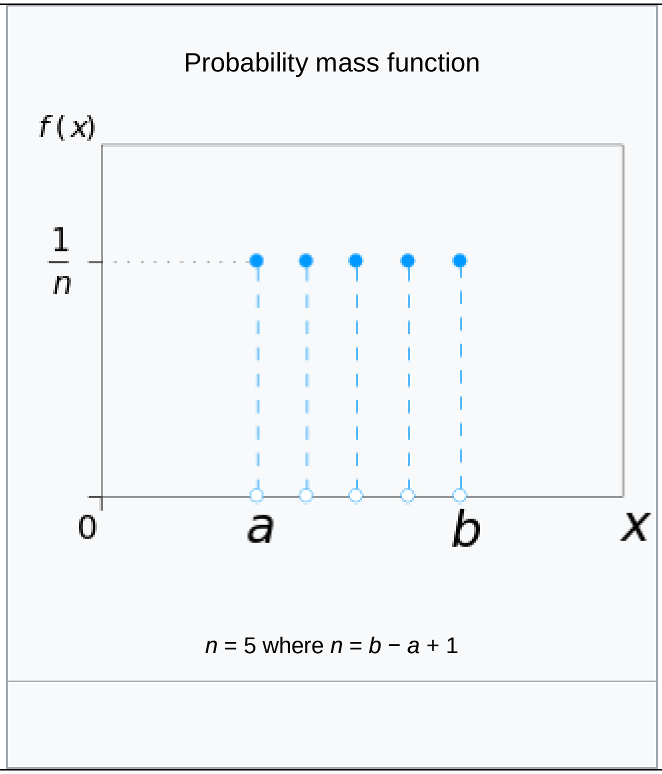
Table of Probability Distributions

(This material was taken primarily from Wikipedia, https://en.wikipedia.org/wiki/Main_Page where you can search for the distributions included in this table. Additional material was taken from the textbook, Ugarte, M.D., Militino, A.F. & Arnholt, A.T. 2016. **Probability and Statistics with R: Second Edition**. Boca Raton, FL: CRC Press, Taylor & Francis Group, A Chapman & Hall Book. Other sources are cited within the table.)

Type	Name	Conditions, Assumptions, Requirements	Uses	Examples
Univariate Discrete				
	Uniform	A symmetric distribution representing a finite number of values that are equally likely.	A simple example is that of a single, fair die. The possible values, 1 through 6, are equally likely. However be careful, the sum of two dice does not have equal probability.	

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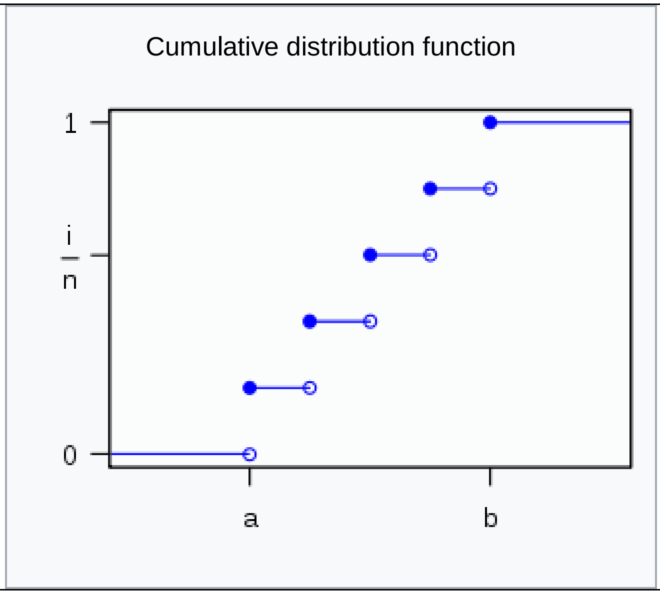
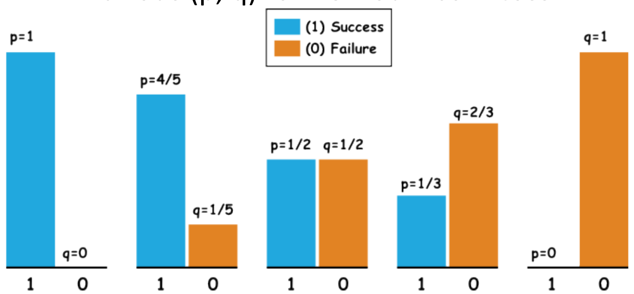
				
	Bernoulli	A random variable equals 1 (one) with probability p and equals 0 with probability $q = 1 - p$	Special case of the Binomial distribution. The probability distribution of an experiment that asks a yes-no question or involves a situation such as a coin toss.	Various (p, q) for Bernoulli coin-toss  (Acquired from: https://commons.wikimedia.org/wiki/File:Bernoulli_coin_toss.png March 28, 2017)
	Binomial	Used to model the number of successes in a	Probability of the number of successes in a sequence of n independent experiments	

