FRE-GY 7851
Interest Rate Derivatives

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Assignment 4
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Contents

1 Use the Focker –Plank equation (aka Forward Kolmogorov Equation) to infer the steady state $(t \longrightarrow \infty)$ probability density function of the Vasicek and the Cox Ingersoll Ross models. Give the formulation and plot those pdf.

Fokker-Planck PDE is a partial differential equation which has as its solution the conditional density of probability distribution of the value of a stochastic process.

Consider the following stochastic process :

$$dx = \mu(x,t)dt + \sigma(x,t)dW$$

and define $g(x,t)$ as a conditional density of the value of the proces x at time t, given that the value of process at t = 0 is x_0 Then the function $g(x,t)$ is a solution to the Fokker-Planck PDE below :

$$\frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 g) - \frac{\partial}{\partial r}(\mu g)$$

with initial condition $g(x,0) = \delta(x - x_0)$

1.1 The Vasicek Model

Now let us consider the stochastic process under the Vasicek model :

$$dr = \kappa(\theta - r)dt + \sigma dW$$

where $r(t)$ represents the instantaneous interest rate with $r(0) > 0$, $\kappa > 0$ is the rate of mean-reversion, $\sigma > 0$ is the volatility term, $\theta > 0$ is the long-term mean level and $W(t)$ is a standard Brownian motion. Here, σ, κ, θ are all constant. Note that, as mentioned earlier, when $\theta = 0$, the model is reduced to the Ornstein-Uhlenbeck (OU) model. In other words, OU model can be considered a special case of the Vasicek model or vice versa where Vasicek model that is constructed by adding a drift term into the OU-SDE given. If we do a change of variable where $y_t = r_t - \theta$, the corresponding equation becomes the OU process as below :

$$dy = -\kappa y dt + \sigma dW$$

Once we apply the Fokker Plank PDE to the above process we get :

$$\frac{\partial g}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial y^2} + \kappa \frac{\partial g}{\partial y}$$

In order to obtain a stationary solution, we consider that $t \to \infty$. This *Limiting density* $f(y) := \lim_{t \to \infty} g(y, t)$ satisfies stationary Fokker-Planck equation given below. Note that $\frac{\partial g}{\partial t} = 0$ for stationary or "steady state" solution.

$$0 = \frac{\sigma^2}{2} \frac{d^2 f}{dy^2} + \kappa \frac{df}{dy}$$

Solving the above ODE in f and y with boundary condition $\int_{-\infty}^{+\infty} f(y) = 1$, we get the below closed form solution,

$$f(y) = \sqrt{\frac{\kappa}{\pi \sigma^2}} exp \frac{-\kappa y^2}{\sigma^2}$$

post change of variable back to r_t, we get the steady state probability density function of r_t is $f_s s(r)$ which is defined as,

$$f_s s(r) = \sqrt{\frac{\kappa}{\pi \sigma^2}} exp \frac{-\kappa (r - \theta)^2}{\sigma^2}$$

which is a normal distribution. **Thus we conclude that the steady state** $(t \longrightarrow \infty)$ **probability density function of the Vasicek model is a normal distribution with mean** θ **and variance** $\frac{\sigma^2}{2\kappa}$

