text{send})$. Correspondingly, the RECV rule takes operations in the send buffer of replica q and adds them to the receive buffer $A_p(\text{recv})$ of replica p. This is the only rule that involves more than one replica, and it models all network communication.

Once an operation appears in the receive buffer $A_p({\rm recv})$, the rule APPLY-REMOTE may apply. Under the preconditions that the operation has not already been processed and that its causal dependencies are satisified, APPLY-REMOTE evaluates the operation in the same way as APPLY-LOCAL, and adds the operation ID to the set of processed operations $A_p({\rm ops})$.

The actual document modifications are performed by evaluating the operations, which we discuss next.

4.3 Applying operations

Figure 10 gives the rules that evaluate an operation op within a context ctx, and produce an updated context ctx'. The context is initially the replica state A_p , but may refer to subtrees of the state as rules are recursively applied. These rules are used by APPLY-LOCAL and APPLY-REMOTE to perform the state updates on a document.

When the operation's cursor refers to a tree node that is not the document root doc, the DESCEND rule first applies. It recursively descends the document tree according to the vector of keys given in the cursor. If the tree node already exists in the local replica state, CHILD-GET finds it, otherwise CHILD-MAP and CHILD-LIST create an empty map or list respectively.

The DESCEND rule also invokes ADD-ID_{1,2} at each tree node along the path described by the cursor, adding the operation ID to the presence set pres(k) to indicate that the subtree includes a mutation made by this operation.

The remaining rules in Figure 10 apply when the vector of keys in the cursor is empty, i.e. when descended to the context of the tree node to which the mutation applies. The ASSIGN rule handles assignment of a primitive value to a register, EMPTY-MAP handles assignment where the value is the empty map literal {}, and EMPTY-LIST handles assignment of the empty list []. These three rules for assign have a similar structure: first clearing the prior value at the cursor (as discussed in the next section), then adding the operation ID to the presence set, and finally incorporating the new value into the tree of local state.

The INSERT_{1,2} rules handle insertion of a new element into an ordered list. In this case, the cursor refers to the list element *prev*, and the new element is inserted after that position in the list. INSERT₁ performs the insertion by manipulating the linked list structure. INSERT₂ handles the case of multiple replicas concurrently inserting list elements at the same position, and uses the ordering relation < on Lamport timestamps to consistently determine the insertion point. Our approach for handling insertions is based on the RGA algorithm [26]. We show later that these rules ensure all replicas converge towards the same state.

$$DESCEND = \frac{ctx, k_1 \Rightarrow child \quad child, \operatorname{op}(id, deps, \operatorname{cursor}((k_2, \dots, k_{n-1}), k_n), \operatorname{mut}) \Rightarrow \operatorname{child}' \quad \operatorname{ctx}, \operatorname{addId}(k_1, id, \operatorname{mut}) \Rightarrow \operatorname{ctx}'}{\operatorname{ctx}, \operatorname{op}(id, deps, \operatorname{cursor}((k_1, k_2, \dots, k_{n-1}), k_n), \operatorname{mut}))} \Rightarrow \operatorname{ctx}'[k_1 \mapsto \operatorname{child}']$$

$$CHILD-GET = \frac{k \in \operatorname{dom}(\operatorname{ctx})}{\operatorname{ctx}, k \Rightarrow \operatorname{ctx}(k)} \qquad CHILD-MAP = \frac{\operatorname{mapT}(k) \notin \operatorname{dom}(\operatorname{ctx})}{\operatorname{ctx}, \operatorname{mapT}(k) \Rightarrow \{\}} \qquad CHILD-LIST = \frac{\operatorname{listT}(k) \notin \operatorname{dom}(\operatorname{ctx})}{\operatorname{ctx}, \operatorname{listT}(k) \Rightarrow \{\operatorname{next}(\operatorname{head}) \mapsto \operatorname{tail}\}}$$

