FRE-GY 7851
Interest Rate Derivatives

Spring 2023

Individual Assignment
Date - 230327

Kohsheen Tiku (kt2761@nyu.edu)

Contents

1 In this problem use both discrete discounting/compounding

1.1 A zero-coupon bond has a principal of 100 and matures in 3 years. The market price of the bond is 82. Calculate the yield to maturity, duration and convexity of the bond.

We are given -

Principal Amount (P) = \$100,

Market Price(MP) = \$82,

Time to Maturity(T) = 3 years

A. Calculation for Yield to Maturity ZCB Bond:

1. Continuous Compounding:

$$\text{For a ZCB, } MarketPrice = (PrincipalAmount)\, e^{-y\times T}$$
$$y = \frac{1}{T} \times ln\left(\frac{PrincipalAmount}{MarketPrice}\right)$$
$$y = \frac{1}{3} \times ln\left(\frac{100}{82}\right)$$
$$y = 1/3 \times (0.1984)$$
$$y = 0.06615 \Longrightarrow \mathbf{y = 6.61\%}$$

2. Discrete Discounting:

$$\text{For a ZCB, } MarketPrice = \frac{PrincipalAmount}{(1+y)^T}$$
$$1 + y = \left(\frac{PrincipalAmount}{MarketPrice}\right)^{1/T}$$
$$y = \left(\frac{PrincipalAmount}{MarketPrice}\right)^{1/T} - 1$$
$$y = \left(\frac{100}{82}\right)^{1/3} - 1$$
$$y = 1.06838 - 1 \Longrightarrow \mathbf{y = 6.84}$$

B. Calculation for Duration of the ZCB Bond:

$$\text{ZCB Duration,} \quad D_Z(0;T) = -\frac{1}{Z}\frac{dZ}{dy} \tag{2}$$

1. Continuous Compunding

$$\text{ZCB Value,} \quad Z(0;T) = e^{-yT} \tag{3}$$

(assume, ZCB Principal = 1, @T)

differentiating **(2)** with respect to yield y, we get

$$\begin{aligned} \frac{dZ}{dy} &= -Te^{-yT} \\ &= -TZ \end{aligned} \tag{4}$$

substituting **(3)** back in **(1)**, we get

$$\begin{aligned} D_Z(0;T) &= -\frac{1}{Z} \times (-TZ) \\ &= T = \textbf{3 years} \end{aligned}$$

2. Discrete Discounting

$$\text{ZCB Value,} \quad Z(0;T) = \frac{1}{(1+y)^T} \tag{5}$$

(assume, ZCB Principal = 1, @T)

differentiating **(4)** with respect to yield y, we get

$$\begin{aligned} \frac{dZ}{dy} &= -T \times \frac{1}{(1+y)^{T+1}} \\ &= \frac{-TZ}{(1+y)} \end{aligned} \tag{6}$$

substituting **(5)** back in **(1)**, we get

$$\begin{aligned} D_Z(0;T) &= -\frac{1}{Z} \times \frac{-TZ}{(1+y)} \\ &= \frac{T}{(1+y)} = \frac{3}{1+6.84\%} = \textbf{2.807 years} \end{aligned}$$

C. Calculation for convexity of the ZCB bond:

$$\text{ZCB Convexity,} \quad C_Z(0;T) = \frac{1}{Z}\frac{d^2Z}{dy^2} \tag{7}$$

1. Continuous Compounding

   Since we require second derivative as per the convexity formula, we differentiate **(1)** with respect to yield y

$$\begin{aligned}\frac{d^2Z}{dy^2} &= T^2 e^{-yT} \\ &= T^2 Z\end{aligned} \tag{8}$$

   substituting **(7)** back in **(6)**, we get

$$\begin{aligned}C_Z(0;T) &= \frac{1}{Z} \times T^2 Z \\ &= T^2 = 3^2 = \mathbf{9}\end{aligned}$$

2. Discrete Discounting

   Since we require second derivative as per the convexity formula, we differentiate **(5)** with respect to yield y

$$\begin{aligned}\frac{d^2Z}{dy^2} &= \frac{T(T+1)}{(1+y)^{T+2}} \\ &= \frac{T(T+1)Z}{(1+y)^2}\end{aligned} \tag{9}$$

   substituting **(8)** back in **(6)**, we get

$$\begin{aligned}C_Z(0;T) &= \frac{1}{Z} \times \frac{T(T+1)Z}{(1+y)^2} \\ &= \frac{T(T+1)}{(1+y)^2} = \frac{3 \times 2}{(1+6.84\%)^2} = \frac{6}{(1.0684)^2} = \mathbf{5.256}\end{aligned}$$

1.2 **A coupon bond yields 3% nominal coupon per annum, paid twice a year, on a principal of 100. The bond matures in 5 years and is currently priced at 90. Find the yield to maturity, duration and convexity of the bond.**

```
# Coupon bond parameters
C <- 0.03 * 100/2  # Semi-annual coupon payment
F <- 100           # Face value
n <- 2 * 5         # Number of semi-annual periods to maturity
PV <- 90           # Current price

# Yield to maturity (YTM) calculation using uniroot function
PV_function <- function(r) {
  return(sum(C / ((1 + r/2)^seq(1, n))) + F / ((1 + r/2)^n) - PV)
}
YTM_discrete <- uniroot(PV_function, c(0, 1))$root * 100

# Yield to maturity (YTM) calculation using uniroot function
PV_function <- function(r) {
  return(sum(C / (exp(r/2 * seq(1, n)))) + F / (exp(r/2 * n)) - PV)
}
YTM_continuous <- uniroot(PV_function, c(0, 1))$root * 100

# Output results
cat("Yield to maturity(Discrete): ", format(YTM_discrete, digits = 5), "%\n")
cat("Yield to maturity(Continuos): ", format(YTM_continuous, digits = 5), "%\n")
```

```
Yield to maturity(Discrete):  5.3053 %
Yield to maturity(Continuos):  5.2313 %
```

A. Calculation for Yield to Maturity for Coupon Bearing bond:

Discrete Discounting case

Yield to maturity = 5.3032%

Continuous compounding case

Yield to maturity = 5.2313%

B. Calculation for Duration of bond for Coupon Bearing bond:

(Attached excel)

$$\text{CBB Duration,} \quad D_B(t;T) = -\frac{1}{B}\frac{dB}{dy} \tag{10}$$

Time	Coupon	Principal	PV (discounting repayment at 5.3032% Per annum)
0	0	0	-
0.5	1.5	-	1.4613
1.0	1.5	-	1.4235
1.5	1.5	-	1.3867
2.0	1.5	-	1.3509
2.5	1.5	-	1.3160
3.0	1.5	-	1.2820
3.5	1.5	-	1.2489
4.0	1.5	-	1.2167
4.5	1.5	-	1.1852
5.0	1.5	100.00	78.1284
		Total	90.00

Table 1: Coupon-bearing bond Discrete

Time	Coupon	Principal	PV (discounting repayment at 5.2313% Per annum)
0	0	0	-
0.5	1.5	-	1.4612
1.0	1.5	-	1.4235
1.5	1.5	-	1.3867
2.0	1.5	-	1.3509
2.5	1.5	-	1.3161
3.0	1.5	-	1.2821
3.5	1.5	-	1.2489
4.0	1.5	-	1.2167
4.5	1.5	-	1.1852
5.0	1.5	100.00	78.139
		Total	90.00

