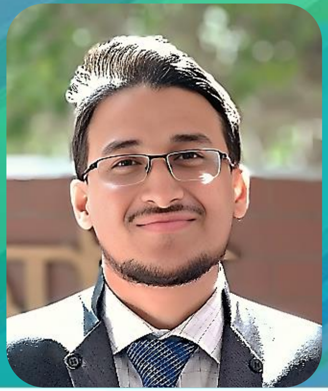
MATH 2161: Matrices and Vector Analysis



Md. Kawsar Ahmed Asif
Lecturer in Mathematics
Department of General Education
Canadian University of Bangladesh
Former Lecturer, World University of Bangladesh

Lesson 03

Lecture Outline

- Elementary Row Operations
- Row Echelon Form
- Application of Elementary Row Operations
- Solution of System of Linear Equations

Elementary Row Operations

Interchange of Two Rows

$$\begin{bmatrix}1&2&3&4\\2&1&2&3\\5&5&1&0\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix}5&5&1&0\\2&1&2&3\\1&2&3&4\end{bmatrix}$$

Multiply Each Element in a Row by a Non-Zero Number

$$\begin{bmatrix}1&2&3&4\\2&1&2&3\\5&5&1&0\end{bmatrix} \cdot 3 \xrightarrow{R_2 \rightarrow 3 \times R_2} \begin{bmatrix}1&2&3&4\\6&3&6&9\\5&5&1&0\end{bmatrix}$$

Multiply a Row by a Non-Zero Number and Add the Result to Another Row

$$\begin{bmatrix}1&2&3&4\\2&1&2&3\\5&5&1&0\end{bmatrix} \cdot 2 \xrightarrow{R_3 \rightarrow R_3 + 2R_1} \begin{bmatrix}1&2&3&4\\2&1&2&3\\7&9&7&8\end{bmatrix}$$

Effect on Determinant due to Row Operation

Row Operation	Effect on Determinant
Interchange two rows	Change the sign
Multiply a row by a constant	Multiply by the constant
Add a multiple of a row to another row	No change

Row Echelon Form (REF)

A matrix is said to be in *row echelon form* if:

1) First non-zero element in each row, called the *leading entry*, is 1.
2) Each leading entry is in a column to the right of the leading entry in the previous row.
3) Rows with all *zero* elements, if any, are below rows having a non-zero element.

Example: The following matrices are in *row echelon form*.

$$A_{\text{ref}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

❖ Row operations used to convert A into A_{ref} is called as Gaussian Elimination.

More Examples: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Application of Elementary Row Operations

Rank of Matrix (Using Elementary Row Transformations)

The number of *linearly independent rows* of a matrix is called the *rank* of the matrix. That is, the number of *non-zero rows in echelon form* is called the *rank* of a matrix. It is denoted by ρ.

Problem: Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ by elementary row operations.

Solution: The rank of the matrix is equal to the number of *non-zero rows* in the matrix after reducing it to the *row echelon form* using elementary transformations over the rows of the matrix.

Application of Elementary Row Operations

Matrix A	Elementary Transformation
$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$	
$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 3 & 5 & 7 \end{bmatrix}$	$R_2{}' \rightarrow R_2 - 2 \times R_1$
$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$	$R_3{}' \rightarrow R_3 - 3 \times R_1$
$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$	$R_3{}' \rightarrow R_3 - 1 \times R_2$

The above matrix is in *echelon form* and the number of *non-zero rows* are 2. Hence the rank of the matrix A is $\rho(A) = 2$.

Application of Elementary Row Operations

Example. Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ by using the row echelon form.

Solution: Now we will apply elementary transformations.

Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$	$R_2' \to R_2 - 2 \times R_1$ $R_3' \to R_3 - 3 \times R_1$
$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$	$R_3' \to R_3 - 2 \times R_2$
$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}$	$R_2' \to \dfrac{R_2}{-3}$

The above matrix is in *row echelon form* and the number of *non-zero rows* are 2. Hence the rank of the matrix A is $\rho(A) = 2$.

Solution of System of Linear Equations (Using Cramer's Rule)

One of the application of the *determinants* is to solve a system of linear equations in which number of variables are equal to the number of equations and the *coefficient matrix* of the system of equations is non-singular. This method enables us to determine solution *directly* without computing the *inverse of the matrix*.

Let us consider the following linear equations:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad \text{Or,} \quad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Let D be the determinant of the *co-efficient* of the variables x and y such that

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Further, let $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ be the determinant obtained from D by replacing the *first column* by the elements c_1, c_2 and replacing the *second column* by the elements c_1, c_2 respectively. Thus the values of x and y can be expressed in the form of determinant as:

$$\frac{x}{D_x} = \frac{y}{D_y} = \frac{1}{D} \qquad \therefore\ x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}, \qquad \text{provided } D \neq 0$$

Solution of System of Linear Equations (Using Cramer's Rule)

Example: Use Cramer's rule to solve the following system of linear equations:

$$\begin{aligned} 5x - 6y + 4z &= 15 \\ 7x + 4y - 3z &= 19 \\ 2x + y + 6z &= 46 \end{aligned}$$

Solution: The above system of linear equations can be written in matrix form as:

$$\begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix} \Rightarrow AX = B$$

where, $A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$D = |A| = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix} = +5\big(24 - (-3)\big) - (-6)\big(42 - (-6)\big) + 4(7 - 8) = 419$$

Since $D \neq 0$, we can apply Cramer's rule and the system is *consistent* with *unique solution*.

Solution of System of Linear Equations (Using Cramer's Rule)

$$D_x = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix} = 15\{24 - (-3)\} - (-6)\{114 - (-138)\} + 4(19 - 184) = \mathbf{1257}$$

$$D_y = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix} = 5\{114 - (-138)\} - 15\{42 - (-6)\} + 4(322 - 38) = \mathbf{1676}$$

$$A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}, B = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$D_z = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix} = 5(184 - 19) - (-6)(322 - 38) + 15(7 - 8) = \mathbf{2514}$$

Therefore,

$$x = \frac{D_x}{D} = \frac{1257}{419} = \mathbf{3}, \qquad y = \frac{D_y}{D} = \frac{1676}{419} = \mathbf{4}, \qquad z = \frac{D_z}{D} = \frac{2514}{419} = \mathbf{6}$$

$\therefore$ Solution of the given system: $x = 3, y = 4$ and $z = 6$.

Next Lecture

- Linear Equations
- System of Linear Equations
- Solution of System of Linear Equations
- Solution of Linear Systems by Matrix Method
- Characteristic Vector and Characteristic Root