FRE-GY 7851
Interest Rate Derivatives

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Assignment 3
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Shweta Pandey (sp6922@nyu.edu),
Kohsheen Tiku (kt2761@nyu.edu),
Prateek Sridhar Kumar(ps4146@nyu.edu)

Contents

1 PDE - solve the B & S PDE, in the Euro-Call equity case

Black & Scholes PDE(Partial Differential equation):

$$\frac{\partial V}{\partial t} + \underbrace{\frac{\sigma^2 S^2}{2}}_{\text{diffusion term}} \frac{\partial^2 V}{\partial S^2} + \underbrace{rS}_{\text{risk-free drift}} \frac{\partial V}{\partial S} = \underbrace{rV}_{\text{discount term}} \tag{1}$$

where $V = V(s,t) =$ Value of an option with underlying asset S at time t and $S = S(t,w) =$ asset S (in our case, a stock) that follows a geometric brownian motion:

$$dS = \mu S dt + \sigma S dX, \text{where } X \text{ is the Weiner Process.} \tag{2}$$

We need to apply the final condition of option payoff on the PDE in order to get the Black Scholes formulae for Euro-Call in the equity case.

Consider the option to be a European Call option on underlying stock S, we have below boundary conditions:

$$\begin{aligned} V(0,t) &= 0, \quad 0 \leqslant t \leqslant T \\ V(S,T) &= (S(T)-K)^+ = \max(S(T)-K,0), \quad S >= 0 \end{aligned} \tag{3}$$

Rewriting (1), let

$$d(V) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} - rV = 0$$

We will first do a bit of manipulation for ease of solving.

Step 1

The following change of variables transforms the Black-Scholes boundary value problem into a standard boundary value problem for the heat equation.

$$x = \ln\left(\frac{S}{K}\right) \in \mathbb{R}, \quad \tau = T - t \in [0,T] \text{ (or in other words, } s = Ke^x, \quad t = T - \tau \text{)}$$

and a new function $Z(x, c)$,

$$Z(x, \tau) = V\left(Ke^x, T - \tau\right)$$
$$= V(s, t)$$

Hawing performed a change of variable, the partial derivatives are

$$\frac{\partial z}{\partial x} = \frac{\partial V}{\partial x} \times K \times e^x, \quad \frac{\partial z}{\partial c} = -\frac{\partial V}{\partial \tau}$$
$$\frac{\partial^2 z}{\partial x^2} = Ke^x \frac{\partial V}{\partial x} + K^2 e^{2x} \frac{\partial^2 V}{\partial x^2}$$

Rewriting $L(v)$ post transformation,

$$\Rightarrow -\frac{\partial z}{\partial \tau} + \frac{1}{2}\sigma^2 \times K^2 e^{2x} \left\{ \frac{1}{K^2 e^{2x}} \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial x} \right) \right\} + rKe^x \times \frac{1}{Ke^x} \frac{\partial z}{\partial x} - rz = 0$$
$$\Rightarrow \frac{\partial z}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 z}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r \right) \frac{\partial z}{\partial x} + rz = 0$$

The boundary condition becomes,

$$Z(x, 0) = V\left(Ke^x, T\right)$$

Step 2

For transformation to Heat equation, now we introduce a new function,

$$u(x, \tau) = e^{\alpha x + \beta \tau} \times Z(x, \tau)$$

where the constants $\alpha, \beta \in \mathbb{R}$ are chosen so that the PDE for u is the heat equation.

PDE for u:

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + A \frac{\partial u}{\partial x} + Bu = 0$$

where

$$
\begin{aligned}
A &= \alpha\sigma^2 + \frac{\sigma^2}{2} - r, \\
B &= (1+\alpha)r - \beta - \frac{\alpha^2\sigma^2 + \alpha\sigma^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
u(x,0) &= e^{\alpha x} z(x,0) \qquad , x > 0 \\
&= e^{\alpha x} V\left(Ke^x, T\right), \\
&= e^{\alpha x}\left(Ke^x - K\right), \\
u(x,0) &= 0 \qquad , x = 0
\end{aligned}
\tag{4}
$$

Since we want to simplify solving Black Scholes equation, we wish to set α & β sit $A = B = 0$.

