The Spin Statistics Theorem

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Here we seek to prove the spin statistics theorem, its implications, and how it has been extended. This paper was prepared as part of PHYS G6036 Statistical Mechanics with Ana Asenjo Garcia in the Fall of 2019.

I. INTRODUCTION

The Spin Statistics Theorem (SST) was first proven by Wolfgang Pauli in 1940 [1] and was subsequently proven by Feynman (1949) [2] using the techniques he developed for QED and by Weinberg (1964) [3] requiring that the quantized fields should either commute or anticommute for spacelike separations. Since then, other proofs have emerged, most notably from the field of axiomatic quantum field theory [4]. As this latter proof seems to suggest, the idea behind the SST results from fundamental assumptions we make to understand the world. It has significance in various fields of physics aside from a philosophical understanding of QFT as we will see when we discuss Anyons.

In this paper, we seek to prove the SST and establish its implications. In Section II, we begin by introducing the fundamental axioms that Jost [4] used to prove the CPT and SST Theorem. In Section III, we introduce our own based in representation theory and in Section IV, we highlight its implications. We end with the Section V on the formal mathematics needed to understand our proof.

II. AXIOMS OF QUANTUM FIELD THEORY

While the details of the axiomatic proof are out of the scope of this paper, we introduce the axioms of relativistic QFT as stated by R.F. Streator and A.S. Wightman in *"PCT, Spin and Statistics, and All That"* [5]. We however note the importance of knowing the assumptions required to derive the SST and CPT theorem and the possible ways in which their proofs may break.

(1) We assume that states of the theory are unit rays in a Lorentz and translation invariant Hilbert Space $\mathcal{H}$ which transforms by continuous unitary representations of the inhomogenous SL(2,$\mathbb{C}$) group. By Lorentz and translation covariance, if U is Lorentz or translation symmetry, for some $\Psi, \Phi \in \mathcal{H}$, the transition probabilities are preserved under U.

$$\left| \langle U\Phi | U\Psi \rangle \right|^2 = \left| \langle \Phi | \Psi \rangle \right|^2$$

(2) We assume the existence of a Lorentz invariant vacuum state, Ψ_0, that is unique up to a phase factor such that for any $U(a, A)$ unitary operator in the inhomogenous SL(2,$\mathbb{C}$) group, $U(a, A)\Psi_0 = \Psi_0$. We in addition assume that all physical states may be constructed by acting with field operators on Ψ_0.

(3) Physical quantities are determined by polynomials of field operators $\phi(x)$ acting on $\mathbb{H}$. Field operators are defined by how they transform under the inhomogenous SL(2,$\mathbb{C}$) and can therefore be considered to be irreducible representations of the Poincare group. Streator and Wightman discuss the need for "smeared fields" to define operators well but this level of rigor is unnecessary for our review.

(4) We require the energy of fields to be positive. This is tied to the notion that we want stability. If there is no lowest energy ground state, it would allow for an abundance of positive and negative energy states to proliferate.

(5) We require causality where we say that field operators commute/anticommute when they cannot be connected by light signals, i.e. they are causally disconnected. This is also known as locality. This implies that for space-like separated points $(x - y)^2 < 0$ fields, $[\phi(x), \phi(y)]_\pm = 0$ where $[]_+$ denotes the anticommutator for fermions and $[]_-$ denotes the commutator for bosons.

These axioms are all that are needed to prove both the CPT and SST in a rigorous setting. We note however that these axioms do not encompass all QFTs that are present in the literature and therefore the proof by Jost in AQFT may not always apply. However, many extensions of the CPT and SST have been similarly proven in the literature.

III. THE SPIN STATISTICS THEOREM

A. The Theorem

The SST can morally be stated by saying that integer spins follow Bose-Einstein statistics and half-integer spins follow Fermi-Dirac statistics. However, by Wightman and Streator[5], the Spin Statistics Theorem can be more rigorously formulated by considering that one may arrive at Bose-Einstein (Fermi-Dirac) statistics by considering a field which commutes (anti-commutes) for space-like separations. Thus, the SST states that a non-trivial integer (half-integer) spin field cannot anti-

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commute (commute) for space-like separated points:

For a general irreducible spinor field of integer spin, if $[\phi(x), \phi^*(y)]_+ = 0$ for $(x-y)^2 < 0$ then $\phi(x)\Psi_0 = 0$. Similarly, for a general irreducible spinor field of half-integer spin, if $[\phi(x), \phi^*(y)]_- = 0$ for $(x-y)^2 < 0$, then $\phi(x)\Psi_0 = 0$.

