Introduction to Real Analysis
– MAT337 Course Notes

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Contents

1 Construction of Real Numbers

1.1 Decimal Expansion

Definition 1.1.1 Let $r \in \mathbb{R}^+$. Then r is called

1. Terminating DE if $r = q.d_1 \dots d_n 0$
2. Repeating DE if $r = q.d_1 \dots d_k d_{k+1} \dots d_n d_{k+1} \dots d_n d_{k+1} \dots$

Proposition 1.1.2 $x = \frac{l}{m}$ is rational if x has a DE that is either terminating or repeating.
proof:
Let $x \in \mathbb{R}^+$.
$\Rightarrow$:
1.Assume x has a DE that is terminating, then

$$x = q.d_1 \dots d_n 0 = q + \sum_{m=1}^{n} \frac{d_m}{10^m} \in \mathbb{Q}$$

2.Assume x has a DE that is repeating, then

$$\begin{aligned} x &= q.d_1 \dots d_k \overline{d_{k+1} \dots d_n} \\ &= q.d_1 \dots d_k 0 + 0.0 \dots 0\overline{d_{k+1} \dots d_n} \end{aligned}$$

We know that the former number $\in \mathbb{Q}$, so we only need to show the rationality of the latter number.

$$\begin{aligned} 0.0 \dots 0\overline{d_{k+1} \dots d_n} &= 10^{-k}\left(\sum_{m=1}^{n} \sum_{l=0}^{\infty} \frac{d'_m}{10^{nl+m}}\right) \\ &\qquad \text{(denote } d'_0, \dots, d'_n \text{ be the repeated digits)} \\ &= 10^{-k} \sum_{m=1}^{n} d'_m 10^{-m} \left(\sum_{l=0}^{\infty} 10^{nl+m}\right) && \text{(decompose)} \\ &= 10^{-k} \sum_{m=1}^{n} d'_m 10^{-m} (1 - 10^{-n})^{-1} && \text{(geometric series)} \\ &= \sum_{m=1}^{n} \frac{d'_m 10^n}{10^{m+k}(10^n - 1)} && \text{(make it look nicer)} \\ &\in \mathbb{Q} \end{aligned}$$

$\Leftarrow$: Assume $x \in \mathbb{Q}$, we'll show that its DE is either terminating or repeating.
<u>**Idea**</u>
By Euclidean division we write

$$l = qm + r_0$$

where $r_0 < m$.
$\rightarrow \frac{l}{m} = q + \frac{r_0}{m}$
$\rightarrow q \leq \frac{l}{m} < q + 1$

Again by ED,

$$10r_0 = d_1 m + r_1$$

$\rightarrow \frac{r_0}{m} = \frac{d_1}{10} + \frac{r_1}{10m} \rightarrow \frac{l}{m} = q + \frac{r_0}{m} = q + \frac{d_1}{10} + \frac{r_1}{10m}$

Repeat this using induction.
<u>**Base Case:**</u>
$\frac{l}{m} = q + \frac{d_1}{10} + \frac{r_1}{10m}$

<u>**Inductive Step:**</u>
Assume $\frac{l}{m} = q + \frac{d_1}{10} + \ldots + \frac{r_n}{10^n m}$.
By ED,

$$10r_n = d_{n+1} m + r_{n+1}$$

$\rightarrow \frac{r_n}{m10^n} = \frac{d_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}m}$
$\rightarrow \frac{l}{m} = q + \frac{d_1}{10} + \ldots + \frac{r_{n+1}}{10^{n+1}m}$
<u>**Case 1**</u> $r_h = 0$ for some $h > 0 \Rightarrow$ then DE is terminating
<u>**Case 2**</u> $r_h > 0 \, \forall l > 0$
WTS DE is repeating.
$r_h \in \{0, \ldots, m-1\} \, \forall h > 0$
Fix h, then $\exists n$ s.t. $r_n = r_h$ for $n > h$

$$\text{Then } \begin{cases} 10r_n = d_{n+1}m + r_{n+1} \\ 10r_h = d_{h+1}m + r_{h+1} \end{cases} \implies \begin{matrix} d_{n+1} = d_{h+1} \\ r_{n+1} = r_{h+1} \end{matrix} \quad \text{(by uniqueness of ED)}$$

$\implies$ ED is repeating. ■

Definition 1.1.3 $x \in \mathbb{R}$ is called <u>irrational</u> if $\nexists \frac{l}{m}$ such that $x = \frac{l}{m}$. Denote as $x \in \mathbb{Q}^C$.