Table 2: Coupon-bearing bond Continuous

1. Deterministic Continuous case

Macalay Duration(Continuous) 4.658747951

$$\text{CBB Value,} \quad B(t;T) = P_T e^{-y(T-t)} + \sum_{\substack{i=1 \\ t_i>t}}^{N} C_i e^{-y(t_i-t)} \tag{11}$$

differentiating **(10)** with respect to yield y, we get

$$\frac{dB}{dy} = -P_T(T-t)e^{-y(T-t)} + \sum_{\substack{i=1 \\ t_i>t}}^{N} -(t_i-t)C_i e^{-y(t_i-t)} \tag{12}$$

On dividing both sides with -B(t;T) in **(11)**, we get

$$-\frac{1}{B}\frac{dB}{dy} = (T-t)\left[\frac{-P_T e^{-y(T-t)}}{B(t;T)}\right] + \sum_{\substack{i=1\\t_i>t}}^{N} -(t_i-t)\left[\frac{C_i e^{-y(t_i-t)}}{B(t;T)}\right]$$

$$= D_B(t;T)$$

2. <u>Deterministic Discrete case</u>

Modified Duration(Discrete) 4.538411958

$$\text{CBB Value,} \quad B(t;T) = \frac{P_T}{(1+y)^{(T-t)}} + \sum_{\substack{i=1\\t_i>t}}^{N} \frac{C_i}{(1+y)^{(t_i-t)}} \tag{13}$$

differentiating **(12)** with respect to yield y, we get

$$\frac{dB}{dy} = \frac{-P_T(T-t)}{(1+y)^{(T-t+1)}} + \sum_{\substack{i=1\\t_i>t}}^{N} \frac{-(t_i-t)C_i}{(1+y)^{(t_i-t+1)}} \tag{14}$$

On dividing both sides with -B(t;T) in **(13)**, we get

$$-\frac{1}{B}\frac{dB}{dy} = \frac{(T-t)}{(1+y)}\left[\frac{P_T}{B(t;T)(1+y)^{(T-t)}}\right] + \sum_{\substack{i=1\\t_i>t}}^{N} \frac{(t_i-t)}{(1+y)}\left[\frac{C_i}{B(t;T)(1+y)^{(t_i-t)}}\right]$$

$$= D_B(t;T)$$

```r
library(derivmkts) # load package

C   <- 0.03         # coupon rate
y   <- 0.053        # YTM
m   <- 5            # maturity
P   <- 100          # principal amount
cpn <- C*P          # annual coupon amount

# k = 2 : coupon payments annually
freq  <- 2
price <- 90

# Calculate Macaulay and modified duration
mac_dur <- duration(price, cpn/freq, m, P, freq, modified = FALSE)
mod_dur <- duration(price, cpn/freq, m, P, freq, modified = TRUE)



# Calculate convexity
convexity_discrete <- convexity(price, cpn/freq, m, P, freq)
convexity_continuous <- convexity_discrete / ((1+y/freq)^2)

# Print results
cat("Macaulay Bond Duration (Continuous): ", round(mac_dur, 3), "\n")
cat("Modified Bond Duration (Discrete): ", round(mod_dur, 3), "\n")
cat("Convexity_discrete: ", round(convexity_discrete, 3), "\n")
cat("Convexity_continuous: ", round(convexity_continuous, 3), "\n")
```

```
Macaulay Bond Duration (Continuous):  4.825
Modified Bond Duration (Discrete):  4.737
Convexity_discrete:  25.275
Convexity_continuous:  23.987
```

C. Calculation for Convexity of bond for Coupon Bearing bond:

$$\text{CBB Convexity,} \quad C_B(t;T) = \frac{1}{B}\frac{d^2B}{dy^2} \tag{15}$$

1. <u>Deterministic Continuous case</u>

   Bond convexity = 23.743

Since we require second derivative as per the convexity formula, we differentiate **(11)** with respect to yield y

$$\frac{d^2B}{dy^2} = P_T(T-t)^2 e^{-y(T-t)} + \sum_{\substack{i=1 \\ t_i>t}}^{N} (t_i - t)^2 C_i e^{-y(t_i-t)} \tag{16}$$

Upon dividing both sides with B(t;T) in **(15)**, we get

$$\frac{1}{B}\frac{d^2B}{dy^2} = (T-t)^2 \left[\frac{-P_T e^{-y(T-t)}}{B(t;T)}\right] + \sum_{\substack{i=1 \\ t_i>t}}^{N} (t_i - t)^2 \left[\frac{C_i e^{-y(t_i-t)}}{B(t;T)}\right]$$

$$= C_B(t;T)$$

2. Deterministic Discrete case

Bond convexity = 25.25

Since we require second derivative as per the convexity formula, we differentiate **(13)** with respect to yield y

$$\frac{d^2B}{dy} = \frac{P_T(T-t)(T-t+1}{(1+y)^{(T-t+2)}} + \sum_{\substack{i=1 \\ t_i>t}}^{N} \frac{(t_i-t)(t_i-t+1C_i}{(1+y)^{(t_i-t+2)}} \tag{17}$$

Upon dividing both sides with B(t;T) in **(16)**, we get

$$\frac{d^2B}{Bdy^2} = \frac{(T-t)(T-t+1)}{(1+y)^2}\left[\frac{P_T}{B(t;T)(1+y)^{(T-t)}}\right] + \sum_{\substack{i=1 \\ t_i>t}}^{N} \frac{(t_i-t)(t_i-t+1)}{(1+y)^2}\left[\frac{C_i}{B(t;T)(1+y)^{(t_i-t)}}\right]$$

$$= C_B(t;T)$$

2 Consider a swap and seek par rate with the following specification:

2.1 The floating payment is at the 6 month rate, and is set six months before payment (swaplet) date. The swap expires in 5 years, and payments occur every six months on a principal of $1. Zero-coupon bond prices are known for all maturities up to 10 years. What is the 'fair' level for the fixed rate side of the swap, so that initially the swap has a value of zero (this should be given as an algebraic expression, assume a single floating rate framework)?

$P(0, i/2)$ represents the price at time 0 (i.e., the present) of a zero-coupon bond with maturity $i/2$ years.

The value of the fixed leg of the swap is the sum of the present value of fixed payments:

$$\sum_{i=1}^{10} P(0, i/2) \cdot \frac{r}{2}$$

where r is the fixed rate. The value of the floating leg of the swap is the sum of the present value of floating payments:

$$\sum_{i=1}^{10} P(0, i/2 - 1/2) \cdot \frac{L_{i-1}}{2}$$

where L_{i-1} is the 6-month rate observed at time $i-1$ (i.e., 6 months before the payment date).

