*NOTE: I assume we are using the metric system for all calculations. The final output of our program (temperature) can easily be converted to English units after all the metric calculations are finished.*

There are 4 types of approximations we'll use:

- *Lumped Capacitance.* Use this when $Bi < 0.1$
- *One-Term Approximation.* Use this when $Fo > 0.2$
- *Multiple-Term Approximation.* Use this when $Bi > 0.1$ and $0.05 < Fo < 0.2$
- *Semi-Infinite Approximation.* Use this when $Fo < 0.05$

$Bi$ is the Biot number and $Fo$ is the Fourier number. Both are unitless quantities.

$$Bi_{plane\ wall} = \frac{h * L}{k}$$

$$Bi_{cylinder,\ sphere} = \frac{h * r_o}{k}$$

$h$ is the heat transfer coefficient (units: $W/m^2K$. It is something that governs the interaction between the object and its surrounding environment. We do not calculate this in our library. We use previously saved data relevant to a wide range of common scenarios.

$L$ (units: $m$) is half a plane wall's thickness.

$r_o$ is a cylinder or sphere's radius.

$k$ is the conduction coefficient (units: $W/mK$) It is something that governs how well heat travels through an object. We do not calculate this in our library. We use previously saved data relevant to a wide range of common scenarios.

$$Fo_{plane\ wall} = \frac{\alpha * t}{L^2}$$

$$Fo_{cylinder,\ sphere} = \frac{\alpha * t}{r_o^2}$$

$\alpha$ is the thermal diffusivity (units: $m^2/s$). "It measures the rate of transfer of heat of a material from the hot side to the cold side." (Wikipedia).

$t$ is time (units: $s$).

**Lumped Capacitance** (Use this whenever Bi <0.1, even if other models are also appropriate)

This is the simplest approximation. Every point in the object is assumed to be at the same temperature. The temperature of the object as a whole changes over time.

$$T(t) = T_{avg} = exp\left[ -\left(\frac{h * A_s}{\rho * V * c}\right)t \right](T_i - T_\infty) + T_\infty$$

$T(t)$ (units: $K$) is the final temperature we're solving for.

$T_i$ (units: $K$) is the object's initial temperature.

$T_\infty$ (units: $K$) is the temperature of the surroundings (assumed uniform and unchanging).

$A_s$ (units: $m^2$) is the surface area of the object.

$\rho$ (units: $kg/m^3$) is the density of the object.

$V$ (units: $m^3$) is the volume of the object.

$c$ (units: $J/K$) is "the ratio of the heat added to (or removed from) an object to the resulting temperature change" (Wikipedia). We do not calculate this in our library. We use previously saved data relevant to a wide range of common scenarios.

$$t = \frac{\rho * V * c}{h * A_s} * ln\left(\frac{T_i - T_\infty}{T(t) - T_\infty}\right)$$

**One-Term Approximation** (Use this when Fo >0.2, but use Lumped Capacitance if Bi <0.1)

This model requires significantly more involved computation. Unlike with the lumped capacitance model, each of the 3 shapes (plane wall, infinite cylinder, sphere) now has a different equation for its temperature, but each of the 3 are similar.

PLANE WALL:

$$T(x,t)_{plane\ wall} = C_1 * exp(-\zeta_1^2 * Fo) * cos\left(\zeta_1 * \frac{x}{L}\right)(T_i - T_\infty) + T_\infty$$

$T(x,t)$ (units: $K$) is the final temperature we're solving for. Unlike with Lumped Capacitance, there is a dependency on time as well as on $x$ now.

$T_i$ (units: $K$) is the object's initial temperature.

$T_\infty$ (units: $K$) is the temperature of the surroundings (assumed uniform and unchanging).

$x$ (units: $m$) is the distance from the centerline of the plane wall to the point in consideration.

$L$ (units: $m$) is half the plane wall's thickness

$C_1$ is a unitless coefficient:

$$C_1 = \frac{4sin(\zeta_1)}{2\zeta_1 + sin(2\zeta_1)}$$

where $\zeta_1$ is a unitless eigenvalue equal to the positive root of the transcendental equation:

$$\zeta_1 * tan(\zeta_1) = Bi$$

If possible, it is preferable to solve for $\zeta_1$ within the program. Otherwise, we can use the approach where we store $\zeta_1$ corresponding to specific $Bi$ and interpolate between stored values.