Notice that the notion of a trivial field has been implemented by the axiomatically provable statement that $\phi(x)\Psi_0 = 0 \implies \phi(x) = 0$ for all x.

B. Proof of the Main Theorem

1. The Setup

Consider the Hilbert space $\mathcal{H}$ of our possibly interacting QFT which has unitary projective representation of the Poincare group or equivalently a unitary representation of its double cover. This representation gives a representation of the Lie algebra which has generators,

$$\hat{P}_i \qquad \hat{K}_i \qquad \hat{J}_i \tag{1a}$$

acting as operators on $\mathcal{H}$ and satisfying the commutation relations shown in the appendix. We assume there is an (interacting) vacuum state $|\Omega\rangle$ is is invariant under the action of the Poincaré group. A particle is given by a multiplet of field operators $\hat{\phi}_i(x)$ in a finite-dimensional irreducible $\mathrm{Spin}\,(1,3)$-representation $S : \mathrm{Spin}\,(1,3) \to W$. Explicitly, this multiplet transforms under a Lorentz transformation Λ (or more generally an element of $\mathrm{Spin}\,(1,3)$) via,

$$U(\Lambda)^{-1}\hat{\phi}_i(x)U(\Lambda) = S_{ij}(\Lambda)\hat{\phi}_j(\Lambda^{-1}x) \tag{2a}$$

where $\Lambda \in \mathrm{Spin}\,(1,3)$ acts on x via the covering projection $\pi : \mathrm{Spin}\,(1,3) \to \mathrm{SO}(1,3)$ which is thus acts as a matrix. Note that the representation W must be of the form (μ,ν) by the classification of projetive $\mathrm{SO}(1,3)$-reps and cannot be unitary because W is finite dimensional. Furthermore, the Poincare group translations act via,

$$e^{i\hat{P}_\mu x^\mu}\hat{\phi}(y^\mu)e^{-i\hat{P}_\mu x^\mu} = \hat{\phi}(x^\mu + y^\mu)$$

2. The Källén-Lehmann Spectral Resolution

We must compute the two-point correlation functions,

$$\langle\Omega|\,\hat{\phi}_i(x)\hat{\phi}_j^\dagger(y)\,|\Omega\rangle \tag{3a}$$

$$\langle\Omega|\,\hat{\phi}_j^\dagger(y)\hat{\phi}_i(x)\,|\Omega\rangle \tag{3b}$$

We now use a resolution of unity. However, since $\hat{\phi}$ is a charged field, by selection rules, we need only consider single particle states in the resolution,

$$\langle\Omega|\,\hat{\phi}_i(x)\hat{\phi}_j^\dagger(y)\,|\Omega\rangle = \sum_{\lambda,s}\int_{p^2=m_\lambda^2}\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2p_0} \tag{4a}$$

$$\cdot\,\langle\Omega|\,\hat{\phi}_i(x)\,|\lambda_{p,s}\rangle\,\langle\lambda_{p,s}|\,\hat{\phi}_j^\dagger(y)\,|\Omega\rangle \tag{4b}$$

where $|\lambda_s\rangle$ is a W-multiplet of single particle rest states of mass m_λ and $|\lambda_{p,s}\rangle = U(B_p)\,|\lambda_s\rangle$ where B_p is the Lorentz transformation which performs a boost to momentum p. Since $\hat{P}_i\,|\lambda_s\rangle = 0$ and $[\hat{P}_i, U(B_p)] = p$. Therefore,

$$\langle\Omega|\,\hat{\phi}_i(x)\,|\lambda_{p,s}\rangle = \langle\Omega|\,e^{i\hat{P}_\mu x^\mu}\hat{\phi}_i(0)e^{-i\hat{P}_\mu x^\mu}\,|\lambda_{p,s}\rangle \tag{5a}$$

