MAT224 Linear Algebra II
Lecture Notes

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Contents

1 Vector Spaces **2**

1.1 Bases And Dimension (Jan 17) 2

2 Linear Transformations **6**

2.1 Linear Tranformations 6

2.2 Linear Transformations Between Finite-Dimensional Vector Spaces 7

2.3 Kernel and Image 9

1 Vector Spaces

1.1 Bases And Dimension (Jan 17)

Definition A subset S of vector space V is called a *basis* of V if V = Span(S) and S is linearly independent.

Examples

1. the standard basis S = $\{e_1,...,e_n\}$ in $\mathbb{R}^n$, since every vector $(a_1,...,a_n) \in \mathbb{R}^n$ may be written as the linear combination $(a_1,...,a_n) = a_1e_1 + ... + a_ne_n$

2. The vector space $\mathbb{R}^n$ has many other bases as well. e.g., in $\mathbb{R}^2$, consider the set $S = \{(1,2),(1,-1)\}$, which is l.i.

3. Let $V = P_n(\mathbb{R})$ and consider $S = \{1, x, x^2, ..., x^n\}$, which is a basis of V.
   proof: It is clear that S spans V. For independence, consider
   $$a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n = \mathbf{0}$$
   Take the derivative of both sides,
   $$\frac{d^n}{dx^n}(a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n) = \frac{d^n}{dx^n}(0)$$
   $$n!a_n = 0 \implies a_n = 0$$
   Similarly, we have $a_i = 0$ for all i, as wanted.

4. The empty subset, $\emptyset$, is a basis of the vector space consisting only of a zero vector, $\{\mathbf{0}\}$.

Theorem 1.6.3 Let V be a vector space, and let S be a nonempty subset of V. Then S is a basis of V iff every vector $\mathbf{x} \in V$ may be written uniquely as a linear combination of the vectors in S.
Proof: $\rightarrow$: Assume S is a basis of V, then given $\mathbf{x} \in V$, there are scalars $a_i \in \mathbb{R}$ and vectors $x_i \in S$ s.t. $\mathbf{x} = a_1x_1 + ... + a_nx_n$. To show this linear combination is unique, consider a possible second linear combination of vectors in S which also adds up to $\mathbf{x}$: $x = b_1x_1 + ... + b_nx_n$. Subtracting these two expressions for $\mathbf{x}$, we find that

$$\mathbf{0} = a_1x_1 + ... + a_nx_n - (b_1x_1 + ... + b_nx_n)$$

$$= (a_1 - b_1)x_1 + ... + (a_n - b_n)x_n$$

Since S is linearly independent, the equation implies that $a_i = b_i$ for all i.

$\leftarrow$: Assume every vector $\mathbf{x} \in V$ may be written uniquely as a linear combination of the vectors in S. This implies $Span(S) = V$. We must show that S is l.i. Consider an equation

$$a_1x_1 + ... + a_nx_n = \mathbf{0}$$

Note that it is also the case that

$$0\mathbf{x} = 0(x_1 + ... + x_n) = \mathbf{0}$$

Since we assumed that every $\mathbf{x}$ has a unique representation in S, then it must be true that $a_i = 0$ for all i. Hence S is l.i.

Theorem 1.6.6 Let V be a vector space that has a finite spanning set, and let S be a linearly independent subset of V. Then there exists a basis S' of V, with $S \subset S'$

Lemma 1.6.8 Let S be a linearly independent subset of V and let $x \in V$, but $x \notin S$. Then $S \cup \{\mathbf{x}\}$ is l.i. iff $\mathbf{x} \notin Span(S)$.

Insight the number of vectors in a basis is, in a rough sense, a measure of "how big" the space is.

Theorem 1.6.10 (Basis Theorem) Let V be a vector space and let S be a spanning set for V, which has m elements. Then no linearly independent set in V can have more than m elements.
proof: It suffices to show that every set in V with more than m elements is linearly dependent. Write $S = y_1, ..., y_m$ and suppose $S' = x_1, ..., x_n$ is a subset of V with $n > m$ vectors. Consider an equation

$$(1) a_1x_1 + ... + a_nx_n = \mathbf{0}$$

Our goal is to show that a_i not all 0. Since S spans V, there are scalars b_{ij} s.t. for each i,

$$x_i = b_{i1}y_1 + ... + b_{im}y_m$$

Substituting these equations into (1), we get

$$a_1(b_{11}y_1 + ... + b_{1m}y_m) + ... + a_n(b_{n1}y_1 + ... + b_{nm}y_m) = \mathbf{0}$$

