기초 수학, 202200307 고홍규

No. 수학과

유클리디안 Space

$\mathbb{R}$ = the set of real numbers

$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$ = De Cartesian Product

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{\underbrace{(x,y)}_{\begin{bmatrix} x \\ y \end{bmatrix}} \mid x, y \in \mathbb{R}\}$

$\mathbb{R}^4 = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \mid x, y, z, w \in \mathbb{R} \right\}$

⋮

$\mathbb{R}^n$ = Euclidean Spaces

→ Space

집합

상수곱 / ⊕

Vector Space

유클리드 공간과 흡사. but 정의가 다름. 벡터임

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \longleftarrow (x,y) = \begin{bmatrix} x \\ y \end{bmatrix} \longleftarrow$ vector

⊕ / 상수곱

Vector Space

Column Vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ 이런 모양.

Vector addition $v + w = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$

Scalar multiplication $2v = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$

linear Combination

v, w 있을 때, $cv + dw$ 꼴

u, v, w 있을 때, $cv + dw + ew$ 꼴

→ fill a plane.

Three Dimension $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $w = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ 꼴.

집합 ⊃ 원소 v, u vector

① ⊕ / 상수곱
② 8가지 Vector 성질
③ Vector Space

No.

8가지 Vector 성질

i) $u+v = v+u$
ii) $(u+v)+w = u+(v+w)$
iii) $u+0 = 0+u = u$
iv) $u+(-u) = 0$
v) $c(u+v) = cu+cv$
vi) $(c+d)u = cu+du$
vii) $c(du) = (cd)u$
viii) $1u = u$

	Basis	Dimension
$\mathbb{R}^2$	2	= Basis 개수
$\mathbb{R}^n$	n	n

Basis 벡터들 ⟷ linealy independent ⟷ span{Basis 벡터들} ⟷ V의 차원 Basis 개수

Basis $\beta = \{v_1, \ldots, v_p\}$ 일때

1) β가 linealy independent

⟷ suppose $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$ ... 일때

해가 유일하다. 그건 $(c_1, c_2, \ldots, c_p) = (0, 0, 0 \ldots, 0)$

⟷ 즉, Basis 가 각각 독립적이다.

증명) $c_1 \neq 0$ 가정.

$c_1v_1 + \cdots + c_pv_p = 0$, $c_1 \neq 0$

$c_1v_1 = -c_2v_2 - \cdots - c_pv_p$

$v_1 = \frac{-c_2}{c_1}v_2 \cdots - \frac{c_p}{c_1}v_p$, 즉 v_1은 $v_2 v_3, \ldots, v_p$로 나타낸 linear combination

$c_1v_1 + c_2v_2 + \cdots c_pv_p = 0$ 에서

v_1은 v_2 v_3 ... v_p로 옮김 → v_1이 의미 X → v_1 불필요.

$\therefore c_1 = 0$

V의 dimension = Basis의 개수

2) $V = \mathrm{span}\{v_1, \ldots, v_p\}$

⟷ $V = c_1b_1 + c_2b_2 + \cdots + c_pb_p$ 로 표현 가능한 공간이다.

No.

C[a,b]에서의 Inner product

$C[a,b] = \{ f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ 는 연속 함수} \}$ 일때,

$(f+g)(x) = f(x)+g(x)$: 연속 [a,b] 에서

$(cf)(x) = c \cdot f(x)$: 연속

$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$

$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f(x)^2\,dx}$

- $\langle f, g \rangle = \langle g, f \rangle$
- $\langle af+bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$
- $\langle f, f \rangle \geq 0$, $\langle f, f \rangle = 0 \rightarrow f = 0$

Fourier Series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nx)$

여기서 $\{1, \cos t, \cos 2t, \cdots \cos nt, \sin t, \cdots \sin nt\}$

의 원소들은 서로 Orthogonal (수직) $\left(\int_0^{2\pi} f(t)g(t)\,dt = 0\right)$

Matrix

$4x_1 - 5x_2 = -13$
$-2x_1 + 3x_2 = 9$

$\Rightarrow \overset{2\times2}{\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}} \overset{2\times1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$

$= \begin{bmatrix} 4x_1 - 5x_2 \\ -2x_1 + 3x_2 \end{bmatrix}$

$\overset{2}{\begin{bmatrix} a & b \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} 2 = ax + by$.

* Identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

* 인버스 $A^{-1}A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -13 \\ 9 \end{bmatrix} \cdot A^{-1}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ 일때

$I \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -13 \\ 9 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$

* 합 / 차

$A - 2B = \begin{bmatrix} 4 & 0 & +5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 & +3 \\ -7 & -7 & -12 \end{bmatrix}$

전체과정

~~Orthogonal~~ Basis = $\{X_1, X_2 \cdots, X_p\}$

$V_1 = X_1$

$V_2 = X_2 - \frac{X_2 \cdot V_1}{V_1 \cdot V_1} \cdot V_1$

$V_3 = X_3 - \frac{X_3 \cdot V_1}{V_1 \cdot V_1} \cdot V_1 - \frac{X_3 \cdot V_2}{V_2 \cdot V_2} V_2$

$\vdots$

$V_p = X_p - \frac{X_p \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{X_p \cdot V_2}{V_2 \cdot V_2} \cdot V_2 - \cdots - \frac{X_p \cdot V_p}{V_p \cdot V_p} \cdot V_p$

$\therefore \{V_1, V_2 \cdots V_p\}$ = orthogonal Basis $\longrightarrow$

QR factorization Matrices

$A = Q \cdot R$

m×n → Basis (linearly Independent)

m×n → 서로 Orthogonal Basis.

n×n

Inner Product Space

1) $\langle u\ v \rangle = \langle v\ u \rangle$

2) $\langle u+v, w \rangle = \langle u\ w \rangle + \langle v\ w \rangle$

3) $\langle cu, v \rangle = c\langle u, v \rangle$

4) $\langle u, u \rangle \geq 0$

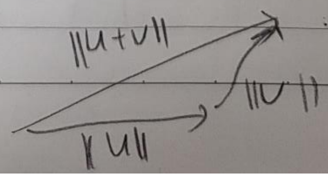
$\langle u, u \rangle = 0 \longleftrightarrow u = 0$

ex) $u = (u_1\ v_2)$, $v = (v_1\ v_2)$ 일 때

$\langle u\ v \rangle = 4u_1v_1 + 5u_1v_2$

Cauchy-Schwartz Inequality $|\langle u\ v \rangle| \leq \|u\|\|v\|$

Triangle Inequality $\|u+v\| \leq \|u\| + \|v\|$



No.

Unit vector $\|u\| = 1$

normalizing ⌊→ V: 벡터 가정

$u = \frac{V}{\|V\|}$ $\qquad \|u\|^2 = u\cdot u = \frac{V}{\|V\|}\cdot\frac{V}{\|V\|} = \frac{V\cdot V}{\|V\|^2} = \frac{\|V\|^2}{\|V\|^2} = 1$

각(angle) θ $\quad u\cdot v = \|u\|\|v\|\cos\theta$

서로 수직

$u\cdot v = 0 \longleftrightarrow$ u와 v는 orthogonal

$(u\neq 0)(v\neq 0)$

Orthogonal Basis

Basis 인데

각각 서로 수직인 것 $\qquad$ ex) $\left\{\begin{bmatrix}2\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

① Orthogonal Basis 집합 찾기 $\qquad \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}-1\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

② 그걸로 $\overset{암}{y} = \overset{모름}{c_1}v_1 + \cdots + \overset{모름}{c_p}v_p$ 설치.

③ c_1을 찾고 싶다?

$u_1\cdot y = (c_1u_1 + \cancel{c_2u_2 + \cdots + c_pu_p})\cdot u_1$

$u_1\cdot y = c_1\times(u_1\cdot u_1)$, $\quad\therefore\ c_1 = \frac{y\cdot u_1}{u_1\cdot u_1}$

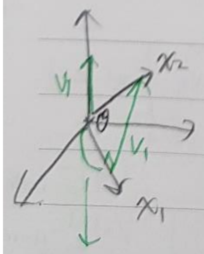
Gram-Schmidt process

Basis 인데

서로 수직 X 인 것들을 $\xrightarrow{과정}$ ① 내적

서로 수직 O 되게 만드는 기술. $\qquad$ ② 정사영부분 proj 제거

③ 수직인 부분만 추려냄



ex). $v_2 = x_2 - p = x_2 - \frac{x_2\cdot x_1}{x_1\cdot x_1}x_1$

$p = \overbrace{\|x_2\|\cos\theta}^{크기}\cdot\overbrace{\frac{x_1}{\|x_1\|}}^{방향}$

$= \frac{x_1\cdot x_2}{\|x_1\|}\cdot\frac{x_1}{\|x_1\|} = \frac{x_2\cdot x_1}{x_1\cdot x_1}\cdot x_1$

$\|x_2\|\cos\theta$

집합 $\xrightarrow{}$ Space $\xrightarrow{}$ Product

(+)/상수곱 8가지성질 4가지 성질

No.