$$= \langle\Omega|\,\hat{\phi}_i(0)\,|\lambda_{p,s}\rangle\,e^{-ip\cdot x} \tag{5b}$$

using the fact that $e^{i\hat{P}_\mu x^\mu}\,|\lambda_{p,s}\rangle = e^{ip_\mu x^\mu}\,|\lambda_{p,s}\rangle$ and the vacuum is invariant under the Poincare group. Now,

$$\langle\Omega|\,\hat{\phi}_i(x)\,|\lambda_{p,s}\rangle = \langle\Omega|\,\hat{\phi}_i(0)U(B_p)\,|\lambda_s\rangle\,e^{-ip\cdot x} \tag{6a}$$

$$= \langle\Omega|\,\hat{\phi}_i(0)U(B_p)\,|\lambda_s\rangle\,e^{-ip\cdot x} \tag{6b}$$

Again, using the fact that $|\Omega\rangle$ is Poincaré invariant we find,

$$\langle\Omega|\,\hat{\phi}_i(x)\,|\lambda_{p,s}\rangle = \langle\Omega|\,U^\dagger(B_p)\hat{\phi}_i(0)U(B_p)\,|\lambda_s\rangle\,e^{-ip\cdot x} \tag{7a}$$

$$= S_{ik}(B_p)\,\langle\Omega|\,\hat{\phi}_k(0)\,|\lambda_s\rangle\,e^{-ip\cdot x} \tag{7b}$$

Furthermore, since $\hat{\phi}_k$ and $|\lambda_s\rangle$ are both W-multiplets, by the Wigner-Ekart theorem we have,

$$\langle\Omega|\,\hat{\phi}_k(0)\,|\lambda_s\rangle = C_\lambda\delta_{ks} \tag{8}$$

for some constant $C_\lambda \in \mathbb{C}$. Thus,

$$\langle\Omega|\,\hat{\phi}_i(x)\,|\lambda_{p,s}\rangle = C_\lambda S_{is}(B_p)e^{-ip\cdot x} \tag{9}$$

Noting that,

$$\langle\lambda_{p,s}|\,\hat{\phi}_j^\dagger(y)\,|\Omega\rangle = \overline{\langle\Omega|\,\hat{\phi}_j(y)\,|\lambda_{p,s}\rangle} = C_\lambda^* S_{js}^*(B_p)e^{ip\cdot y} \tag{10}$$

Therefore, defining $Z_\lambda = |C_\lambda|^2 \geq 0$ we have,

$$\langle\Omega|\,\hat\phi_i(x)\hat\phi_j^\dagger(y)\,|\Omega\rangle = \sum_\lambda \int_{p^2=m_\lambda^2} \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{-ip\cdot(x-y)}}{2p_0} Z_\lambda \sum_s S_{is}(B_p)S^*_{js}(B_p) \tag{11a}$$

$$= \sum_\lambda \int_{p^2=m_\lambda^2} \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{-ip\cdot(x-y)}}{2p_0} Z_\lambda \left(S(B_p)S^\dagger(B_p)\right)_{ij} \tag{11b}$$

A similar computation shows that,

$$\langle\Omega|\,\hat\phi_j^\dagger(y)\hat\phi_i(x)\,|\Omega\rangle = \sum_{\bar\lambda,s} \int_{p^2=m_{\bar\lambda}^2} \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2p_0} \langle\Omega|\,\hat\phi_j^\dagger(y)\,|\bar\lambda_{p,s}\rangle\langle\bar\lambda_{p,s}|\,\hat\phi_i(x)\,|\Omega\rangle \tag{12a}$$

$$= \sum_{\bar\lambda,s} \int_{p^2=m_{\bar\lambda}^2} \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{-ip\cdot(y-x)}}{2p_0} \langle\Omega|\,S^*_{jk}(B_p)\hat\phi_k^\dagger(0)\,|\bar\lambda_s\rangle\langle\bar\lambda_s|\,S_{i\ell}(B_p)\hat\phi_\ell(0)\,|\Omega\rangle \tag{12b}$$

$$= \sum_{\bar\lambda} \int_{p^2=m_{\bar\lambda}^2} \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{-ip\cdot(y-x)}}{2p_0} Z_{\bar\lambda} \left(S(B_p)S^\dagger(B_p)\right)_{ij} \tag{12c}$$