INFINITE CYLINDER:

$$T(r,t)_{cylinder} = C_1 * exp(-\zeta_1^2 * Fo) * J_0\left(\zeta_1 * \frac{r}{r_o}\right)(T_i - T_\infty) + T_\infty$$

$T(r,t)$ (units: $K$) is the final temperature we're solving for. Unlike with Lumped Capacitance, there is a dependency on time as well as on $r$ now.

$T_i$ (units: $K$) is the object's initial temperature.

$T_\infty$ (units: $K$) is the temperature of the surroundings (assumed uniform and unchanging).

$r$ (units: $m$) is the distance from the axis of the cylinder to the point in consideration.

$r_o$ (units: $m$) is the cylinder's radius.

$C_1$ is a unitless coefficient:

$$C_1 = \frac{2J_1(\zeta_1)}{\zeta_1 * [J_0^2(\zeta_1) + J_1^2(\zeta_1)]}$$

where $\zeta_1$ is a unitless eigenvalue equal to the positive root of the transcendental equation:

$$\zeta_1 * \frac{J_1(\zeta_1)}{J_0(\zeta_1)} = Bi$$

If possible, it is preferable to solve for $\zeta_1$ within the program. Otherwise, we can use the approach where we store $\zeta_1$ for specific $Bi$s and interpolate between stored values.

"The quantities $J_1$ and $J_0$ are Bessel functions of the first kind" (Incropra). This is where we use C++17's special math library: http://en.cppreference.com/w/cpp/numeric/special_math/cyl_bessel_j

SPHERE:

$$T(r,t)_{sphere} = C_1 * exp(-\zeta_1^2 * Fo) * \frac{1}{\zeta_1 * (\frac{r}{r_o})} sin\left(\zeta_1 * \frac{r}{r_o}\right)(T_i - T_\infty) + T_\infty$$

$T(r,t)$ (units: $K$) is the final temperature we're solving for. Unlike with Lumped Capacitance, there is a dependency on time as well as on $r$ now.

$T_i$ (units: $K$) is the object's initial temperature.

$T_\infty$ (units: $K$) is the temperature of the surroundings (assumed uniform and unchanging).

$r$ (units: $m$) is the distance from the center of the sphere to the point in consideration.

$r_o$ (units: $m$) is the sphere's radius.

$C_1$ is a unitless coefficient:

$$C_1 = \frac{4[sin(\zeta_1) - \zeta_1 cos(\zeta_1)]}{2\zeta_1 - sin(2\zeta_1)}$$

where $\zeta_1$ is a unitless eigenvalue equal to the positive root of the transcendental equation:

$$1 - \zeta_1 cot(\zeta_1) = Bi$$

If possible, it is preferable to solve for $\zeta_1$ within the program. Otherwise, we can use the approach where we store $\zeta_1$ for specific $Bi$s and interpolate between stored values.

For all geometries, the average temperature using one-term approximation is:

$$T_{avg} = C_1 * exp(-\zeta_1^2 * Fo)$$

**Multiple-Term Approximation** (Use this when Bi >0.1 and 0.05 <Fo <0.2)

The "One-Term Approximation" of the previous section was:

$$T(position, t) = C_1 * exp(-\zeta_1^2 * Fo) * fn(\zeta_1 * position^*)(T_i - T_\infty) + T_\infty$$

where *position* is either the $x$ or $r$ specified by the user. $position^*$ is the relative position – either the $x/L$ or $r/r_o$ dependent on whether a plane or cylinder/sphere is being considered.

$fn$ is a function. For a plane wall it was *cos*, for a cylinder it was $J_0$, and for a sphere, it was *sin*. The sphere also has the *sin* be multiplied by $\frac{1}{\zeta_1 * (\frac{r}{r_o})}$ (see previous page).