Please see the plot for the same in Figure 1 below -

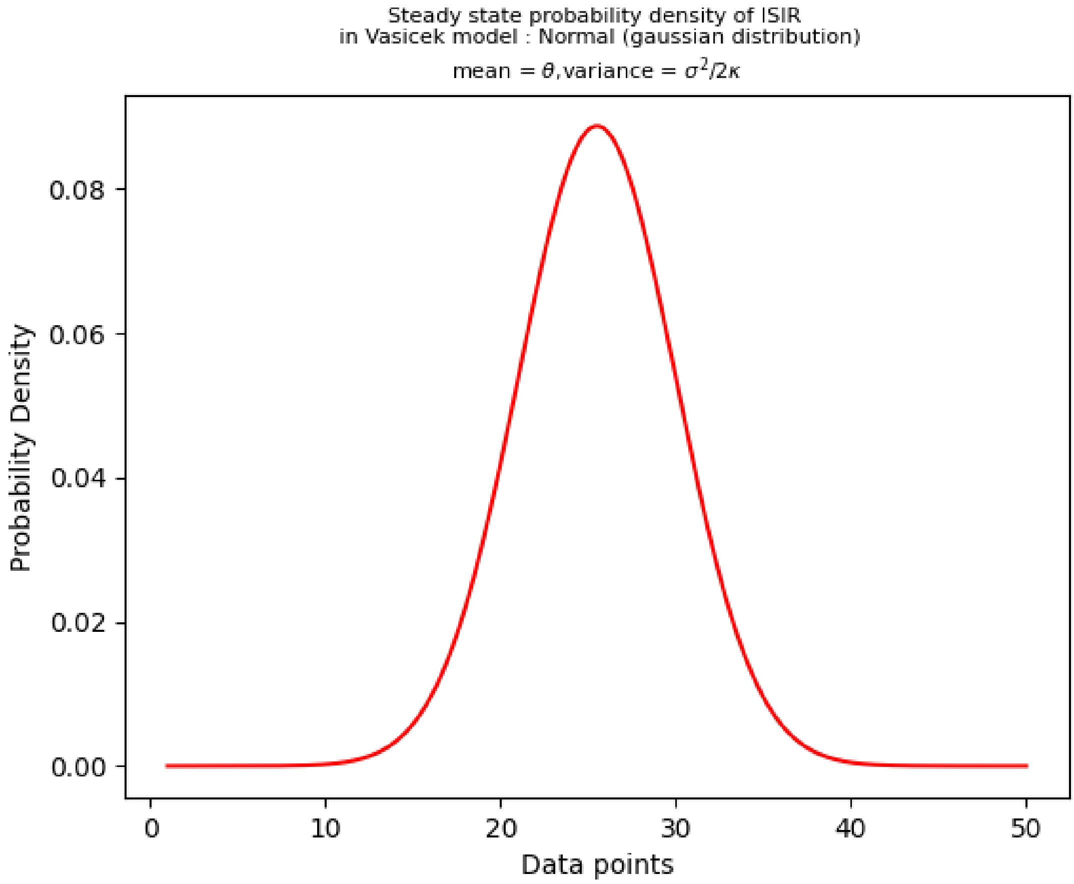

Figure 1: Steady state PDF for Vasicek Model

1.2 The Cox Ingersoll Ross Model

Now let us consider the stochastic process under the CIR model :

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dW$$

where $r(t)$ represents the instantaneous interest rate with $r(0) > 0$, $\kappa > 0$ is the rate of mean-reversion, $\sigma > 0$ is the volatility coefficient, $\theta > 0$ is the long-term mean level and $W(t)$ is a standard Brownian motion. Here, σ, κ, θ are all constant.

Once we apply the Fokker Plank PDE to the above process we get :

$$\frac{\partial g}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2}(rg) - \kappa\frac{\partial}{\partial r}((\theta - r)g)$$

In order to obtain a stationary solution, we consider that $t \to \infty$. This *Limiting density* $f(r) := \lim_{t\to\infty} g(r,t)$ satisfies stationary Fokker-Planck equation given below. Note that $\frac{\partial g}{\partial t} = 0$ for stationary or "steady state" solution.

$$0 = \frac{\sigma^2}{2}\frac{d^2}{dr^2}(rf) - \kappa\frac{df}{dr}((\theta - r)f)$$