3. *The Representation Function*

Let H_{m_λ} denote the hyperbola $\{p^2 = m_\lambda^2\} \subset \mathbb{R}^{1,3}$. Then we define a matrix of functions $f_{ij} : H_{m_\lambda} \to \mathbb{C}$ via,

$$f_{ij}(p) = (S(B_p)S^\dagger(B_p))_{ij} \tag{13}$$

I claim that $V = \mathrm{Span}(f_{ij}) \subset \mathcal{C}^\infty(H_{m_\lambda})$ is a finite-dimensional $\mathrm{Spin}\,(1,3)$-representation. Consider the transformed, $f(\Lambda p) = S(B_{\Lambda p})S^\dagger(B_{\Lambda p})$, with a Lorentz transformation $T = B_p^{-1}\Lambda^{-1}B_{\Lambda p}$ which acts on the time Killing vector as follows,

$$B_p^{-1}\Lambda^{-1}B_{\Lambda p} m_\lambda e_t = B_p^{-1}\Lambda^{-1}(\Lambda p) = B_p^{-1}p = m_\lambda e_t \tag{14}$$

so T preserves the time direction. Therefore, T must be a rotation (see Lemma V.1) so $T = R_\Lambda$. Therefore,

$$B_{\Lambda p} = \Lambda B_p R_\Lambda \tag{15}$$

This implies that,

$$f(\Lambda p) = S(\Lambda)S(B_p)S(R_\Lambda)S^\dagger(R_\Lambda)S^\dagger(B_p)S^\dagger(\Lambda) \tag{16a}$$

$$= S(\Lambda)S(B_p)S^\dagger(B_p)S^\dagger(\Lambda) \tag{16b}$$

$$= S(\Lambda)\cdot f(p)\cdot S^\dagger(\Lambda) \tag{16c}$$

Where I have used the fact that $\mathrm{SO}(3) \subset \mathrm{SO}(1,3)$ is compact and thus we may choose our representation restricted to this compact subgroup $S|_{\mathrm{SO}(3)}$ to be unitary. We have therefore, shown that V transforms in the $W \otimes \overline{W}$ representation which is finite and decomposes into irreducible $\mathrm{SO}(1,3)$-reps as,

$$W\otimes\overline{W} = (\mu,\nu)\otimes(\nu,\mu) = \sum_{|\mu-\nu|\le j_+,j_+\le\mu+\nu} (j_+,j_-) \tag{17}$$

However, V is a representation consisting of Lorentz covariant functions and therefore by general theory V decomposes as a sum of irreducible harmonic functions on the hyperbola in representations (J,J) for $J \in \{0,\frac{1}{2},1,\frac{3}{2},\cdots\}$ so we in fact have only the following direct summands in the decomposition of V,

$$V \hookrightarrow \bigoplus_{|\mu-\nu|\le J\le\mu+\nu} (J,J) \tag{18}$$

This implies the following harmonic decomposition,

$$f_{ij}(p^\mu) = \sum_{J=|\mu-\nu|}^{\mu+\nu} f_J \sum_{-J\le m_+,m_-\le J} \begin{pmatrix} & i & j \\ J & m_+ & m_- \end{pmatrix} m_\lambda^{2J}\mathcal{Y}^J_{m_+,m_-}(p^\mu/m_\lambda) \tag{19}$$

where these symbols are the generalizations of the Clebsch-Gordon decomposition coefficients. The harmonics $\mathcal{Y}^J_{m_+,m_-}(x)$ are homogeneous polynomials of degree $2J$ [6]. Therefore, we may analytically continue p off-shell i.e. over $H_{m_\lambda} \subset \mathbb{R}^{1,3} \subset \mathbb{C}^4$ to both arbitrary real and complex momenta by the definition,

$$f_{ij}(p) = \sum_{J=|\mu-\nu|}^{\mu+\nu} f_J \sum_{m_+,m_-} \begin{pmatrix} i & j \\ J\, m_+ & m_- \end{pmatrix} \mathcal{Y}^J_{m_+,m_-}(p) \quad (20)$$

