1 Generation from Mixture Distribution

Let K be a fixed positive integer and $\pi_1, \pi_2, \ldots, \pi_K$ be non-negative real numbers such that $\sum_{i=1}^K \pi_i = 1$. Also, let $f_1, f_2, \ldots, f_K$ be K PDFs. Then, it is easy to see that

$$f(x) = \sum_{i=1}^{K} \pi_i f_i(x), \quad x \in \mathbb{R} \tag{1}$$

is a PDF. The corresponding CDF is given by

$$F(x) = \sum_{i=1}^{K} \pi_i F_i(x), \quad x \in \mathbb{R},$$

where F_i is the CDF corresponding to PDF f_i. The PDF given in (1) is called **mixture PDF** and corresponding distribution is called **mixture distribution**. In this section, we will discuss the method of generation from PDF of the form (1). Let $q_0 = 0$ and $q_k = \pi_1 + \pi_2 + \ldots + \pi_k$ for $k = 1, 2, \ldots, K$. Now, the following algorithm can be used to generate random numbers from the PDF (1).

Algorithm 1 Generation from mixture distribution

1: generate U from $U(0, 1)$
2: **if** $q_{k-1} < U \leq q_k$ **then**
3: Generate X from f_k.
4: **end if**
5: **return** X

Let us discuss the justification of the algorithm. For $x \in \mathbb{R}$, we have

$$\begin{aligned}
P\left(X \leq x\right) &= \int_0^1 P\left(X \leq x | U = u\right) du \\
&= \sum_{k=1}^{K} \int_{q_{k-1}}^{q_k} P\left(X \leq x | U = u\right) du \\
&= \sum_{k=1}^{K} \int_{q_{k-1}}^{q_k} F_k(x) du \\
&= \sum_{k=1}^{K} F_k(x) \left(q_k - q_{k-1}\right) \\
&= \sum_{k=1}^{K} \pi_k F_k(x).
\end{aligned}$$

2 Generating Sample Paths of a Brownian Motion

By a **standard one-dimensional Brownian Motion** on $[0, T]$, we mean a **stochastic process** $\{W(t) : 0 \leq t \leq T\}$ with the following properties :

1. $W(0) = 0$.
2. The **mapping** $t \mapsto W(t)$ is, with probability 1, a **continuous function** on $[0, T]$.
3. The **increments** $\{W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_k) - W(t_{k-1})\}$ are **independent** for any k and any $0 \leq t_0 < t_1 < \cdots < t_k \leq T$.
4. $W(t) - W(s) \sim \mathcal{N}(0, t - s)$ for any $0 \leq s < t \leq T$.

A consequence of (1) and (4) is that

$$W(t) \sim \mathcal{N}(0, t) \text{ for } 0 < t \leq T.$$

For constants μ and $\sigma > 0$, we call a process $X(t)$, a Brownian motion with **drift** μ and **diffusion coefficient** σ^2 (abbreviated $X \sim BM(\mu, \sigma^2)$) if $(X(t) - \mu t)/\sigma$ is a standard Brownian motion. Thus we may construct X from a standard Brownian Motion from W by setting,

$$X(t) = \mu t + \sigma W(t).$$

In discussing the **simulation of** $BM(0,\, 1)$, we mostly focus on simulating values

$$\{W(t_0),\, W(t_1),\, W(t_2),\, \ldots,\, W(t_n)\} \text{ or } \{X(t_0),\, X(t_1),\, X(t_2),\, \ldots,\, X(t_n)\}$$

at a fixed set of times $0 = t_0 < t_1 < t_2 < \cdots < t_n$. Because Brownian Motion has **independent normally distributed increments**, simulating the $W(t_i)$ or $X(t_i)$ from their increments is straightforward. Let $Z_1, Z_2, Z_3, \ldots, Z_n$ be independent standard normal variables generated using any of the standard methods. For a standard Brownian Motion we set : $t_0 = 0$ and $W(0) = 0$. Subsequent values can be **generated** using:

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} \cdot Z_{i+1} \;,\; i = 0, 1, \ldots, n - 1.$$

For $X \sim BM(\mu, \sigma^2)$ with constants μ and σ and given $X(0)$ we set,

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} \cdot Z_{i+1} \;,\; i = 0, 1, \ldots, n - 1.$$

The methods above are exact at the time points $t_1, t_2, \ldots, t_n$, but subject to discretization error, compared to the true Brownian motion.