This will reduce the PDE to

$$
\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0
$$

which can be easily solved.

$$
\alpha = \frac{r}{\sigma^2} - \frac{1}{2}, \quad \beta = \left(\frac{1}{2} + \frac{r}{\sigma^2}\right) r - \frac{\alpha^2\sigma^2 + \alpha\sigma^2}{2}
$$

$$
\begin{aligned}
\beta &= \frac{r}{2} + \frac{r^2}{\sigma^2} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 \frac{\sigma^2}{2} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right)\frac{\sigma^2}{2} \\
\beta &= \frac{r}{2} + \frac{r^2}{\sigma^2} - \frac{r^2}{2\sigma^2} - \frac{\sigma^2}{8} + \frac{r}{2} - \frac{r}{2} + \frac{\sigma^2}{4} \\
\beta &= \frac{r}{2} + \frac{r^2}{2\sigma^2} + \frac{\sigma^2}{8}
\end{aligned}
$$

Step 3: Possible PDE solution using α and β values in step 2, we reduce the PDE to

$$
\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0 \tag{5}
$$

solution of the PDE in (5) can be given by Green's formula

$$u(x,\tau) = \frac{1}{\sqrt{2\sigma^2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2\tau}e} u(s,0)ds \tag{6}$$

We evaluate the integral in (6) and then perform backward substitution $u(x,\tau) \longrightarrow Z(x,\tau) \longrightarrow V(s,t)$.

Solving (6) using payoff function in (4), we get,

$$u(x,\tau) = \frac{1}{\sqrt{2\sigma^2\pi\tau}} \int_{0}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2 c}} e^{\alpha x} \left(Ke^{x} - K\right) ds \tag{7}$$

Note that the CDF for normal distribution in continuous domain is,

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-m^2/2} dm \tag{8}$$

The Green function has an interesting property wherein it behaves like a **Dirac Delta** function when x = s. Delta function has below important property.

$$g(x) = \int_{-\infty}^{+\infty} \delta(s-x) g(s) ds$$

Thus the integration with respect to s in (7) is similar to a summation.

Retracing our steps to write our solution in terms of the original variables, we get

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{9}$$

where

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

2 Binomial tree

Take the continuous formulation for a Euro Call and the boundary condition:

$$\begin{cases} V(S,t) = D\left(qV\left(S+\delta S_{+}, t+\delta t\right) + (1-q)V\left(S+\delta S_{-}, t+\delta t\right)\right) \quad (10.1) \\ V(i,N) = [S(i,N)-K]^{+} = \left[S_0 u^i v^{N-i} - K\right]^{+} \quad (10.2) \end{cases} \tag{10}$$

And show by induction that the Call value is:

$$V_0 = D^N \sum_{k=0}^{N} C_K^N q^k (1-q)^{N-k} V(k,\ \mathrm{N})$$

Solution:

$$q = \frac{1/D - v}{u-v}, \quad D = e^{-r\delta t}, \quad \delta = \frac{T-t}{N}$$

$$S + \delta S_{+} = uS, \quad S + \delta S_{-} = vS$$

$V(S,t)$ can be rewritten in discrete formulation with V expressed with

$$t_{j+1} - t_j = \delta t = \frac{T-t}{N}$$

$$\underbrace{\forall j \in N_{N-1}}_{\text{for any back step [0,N-1]}}, \quad \underbrace{\forall i \in N_j}_{\text{for node level i at time step j}}, \quad \underbrace{V(i,j)}_{\text{Value of derivative at i,j}},$$

$$V(i,j) = D[qV(i+1,j+1) + (1-q)V(i,j+1)] \tag{11}$$

Together with the boundary condition (10.2), we can evaluate V at any point on the lattice. Note that, (10.2) gives the value of the option at node i at maturity (time step N). The value of the option 1 step before maturity, can then be written as

$$\begin{aligned} &V(i,N-1) = D(qV(i+1,N) + (1-q)V(i,N)) \\ &\forall i \in N_{N-1} \text{ i.e } i \in [0, N-1] \end{aligned} \tag{12}$$

now lets assume for $N = 1$ (ie 1 period model),

$$
\begin{aligned}
V(0,0) = V_0 &= D(qV(1,1) + (1-q)V(0,1)) \\
&= D(qV_u + (1-q)V_v) \rightarrow \text{Value of option 1 step before expiration}
\end{aligned} \tag{13}
$$