Solving the above ODE in f and r with boundary condition $\int_{-\infty}^{+\infty} f(r) = 1$, we get the steady state probability density function of r_t is $f_s s(r)$ which is defined as,

$$f_s s(r) = K r^{\frac{2\kappa\theta}{\sigma^2}} exp\left(\frac{-2\kappa r}{\sigma^2}\right)$$

which is the gamma distribution. **Thus we conclude that the steady state** $(t \longrightarrow \infty)$ **probability density function of the Cox Ross Ingersoll model is a gamma distribution with mean** θ **and variance** $\frac{\sigma^2\theta}{2\kappa}$

Please see the plot for the same in Figure 2 below -

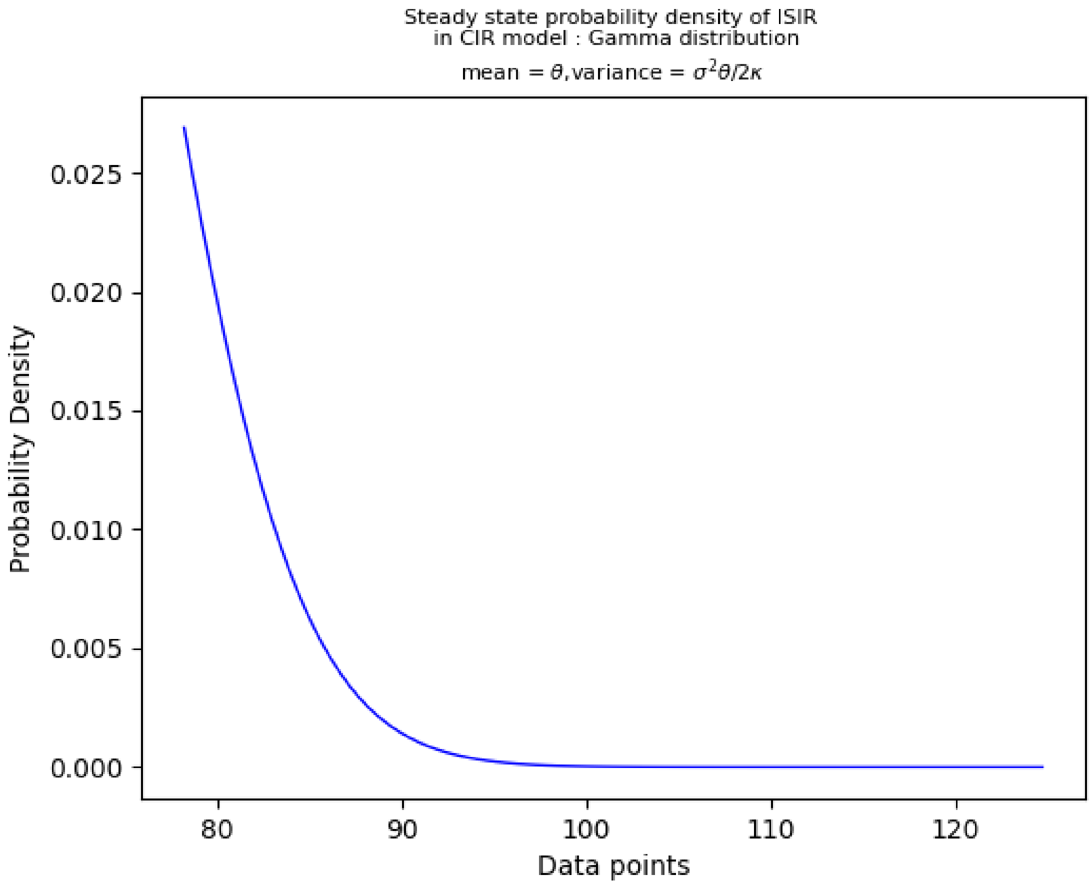

Figure 2: Steady state PDF for CIR Model

2 Can interest rates become negative with Vasicek? What about Cox Ingersoll Ross?

The Vasicek model allows for negative interest rates whereas the Cox Ingersoll Ross (CIR) model does not allow for negative interest rates. In the Vasicek specification (referred to as the normal model), volatility is independent of the level of the short rate. In the normal model, it is possible for negative interest rates to be generated, given the fact that if r_t were to tend to, the volatility term would be unaffected. In the proportional volatility model, volatility is proportional to the short rate. The Cox-Ingersoll-Ross (CIR) specification, referred to for obvious reasons as the square-root model, makes the volatility proportional to the square root of the short rate. Negative interest rates are not possible in the square-root model. This was solidified by Feller in his 1951 paper in the form of the "Feller condition". if the condition $\sigma^2 \geq 2\kappa\theta$ is satisfied, the CIR process can never hit zero. Otherwise, the interest rate $r(t)$ can get a zero value.

3 Demonstrate the Vasicek & Ho & Lee solutions for V

The RN ISIR for Vasicek (1977) is defined as:

$$dr = (\eta - \gamma r)dt + \sqrt{\beta}dX$$

Which gives the following Vasicek BPE:

$$\frac{\partial V}{\partial t} + \frac{\beta}{2}\frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r)\frac{\partial V}{\partial r} = rV$$