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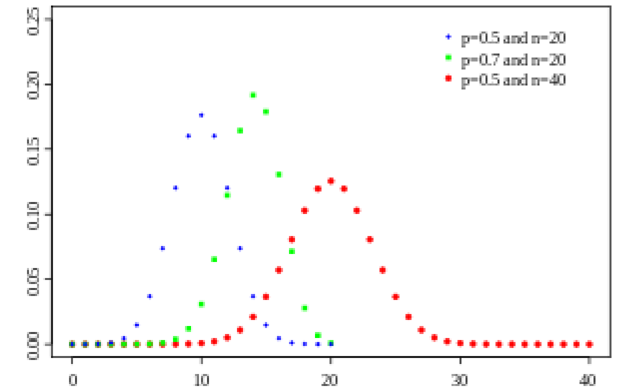
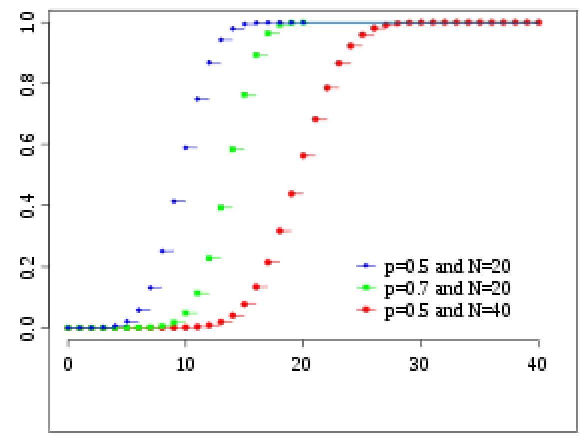
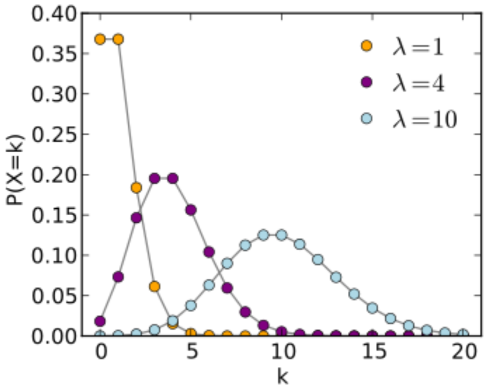
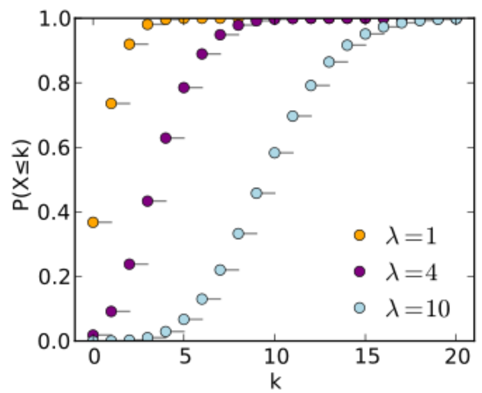
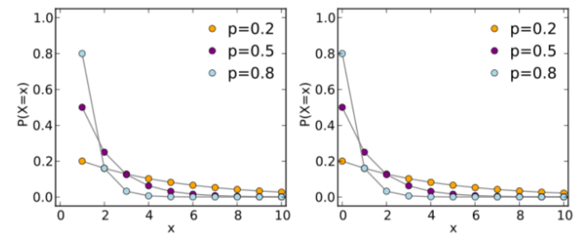
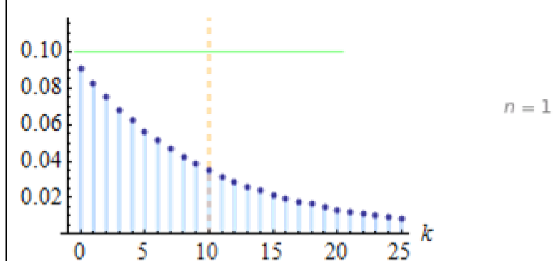
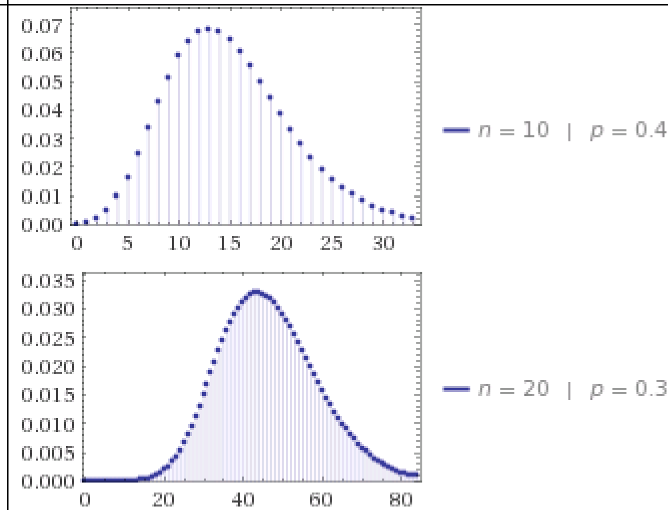
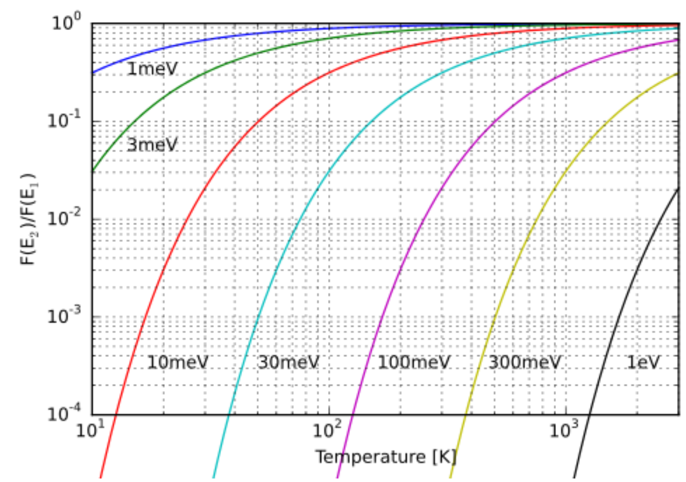
		sample of size n drawn <u>with replacement</u> from a population of size N.	yielding yes-no type results. A single success/failure experiment is called a Bernoulli trial or Bernoulli experiment. And, where $n = 1$ the binomial distribution is the special case called the Bernoulli distribution. When sampling is carried out without replacement the draws are not independent and the result is represented by the hypergeometric distribution.	Probability mass function  Cumulative distribution function 
	Poisson	For a given number of events over a fixed interval of time and/or space	Has been used to model events such as the number of patients arriving in an emergency room.	

