MAT224 Linear Algebra II
Lecture Notes

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Contents

1 Vector Spaces

1.1 Vector Spaces

Definition 1.1.1 A (real) vector space is a set V (whose elements are called vectors) together with

1. an operation called vector addition, which for each pair of vectors $\mathbf{x}, \mathbf{y} \in V$ produced another vector in V denoted $\mathbf{x} + \mathbf{y}$, and
2. an operation called multiplication by a scalar (a real number), which for each vector $\mathbf{x} \in V$, and each scalar $c \in \mathbb{R}$ produced another vector in V denoted $c\mathbf{x}$

Furthermore, the two operations must satisfy the following axioms: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \forall c, d \in \mathbb{R}$,

1. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
3. $\exists \mathbf{0} \in V$ s.t. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ (additive identity)
4. $\exists -\mathbf{x} \in V$ s.t. $\mathbf{x} + -\mathbf{x} = \mathbf{0}$ (additive inverse)
5. $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
6. $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$
7. $(cd)\mathbf{x} = c(d\mathbf{x})$
8. $1\mathbf{x} = \mathbf{x}$

Smooth functions C^∞
Most functions are not smooth.

1.2 Subspaces

Example $C^\infty(\mathbb{R}) < C^k(\mathbb{R}) <$ Differentiable functions $< C(\mathbb{R}) < F(\mathbb{R})$

Definition Let V be a vector space and Let $W \subseteq V$ be a subset. Then W is a (vector) subspace of V if W is a vector space itself under the operations of vector sum and scalar multiplication from V.

Theorem 1.2.8 Let V be a vector space and Let $W \subseteq V$ be a nonempty subset of V. Then W is a subspace of V iff $\forall \mathbf{x}, \mathbf{y} \in W$, and all $c \in \mathbb{R}$, we have $c\mathbf{x} + \mathbf{y} \in W$.
proof: $\rightarrow$: If W is a subspace of V, then $\forall \mathbf{x}, \mathbf{y} \in W$ and $c \in \mathbb{R}, c\mathbf{x} + \mathbf{y} \in W$ holds since W itself is a real vector space.
$\leftarrow$: If $\forall \mathbf{x}, \mathbf{y} \in W$, and all $c \in \mathbb{R}$, we have $c\mathbf{x} + \mathbf{y} \in W$
Can have $c = 1$, so $\mathbf{x} + \mathbf{y} \in W$ (close under addition)
$c = -1$ and $\mathbf{y} = \mathbf{x}$, so $-\mathbf{x} + \mathbf{x} = \mathbf{0} \in W$ (additive identity)
$\mathbf{y} = \mathbf{0}$, so $c\mathbf{x} \in W$ (close under scalar multiplication)
These implies all the axioms. ∎

Examples

1. $W = \{f \in C(\mathbb{R}) | f(\pi) = 0\}$. W subspace of $C(\mathbb{R})$? -Yes
2. $W = \{f \in C(\mathbb{R}) | f(e) = e\}$. W subspace of $C(\mathbb{R})$? -No, not close under addition
3. $W = \{(x_1, ..., x_n) | x_i \geq 0 \forall i\}$. W subspace of $C(\mathbb{R})$? -No, there is no additive inverse for each item in W.

Theorem 1.2.13 Let V be a vector space. Then the intersection of any collection of subspaces of V is a subspace of V.
proof: Consider any collection of subspace of V. Note that the intersection of the subspaces is not empty since at least the zero vector from V is in it. Now let $\mathbf{x}, \mathbf{y}$ be any two vectors in the intersection, so they are in every single subspace in the collection. Therefore $c\mathbf{x} + \mathbf{y}$ is also in every single subspace in the collection, so that it is in the intersection as well. Hence the intersection is a subspace of V. ∎

Application The set of all solutions of any simultaneous system of equations is a subspace of $\mathbb{R}^n$.