The One-Term Approximation is the first term of the infinite series:

$$T(position, t) = \sum_{n=1}^{\infty} \left[C_n * exp(-\zeta_n^2 * Fo) * fn(\zeta_n * position^*)\right](T_i - T_\infty) + T_\infty$$

When in the range of $Fo$ or $Bi$ where the One-Term Approximation isn't appropriate, it is necessary to use more terms. Use $n$ terms where $n$ satisfies:

$$\frac{C_n * exp(-\zeta_n^2 * Fo)}{C_{n-1} * exp(-\zeta_{n-1}^2 * Fo)} < \epsilon$$

$\epsilon$ is a ratio you specify (0.01) after which adding successive terms has little effect on the calculated temperature.

For all geometries, the average temperature using multiple-term approximation is:

$$T_{avg} = \sum_{n=1}^{\infty} C_n * exp(-\zeta_n^2 * Fo)$$

As with the temperature at a single point, the number of terms used to calculate the average temperature of the entire object is $n$, where $n$ satisfies:

$$\frac{C_n * exp(-\zeta_n^2 * Fo)}{C_{n-1} * exp(-\zeta_{n-1}^2 * Fo)} < \epsilon$$

**Semi-Infinite Approximation** (Use this when Fo <0.05, but use Lumped Capacitance if Bi <0.1)

Like with lumped capacitance, there is only one equation for the temperature, however it is quite long:

$$T(x,t) = \left\{ erfc\left(\frac{x}{2\sqrt{\alpha * t}}\right) - \left[exp\left(\frac{h*x}{k} + \frac{h^2 * \alpha * t}{k^2}\right)\right]\left[erfc\left(\frac{x}{2\sqrt{\alpha * t}} + \frac{h * \sqrt{\alpha * t}}{k}\right)\right]\right\}(T_\infty - T_i) + T_i$$

Because the semi-infinite solid has infinite depth, its average temperature remains unchanged from its initial temperature.

$$T_{avg} = T_i$$

$T(x,t)$ (units: $K$) is the final temperature we're solving for. Unlike with Lumped Capacitance, there is a dependency on time as well as on $x$ now.

$T_i$ (units: $K$) is the object's initial temperature.

$T_\infty$ (units: $K$) is the temperature of the surroundings (assumed uniform and unchanging).

$x$ (units: $m$) is the distance from the SURFACE of the object to the point in consideration.

$h$ is the heat transfer coefficient (units: $W/m^2K$. It is something that governs the interaction between the object and its surrounding environment. We do not calculate this in our library. We use previously saved data relevant to a wide range of common scenarios.

$k$ is the conduction coefficient (units: $W/mK$) It is something that governs how well heat travels through an object. We do not calculate this in our library. We use previously saved data relevant to a wide range of common scenarios.

$\alpha$ is the thermal diffusivity (units: $m^2/s$). "It measures the rate of transfer of heat of a material from the hot side to the cold side." (Wikipedia).

$erfc$ is the complementary error function (http://en.cppreference.com/w/cpp/numeric/math/erfc)

**Solving for Temperatures in More Complicated 3D Objects**

So how does all this information relate to the 3D geometries? There are 9 geometries that our library must find the temperatures of (10 counting the sphere).

