MAT237 Homework 2.3

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Question 1

First I'll prove the following claim:

Claim: $\chi_{T_j}(\mathbf{u}) = \chi_{R_j}(G(\mathbf{u}))$
where T_j is the pre-image of R_j under $\mathbf{G}$, for $j = 1, \ldots, J$
proof: Let $\mathbf{u} \in T_j$, then $\mathbf{G}(\mathbf{u}) \in R_j$ (By the property of function $\mathbf{G}$)
Then

$$\begin{aligned}
\chi_{T_j}(\mathbf{u}) &= \begin{cases} 1 & \mathbf{u} \in T_j \\ 0 & \mathbf{u} \notin T_j \end{cases} \\
&= \begin{cases} 1 & \mathbf{G}(\mathbf{u}) \in R_j \\ 0 & \mathbf{G}(\mathbf{u}) \notin R_j \end{cases} \\
&= \chi_{R_j}(\mathbf{G}(\mathbf{u}))
\end{aligned}$$

$\blacksquare$

Next I'll prove that the Change of Variables formula holds if f has the form $f(\mathbf{x}) = \sum_{j=1}^{J} a_j \chi_{R_j}(\mathbf{x})$, where $a_j \in \mathbb{R}$ and R_j is a rectangle in R, for $j = 1, \ldots, J$

$$
\begin{aligned}
\iint_R f(\mathbf{x}) d^2\mathbf{x} &= \iint_R \sum_{j=1}^{J} a_j \chi_{R_j}(\mathbf{x})\, d^2\mathbf{x} \\
&= \sum_{j=1}^{J} \iint_R a_j \chi_{R_j}(\mathbf{x})\, d^2\mathbf{x} \\
&= \sum_{j=1}^{J} a_j \iint_{R_j} d^2\mathbf{x} \\
&= \sum_{j=1}^{J} a_j \iint_{T_j} |\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} \qquad \text{(By assumption (2))} \\
&= \sum_{j=1}^{J} a_j \iint_{T} \chi_{T_j}(\mathbf{u}) |\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} \\
&= \sum_{j=1}^{J} a_j \iint_{T} \chi_{R_j}(\mathbf{G}(\mathbf{u})) |\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} \qquad \text{(by Claim)} \\
&= \iint_{T} \sum_{j=1}^{J} a_j \chi_{R_j}(\mathbf{G}(\mathbf{u})) |\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} \\
&= \iint_{T} f(\mathbf{G}(\mathbf{u}))\, |\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u}
\end{aligned}
$$

as we wanted. ■

Question 2

Formulas

1. $f_P(\mathbf{x}) = \sum_{j=1}^{J} m_j \chi_{R_j}(\mathbf{x})$, where $m_j := \inf\{f(\bar{\mathbf{x}}) : \bar{\mathbf{x}} \in R_j\}$

2. $F_P(\mathbf{x}) = \sum_{j=1}^{J} M_j \chi_{R_j}(\mathbf{x})$, where $M_j := \sup\{f(\bar{\mathbf{x}}) : \bar{\mathbf{x}} \in R_j\}$

Proof

that f_P and F_P has the properties:

Step 1 It is obvious that both functions have the form $\sum_{j=1}^{J} a_j \chi_{R_j}$, where every R_j is a rectangle of P

Step 2 WTS: $f_P(\mathbf{x}) \leq f(\mathbf{x}) \leq F_P(\mathbf{x})$ for all $\mathbf{x} \in R$

Let $\mathbf{x} \in R$, then $\exists R_i \in \{R_1, \ldots, R_J\}$ s.t. $\mathbf{x} \in R_i$

$$\begin{aligned} f_P(\mathbf{x}) &= \sum_{j=1}^{J} m_j \chi_{R_j}(\mathbf{x}) \\ &= m_i && (\text{Since } \mathbf{x} \in R_i) \\ &\leq f(\mathbf{x}) && (\text{Since } m_i \leq f(\mathbf{x})) \end{aligned}$$

The proof of $f(\mathbf{x}) \leq F_P(\mathbf{x})$ is almost identical.

Step 3 WTS: $\iint_R f_P \, dA = s_P f$, $\iint_R F_P \, dA = S_P f$

Since the number of rectangles R_j is finite, then f_P is discontinuous on a set of zero content, hence f_P is integrable.

Hence

$$\begin{aligned}
\iint_R f_P \, dA &= \iint_R \sum_{j=1}^{J} m_j \chi_{R_j} \, dA \\
&= \sum_{j=1}^{J} \iint_R m_j \chi_{R_j} \, dA \\
&= \sum_{j=1}^{J} \iint_{R_j} m_j \, dA \\
&= \sum_{j=1}^{J} Area(R_j) \cdot m_j \\
&= s_P f
\end{aligned}$$

The proof of $\iint_R F_P \, dA = S_P f$ is almost identical. ■

Question 3

WTS: $\forall$ partition P of R,

$$s_P f \leq \iint_T f(\mathbf{G}(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u}$$

$$\begin{aligned}
s_p f &= \iint_R f_P \, dA && \text{(by Q2)} \\
&= \iint_R f_P(\mathbf{x}) \, d^2\mathbf{x} \\
&= \iint_T f_P(G(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} && \text{(by Q1)} \\
&\leq \iint_T f(G(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} \\
& \qquad \text{(by online notes S4.2 Theorem 1.3 since } f_P \leq f\text{)}
\end{aligned}$$

The proof of $\iint_T f(\mathbf{G}(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} \leq S_P f$ is almost identical. ■

Question 4

Suppose f and $\mathbf{G}$ are functions satisfying all the hypothesis of Theorem 1.
Let P be a partition of R, then by Question 3,

$$s_P f \leq \iint_T f(\mathbf{G}(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} \leq S_P f$$

Since P is arbitrary, then

$$\iint_R f(\mathbf{x})\, d^2\mathbf{x} = \sup_P s_P f \leq \iint_T f(\mathbf{G}(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u} \leq \inf_P S_P f = \iint_R f(\mathbf{x})\, d^2\mathbf{x}$$

$$\implies \iint_R f(\mathbf{x})\, d^2\mathbf{x} = \iint_T f(\mathbf{G}(\mathbf{u}))|\det D(\mathbf{G}(\mathbf{u})|\, d^2\mathbf{u}$$

■