Collecting terms and rearranging,

$$(a_1b_{11} + ... + a_nb_{n1})y_1 + ... + (a_1b_{1m} + ... + a_nb_{nm})y_m = \mathbf{0}$$

Since S is l.i., this is equivalent to solving the system

$$b_{11}a_1 + ... + b_{n1}a_n = 0$$

$$.$$

$$.$$

$$.$$

$$b_{1m}a_1 + ... + b_{nm}a_n = 0$$

But this is a system with n unknowns and m equations and $n > m$, so there must exist a non-trivial solution $\{a_1, ..., a_n\}$, which is what we wanted to show. QED

Corollary 1.6.11 Let V be a vector space and let S and S' be two bases of V, with m and m' elements, respectively. Then m = m'.
proof:
Since S is a spanning set of V and S' is l.i., we have $m' \leq m$. Since S' is a spanning set of V and S is l.i.m we have $m \leq m'$. Hence $m = m'$. QED

Definitions 1.6.12

1. If V is a vector space with some finite basis(possibly empty), we say V is *finite-dimentional*.
2. Let V be a finite-dimensional vector space. The dimension of V, denoted dim(V), is the number of vectors in a (hence any) basis of V.
3. If $V = \{\mathbf{0}\}$, we define dim(V) = 0.
4. dim($span\{(1,2,3),(4,5,6),(7,8,9)\}$) = 2

Examples

1. For each n, dim($\mathbb{R}^n$) = n, since the standard basis contains n vectors.
2. dim($P_n(\mathbb{R})$) = $n + 1$, since a basis for $P_n(\mathbb{R})$ contains n + 1 functions.
3. The vector spaces $P(\mathbb{R}), C^1(\mathbb{R})$ and $C(\mathbb{R})$ are not finite-dimensional. We say that such spaces are *infinite-dimentional*.

Corollary 1.6.14 Let W be a subspace of a finite-dimensional vector space V. Then $dim(W) \leq dim(V)$. Furthermore, $dim(W) = dim(V)$ iff $W = V$.

Corollary 1.6.15 Let W be a subspace of $\mathbb{R}^n$ defined by a system of homogeneous linear equations. Then dim(W) is equal to the number of free variables in the corresponding echelon form system.

Theorem 1.6.18 Let W_1 and W_2 be finite-dimensional subspaces of a vector space V. Then

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$$

Remark Analogous to the Principle of Inclusion-Exclusion
proof: Result obvious if either W_1 or W_2 is $\{\mathbf{0}\}$.
Therefore, we assume that neither W_1 nor W_2 is $\{\mathbf{0}\}$. Starting from a basis S of $W_1 \cap W_2$. We can always find sets T_1 and T_2 (disjoint from S) such that $S \cup T_1$ is a basis for W_1 and $S \cup T_2$ is a basis for W_2. We claim that $U = S \cup T_1 \cup T_2$ is a basis for $W_1 + W_2$, since

$$U = S \cup T_1 \cup T_2 = (S \cup T_1) \cup (S \cup T_2)$$

$$Span(U) = Span((S \cup T_1) \cup (S \cup T_2)) = W_1 + W_2$$

It remains to prove that U is linearly independent. Any potential linear dependence among the vectors in U must have the form

$$\mathbf{v} + \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}$$

where $\mathbf{v} \in Span(S) = W_1 \cap W_2, \mathbf{w}_1 \in Span(T_1) \subset W_1, \mathbf{w}_2 \in Span(T_2) \subset W_2$. (slice the linear combination into the sum of the vectors from 3 vector spaces). It suffices to prove that in any such potential linear dependence, we must have $\mathbf{v} = \mathbf{w}_1 = \mathbf{w}_2 = \mathbf{0}$ (each vector is a lin comb, and equals $\mathbf{0}$). Consider $\mathbf{w}_2 = -\mathbf{v} - \mathbf{w}_1$. Since $-\mathbf{v} - \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$, we must have $\mathbf{w}_2 \in W_1 \cap W_2$. By definition, $\mathbf{w}_2 \in Span(T_2)$ But $S \cap T_2 = \emptyset$, hence $Span(S) \cap Span(T_2) = \{\mathbf{0}\}$. Therefore we must have $\mathbf{w}_2 = \mathbf{0}$. So then $-\mathbf{v} = \mathbf{w}_1 \in W_1 \cap W_2$. Since $S \cap T_1 = \emptyset$, $Span(S) \cap Span(T_1) = \{\mathbf{0}\}$ and we have $\mathbf{w}_1 = \mathbf{0}$, so $\mathbf{v} = \mathbf{0}$ as well. QED

Excercises for 1.4 1.(k), 7

Exercises for 1.6 1.(d)(e)(f), 3, 4, 16

2 Linear Transformations

2.1 Linear Tranformations

A function T from V to W is denoted by $T : V \to W$. The vector $\mathbf{w} = T(\mathbf{v})$ in W is called the image of $\mathbf{v}$ under the function T. Loosely speaking, we want our functions to turn the algebraic operations of addition and scalar multiplication in V into addition and scalar multiplication in W.