Proposition 1.1.4 $x \in \mathbb{Q}^C \iff$ the decimal expansion of x neither terminates nor repeats.

Fact 1 $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^C$.

1.2 Properties of Supremum and Infimum

Proposition 1.2.1 Every nonempty bounded above set S has a supremum.

<u>*proof:*</u>
Since S is bounded above, $\exists M \in \mathbb{R}, \exists m_0, m_1 \in \mathbb{N}, M = m_0.m_1, M \geq s\ \forall s \in S$.
Pick $s' = s_0.s_1 \in S$. Since $M \geq s'$, then $m_0 + 1 > s_0$.
Find the smallest integer $a_0 \in \{s_0, s_0+1, \ldots, m_0+1\}$ that a_0+1 is an upper bound for S.
Let $x_0 \in S$ s.t. $a_0 - 1 < x_0$.
Let $y_1 = a_0 + \frac{a_1}{10}$ where $a_1 \in \{0, 1, \ldots, 9\}$ is the smallest integer s.t. y_1 is an upper bound for S.
Let $x_1 \in S$ s.t. $a_0.a_1 - 0.1 \leq x_1 \leq a_0.a_1$
Let $y_2 = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2}$ where $a_2 \in \{0, 1, \ldots, 9\}$ is the smallest integer s.t. y_2 an upper bound for S.
Let $x_2 \in S$ s.t. $a_0.a_1a_2 - 0.01 \leq x_2 \leq a_0.a_1a_2$
...
Let $y_n = a_0 + \frac{a_1}{10} + \ldots + \frac{a_n}{10^n}$ where $a_n \in \{0, 1, \ldots, 9\}$ is the smallest integer s.t. y_n an upper bound for S.
Let $x_n \in S$ s.t. $a_0.a_1 \ldots a_n - \frac{1}{10^n} \leq x_n \leq y_n$

Claim: $L = a_0.a_1a_2\ldots$ is the supremum for S.
<u>*proof:*</u>
prove upper bound: Let $s = s_0.s_1 \ldots \in S$. There are 3 cases:

1. $s_i = a_i\, \forall i$ so that $s = L$

2. $\exists k \in \mathbb{N}, \forall i < k, s_i = a_i$ but $s_k > a_k$

$$\begin{aligned} y_k &= a_0.a_1 \ldots a_{k-1}a_k0 \\ &< a_0.a_1 \ldots a_{k-1}s_k0 \\ &= s_0.s_1 \ldots s_{k-1}s_k0 \\ &\leq s_0.s_1 \ldots s_{k-1}s_ks_{k+1} \\ &= s \in S \end{aligned}$$

   Since y_k is an upper bound for S, this cannot happen.

3. $s_i = a_i \ \forall i < k$ but $s_k < a_k$
   $\implies y_k > s \implies L > y_k > s$

prove subsequence property: $\forall \epsilon > 0$, WTS $\exists s_\epsilon \in S$ s.t. $L - \epsilon \leq s_\epsilon \leq L$
Let $\epsilon > 0$. Pick $n > 0$ s.t. $\frac{1}{10^n} < \epsilon$, so then

$$L - \epsilon \leq L - \frac{1}{10^n} \leq x_n \leq y_n < L$$

Choose $s_\epsilon = x_n$. ■

Proposition 1.2.2 Supremum is unique. *proof:*
Assume for a set $S \in \mathbb{R}$, there are two supremums $u, v \in \mathbb{R}$.
Let $\epsilon = u - v > 0$. Then by definition of supremum, $\exists s_\epsilon \in S$ s.t. $u - \epsilon = v < s_\epsilon$
$\implies$ contradiction: v is not a supremum of S. ■

Proposition 1.2.3 For bounded above set A and $c \geq 0$,

$$\sup(cA) = c\sup(A)$$

proof:
Let $M = \sup(A)$.
Upper bound property: $\forall s \in cA, s/c \in A \implies s/c \leq M \implies s \leq cM$
Subsequence property: Let $\epsilon > 0$, then take $\epsilon * = \frac{\epsilon}{c}$.
By the definition of $\sup(A)$, $\exists s_{\epsilon *} \in A, M - \epsilon * \leq s_{\epsilon *}$
Choose $s_\epsilon = cs_{\epsilon *}$, then
$M - \epsilon * \leq s_{\epsilon *} \implies cM - \epsilon \leq s_\epsilon$ as wanted. ■