Model Predictive Control - EL2700

Computer Exercise 1: Quadratic Programs and CasADi solvers

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Task 1: A simple Linear Programming problem

In this first task we will do a warm-up exercise with Linear Programming. In linear programming, both the cost function and constraints are linear with respect to the optimization variable, taking the form

$$\begin{aligned} \min_{x} \quad & g^T x && (1)\\ \text{subject to:} \quad & x \geq x_{ub} && (2)\\ & Ax \leq b && (3) \end{aligned}$$

Note, however, that the solver we use is more generic (capable of solving Quadratic Programming problems), and therefore we might need to adjust our implementation to the form that the solver accepts. In our case, we will use CasADi's `conic` function, which solves optimization problems of the form

$$\begin{aligned} \min_{x} \quad & \frac{1}{2}x^T H x + g^T x && (4)\\ \text{subject to:} \quad & x_{lb} \leq x \leq x_{ub} && (5)\\ & a_{lb} \leq Ax \leq a_{ub} && (6) \end{aligned}$$

Taking this into account, lets take a look at the problem.

Airline Sitting Management MPC Airline (MPCA) airline is planning how many tickets should be sold from first class and second class to maximize profit. To be profitable, MPCA needs to sell at least 20 tickets from first class and 35 tickets from second class. Knowing that MPCA has a profit of 2000 SEK for second class tickets and 1500 SEK for first class tickets, and its airplanes can take a maximum of 130 passengers, how many tickets of each kind should be sold to maximize the profit?

Task 2: Hanging Chain[1]

In this problem we will formulate our first QP problem and solve it using CasADi's `conic` class. This problem deals with the hanging chain optimization problem. Our goal is to obtain the final position of each link of the chain by solving an optimization problem.

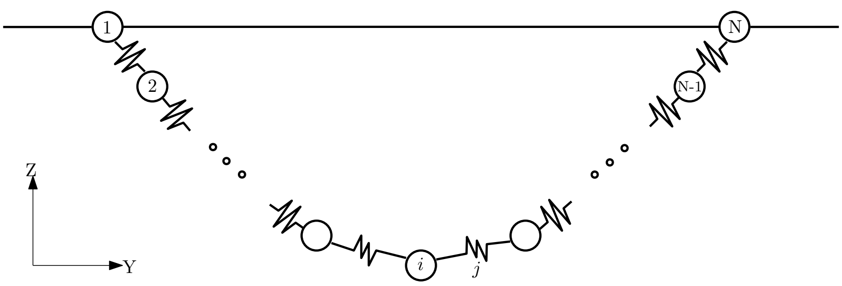

Figure 1: Hanging chain setup. Note that mass links 1 and N are fixed.

Consider the chain in Figure 1 . The chain is composed by N mass links coupled together via $N-1$ springs. Let the position of each mass element be denoted by $x_i, i = 1, \ldots, N$. Moreover, let m_i be the mass of each the ith element, k_i the spring constant of spring i, and g_c the gravity acceleration. Note that the mass elements 1 and N are fixed.

Q1: What is the kinetic and potential energy for each mass-spring system pair?

$$x_i = \begin{bmatrix} y_i & z_i \end{bmatrix} P_{spring} = \frac{1}{2}k_i((y_i - y_{i+1})^2 + (z_i - z_{i+1})^2)^2 \quad (7)$$

$$P_{gravity} = g_c m_i z_i \quad (8)$$

[1] Adapted from Joel Andersson, Joris Gillis and Moritz Diehl's "Equilibrium position for a hanging chain"

Q2: What is the total kinetic and potential energy for the whole chain?

$$P_{total} = \frac{1}{2}\sum_{i=1}^{N-1} k_i((y_i - y_{i+1})^2 + (z_i - z_{i+1})^2) + g_c \sum_{i=1}^{N} m_i z_i \tag{9}$$

To find out the final shape of the hanging chain, we will formulate a Quadratic Programming problem. To this end we will minimize the chain potential energy, while keeping in mind that the chain hangs by two fixed points.