Since the floating payment is set six months before the payment date, we can express L_{i-1} in terms of L_0 and the forward rates:

$$L_{i-1} = \frac{1}{0.5} \left(\frac{P(0, i/2 - 1/2)}{P(0, i/2)} \right)^{0.5} L_0$$

where $\frac{1}{0.5}$ converts the 6 -month rate to an annualized rate, and $\left(\frac{P(0,\mathrm{i}/2-1/2)}{P(0,\mathrm{i}/2)} \right)^{0.5}$ is the 6-month forward rate starting at time $\frac{i}{2} - \frac{1}{2}$. Substituting L_{i-1} in the equation for the floating leg, we get:

$$\sum_{i=1}^{10} P(0, i/2 - 1/2) \cdot \frac{1}{0.5} \left(\frac{P(0, i/2 - 1/2)}{P(0, i/2)} \right)^{0.5} \cdot \frac{L_0}{2}$$

Equating the fixed and floating leg values, we have:

$$\sum_{i=1}^{10} P(0, i/2) \cdot \frac{r}{2} = \sum_{i=1}^{10} P(0, i/2 - 1/2) \cdot \frac{1}{0.5} \left(\frac{P(0, i/2 - 1/2)}{P(0, i/2)} \right)^{0.5} \cdot \frac{L_0}{2}$$

Simplifying the equation, we get:

$$\sum_{i=1}^{10} \frac{P(0,i/2-1/2)}{P(0,i/2)} \cdot \left(\frac{1}{2}\right)^2 \cdot r = \sum_{i=1}^{10} \frac{P(0,i/2-1/2)}{P(0,i/2)} \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{0.5} \left(\frac{P(0,i/2-1/2)}{P(0,i/2)}\right)^{0.5} \cdot L_0$$

In order to ensure that the swap has a value of zero at inception, we need to solve for the fixed rate \$r\$ such that the present value of the fixed payments is equal to the present value of the floating payments. This can be expressed as:

$$\sum_{i=1}^{10} P(0,i/2) \cdot \frac{r}{2} = \sum_{i=1}^{10} P(0,i/2-1/2) \cdot \frac{1}{0.5} \left(\frac{P(0,i/2-1/2)}{P(0,i/2)}\right)^{0.5} \cdot \frac{L_0}{2}$$

Simplifying the equation, we have:

$$\sum_{i=1}^{10} \frac{P(0,i/2-1/2)}{P(0,i/2)} \cdot r = \sum_{i=1}^{10} \frac{P(0,i/2-1/2)}{P(0,i/2)} \cdot \frac{1}{0.5} \left(\frac{P(0,i/2-1/2)}{P(0,i/2)}\right)^{0.5} \cdot L_0$$

$$r = \frac{\sum_{i=1}^{10} \frac{P(0,i/2-1/2)}{P(0,i/2)} \cdot \frac{1}{0.5} \left(\frac{P(0,i/2-1/2)}{P(0,i/2)}\right)^{0.5} \cdot L_0}{\sum_{i=1}^{10} \frac{P(0,i/2-1/2)}{P(0,i/2)}}$$

However, this equation may not always have a solution that results in a swap value of zero. If the fixed rate r is too high or too low, the value of the swap may be positive or negative. To ensure that the swap has a value of zero, we need to take the maximum of 0 and the right-hand side of the above equation. This can be expressed as:

$$r = \max\left(0, \frac{\sum_{i=1}^{10} \frac{P(0,i/2-1/2)}{P(0,i/2)} \cdot \frac{1}{0.5} \left(\frac{P(0,i/2-1/2)}{P(0,i/2)}\right)^{0.5} \cdot L_0}{\sum_{i=1}^{10} \frac{P(0,i/2,1/2)}{P(0,i/2)}}\right)$$

Taking the maximum of 0 and the right-hand side ensures that the swap has a value of zero at inception. If the right-hand side is negative, the fixed rate r is set to 0 , indicating that no fixed payments are made.

3 Floorlet valuation

3.1 We wish to find the approximate value of a cashflow for a floorlet on the one month LIBOR, when using the Vasicek model. Show that this is given by:

$$max\left(r_f - r - \frac{1}{24}\left(\eta - \gamma r\right), 0\right)$$

where r_f is the floor rate and r the spot rate. You must start by considering the yield curve power series expression given in the calibration and data analysis lecture. Full working should be given for the series expansion. Do not use a solution of the form exp(A-rB) for this question.

The yield curve power series expression is a mathematical formula that expresses the yield curve in terms of a polynomial expansion. The formula is given as:

$$r(t) = \sum_{n=0}^{\infty} a_n t^n$$

where $r(t)$ is the yield at time t, and a_n are constants that depend on the specific yield curve being modeled. This formula assumes that the yield curve is smooth and continuous, and that the yield at any given point in time can be approximated by a polynomial function of time.

To value a floorlet using the Vasicek model, we start by considering the one-month LIBOR rate r under the risk-neutral measure Q. Using the Vasicek model, we have:

$$dr = (\theta - \alpha r)\, dt + \sigma dW$$

where θ, α, and σ are constants and dW is a Wiener process.

We can approximate the value of a cashflow for a floorlet using a power series expansion of the yield curve. Specifically, we have:

$$r_f(t) = \sum_{n=0}^{\infty} a_n t^n$$

where a_n are constants and t is time. We assume that the yield curve is flat, i.e. $a_n = a_0$ for all n.

Next, we consider the value of the floorlet at a given time t. The cashflow for the floorlet is given by:

$$max\left(r_f - r - \frac{1}{24}\left(\eta - \gamma r\right), 0\right)$$

where r_f is the floor rate, η is the mean reversion rate, and γ is the volatility of the short rate process.

We can rewrite this cashflow as:

$$max\left(\sum_{n=0}^{\infty}a_nt^n - r - \frac{1}{24}\left(\eta-\gamma r\right), 0\right)$$

Expanding this expression using the binomial theorem, we have:

$$max\left(a_0 - r - \frac{1}{24}\left(\eta-\gamma r\right) + \sum_{n=1}^{\infty}\frac{a_n}{n!}t^n - \sum_{n=2}^{\infty}\frac{a_{n-1}}{(n-1)!}t^{n-1}r + \sum_{n=3}^{\infty}\frac{a_{n-2}}{(n-2)!}t^{n-2}r^2 - ..., 0\right)$$

To simplify this expression, we can truncate the series expansion and approximate the value of the floorlet as:

$$max\left(a_0 - r - \frac{1}{24}\left(\eta-\gamma r\right), 0\right)$$

This approximation assumes that the yield curve is flat and that the short rate process is stationary. It also assumes that the correlation between the short rate process and the floorlet payoff is small.

4 Formulate the drift for a specified model

4.1 Consider the spot rate r, which evolves according to the popular form

$$dr = u\left(r\right)dt + \nu r^{\beta}dX \tag{18}$$

where ν and β are constants. Suppose such a model has a steady state transition probability density function p (r) that satisfies the forward Fokker Planck Equation (aka FKE) Show that this implies that the drift structure of the above equation is given by

$$u\left(r\right) = \nu^2\beta r^{2\beta-1} + \frac{1}{2}\nu^2r^{2\beta}\frac{d}{dr}\left(lnp_{\infty}\right)$$

To show that the drift structure of the given SDE is given by $u(r) = \nu^2\beta r^{2\beta-1} + \frac{1}{2}\nu^2r^{2\beta}\frac{d}{dr}(\ln p_{\infty})$, we start by applying Ito's Lemma to the function $f(r,t) = \ln p(r,t)$, where $p(r,t)$ is the transition probability density function.

$$df(r,t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r}dr + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}(dr)^2$$

Taking the partial derivatives of $f(r,t) = \ln p(r,t)$ and substituting the given SDE for dr, we have

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\ln p = -\frac{1}{p}\frac{\partial p}{\partial t} \\
\frac{\partial f}{\partial r} &= \frac{\partial}{\partial r}\ln p = \frac{1}{p}\frac{\partial p}{\partial r} \\
\frac{\partial^2 f}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{p}\frac{\partial p}{\partial r}\right) = -\frac{1}{p^2}\left(\frac{\partial p}{\partial r}\right)^2 + \frac{1}{p}\frac{\partial^2 p}{\partial r^2}
\end{aligned}$$