$$CHILD-REG = \frac{\operatorname{regT}(k) \notin \operatorname{dom}(\operatorname{ctx})}{\operatorname{ctx}, \operatorname{regT}(k) \Rightarrow \{\}} \qquad PRESENCE_1 = \frac{\operatorname{pres}(k) \in \operatorname{dom}(\operatorname{ctx})}{\operatorname{ctx}, \operatorname{pres}(k) \Rightarrow \operatorname{ctx}(\operatorname{pres}(k))} \qquad PRESENCE_2 = \frac{\operatorname{pres}(k) \notin \operatorname{dom}(\operatorname{ctx})}{\operatorname{ctx}, \operatorname{pres}(k) \Rightarrow \{\}}$$

$$ADD-ID_1 = \frac{\operatorname{mut} \neq \operatorname{delete}}{\operatorname{ctx}, \operatorname{addId}(k_{tag}, id, \operatorname{mut}) \Rightarrow \operatorname{ctx}[\operatorname{pres}(k) \mapsto \operatorname{pres} \setminus \{id\}]} \qquad ADD-ID_2 = \frac{\operatorname{mut} = \operatorname{delete}}{\operatorname{ctx}, \operatorname{addId}(k_{tag}, id, \operatorname{mut}) \Rightarrow \operatorname{ctx}}$$

$$ASSIGN = \frac{\operatorname{val} \neq \{1 \land \land val \neq \{\}}{\operatorname{ctx}, \operatorname{clear}(\operatorname{deps}, \operatorname{regT}(k)) \Rightarrow \operatorname{ctx}', \operatorname{pres}} \qquad \operatorname{ctx}', \operatorname{addId}(\operatorname{regT}(k), id, \operatorname{assign}(\operatorname{val})) \Rightarrow \operatorname{ctx}'' = \operatorname{ctx}'', \operatorname{regT}(k) \Rightarrow \operatorname{child}$$

$$\operatorname{ctx}, \operatorname{op}(id, \operatorname{deps}, \operatorname{cursor}(\langle\rangle, k), \operatorname{assign}(\operatorname{val})) \Rightarrow \operatorname{ctx}'' = \operatorname{regT}(k) \Rightarrow \operatorname{child}$$

$$\operatorname{ctx}, \operatorname{op}(id, \operatorname{deps}, \operatorname{cursor}(\langle\rangle, k), \operatorname{assign}(\operatorname{val})) \Rightarrow \operatorname{ctx}'' = \operatorname{ctx}'' = \operatorname{ctx}'', \operatorname{addId}(\operatorname{listT}(k), \operatorname{id}, \operatorname{assign}(\operatorname{val})) \Rightarrow \operatorname{ctx}'' = \operatorname{ctx}'', \operatorname{mapT}(k) \Rightarrow \operatorname{child}$$

$$\operatorname{ctx}, \operatorname{op}(id, \operatorname{deps}, \operatorname{cursor}(\langle\rangle, k), \operatorname{assign}(\operatorname{val})) \Rightarrow \operatorname{ctx}'' = \operatorname{$$

Fig. 10. Rules for applying insertion and assignment operations to update the state of a replica.

$$\text{Delete} \frac{ctx, \operatorname{clearElem}(deps, k) \Rightarrow ctx', pres}{ctx, \operatorname{op}(id, deps, \operatorname{cursor}(\lozenge), k), \operatorname{delete}) \Rightarrow ctx'} }{ctx, \operatorname{op}(id, deps, \operatorname{cursor}(\lozenge), k), \operatorname{delete}) \Rightarrow ctx'}$$

$$\text{CLEAR-ELEM} \frac{ctx, \operatorname{clearAny}(deps, k) \Rightarrow ctx', pres_1}{ctx, \operatorname{clearElem}(deps, k) \Rightarrow ctx'[\operatorname{pres}(k) \Rightarrow pres_2] \quad pres_3 = pres_1 \cup pres_2 \setminus deps} }{ctx, \operatorname{clear}(deps, \operatorname{map}T(k)) \quad ctx_1, \operatorname{clear}(deps, \operatorname{list}T(k)) \quad ctx_2, \operatorname{clear}(deps, \operatorname{reg}T(k)) }{ctx, \operatorname{clearAny}(deps, k) \Rightarrow ctx_2, pres_3} \Rightarrow ctx_3, pres_3}$$