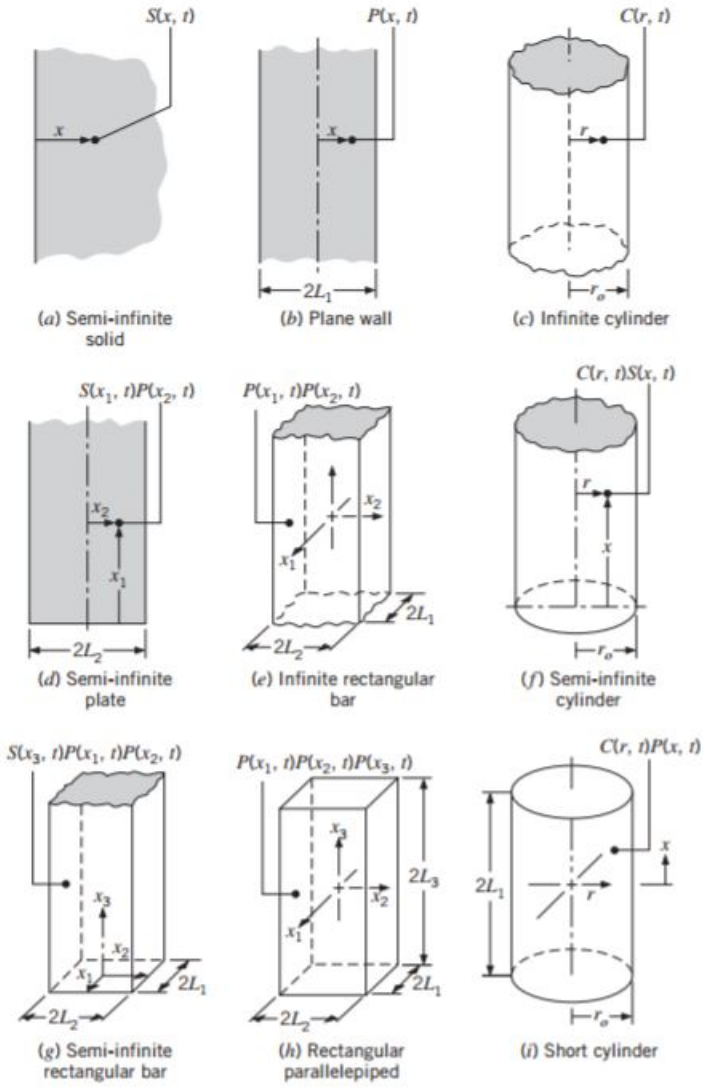


FIGURE 5S.11 Solutions for multidimensional systems expressed as products of one-dimensional results.

The 3 geometries in the first row (plus the sphere) we have already solved for the temperature of. The Semi Infinite Approximation (method 4) is always used for the "Semi-Infinite Solid".

The user should not be allowed to specify among all 10 geometries. Instead, they should only be allowed to specify 7 of them. The ones they should be prevented from choosing are the semi-infinite cylinder, the semi-infinite rectangular bar, and the semi-infinite solid. The first 2 geometries should be handled internally because the values for the Biot number determine whether the short cylinder is semi-infinite and whether

the rectangular parallelepiped is a semi-infinite rectangular bar – the user could easily give contradictory information. The semi-infinite solid is not a valid geometry; it is a model used for approximation that happens to also have a physical representation.

To obtain the temperatures in the remaining 2 rows requires taking products of combinations of the results in the first row (S, P, and C). To be as unambiguous as possible, I will detail the procedure used to solve for the most complicated shape, the rectangular parallelepiped.

As seen in the above figure, the rectangular parallelepiped is represented as the product of 3 orthogonal plane walls:

$$P(x_1,t)P(x_2,t)P(x_3,t)$$

The $Bi$ and $Fo$ of each of the 3 plane walls is calculated to determine which of the 4 approximations is appropriate for each. It is possible that all 3 could have different approximations be appropriate for them. Regardless of the approximations used, the equation for temperature of the rectangular parallelepiped is:

$$T(x_1,x_2,x_3,t) = \theta_1^* * \theta_2^* * \theta_3^*(T_i - T_\infty) + T_\infty$$

Notice that the temperature is dependent on not 1, but 3 values of $x$ now.

To keep the above equation compact and general, $\theta^*$ is used. $\theta_n^* = \dfrac{T(x_n,t) - T_\infty}{T_i - T_\infty}$. $\theta^*$ was what appeared on the left hand side of all the equations for temperature up until now. To make things explicit, the denominator of $\theta^*$ was multiplied to both sides of the equations and then $T_\infty$ was added to both sides. For example, the equation for One-Term Approximation of a sphere was:

$$T(r,t)_{sphere} = C_1 * exp(-\zeta_1^2 * Fo) * \frac{1}{\zeta_1 * (\frac{r}{r_o})} sin\left(\zeta_1 * \frac{r}{r_o}\right)(T_i - T_\infty) + T_\infty$$

By subtracting $T_\infty$ from both sides and then dividing by $T_i - T_\infty$ from both sides, the equation becomes:

$$\frac{T(r,t)_{sphere} - T_\infty}{T_i - T_\infty} = C_1 * exp(-\zeta_1^2 * Fo) * \frac{1}{\zeta_1 * (\frac{r}{r_o})} sin\left(\zeta_1 * \frac{r}{r_o}\right)$$

The left hand side of this equation is $\theta^*$. Of course, you do not have the value for the LHS of temperatures of the three planes – you must instead use the RHS appropriate to the best approximation for each.