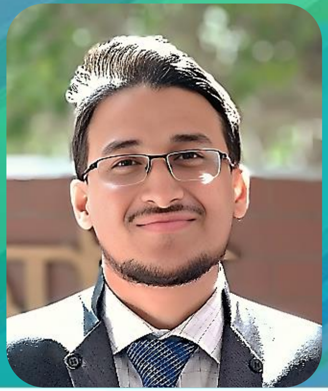
MATH 2161: Matrices and Vector Analysis



Md. Kawsar Ahmed Asif
Lecturer in Mathematics
Department of General Education
Canadian University of Bangladesh
Former Lecturer, World University of Bangladesh

Lesson 01

Lecture Outline

- Introduction to Matrices
- Basic Properties of Matrices
- Diagonal and Trace of Matrix
- Special Types of Matrices
- Matrix Operations
- Properties of Matrix Operations

Introduction to Matrices

Matrix: A matrix (plural *matrices*) is a rectangular array of numbers ($real$ or $complex$) arranged in rows and columns, enclosed by a pairs of brackets (( ) or [ ]) and generally denoted by capital letters (A, B, X or Y).

In other words, it is an ordered rectangular arrangement of numbers or functions which are represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Row

Column

Basic Properties of Matrices

- The numbers in the array are called the *entries* or the *elements* of the matrix.
- Number of rows and columns that a matrix has is called its *dimension* or *order*.
- Matrix is enclosed by [ ] or ( ).
- Matrix is also represented as $A = [a_{ij}]$, where i and j are the row and column number.
- Symbolically depicted as: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$
- First subscript refers to *row number (m)* and Second subscript refers to *column number (n)*.

Main Diagonal of Matrix

Main Diagonal: The diagonal from the top left corner to the bottom right corner of a square matrix is called the *main diagonal* or *leading diagonal* or *principal diagonal*.

In other words, it consists of the elements where the *row index* is equal to the *column index*. The main diagonal divides the matrix into two triangular regions: the upper triangular region and the lower triangular region.

If the matrix is A, then its *main diagonal* are the elements who's row number and column number are equal, a_{ii}.

$$A = \begin{bmatrix} \mathbf{1} & 8 & -1 & 9 \\ 0 & \mathbf{7} & 0 & 8 \\ 6 & 3 & \mathbf{5} & -2 \\ 2 & 4 & -2 & \mathbf{4} \end{bmatrix}_{4\times 4}$$

Anti-Diagonal of Matrix

Anti-Diagonal: The *anti-diagonal* of an order n square matrix A is the collection of entries a_{ij} such that $i + j = n + 1$ for all $1 \leq i, j \leq n$. That is, it runs from the top-right corner to the bottom-left corner of the matrix.

$$A = \begin{bmatrix} 1 & 8 & -1 & \mathbf{9} \\ 0 & 7 & \mathbf{0} & 8 \\ 6 & \mathbf{3} & 5 & -2 \\ \mathbf{2} & 4 & -2 & 4 \end{bmatrix}_{4\times 4}$$

In other words, the diagonal opposite to the *main diagonal* is called *anti-diagonal*, *counter diagonal* or *trailing diagonal*.

Off-Diagonal of Matrix

Off-Diagonal: The *off-diagonal* of a matrix includes all the elements that are not part of the *main diagonal*. It encompasses all the entries that are above or below the *main diagonal*, regardless of whether they are part of the anti-diagonal or not. In other words, the *off-diagonal* elements are the ones for which the *row index* is not equal to the *column index*.

$$A = \begin{bmatrix} 1 & \mathbf{8} & \mathbf{-1} & \mathbf{9} \\ \mathbf{0} & 7 & \mathbf{0} & \mathbf{8} \\ \mathbf{6} & \mathbf{3} & 5 & \mathbf{-2} \\ \mathbf{2} & \mathbf{4} & \mathbf{-2} & 4 \end{bmatrix}_{4\times 4}$$

In summary, while the *anti-diagonal* is a specific diagonal that is opposite to the *main diagonal*, the *off-diagonal* includes all the elements that are not part of the *main diagonal*, regardless of their position or relation to the *anti-diagonal*.

Trace of Matrix

Trace: If A is a square matrix, then the *trace* of A, denoted by $tr(A)$, is defined to be the sum of the entries on the *main diagonal* of A. The trace of A is undefined if A is not a square matrix.