Corollary 1.2.14 Let $a_{ij} (1 \leq i \leq m, 1 \leq j \leq n)$ be any real numbers and let $W = \{(x_1, ..., x_n) \in \mathbb{R}^n | a_{i1}x_1 + \ldots + a_{in}x_n = 0 \text{ for all } i, 1 \leq i \leq m\}$. Then W is a subspace of $\mathbb{R}^n$.

1.3 Linear Combinations

Definition 1.3.1 Let S be a subset of a vector space V.

1. A *linear combination* of vectors in S is any sum $a_1\mathbf{x}_1 + \ldots + a_n\mathbf{x}_n$ where the $a_i \in \mathbb{R}$, and the $\mathbf{x}_i \in S$
2. If $S \neq \emptyset$, the set of all linear combinations of vectors in S is called the *span* of S, and denoted Span(S). If $S = \emptyset$, we define Span(S) $= \{\mathbf{0}\}$. (Remark: It is a mathematician convention)
3. If W = Span(S), we say S spans (or generates) W.

Theorem 1.3.4 Let V be a vector space and let S be any subset of V. Then Span(S) is a subspace of V.

proof: Span(S) is non-empty by definition. Let $\mathbf{x}, \mathbf{y} \in Span(S)$, then they are linear combinations of vectors in S. Check that $c\mathbf{x} + \mathbf{y}$ is also a linear combination of vectors in S, so $c\mathbf{x} + \mathbf{y} \in Span(S)$. Hence Span(S) is a subspace of V. ■

Definition Let W_1 and W_2 be subspaces of a vector space V. The *sum* of W_1 and W_2 is the set

$$W_1 + W_2 = \{\mathbf{x} \in V | \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \text{ for some } \mathbf{x}_1 \in W_1 \text{ and } \mathbf{x}_2 \in W_2\}$$

Proposition 1.3.8 The basis of sum is the union of two bases Let $W_1 = Span(S_1)$ and $W_2 = Span(S_2)$ be subspaces of a vector space V. Then $W_1 + W_2 = Span(S_1 \cup S_2)$

Theorem 1.3.9 Let W_1 and W_2 be subspaces of a vector space V. Then $W_1 + W_2$ is also a subspace of V.

Proposition 1.3.11 $W_1 + W_2$ is the smallest subspace containing $W_1 \cup W_2$: Let W_1 and W_2 be subspaces of a vector space V and let W be a subspace of V such that $W_1 \cup W_2 \subseteq W$. Then $W_1 + W_2 \subseteq W$

Remark $W_1 \cup W_2$ is a subspace of V iff one is contained in another.

1.4 Linear Dependence and Linear Independence

Definitions 1.4.2 Let V be a vector space, and let S be a subset of V.

1. A *linear dependence* among the vectors of S is an equation

$$a_1\mathbf{x}_1 + \ldots + a_n\mathbf{x}_n = \mathbf{0}$$

   where the $x_i \in S$, and the $a_i \in \mathbb{R}$ are not all zero (i.e., at least one of the $a_i \neq \mathbf{0}$

2. the set S is said to be *linearly dependent* if there exists a linear dependence among the vectors in S.

Fact Let S be a set. If $\mathbf{0} \in S$, then S is dependent.

Definition 1.4.4 A subset S of a vector space V is *linearly independent* if whenever we have $a_i \in \mathbb{R}$ and $\mathbf{x}_i \in S$ such that $a_1\mathbf{x}_1 + \ldots + a_n\mathbf{x}_n = \mathbf{0}$, then $a_i = 0$ for all i.

Example In any vector space the empty subset $\emptyset$ is linearly independent.

Proposition 1.4.7

1. Let S be a linearly independent subset of a vector space V, and let S' be another subset of V that contains S. Then S' is also linearly dependent.

2. Let S be linearly independent subset of a vector space V and let S' be another subset of V that is contained in S. Then S' is also linearly independent.

1.5 Interlude on Solving Systems of Linear Equations (MAT223)

1.6 Bases And Dimension (Jan 17)

Definition A subset S of vector space V is called a *basis* of V if V = Span(S) and S is linearly independent.