$$\frac{ctx, \operatorname{clearAny}(deps, k) \Rightarrow ctx_3, pres_1 \cup pres_2 \cup pres_3} }{ctx, \operatorname{clearAny}(deps, k) \Rightarrow ctx_3, pres_1 \cup pres_2 \cup pres_3} }$$

$$\frac{k \notin \operatorname{dom}(ctx)}{ctx, \operatorname{clear}(deps, k) \Rightarrow ctx_3, pres_1 \cup pres_2 \cup pres_3} }$$

$$\frac{k \notin \operatorname{dom}(ctx)}{ctx, \operatorname{clear}(deps, k) \Rightarrow ctx_3, pres_1 \cup pres_2 \cup pres_3} }$$

$$\frac{k \notin \operatorname{dom}(ctx)}{ctx, \operatorname{clear}(deps, k) \Rightarrow ctx_3, pres_1 \cup pres_2 \cup pres_3} }$$

$$\frac{k \notin \operatorname{dom}(ctx)}{ctx, \operatorname{clear}(deps, k) \Rightarrow ctx_3, pres_1 \cup pres_2 \cup pres_3} }$$

$$\frac{\operatorname{dom}(ctx)}{ctx, \operatorname{clear}(deps, k) \Rightarrow ctx_3, pres_1 \cup pres_2 \cup pres_3} }$$

$$\frac{\operatorname{dom}(ctx)}{ctx, \operatorname{clear}(deps, t) \Rightarrow ctx_3, pres_1 \cup pres_2} }$$

$$\frac{\operatorname{dom}(ctx)}{ctx, \operatorname{clearMap}(deps, k), \operatorname{ctx'}, \operatorname{clear}(deps, next), \operatorname{ctx'}(\operatorname{clear}(deps, next), \operatorname{ctx'}(\operatorname{clear}(deps, next), \operatorname{ctx'}(\operatorname{clear}(deps, next), \operatorname{ctx'}(\operatorname{clear}(deps, k), \operatorname{ctx'}, \operatorname{clear}(deps, next), \operatorname{ctx'}(\operatorname{clear}$$

Fig. 11. Rules for applying deletion operations to update the state of a replica.

4.3.1 Clearing prior state

Assignment and deletion operations require that prior state (the value being overwritten or deleted) is cleared, while also ensuring that concurrent modifications are not lost, as illustrated in Figure 2. The rules to handle this clearing process are given in Figure 11. Intuitively, the effect of clearing something is to reset it to its empty state by undoing any operations that causally precede the current operation, while leaving the effect of any concurrent operations untouched.

A delete operation can be used to delete either an element from an ordered list or a key from a map, depending on what the cursor refers to. The DELETE rule shows how this operation is evaluated by delegating to CLEAR-ELEM. In turn, CLEAR-ELEM uses CLEAR-ANY to clear out any data with a given key, regardless of whether it is of type mapT, listT or regT, and also updates the presence set to include any nested operation IDs, but exclude any operations in deps.

The premises of CLEAR-ANY are satisfied by CLEAR-MAP₁, CLEAR-LIST₁ and CLEAR-REG if the respective key appears in ctx, or by CLEAR-NONE (which

does nothing) if the key is absent.

As defined by the ASSIGN rule, a register maintains a mapping from operation IDs to values. CLEAR-REG updates a register by removing all operation IDs that appear in *deps* (i.e. which causally precede the clearing operation), but retaining all operation IDs that do not appear in *deps* (from assignment operations that are concurrent with the clearing operation).

Clearing maps and lists takes a similar approach: each element of the map or list is recursively cleared using clearElem, and presence sets are updated to exclude *deps*. Thus, any list elements or map entries whose modifications causally precede the clearing operation will end up with empty presence sets, and thus be considered deleted. Any map or list elements containing operations that are concurrent with the clearing operation are preserved.

4.4 Convergence

As outlined in Section 1.2, we require that all replicas automatically converge towards the same state – a key requirement of a CRDT. We now formalize this notion, and show that the rules in Figures 8 to 11 satisfy this requirement.

