FRE-GY 6971
Fixed Income Quant Trading

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Assignment 3

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1 Read Chapters 2-3 of the Fisher & Gilles paper.

1-Factor Affine model:

$$dr(t) = \mu - \kappa r(t)dt + \sqrt{\gamma r(t) + \sigma}dW(t)$$

Prove that

$$p(t,T) = V(r,t;T) = e^{A(t;T)-rB(t;T)}$$

where functions A & B satisfy the Ricatti equations:

$$\begin{aligned} \frac{dA}{dt} - \mu B - \frac{1}{2}\sigma B^2 = 0 \\ \frac{dB}{dt} - \kappa B - \frac{1}{2}\gamma B^2 + 1 = 0 \end{aligned}$$

Solution:

$$dr(t) = \mu - \kappa r(t)dt + \sqrt{\gamma r(t) + \sigma}dW(t)$$

$$u_r = \mu - \kappa r(t)$$

$$\sigma_r = \sqrt{\gamma r(t) + \sigma}$$

Lets consider $p(t,T) = e^{A-rB}$ to be a possible solution, then

$$p_t = (A' - rB')p$$

$$p_r = -Bp$$

$$p_{rr} = (-B)^2 p$$

$p(t,T) = p(r(t),t)$

Applying Ito's Lemma,

$$\begin{aligned} dp &= p_t dt + p_r dr + \frac{1}{2}p_{rr}drdr \\ &= p_t dt + p_r[(\mu - \kappa r(t))\,dt + \sqrt{\gamma r(t) + \sigma}dW] + \frac{1}{2}p_{rr}(\gamma r(t) + \sigma)dt \\ &= [p_t + p_r\,(\mu - \kappa r(t)) + \frac{1}{2}p_{rr}(\gamma r(t) + \sigma)]dt + [\sqrt{\gamma r(t) + \sigma}]dW \end{aligned}$$

Discounted asset price : $u_p - r_t = 0$

$$
\begin{aligned}
&p_t + p_r\left(\mu - \kappa r(t)\right) + \frac{1}{2}p_{rr}(\gamma r(t) + \sigma) = rp \\
&(A' - rB')p + -Bp\left(\mu - \kappa r(t)\right) + \frac{1}{2}(-B)^2 p(\gamma r(t) + \sigma) = rp \\
&(A' - rB') + -B\left(\mu - \kappa r(t)\right) + \frac{1}{2}(-B)^2(\gamma r(t) + \sigma) - r = 0 \\
&(A' - \mu B + \frac{1}{2}(-B)^2\sigma) + r(-B' + \kappa B + \frac{1}{2}(-B)^2\gamma - 1) = 0
\end{aligned}
$$

From this we get,

$$
\begin{aligned}
&A' - \mu B + \frac{1}{2}B^2\sigma = 0 \\
&OR
\end{aligned}
$$

$$
\begin{aligned}
-B' + \kappa B + \frac{1}{2}B^2\gamma - 1 = 0 \\
B' - \kappa B - \frac{1}{2}B^2\gamma + 1 = 0
\end{aligned}
$$

2 Implement the solver for the Ricatti equations and find the solution for a given set of parameters. You can use scipy.integrate.odeint, for instance.

Done in jupyter notebook

2.1 Find the analytical solution for the zero-coupon bond price, starting from the pricing formula for the zero-coupon bond we derived in class, and compare the result with the solution of the above system of ODEs.

$$
\begin{aligned}
\frac{dA}{dt} - \mu B - \frac{1}{2}\sigma B^2 = 0 \\
\frac{dB}{dt} - \kappa B - \frac{1}{2}\sigma B^2 + 1 = 0
\end{aligned}
$$

Since the values are as follows:

r0 = 0.04

mu = 0.0025

kappa = 0.05

gamma = 0

sigma = 0.01

The equations are transformed to :

$$\frac{dA}{dt} - \mu B - \frac{1}{2}\sigma B^2 = 0$$
$$\frac{dB}{dt} - \kappa B + 1 = 0$$

To obtain the analytical solution for A(t,T) and B(t,T), we can solve the above system of differential equations analytically.

Integrating the second equation, we get:

$B(t,T) = \frac{1}{\kappa}(1 - e^{-\kappa(T-t)})$

Taking T as 30 years and t = 0(for present value),

We get

$$B(t,T) = \frac{1}{0.05}(1 - e^{-0.05*30})$$
$$= 15.537$$

Next, we substitute this expression for B(t,T) in the first equation, and solve for A(t,T):

Solve for $A(t;T)$ using the first ODE and the boundary condition $A(T;T) = 0$:

Substituting the expression for $B(t;T)$ from above into the first ODE with $\dot{A}(t)$, we get:

$\dot{A}(t) + \frac{\sigma}{2}B^2(t) - \mu B(t) = 0$

$\dot{A}(t) = \mu B(t) - \frac{\sigma}{2}B^2(t)$

$\dot{A}(t) = \mu\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) - \frac{\sigma}{2}(\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right))^2$

$\int_t^T \dot{A}(t) = \int_t^T \mu\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) - \frac{\sigma}{2}(\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right))^2$

