Algorithm Efficiency Analysis

Objectives

At the end of the class, students are expected to be able to do the following:

- Know how to measure algorithm efficiency.
- Know the meaning of big O notation and determine the notation in algorithm analysis.

Algorithm analysis

- Study the efficiency of algorithms when the input size grow based on the number of steps, the amount of computer time and space.
- Is a major field that provides tools for evaluating the efficiency of different solutions.
- What is an efficient algorithm?
  - ✓ Faster is better (Time) - How do you measure time? Wall clock? Computer clock?
  - ✓ Less space demanding is better - But if you need to get data out of main memory it takes time

Analysis of algorithms

Algorithm analysis should be independent of :

- Specific implementations and coding tricks (programming language, control statements - Pascal, C, C++, Java)
- Specific Computers (hardware chip, OS, clock speed)
- Particular sets of data (string, int, float).

But size of data should matter

Analysis of algorithms

For a particular problem size, we may be interested in:

- **Worst-case efficiency**: Longest running time for any input of size n

  A determination of the maximum amount of time that an algorithm requires to solve problems of size n

- **Best-case efficiency**: Shortest running time for any input of size n

  A determination of the minimum amount of time that an algorithm requires to solve problems of size n

- **Average-case efficiency**: Average running time for all inputs of size n

  A determination of the average amount of time that an algorithm requires to solve problems of size n.

Complexity of algorithm

- Complexity time can be represented by **big 'O' notation**
- Big 'O' notation is denoted as: **O(*acc*)**

  whereby:

  ✓ **O** - order

  ✓ ***acc*** - class of algorithm complexity
- Big O notation example:

  $O(1)$, $O(\log_x n)$, $O(n)$, $O(n \log_x n)$, $O(n^2)$

Big O notation

Notation	Execution time/ number of step
O(1)	Constant function, independent of input size, n. Example: Finding the first element of a list.
$O(\log_x n)$	Problem complexity increases slowly as the problem size increases. Squaring the problem size only doubles the time. Characteristic: Solve a problem by splitting into constant fractions of the problem (e.g., throw away ½ at each step)
O(n)	Problem complexity increases linearly with the size of the input, n Example: counting the elements in a list.

Big O notation

Notation	Execution time/ number of step
$O(n\log_x n)$	Log-linear increase - Problem complexity increases a little faster than n Characteristic: Divide problem into sub problems that are solved the same way. Example: Merge sort
$O(n^2)$	Quadratic increase. Problem complexity increases fairly fast, but still manageable Characteristic: Two nested loops of size n
$O(n^3)$	Cubic increase. Practical for small input size, n.
$O(2^n)$	Exponential increase - Increase too rapidly to be practical Problem complexity increases very fast Generally unmanageable for any meaningful n Example: Find all subsets of a set of n elements