with the redemption condition $V(r, T; T) = 1$

Assume a solution of the form (aka "affine form solution"):

$$V(r, t; T) = e^{A(t;T) - rB(t;T)}$$

Take partial derivatives:

$$\begin{aligned}
\frac{\partial V}{\partial t} &= (\dot{A}(t) - r\dot{B}(t))V \\
\frac{\partial V}{\partial r} &= -B(t)V \\
\frac{\partial^2 V}{\partial r^2} &= B^2(t)V
\end{aligned}$$

Substitute into the Vasicek BPE and simplify:

$$
\begin{aligned}
&(\dot{A}(t) - r\dot{B}(t))V + \frac{\beta}{2}B^2(t)V + (\eta - \gamma r)(-B(t)V) = rV \\
&(\dot{A}(t) - r\dot{B}(t)) + \frac{\beta}{2}B^2(t) - (\eta - \gamma r)B(t) = r \\
&\left(\dot{A}(t) + \frac{\beta}{2}B^2(t) - \eta B(t)\right) - (\dot{B}(t) - \gamma B(t) + 1)r = 0
\end{aligned}
$$

For the above equation to be 0, we need to set the below expressions to zero $\forall r$, which yields the two ODEs:

$$
\begin{aligned}
\dot{A}(t) + \frac{\beta}{2}B^2(t) - \eta B(t) = 0 \\
\dot{B}(t) - \gamma B(t) + 1 = 0
\end{aligned}
$$

The boundary condition gives :

$$
\begin{aligned}
V(r, T; T) = 1 \\
\implies expA(T) - rB(T) = 1 \\
\implies A(T) - rB(T) = 0 \\
\implies A(T) = B(T) = 0
\end{aligned}
$$

Solve for $B(t;T)$ using the second ODE and the boundary condition $B(T;T) = 0$:

$$
\begin{aligned}
&\dot{B}(t) - \gamma B(t) + 1 = 0 \\
&\dot{B}(t) = \gamma B(t) - 1
\end{aligned}
$$

$$
\text{Let, } y = \gamma B(t) - 1 \quad \text{then,} \quad dy = \gamma \dot{B}(t)
$$

We can rewrite the above equation in the form,

$$
\begin{aligned}
\frac{1}{\gamma}\int_t^T \frac{\gamma\dot{B}t}{\gamma B(t)-1} &= \int_t^T 1dt \\
\implies \frac{1}{\gamma}\int_t^T \frac{dy}{y} &= (T-t) \\
\implies ln(y)|_t^T &= \gamma(T-t) \\
\implies ln(y(T)) - ln(y(t)) &= \gamma(T-t) \\
\implies \frac{y(T)}{y(t)} &= exp^{\gamma(T-t)} \\
\implies \frac{-1}{\gamma B(t)-1} &= e^{\gamma(T-t)} \\
\implies \frac{1}{\gamma B(t)-1} &= -e^{\gamma(T-t)} \\
\implies (\gamma B(t)-1) &= -e^{-\gamma(T-t)} \\
B(t;T) &= \frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right)
\end{aligned}
$$