Definition 2.1.1 A function $T : V \to W$ is called a *linear mapping* or a *linear transformation* if it satisfies

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v} \in$ V
2. $T(a\mathbf{v}) = aT(\mathbf{v})$ for all $a \in \mathbb{R}$ and $\mathbf{v} \in V$

V is called the *domain* of T and W is called the *target* of T.
We say that a linear transformation preserves the operations of addition and scalar multiplication.

Property A linear mapping always takes the zero vector in the domain vector space to the zero vector in the target vector space.

Proposition 2.1.2 A function $T : V \to W$ is a linear transformation if and only if for all a and $b \in \mathbb{R}$ and all $\mathbf{u}$ and $\mathbf{v} \in V$

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

Corollary 2.1.3 A function $T : V \to W$ is a linear transformation if and only if for all $a_1, .., a_k \in \mathbb{R}$ and for all $\mathbf{v}_1, ..., \mathbf{v}_k \in V$:

$$T(\sum_{i=1}^{k} a_i\mathbf{v}_i) = \sum_{i=1}^{k} a_iT(\mathbf{v}_i)$$

Examples

1. Let V be any vector space, and let W = V. The *underidentity transformation* $I : V \to V$ is defined by I($\mathbf{v}$) = $\mathbf{v}$ for all $\mathbf{v} \in$ V.
2. Let V and W be any vector spaces, and let $T : V \to W$ be the mapping that takes every vector in V to the zero vector in W:

$$T(\mathbf{v}) = \mathbf{0}_W$$

for all $\mathbf{v} \in$ V. T is called <u>*zero transformation*</u>.

3. $T(\boldsymbol{x}) = a_1x_1 + ... + a_nx_n$
4. *Differentiation, definite integration*

Remark The inner product plays a crucial role in linear algebra in that it provides a bridge between algebra and geometry, which is the heart of the more advanced material that appears later in the text.

Proposition 2.1.14 If $T : V \to W$ is a linear transformation and V is finite-dimensional, then T is uniquely determined by its values on the members of a basis of V.
<u>*proof:*</u> Show that if S and T are linear transformations that take the same values on each member of a basis for V, then in fact S = T.

$$\begin{aligned} T(v) &= T(a_1v_1 + ... + a_kv_k) \\ &= a_1T(v_1) + ... + a_kT(v_k) \\ &= a_1S(v_1) + ... + a_kS(v_k) \\ &= S(a_1v_1 + ... + a_kv_k) \\ &= S(v) \end{aligned}$$

Therefore, S and T are equal as mappings from V to W. ■

2.2 Linear Transformations Between Finite-Dimensional Vector Spaces

Proposition 2.2.1 Let $T : V \to W$ be a linear transformation between the finite-dimensional vector spaces V and W. If $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a basis for V and $\{\mathbf{w}_1, ..., \mathbf{w}_l\}$ is a basis for W, then $T : V \to W$ is uniquely determined by the $l \cdot k$ scalars used to express $T(\mathbf{v}_j), j = 1, ..., k$, in terms of $\mathbf{w}_1, ..., \mathbf{w}_l$.

Definition 2.2.6 Let $T : V \to W$ be a linear transformation between the finite-dimensional vector spaces V and W, and let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ and $\beta = \{\mathbf{w}_1, ..., \mathbf{w}_l\}$, respectively, be any bases for V and W. Let $a_ij, 1 \leq i \leq l$ and $1 \leq j \leq k$ be the $l \cdot k$ scalars that determine T with respect to the bases α and β. The matrix whose entries are the scalars a_ij, $1 \leq i \leq l$ and $1 \leq j \leq k$, is called the *matrix of the linear transformation T with respect to the bases α for V and β for W*. This matrix is denoted by $[T]_\alpha^\beta$.

Remark The basis vectors in the domain and target spaces are written in some particular order.