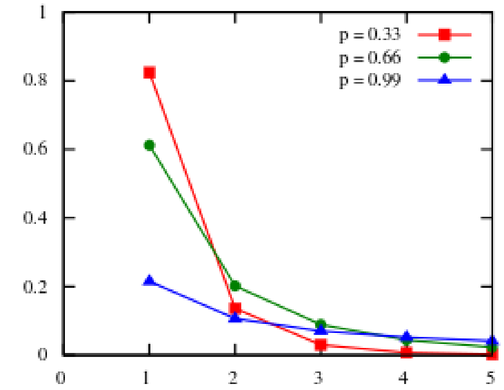
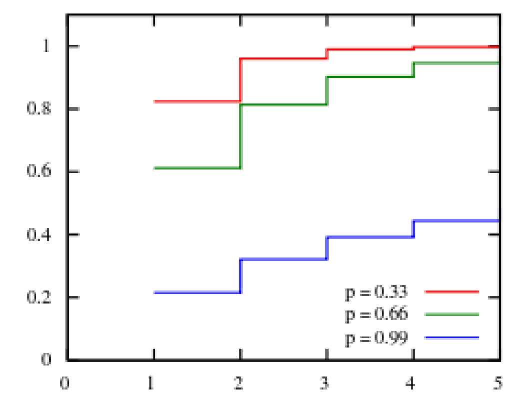
Type	Name	Conditions, Assumptions, Requirements	Uses	Examples
		if the events have a known average rate and are independent.		Probability mass function  The horizontal axis is the index k, the number of occurrences. λ is the expected number of occurrences. The vertical axis is the probability of k occurrences given λ. The function is defined only at integer values of k. The connecting lines are only guides for the eye. Cumulative distribution function  The horizontal axis is the index k, the number of occurrences. The CDF is discontinuous at the integers of k and flat everywhere else because a variable that is Poisson distributed

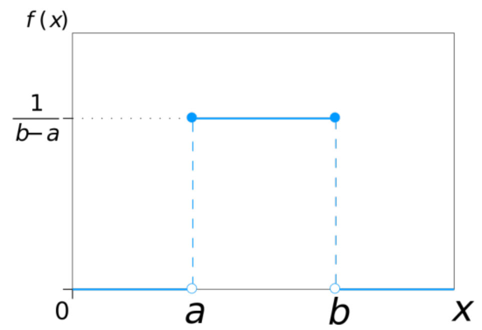
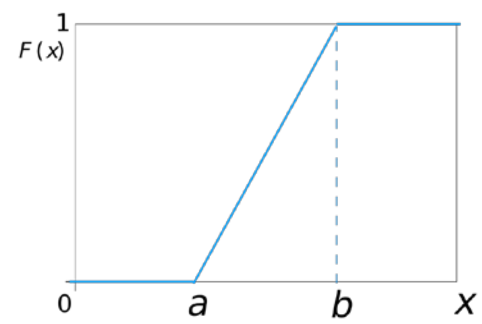
Type	Name	Conditions, Assumptions, Requirements	Uses	Examples
				takes on only integer values.
	Geometric	An experiment involves a sequence of independent trials for which there are only 2 (two) possible outcomes, e.g. success or failure.	For these experiments the (geometric) random variable is the count of failures before the first success. For example, if any of a family of drugs has a probability of $p = 0.6$ of being effective for a particular patient what is the probability that a physician will prescribe the drug from the family of drugs that is effective on the first trial? What is the probability that the physician will prescribe the effective drug on the second trial? and so on.	There are actually 2 (two) distinct geometric probability distributions. • Often called the "Shifted" Geometric distribution, this is the probability distribution for the number X of Bernoulli trials required to achieve one success over the set of real numbers (trials) beginning with the number 1, i.e. $\{1, 2, 3, \dots\}$. • This is the distribution, $Y = X - 1$, representing the number of failures before the first success. This distribution begins with 0 (zero). 
	Negative Binomial	Represents the number of successes in a sequence of independent and identically distributed Bernoulli trials that occur prior to a specified/ known	Also called the Pascal distribution. The Pascal and Polya distributions are special cases of the negative binomial distribution. For situations in which events are not independent, e.g. a tornado outbreak due to the	