Solve for $A(t;T)$ using the first ODE and the boundary condition $A(T;T)=0$:

Substituting the expression for $B(t;T)$ from above into the first ODE with $\dot{A}(t)$, we get:

$$
\begin{aligned}
&\dot{A}(t) + \frac{\beta}{2}B^2(t) - \eta B(t) = 0 \\
&\dot{A}(t) = \eta B(t) - \frac{\beta}{2}B^2(t) \\
&\dot{A}(t) = \eta\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right) - \frac{\beta}{2}(\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right))^2 \\
&\int_t^T \dot{A}(t) = \int_t^T \eta\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right) - \frac{\beta}{2}(\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right))^2 \\
&A(t) = \int_t^T \eta\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right)dt - \frac{\beta}{2}(\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right))^2 dt \\
&= \int_t^T \eta\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right)dt - \frac{\beta}{2}(\frac{1}{\gamma^2}\left(1-2e^{-\gamma(T-t)}+e^{-\gamma(2(T-t))}\right))dt \\
&= \eta\frac{1}{\gamma}\left((T-t) - \frac{1}{\gamma}\times(1-e^{-\gamma(T-t)})\right) - \frac{\beta}{2}(\frac{1}{\gamma^2}\left((T-t) - (\frac{2}{\gamma}(1-e^{-\gamma(T-t)}) - (\frac{1}{2\gamma}(1-e^{-2\gamma(T-t)})\right)) \\
&A(t) = (-T+t+(1-e^{-\gamma(T-t)}))\left(\frac{\eta}{\gamma}-\frac{\beta}{2\gamma^2}\right) - \frac{\beta}{4\gamma}\left(\frac{1-e^{-\gamma(T-t))}}{\gamma}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
A(t) &= (t-T+B(t))\left(\frac{\eta}{\gamma}-\frac{\beta}{2\gamma^2}\right) - \frac{\beta}{4\gamma}(B(t))^2 \\
A(t;T) &= \frac{1}{\gamma^2}(B(t;T)-T+t)\left(\eta\gamma-\frac{\beta}{2}\right) - \frac{\beta}{4\gamma}(B(t;T))^2
\end{aligned}
$$

Hence, we have A(t;T) and B(t;T) for the Vasicek model as follows:

$$
\begin{aligned}
B(t;T) &= \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right) \\
A(t;T) &= \frac{1}{\gamma^2}(B(t;T) - T + t)\left(\eta\gamma - \frac{\beta}{2}\right) - \frac{\beta B^2(t;T)}{4\gamma}
\end{aligned}
$$

Ho & Lee

Assume that the short-term interest rate follows below stochastic process -

$$dr = \eta(t)dt + \sqrt{\beta}dX_t$$

where $\eta(t)$ is the drift term and β is the volatility parameter, and X_t is a Brownian motion.

Hence we get the following BPE (Bond pricing equation):

$$\frac{\partial V}{\partial t} + \frac{\beta}{2}\frac{\partial^2 V}{\partial r^2} + \eta(t)\frac{\partial V}{\partial r} - rV = 0$$

The boundary condition gives:

$$
\begin{aligned}
V(r,T;T) = 1 \quad \text{and} \quad e^{A(T)-rB(T)} &= 1 \\
A(T) - rB(T) &= 0 \\
A(T) = B(T) &= 0
\end{aligned}
$$

We seek a solution for the bond price of the affine form:

$$V(r,t;T) = e^{A(t;T)-rB\left(\mathrm{t}^{(T)}\right)}$$

where $\mathrm{A(t;T)}$ and $\mathrm{B(t;T)}$ are functions to be determined.

