1 Quasi-Monte Carlo

Quasi-Monte Carlo approximates the integral $\int_{[0,\,1)^m} f(\boldsymbol{x})d\boldsymbol{x}$ using

$$\int_{[0,\,1)^m} f(\boldsymbol{x})d\boldsymbol{x} \approx \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{x}_i)$$

for **carefully** and **deterministically chosen points** $\boldsymbol{x}_1,\,\ldots,\,\boldsymbol{x}_n$. Quasi-Monte Carlo **differ from** ordinary Monte Carlo in by **making no attempt to mimic randomness**. Quasi-Monte Carlo try to increase accuracy specifically by **generating points** that are **too evenly distributed** to be random.

1.1 Sequences of Numbers with Low Discrepancy

One difficulty with random numbers is that they may **fail to distribute** uniformly. Here "uniform" is **not meant in the stochastic** sense of the distribution $U(0,1)$, but has the meaning of **equidistributiveness**. The aim is to generate numbers for which the deviation from uniformity is minimal. This **deviation** is called "**discrepancy**".

It would be desirable to find a **compromise**, picking samples points in some schemes such that the **fineness advances but clustering is avoided**. The sample points should **fill the integration domain** as **uniformly as possible**. Let $Q \subseteq [0,1]^m$ be an arbitrary axially parallel m-dimensional rectangle in the unit cube $[0,1]^m$ of $\mathbb{R}^m$. This Q is thus a **product** of m-intervals. Suppose a set of points $x_1, x_2, \ldots, x_M \in [0,1]^m$. Let # denote the number of points, then the **goal** is,

$$\frac{\#\text{ of } x_i \in Q}{\#\text{ of all points}} \cong \frac{\text{vol}(Q)}{\text{vol}([0,1]^m)}.$$

for **as many rectangles** as possible.

Given a collection $\mathcal{A}$ of subsets of $[0,\,1)^m$, the **discrepancy of the point set** $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$ relative to $\mathcal{A}$ is defined by

$$D(\boldsymbol{x}_1,\,\ldots,\,\boldsymbol{x}_N;\mathcal{A}) := \sup_{Q\in\mathcal{A}}\left|\frac{\#\text{ of } \boldsymbol{x}_i \in Q}{N} - \text{vol}(Q)\right|.$$

Taking $\mathcal{A}$ be the collection of rectangles in $[0,\,1)^m$ of the form

$$\prod_{j=1}^{m}[u_j,\,v_j), \qquad 0 \le u_j < v_j < 1,$$

yields **ordinary discrepancy** $D(\boldsymbol{x}_1,\,\ldots,\,\boldsymbol{x}_N)$. Restricting $\mathcal{A}$ to rectangles of the form

$$\prod_{j=1}^{m}[0,\,v_j), \qquad 0 < u_j < 1$$

defines the **star discrepancy** $D^*(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$. It can be shown that

$$D^*(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \le D(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \le 2^m D^*(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N).$$

It is customary to reserve the informal term "**low discrepancy**" for **methods that achieve a** star discrepancy of $O((\log n)^m/n)$. The following introduce low discrepancy sequences, *viz.*, Van Der Corput sequence and Halton sequence.

Van Der Corput Sequence

It is an **one-dimensional** low-discrepancy sequence. For $i = 1, 2, \ldots$ let

$$i = \sum_{k=0}^{\infty} d_k b^k,$$

be the **expression in base** b (integer ≥ 2) with digits $d_k \in \{0, 1, \ldots, b-1\}$. In the expression of i, **all but finitely many of the coefficients** d_k equal to **zero**. Then the **radical inverse function** is defined by

$$\phi_b(i) := \sum_{k=0}^{\infty} d_k b^{-k-1} = \sum_{k=0}^{\infty} \frac{d_k}{b^{k+1}}.$$

The base-b **Van Der Corput sequence** is given by

$$x_i := \phi_b(i).$$

Halton Sequence

It is a **multi-dimensional** low-discrepancy sequence. Let $p_1, p_2, \ldots, p_m$ be a **relatively prime** integers. The **Halton sequence** in m dimension is defined as follows:

$$\boldsymbol{x}_i := (\phi_{p_1}(i), \phi_{p_2}(i), \ldots, \phi_{p_m}(i)), i = 1, 2, \ldots$$

Usually one takes $p_1, p_2, \ldots, p_m$ to the **first** m **prime** numbers.