<table>
<tr><th>Type</th><th>Name</th><th>Conditions, Assumptions, Requirements</th><th>Uses</th><th>Examples</th></tr>
<tr><td></td><td></td><td>number of failures.</td><td>same severe weather system, the Polya distribution produces a more accurate model than the Poisson distribution. Such events are often called “contagious” events because of their highly, positively correlated occurrences. Occurrences of events that are truly independent have a positive covariance term.</td><td>
</td></tr>
<tr><td></td><td>Hyper-geometric</td><td>Describes the probability of k successes in n draws without replacement from a finite population of size N with exactly K successes where each draw is either a success or a failure.</td><td>Represents the classic case of an urn with 2 (two) colored balls (or marbles), e.g. one green and one red (or one black and one white). N is the total number of balls, K the number of green balls. Therefore, N – K is the number of red balls. Drawing either green (or black) represents success whereas drawing either red (or white) represents failure. Because the balls are not replaced as they are drawn</td><td>Contingency Table
<table>
<tr><th></th><th>drawn</th><th>not drawn</th><th>total</th></tr>
<tr><td>green marbles</td><td>k</td><td>$K - k$</td><td>K</td></tr>
<tr><td>red marbles</td><td>$n - k$</td><td>$N + k - n - K$</td><td>$N - K$</td></tr>
<tr><td>total</td><td>n</td><td>$N - n$</td><td>N</td></tr>
</table></td></tr>
</table>

<table>
<tr><th>Type</th><th>Name</th><th>Conditions, Assumptions, Requirements</th><th>Uses</th><th>Examples</th></tr>
<tr><td></td><td></td><td></td><td>the draws are not independent. As an aside, the card game Texas Hold 'em uses this distribution to compute the probability of drawing desired cards.</td><td>Relationship to other distributions
<table>
<tr><th></th><th>With replacements</th><th>No replacements</th></tr>
<tr><td>Given number of draws</td><td>binomial distribution</td><td>hypergeometric distribution</td></tr>
<tr><td>Given number of failures</td><td>negative binomial distribution</td><td>negative hypergeometric distribution</td></tr>
</table></td></tr>
<tr><td></td><td>Boltzmann</td><td>In statistical mechanics, gives the probability that a system will be in a certain state based on the state's energy and the system's temperature.</td><td>Has been used in physics, mathematics and economics. For example, in economics the Boltzmann distribution has been used to allocate permits for emissions trading among countries.</td><td>Frequency Distribution (Boltzmann Distribution) for Particles in a System over Various Possible States


Occupation probability following a Boltzmann distribution depending on the energy</td></tr>
</table>

Type	Name	Conditions, Assumptions, Requirements	Uses	Examples
				difference and temperature.
	Logarithmic (series)	Also called the log-series distribution, is derived from the Maclaurin series expansion $-\ln(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \cdots$, which is also a Taylor series expansion when centered at 0 (zero).	Used by R. A. Fisher to describe relative species abundance.	Probability mass function  The function is only defined at integer values. The connecting lines are merely guides for the eye. Cumulative distribution function 

Type	Name	Conditions, Assumptions, Requirements	Uses	Examples
Univariate Continuous				
	Uniform	Also called the rectangular distribution, is a family of symmetric probability distributions. Over the interval it is defined, the pdf for the continuous uniform distribution is given by: $f(x) = \begin{cases} \frac{1}{b-a}\ for\ a \leq x \leq b \\ 0\ \ for\ x < a\ or\ x > b \end{cases}$	Step-function; pseudo-random number generation with a uniform distribution; etc.	Probability Density Function  Cumulative Distribution Function 
	Exponential	Describes the time between events in a Poisson process. Involves a rate parameter, typically λ. The exponential distribution is a special case of the gamma distribution. It is the continuous analogue of	There are many uses of this all centering around arrivals, e.g. arrivals of phone calls in a call center, the service times in queuing theory, or time for radioactive particle decays.	