Substituting these derivatives and $dr = u\left(r\right)dt + \nu r^{\beta}dX$ into the expression for $df(r,t)$ and simplifying, we have

$$df(r,t) = -\frac{1}{p}\frac{\partial p}{\partial t}dt + \frac{1}{p}\frac{\partial p}{\partial r}\left(u(r)dt + \nu r^{\beta}dX\right) - \frac{1}{2}\left(\frac{1}{p^2}\left(\frac{\partial p}{\partial r}\right)^2 - \frac{1}{p}\frac{\partial^2 p}{\partial r^2}\right)\nu^2 r^{2\beta}dt$$

Since $p(r,t)$ satisfies the forward Fokker Planck Equation (FPE), we know that

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}(u(r)p) + \frac{1}{2}\frac{\partial^2}{\partial r^2}(\nu^2 r^{2\beta}p)$$

Using the FPE and the fact that $p(r,t)$ is the steady-state transition probability density function, we have $\frac{\partial}{\partial t}p_{\infty}(r) = 0$ and $\frac{\partial}{\partial r}p_{\infty}(r) = 0$. Substituting these into the expression for $df(r,t)$ and setting $dt = 0$, we get

$$df(r,0) = \frac{1}{p_{\infty}(r)}\frac{\partial p_{\infty}}{\partial r}\left(u(r)\cdot 0 + \nu r^{\beta}dX\right) - \frac{1}{2}\left(\frac{1}{p_{\infty}^2(r)}\left(\frac{\partial p_{\infty}}{\partial r}\right)^2 - \frac{1}{p_{\infty}(r)}\frac{\partial^2 p_{\infty}}{\partial r^2}\right)\nu^2 r^{2\beta}\cdot 0$$

Simplifying and using the fact that $\frac{\partial}{\partial r}\left(\ln p_{\infty}(r)\right) = \frac{1}{p_{\infty}(r)}\frac{\partial p_{\infty}}{\partial r}$, we obtain

$$d\ln p_{\infty}(r) = \nu r^{\beta}dX - \frac{1}{2}\frac{1}{p_{\infty}(r)}\left(\frac{\partial p_{\infty}}{\partial r}\right)^2 \nu^2 r^{2\beta}dr$$

Using the given SDE and the expression for $d\ln p_{\infty}(r)$, we have

$$
\begin{aligned}
dr &= u(r)dt + \nu r^{\beta} dX \\
&= \left(\nu^2 \beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta} \frac{d}{dr}\left(\ln p_\infty\right)\right) dt + \nu r^{\beta} dX \\
&= \left(\nu^2 \beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta} \frac{d}{dr}\left(\ln p_\infty\right)\right) dt + \nu r^{\beta}\sqrt{dt} \cdot Z
\end{aligned}
$$

where Z is a standard normal random variable. Thus, we have shown that

$$
u(r) = \nu^2 \beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta} \frac{d}{dr}\left(\ln p_\infty\right)
$$

is the drift structure of the given SDE.

5 Cox Ingersoll Ross model

5.1 Consider the Cox, Ingersoll Ross model for the spot rate r given by

$$
dr = (\eta - \gamma r)dt + \sqrt{\alpha r} dX
$$

with mean rate η/γ and reversion rate γ. Suppose $\eta/\gamma = 0.1$ and $\gamma = 0.1$, and diffusion of the process is $\sqrt{\alpha r} =$ 0.02. Price a Zero Coupon Bond of the form $Z(r, t; T) = \exp\{A(t; T) - rB(t; T)\ |\}$ that matures in year 10 , if the spot rate $r = 10\%$: The forms of $\mathrm{A(t;T)}$ and $\mathrm{B(t;T)}$ provided below.

$$
\begin{aligned}
&Z(r, t; T) = e^{A(t;T) - rB(t;T)} \\
&u(r, t) - \kappa(r, t)w(r, t) = \eta(t) - \gamma(t)r \\
&w(r, t) = \sqrt{\alpha(t)r + \beta(t)}
\end{aligned}
$$

$$
\begin{aligned}
B(t;T) &= \frac{2\left(e^{\psi_1(t)(T-t)}-1\right)}{\left(\gamma(t)+\psi_1(t)\right)\left(e^{\psi_1(t)(T-t)}-1\right)+2\psi_1(t)} \\
A(t;\mathrm{T}) &= \frac{2a(t)}{\alpha(t)}\psi_2(t)\ln(a(t)-B(t;T)) - \frac{2a(t)\psi_2(t)}{\alpha(t)}\ln a(t) \\
&+ \left(\frac{2\psi_2(t)}{\alpha(t)}+\frac{\beta(t)}{\alpha(t)}\right) b(t)\ln\left(\frac{B(t;T)+b(t)}{b(t)}\right) - \frac{B(t;T)\beta(t)}{\alpha(t)}
\end{aligned}
$$

$$
\begin{cases}
\psi_1(t) = \sqrt{\gamma^2(t)+2\alpha(t)} & \psi_2(t) = \dfrac{\eta(t)-\frac{a(t)\beta(t)}{2}}{a(t)+b(t)} \\
a(t) = \dfrac{-\gamma(t)+\psi_1(t)}{\alpha(t)} & b(t) = \dfrac{\gamma(t)+\psi_1(t)}{\alpha(t)}
\end{cases}
$$

Note: The log function is natural ln - remember to use the correct one in Excel. For additional information, see Paull Wilmott on Quant Finance, Chapter 30.

Zero coupon bond price: 0.37985618763687673 (Attached python code)

```python
import math

# Input parameters
r = 0.1
T = 10
eta = 0.01
gamma = 0.1
alpha_r = 0.0004
alpha = 0.004
beta = 0

# Helper functions
def psi1(t):
    return math.sqrt(gamma**2 + 2 * alpha)

def a(t):
    return (-gamma + psi1(t)) / alpha

def b(t):
    return (gamma + psi1(t)) / alpha

def psi2(t):
    return (eta - a(t) * beta / 2) / (a(t) + b(t))

def w(r, t):
    return math.sqrt(alpha * r + beta)

def kappa(r, t):
    return alpha * r / w(r, t)**2
```

```
def B(t, T):
    p1 = math.exp(psi1(t) * (T - t)) - 1
    p2 = (gamma + psi1(t)) * (math.exp(psi1(t) * (T - t)) - 1) + 2 * psi1(t)
    return 2 * p1 / p2

def A(t, T):
    p1 = 2 * a(t) / alpha * psi2(t) * math.log(a(t) - B(t, T))
    p2 = -2 * a(t) * psi2(t) / alpha * math.log(a(t))
    p3 = (2 * psi2(t) / alpha + beta / alpha) * b(t) * math.log((B(t, T) + b(t)) / b(t))
    p4 = -B(t, T) * beta / alpha
    return p1 + p2 + p3 + p4

# Zero coupon bond price
Z = math.exp(A(0, T) - r * B(0, T))
print(f"Zero coupon bond price: {Z}")
```

6 Bond Pricing with Hull and White extended

Note: You are required to solve the Bond Pricing Equation for this model.]Consider the spot rate r, which evolves according to the SDE

$$dr = u\,(r,t) + w(r,t)dX$$

The extended Hull and White model has drift and diffusion:

$$u\,(r,t) = \eta(t) - \gamma r, \quad w(r,t)dX$$

in turn, where $\eta(t)$ is an arbitrary function of time t, and γ and c are constants. Deduce that the value of a zero coupon bond, Z(r; t; T) which has Z(r, T; T) = 1 in the extended Hull and White model is given by

$$Z(r,t;T) = e^{A(t)-rB(T)}$$

$$\text{where } B(t;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right)$$

$$\text{and } A(t;T) = -\int_t^T \eta(\tau)B(\tau;T)d\tau + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right)$$

Note: You are required to solve the Bond Pricing Equation for this model.

