Introduction to Real Analysis – MAT337 Course Notes

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1 Construction of Real Numbers

1.1 Decimal Expansion

Definition 1 Let $r \in \mathbb{R}^+$. Then r is called

1. Terminating DE if $r = q.d_1 \dots d_n 0$
2. Repeating DE if $r = q.d_1 \dots d_k d_{k+1} \dots d_n d_{k+1} \dots d_n d_{k+1} \dots$

Proposition 1 $x = \frac{l}{m}$ is rational if x has a DE that is either terminating or repeating.
proof:
Let $x \in \mathbb{R}^+$.
$\Rightarrow$:
1.Assume x has a DE that is terminating, then

$$x = q.d_1 \dots d_n 0 = q + \sum_{m=1}^{n} \frac{d_m}{10^m} \in \mathbb{Q}$$

2.Assume x has a DE that is repeating, then

$$\begin{aligned} x &= q.d_1 \dots d_k \overline{d_{k+1} \dots d_n} \\ &= q.d_1 \dots d_k 0 + 0.0 \dots 0\overline{d_{k+1} \dots d_n} \end{aligned}$$

We know that the former number $\in \mathbb{Q}$, so we only need to show the rationality of the latter number.

$$\begin{aligned} 0.0 \dots 0\overline{d_{k+1} \dots d_n} &= 10^{-k}\left(\sum_{m=1}^{n} \sum_{l=0}^{\infty} \frac{d'_m}{10^{nl+m}}\right) \\ &\qquad \text{(denote } d'_0, \dots, d'_n \text{ be the repeated digits)} \\ &= 10^{-k} \sum_{m=1}^{n} d'_m 10^{-m} \left(\sum_{l=0}^{\infty} 10^{nl+m}\right) && \text{(decompose)} \\ &= 10^{-k} \sum_{m=1}^{n} d'_m 10^{-m} (1 - 10^{-n})^{-1} && \text{(geometric series)} \\ &= \sum_{m=1}^{n} \frac{d'_m 10^n}{10^{m+k}(10^n - 1)} && \text{(make it look nicer)} \\ &\in \mathbb{Q} \end{aligned}$$

$\Leftarrow$: Assume $x \in \mathbb{Q}$, we'll show that its DE is either terminating or repeating.
Idea
By Euclidean division we write

$$l = qm + r_0$$

where $r_0 < m$.
$\rightarrow \frac{l}{m} = q + \frac{r_0}{m}$
$\rightarrow q \leq \frac{l}{m} < q + 1$

Again by ED,

$$10r_0 = d_1 m + r_1$$

$\rightarrow \frac{r_0}{m} = \frac{d_1}{10} + \frac{r_1}{10m} \rightarrow \frac{l}{m} = q + \frac{r_0}{m} = q + \frac{d_1}{10} + \frac{r_1}{10m}$

Repeat this using induction.
Base Case:
$\frac{l}{m} = q + \frac{d_1}{10} + \frac{r_1}{10m}$

Inductive Step:
Assume $\frac{l}{m} = q + \frac{d_1}{10} + \ldots + \frac{r_n}{10^n m}$.
By ED,

$$10r_n = d_{n+1}m + r_{n+1}$$

$\rightarrow \frac{r_n}{m10^n} = \frac{d_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}m}$
$\rightarrow \frac{l}{m} = q + \frac{d_1}{10} + \ldots + \frac{r_{n+1}}{10^{n+1}m}$
Case 1 $r_h = 0$ for some $h > 0 \Rightarrow$ then DE is terminating
Case 2 $r_h > 0 \, \forall l > 0$
WTS DE is repeating.
$r_h \in \{0, \ldots, m-1\} \, \forall h > 0$
Fix h, then $\exists n$ s.t. $r_n = r_h$ for $n > h$

$$\text{Then } \begin{cases} 10r_n = d_{n+1}m + r_{n+1} \implies & d_{n+1} = d_{h+1} \\ 10r_h = d_{h+1}m + r_{h+1} & r_{n+1} = r_{h+1} \end{cases} \quad \text{(by uniqueness of ED)}$$

$\implies$ ED is repeating. ∎

Definition 2 $x \in \mathbb{R}$ is called irrational if $\nexists \frac{l}{m}$ such that $x = \frac{l}{m}$. Denote as $x \in \mathbb{Q}^C$.

Proposition 2 $x \in \mathbb{Q}^C \iff$ the decimal expansion of x neither terminates nor repeats.

Fact 1 $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^C$

1.2 Definition and Existence of Supremum and Infimum

Trivial.

1.3 Construction of Real Numbers Using Cauchy Sequence

Next lecture.