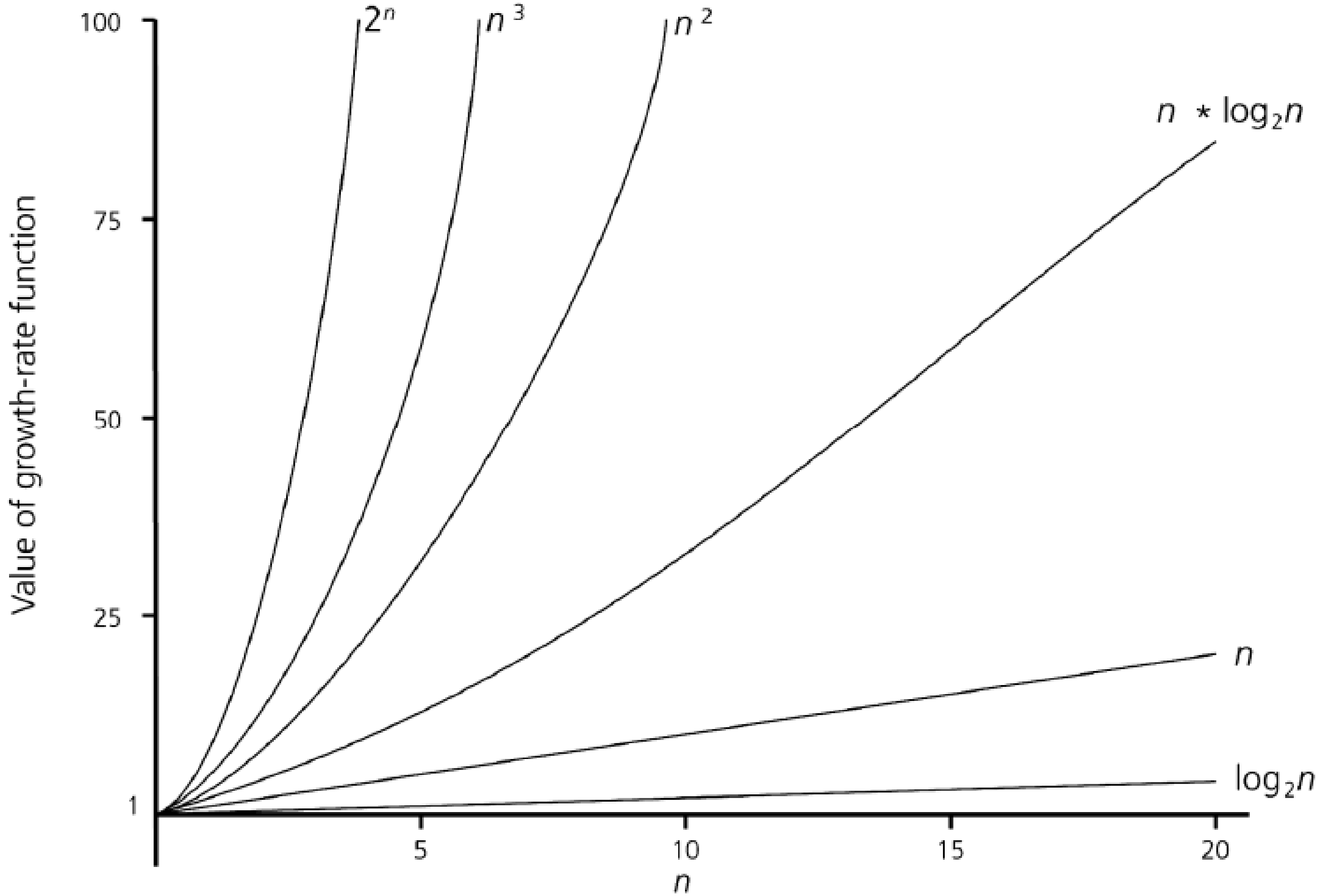
Order-of-Magnitude Analysis and Big O Notation

A comparison of growth-rate functions in tabular form

Function	n = 10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Order-of-Magnitude Analysis and Big O Notation

A comparison of growth-rate functions in graphical form



Order of increasing complexity

$$O(1) < O(\log_x n) < O(n) < O(n \log_2 n) < O(n^2) < O(n^3) < O(2^n)$$

Notasi	n = 8	n = 16	n = 32
$O(\log_2 n)$	3	4	5
$O(n)$	8	16	32
$O(n \log_2 n)$	24	64	160
$O(n^2)$	64	256	1024
$O(n^3)$	512	4096	32768
$O(2^n)$	256	65536	4294967296

Big O Notation

Example of algorithm (only for cout operation):

Notation	Code
O(1)	`int counter = 1;` `cout << "Arahan cout kali ke " << counter << "\n";`

$O(\log_x n)$	`int counter = 1; int i;`
	`for (i = x; i <= n; i = i * x) { // x must be > than 1`
	`    cout << "Arahan cout kali ke " << counter << "\n";`
	`    counter++;`
	`}`

O(n)	`int counter = 1; int i;`
	`for (i = 1; i <= n; i++) {`
	`    cout << "Arahan cout kali ke " << counter << "\n";`
	`    counter++;`
	`}`

Big O Notation

Example of algorithm (only for cout operation):

<table>
<tr><th>Notation</th><th>Code</th></tr>
<tr><td>$O(n\log_x n)$</td><td><pre>int counter = 1; int i, j = 1;
for (i = x; i <= n; i = i * x) { // x must be > than 1
      while (j <= n) {
            cout << "Arahan cout kali ke " << counter << "\n";
            counter++;
            j++;
      }
}</pre></td></tr>
</table>

Big O Notation

Example of algorithm (only for cout operation):

Notation	Code

$O(n^2)$	int counter = 1; int i, j;
	for (**i = 1**; i <= n; **i++**) {
	    for (**j = 1**; j <= n; **j++**) {
	        cout << "Arahan cout kali ke " <<
	        counter << "\n";
	        counter++;
	    }
	}

