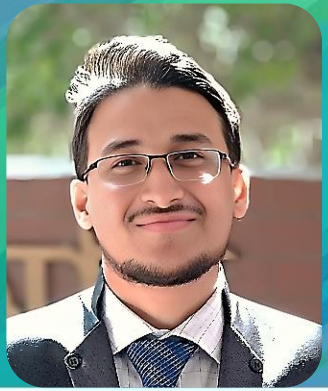
MATH 2161: Matrices and Vector Analysis



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Lesson 02

Lecture Outline

- Matrix Multiplication
- Minors and Cofactors
- Determinant of Matrix
- Inverse of Matrix
- Rank of Matrix

Matrix Multiplication

Problem: If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, then show that A satisfies the equation $A^3 - 4A^2 - 3A + 11I = 0$.

Solution: $A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

and $A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix}$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

Matrix Multiplication

Now $A^3 - 4A^2 - 3A + 11I = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - \begin{bmatrix} 36 & 28 & 20 \\ 4 & 16 & 4 \\ 32 & 36 & 36 \end{bmatrix} - \begin{bmatrix} 3 & 9 & 6 \\ 6 & 0 & -3 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6+0 & 5-16+0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore\ A^3 - 4A^2 - 3A + 11I = 0 \quad (Proved)$

Idempotent and Involutory Matrix

Idempotent Matrix: A square matrix A is said to be an *idempotent* matrix if $A^2 = A$.

For examples, $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are idempotent matrices.

Involutory Matrix: A square matrix A is called an *involutory* matrix if $A^2 = I$.

For examples, $A = \begin{bmatrix} 4 & 3 \\ -5 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ are involutory matrices.

Minors and Cofactors

If A is a square matrix, then the *minor of entry* a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i-th row and j-th column are deleted from A. The number $(-1)^{i+j} \cdot M_{ij}$ is denoted by C_{ij} and is called the *cofactor of entry* a_{ij}.

Example: To illustrate this definition, consider the following 3 by 3 matrix. $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{bmatrix}$

(a) The minor of entry a_{11} is $M_{11} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 1 & 4 \end{vmatrix} = 12 - 6 = 6$

The cofactor of a_{11} is $C_{11} = (-1)^{1+1} \cdot M_{11} = M_{11} = 6$

(b) The minor of entry a_{21} is $M_{21} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2$

The cofactor of a_{21} is $C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = -2$

Minors and Cofactors

Find all the minors and cofactors of the matrix

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$
$M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 40 - 24 = 16$	$M_{12} = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 16 - 6 = 10$	$M_{13} = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 8 - 5 = 3$
$C_{11} = (-1)^{1+1} M_{11} = 16$	$C_{12} = (-1)^{1+2} M_{12} = -10$	$C_{13} = (-1)^{1+3} M_{13} = 3$
$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$
$M_{21} = \begin{vmatrix} 1 & -4 \\ 4 & 8 \end{vmatrix} = 8 + 16 = 24$	$M_{22} = \begin{vmatrix} 3 & -4 \\ 1 & 8 \end{vmatrix} = 24 + 4 = 28$	$M_{23} = \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 12 - 1 = 11$
$C_{21} = (-1)^{2+1} M_{21} = -24$	$C_{22} = (-1)^{2+2} M_{22} = 28$	$C_{23} = (-1)^{2+3} M_{23} = -11$
$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$
$M_{31} = \begin{vmatrix} 1 & -4 \\ 5 & 6 \end{vmatrix} = 6 + 20 = 26$	$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 18 + 8 = 26$	$M_{33} = \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} = 15 - 2 = 13$
$C_{31} = (-1)^{3+1} M_{31} = 26$	$C_{32} = (-1)^{3+2} M_{32} = -26$	$C_{33} = (-1)^{3+3} M_{33} = 13$

Determinant of Matrix

The determinant of a matrix is a number that is specially defined only for *square matrices*. Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equations. For every square matrix $A = [a_{ij}]$ of order n, we can associate a number called *determinant* of square matrix. It is denoted by $|A|$ or $\det(A)$.

Evaluating Determinants

Order One:

$$A = [a] \quad \therefore |A| = |a| = a$$

Order Two:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \therefore |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

Order Three:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \therefore |A| = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array} = +a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant using Cofactor Expansion

Example: Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ by cofactor expansion.

Along the First *Row*:

$$\begin{array}{c} \quad + \quad\quad - \quad\quad + \\ \det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} \end{array} = +3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(8-12) - 1(4-15) + 0(-8+20)$$

$$= 3(-4) - 1(-11) + 0 = \mathbf{-1}$$

Along the First *Column*:

$$\det(A) = \begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(8-12) - (-2)(-2-0) + 5(3+0)$$

$$= 3(-4) - (-2)(-2) + 5(3) = \mathbf{-1}$$

Singular and Non-Singular Matrix

Singular and Non-Singular Matrices: Let D be the determinant of the square matrix A, then if

(a) $D = 0$, the matrix A is called a *singular* matrix

(b) $D \neq 0$, the matrix A is called a *non-singular* matrix

For examples, $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

Where, A & B are *singular matrices* and C & D are *non-singular matrices*.

Adjoint and Inverse of Matrix

Adjoint or Adjugate Matrix: Let $A = [a_{ij}]$ be a square matrix of order n. The *adjoint* of the matrix A is the transpose of the *cofactor matrix* of A. It is denoted by $adj(A)$.

$$adj(A) = [cof(A)]^T$$

Inverse Matrix: If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be *invertible* and B is called an *inverse* of A.

Inverse of a Matrix Using Its Adjoint: If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \cdot adj(A) = \frac{adj(A)}{|A|}$$

Inverse of Matrix

Problem: Find the inverse of the matrix $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$, using its adjoint.

Solution: Given, $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

Let $D = |A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix}$

$$= -1(5+0) - 2(10-0) - 3(-4-4)$$

$$= -5 - 20 + 24 = -1 \neq 0$$

So A is non-singular matrix and hence A^{-1} exists.

Inverse of Matrix

Cofactors of $-1 = A_{11} =$ $\begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 5$	Cofactors of $2 = A_{12} =$ $(-1)\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = -10$	Cofactors of $-3 = A_{13} =$ $\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -8$
Cofactors of $2 = A_{21} =$ $(-1)\begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} = -4$	Cofactors of $1 = A_{22} =$ $\begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} = 7$	Cofactors of $0 = A_{23} =$ $(-1)\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} = 6$
Cofactors of $4 = A_{31} =$ $\begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 3$	Cofactors of $-2 = A_{32} =$ $(-1)\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -6$	Cofactors of $5 = A_{33} =$ $\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5$

$$\therefore adj(A) = [cof(A)]^T = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot adj(A) = \frac{1}{-1}\begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

Rank of Matrix

Let A be an $m \times n$ matrix. The *rank* of A is the *maximal order* of a non-zero minor of A.

Example: Find the rank of the matrix $\begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$ $\because$ The order of A is 3×3. $\therefore \rho(A) \leq 3$.

Consider the third order minor $\begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 5\begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix} - 3\begin{vmatrix} 1 & -4 \\ -2 & 8 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix}$

$$= 5(16 - 16) - 3(8 - 8) + 0(-4 + 4) = 0$$

Since the third order minor vanishes, therefore $\rho(A) \neq 3$ and $\rho(A) \leq 2$.

Consider a second order minor $\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7 \neq 0$

There is a minor of order 2, which is not zero. $\therefore$ $\rho(A) = 2$.

Next Lecture

- Elementary Row Operations
- Row Echelon Form
- Application of Elementary Row Operations
- Solution of System of Linear Equations