Now, since $\mathcal{Y}^J_{m_+,m_-}$ has degree $2J$ we have $\mathcal{Y}^J_{m_+,m_-}(-p) = (-1)^{2J}(p)$ and furthermore, since $|\mu-\nu| \leq J \leq \mu+\nu$ indexes by one, each possible $2J$ has the same parity as $2(\mu+\nu) = 2s$ where s is the spin of the representation W. This follows from the fact that,

$$\hat{J}_+ + \hat{J}_- = \hat{J} \quad (21)$$

so when W restricts to $\mathrm{SO}(3) \hookrightarrow \mathrm{SO}(1,3)$ it decomposes into irreps with spin $|\mu-\nu| \leq s \leq \mu+\nu$. Therefore, we again have that $(-1)^{2J} = (-1)^{2s} = (-1)^{2(\mu+\nu)}$ where $2s$ gives the parity for the possible spin states of the particle. Applying this discussion,

$$f_{ij}(-p) = \sum_{J=|\mu-\nu|}^{\mu+\nu} f_J \sum_{m_+,m_-} \begin{pmatrix} i & j \\ J\, m_+ & m_- \end{pmatrix} \mathcal{Y}^J_{m_+,m_-}(-p) \quad (22a)$$

$$= \sum_{J=|\mu-\nu|}^{\mu+\nu} f_J \sum_{m_+,m_-} \begin{pmatrix} i & j \\ J\, m_+ & m_- \end{pmatrix} (-1)^{2J}\, \mathcal{Y}^J_{m_+,m_-}(p) \quad (22b)$$

$$= (-1)^{2s} f_{ij}(p) \quad (22c)$$

4. Completing the Proof

The derived relations hold for $f_{ij}(p)$ off-shell so we need to modify the two-point correlation functions to take p^μ off-shell. Using the polynomial nature of the functions $f_{ij}(p)$,

$$\langle\Omega|\, \hat{\phi}_i(x)\hat{\phi}^\dagger_j(y)\, |\Omega\rangle = \sum_\lambda Z_\lambda\, f_{ij}(+i\partial_x) D_\lambda(x-y) \quad (23a)$$

$$\langle\Omega|\, \hat{\phi}^\dagger_j(y)\hat{\phi}_i(x)\, |\Omega\rangle = \sum_{\bar{\lambda}} Z_{\bar{\lambda}}\, f_{ij}(-i\partial_x) D_{\bar{\lambda}}(y-x) \quad (23b)$$

where,

$$D_\lambda(x-y) = \int\limits_{p^2=m_\lambda^2} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{-ip\cdot(x-y)}}{2p_0} \quad (24)$$

When x and y are space-like separated i.e. $(x-y)^2 < 0$ then $D(x-y) = D(y-x)$ [7, p.28]. In this case, applying our result of the previous section,

$$\langle\Omega|\, \hat{\phi}^\dagger_j(y)\hat{\phi}_i(x)\, |\Omega\rangle = \sum_\lambda Z_\lambda\, f_{ij}(-i\partial_x) D_\lambda(x-y) \quad (25a)$$

$$= (-1)^{2s} \sum_{\bar{\lambda}} Z_{\bar{\lambda}}\, f_{ij}(+i\partial_x) D_{\bar{\lambda}}(x-y) \quad (25b)$$

From relativistic causality arguments we know that these field operators must either commute or anti-commute. Since $Z_\lambda \geq 0$ the value of $(-1)^{2s}$ fixes the sign of this relation proving our theorem. In fact, we see slightly more. In order that these correlation functions actually commute or anti-commute, there must be a spectral symmetry $\lambda \leftrightarrow \bar{\lambda}$ reversing charge but preserving mass and spin. Thus, the spin-statistics relationship also establishes the existence and spectral symmetry of anti-particles.

IV. EXTENSIONS

The SST is a remarkable theorem that is essential in all of quantum mechanics and the discovery of new phenomena. We present here how it can easily lead to the CPT Theorem and the existence of anyons.

A. CPT Theorem

The CPT Theorem states that given the operators charge conjugation C, parity inversion P, and time reversal T, their combination CPT is a symmetry of any relativistic QFT. It too has an abundance of proofs from Lüders [8], Pauli [9], and Jost [4], the last of whom proved it in the context of axiomatic QFT. It is important to note that both the CPT and SST are distinct theorems that can be independently proved using only the axioms mentioned in Section I. We illustrate how the CPT emerges out of the SST.