Big O Notation

Example of algorithm (only for cout operation):

Notation	**Code**

$O(n^3)$	int counter = 1; int i, j, k;
	for (**i = 1**; i <= n; **i++**) {
	    for (**j = 1**; j <= n; **j++**) {
	        for (**k = 1**; k <= n; **k++**) {
	            cout << "Arahan cout kali ke " <<
	            counter << "\n";
	            counter++;
	        }
	    }
	}

Big O Notation

Example of algorithm (only for cout operation):

<table>
<tr><th>Notation</th><th>Code</th></tr>
<tr><td>$O(2^n)$</td><td>

```
int counter = 1; int i = 1, j = 1;
while (i <= n) {
     j = j * 2;
     i++;
}

for (i = 1; i <= j; i++) {
     cout << "Arahan cout kali ke " << counter
     << "\n";
     counter++;
}
```

</td></tr>
</table>

Determine the complexity time of algorithm

Can be determined

- Theoretically - by calculation

  The complexity time is related to the number of steps/ operations

  ✓ Count the number of steps and then find the class of complexity @

  ✓ Find the complexity time for each steps and then count the total

- Practically – by experiment or implementation

  ✓ Implement the algorithms in any programming language and run the programs

  ✓ Depend on the compiler, computer, data input and programming style.

Determine the number of steps

It can be expressed by **summation series**

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + . . . + f(n) = n$$

where:

f(i) – Statement executed in the loop

Example 1: If ***n = 5, i = 1***

$$\sum_{i=1}^{5} f(i) = f(1) + f(2) + f(3) + f(4) + f(5) = 5$$

The statement that represented by ***f(i)*** will be **repeated 5 times**

Determine the number of steps

Example 2: If ***n = 5, i = 3***

$$\sum_{i=3}^{5} f(i) = f(3) + f(4) + f(5) = 3$$

The statement that represented by ***f(i)*** will be **repeated 3 times**

Example 3: If ***n = 1, i = 1***

$$\sum_{i=1}^{1} f(i) = f(1) = 1$$

The statement that represented by ***f(i)*** will be **executed only once**

Determine the number of steps

Example 1

Statements	Number of steps
int counter = 1;	$\sum_{i=1}^{1} f(i) = 1$
int i = 0;	$\sum_{i=1}^{1} f(i) = 1$
for (i = 1; i <= n; i++) {	$\sum_{i=1}^{n} f(i) = n$
cout << "Arahan cout kali ke " << counter << "\n";	$\sum_{i=1}^{n} f(i) \sum_{i=1}^{1} f(i) = n.1 = n$
counter++;	$\sum_{i=1}^{n} f(i) \sum_{i=1}^{1} f(i) = n.1 = n$
}	0
Total Steps	**2 + 3n**
Complexity Time	**O(n)**

Determine the number of steps

Example 2

Algorithm	Number of steps
void sample4 ( ) {	0
for (int a=2; a<=n; a++)	n – 2 + 1 = n - 1
cout << “Example of step calculation”;	(n – 1).1 = n - 1
}	0
Total Steps	**2(n – 1)**
Complexity Time	**O(n)**

Determine the number of steps

Example 3

Algorithm	Number of steps
void sample5 ( ) {	0
for (int a=1; a<=n-1; a++)	n - 1 - 1 + 1 = n - 1
cout << “Example of step calculation”;	(n – 1).1 = n - 1
}	0
Total Steps	2(n – 1)
Complexity Time	O(n)

Determine the number of steps

Example 4

Algorithm	Number of steps
void sample6 ( ) {	0
for (int a=1; a<=n; a++)	n - 1 + 1 = n
for (int b=1; b<=n; b++)	n.(n - 1 + 1) = n.n
cout << “Example of step calculation”;	n.n.1 = n.n
}	0
Total Steps	**n + 2n²**
Complexity Time	**O(n²)**

Determine the number of steps

Example 5

Algorithm	Number of steps
void sample6 ( ) {	0
for (int a=1; a<=n; a++)	n - 1 + 1 = **n**
for (int b=1; b<=a; b++)	**n(n + 1) / 2**
cout << "Example of step calculation";	**(n(n + 1) / 2).1 = n(n+1)/2**
}	0
Total Steps	**$n + n^2 + n = 2n + n^2$**
Complexity Time	**$O(n^2)$**

