

1 Generation from Univariate Normal Distribution

Recall that the PDF and CDF of univariate normal distribution with mean μ ($\mu \in \mathbb{R}$) and variance σ^2 ($\sigma > 0$) are

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad \text{for } x \in \mathbb{R}$$

and

$$\Phi_{\mu,\sigma}(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^x e^{-(t-\mu)^2/2\sigma^2} dt \quad \text{for } x \in \mathbb{R},$$

respectively, where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for } x \in \mathbb{R}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt \quad \text{for } x \in \mathbb{R}.$$

Note that $\phi(\cdot)$ and $\Phi(\cdot)$ are PDF and CDF of **standard normal distribution**. The word “**standard**” indicates that the mean is 0 and the variance is 1. If $Z \sim N(0, 1)$, then $\mu + \sigma Z \sim N(\mu, \sigma^2)$. Thus, given a method for generating samples Z_1, Z_2, \dots from the standard normal distribution, we can generate samples X_1, X_2, \dots from $N(\mu, \sigma^2)$ by setting $X_i = \mu + \sigma Z_i$. It, therefore, suffices to consider methods for sampling from $N(0, 1)$.

We now discuss algorithms for generating univariate normal distribution. We assume the availability of a sequence U_1, U_2, \dots of independent random variables uniformly distributed on the unit interval $(0, 1)$ and consider methods for transforming these uniform random variables to normally distributed random variables.

1.1 Using Acceptance Rejection Method

To obtain an **upper bound** of $\phi(x)$, we need to find an **lower bound** of $\frac{x^2}{2}$. Notice that

$$\frac{1}{2} (|x| - 1)^2 = \frac{x^2}{2} - |x| + \frac{1}{2} \geq 0.$$

Thus, $\frac{x^2}{2} \geq |x| - \frac{1}{2}$ for all $x \in \mathbb{R}$. This shows that, for all $x \in \mathbb{R}$,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \leq \frac{1}{\sqrt{2\pi}} e^{-|x| + \frac{1}{2}} = cg(x),$$

where $c = \sqrt{\frac{2e}{\pi}}$ and $g(x) = \frac{1}{2}e^{-|x|}$ for all $x \in \mathbb{R}$. Therefore, we can use acceptance rejection method if we can generate from g easily. To generate from $g(\cdot)$ we can use the following result. The PDF of $X = ZY$ is $g(\cdot)$ if Y and Z are independent random variables such that $Y \sim \text{Exp}(1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. To see it, notice that the CDF of X is

$$\begin{aligned} P(X \leq x) &= P(X \leq x | Z = 1) P(Z = 1) + P(X \leq x | Z = -1) P(Z = -1) \\ &= \frac{1}{2} P(ZY \leq x | Z = 1) + \frac{1}{2} P(ZY \leq x | Z = -1) \\ &= \frac{1}{2} P(Y \leq x) + \frac{1}{2} P(Y \geq -x) \\ &= \begin{cases} \frac{1}{2}e^x & \text{if } x < 0 \\ 1 - \frac{1}{2}e^{-x} & \text{if } x \geq 0. \end{cases} \end{aligned}$$

This shows that the PDF of X is same as $g(\cdot)$. This result tells us the method of generation from g . **First, generate Y from exponential distribution with mean 1 and then assign a random sign to it.** Therefore, we have the Algorithm 1.

Algorithm 1 Generation of $N(0, 1)$ by rejecting from Laplace PDF

```

1: repeat
2:   generate  $U$  from  $U(0, 1)$  ▷ Will be used to generate from  $\text{Exp}(1)$ 
3:    $X \leftarrow -\ln(U)$  ▷  $X \sim \text{Exp}(1)$ 
4:   generate  $V$  from  $U(0, 1)$  ▷ Will be used to generate random sign
5:   if  $V < \frac{1}{2}$  then ▷ Assigning signs with equal probability
6:      $X \leftarrow -X$ 
7:   end if
8:   generate  $W$  from  $U(0, 1)$  ▷ Will be used to implement acceptance rejection step
9: until  $We^{\frac{1}{2}-|X|} \leq e^{-\frac{X^2}{2}}$ 
10: return  $X$ 

```

Note that the condition at Line 9 of the Algorithm 1 **does not depend on the sign** of X . Thus, we may assign the sign if the candidate is accepted. Moreover, $We^{\frac{1}{2}-|X|} \leq e^{-\frac{X^2}{2}}$ is equivalent to $-2\ln W \geq (|X| - 1)^2$. Thus, Algorithm 1 can be modified to obtain a better algorithm given below.