Examples

1. the standard basis S = $\{e_1,...,e_n\}$ in $\mathbb{R}^n$, since every vector $(a_1, ..., a_n) \in \mathbb{R}^n$ may be written as the linear combination $(a_1, ..., a_n) = a_1e_1 + ... + a_ne_n$

2. The vector space $\mathbb{R}^n$ has many other bases as well. e.g., in $\mathbb{R}^2$, consider the set $S = \{(1, 2), (1, -1)\}$, which is l.i.

3. Let $V = P_n(\mathbb{R})$ and consider $S = \{1, x, x^2, ..., x^n\}$, which is a basis of V.
   proof: It is clear that S spans V. For independence, consider
   $$a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n = \mathbf{0}$$
   Take the derivative of both sides,
   $$\frac{d^n}{dx^n}(a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n) = \frac{d^n}{dx^n}(0)$$
   $$n!a_n = 0 \implies a_n = 0$$
   Similarly, we have $a_i = 0$ for all i, as wanted.
4. The empty subset, $\emptyset$, is a basis of the vector space consisting only of a zero vector, $\{\mathbf{0}\}$.

Theorem 1.6.3 Let V be a vector space, and let S be a nonempty subset of V. Then S is a basis of V iff every vector $\mathbf{x} \in V$ may be written uniquely as a linear combination of the vectors in S.
Proof: $\rightarrow$: Assume S is a basis of V, then given $\mathbf{x} \in V$, there are scalars $a_i \in \mathbb{R}$ and vectors $x_i \in S$ s.t. $\mathbf{x} = a_1x_1 + ... + a_nx_n$. To show this linear combination is unique, consider a possible second linear combination of vectors in S which also adds up to $\mathbf{x}$: $x = b_1x_1 + ... + b_nx_n$. Subtracting these two expressions for $\mathbf{x}$, we find that
$$\mathbf{0} = a_1x_1 + ... + a_nx_n - (b_1x_1 + ... + b_nx_n)$$
$$= (a_1 - b_1)x_1 + ... + (a_n - b_n)x_n$$
Since S is linearly independent, the equation implies that $a_i = b_i$ for all i.

$\leftarrow$: Assume every vector $\mathbf{x} \in V$ may be written uniquely as a linear combination of the vectors in S. This implies $Span(S) = V$. We must show that S is l.i. Consider an equation
$$a_1x_1 + ... + a_nx_n = \mathbf{0}$$
Note that it is also the case that
$$0\mathbf{x} = 0(x_1 + ... + x_n) = \mathbf{0}$$
Since we assumed that every $\mathbf{x}$ has a unique representation in S, then it must be true that $a_i = 0$ for all i. Hence S is l.i.

Theorem 1.6.6 Let V be a vector space that has a finite spanning set, and let S be a linearly independent subset of V. Then there exists a basis S' of V, with $S \subset S'$

Lemma 1.6.8 Let S be a linearly independent subset of V and let $x \in V$, but $x \notin S$. Then $S \cup \{\mathbf{x}\}$ is l.i. iff $\mathbf{x} \notin Span(S)$.

Insight the number of vectors in a basis is, in a rough sense, a measure of "how big" the space is.

Theorem 1.6.10 (Basis Theorem) Let V be a vector space and let S be a spanning set for V, which has m elements. Then no linearly independent set in V can have more than m elements.
proof: It suffices to show that every set in V with more than m elements is linearly dependent. Write $S = y_1, ..., y_m$ and suppose $S' = x_1, ..., x_n$ is a subset of V with $n > m$ vectors. Consider an equation

$$(1) a_1 x_1 + ... + a_n x_n = \mathbf{0}$$

Our goal is to show that a_i not all 0. Since S spans V, there are scalars b_{ij} s.t. for each i,

$$x_i = b_{i1} y_1 + ... + b_{im} y_m$$

Substituting these equations into (1), we get

$$a_1(b_{11} y_1 + ... + b_{1m} y_m) + ... + a_n(b_{n1} y_1 + ... + b_{nm} y_m) = \mathbf{0}$$