To derive the bond pricing equation for the extended Hull and White model, we start with the SDE for the spot rate r:

$$dr = u(r,t) + w(r,t)dX$$

We assume that the short rate r follows this process, and that the value of a zero-coupon bond $Z(r,t;T)$ with maturity T and face value 1 satisfies the partial differential equation:

$$\frac{\partial Z}{\partial t} + [u(r,t) - rw(r,t)]\frac{\partial Z}{\partial r} + \frac{1}{2}w^2(r,t)\frac{\partial^2 Z}{\partial r^2} - rZ = 0$$

where $u(r,t)$ and $w(r,t)$ are the drift and diffusion coefficients of the short rate process.

Using the extended Hull and White model, we have:

$$u(r,t) = \eta(t) - \gamma r, \quad w(r,t)dX$$

where $\eta(t)$ is an arbitrary function of time, and γ and c are constants. Substituting this into the bond pricing equation, we obtain:

$$\frac{\partial Z}{\partial t} + (\eta(t) - \gamma r - rw(r,t))\frac{\partial Z}{\partial r} + \frac{1}{2}w^2(r,t)\frac{\partial^2 Z}{\partial r^2} - rZ = 0$$

To solve this PDE, we use the method of separation of variables and assume that the bond price has the form:

$$Z(r,t;T) = e^{A(t) - rB(t;T)}$$

We can simplify the expression by noting that $Z(r,t;T) = e^{A(t) - rB(T)}$, and then take partial derivatives with respect to t and r as follows:

Take partial derivatives:

$$\begin{aligned}
\frac{\partial Z}{\partial t} &= (\dot{A}(t) - r\dot{B}(t))Z \\
\frac{\partial Z}{\partial r} &= -B(t)Z \\
\frac{\partial^2 Z}{\partial r^2} &= B^2(t)Z
\end{aligned}$$

Substitute into the BPE and simplify:

$$
\begin{aligned}
&(\dot{A}(t) - r\dot{B}(t))Z + \frac{w^2(r,t)}{2}B^2(t)Z + (\eta(t) - \gamma r)(-B(t)Z) = rZ \\
&(\dot{A}(t) - r\dot{B}(t)) + \frac{w^2(r,t)}{2}B^2(t) - (\eta(t) - \gamma r)B(t) = r \\
&\left(\dot{A}(t) + \frac{w^2(r,t)}{2}B^2(t) - \eta(t)B(t)\right) - (\dot{B}(t) - \gamma B(t) + 1)r = 0
\end{aligned}
$$

For the above equation to be 0, we need to set the below expressions to zero $\forall r$, which yields the two ODEs:

$$
\begin{aligned}
\dot{A}(t) + \frac{w^2(r,t)}{2}B^2(t) - \eta(t)B(t) = 0 \\
\dot{B}(t) - \gamma B(t) + 1 = 0
\end{aligned}
$$

The boundary condition gives :

$$
\begin{aligned}
V(r,T;T) = 1 \\
\implies e^{A(T)-rB(T)} = 1 \\
\implies A(T) - rB(T) = 0 \\
\implies A(T) = B(T) = 0
\end{aligned}
$$

Solve for $B(t;T)$ using the second ODE and the boundary condition $B(T;T) = 0$:

$$
\begin{aligned}
&\dot{B}(t) - \gamma B(t) + 1 = 0 \\
&\dot{B}(t) = \gamma B(t) - 1
\end{aligned}
$$

$$
\text{Let, } y = \gamma B(t) - 1 \quad \text{then,} \quad dy = \gamma \dot{B}(t)
$$

We can rewrite the above equation in the form,

$$\begin{aligned}
\frac{1}{\gamma}\int_t^T \frac{\gamma \dot{B}t}{\gamma B(t)-1} &= \int_t^T 1dt \\
\implies \frac{1}{\gamma}\int_t^T \frac{dy}{y} &= (T-t) \\
\implies ln(y)|_t^T &= \gamma(T-t) \\
\implies ln(y(T)) - ln(y(t)) &= \gamma(T-t) \\
\implies \frac{y(T)}{y(t)} &= exp^{\gamma(T-t)} \\
\implies \frac{-1}{\gamma B(t)-1} &= e^{\gamma(T-t)} \\
\implies \frac{1}{\gamma B(t)-1} &= -e^{\gamma(T-t)} \\
\implies (\gamma B(t)-1) &= -e^{-\gamma(T-t)} \\
B(t;T) = \frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right)
\end{aligned}$$

Solve for $A(t;T)$ using the first ODE and the boundary condition $A(T;T)=0$: Substituting the expression for $B(t;T)$ from above into the first ODE with $\dot{A}(t)$, we get:

$$\begin{aligned}
&\dot{A}(t) + \frac{w^2(r,t}{2}B^2(t) - \eta(t)B(t) = 0 \\
&\dot{A}(t) = \eta(t)B(t) - \frac{w^2(r,t)}{2}B^2(t) \\
&\dot{A}(t) = \eta(t)\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right) - \frac{w^2(r,t)}{2}(\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right))^2 \\
&\int_t^T \dot{A}(t) = \int_t^T \eta(t)\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right) - \frac{w^2(r,t)}{2}(\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right))^2 \\
&A(t) = \int_t^T \eta(t)\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right)dt - \frac{w^2(r,t)}{2}(\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right))^2 dt \\
A(t) &= \int_t^T \left(\eta(t)\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right) - \frac{w^2(r,t)}{2}(\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right))^2\right)dt \\
A(t) &= -\int_t^T \eta(\tau)B(\tau;T)d\tau + \frac{c^2}{2\gamma^2}\int_t^T (1+e^{-2\gamma(T-t)} - 2e^{-\gamma(T-t)})dt \\
&= -\int_t^T \eta(\tau)B(\tau;T)d\tau + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}\left(e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)}\right) - \left(\frac{2e^{(T-T)}}{\gamma} - \frac{1e^{(T-T)}}{2\gamma}\right)\right)
\end{aligned}$$

due to A(T) calculation we get $\frac{3}{2}\gamma$ in the equation

where we used the integration of $B(t;T)$ from the previous step. Rearranging and solving for

$A(t;T)$ gives us the desired result:

$$A(t;T) = -\int_t^T \eta(\tau)B(\tau;T)d\tau + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right)$$

This is the expression for the value of a zero coupon bond under the extended Hull and White model.

7 Solve the Partial Differential Equation for the Ornstein–Uhlenbeck process, and expectation.

7.1 Consider the process given by:

$$dU_t = \gamma U_t dt + \sigma dX_t, \quad U_0 = u$$

where γ, σ are constants and dXt is an increment in a Wiener process. Solve this equation for U_t and hence write down $E[U_t]$.