Definition 1 (valid execution). A valid execution is a set of operations generated by a set of replicas $\{p_1, \ldots, p_k\}$, each reducing a sequence of commands $\langle cmd_1; \ldots; cmd_n \rangle$ without getting stuck.

A reduction gets stuck if there is no application of rules in which all premises are satisfied. For example, the $\mathsf{NEXT}_{2,3}$ rules in Figure 8 get stuck if next tries to iterate past the end of a list; in a real implementation this would be a runtime error. By constraining valid executions to those that do not get stuck, we ensure that operations only refer to list elements that actually exist.

Note that it is valid for an execution to never perform any network communication, either because it never invokes the yield command, or because the nondeterministic execution of yield never applies the RECV rule. We need only a replica's local state to determine whether reduction gets stuck.

Definition 2 (history). A history is a sequence of operations in the order it was applied at one particular replica p by application of the rules APPLY-LOCAL and APPLY-REMOTE.

Since the evaluation rules sequentially apply one operation at a time at a given replica, the order is well-defined. Even if two replicas p and q applied the same set of operations, i.e. if $A_p(\mathsf{ops}) = A_q(\mathsf{ops})$, they may have applied any concurrent operations in a different order. Due to the premise $op.deps \subseteq A_p(\mathsf{ops})$ in APPLY-REMOTE, histories are consistent with causality: if an operation has causal dependencies, it appears at some point after those dependencies in the history.

Definition 3 (document state). The document state of a replica p is the subtree of A_p containing the document: that is, $A_p(\mathsf{mapT}(\mathsf{doc}))$ or $A_p(\mathsf{listT}(\mathsf{doc}))$ or $A_p(\mathsf{regT}(\mathsf{doc}))$, whichever is defined.

 A_p contains variables defined with let, which are local to one replica, and not part of the replicated state. The definition of document state excludes these variables.

Theorem. For any two replicas p and q that participated in a valid execution, if $A_p(\mathsf{ops}) = A_q(\mathsf{ops})$, then p and q have the same document state.

This theorem is proved in the appendix. It formalizes the safety property of convergence: if two replicas have processed the same set of operations, possibly in a different order, then they are in the same state. In combination with a liveness property, namely that every replica eventually processes all operations, we obtain the desired notion of convergence: all replicas eventually end up in the same state.

The liveness property depends on assumptions of replicas invoking yield sufficiently often, and all nondeterministic rules for yield being chosen fairly. We will not formalize the liveness property in this paper, but assert that it can usually be provided in practice, as network interruptions are usually of finite duration.

5 Conclusions and further work

In this paper we demonstrated how to compose CRDTs for ordered lists, maps and registers into a compound CRDT with a JSON data model. It supports arbitrarily nested lists and maps, and it allows replicas to make arbitrary changes to the data without waiting for network communication. Replicas asynchronously send mutations to other replicas in the form of operations. Concurrent operations are commutative, which ensures that replicas converge towards the same state without requiring application-specific conflict resolution logic.

This work focused on the formal semantics of the JSON CRDT, represented as a mathematical model. We have also created a practical implementation and will report on its performance characteristics in follow-on work. In fact, we wrote the LATEX source text of this paper using an experimental collaborative text editor that is based on our implementation of this CRDT. We are making the character-by-character editing trace of this document available as supplemental data of this paper, in order to provide a dataset for the empirical study of collaborative editing algorithms.

Our principle of not losing input due to concurrent modifications appears reasonable, but as illustrated in Figure 4, it leads to merged document states that may be surprising to application programmers who are more familiar with sequential programs. Further work will be needed to understand the expectations of application programmers and to design data structures that are minimally surprising under concurrent modification. It may turn out that a schema language will be required to support more complex applications. A schema language could also support semantic annotations, such as indicating that a number should be treated as a counter rather than a register.