Collecting terms and rearranging,

$$(a_1 b_{11} + ... + a_n b_{n1}) y_1 + ... + (a_1 b_{1m} + ... + a_n b_{nm}) y_m = \mathbf{0}$$

Since S is l.i., this is equivalent to solving the system

$$b_{11} a_1 + ... + b_{n1} a_n = 0$$

$$\vdots$$

$$b_{1m} a_1 + ... + b_{nm} a_n = 0$$

But this is a system with n unknowns and m equations and $n > m$, so there must exist a non-trivial solution $\{a_1, ..., a_n\}$, which is what we wanted to show. QED

Corollary 1.6.11 Let V be a vector space and let S and S' be two bases of V, with m and m' elements, respectively. Then m = m'.
proof:
Since S is a spanning set of V and S' is l.i., we have $m' \leq m$. Since S' is a spanning set of V and S is l.i.m we have $m \leq m'$. Hence $m = m'$. QED

Definitions 1.6.12

1. If V is a vector space with some finite basis(possibly empty), we say V is *finite-dimentional*.
2. Let V be a finite-dimensional vector space. The dimension of V, denoted dim(V), is the number of vectors in a (hence any) basis of V.
3. If $V = \{\mathbf{0}\}$, we define dim(V) = 0.
4. $\dim(span\{(1,2,3),(4,5,6),(7,8,9)\}) = 2$

Examples

1. For each n, $\dim(\mathbb{R}^n)$ = n, since the standard basis contains n vectors.
2. $\dim(P_n(\mathbb{R})) = n+1$, since a basis for $P_n(\mathbb{R})$ contains n + 1 functions.
3. The vector spaces $P(\mathbb{R}), C^1(\mathbb{R})$ and $C(\mathbb{R})$ are not finite-dimensional. We say that such spaces are *infinite-dimentional*.

Corollary 1.6.14 Let W be a subspace of a finite-dimensional vector space V. Then $dim(W) \leq dim(V)$. Furthermore, $dim(W) = dim(V)$ iff $W = V$.

Corollary 1.6.15 Let W be a subspace of $\mathbb{R}^n$ defined by a system of homogeneous linear equations. Then dim(W) is equal to the number of free variables in the corresponding echelon form system.

Theorem 1.6.18 Let W_1 and W_2 be finite-dimensional subspaces of a vector space V. Then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Remark Analogous to the Principle of Inclusion-Exclusion
proof: Result obvious if either W_1 or W_2 is $\{\mathbf{0}\}$.
Therefore, we assume that neither W_1 nor W_2 is $\{\mathbf{0}\}$. Starting from a basis S of $W_1 \cap W_2$. We can always find sets T_1 and T_2 (disjoint from S) such that $S \cup T_1$ is a basis for W_1 and $S \cup T_2$ is a basis for W_2. We claim that $U = S \cup T_1 \cup T_2$ is a basis for $W_1 + W_2$, since

$$U = S \cup T_1 \cup T_2 = (S \cup T_1) \cup (S \cup T_2)$$

$$Span(U) = Span((S \cup T_1) \cup (S \cup T_2)) = W_1 + W_2$$

Next, prove that U is linearly independent. Any potential linear dependence among the vectors in U must have the form