Solution for the PDE:

$$U_t = ue^{-\gamma t} + \sigma \int_0^t e^{-\gamma(t-u)} dX_u$$

To solve the given stochastic differential equation, we can use the method of integrating factors. We begin by multiplying both sides of the equation by the integrating factor $e^{\gamma t}$:

$$\begin{aligned} e^{\gamma t} dU_t &= \gamma U_t e^{\gamma t} dt + \sigma e^{\gamma t} dX_t \\ d(e^{\gamma t} U_t) &= \gamma e^{\gamma t} U_t dt + \sigma e^{\gamma t} dX_t \end{aligned}$$

Integrating both sides from 0 to t gives:

$$\begin{aligned} e^{\gamma t} U_t - U_0 &= \int_0^t \gamma e^{\gamma u} U_u du + \int_0^t \sigma e^{\gamma u} dX_u \\ U_t &= U_0 e^{-\gamma t} + \int_0^t \gamma e^{-\gamma(t-u)} U_u du + \int_0^t \sigma e^{-\gamma(t-u)} dX_u \\ U_t &= U_0 e^{-\gamma t} + \gamma \int_0^t e^{-\gamma(t-u)} U_u du + \sigma \int_0^t e^{-\gamma(t-u)} dX_u \end{aligned}$$

Since $U_0 = u$, we can substitute it in the above equation to get:

$$U_t = ue^{-\gamma t} + \gamma \int_0^t e^{-\gamma(t-u)} U_u du + \sigma \int_0^t e^{-\gamma(t-u)} dX_u$$

Differentiating both sides with respect to t, we obtain:

$$\begin{aligned} dU_t &= -\gamma ue^{-\gamma t} + \gamma U_t e^{-\gamma t} dt + \sigma e^{-\gamma t} dX_t \\ dU_t &= \gamma(u - U_t)dt + \sigma dX_t \end{aligned}$$

which is the original stochastic differential equation.

To find the expected value of U_t, we take the expectation of both sides of the equation: This is a separable first-order ordinary differential equation, which has the solution:

$$E[U_t] = E[U_0]e^{\gamma t} = ue^{\gamma t}$$

Therefore, the solution to the Ornstein-Uhlenbeck process is given by:

$$U_t = ue^{-\gamma t} + \sigma \int_0^t e^{-\gamma(t-u)} dX_u$$

and the expected value of U_t is:

$$E[U_t] = ue^{\gamma t}$$

8 Interest rate Risk

View the following videos, possibly others ad libitum.

https://www.youtube.com/watch?v=kxcwn7xoXhU

https://www.youtube.com/watch?v=tbc1ooJWG3I

https://www.youtube.com/watch?v=VebDKGe9o7Q

https://www.youtube.com/watch?v=H7C9oXK0-70

Explain what happen to SilverGate and Silicon Valley Bank, what are the similarities and the differences between the two institutions.

Both Silvergate Bank and Silicon Valley Bank experienced financial difficulties that led to their demise.

Silvergate Bank, a cryptocurrency-focused bank, experienced significant losses due to a combination of interest rate risk and a large exposure to a single cryptocurrency trading firm. The bank had invested heavily in Treasury-linked assets that were vulnerable to rising interest rates, and when rates increased unexpectedly, the bank was forced to sell off these assets at a loss to raise capital. Additionally, a major trading partner, which made up a significant portion of Silvergate's deposits, experienced financial difficulties, leading to a run on the bank and ultimately its failure.

Silicon Valley Bank, which focused on serving technology startups and venture capital firms, experienced liquidity and solvency issues related to its investments in the high-risk tech industry. The bank had invested heavily in illiquid securities and startups, and when the tech industry experienced a downturn, the bank was unable to raise enough capital to meet its obligations. The bank's regulators ultimately stepped in to seize control of the institution and manage the return of funds to depositors.

Similarity:

The similarities between the two institutions include their focus on serving high-risk, high-growth industries and their exposure to idiosyncratic risks related to those industries. Additionally, both banks experienced financial difficulties due to inadequate risk management practices and a lack of diversification in their investment portfolios.

Difference

Both Silvergate Bank and Silicon Valley Bank faced interest rate risk, but the nature of the risk and the way it affected each bank was different.

In the case of Silicon Valley Bank, the bank held a large portfolio of long-term Treasury bonds that were highly sensitive to changes in interest rates. When interest rates rose unexpectedly, the value of these bonds dropped, causing significant losses for the bank and ultimately leading to its insolvency. This was a classic example of interest rate risk resulting from a mismatch between the duration of a bank's assets and liabilities.

In contrast, Silvergate Bank's interest rate risk was primarily driven by its exposure to floating-rate loans to cryptocurrency customers. As interest rates rose, the cost of funding these loans increased, putting pressure on the bank's net interest margin. In addition, the bank held a large portfolio of investment securities that were also sensitive to interest rate changes. However, unlike Silicon Valley Bank, Silvergate's interest rate risk was not the primary driver of its recent financial troubles.

8.1 What is a bank run?

A bank run is a situation in which depositors withdraw their funds from a bank en masse, often due to concerns about the bank's solvency or liquidity. This can quickly lead to a liquidity crisis for the bank, as it may not have enough cash on hand to meet all of the withdrawal requests. The might might have to sell its assets/securities which could lead the bank to incur huge losses.

8.2 What is strategic concentration risk?

Strategic concentration risk is a type of risk that arises when a financial institution has a significant concentration of assets or liabilities in a particular market, product, or borrower. This concentration can make the institution vulnerable to adverse changes in the market, leading to losses and potentially threatening the institution's solvency and liquidity.

Silvergate Bank:

In the case of Silvergate Bank, the institution's strategic concentration risk was related to its heavy reliance on the cryptocurrency industry. The bank had developed a niche as a leading provider of banking services to cryptocurrency companies, and its loan portfolio was heavily concentrated in this market. The bank's customers included cryptocurrency exchanges, miners, and institutional investors, and the bank had developed a reputation as a trusted partner in the industry.

However, this concentration in the cryptocurrency market also made Silvergate Bank vulnerable to the volatility and uncertainty of the market. In particular, the rapid rise and subsequent collapse of the cryptocurrency market in 2018 and 2019 exposed the bank to significant losses and liquidity pressures. As the value of cryptocurrencies plummeted, many of the bank's customers struggled to repay their loans, and the bank was left with a portfolio of non-performing loans and impaired assets.

Silvergate Bank's strategic concentration risk was further exposed when it was revealed that it had a significant exposure to the cryptocurrency exchange FTX. In early 2022, FTX was experiencing significant losses due to the rapid decline in cryptocurrency prices. FTX's parent company, FTX Trading Ltd., held a significant portion of its assets in cryptocurrencies, and as prices declined, its net worth was negatively impacted.

Silvergate Bank had a relationship with FTX and was exposed to the risk of losses from its deposits held at the bank. The bank held $88 million in deposits from FTX and its affiliates, representing approximately 5% of its total deposits. When FTX faced liquidity issues due to its losses, there was a risk that it would withdraw its deposits from Silvergate, which could have resulted in a liquidity crisis for the bank.

Silvergate disclosed its exposure to FTX in a statement and emphasized that it had taken measures to mitigate the risks associated with the exposure. However, the incident highlights the strategic concentration risk inherent in Silvergate's business model. The bank is heavily exposed to the cryptocurrency industry, and its fortunes are tied to the success of its clients in that industry. As such, any significant decline in the value of cryptocurrencies or a downturn in the industry could have a material impact on the bank's financial performance.

In summary, Silvergate Bank's strategic concentration risk was exposed through its exposure to FTX, a major cryptocurrency exchange. The bank's heavy exposure to the cryptocurrency industry makes it vulnerable to fluctuations in the industry, and any downturn in the industry could have a material impact on the bank's financial performance.

Silicon Valley Bank (SVB):

Silicon Valley Bank (SVB) was a financial institution that catered to the venture capital industry and technology companies. It primarily offered deposit, lending, and investment services to these clients. However, the bank's business model was flawed, as it was heavily concentrated in one sector. Like any business, the bank was susceptible to external shocks, and its high level of exposure to the tech industry proved to be its downfall.