The CRDT defined in this paper supports insertion, deletion and assignment operations. In addition to these, it would be useful to support a *move* operation (to change the order of elements in an ordered list, or to move a subtree from one position in a document to another) and an *undo* operation. Moreover, garbage collection (tombstone removal) is required in order to prevent unbounded growth of the datastructure. We plan to address these missing features in future work.

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APPENDIX

PROOF OF CONVERGENCE

Theorem 1. For any two replicas p and q that participated in a valid execution, if $A_p(\mathsf{ops}) = A_q(\mathsf{ops})$, then p and q have the same document state.

Proof. Consider the histories H_p and H_q at p and q respectively (see Definition 2). The rules APPLY-LOCAL and APPLY-REMOTE maintain the invariant that an operation is added to $A_p(\mathsf{ops})$ or $A_q(\mathsf{ops})$ if and only if it was applied to the document state at p or q. Thus, $A_p(\mathsf{ops}) = A_q(\mathsf{ops})$ iff H_p and H_q contain the same set of operations (potentially ordered differently).

The history H_p at replica p is a sequence of n operations: $H_p = \langle o_1, \dots, o_n \rangle$, and the document state at p is derived from H_p by starting in the empty state and applying the operations in order. Likewise, the document state at q is derived from H_q , which is a permutation of H_p . Both histories must be consistent with causality, i.e. for all i with $1 \leq i \leq n$, we require $o_i.deps \subseteq \{o_j.id \mid 1 \leq j < i\}$. The causality invariant is maintained by the APPLY-* rules.

We can prove the theorem by induction over the length of history n.

Base case: An empty history with n=0 describes the empty document state. The empty document is always the same, and so any two replicas that have not executed any operations are by definition in the same state.

Induction step: Given histories H_p and H_q of length n, such that $H_p = \langle o_1, \dots, o_n \rangle$ and H_q is a permutation of H_p , and such that applying H_p results in the same document state as applying H_q , we can construct new histories H'_p and H'_q of length n+1 by inserting a new operation o_{n+1} at any causally ready position in H_p or H_q respectively. We must then show that for all the histories H'_p and H'_q constructed this way, applying the sequence of operations in order results in the same document state.

In order to prove the induction step, we examine the insertion of o_{n+1} into H_p and H_q . Each history can be split into a prefix, which is the minimal subsequence $\langle o_1,\ldots,o_j\rangle$ such that $o_{n+1}.deps\subseteq \{o_1.id,\ldots,o_j.id\}$, and a suffix, which is the remaining subsequence $\langle o_{j+1},\ldots,o_n\rangle$. The prefix contains all operations that causally precede o_{n+1} , and possibly some operations that are concurrent with o_{n+1} ; the suffix contains only operations that are concurrent with o_{n+1} . The earliest position where o_{n+1} can be inserted into the history is between the prefix and the suffix; the latest position is at the end of the suffix; or it could be inserted at any point within the suffix.

We need to show that the effect on the document state is the same, regardless of the position at which o_{n+1} is inserted, and regardless of whether it is inserted into H_p or H_q . We do this in Lemma 8 by showing that o_{n+1} is commutative with respect to all operations in the suffix, i.e. with respect to any operations that are concurrent to o_{n+1} .

Before we can prove the commutativity of operations, we must first define some more terms and prove some preliminary lemmas.

Definition 4 (appearing after). In the ordered list ctx, list element k_j appears after list element k_1 if there exists a (possibly

empty) sequence of list elements k_2, \ldots, k_{j-1} such that for all i with $1 \leq i < j$, $ctx(\mathsf{next}(k_i)) = k_{i+1}$. Moreover, we say k_j appears immediately after k_1 if that sequence is empty, i.e. if $ctx(\mathsf{next}(k_1)) = k_j$.

The definition of *appearing after* corresponds to the order in which the NEXT rules iterate over the list.

Lemma 2. If k_2 appears after k_1 in an ordered list, and the list is mutated according to the evaluation rules, k_2 also appears after k_1 in all later document states.

Proof. The only rule that modifies the next pointers in the context is INSERT₁, and it inserts a new list element between two existing list elements (possibly head and/or tail). This modification preserves the appears-after relationship between any two existing list elements. Since no other rule affects the list order, appears-after is always preserved. □

Note that deletion of an element from a list does not remove it from the sequence of next pointers, but only clears its presence set pres(k).