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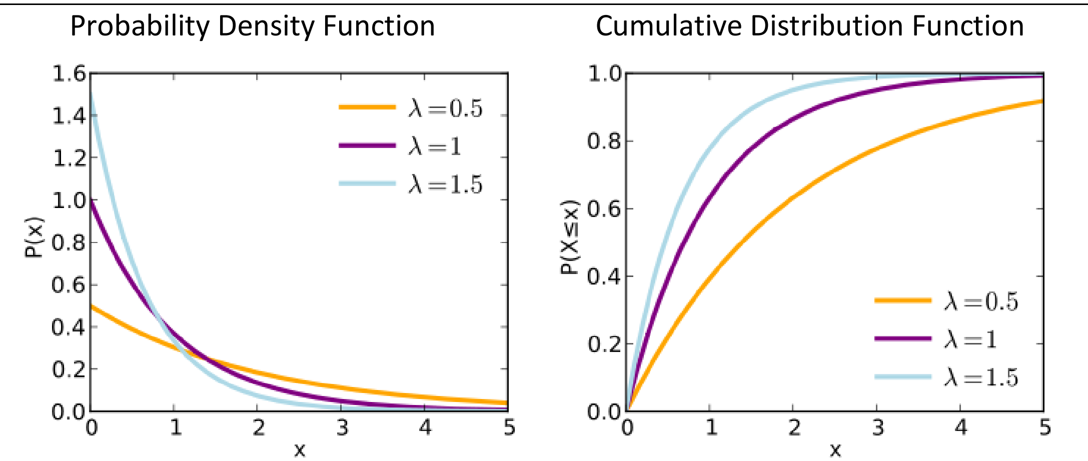
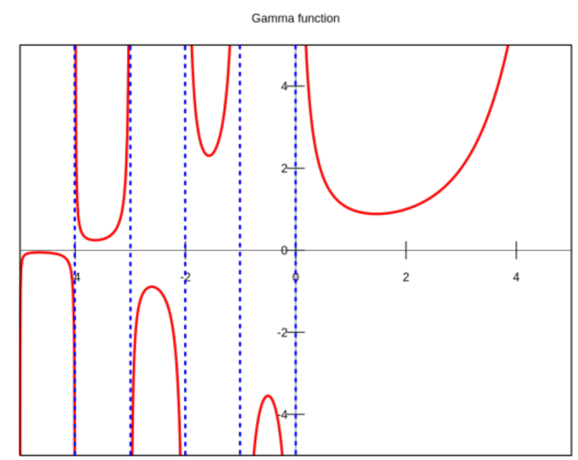
		the geometric distribution.		
	Gamma	The gamma function is an extension of the factorial function with its argument shifted by -1. The gamma function is defined for all real numbers and for all complex numbers except for non-positive integers. If n is a positive integer the gamma function is given by: $\Gamma(n) = (n-1)!$ For example, $\Gamma(1) = 1, \Gamma(2) =$	The gamma function is used as the basis for many other distributions, e.g. the Beta distribution. The gamma function has been used to interpolate a factorial function to non-integer values.	Gamma function along part of the real axis. 