The CPT action for the case of a scalar field, photon spin 1 field, and spin 1/2 field is,[10] $\phi(x) \to \phi^\dagger(-x), A_\mu(x) \to -A^\dagger_\mu(-x), \psi \to -\gamma_5\psi^{\dagger T}(-x)$. Thus, has a relativistic QFT Lagrangian is a Lorentz scalar, we need only consider the various bilinear spinor forms such that they have zero, one, or two Lorentz indices. We show only one example of a CPT transformation of a Weyl and Dirac spinor. $\overline{\chi}(x)\psi(x) \to \chi^T(-x)\gamma_5\gamma^{0*}\gamma_5\psi^{\dagger T}(-x) = -\chi^T(-x)\gamma^{0*}\psi^{\dagger T}(-x) = -\chi^{\dagger T\dagger}\gamma^{0*}\psi^{\dagger T}(-x) = -\left[\psi^T(-x)\gamma^{0T}\chi^{T\dagger}(-x)\right]^\dagger = \left[\left\{\chi^\dagger(-x)\gamma^0\psi(-x)\right\}\right]^\dagger = (\overline{\chi}(-x)\psi(-x))^\dagger$ where we note that we have used the SST in anticommuting the fermion fields. Checking every combination of spinor bilinear form suffices to show that every relativistic QFT preserves CPT symmetry.

B. Anyons

The proof above only considers the case $d = 4$, i.e. the case of 3 spatial dimensions. This case is the most applicable since most physical systems have 3 spatial dimensions. However, there are also some examples of 2-dimensional materials, where the system is so constrained in one of the three dimensions that the particles within the system are best modeled with a 2+1 dimensional field theory. This leads us to consider how the Spin-Statistics theorem for $d = 3$.

The key idea in generalizing the spin-statistics theorem to $d = 3$ is that we have an equation

$$\phi_i^\dagger(x)\phi_j(y) = (-1)^{2j}\phi_j(x)\phi_i^\dagger(y) \tag{26}$$

Where the quantity j is related to the representation theory of $\mathrm{SO}(3) = \mathrm{SO}(d-1)$ for $d = 4$. In the case of $d = 4$ the allowed values of j are $j = \frac{1}{2}n$ for $n \in \mathbb{Z}_{>0}$, meaning that $(-1)^{2j}$ can be either 1 (bosons) or -1 (fermions). However, for $d = 3$ (i.e. 2 spatial dimensions) the story is quite different. Here $\mathrm{SO}(d-1) = \mathrm{SO}(2) = S^1$ is the circle group, whose universal cover is $\mathbb{R}$. This means that any $j \in \mathbb{R}$ is possible, allowing any possible phase to appear in (26). Thus in some sense 2 dimensional particles can pick up any phase when swapping two identical particles, and hence these particles are called “anyons.”

An interesting note here is the picture of anyons only works in a nonrelativistic regime; if the particles we’re studying are relativistic then we must consider the full representations of $\mathrm{SO}(1, 2)$, for which the quantization of $(-1)^{2j}$ to ± 1 still holds.[11][6]

V. FORMALISM

We present an introduction to the representation theory relevant to the proof of the SST. More information can be found in the literature.[3][5][12]

A. Definition of a particle

We give three equivalent definitions of a particle: (1) as a projective representation of $\mathrm{SO}(1, 3)$, (2) as a representation of the universal cover of $\mathrm{SO}(1, 3)$, namely $\mathrm{SL}(2, \mathbb{C})$, and (3) as a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $\mathrm{SO}(1, 3)$. We will show later (Section V C) that this leads to a classification of all possible irreducible representations by two integers or half integers j_+ and j_-. The sum $j = j_+ + j_-$ classifies the total spin of the particle; for massive particles j is the spin corresponding projective $\mathrm{SO}(3)$ representation obtained by restricting to the rotations of the particle in its rest frame. Note that j will be either an integer or a half-integer; analogously, $2j$ will be either even or odd, yielding the phase factor $(-1)^{2j}$ which is $+1$ for particles with integral spin and -1 for particles with half-integral spin.

