- Möbius Function

$$\mu(n) = \begin{cases} 0 & \exists d \in \mathbb{N}, d>1 \text{ s.t. } d^2 \mid n \\ (-1)^k, & n = p_1 \cdots p_k \end{cases}$$

- Dirichlet Convolution

$$(f*g)(n) = \sum_{d \mid n} f(d)\, g\left(\frac{n}{d}\right)$$

ex) $$(\mu * 1)(n) = \sum_{d \mid n} \mu(d) = \varepsilon \begin{cases} 0 & (n \neq 1) \\ 1 & (n = 1) \end{cases}$$

$$(f * \varepsilon)(n) = \sum_{d \mid n} f(d)\, \varepsilon\left(\frac{n}{d}\right) = f(n)$$

$$= (\varepsilon * f)(n)$$

- Xudyh's Sieve

$S_f(n) := \sum_{i=1}^{n} f(i)$ → 빠르게 계산원함

$$S_{f*g}(n) = \sum_{i=1}^{n} \sum_{d \mid i} g(d) f\left(\frac{i}{d}\right)$$

$$= \sum_{u=1}^{n} \sum_{v=1}^{[n/u]} f(v) g(u) \quad (i = uv)$$

$$= g(1) S_f(n) + \sum_{u=2}^{n} g(u) S_f\left(\left[\frac{n}{u}\right]\right)$$

$$\rightarrow S_f(n) = \frac{1}{g(1)} \left\{ \underbrace{S_{f*g}(n)}_{\text{쉽게 계산 가능 가정}} - \sum_{u=2}^{n} g(u) S_f\left(\left[\frac{n}{u}\right]\right) \right\}$$

- Time Complexity 계산

$\left[\frac{n}{u}\right]$은 최대 $2\sqrt{n}$개 ($\sqrt{n}$보다 큰것 : $\sqrt{n}$개, $1 \sim \sqrt{n}$)

$$\rightarrow T(n) = O(\sqrt{n}) + \sum_{i=1}^{\sqrt{n}} \sqrt{i} + \sum_{i=1}^{\sqrt{n}} \sqrt{\left[\frac{n}{i}\right]}$$

$$\approx O(\sqrt{n}) + \int_0^{\sqrt{n}} \sqrt{x}\,dx + \int_0^{\sqrt{n}} \sqrt{\frac{n}{x}}\,dx$$

$$= O(\sqrt{n}) + \frac{2}{3} n^{\frac{3}{4}} + 2 n^{\frac{3}{4}} = O\left(n^{\frac{3}{4}}\right)$$

- Optimization

미리 k개를 전처리

$$\rightarrow T(n) = O(\sqrt{n}) + O(k) + \int_0^{\frac{n}{k}} \sqrt{\frac{n}{x}}\,dx$$

$$= O(\sqrt{n}) + O(k) + \frac{2n}{\sqrt{k}}$$

$O(k) = O\left(\frac{n}{\sqrt{k}}\right) \rightarrow k = n^{\frac{2}{3}} \rightarrow n^{\frac{2}{3}}$개정도 전처리 후 계산

· ex) $\sum_{i=1}^{n} \phi(i) = ?$

$(\phi * 1)(n) = \sum_{d|n} \phi(d) = n$

$S_{\phi * 1}(n) = S_\phi(n) + \sum_{u=2}^{n} S_\phi\left(\left[\frac{n}{u}\right]\right)$

$\rightarrow S_f(n) = \frac{n(n+1)}{2} - \sum_{u=2}^{n} S_\phi\left(\left[\frac{n}{u}\right]\right)$

$\therefore \sum_{i=1}^{n} \phi(i)$: $O(n^{\frac{2}{3}})$에 계산가능

다른방법으로 이식 유도하기

$S_\phi(n) : |\{(a,b) \mid gcd(a,b)=1,\ 1 \le a \le b \le n\}|$

$|\{(a,b) \mid gcd(a,b)=g,\ 1 \le a \le b \le n\}|$

$= |\{(a,b) \mid gcd(a,b)=1,\ 1 \le a \le b \le \left[\frac{n}{g}\right]\}|$

$= S_\phi\left(\left[\frac{n}{g}\right]\right)$

$\rightarrow \sum_{i=1}^{n} S_\phi\left(\left[\frac{n}{i}\right]\right) = \frac{n(n+1)}{2}$

$\therefore S_\phi(n) = \frac{n(n+1)}{2} - \sum_{i=2}^{n} S_\phi\left(\left[\frac{n}{i}\right]\right)$

역시 $O(n^{\frac{2}{3}})$에 계산 가능