Q1. $a, b \neq 0$, $gcd(a,b) \mid c$

$ax_0 + by_0 = c$ for some $x_0, y_0 \in \mathbb{Z}$.

$S = \{ (u,v) \in \mathbb{Z} \times \mathbb{Z} \mid au + bv = c \}$

$T = \left\{ \left(x_0 - \frac{b}{gcd(a,b)} k , \ y_0 + \frac{a}{gcd(a,b)} k \right) \mid k \in \mathbb{Z} \right\}$

i) $T \subseteq S$ 증명.

Let $(u,v) \in T$

$\rightarrow \ u = x_0 - \frac{b}{gcd(a,b)} k$, $v = y_0 + \frac{a}{gcd(a,b)} k$

$au + bv = ax_0 - \cancel{\frac{ab}{gcd(a,b)} k} + by_0 + \cancel{\frac{ab}{gcd(a,b)} k}$

$= ax_0 + by_0 = c$

$\rightarrow (u,v) \in S$

$\rightarrow T \subseteq S$ ■

ii) $S \subseteq T$ 증명

Let $(u,v) \in S$

$\rightarrow \ au + bv = c$

$\rightarrow \ au + bv = ax_0 + by_0$ for some $x_0, y_0 \in \mathbb{Z}$

$\rightarrow \ a(u - x_0) = b(y_0 - v)$ ... (1)

Let $d = gcd(a,b)$, $d \mid a$ $\begin{cases} a \neq 0 \\ a = dq_1 & (q_1 \in \mathbb{Z}) \end{cases}$

$d \mid b$ $\begin{cases} b \neq 0 \\ b = dq_2 & (q_2 \in \mathbb{Z}) \end{cases}$

$d \mid c$ $\begin{cases} c \neq 0 \\ c = dQ & (Q \in \mathbb{Z}) \end{cases}$

(1)을 d로 나누면

$$\frac{a}{d}(u-x_0) = \frac{b}{d}(y_0 - v)$$

$$\rightarrow q_1(u-x_0) = q_2(y_0 - v) \cdots (2)$$

$$\cancel{d} = \gcd(a,b) = \gcd(d \cdot q_1, d \cdot q_2) = \cancel{d} \cdot \gcd(q_1, q_2) \quad (\because \text{Lemma } 47)$$

$\rightarrow \gcd(q_1, q_2) = 1$, 즉 q_1과 q_2는 서로소이다 $\cdots (3)$

$$\left(\gcd(q_1, q_2) = 1, \quad \begin{matrix} u - x_0 \in \mathbb{Z} \\ y_0 - v \in \mathbb{Z} \\ q_1, q_2 \in \mathbb{Z} \end{matrix}, \quad \begin{matrix} q_2 \nmid q_1 \\ q_1 \nmid q_2 \end{matrix} \right) \text{ --- } (4)$$

(2) 에서

$q_1(u - x_0) = q_2(y_0 - v)$, (4)에 의해 $q_2 \nmid q_1$

$$\underset{\in \mathbb{Z}}{q_1} \underset{\in \mathbb{Z}}{\frac{(u - x_0)}{q_2}} = \underset{\in \mathbb{Z}}{(y_0 - v)}$$

$(q_1, \ y_0 - v \in \mathbb{Z})$

$\rightarrow q_2 \mid u - x_0$ $\qquad \longrightarrow \dfrac{u - x_0}{q_2} \in \mathbb{Z}.$

$u - x_0 = q_2 \cdot k_2 \quad (k_2 \in \mathbb{Z})$

$\rightarrow q_1 \cdot k_2 = y_0 - v$

$v = y_0 + q_1 \cdot k_2 \quad \left(k_2 \in \mathbb{Z}, \ q_1 = \dfrac{a}{d}\right)$

$v = y_0 + \dfrac{a}{\gcd(a,b)} k_2 \quad (k_2 \in \mathbb{Z})$ ——— (I)

(2)에서

$q_1(u - x_0) = q_2(y_0 - v)$, (4)에 의해 $q_1 \nmid q_2$

$u - x_0 = q_2 \cdot \dfrac{(y_0 - v)}{q_1} = -q_2 \cdot \dfrac{(v - y_0)}{q_1}$

$\rightarrow q_1 \mid (v - y_0)$ $\qquad \left(\because \ u - x_0, \ q_2 \in \mathbb{Z} \quad \rightarrow \dfrac{v - y_0}{q_1} \in \mathbb{Z} \right)$

$v - y_0 = q_1 k_1 \quad (k_1 \in \mathbb{Z})$

$\rightarrow u - x_0 = -q_2 \cdot k_1 \quad (k_1 \in \mathbb{Z})$

$u = x_0 - q_2 k_1 \quad \left(k_1 \in \mathbb{Z}, \ q_2 = \dfrac{b}{d} \right)$

$u = x_0 - \dfrac{b}{\gcd(a,b)} k_1 \quad (k_1 \in \mathbb{Z})$ ——— (II)

$\therefore$ (I), (II) 에 의해 $(u, v) \in T \longrightarrow S \subseteq T$.