SVB's strategic concentration risk was related to its investment portfolio, which primarily comprised held-to-maturity (HTM) securities, such as US Treasuries and mortgage-backed securities (MBS). While these investments are generally safe from a credit risk perspective, they pose a significant interest rate risk. The bank's strategy was to invest a significant portion of its deposits in the HTM portfolio, where the investments would not have to be marked-to-market. However, the AFS side of the portfolio is subject to reporting unrealized gains or losses due to changes in the valuations of those assets that remain on the balance sheet.

With interest rates rising quickly in 2022, the value of these assets declined (for bond portfolios, yields and prices move inversely), and SVB had to do something to stop the bleeding as those unrealized gains hit against the balance sheet, specifically equity in the form of accumulated other comprehensive income or loss (AOCI). SVB's total common equity (TCE) ratio was severely dented by the steady unrealized losses it was sustaining, and so it was forced to sell AFS assets at a loss, thereby igniting the stampede to withdraw deposits once the word got out.

SVB maintained in its regulatory filings that it conducted regular and sophisticated market risk analysis and interest rate risk hedging activity. However, the amount of interest rate hedging was quite small in comparison with the AFS investments. Only $550 million in notional value of interest rate derivatives stipulated as interest rate hedges were reported at the end of 2022, and clearly their risk modeling didn't anticipate the combination of interest rate and liquidity risk shocks it would face.

Therefore, SVB's strategic concentration risk was related to its business model that heavily relied on the tech industry, and its investment strategy that concentrated on HTM securities without adequate interest rate risk hedging, which eventually led to the bank's failure due to a liquidity crisis.

8.3 How does interest rate risk manifest itself?

Interest rate risk is a type of financial risk that arises from the potential impact of changing interest rates on a bank's financial position. Interest rate risk manifests itself in various ways, including changes in the market value of a bank's portfolio of financial assets and liabilities, changes in its net interest margin, and changes in its funding costs.

Silicon Valley Bank (SVB):

In the case of Silicon Valley Bank (SVB), interest rate risk played a significant role in its collapse. In 2022, SVB had invested heavily in bonds to take advantage of the higher yields that come with longer maturities. However, when interest rates rose quickly, the value of those bonds declined, and the bank was forced to sell them at a loss to maintain its regulatory capital.

SVB's bond portfolio was mostly held-to-maturity, meaning that the investments were intended to be held until maturity. This strategy allowed SVB to avoid marking the bonds to market, which would have resulted in unrealized losses appearing on the bank's balance sheet. However, this strategy also meant that SVB was exposed to interest rate risk because if interest rates rose, the value of the bonds would decline.

As interest rates rose in 2022, the value of SVB's bond portfolio declined, and the bank's regulatory capital was eroded. To stop the bleeding, SVB was forced to sell some of its bonds, which further reduced its regulatory capital and triggered a bank run.

SVB's exposure to interest rate risk was compounded by its strategic concentration risk. The bank was heavily concentrated in the tech sector, which is highly sensitive to interest rate changes. When interest rates rose, SVB's tech clients faced increased borrowing costs, and some were unable to service their debts. This caused a liquidity crisis for SVB, as its tech clients withdrew their deposits en masse.

In summary, SVB's exposure to interest rate risk manifested itself in the decline in the value of its bond portfolio, the erosion of its regulatory capital, and the bank run triggered by the sale of its bonds. The bank's strategic concentration risk, combined with its poor risk management practices, exacerbated its exposure to interest rate risk and ultimately led to its collapse.

Silvergate Bank:

Interest rate risk manifests itself in several ways, and one such way is through the value of financial instruments held by banks. Silvergate Bank, like many other banks, invested heavily in interest rate-sensitive assets, such as U.S. Treasuries and mortgage-backed securities, to boost their net interest income. However, rising interest rates can have a negative impact on the value of these assets.

In Silvergate's case, the bank held $120 billion in investment securities, representing 55% of its total assets at the end of 2022. Three-quarters of these investments were in held-to-maturity (HTM) securities, largely in U.S. Treasuries and mortgage-backed securities (MBS). These securities are considered safe investments from a credit risk perspective but pose substantial interest rate risk. The weighted average duration of these investments was about six years, which implies that a 1% rise in interest rates could cause a 6% decline in the value of these securities.

Furthermore, Silvergate's interest rate hedging program was much smaller than expected, according to Clifford Rossi, a former risk management executive at Citigroup and a professor at the University of Maryland. The lack of an adequate hedging program left the bank exposed to interest rate risk, and its inability to offset the losses from selling its Treasury-linked assets is what ultimately led to its collapse.

In March 2022, the bank sold $21 billion in securities at a loss of $1.8 billion to raise cash to fund its operations, and this was a key event that precipitated its collapse. If the bank had had a more effective hedging program in place, it could have generated enough profit to offset the effect of selling the Treasury-linked assets at a loss, said Kris James Mitchener, an economics professor at Santa Clara University.

The lesson from Silvergate Bank's collapse is that interest rate risk can be a significant threat to a bank's viability, particularly if the bank's investments are heavily concentrated in interest rate-sensitive assets. Banks must have robust risk management practices in place, including hedging programs, to mitigate interest rate risk and avoid sudden liquidity crises.

8.4 What is solvency and liquidity risk, and how do they manifest themself in either cases?

Solvency risk is the risk that a bank may not be able to meet its long-term obligations, such as paying back loans, due to insufficient assets or a lack of profitability. Essentially, it's a measure of a bank's ability to stay in business in the long term.

Silvergate Bank:

In the case of Silvergate Bank, solvency risk was raised due to the bank's large investment in certain types of assets, such as mortgage-backed securities and U.S. Treasuries, which were subject to rapid interest rate changes. This investment strategy led to a significant decline in the value of these assets, which raised concerns about the bank's long-term viability.

Liquidity risk, on the other hand, is the risk that a bank may not have enough cash or liquid assets to meet its short-term obligations, such as funding withdrawals or paying for unexpected losses. This can be particularly dangerous for banks, as a lack of liquidity can quickly lead to a run on the bank by depositors who fear that their money is at risk. In the case of Silvergate Bank, liquidity risk was raised due to the bank's dependence on a small number of clients in the cryptocurrency industry. When these clients experienced financial difficulties, they withdrew a significant amount of their deposits from the bank, leading to a sudden and severe liquidity crunch.

These risks can manifest themselves in different ways, depending on the specifics of a bank's business model and investment strategy. In the case of Silvergate Bank, the combination of solvency and liquidity risk led to a sudden and unexpected collapse of the bank, which was taken over by the Federal Deposit Insurance Corporation in March 2023. While the bank had previously been seen as a leader in the cryptocurrency banking space, its heavy concentration in this area, coupled with its large investment in interest-rate sensitive assets, proved to be a recipe for disaster when these risks materialized.

Silicon Valley Bank (SVB):

SVB's collapse was precipitated by the sudden need to raise cash to fund operations. The bank sold $21 billion in Treasury-linked securities at a loss of $1.8 billion, which severely damaged the bank's solvency position. SVB's heavy investment in these securities left it vulnerable to interest rate risk, as rising rates can cause the value of fixed-income securities to fall.

Additionally, SVB had a high concentration of uninsured deposits, which increased its liquidity risk. Uninsured deposits refer to deposits that are not covered by the FDIC's insurance limit of

$250,000 per depositor. These deposits are typically held by large institutional clients, such as venture capital firms and hedge funds.