For a 2 period market model, we will similarly have,

Value of option 1 step before expiry in 2 period model.

$$
\begin{aligned}
V(0,1) &= D(qV(1,2) + (1-q)V(0,2) \\
V(1,1) &= D(qV(2,2) + (1-q)V(1,2))
\end{aligned}
$$

Value of option, 2 steps before expiry in a 2 step period market model.

$$
\begin{aligned}
V_0 &= D(qV(1,1) + (1-q)V(0,1)) \\
&= D^2(q^2V(2,2) + 2q(1-q)V(1,2) + (1-q)^2V(0,2)) \\
V_0 &= D^2 \sum_{k=0}^{2} C_k^2 q^k (1-q)^{2-k} V(k,2)
\end{aligned}
$$

Consider value of option h steps before maturity/ expiration in N step market model. Generalizing we get,

$$
\begin{aligned}
V(i, N-h) &= D^h \sum_{k=0}^{h} C_k^h q^k (1-q)^{h-k} V(i+k, N) \\
&\forall i \in N_{N-h}
\end{aligned} \tag{14}
$$

Now by backward induction, we can say that value of option N steps before maturity in N step market model = Value at time 0 i.e

$$
V_0 = D^N \sum_{k=0}^{N} C_k^N q^k (1-q)^{N-k} V(k, N)
$$

3 Martingale

3.1 Assume the deterministic case for the RFA: $A(t) = \exp(rt)$. Apply the proper change of measure in the GBM, from real world to risk neutral, and show that under RN measure, the normalized log variable of S, say Y, follows a normal distribution such that:

$$\begin{cases} \mathbb{E}\left[Y_{t:T}\right] = \left(r - \frac{\sigma^2}{2}\right)(T-t) \\ \mathbb{V}\left[Y_{t;T}\right] = \sigma^2(T-t) \end{cases}$$

Solution:

Risk free asset (RFA) is defined in the deterministic case as :

$$A(t) = \exp(rt) \tag{15}$$

under the real world probability measure, a stock following a GBM will satisfy:

$$\forall t \in [0,T] \quad \frac{dS_t}{S_t} = \mu dt + \sigma dX_t^{\mathbb{P}} \tag{16}$$

we know that solution of (16) is

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t} + \sigma X_t^{\mathbb{P}} \tag{17}$$

Normalized log variable of S is:

$$Y_{0,t} = \log \frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma X_t^{\mathbb{P}}$$

To change the probability measure from real world to Risk Neutral we mill use Girsanov theorem.

We know from the corollary of Girsanov theorem that

$$X_t^P = X_t^Q + \int_0^t \theta_s ds \tag{18}$$

where driftless ABM X_t^Q on (Ω, F, Q) is a martingale.

Applying (18) on normalized log variable equation we get,

$$\begin{aligned}
Y_{0;t} &= \log \frac{S_t}{S_0} \\
&= \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma X_t \mathbb{P} \\
&= \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma X_t^Q + \sigma \int_0^t \theta_s ds \\
&= \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma X_t^Q + \sigma \int_0^t \frac{r-\mu}{\sigma} ds \\
&= \left(\mu - \frac{\sigma^2}{2}\right) + \sigma X_t^Q + (r-\mu)t \\
&= \left(r - \frac{\sigma^2}{2}\right) t + \sigma X_t^Q
\end{aligned} \tag{19}$$

$$Y_{t;T} = \log\left(\frac{S_T}{S_t}\right) = \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(X_T - X_t\right) \tag{20}$$

$$E^Q\left[Y_{t;T}\right] = \left(r - \frac{\sigma^2}{2}\right)(T-t) + 0 \text{ (Brownian Motion increments have expectation 0)} \tag{21}$$