Definition of coordinate vectors If $\mathbf{v} = a_1\mathbf{v}_1 + ... + a_k\mathbf{v}_k$ and $\mathbf{w} = b_1\mathbf{w}_1 + ... + b_l\mathbf{w}_l$, we can express $\mathbf{v}$ and $\mathbf{w}$ in coordinates, respectively, as a $k \times 1$ matrix and as an $l \times 1$ matrix, with respect to the chosen bases. These coordinate vectors will be denoted by $[\mathbf{v}]_\alpha$ and $[\mathbf{w}]_\beta$, respectively. Thus $[\mathbf{v}]_\alpha = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$ and $[\mathbf{w}]_\beta = \begin{bmatrix} b_1 \\ \vdots \\ b_l \end{bmatrix}$

Proposition 2.2.15 Let $T : V \to W$ be a linear transformation between vector spaces V of dimension k and W of dimension l. Let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for V and $\beta = \{\mathbf{w}_1, ..., \mathbf{w}_l\}$ be a basis for W. Then for each $\mathbf{v} \in V$,

$$[T(\mathbf{v})]_\beta = [T]_\alpha^\beta [\mathbf{v}]_\alpha$$

proof: Let $\mathbf{v} = x_1\mathbf{v}_1 + ... + x_k\mathbf{v}_k \in V$. Then if $T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + ... + a_{lj}\mathbf{w}_l$

$$\begin{aligned} T(\mathbf{v}) &= \sum_{j=1}^{k} x_j T(\mathbf{v}_j) \\ &= \sum_{j=1}^{k} x_j (\sum_{i=1}^{l} a_{ij}\mathbf{w}_i) \\ &= \sum_{i=1}^{l} (\sum_{j=1}^{k} x_j a_{ij}) \mathbf{w}_i \end{aligned}$$

Thus, the ith coefficient of T($\mathbf{v}$) in terms of β is $\sum_{j=1}^{k} x_j a_{ij}$ and $[T(\mathbf{v})]_\beta = \begin{bmatrix} \sum_{j=1}^{k} x_j a_{1j} \\ \vdots \\ \sum_{j=1}^{k} x_j a_{lj} \end{bmatrix}$ which is precisely $[T(\mathbf{v})]_\beta = [T]_\alpha^\beta [\mathbf{v}]_\alpha$. ■

Proposition 2.2.19 Let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for V and $\beta = \{\mathbf{w}_1, ..., \mathbf{w}_l\}$ be a basis for W, and let $\mathbf{v} = x_1\mathbf{v}_1 + ... + x_k\mathbf{v}_k \in V$

1. If A is an $l \times k$ matrix, then the function

$$T(\mathbf{v}) = \mathbf{w}$$

where $[\mathbf{w}]_\beta = A[\mathbf{v}]_\alpha$ is a linear transformation.

2. If $A = [S]_\alpha^\beta$ is the matrix of a transformation $S : V \to W$, then the transformation T constructed from $[S]_\alpha^\beta$ is equal to S.

3. If T is the transformation of (1) constructed from A, then $[T]_\alpha^\beta = A$

Proposition 2.2.20 Let V and W be finite-dimensional vector spaces. Let α be a basis for V and β a basis for W. Then the assignment of a matrix to a linear transformation from V to W given by T goes to $[T]_\alpha^\beta$ is injective and surjective.

Notes

1. When proving a function T is not a linear transformation, can consider $T(\mathbf{0}) \neq \mathbf{0}$.

2.3 Kernel and Image

Definition 2.3.1 The *kernel* of T, denoted $Ker(T)$, is the subset of V consisting of all vectors $\mathbf{v} \in V$ such that $T(\mathbf{v}) = 0$.

Proposition 2.3.2 Let $T : V \to W$ be a linear transformation. Ker(T) is a subspace of V.

Examples

1. Let $V = P_3(\mathbb{R})$. Define $T : V \to V$ by $T(p(x)) = \frac{d}{dx}p(x)$. Ker(T) only consists constant polynomials.

2. Let $V = W = \mathbb{R}^2$. Let T be a rotation R_θ. Then $Ker(T) = \{\mathbf{0}\}$.

Proposition 2.3.7 Let $T : V \to W$ be a linear transformation of finite-dimensional vector spaces, and let α and β be bases for V and W, respectively. Then $\mathbf{x} \in Ker(T)$ ife lf the coordinate vector of $\mathbf{x}$, $[\mathbf{x}]_\alpha$, satisfies the system of equations

$$a_{11}x_1 + ... + a_{1k}x_k = 0$$

$$\vdots$$

$$a_{l1}x_1 + ... + a_{lk}x_k = 0$$

where the coefficients a_{ij} are the entries of the matrix $[T]_\alpha^\beta$.

Proposition 2.3.8 Let V be a finite-dimensional vector space, and let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for V. Then the vectors $\mathbf{x}_1, ..., \mathbf{x}_m \in V$ are linearly independent iff there corresponding coordinate vectors $[\mathbf{x}_1]_\alpha, ..., [\mathbf{x}_m]_\alpha$ are linearly independent.