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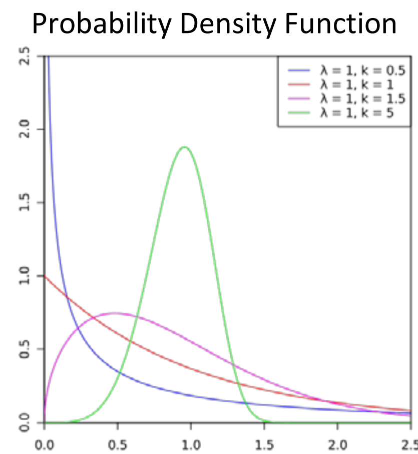
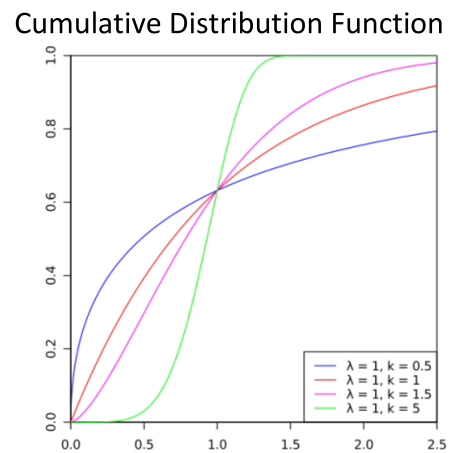
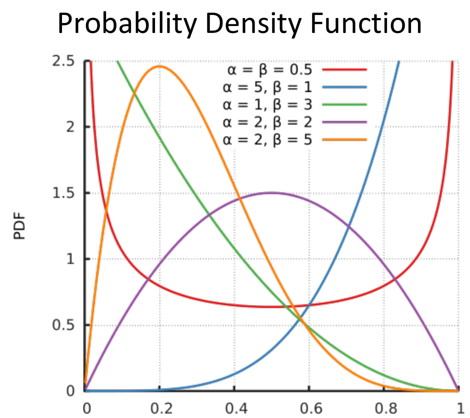
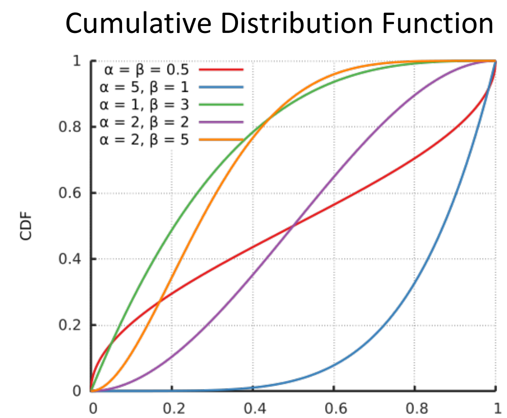
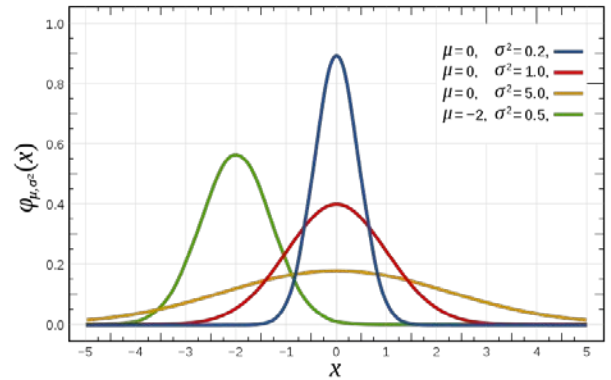
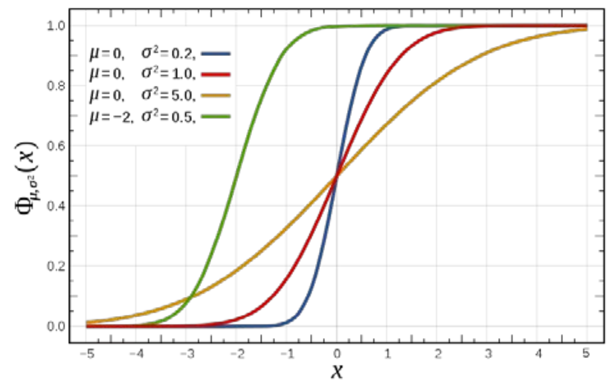
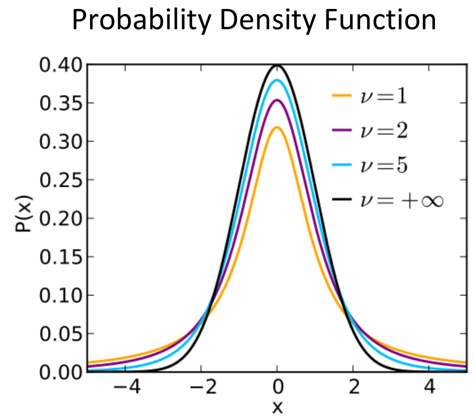
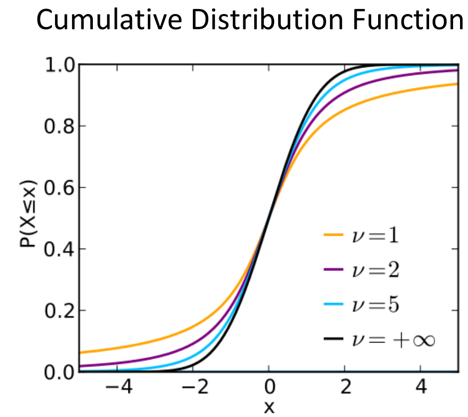
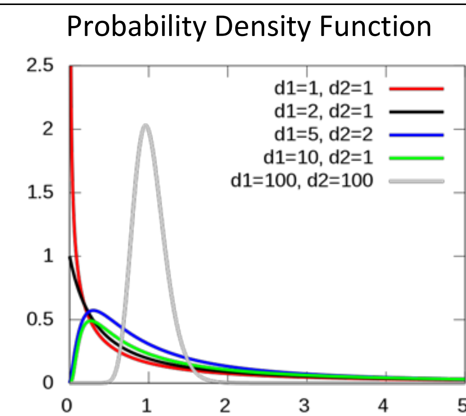
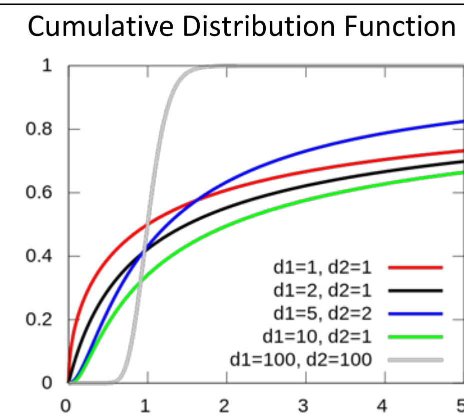
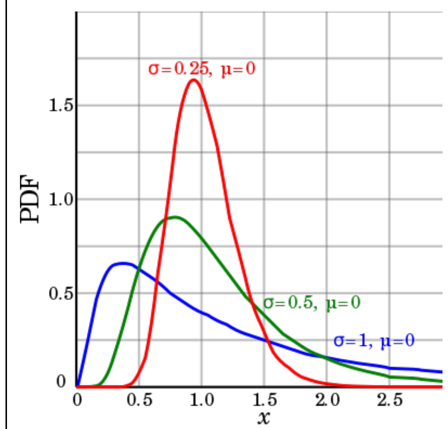
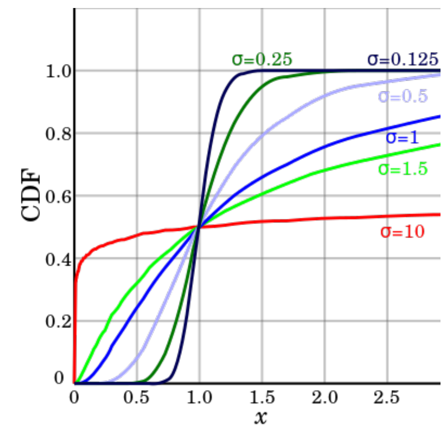
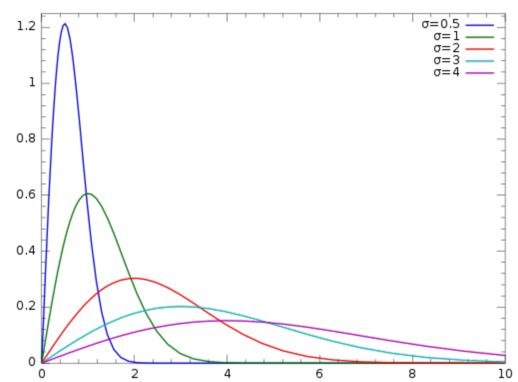
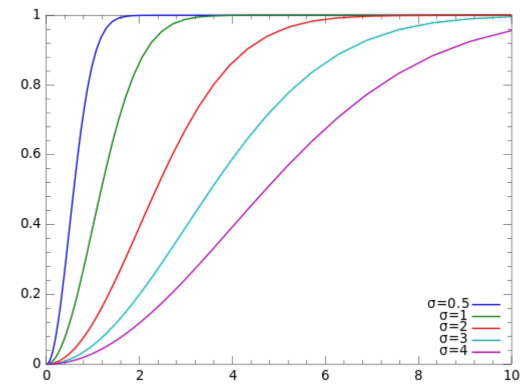
		$1, \Gamma(3) = 2, \Gamma(4) = 6, \Gamma(5) = 24$, and so on. For complex numbers with a positive real part the gamma function is given by: $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$		
	Weibull	Parameters include: $\lambda \in (0, +\infty)$ $scale$ $k \in (0, +\infty)$ $shape$ Support: $x \in [0, +\infty)$ The PDF is given by: $f(x) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k} & x \\ 0 & x \end{cases}$ The CDF is given by: $\begin{cases} 1 - e^{-(x/\lambda)^k} & x \geq 0 \\ 0 & x < 0 \end{cases}$ Mean: $\lambda\Gamma(1 + 1/k)$ Median: $\lambda(\ln 2)^{1/k}$ Mode: $\begin{cases} \lambda\left(\frac{k-1}{k}\right)^{1/k} & k > 1 \\ 0 & k \leq 1 \end{cases}$	The Weibull distribution is used for many applications including: in hydrology to model extreme events such as annual maximum one-day rainfalls, the distribution of particle sizes generated by grinding/milling and crushing operations, survival analysis, reliability engineering and failure analysis, industrial engineering for manufacturing and delivery times, wind speed distributions for the wind power industry, and more.	