Example: The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$tr(A) = a_{11} + a_{22} + a_{33} \quad \text{and} \quad tr(B) = (-1) + 5 + 7 + 0 = 11$$

Submatrix of a Matrix

Sub-Matrix: A matrix which is obtained from a given matrix by deleting any number of rows and/or columns is called a *sub-matrix* of the given matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

submatrices

For example, if $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$, a few submatrices of A are

$$[1], \quad [2], \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [1 \quad 5], \quad \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}, \quad A$$

However, the matrices $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ are not submatrices of A.

Special Types of Matrices with Examples

Rectangular Matrix: A matrix having unequal number of rows and columns $(m \neq n)$ is called *rectangular matrix*.

For examples, $\begin{bmatrix} 1 & -1 & 2 & 7 \\ -2 & 3 & 5 & 0 \end{bmatrix}_{2\times 4}$ and $\begin{bmatrix} 1-i & 2i & 3 \\ 0 & -2 & 4i \end{bmatrix}_{2\times 3}$ are rectangular matrices.

Square Matrix: A matrix with the equal number of rows and columns $(m = n)$ is called a *square matrix*.

For examples, $\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}_{2\times 2}$ and $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & -5 \\ 7 & -2 & 6 \end{bmatrix}_{3\times 3}$ are square matrices.

Special Types of Matrices with Examples

Horizontal Matrix: If in a matrix the number of columns is more than the number of rows $(n > m)$, then it is called a *horizontal matrix*.

For examples, $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 5 & 7 & 9 \end{bmatrix}$ are horizontal matrices.

Vertical Matrix: If in a matrix the number of rows is more than the number of columns $(m > n)$, then it is called a *vertical matrix*.

For examples, $\begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 6 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{bmatrix}$ are vertical matrices.

Special Types of Matrices with Examples

Row Matrix: A matrix having only one row is called a *row matrix*.

For examples, $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 & -1 & 0 \end{bmatrix}$ are row matrices. These are also called row vector.

Column Matrix: A matrix having only one column is called a *column matrix*.

For examples, $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} i \\ 4 \\ 0 \\ -2 \end{bmatrix}$ are column matrices. These are also called column vector.

Special Types of Matrices with Examples

Diagonal Matrix: A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a *diagonal matrix*.

For examples, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$ are diagonal matrices.

Scalar Matrix: A diagonal matrix whose diagonal elements are all equal is called a *scalar matrix*.

For examples, $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}_{2\times2}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times3}$ are scalar matrices.

Identity or Unit Matrix: A square matrix whose elements $a_{ij} = 0$, if $i \neq j$ and $a_{ij} = 1$, if $i = j$ is called an *identity matrix*.

For examples, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$ is an identity matrix of order 3.

Special Types of Matrices with Examples

Zero or Null Matrix: A matrix in which every element is zero is called a *zero* or *null matrix*.

For examples, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices.

Equal Matrix: Two matrices are said to be *equal* if

(a) Each matrix has same number of rows and columns.
(b) Corresponding elements within each matrix are equal.

For example,

$$A = \begin{bmatrix} 11 & x \\ y & 44 \end{bmatrix} \qquad B = \begin{bmatrix} 11 & 22 \\ 33 & 44 \end{bmatrix} \qquad C = \begin{bmatrix} l & m & n \\ o & p & q \\ r & s & t \end{bmatrix}$$

A and B matrices are equal. Matrix C is not equal to A or B.

Special Types of Matrices with Examples

Upper Triangular Matrix: A square matrix in which all the elements *below* the diagonal are ***zero*** is known as the *upper triangular matrix*.

For examples, $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$ and $\begin{bmatrix} 3 & -5 & 7 \\ 0 & 4 & 1 \\ 0 & 0 & 9 \end{bmatrix}$ are upper triangular matrices.

Lower Triangular Matrix: A square matrix in which all the elements *above* the diagonal are ***zero*** is known as the *lower triangular matrix*.

For examples, $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & -2 & 7 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 0 \\ -5 & 4 & 0 \\ 7 & 1 & 9 \end{bmatrix}$ are lower triangular matrices.

Special Types of Matrices with Examples

Commutative Matrices: If A and B are two square matrices such that $AB = BA$, then A and B are said to be *commutative matrices* or are said to commute. If $AB = -BA$, then the matrices A and B are said to be *anti-commutative matrices*.