Take partial derivatives:

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= (\dot{A}(t) - r\dot{B}(t))V \\
\frac{\partial V}{\partial r} &= -B(t)V \\
\frac{\partial^2 V}{\partial r^2} &= B^2(t)V
\end{aligned}
$$

Plugging these expressions into the Ho & Lee BPE and simplifying, we get:

$$\left(\dot{A}(t) - r\dot{B}(t)\right)V + \frac{\beta}{2}B^2(t)V + \eta(t)\left(-B(t)V\right) = rV$$

Cancelling V on both sides, we get:

$$\left(\dot{A}(t) - r\dot{B}(t)\right) + \frac{\beta}{2}B^2(t) + \eta(t)\left(-B(t)\right) = r$$
$$\left(\dot{A}(t) + \frac{\beta}{2}B^2(t) - \eta(t)B(t)\right) - (\dot{B}(t) + 1)r = 0$$

Setting the coefficient to zero, we can solve for $B(t)$ as:

$$\dot{B}(t) + 1 = 0$$
$$\dot{B}(t) = -1$$

Integrating both sides we get,

$$\int_t^T \dot{B}(t) = \int_t^T -1dt$$

Hence, we get the $B(t)$ as:

$$B(t) = T - t \tag{1}$$

Setting the coefficient to zero, we can solve for $A(t)$

$$\dot{A}(t) + \frac{\beta}{2}B^2(t) - \eta(t)B(t) = 0$$
$$\dot{A}(t) = -\frac{\beta}{2}B^2(t) + \eta(t)B(t)$$

Substituting value of $B(t)$ as $(T - t)$ we get,

$$\dot{A(t)} = -\frac{\beta}{2}(T - t)^2 + \eta(t)(T - t)$$

Integrating both sides we get,

$$\int_t^T \dot{A}(t) = \int_t^T -\frac{\beta}{2}(T - s)^2 ds + \int_t^T \eta(t)(T - s)ds$$
$$-A(t) = -\frac{\beta}{6}(T - t)^3 + \int_t^T \eta(t)(T - s)ds$$

Hence, we get the $A(t)$ as:

$$A(t) = \frac{\beta}{6}(T-t)^3 - \int_t^T \eta(t)(T-s)ds \tag{2}$$

Therefore, the Ho & Lee solution for V is:

$$\begin{aligned} V(r,t;T) &= e^{A(t;T)-rB(t;T)} \\ B(t;T) &= (T-t) \\ A(t;T) &= \frac{1}{6}\beta(T-t)^3 - \int_t^T \eta(t)(T-s)ds \end{aligned}$$

4 Solve for V the BPE with BC below:

$$\frac{\partial V}{\partial t} + \frac{\omega^2}{2}\frac{\partial^2 V}{\partial r^2} + (u-\kappa w)\frac{\partial V}{\partial r} = rV \quad \textbf{and} \quad V(r,T;T) = 1$$

assuming w **constant,** $\mathbf{u} - \mathrm{KW} = 1$ **and V affine form** $V(r,t;T) = e^{A(t,T)-rB(t,T)}$

Solution: To solve for \$$V(r,t;T)$\$, we first assume an affine form for the solution: $\leftrightarrow \infty$

$$V(r,t;T) = e^{A(t;T)-rB(t;T)}$$

We then substitute this solution into the given PDE and boundary condition, which gives:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\omega^2}{2}\frac{\partial^2 V}{\partial r^2} + (u-\kappa w)\frac{\partial V}{\partial r} &= rV \\ V(r,T;T) &= 1 \end{aligned}$$

Take partial derivatives:

$$\begin{aligned} \frac{\partial V}{\partial t} &= (\dot{A}(t) - r\dot{B}(t))V \\ \frac{\partial V}{\partial r} &= -B(t)V \\ \frac{\partial^2 V}{\partial r^2} &= B^2(t)V \end{aligned}$$