$\mathrm{Var}[Y_{t;T}] = 0 + \sigma^2(T-t)$ (Variance of deterministic term $=0$,and Brownian Motion increments have variance a

as X_t is a ABM under Q from (17), $Y_{t;T}$ is normally distributed as

$$N\left(\left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$$

3.2 From Feynman Kac, show that if g is the payoff function, then:

$$V_t = e^{-r(T-t)} \int_{-\infty}^{+\infty} g\left(S_t e^y\right) p(y) dy$$

Feynman Kac states that,

$$V_t = A_t \mathbb{E}_t^Q \left[\frac{V_T}{A_T} \right] \tag{22}$$

let $g(\cdot)$ be the payoff function, then

$$\begin{aligned} V_t &= \frac{A_t}{A_T} E_t^Q \left[V_T\right] \\ &= e^{-r(T-t)} E_t^Q \left[V_T\right] \end{aligned}$$

now we know that value of a derivative depends on the underlying stock price

$$V_T = g\left(S_T\right) \tag{23}$$

From(20),

$$\begin{aligned} Y_{t;T} &= \log \frac{S_T}{S_t} \\ S_T &= S_t e^{Y_{t;T}} \end{aligned} \tag{24}$$

Putting (24) back into our expression for V_T in (23) and (22)

$$\begin{aligned} V_T &= g\left(S_t e^{Y_{t;T}}\right) \\ V_t &= e^{-(T-t)} E_t^Q \left[g\left(S_t e^{Y_{t;T}}\right)\right] \end{aligned}$$

let the stochastic variable Y have probability distribution $p(y)$,

$$V_t = e^{-r(T-t)} \int_{-\infty}^{\infty} g\left(S_t e^y\right) p(y) dy \tag{25}$$

Hence proved

3.3 Expressing Y with a SNRV $Z(0,1)$, show that for a Euro-Call of strike K:

$$V_t = e^{-r(T-t)} \int_{-\infty}^{+\infty} \left[S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma z\sqrt{T-t}} - K \right]^+ \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

Solution

For euro-call, payoff function is,

$$\left[S_T - K^+\right]$$

If y is expressed with $N(0,1)$, euro-call payoff can be rewritten as

$$\begin{aligned} [S_t e^{y_{t;T}} - K^+] &= S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(x_T - x_t)} \\ &= \left[S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma Z\sqrt{(T-t)}} - K \right]^+ \end{aligned} \tag{26}$$

where $Z \sim N(0,1)$. Note that we define Z as

$$Z = \frac{X_T - X_t}{\sqrt{T-t}}$$

Substituting in (25) with payoff function and associated PDF obtained in (26)

$$V_t = e^{-r(T-t)} \int_{-\infty}^{\infty} \left[S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma z\sqrt{T-t}} - K^+ \right]^+ \times \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \tag{27}$$

$(S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma z\sqrt{T-t}} - K)^+$ is >0 if and only if

$$
\begin{aligned}
&\implies (S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma z\sqrt{T-t}} - K)^+ > 0 \\
&\implies \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma z\sqrt{T-t} > \log\frac{K}{S_t} \\
&\implies \sigma\sqrt{(T-t)} > \log(k/s) - \left(r - \frac{\sigma^2}{2}\right)(T-t) \\
&\implies Z > -\frac{\log(s/k) + \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Consequently we have,

$$
V_t = e^{-r(T-t)} \int_{-d_{2(\tau,s)}}^{\infty} \left(e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma^2} - K\right)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz
$$

$$
\text{where } d_2(\tau,s) = \frac{\ln\left(\frac{S}{k}\right) + \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}
$$

substituting y by $-y$, we get

$$
\begin{aligned}
V_t &= e^{-r(T-t)} \int_{-\infty}^{d_2(\tau,s)} \left(S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)-\sigma z\sqrt{(T-t)}} \text{ - } K\right)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz \\
&= \int_{-\infty}^{d_2(\tau,s)} S_t e^{-\sigma Z\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)}\frac{e^{-Z^2/2}}{\sqrt{2\pi}}dz - e^{-r(T-t)}\int_{-\infty}^{d_2(\tau,s)} K\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz \\
&= \int_{-\infty}^{d_1(r,s)} S_t\frac{e^{-\frac{1}{2}(z+\sigma\sqrt{T-t})^2}}{\sqrt{2\pi}}dz - e^{-r(T-t)}\int_{-\infty}^{d_2(\tau,s)} K\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz
\end{aligned}
$$

