Q9. $100 = 2^2 \cdot 5^2$

$\varphi(100) = 100\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 40$ ($\because$ Thm 8.9)

$\gcd(2023, 2^2 5^2) = 1$ because $2 \nmid 2023$ and $5 \nmid 2023$

We then obtain by Euler's theorem that

$2023^{40} \equiv 1 \pmod{100}$ ---- (1)

We derive that

$2023^{2024} \equiv 2023^{40 \cdot 50 + 24} \pmod{100}$

$\equiv (2023^{40})^{50} \cdot 2023^{24} \pmod{100}$

$\equiv 1^{50} \cdot 2023^{24} \pmod{100}$ by (1)

$\equiv 2023^{24} \pmod{100}$

추가로, 2023^{24}의 마지막 두자리수를 찾자.

2023^{24}의 마지막 두자리수는 23^{24}의 마지막 두자리수와 같다.

$2023 \equiv 23 \pmod{100}$
$\to 2023^{24} \equiv 23^{24} \pmod{100}$

$23^2 = 529$, $529 \equiv 29 \pmod{100}$ $\to 23^2 \equiv 29 \pmod{100}$

$23^4 = (23^2)^2$, $(23^2)^2 \equiv 29^2 \pmod{100}$
$29^2 = 841$, $841 \equiv 41 \pmod{100}$
$\to 23^4 \equiv 41 \pmod{100}$

$23^8 = (23^4)^2$, $(23^4)^2 \equiv 41^2 \pmod{100}$
$41^2 = 1681$, $1681 \equiv 81 \pmod{100}$
$\to 23^8 \equiv 81 \pmod{100}$

$23^{16} = (23^8)^2$, $(23^8)^2 \equiv 81^2 \pmod{100}$
$81^2 = 6561$, $6561 \equiv 61 \pmod{100}$
$\to 23^{16} \equiv 61 \pmod{100}$

$\to 23^{24} = 23^{16} \cdot 23^8$, $23^{16} \cdot 23^8 \equiv 61 \cdot 81 \pmod{100}$

$61 \times 81 = 4941$, $4941 \equiv 41 \pmod{100}$

$23^{24} \equiv 41 \pmod{100}$

따라서 $2023^{2024} \equiv 2023^{24} \equiv 23^{24} \equiv 41 \pmod{100}$

$2023^{2024} - 41 = 100q$. ($q \in \mathbb{Z}$)

$2023^{2024} = 100q + 41$

Thus, the last two digits of 2023^{2024} are $\therefore$ 4, 1 ▨