Standard Basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ 이런 거.

+) 보이기 Span 증명하자.

$\leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 v_1 + c_2 v_2 + c_3 v_3$ 보이자.

linear independent 증명하자

$\leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = a_1 v_1 + a_2 v_2 + a_3 v_3 \rightarrow a_1 = a_2 = a_3 = 0$ 보이자.

Inner Product (내적) Inner product ⊃ dot product

$v = (v_1, v_2)$

$w = (w_1, w_2)$

$\rightarrow v \cdot w = v_1 w_1 + v_2 w_2$

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Inner Product 기본성질 4가지.

1) $u \cdot v = v \cdot u$

2) $(u+v) \cdot w = u \cdot w + v \cdot w$

3) $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

4) $u \cdot u \geq 0$

$u \cdot u = 0 \longleftrightarrow u = 0$

(Positive Definite)

$$u \cdot u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \underset{\substack{\| \\ 0}}{u_1^2} + \underset{\substack{\| \\ 0}}{u_2^2} + \cdots + \underset{\substack{\| \\ 0}}{u_N^2} \geq 0$$

Vector 길이 (norm) $\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

$\|v\|^2 = v \cdot v$

$v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$ 일 때

$\sqrt{x_1^2 + \cdots + x_N^2}$: V의 L^2-norm

distance $dist(u, v) = \|u - v\|$

$|x_1| + |x_2| + \cdots + |x_N|$: V의 L^1-norm

No.

Matrix 곱 $(AB)_{ij} = AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \\ \vdots & & \\ a_{i1} & a_{i2} \cdots & a_{in} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{1j} & b_{1l} \\ b_{21} & b_{2j} & \vdots \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{nj} & b_{nl} \end{bmatrix}$

$\underbrace{m \times \boxed{n} \quad \boxed{n} \times l}_{m \times l}$

Transpose $a_{ij} \to a_{ji}$ 행과 열 맞바꾸기.

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A+B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $A = A^T \longrightarrow$ A는 symmetric. ex) $A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$

Identity Matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 이런거.

$\underset{m \times n}{A} \underset{n \times n}{I} = A$

Diagonal Matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ 이런거.

※ $AB \neq BA$ 주의

Subspace $\mathbb{R}^3 \leftarrow$ Vector Space

$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \ x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$

그럼 H도 Vector Space? $\xrightarrow{\text{+/상수곱 체크}}$ 확인하자.

① + $u, v \in H$ $\to u+v \in H$ 맞는가? ② 상수곱 $u \in H$ $cu \in H$ 맞는가?

ex) $\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix} \in H$ $\quad c \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ 0 \end{bmatrix} \in H$

No.

linear transformation

1) $T(u+v) = T(u) + T(v)$ $\forall u, v \in V$

2) $T(cu) = cT(u)$ $\forall u, v \in V$, $\forall c$

kernel.

$L: V \to W$; linear transformation

$\ker L = \{v \in V \mid Lv = 0\}$

Dimension Formula

$L: V \to W$: linear transformation

$\dim V = \dim \ker V + \dim L(V)$

$= L + \text{rank} L$

Determinants

$\det A = 0 \longleftrightarrow A$ is not invertible

$\longleftrightarrow \exists x_0$ s.t. $Ax_0 = 0$ $(x_0 \neq 0)$

$\longleftrightarrow \ker(A) \neq \{0\}$

$T(S)$의 area $=$ $\det(A) \cdot \{S$의 area$\}$

$T(S)$의 volume $=$ $\det(A) \cdot \{S$의 volume$\}$

Classical adjoint

$\text{adj}(A) \in \mathbb{R}^{n \times n}$,

$(\text{adj}(A))_{ij} = (-1)^{i+j} |A_{\setminus j \setminus i}|$

$A^{-1} = \frac{1}{|A|} \text{adj}(A)$

norm of vector $\|x\|$

$\|x\|^2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

General formula of determinant

$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{\setminus i \setminus j}|$

$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{\setminus i \setminus j}|$