$$\mathbf{v} + \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}$$

where $\mathbf{v} \in Span(S) = W_1 \cap W_2, \mathbf{w}_1 \in Span(T_1) \subset W_1, \mathbf{w}_2 \in Span(T_2) \subset W_2$. (slice the linear combination into the sum of the vectors from 3 vector spaces). It suffices to prove that in any such potential linear dependence, we must have $\mathbf{v} = \mathbf{w}_1 = \mathbf{w}_2 = \mathbf{0}$ (each vector is a lin comb, and equals $\mathbf{0}$).
Consider $\mathbf{w}_2 = -\mathbf{v} - \mathbf{w}_1$. Since $-\mathbf{v} - \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$, we must have $\mathbf{w}_2 \in W_1 \cap W_2$. By definition, $\mathbf{w}_2 \in Span(T_2)$ But $S \cap T_2 = \emptyset$, hence $Span(S) \cap Span(T_2) = \{\mathbf{0}\}$. Therefore we must have $\mathbf{w}_2 = \mathbf{0}$. So then $-\mathbf{v} = \mathbf{w}_1 \in W_1 \cap W_2$. Since $S \cap T_1 = \emptyset$, $Span(S) \cap Span(T_1) = \{\mathbf{0}\}$ and we have $\mathbf{w}_1 = \mathbf{0}$, so $\mathbf{v} = \mathbf{0}$ as well. So U is independent.

$$\begin{aligned} |U| &= |S| + |T_1| + |T_2| \\ &= \dim W_1 \cap W_2 + (\dim W_1 - \dim W_1 \cap W_2) + (\dim W_2 - \dim W_1 \cap W_2) \\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \end{aligned}$$

Exercises for 1.4 1.(k), 7

Exercises for 1.6 1.(d)(e)(f), 3, 4, 16

2 Linear Transformations

2.1 Linear Tranformations

A function T from V to W is denoted by $T : V \to W$. The vector $\mathbf{w} = T(\mathbf{v})$ in W is called the image of $\mathbf{v}$ under the function T. Loosely speaking, we want our functions to turn the algebraic operations of addition and scalar multiplication in V into addition and scalar multiplication in W.

Definition 2.1.1 A function $T : V \to W$ is called a *linear mapping* or a *linear transformation* if it satisfies

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v} \in$ V
2. $T(a\mathbf{v}) = aT(\mathbf{v})$ for all $a \in \mathbb{R}$ and $\mathbf{v} \in V$

V is called the *domain* of T and W is called the *target* of T.
We say that a linear transformation preserves the operations of addition and scalar multiplication.

Property A linear mapping always takes the zero vector in the domain vector space to the zero vector in the target vector space.

Proposition 2.1.2 A function $T : V \to W$ is a linear transformation if and only if for all a and $b \in \mathbb{R}$ and all $\mathbf{u}$ and $\mathbf{v} \in V$

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

Corollary 2.1.3 A function $T : V \to W$ is a linear transformation if and only if for all $a_1, .., a_k \in \mathbb{R}$ and for all $\mathbf{v}_1, ..., \mathbf{v}_k \in V$:

$$T(\sum_{i=1}^{k} a_i\mathbf{v}_i) = \sum_{i=1}^{k} a_iT(\mathbf{v}_i)$$

Examples

1. Let V be any vector space, and let W = V. The *underidentity transformation* $I : V \to V$ is defined by I($\mathbf{v}$) = $\mathbf{v}$ for all $\mathbf{v} \in$ V.
2. Let V and W be any vector spaces, and let $T : V \to W$ be the mapping that takes every vector in V to the zero vector in W:

$$T(\mathbf{v}) = \mathbf{0}_W$$

for all $\mathbf{v} \in$ V. T is called *zero transformation*.

3. $T(\boldsymbol{x}) = a_1x_1 + ... + a_nx_n$

4. *Differentiation, definite integration*

Remark The inner product plays a crucial role in linear algebra in that it provides a bridge between algebra and geometry, which is the heart of the more advanced material that appears later in the text.