First, lets take another look at the QP formulation in CasADi:

$$\begin{aligned}
\min_{x} \quad & \frac{1}{2}x^T H x + g^T x && (10)\\
\text{subject to:} \quad & x_{lb} \le x \le x_{ub} && (11)\\
& a_{lb} \le Ax \le a_{ub} && (12)
\end{aligned}$$

Q3: How would you translate the problem we want to solve into the QP problem in CasADi's framework? Consider at first $N = 3$ to solve it "by hand".

$$\frac{1}{2}\sum_{i=1}^{N-1} k_i((y_i - y_{i+1})^2 + (z_i - z_{i+1})^2) + g_c \sum_{i=1}^{N} m_i z_i = \tag{13}$$

$$\frac{1}{2}k\sum_{i=1}^{N-1}(y_i - y_{i+1})^T(y_i - y_{i+1}) + \frac{1}{2}k\sum_{i=1}^{N-1}(z_i - z_{i+1})^T(z_i - z_{i+1}) + g_c m \sum_{i=1}^{N} z_i \tag{14}$$

The gravity potential is already in a good form, lets focus on the spring potential

$$\frac{1}{2}ky_1^Ty_1 + \frac{1}{2}ky_2^Ty_2 - \frac{1}{2}ky_1^Ty_2 - \frac{1}{2}ky_2^Ty_1 + \tag{15}$$

$$\frac{1}{2}kz_1^Tz_1 + \frac{1}{2}kz_2^Tz_2 - \frac{1}{2}kz_1^Tz_2 - \frac{1}{2}kz_2^Tz_1 + \tag{16}$$

$$\frac{1}{2}ky_2^Ty_2 + \frac{1}{2}ky_3^Ty_3 - \frac{1}{2}ky_2^Ty_3 - \frac{1}{2}ky_3^Ty_2 + \tag{17}$$

$$\frac{1}{2}kz_2^Tz_2 + \frac{1}{2}kz_3^Tz_3 - \frac{1}{2}kz_2^Tz_3 - \frac{1}{2}kz_3^Tz_2 = \tag{18}$$

$$\frac{1}{2}k\Big[y_1^Ty_1 + 2y_2^Ty_2 + y_3^Ty_3 - y_1^Ty_2 - y_2^Ty_1 - y_2^Ty_3 - y_3^Ty_2 + \tag{19}$$

$$z_1^Tz_1 + 2z_2^Tz_2 + z_3^Tz_3 - z_1^Tz_2 - z_2^Tz_1 - z_2^Tz_3 - z_3^Tz_2\Big] = \tag{20}$$

$$\frac{1}{2}kx^THx \tag{21}$$

where

$$x = \begin{bmatrix} y_1 & z_1 & y_2 & z_2 & y_3 & z_3 \end{bmatrix} \tag{22}$$

$$H = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \tag{23}$$

Q4: Consider. $m_i = 4[kg], k_i = 1000[N]$, $N = 10$, $g_c = 9.81[m/s^2]$, $x_1 = \begin{bmatrix} -2 & 1 \end{bmatrix}$ and $x_N = \begin{bmatrix} 2 & 1 \end{bmatrix}$. Calculate the QP problem matrices using Numpy and CasADi variables. **Notes:**

- you should implement it as a function of N, such that it would be easy to add or remove links;

At this point we are ready to solve our problem. To that end, we will use CasADi's `conic` interface, which takes the following form

```
qp = {'h': H.sparsity(),  'a':  A.sparsity() }
S = conic('S','qpoases',qp)
r = S(h=H, g=g, a=A, lbx=x_lb, ubx=x_ub, lba=a_lb, uba=a_ub)
x_opt = r['x']
```

Q6: We would like our chain to stop at the ground, where the height is $0.5[m]$. How should we do it?

Q7: Now we would like to consider a ground plane that is tilted, for instance $z - 0.1y >= 0.5$. How should we introduce this constraint?

Good Luck!