For examples, $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix}$ are commutative matrices.

$\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$ are anti-commutative matrices.

Transpose Matrix: If the rows and columns in a matrix A are interchanged, the new matrix is called the *transpose* of the original matrix A. The transposed matrix is denoted by A^T.

For example, if $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$, then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$.

Special Types of Matrices with Examples

Symmetric Matrix: A matrix equal to its transpose, i. e. a square matrix such that $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$ is said to be *symmetric*. In a short, a square matrix A will be symmetric if $A^T = A$.

For examples, $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \\ -3 & 7 & 3 \end{bmatrix}$ are symmetric matrices.

Skew-Symmetric Matrix: A matrix equal to the negative of its transpose, i. e. a square matrix in which $a_{ij} = -a_{ji}$ and $a_{ii} = 0$ is said to be *skew-symmetric*. In a short, a square matrix A will be skew-symmetric if $A^T = -A$.

For examples, $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ are skew-symmetric matrices.

Matrix Operations

[1] Addition and Subtraction of Matrices

If A and B are the matrices of same order, then the *addition* or *subtraction* of A and B can be obtained by adding or subtracting the corresponding elements of A and B. Matrices of different order cannot be added or subtracted.

For example, consider matrix A and matrix B as below:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix}_{2\times 3}$$

Thus, $A + B = \begin{bmatrix} 1+5 & 2+6 & 3+7 \\ 7+3 & 8+4 & 9+5 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 10 \\ 10 & 12 & 14 \end{bmatrix}$

And, $A - B = \begin{bmatrix} 1-5 & 2-6 & 3-7 \\ 7-3 & 8-4 & 9-5 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -4 \\ 4 & 4 & 4 \end{bmatrix}$

Matrix Operations

[2] Scalar Multiplication of Matrices

If A is any matrix and k is any scalar, then the product kA or Ak is the matrix obtained by multiplying each element of A by k.

For example, consider matrix A and scalar k as below:

$$\text{Scalar, } k = 5 \text{ and Matrix, } A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$\text{Then, } kA = 5A = 5\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 30 & 25 \end{bmatrix}$$

Matrix Operations

[3] Matrix Multiplication

- Matrix product AB is defined only when the number of columns of A is equal to number of rows of B.

- Suppose, A is a matrix of order $i \times j$ and B is a matrix of order $j \times k$. Then the matrix product AB results in a matrix, say C of order $i \times k$.

- Each element in C can be computed according to: $C_{ik} = \sum_j A_{ij}\, B_{jk}$

  where, C_{ik} is $(i,k)^{th}$ element of C
  A_{ij} is $(i,j)^{th}$ element of A
  B_{jk} is $(j,k)^{th}$ element of B
  $\sum_j$ is summation sign, which indicates that the $a_{ij}\, b_{jk}$ terms should be summed over j.

Matrix Operations

For example, $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}_{2\times 3}$ $\quad B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \\ 10 & 11 \end{bmatrix}_{3\times 2}$

Suppose, $AB = C$ of order 2×2. Using formula $C_{ik} = \sum_j A_{ij}B_{jk}$, we get

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \times \begin{bmatrix} 6 & 7 \\ 8 & 9 \\ 10 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 8 + 20 & 0 + 9 + 22 \\ 18 + 32 + 50 & 21 + 36 + 55 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 31 \\ 100 & 112 \end{bmatrix}_{2\times 2} = C \text{ (Resultant Matrix)}$$

Properties of Matrix Addition, Scalar and Matrix Multiplication

Properties of Matrix Addition

For three matrices A, B and C of same order,

Commutative Law: $A + B = B + A$

Associative Law: $(A + B) + C = A + (B + C)$

Additive Identity: $A + 0 = 0 + A = A$

Properties of Scalar Multiplication

For three matrices A, B, C and scalars c, k

Associative Law: $(ck)A = c(kA)$

Distributive Law: $k(A + B) = kA + kB$

Scalar Identity: $1A = A$

Properties of Matrix Multiplication

For three matrices A, B and C

Associative Law: $A(BC) = (AB)C$

Distributive Law: $A(B + C) = AB + AC$

Multiplicative Identity: $AI = IA = A$

Next Lecture

- Matrix Multiplication
- Minors and Cofactors
- Determinant of Matrix
- Inverse of Matrix
- Rank of Matrix