Proposition 2.1.14 If $T : V \to W$ is a linear transformation and V is finite-dimensional, then T is uniquely determined by its values on the members of a basis of V.
proof: Show that if S and T are linear transformations that take the same values on each member of a basis for V, then in fact S = T.

$$\begin{aligned} T(v) &= T(a_1v_1 + ... + a_kv_k) \\ &= a_1T(v_1) + ... + a_kT(v_k) \\ &= a_1S(v_1) + ... + a_kS(v_k) \\ &= S(a_1v_1 + ... + a_kv_k) \\ &= S(v) \end{aligned}$$

Therefore, S and T are equal as mappings from V to W. ■

2.2 Linear Transformations Between Finite-Dimensional Vector Spaces

Proposition 2.2.1 Let $T : V \to W$ be a linear transformation between the finite-dimensional vector spaces V and W. If $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a basis for V and $\{\mathbf{w}_1, ..., \mathbf{w}_l\}$ is a basis for W, then $T : V \to W$ is uniquely determined by the $l \cdot k$ scalars used to express $T(\mathbf{v}_j), j = 1, ..., k$, in terms of $\mathbf{w}_1, ..., \mathbf{w}_l$.

Definition 2.2.6 Let $T : V \to W$ be a linear transformation between the finite-dimensional vector spaces V and W, and let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ and $\beta = \{\mathbf{w}_1, ..., \mathbf{w}_l\}$, respectively, be any bases for V and W. Let $a_{ij}, 1 \le i \le l$ and $1 \le j \le k$ be the $l \cdot k$ scalars that determine T with respect to the bases α and β. The matrix whose entries are the scalars a_{ij}, $1 \le i \le l$ and $1 \le j \le k$, is called the *matrix of the linear transformation T with respect to the bases α for V and β for W*. This matrix is denoted by $[T]_\alpha^\beta$.

Remark The basis vectors in the domain and target spaces are written in some particular order.

Definition of coordinate vectors If $\mathbf{v} = a_1\mathbf{v}_1 + ... + a_k\mathbf{v}_k$ and $\mathbf{w} = b_1\mathbf{w}_1 + ... + b_l\mathbf{w}_l$, we can express $\mathbf{v}$ and $\mathbf{w}$ in coordinates, respectively, as a $k \times 1$ matrix and as an $l \times 1$ matrix, with respect to the chosen bases. These coordinate vectors will be denoted by $[\mathbf{v}]_\alpha$ and $[\mathbf{w}]_\beta$, respectively. Thus $[\mathbf{v}]_\alpha = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$ and $[\mathbf{w}]_\beta = \begin{bmatrix} b_1 \\ \vdots \\ b_l \end{bmatrix}$

Proposition 2.2.15 Let $T : V \to W$ be a linear transformation between vector spaces V of dimension k and W of dimension l. Let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for V and $\beta = \{\mathbf{w}_1, ..., \mathbf{w}_l\}$ be a basis for W. Then for each $\mathbf{v} \in V$,

$$[T(\mathbf{v})]_\beta = [T]_\alpha^\beta [\mathbf{v}]_\alpha$$

<u>*proof:*</u> Let $\mathbf{v} = x_1\mathbf{v}_1 + ... + x_k\mathbf{v}_k \in V$. Then if $T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + ... + a_{lj}\mathbf{w}_l$

$$\begin{aligned} T(\mathbf{v}) &= \sum_{j=1}^{k} x_j T(\mathbf{v}_j) \\ &= \sum_{j=1}^{k} x_j (\sum_{i=1}^{l} a_{ij}\mathbf{w}_i) \\ &= \sum_{i=1}^{l} (\sum_{j=1}^{k} x_j a_{ij})\mathbf{w}_i \end{aligned}$$

Thus, the ith coefficient of T($\mathbf{v}$) in terms of β is $\sum_{j=1}^{k} x_j a_{ij}$ and $[T(\mathbf{v})]_\beta = \begin{bmatrix} \sum_{j=1}^{k} x_j a_{1j} \\ \vdots \\ \sum_{j=1}^{k} x_j a_{lj} \end{bmatrix}$ which is precisely $[T(\mathbf{v})]_\beta = [T]_\alpha^\beta [\mathbf{v}]_\alpha$. ■

Proposition 2.2.19 Let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for V and $\beta = \{\mathbf{w}_1, ..., \mathbf{w}_l\}$ be a basis for W, and let $\mathbf{v} = x_1\mathbf{v}_1 + ... + x_k\mathbf{v}_k \in V$

1. If A is an $l \times k$ matrix, then the function

$$T(\mathbf{v}) = \mathbf{w}$$

   where $[\mathbf{w}]_\beta = A[\mathbf{v}]_\alpha$ is a linear transformation.