Lemma 3. If one replica inserts a list element k_{new} between k_1 and k_2 , i.e. if k_{new} appears after k_1 in the list and k_2 appears after k_{new} in the list on the source replica after applying APPLY-LOCAL, then k_{new} appears after k_1 and k_2 appears after k_{new} on every other replica where that operation is applied.

Proof. The rules for generating list operations ensure that k_1 is either head or an operation identifier, and k_2 is either tail or an operation identifier.

When the insertion operation is generated using the Make-Op rule, its operation identifier is given a counter value ctr that is greater than the counter of any existing operation ID in $A_p(\mathsf{ops})$. If k_2 is an operation identifier, we must have $k_2 \in A_p(\mathsf{ops})$, since both Apply-Local and Apply-Remote add operation IDs to $A_p(\mathsf{ops})$ when applying an insertion. Thus, either $k_2 < k_{new}$ under the ordering relation < for Lamport timestamps, or $k_2 =$ tail.

When the insertion operation is applied on another replica using APPLY-REMOTE and INSERT_{1,2}, k_2 appears after k_1 on that replica (by Lemma 2 and causality). The cursor of the operation is $\operatorname{cursor}(\langle \ldots \rangle, k_1)$, so the rules start iterating the list at k_1 , and therefore k_{new} is inserted at some position after k_1 .

If other concurrent insertions occurred between k_1 and k_2 , their operation ID may be greater than or less than k_{new} , and thus either INSERT $_1$ or INSERT $_2$ may apply. In particular, INSERT $_2$ skips over any list elements whose Lamport timestamp is greater than k_{new} . However, we know that $k_2 < k_{new} \lor k_2 = \text{tail}$, and so INSERT $_1$ will apply with $next = k_2$ at the latest. The INSERT $_{1,2}$ rules thus never iterate past k_2 , and thus k_{new} is never inserted at a list position that appears after k_2 .

Definition 5 (common ancestor). *In a history H, the* common ancestor *of two concurrent operations* o_r *and* o_s *is the latest document state that causally precedes both* o_r *and* o_s .

The common ancestor of o_r and o_s can be defined more formally as the document state resulting from applying a sequence of operations $\langle o_1, \ldots, o_j \rangle$ that is the shortest prefix of H that satisfies $(o_r.deps \cap o_s.deps) \subseteq \{o_1.id, \ldots, o_j.id\}$.

Definition 6 (insertion interval). Given two concurrent operations o_r and o_s that insert into the same list, the insertion interval of o_r is the pair of keys $(k_r^{\rm before}, k_r^{\rm after})$ such that o_r id appears after $k_r^{\rm before}$ when o_r has been applied, $k_r^{\rm after}$ appears after o_r id when o_r has been applied, and $k_r^{\rm after}$ appears immediately after $k_r^{\rm before}$ in the common ancestor of o_r and o_s . The insertion interval of o_s is the pair of keys $(k_s^{\rm before}, k_s^{\rm after})$ defined similarly.

It may be the case that $k_r^{\rm before}$ or $k_s^{\rm before}$ is head, and that $k_r^{\rm after}$ or $k_s^{\rm after}$ is tail.

Lemma 4. For any two concurrent insertion operations o_r, o_s in a history H, if $o_r.cur = o_s.cur$, then the order at which the inserted elements appear in the list after applying H is deterministic and independent of the order of o_r and o_s in H.