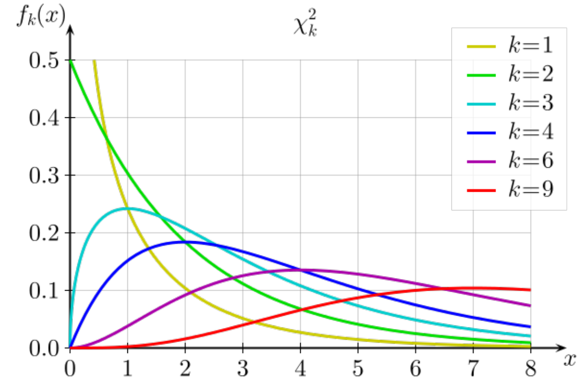
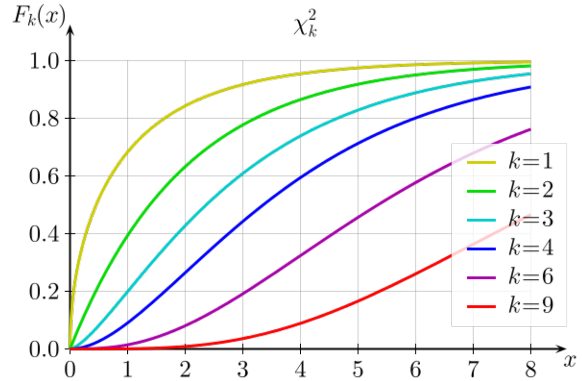
	Beta	Is defined on the interval $[0, 1]$ and parameterized by two positive shape parameters, $\alpha\ and\ \beta$, i.e. notation $Beta(\alpha, \beta)$. Parameters: $\alpha > 0$ shape (real) $\beta > 0$ shape (real) Support: $x \in [0,1]\ or\ x \in (0,1)$ The PDF is given by: $\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ The CDF is given by: $I_x(\alpha, \beta)$ (the regularized incomplete beta function)	In the limit when both shape parameters, α and β approach zero, the Beta distribution approaches the Bernoulli distribution. Used in Bayesian inference as part of a family of conjugate prior probability distributions for the binomial (including the Bernoulli) and geometric distributions. Used for continuous wavelets known as Beta wavelets. Used to model events (e.g. in project management) that are constrained over an interval with specified minimum and maximum values, e.g. PERT, CPM, etc.	

	Gaussian (Normal)	Notation: $\mathcal{N}(\mu, \sigma^2)$ Parameters: $\mu \in \mathbb{R}$ =mean $\sigma^2 > 0$ = variance The PDF is given by: $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ The CDF is given by: $\frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$	The most commonly used distribution, the Gaussian distribution is often used to represent real-valued random variables whose distribution is not a priori known, e.g. in the natural and social sciences. Also known as the “Bell curve”.	Probability Density Function  Cumulative Distribution Function 
	Student’s t	Parameters: $\upsilon > 0$ degrees of freedom (real). The PDF is given by:	Assessing the statistical significance of two sample means or generating confidence intervals for the difference between two population means. Also used in linear regression analyses.	