2. If $A = [S]_\alpha^\beta$ is the matrix of a transformation $S : V \to W$, then the transformation T constructed from $[S]_\alpha^\beta$ is equal to S.

3. If T is the transformation of (1) constructed from A, then $[T]_\alpha^\beta = A$

Proposition 2.2.20 Let V and W be finite-dimensional vector spaces. Let α be a basis for V and β a basis for W. Then the assignment of a matrix to a linear transformation from V to W given by T goes to $[T]_\alpha^\beta$ is injective and surjective.

Notes

1. When proving a function T is not a linear transformation, can consider $T(\mathbf{0}) \neq \mathbf{0}$.

2.3 Kernel and Image

Definition 2.3.1 The *kernel* of T, denoted $Ker(T)$, is the subset of V consisting of all vectors $\mathbf{v} \in V$ such that $T(\mathbf{v}) = 0$.

Remark Kernel is different from null spaces. A null space is about a matrix, and it is something in $\mathbb{R}^n$.

Proposition 2.3.2 Let $T : V \to W$ be a linear transformation. Ker(T) is a subspace of V.

Examples

1. Let $V = P_3(\mathbb{R})$. Define $T : V \to V$ by $T(p(x)) = \frac{d}{dx}p(x)$. Ker(T) only consists constant polynomials.

2. Let $V = W = \mathbb{R}^2$. Let T be a rotation R_θ. Then $Ker(T) = \{\mathbf{0}\}$.

Proposition 2.3.7 Let $T : V \to W$ be a linear transformation of finite-dimensional vector spaces, and let α and β be bases for V and W, respectively. Then $\mathbf{x} \in Ker(T)$ ife lf the coordinate vector of $\mathbf{x}$, $[\mathbf{x}]_\alpha$, satisfies the system of equations

$$\begin{gathered} a_{11}x_1 + ... + a_{1k}x_k = 0 \\ \vdots \\ a_{l1}x_1 + ... + a_{lk}x_k = 0 \end{gathered}$$

where the coefficients a_{ij} are the entries of the matrix $[T]_\alpha^\beta$.

Remark This says

$$x \in \ker(T) \iff [x]_\alpha \in Nul[T]_\alpha^\beta$$

Proposition 2.3.8 Independence is Basis-Independent Let V be a finite-dimensional vector space, and let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for V. Then the vectors $\mathbf{x}_1, ..., \mathbf{x}_m \in V$ are linearly independent iff their corresponding coordinate vectors $[\mathbf{x}_1]_\alpha, ..., [\mathbf{x}_m]_\alpha$ are linearly independent.

Definition 2.3.10 The subset of W consisting of all vectors $\mathbf{w} \in W$ for which there exists a $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$ is called the *image* of T and is denoted by $Im(T)$.

Proposition 2.3.11 Let $T : V \to W$ be a linear transformation. The image of T is a subspace of W.

Proposition 2.3.12 If $\{\mathbf{v}_1, ..., \mathbf{v}_m\}$is any set that spans V (in particular, it could be a basis of V), then $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_m)\}$ spans $Im(T)$.

Corollary 2.3.13 If $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a basis for V and $\beta = \{\mathbf{w}_1, ..., \mathbf{w}_l\}$ is a basis for W, then the vectors in W, whose coordinate vectors (in terms of β) are the columns of $[T]_\alpha^\beta$, span $Im(T)$.