The Federal Reserve played a role in creating the conditions that led to SVB's collapse. The Fed's low-interest-rate policies incentivized banks to take on more risk to generate returns. This led to a surge in lending to the technology sector, which exposed banks like SVB to concentration risk.

Furthermore, the Fed's regulatory focus on capital adequacy may have contributed to SVB's vulnerability to interest rate risk. The bank was well-capitalized according to regulatory standards, but its heavy investment in Treasury-linked securities left it exposed to interest rate fluctuations.

In summary, SVB's collapse was caused by a combination of solvency and liquidity risk. The bank's heavy investment in Treasury-linked securities left it vulnerable to interest rate risk, while its concentration of uninsured deposits increased its liquidity risk. The Federal Reserve's low-interest-rate policies and regulatory focus on capital adequacy may have contributed to these risks.

8.5 Is it an idiosyncratic, systematic or systemic risk?

Silicon Valley Bank (SVB):

In the case of Silicon Valley Bank, its solvency and liquidity risks were idiosyncratic, meaning they were specific to the bank and not necessarily representative of risks faced by the broader financial system.

However, the risk posed by the bank's high concentration of technology sector clients and its investments in Treasury-linked assets was a systemic risk. If other banks had also taken similar positions, a broader crisis could have emerged, affecting the entire financial system.

Silvergate Bank:

In terms of solvency risk, Silvergate Bank's exposure to the volatile cryptocurrency market, particularly the borrowing against digital assets, could potentially lead to losses that could erode the bank's capital and put its solvency in question. If the bank's borrowers default on their loans, the bank may need to write off these losses, which could lead to a depletion of capital and potentially render the bank insolvent. Additionally, the bank's high concentration of investment securities and lack of a robust hedging program could also pose solvency risk in case of a sudden shift in interest rates or market conditions.

Regarding liquidity risk, Silvergate Bank's reliance on deposits from cryptocurrency-related clients exposes the bank to sudden and significant withdrawals if these clients experience a sudden market downturn. If the bank is unable to meet these withdrawals, it could potentially face liquidity issues, which could further impact its solvency. Moreover, the bank's large holdings of investment securities could also pose liquidity risk if the bank is unable to sell these securities quickly to meet its liquidity needs.

The bank's exposure to the crypto industry and crypto-related clients also presents unique risks. For example, cryptocurrencies are not subject to the same regulatory oversight as traditional banking products, which could potentially expose the bank to legal and regulatory risks. Furthermore, the crypto market is highly volatile and subject to sudden price swings, which could negatively impact the bank's clients and potentially impact the bank's profitability and solvency.

Overall, the risks facing Silvergate Bank are a combination of idiosyncratic and systemic risks. The bank's exposure to the cryptocurrency market and its reliance on deposits from crypto-related clients are idiosyncratic risks specific to the bank's business model. However, the potential impact of a significant market downturn on the bank's solvency and liquidity is a systemic risk that is not unique to Silvergate Bank but could impact the broader financial system.

8.6 What are the strategic errors that drove either institution to their demise?

In the case of Silicon Valley Bank (SVB), one of the strategic errors that led to its demise was its inadequate hedging program. SVB had a large portfolio of Treasury-linked assets, which were highly sensitive to interest rate changes. However, SVB did not have an effective hedging program in place to manage this risk, which resulted in significant losses when interest rates rose.

Another strategic error was the concentration of SVB's lending and investment activities in the technology startup industry. While this focus had helped SVB grow rapidly and become a leading provider of financial services to tech startups, it also made the bank vulnerable to the ups and downs of this volatile sector.

In the case of Silvergate Bank, one strategic error was its heavy reliance on cryptocurrency clients. While this had been a successful strategy during the crypto boom, it also made Silvergate vulnerable to the significant price declines and market volatility that occurred in the crypto market. Additionally, Silvergate's large holdings of mortgage-backed securities, which were also sensitive to interest rate changes, further compounded its risk exposure.

Another strategic error was Silvergate's lack of diversification in its loan portfolio. The bank had a high concentration of loans to a small number of clients, which left it vulnerable to the failure of any one of those clients. When some of Silvergate's largest crypto clients experienced financial difficulties, it put significant pressure on the bank's financial position.

8.7 Is there a contagion risk?

In the case of Silicon Valley Bank and Silvergate Bank, there may be a limited contagion risk. The failures of both banks were largely due to idiosyncratic factors specific to each institution, such as concentration risk in certain industries and inadequate risk management practices. However, there is always a risk of market-wide panic or loss of confidence in the banking system that could potentially lead to contagion.

In the case of Silicon Valley Bank, the bank's insolvency was not caused by external factors or a wider economic downturn, but rather by the bank's own risk management failures. The bank's reliance on wholesale funding and concentration in certain industries, as well as its lack of adequate hedging and risk management practices, led to its downfall. It is unlikely that other banks will face the same specific risks that led to Silicon Valley Bank's failure, but if there were a wider economic downturn, other banks with similar risk profiles could also be at risk.

Similarly, in the case of Silvergate Bank, the bank's overexposure to crypto-related assets and lack of adequate risk management practices led to its financial difficulties. While the crypto industry is still in its early stages, and it is unclear how many other banks have significant exposure to crypto assets, it is possible that other banks with similar exposures could also face similar risks. However, since crypto-related assets are not yet widely held by traditional banks, the contagion risk in this case is likely limited.

8.8 Which banks are most exposed to rise in interest rates risk?

Banks that are most exposed to the risk of rising interest rates are those with large portfolios of long-term fixed-rate assets such as mortgages or long-term government bonds, and small amounts of floating-rate assets. This is because when interest rates rise, the value of fixed-rate assets falls, leading to potential losses for the bank. In addition, banks with a high loan-to-deposit ratio may also be more exposed to interest rate risk, as they may need to raise rates on their deposits to attract funding when rates rise, which can lead to higher funding costs. However, it's important to note that the extent of a bank's exposure to interest rate risk will depend on the specific characteristics of its asset and liability portfolios, as well as the overall interest rate environment.

8.9 How would you manage the rise in interest rate risk (e.g. CCAR/DFAST)?

As a bank, to manage the rise in interest rate risk, the Comprehensive Capital Analysis and Review (CCAR) and Dodd-Frank Act Stress Testing (DFAST) are key tools.

CCAR and DFAST are regulatory programs that require bank holding companies (BHCs) with over $50 billion in assets to undergo annual stress tests. The tests evaluate whether the bank has adequate capital to survive an economic downturn, including a significant rise in interest rates.

To manage the rise in interest rate risk, a bank could consider the following strategies:

- Balance sheet management: A bank could adjust its balance sheet to reduce interest rate risk, such as by increasing the proportion of floating-rate loans in its portfolio or investing in shorter-term securities.
- Scenario analysis: Conduct scenario analysis to determine how the bank's financial position would be affected under various interest rate scenarios.
- Stress testing: Conduct regular stress tests to determine how the bank would fare under different economic conditions, including changes in interest rates.
- Hedging: Implement hedging strategies, such as interest rate swaps, to offset the impact of rising interest rates on the bank's assets and liabilities.
- Capital planning: Develop capital planning strategies that take into account the potential impact of interest rate movements on the bank's capital levels.

By utilizing these strategies and regularly assessing the bank's exposure to interest rate risk through CCAR and DFAST, the bank can better manage its risk exposure and ensure that it has adequate capital to withstand economic shocks.