Determine the number of steps

To get **n.(n+1)/2**, we used summation series as shown below:

$$\sum_{a=1}^{n} \sum_{b=1}^{n} 1 =$$

$$= n(1 + 2 + 3 + 4 + \ldots + n)$$

$$= \frac{n(n+1)}{2}$$

$$= \frac{n^2 + n}{2}$$

Proof:

If **n = 4**, for inner loop

for (int b=1; b<=a; b++)

It will be repeated:

(1+2+3+4) = 10 times

By using the formula:

n.(n+1)/2 = 4(5)/2 = 10 times

Determine the number of steps

Example 6

Statements	Number of steps
int counter = 1;	$\sum_{i=1}^{1} f(i) = 1$
int i = 0;	$\sum_{i=1}^{1} f(i) = 1$
int j = 1;	$\sum_{i=1}^{1} f(i) = 1$
for (i = 3; i <= n; i = i * 3) {	$\sum_{i=3}^{n} f(i) = f(3)+f(9)+f(27)+\ldots+f(n) = \log_3 n$
while (j <= n) {	$\sum_{i=3}^{n} f(i) \sum_{i=1}^{n} f(i) = n.\log_3 n$
cout << "Arahan cout kali ke " << counter << "\n";	$\sum_{i=3}^{n} f(i) \sum_{i=1}^{n} f(i) \sum_{i=1}^{1} f(i) = n.\log_3 n.1$
counter++;	$\sum_{i=3}^{n} f(i) \sum_{i=1}^{n} f(i) \sum_{i=1}^{1} f(i) = n.\log_3 n.1$
j++;	$\sum_{i=3}^{n} f(i) \sum_{i=1}^{n} f(i) \sum_{i=1}^{1} f(i) = n.\log_3 n.1$
}}	0
Total Steps	$3 + \mathrm{Log}_3 n + 4n \log_3 n$
Complexity Time	$O(n \log_2 n)$

Determine the number of steps

$3 + Log_3 n + 4n\ log_3 n$

- Consider the largest factor: ***$4n\ log_3 n$***
- Remove the coefficient: ***$n\ log_3 n$***
- In **asymptotic classification**, the base of the log can be omitted as shown in this formula:

  $$\log_a n = \log_b n / \log_b a$$

- Then,

  $$\log_3 n = \log_2 n / \log_2 3 = \log_2 n / 1.58$$

- Remove the coefficient 1/1.58
- So, the ***complexity time = $O\ (n\log_2 n)$***

Determine the number of steps

Example 7

Algorithm	Number of steps
void sample8 ( ) {	0
int n, x, i=1;	1
while (i<=n) {	n - 1 + 1 = **n**
x++;	**n.1 = n**
i++;	**n.1 = n**
}	0
Total Steps	**1 + 3n**
Complexity Time	**O(n)**

Determine the number of steps

Example 8

Algorithm	Number of steps
void sample9 ( ) {	0
int n, x, i=1;	1
while (i<=n) {	$1 + \log_2 n$
x++;	$(1 + \log_2 n).1 = 1 + \log_2 n$
i=i*2;	$(1 + \log_2 n).1 = 1 + \log_2 n$
}	0
Total Steps	$1 + 3(1 + \log_2 n)$
Complexity Time	$O(\log_2 n)$

Determine the number of steps

Example 9

Algorithm	Number of steps
void sample9 ( ) {	0
int n, x, i=1;	1
while (i<=n) {	**$1 + \log_4 n$**
x++;	**$(1 + \log_4 n).1 = 1 + \log_4 n$**
i=i*4;	**$(1 + \log_4 n).1 = 1 + \log_4 n$**
}	0
Total Steps	**$1 + 3(1 + \log_4 n)$**
Complexity Time	**$O(\log_2 n)$**

- *While loop iterate from i=1 until i=n ;* ***i increment 4 times at each iteration***
- *Number of iteration for while loop =* ***$(1 + \log_4 n)$***

Determine the number of steps

Using **asymptotic O**, where base for the algorithm is ignored:

$$\log_a n = \log_b n / \log_b a$$

Then,

$$\log_4 n = \log_2 n / \log_2 4 = \log_2 n / 2$$

Therefore,

Complexity Time = O ($\log_2 n$)

Thank You