		$\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}$ where Γ is the Gamma function. The CDF is given by: $\frac{1}{2}+x\Gamma\left(\frac{v+1}{2}\right) x \frac{{}_2F_1\left(\frac{1}{2},\frac{v+1}{2};\frac{3}{2};-\frac{x^2}{v}\right)}{\sqrt{\pi v}\,\Gamma\left(\frac{v}{2}\right)}$ where ${}_2F_1$ is the hypergeometric function.	Mean: $0\ for\ v>1,$ $otherwise\ undefined$ Median: 0 Mode: 0 Variance: $\frac{v}{v-2}\ for\ v>2,$ $\infty\ for\ 1<\ v\leq 2,$ Otherwise undefined.	 
	F-Distribution (see also the "non-central F-distribution")	If a random variable X has an F-distribution with parameters $d_1$andd_2, then the notation is: $X\sim F(d_1,d_2)$. Parameters: $d_1,d_2>0$ degrees of freedom Support: $x\in[0,+\infty)$	Often used as the null distribution of a test statistic, e.g. the $F-test$ in ANOVA.	 

		The PDF is given by: $\frac{\sqrt{\frac{(d_1\,x)^{d_1} d_2^{d_2}}{(d_1\,x+d_2)^{d_1+d_2}}}}{xB\left(\frac{d_1}{2},\frac{d_2}{2}\right)}$ where B is the Beta function. (Note: do not confuse the Beta function and the Beta distribution!) The CDF is given by: $I_{\frac{d_1 x}{d_1 x+d_2}}\left(\frac{d_1}{2},\frac{d_2}{2}\right)$ that is, $F(x;\ d_1, d_2) = I_{\frac{d_1 x}{d_1 x+d_2}}\left(\frac{d_1}{2},\frac{d_2}{2}\right)$ Mean: $\frac{d_2}{d_2-2}$ for $d_2 > 2$		
	Log-normal	If $X \sim \mathcal{N}(\mu, \sigma^2)$ is a normal distribution then $\exp(X) \sim Lognormal(\mu,$. Parameters:	Used for a random variable whose logarithm is normally distributed.	

		$\mu \in (-\infty, +\infty)$, $\sigma > 0$ The PDF is given by: $\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$ The CDF is given by: $\frac{1}{2} + \frac{1}{2} erf\left[\frac{\ln x - \mu}{\sqrt{2}\sigma}\right]$ Mean: $exp\left(\mu + \frac{\sigma^2}{2}\right)$		Probability Density Function  Cumulative Distribution Function 
	Rayleigh	Parameters: scale: $\sigma > 0$ The PDF is given by: $\frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}$ The CDF is given by: $1 - e^{-x^2/(2\sigma^2)}$ Mean: $\sigma\sqrt{\frac{\pi}{2}}$	The Rayleigh distribution is a Chi distribution with two degrees of freedom. A Rayleigh distribution is often observed when the overall magnitude of a vector is related to its directional components, e.g. when the wind velocity is analyzed in two dimensions.	Probability Density Function  Cumulative Distribution Function 

	Chi-squared (see also the "non-central chi-squared distribution")	Notation: $\mathcal{X}^2(k) or\ \mathcal{X}_k^2$ Parameters: $k \in \mathbb{N} > 0$ (known as "degrees of freedom") The PDF is given by: $\frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$ The CDF is given by: $\frac{1}{\Gamma(k/2)} \gamma\left(\frac{k}{2}, \frac{x}{2}\right)$ where $\gamma(s,t)$ is the lower complete gamma function Mean: k	The Chi-squared distribution with k degrees of freedom is the distribution of the sum of squares of k independent standard normal random variables. It is a special case of the gamma distribution and is frequently used in inferential statistics particularly for hypothesis testing.	Probability Density Function  Cumulative Distribution Function 

Multivariate				
	Multinomial	Parameters: $n > 0$ number of trials (integer) $p_1, \ldots, p_k$ event probabilities ($\sum p_i = 1$) The pmf is given by: $$\frac{n!}{x_i! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$ Mean: $$E\{X_i\} = np_i$$	The multinomial distribution is a generalization of the binomial distribution used to model the probability of counts for rolling a k-sided die n times. When $k = 2$ the multinomial distribution is the binomial distribution.	
	Multivariate hyper-geometric distribution	Parameters: $$c \in \mathbb{N} = \{0, 1, \ldots\}$$ $$(K_1, \ldots, K_c) \in \mathbb{N}^c$$ $$N = \sum_{i=1}^{c} K_i$$ $$n \in \{0, \ldots, N\}$$ The pmf is given by: $$\frac{\prod_{i=1}^{c} \binom{K_i}{k_i}}{\binom{N}{n}}$$ Mean: $$E(X_i) = \frac{nK_i}{N}$$	Extension of the model of an urn with green and red marbles extended to include additional colors of marbles. It has the same relationship to the multinomial distribution that the hypergeometric distribution has to the binomial distribution. That is, the multinomial distribution is the "with-replacement" distribution and the multivariate hyper-geometric is the "without replacement"	

			distribution.	
	Multivariate log-normal	If $\boldsymbol{\mathcal{X}} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a multivariate normal distribution then $\boldsymbol{Y} = \exp(\boldsymbol{\mathcal{X}})$ has a multivariate log-normal distribution with mean $E[\boldsymbol{Y}]_i = e^{\mu_i + \frac{1}{2}\Sigma_{ii}}$.		