Rank-Nullity Theorem 2.3.17 If V is finite-dimensional vector space and $T : V \to W$ is a linear transformation, then

$$\dim(Ker(T)) + \dim(Im(T)) = \dim(V)$$

Equivalently,

$$Nullity(T) + Rank(T) = \dim(V)$$

2.4 Applications of the Dimension Theorem

Proposition 2.4.2 A linear transformation $T : V \to W$ is injective iff $\dim(Ker(T)) = 0$, or $\dim(Im(T)) = \dim(V)$.

Remark Analogously, in MAT223 we said that a matrix is one-to-one if all the columns are l.i..

Corollary 2.4.3 A linear mapping $T : V \to W$ on a finite-dimensional vector space V is injective iff $\dim(Im(T)) = \dim(V)$.

Corollary 2.4.4 If $\dim(W) < \dim(V)$ and $T : V \to W$ is a linear mapping, then T is not injective.
proof:

$$dim(Im(T)) \leq dim(W) < dim(V)$$
$$\implies dim(Ker(T)) > 0$$

Proposition 2.4.7 If W is finite-dimensional, then a linear mapping $T : V \to W$ is surjective iff $\dim(Im(T)) = \dim(W)$

Remark Analogously, in MAT223 we said that a matrix is onto if there is a pivot in every row.

Corollary 2.4.8 If V and W are finite-dimensional, with $\dim(V) < \dim(W)$, then there is no surjective linear mapping $T : V \to W$
proof: $\dim(Im(T)) \leq \dim(V) < \dim(W) \implies T$ is not surjective

Corollary 2.4.9 A linear mapping $T : V \to W$ can be surjective iff

$$\dim(V) \geq \dim(W)$$

Proposition 2.4.10 Let $\dim(V) = \dim(W)$. A linear transformation $T : V \to W$ is injective iff it is surjective.

Proposition 2.4.11 Let $T : V \to W$ be a linear transformation, and let $w \in Im(T)$. Let v_1 be any fixed vector with $T(v_1) = w$. Then every vector $v_2 \in T^{-1}(\{w\})$ can be written uniquely as $v_2 = v_1 + u$, where $u \in Ker(T)$

Remark In this situation $T^{-1}(\{w\})$ is a subspace of V iff $w = 0$.

Corollary 2.4.15 Let $T : V \to W$ be a linear transformation of finite-dimensional vector spaces, and let $w \in W$. Then there is a unique vector $v \in V$ such that $T(v) = w$ iff

1. $w \in Im(T)$, and
2. dim(Ker(T)) = 0

Proposition 2.4.16 With notation as before

1. The set of solutions of the system of linear equations $A\mathbf{x} = \mathbf{b}$ is the subset $T^{-1}(\{\mathbf{b}\})$ of $V = \mathbb{R}^n$

2. The set of solutions of the system of linear equations $A\mathbf{x} = \mathbf{b}$ is a subspace of V iff the system is homogeneous, in which case the set of solutions is $Ker(T)$.

Corollary 2.4.17

1. The number of free variables in the homogeneous system $A\mathbf{x} = \mathbf{0}$ (or its echelon form equivalent) is equal to $\dim(Ker(T))$

2. The number of basic variables of the system is equal to $dim(Im(T)$

Definition 2.4.18 Given an inhomogeneous system of equations, $A\mathbf{x} = \mathbf{b}$, any single vector $\mathbf{x}$ satisfying the system (necessarily $\mathbf{x} \neq \mathbf{0}$ is called a particular solution of the system of equations.

Proposition 2.4.19 Let $\mathbf{x}_p$ be a particular solution of the system $A\mathbf{x} = \mathbf{b}$. Then every other solution to $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system of equations $A\mathbf{x} = \mathbf{0}$. Furthermore, given $\mathbf{x}$ and $\mathbf{x}_p$, there is a unique $\mathbf{x}_h$ such that $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

Corollary 2.4.20 The system $A\mathbf{x} = \mathbf{b}$ has a unique solution iff $\mathbf{b} \in Im(T)$ and the only solution to $A\mathbf{x} = \mathbf{0}$ is the zero vector.