Proof. Without loss of generality, assume that $o_r.id < o_s.id$ according to the ordering relation on Lamport timestamps. (If the operation ID of o_r is greater than that of o_s , the two operations can be swapped in this proof.) We now distinguish the two possible orders of applying the operations:

- 1) o_r is applied before o_s in H. Thus, at the time when o_s is applied, o_r has already been applied. When applying o_s , since o_r has a lesser operation ID, the rule INSERT₁ applies with $next = o_r.id$ at the latest, so the insertion position of o_s must appear before o_r . It is not possible for INSERT₂ to skip past o_r .
- 2) o_s is applied before o_r in H. Thus, at the time when o_r is applied, o_s has already been applied. When applying o_r , the rule INSERT $_2$ applies with $next = o_s.id$, so the rule skips past o_s and inserts o_r at a position after o_s . Moreover, any list elements that appear between $o_s.cur$ and o_s at the time of inserting o_r must have a Lamport timestamp greater than $o_s.id$, so INSERT $_2$ also skips over those list elements when inserting o_r . Thus, the insertion position of o_r must be after o_s .

Thus, the insertion position of o_r appears after the insertion position of o_s , regardless of the order in which the two operations are applied. The ordering depends only on the operation IDs, and since these IDs are fixed at the time the operations are generated, the list order is determined by the IDs.

Lemma 5. In an operation history H, an insertion operation is commutative with respect to concurrent insertion operations to the same list.

Proof. Given any two concurrent insertion operations o_r , o_s in H, we must show that the document state does not depend on the order in which o_r and o_s are applied.

Either o_r and o_s have the same insertion interval as defined in Definition 6, or they have different insertion intervals. If the insertion intervals are different, then by Lemma 3 the operations cannot affect each other, and so they have the same effect regardless of their order. So we need only analyze the case in which they have the same insertion interval $(k^{\text{before}}, k^{\text{after}})$.

If $o_r.cur = o_s.cur$, then by Lemma 4, the operation with the greater operation ID appears first in the list, regardless of the order in which the operations are applied. If $o_r.cur \neq o_s.cur$, then one or both of the cursors must refer to a list

element that appears between k^{before} and k^{after} , and that did not yet exist in the common ancestor (Definition 5).

Take a cursor that differs from k^{before} : the list element it refers to was inserted by a prior operation, whose cursor in turn refers to another prior operation, and so on. Following this chain of cursors for a finite number of steps leads to an operation o_{first} whose cursor refers to k^{before} (since an insertion operation always inserts at a position after the cursor).

Note that all of the operations in this chain are causally dependent on $o_{\rm first}$, and so they must have a Lamport timestamp greater than $o_{\rm first}$. Thus, we can apply the same argument as in Lemma 4: if INSERT₂ skips over the list element inserted by $o_{\rm first}$, it will also skip over all of the list elements that are causally dependent on it; if INSERT₁ inserts a new element before $o_{\rm first}$, it is also inserted before the chain of operations that is based on it.

Therefore, the order of o_r and o_s in the final list is determined by the Lamport timestamps of the first insertions into the insertion interval after their common ancestor, in the chains of cursor references of the two operations. Since the argument above applies to all pairs of concurrent operations o_r, o_s in H, we deduce that the final order of elements in the list depends only on the operation IDs but not the order of application, which shows that concurrent insertions to the same list are commutative. \square

Lemma 6. *In a history H, a deletion operation is commutative with respect to concurrent operations.*

Proof. Given a deletion operation o_d and any other concurrent operation o_c , we must show that the document state after applying both operations does not depend on the order in which o_d and o_c were applied.

The rules in Figure 11 define how a deletion operation o_d is applied: starting at the cursor in the operation, they recursively descend the subtree, removing o_d . deps from the presence set pres(k) at all branch nodes in the subtree, and updating all registers to remove any values written by operations in o_d . deps.

If o_c is an assignment or insertion operation, the ASSIGN rule adds $o_c.id$ to the mapping from operation ID to value for a register, and the DESCEND, ASSIGN, EMPTY-MAP and EMPTY-LIST rules add $o_c.id$ to the presence sets $\operatorname{pres}(k)$ along the path through the document tree described by the cursor.

If $o_d.cur$ is not a prefix of $o_c.cur$, the operations affect disjoint subtrees of the document, and so they are trivially commutative. Any state changes by DESCEND and ADD-ID₁ along the shared part of the cursor path are applied using the set union operator \cup , which is commutative.