$A(t) = \int_t^T \mu\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) dt - \frac{\sigma}{2}(\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right))^2 dt$

$= \int_t^T \mu\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) dt - \frac{\sigma}{2}(\frac{1}{\kappa^2}\left(1 - 2e^{-\kappa(T-t)} + e^{-\kappa(2(T-t))}\right))dt$

$= \mu\frac{1}{\kappa}\left((T-t) - \frac{1}{\kappa} \times (1 - e^{-\kappa(T-t)})\right) - \frac{\sigma}{2}(\frac{1}{\kappa^2}\left((T-t) - (\frac{2}{\kappa}(1 - e^{-\kappa(T-t)}) - (\frac{1}{2\kappa}(1 - e^{-2\kappa(T-t)})\right))$

$A(t) = (-T + t + (1 - e^{-\kappa(T-t)}))\left(\frac{\mu}{\kappa} - \frac{\sigma}{2\kappa^2}\right) - \frac{\sigma}{4\kappa}\left(\frac{1 - e^{-\kappa(T-t))}}{\kappa}\right)^2$

$$
\begin{aligned}
A(t) &= (t - T + B(t))\left(\frac{\mu}{\kappa} - \frac{\sigma}{2\kappa^2}\right) - \frac{\sigma}{4\kappa}(B(t))^2 \\
A(t;T) &= \frac{1}{\kappa^2}(B(t;T) - T + t)\left(\mu\kappa - \frac{\sigma}{2}\right) - \frac{\sigma}{4\kappa}(B(t;T))^2
\end{aligned}
$$

Hence, we have A(t;T) and B(t;T) for the Vasicek model as follows:

$$
\begin{aligned}
B(t;T) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \\
A(t;T) &= \frac{1}{\kappa^2}(B(t;T) - T + t)\left(\mu\kappa - \frac{\sigma}{2}\right) - \frac{\sigma B^2(t;T)}{4\kappa}
\end{aligned}
$$

Substituting B as -15.537, we get

$$
\begin{aligned}
A(t,T) &= \frac{1}{0.05^2}(15.537 - 30)(0.0025 * 0.05 - 0.01/2) - 0.01(15.537^2/(4 * 0.05)) \\
&= 16.131
\end{aligned}
$$

Hence the values match

$$
V = e^{(A - rB)} = e^{(16.131 - 0.04 * 15.53)} = 5444113
$$

3 Assume the dynamics are as follows:

$$
dr(t) = \mu - \kappa r(t)^2 dt + \sqrt{\gamma r(t) + \sigma} dW(t)
$$

3.1 Prove that this model is not from the Affine class

$$
\begin{aligned}
& dr(t) = \left(\mu - \kappa r^2(t)\right) dt + \sqrt{r(t) + \sigma} \\
& dW(t) = \mu_r dt + \sigma_r dW(t) \\
& p(t,T) = \tilde{\mathbb{E}}\left[\frac{m(T)}{m(t)} p(T,T)\right] \\
& \tilde{\mathbb{E}}\left[\frac{m(T)}{m(t)}\right] \qquad \gamma(t) = 0 \\
& \frac{dm(t)}{m(t)} = -r(t)dt \qquad \lambda(t) = 0 \text{ here}
\end{aligned}
$$

as per Fisher & Giles, a term structure model falls in affine class if and only if $\mu_r(t,T)$ is affine in r (we do not mention $\lambda\sigma$ condition as $\lambda = 0$)

In this case,

The function $\mu_r(t,T)$ can be obtained by taking the partial derivative of μ with respect to r:

$$\mu_\gamma(t,T) = -2\kappa r(t)$$

which is quadratic in r therefore not affine.

3.2 Is zero coupon bond price $p(t,T)$ still a solution to a system of ODEs? Explain your answer in detail

No, the zero coupon bond price $p(t,T)$ is not a solution to the system of ODEs given in the question. The PDE for this shot rate model is

$$P_t + \frac{1}{2}(\gamma r + \sigma)P_{rr} + \left(\mu - \kappa r(t)^2\right)P_r - rP = 0$$

$$(A' - rB') + \frac{1}{2}(\gamma r + \sigma)B^2 - \left(\mu - \kappa r(t)^2\right)B - r = 0$$

$$\left(A' + \frac{\sigma}{2}B^2 - \mu B\right) + \left(-B' - \frac{(\gamma B^2 - 1)}{2}\right)r + r^2(\kappa B) = 0$$

If this is true for all r

$$\left(A' + \frac{\sigma}{2}B^2 - \mu B\right) = 0$$

OR

$$\left(-B' - \frac{(\gamma B^2 - 1)}{2}\right) = 0$$

we also have a third equation

$$\kappa B' = 0$$

If κ is zero then A and B have a unique solution.

$$KB' = 0 \Rightarrow B = \text{ or } K = 0$$

$$\text{If } B' = 0 \Rightarrow B = \text{ constant } A \text{ as } B(T) = 0 \Rightarrow B = 0 \& A = 0$$

$$\text{If } K = 0, \text{ on } B \text{ \& A have a unique}$$

There are 3 equations and 2 variables.