Algorithm 2 Generation of $N(0, 1)$ by rejecting from Laplace PDF

```

1: repeat
2:   generate  $U$  from  $U(0, 1)$  ▷ Will be used to generate from  $\text{Exp}(1)$ 
3:    $X \leftarrow -\ln(U)$  ▷  $X \sim \text{Exp}(1)$ 
4:   generate  $W$  from  $U(0, 1)$  ▷ Needed to implement acceptance rejection step and assign sign
5: until  $-2\ln(W) \geq (X - 1)^2$ 
6: if  $W < \frac{1}{2}$  then
7:    $X \leftarrow -X$ 
8: end if
9: return  $X$ 

```

2 Box-Muller Method

Perhaps the simplest method to implement (though **not the fastest or necessarily the most convenient**) is the one by Box-Muller. This algorithm generates a sample from a bivariate standard normal distribution, each component of which is thus a univariate standard normal. The algorithm is based on the following result.

Theorem 1. *Let $U_1, U_2 \stackrel{i.i.d.}{\sim} U(0, 1)$. Define*

$$Z_1 = \sqrt{-2 \ln U_1} \cos(2\pi U_2) \quad \text{and} \quad Z_2 = \sqrt{-2 \ln U_1} \sin(2\pi U_2).$$

Then $Z_1, Z_2 \stackrel{i.i.d.}{\sim} N(0, 1)$.

Proof. Note that

$$\begin{aligned} \frac{\partial z_1}{\partial u_1} &= -\frac{\cos(2\pi u_2)}{u_1 \sqrt{-2 \ln u_1}}, \\ \frac{\partial z_2}{\partial u_1} &= -\frac{\sin(2\pi u_2)}{u_1 \sqrt{-2 \ln u_1}}, \\ \frac{\partial z_1}{\partial u_2} &= 2\pi \sin(2\pi u_2) \sqrt{-2 \ln u_1}, \\ \frac{\partial z_2}{\partial u_2} &= 2\pi \cos(2\pi u_2) \sqrt{-2 \ln u_1}. \end{aligned}$$

Thus, the Jacobian of the transformation is

$$J = \det \begin{pmatrix} \frac{\partial z_1}{\partial u_1} & \frac{\partial z_2}{\partial u_1} \\ \frac{\partial z_1}{\partial u_2} & \frac{\partial z_2}{\partial u_2} \end{pmatrix} = -\frac{2\pi}{u_1}.$$

Therefore, the absolute value of the Jacobian of inverse transformation is

$$\frac{1}{|J|} = \frac{u_1}{2\pi} = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2 + z_2^2)}.$$

and hence the JPDP of (Z_1, Z_2) is

$$f_{Z_1, Z_2}(z_1, z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2 + z_2^2)} = \phi(z_1)\phi(z_2) \quad \text{for } z_1 \in \mathbb{R}, z_2 \in \mathbb{R}.$$

This proves the result. □

Now, based on the theorem, we have the Algorithm 3 to generate a pair of independent standard normal random numbers from a pair of independent $U(0, 1)$ random numbers.

Algorithm 3 Box-Muller Method to generate $N(0, 1)$ random numbers

- 1: generate U_1 and U_2 from $U(0, 1)$
 - 2: $R \leftarrow \sqrt{-2 \ln U_1}$
 - 3: $\theta \leftarrow 2\pi U_2$
 - 4: $Z_1 \leftarrow R \cos(\theta)$
 - 5: $Z_2 \leftarrow R \sin(\theta)$
 - 6: **return** (Z_1, Z_2) .
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3 Marsaglia and Bray Method

Marsaglia and Bray developed a modification of the Box-Muller method that reduces computing time by **avoiding evaluation** of the “cos” and “sin” functions. The Marsaglia and Bray method instead uses **acceptance rejection method** to sample points **uniformly in the unit disc** and then transforms these points to normal variables. The algorithm is as follows:

Algorithm 4 Marsaglia and Bray Method to generate $N(0, 1)$ random numbers

```
1: repeat
2:   generate  $U_1$  and  $U_2$  from  $U(0, 1)$ 
3:    $U_1 \leftarrow 2U_1 - 1$  and  $U_2 \leftarrow 2U_2 - 1$   $\triangleright U_i \sim U(-1, 1)$ 
4: until  $U_1^2 + U_2^2 \leq 1$   $\triangleright (U_1, U_2)$  is uniformly distributed on the disc of radius 1 centered at the origin
5:  $Z_1 \leftarrow U_1 \left[ \frac{-2 \ln(U_1^2 + U_2^2)}{U_1^2 + U_2^2} \right]^{\frac{1}{2}}$  and  $Z_2 \leftarrow U_2 \left[ \frac{-2 \ln(U_1^2 + U_2^2)}{U_1^2 + U_2^2} \right]^{\frac{1}{2}}$   $\triangleright Z_i \sim N(0, 1)$  and they are independent
6: return  $(Z_1, Z_2)$ .
```