Now consider the case where $o_d.cur$ is a prefix of $o_c.cur$. Since o_c is concurrent with o_d , we know that $o_c.id \notin o_d.deps$. Therefore, if o_c is applied before o_d in the history, the CLEAR-* rules evaluating o_d will leave any occurrences of $o_c.id$ in the document state undisturbed, while removing any occurrences of operations in $o_d.deps$.

If o_d is applied before o_c , the effect on presence sets and registers is the same as if they had been applied in the reverse order. Moreover, o_c applies in the same way as if o_d had not been applied previously, because applying a deletion only modifies presence sets and registers, without

actually removing map keys or list elements, and because the rules for applying an operation are not conditional on the previous content of presence sets and registers.

Thus, the effect of applying o_c before o_d is the same as applying o_d before o_c , so the operations commute.

Lemma 7. *In a history* H, an assignment operation is commutative with respect to concurrent operations.

Proof. Given an assignment o_a and any other concurrent operation o_c , we must show that the document state after applying both operations does not depend on the order in which o_a and o_c were applied.

The rules ASSIGN, EMPTY-MAP and EMPTY-LIST define how an assignment operation o_a is applied, depending on the value being assigned. All three rules first clear any causally prior state from the cursor at which the assignment is occurring; by Lemma 6, this clearing operation is commutative with concurrent operations, and leaves updates by concurrent operations untouched.

The rules also add $o_a.id$ to the presence set identified by the cursor, and Descend adds $o_a.id$ to the presence sets on the path from the root of the document tree described by the cursor. These state changes are applied using the set union operator \cup , which is commutative.

Finally, in the case where value assigned by o_a is a primitive and the ASSIGN rule applies, the mapping from operation ID to value is added to the register by the expression $child[id \mapsto val]$. If o_c is not an assignment operation or if $o_a.cursor \neq o_c.cursor$, the operations are independent and thus trivially commutative.

If o_a and o_c are assignments to the same cursor, we use the commutativity of updates to a partial function: $child[id_1\mapsto val_1][id_2\mapsto val_2]=child[id_2\mapsto val_2][id_1\mapsto val_1]$ provided that $id_1\neq id_2$. Since operation IDs (Lamport timestamps) are unique, two concurrent assignments add two different keys to the mapping, and their order is immaterial.

Thus, all parts of the process of applying o_a have the same effect on the document state, regardless of whether o_c is applied before or after o_a , so the operations commute. \Box

Lemma 8. Given an operation history $H = \langle o_1, \ldots, o_n \rangle$ from a valid execution, a new operation o_{n+1} from that execution can be inserted at any point in H after o_{n+1} .deps have been applied. For all histories H' that can be constructed this way, the document state resulting from applying the operations in H' in order is the same, and independent of the ordering of any concurrent operations in H.

Proof. H can be split into a prefix and a suffix, as described in the proof of Theorem 1. The suffix contains only operations that are concurrent with o_{n+1} , and we allow o_{n+1} to be inserted at any point after the prefix. We then prove the lemma case-by-case, depending on the type of mutation in o_{n+1} .

If o_{n+1} is a deletion, by Lemma 6 it is commutative with all operations in the suffix, and so o_{n+1} can be inserted at any point within, before, or after the suffix without changing its effect on the final document state. Similarly, if o_{n+1} is an assignment, by Lemma 7 it is commutative with all operations in the suffix.

If o_{n+1} is an insertion, let o_c be any operation in the suffix, and consider the cases of o_{n+1} being inserted before and after o_c in the history. If o_c is a deletion or assignment, it is commutative with o_{n+1} by Lemma 6 or Lemma 7 respectively. If o_c is an insertion into the same list as o_{n+1} , then by Lemma 5 the operations are commutative. If o_c is an insertion into a different list in the document, its effect is independent from o_{n+1} and so the two operations can be applied in any order.

Thus, o_{n+1} is commutative with respect to any concurrent operation in H. Therefore, o_{n+1} can be inserted into H at any point after its causal dependencies, and the effect on the final document state is independent of the position at which the operation is inserted.

This completes the induction step in the proof of Theorem 1, and thus proves convergence of our datatype.