We perform another change of variables

$$
\begin{aligned}
V_t &= \int_{-\infty}^{d_2(\tau,s)+\sigma\sqrt{T-t})} S_t\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz - e^{-r(T-t)}\int_{-\infty}^{d_2(\tau,s)} k\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz \\
&= S_t N\left(d_1\left(T-t, S_t\right)\right) - Ke^{-r(T-t)}N\left(d_2\left(T-t, S_t\right)\right)
\end{aligned}
$$

where $d_1\left(T-t, S_t\right)$ is defined as

$$
d_1(\tau,s) = d_2(\tau,s) + \sigma\sqrt{\tau}
$$

Hence proved

4 Martingale: Show that the market price of risk is a sufficient RV to specify the transition from Real World to the Risk Neutral world - and vice versa.

Market Price of risk is risk premium, aka sharpe ratio

$$\text{Market Price} = \left(\frac{\mu - r}{\sigma}\right) = -\theta$$

where r is the risk free rate, μ, σ^2 are mean & variance of the Stock GBM.

Part I

Transition from real world to risk neutral world.

From Girsanov theorem, we know that for any random process X defined over $(\Omega, \mathcal{F}, \mathbb{P})$, if there exists a θ such that,

$$\mathbb{E}^{\mathbb{P}}\left[\varepsilon\left(\int_0^t \theta_s dX_s\right)\right] = 1$$

where $\varepsilon(.)$ is the **Doléans Exponential** defined below :

$$\varepsilon\left(\int_0^T \theta_s dX_s\right) = \exp\left(\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

,

then there exists a P-equivalent risk neutral measure such that,

$$\frac{dQ}{dP} = \varepsilon\left(\int_0^t \theta_s dX_s\right) = e^{-\frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \theta_s dX_s}$$

& corollary states that

$$\forall t \in [0, T], \quad X_t^{\theta} = X_t^{\mathbb{P}} - \int_0^t \theta_s ds$$

where X_t^Q is driftless ABM (Arithmetic Brownian Motion.

With this theorem in mind, let us evaluate whether sharpe ratio as θ satisfies Girsanov's theorem.

$$
\begin{aligned}
E^{\mathbb{P}}\left[\varepsilon\left(\int_0^T \theta_s dX_s\right)\right] &= E^{\mathbb{P}}\left[e^{-\frac{1}{2}\int_0^T \theta_s^2 ds + \int_0^T \theta_s dX_s}\right] \\
&= E^{\mathbb{P}}\left[e^{-\frac{1}{2}\theta^2 T + \int_0^T dX_s}\right] \quad \text{, as } \theta \text{ is constant,} \\
&= e^{-\frac{1}{2}\theta^2 T} E^{\mathbb{P}}\left[e^{\theta \int_0^\top dX_s}\right] \\
&= e^{-\frac{1}{2}\theta^2 T} \times e^{\frac{1}{2}\theta^2 T} \quad \text{, as expectation of a log-normal random variable} = e^{\mu+\sigma^2/2} \\
&= e^0 \\
&= 1
\end{aligned}
$$

where X is a random variable which in our case we consider to be a Brownian motion

As the condition is satisfied, then there exists a P-equivalent measure for θ such that,

$$
\begin{aligned}
\frac{dQ}{dP} &= \left[\varepsilon\left(\int_0^T \theta_s dX_s\right)\right] \\
&= e^{-\frac{1}{2}\theta^2 \int_0^t ds + \int_0^t \theta dX_s} \\
&= e^{-\frac{1}{2}\theta_t^2 t + \theta_t X_t}
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
X_t^P &= X_t^Q + \int_0^t \theta_s ds \\
&= X_t^Q + \theta t
\end{aligned}
\tag{29}
$$

Part 2

For same θ, we can rewrite (28) & (29) as

$$
\begin{aligned}
\frac{d\mathbb{P}}{dQ} &= e^{\frac{1}{2}\theta_t^2 t - \theta_t X_t} \\
X_t^Q &= X_t^P - \int_0^t \theta_s ds \\
&= X_t^P - \theta_t
\end{aligned}
$$

Hence proved that $\theta =$ Sharpe ratio enables measure transformation from real world to risk neutral world and vice versa.