Substituting these expressions into the PDE and simplifying, we obtain:

$$(\dot{A}(t;T) - r\dot{B}(t;T))V(r,t;T) + \frac{\omega^2}{2}B^2(t;T)V(r,t;T) + (u - \kappa w)(-B(t;T)V(r,t;T)) = rV(r,t;T)$$
$$(\dot{A}(t;T) - r\dot{B}(t;T)) + \frac{\omega^2}{2}B^2(t;T) + (u - \kappa w)(-B(t;T)) = r$$
$$(\dot{A}(t;T) - r\dot{B}(t;T)) + \frac{\omega^2}{2}B^2(t;T) + (1)(-B(t;T)) = r$$
$$(\dot{A}(t;T) + \frac{\omega^2}{2}B^2(t;T) - B(t;T)) - r(\dot{B}(t;T) + 1) = 0$$

Setting the coefficient to zero, we can solve for $B(t)$ as:

$$\dot{B}(t) + 1 = 0$$
$$\dot{B}(t) = -1$$

Integrating both sides we get,

$$\int_t^T \dot{B}(t) = \int_t^T -1dt$$

Hence, we get the $B(t)$ as:

$$B(t) = T - t \tag{3}$$

Setting the coefficient to zero, we can solve for $A(t)$

$$\dot{A}(t) + \frac{\omega^2}{2}B^2(t) - B(t) = 0$$
$$\dot{A}(t) = -\frac{\omega^2}{2}B^2(t) + B(t)$$

Substituting value of $B(t)$ as $(T - t)$ we get,

$$\dot{A(t)} = -\frac{\omega^2}{2}(T - t)^2 + (T - t)$$

Integrating both sides we get,

$$\int_t^T \dot{A}(t) = \int_t^T -\frac{\omega^2}{2}(T - s)^2 ds + \int_t^T (T - s)ds$$
$$-A(t) = -\frac{\omega^2}{6}(T - t)^3 + (T - t)$$

Hence, we get the $A(t;T)$ as:

$$A(t;T) = \frac{\omega^2}{6}(T-t)^3 - (T-t) \tag{4}$$

5 What final condition, or payoff, should we use for the BPE for a swap, cap, floor, ZCB and bond option?

The final condition or payoff for the Bond pricing equation (PDE) varies depending on the financial instrument being priced. Here are the final conditions/payoffs for some common instruments:

Swap: The final condition for a swap is the difference between the floating rate payments and the fixed rate payments, discounted to the present value. The payoff function is linear and depends on the difference between the floating and fixed rates at each payment date.

Cap: The final condition for a cap is the maximum between the difference between the floating rate and the cap rate, and zero, discounted to the present value. The payoff function is nonlinear and depends on the floating rate at each payment date. The payoff to be used for the BPE for Cap, V with rate r at maturity T is :

$$V(r,T) = max(r - r_c, 0)$$

where r is the floating rate (such as LIBOR) and r_c is the cap rate

Floor: The final condition for a floor is the maximum between the difference between the floating rate and the floor rate and zero, discounted to the present value. The payoff function is nonlinear and depends on the floating rate at each payment date. The payoff to be used for the BPE for Floor, V with rate r at maturity T is :

$$V(r,T) = max(r_f - r, 0)$$

where r is the floating rate and r_f is the floor rate

ZCB (Zero Coupon Bond): The final condition for a ZCB is simply its face value, discounted to the present value. The payoff of a Zero Coupon bond is the pull to par at maturity that is $Z(T;T) = 1$ for zero coupon bond with face value of 1

Bond Option: The final condition for a bond option is the maximum between the difference between the bond's strike price and its market value, and zero, discounted to the present value.

The payoff function is nonlinear and depends on the bond's market value at the option's expiry date. The payoff to be used for the BPE for ZCB bond option, V with rate r at maturity T is :

$$V(r, T) = max(Z(r, t; T_B) - K, 0)$$

where $Z(r, t; T_B)$ is the bond's market value and K is the bond's strike price