B. Addition of Angular Momentum

The addition of angular momentum is a standard representation-theoretic fact from quantum mechanics which tells us how the tensor product of two representations of $\mathrm{SU}(2)$ decomposes into irreducible representations. In particular, if we have representations (j_1) and (j_2) with spin j_1 and j_2 respectively, then we have

$$(j_1)\otimes(j_2) = (|j_1-j_2|)\oplus(|j_1-j_2|+1)\oplus\cdots\oplus(j_1+j_2) \tag{27}$$

One can easily see that every term on the right-hand side has $(-1)^{2j} = (-1)^{2j_1}(-1)^{2j_2}$; this means that the phase factor $(-1)^{2j}$ can still be defined on the right-hand-side, even though the representation isn’t irreducible. Following this procedure we can define the phase factor $(-1)^{2j}$ for any composite of particles. For instance, this allows us to apply the spin-statistics theorem to an atom, even if the atom is not living in an single irreducible representation of the rotation group.

C. Representations of the Lorentz Group

The inhomogenous SL(2,$\mathbb{C}$) group is the universal, double cover of the proper orthochronous Lorentz group $SO^+(1,3)$. Recall that proper means orientation preserving $\det\Lambda = 1$ and orthochronous means that the forward light cone is mapped to itself. We may then construct a vector representation of the universal cover of the Poincare group with translations a and homogenous Lorentz transformations Λ_i, $\{a, \Lambda_i\} \in \mathrm{SO}^+(1, 3)$,

$$\{a, \Lambda_i\}\{b, \Lambda_j\} = \{a + b, \Lambda_i\Lambda_j\}$$

The Lie algebra of the Lorentz group is given by generators J_i, $i = 1, 2, 3$ (corresponding to rotations), and K_i, $i = 1, 2, 3$ (corresponding to boosts), with the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{28}$$
$$[J_i, K_J] = i\epsilon_{ijk}K_k \tag{29}$$
$$[K_i, K_J] = -i\epsilon_{ijk}J_k \tag{30}$$

The classification of finite irreducible representations of this Lie algebra can be obtained by considering the following operators:

$$J_i^+ = \tfrac{1}{2}(J_i + iK_i) \tag{31}$$
$$J_i^- = \tfrac{1}{2}(J_i - iK_i) \tag{32}$$

Calculating the commutator of $J_i^\pm$ with itself one finds:

$$[J_i^\pm, J_j^\pm] = i\epsilon_{ijk}J_k^\pm \tag{33}$$
$$[J_i^\pm, J_j^\mp] = 0 \tag{34}$$

Therefore, we identify the complexification of this Lie algebra with a direct sum of two copies of $\mathfrak{su}(2)$. Such an identification classifies all irreducible projective representations of $\mathrm{SO}(1, 3)$ by pairs of irreducible $\mathrm{SU}(2)$ representations (we need not write projective here since $\mathrm{SU}(2)$ is simply connected). We write (μ, ν) for the representation with J^+ in the spin μ irrep and J^- in the spin ν irrep. This defines an irreducible projective $\mathrm{SO}(1, 3)$-rep $\pi_{\mu,\nu} : \mathrm{SO}(1, 3) \to \mathrm{GL}(\mathbb{C}^{(2\mu+1)(2\nu+1)})$. It turns out that this argument classifies all finite dimensional projective irreps of the Lorentz group.

We also need a few other representation-theoretic facts:

Lemma V.1. If a Lorentz transformation $\Lambda \in \mathrm{SO}(1,3)$ preserves the temporal Killing vector e_t then Λ is a rotation.

Proof. We can write any four vector in the form $x^\mu = x^0 e_t + \vec{x}$. Then, $\Lambda x^\mu = x^0 e_t + \Lambda\vec{x}$. Furthermore $e_t \cdot \vec{x} = 0$ and Λ preserves the Minkowski inner product so $e_t \cdot \Lambda\vec{x}$ and thus $\vec{x}$ is also a vector. Furthermore, Λ preserves $(x^0)^2 - \vec{x}^2$ so $\Lambda\vec{x}$ must be a vector with $|\Lambda\vec{x}|^2 = |\vec{x}|^2$ so Λ must be a rotation since $\det\Lambda = 1$. □

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