STA261 Probability and Statistics II
Lecture Notes

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1 Normal Distribution Theory

Theorem: Sum of independent normal random variables Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, ..., n$ and that they are independent random variables. Let $Y = (\Sigma_i a_i X_i) + b$ for some constants $\{a_i\}$ and b. Then

$$Y \sim N((\Sigma_i a_i \mu_i) + b, \Sigma_i a_i^2 \sigma_i^2)$$

Corollary: The distribution of the sample mean of normal random variables Suppose $X_i \sim N(\mu, \sigma^2)$ for $i = 1, 2, ..., n$ and that they are independent random variables, If $\bar{X} = (X_1 + ... + X_n)/n$, then $\bar{X} \sim N(\mu, \sigma^2/n)$

Theorem: The covariance of sums of normal random variables Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, ..., n$ and also that the $\{X_i\}$are independent. Let $U = \Sigma_{i=1}^n a_i X_i$ and $V = \Sigma_{i=1}^n b_i X_i$ for some constants $\{a_1\}$ and $\{b_i\}$. Then $Cov(U, V) = \Sigma_i a_i b_i \sigma^2$. Furthermore, $Cov(U, V) = 0$ if and only if U and V are independent.

2 Expectation and Covariance

2.1 Expectation -Discrete case

Definition of expectation Let X be a discrete random variable, taking on distince values $x_1, x_2, ...$, with $p_i = P(X = x_i)$. Then the *expected value* (or *mean* or *mean value*) of X, written E(X) (or μ_x), is defined by

$$E(X) = \Sigma_i x_i p_i$$

Theorem: expectation involving nested functions

1. Let X be a discrete random variable, and let $g : \mathbb{R} \to \mathbb{R}$ be some function such that the expectation of the random variable $g(X)$ exists. Then
$$E(g(X)) = \Sigma_x g(x) P(X = x)$$

2. Let X and Y be discrete random variables, and let $h : \mathbb{R}^2 \to \mathbb{R}$ be some function such that the expectation of the random variable $h(X, Y)$ exists. Then
$$E(h(X, Y)) = \Sigma_{x,y} h(x, y) P(X = x, Y = y)$$

Theorem: Linearity of expected values Let X and Y be discrete random variables, let a and b be real numbers, and put $Z = aX + bY$. Then

$$E(Z) = aE(X) + bE(Y)$$

Theorem: Expectation of product of independent r.v Let X and Y be discrete random variables that are independent. Then

$$E(XY) = E(X)E(Y)$$

Monotonicity Let X and Y be discrete random variables, and suppose that $X \leq Y$ (Remember that this means $X(s) \leq Y(s)$ for all $s \in S$) Then $E(X) \leq E(Y)$.

2.2 Expectation - Continuous case

Definition of expectation Let X be an absolutely continuous random variable, with density function f_X. Then the *expected value* of X is given by

$$E(x) = \int_{-\infty}^{\infty} x f_X(x) dx$$

Theorem: expectation involving nested functions

1. Let X be a an absolutely continuous random variable with density function f_X, and let $g : \mathbb{R} \to \mathbb{R}$ be some function such that the expectation of the random variable $g(X)$ exists. Then

$$\int_{-\infty}^{\infty} = g(x) f_X(x) dx$$

2. Let X and Y be discrete random variables, and let $h : \mathbb{R}^2 \to \mathbb{R}$ be some function such that the expectation of the random variable $h(X, Y)$ exists. Then

$$E(h(X, Y)) = \int_{-\infty}^{\infty} h(x, y) f_{X,Y}(x, y) dx dy$$

Theorem: Linearity of expected values Let X and Y be jointly absolutely continuous random variables, let a and b be real numbers. Then

$$E(aX + bY) = aE(X) + bE(Y)$$

Monotonicity Let X and Y be jointly continuous random variables, and suppose that $X \leq Y$ (Remember that this means $X(s) \leq Y(s)$ for all $s \in S$) Then $E(X) \leq E(Y)$.

2.3 Variance, Covariance and Correlation

Definition of variance The *variance* of a random variable X is the quantity

$$\sigma_x^2 = Var(X) = E((X - \mu_X)^2)$$

where σ_X is the *standard deviation* of X.

Theorem Let X be any r.v. with $\mu_X = E(X)$ and variance Var(X). Then the following hold true:

1. $Var(X) \geq 0$
2. If a and b are real numbers, $Var(aX + b) = a^2 Var(X)$
3. $Var(X) = E(X^2) - (\mu_X)^2 = E(X^2) - E(X)^2$
4. $Var(X) \leq E(X^2)$

Definition of covariance

$$Cov(X, Y) = E((X - \mu_X)(Y - \mu_Y))$$

Theorem: Linearity of covariance Let X, Y and z be three r.v.s. Let a and b be real numbers. Then

$$Cov(aX + bY.Z) = aCov(X, Z) + bCov(Y, Z)$$

Theorem Let X and Y be r.v.s. Then

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

Theorem If X and Y are independent, then

$$Cov(X, Y) = 0$$

.

Theorem

1. For any r.v.s X and Y,

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

2. More generally, for any r.v.s $X_1, ..., X_n$,

$$Var(\Sigma_i X_i) = \Sigma_i Var(X_i) + 2\Sigma_{i<j} Cov(X_i, X_j)$$

Corollary

1. If X and Y are independent, then $Var(X+Y) = Var(X) + Var(Y)$
2. If $X_1, ...X_n$ are independent, then $Var(\Sigma_{i=1}^n X_i = \Sigma_{i=1}^n Var(X_i)$

Definition The *correlation* of two r.v.s X and Y is given by

$$Corr(X,Y) = \frac{Cov(X,Y)}{Sd(X)Sd(Y)}$$

provided $Var(X) < \infty$ and $Var(Y) < \infty$