

# MAT224 Linear Algebra II

## Lecture Notes

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# 1 Vector Spaces

## 1.1 Vector Spaces

**Definition 1.1.1** A (real) vector space is a set  $V$  (whose elements are called vectors) together with

1. an operation called vector addition, which for each pair of vectors  $\mathbf{x}, \mathbf{y} \in V$  produced another vector in  $V$  denoted  $\mathbf{x} + \mathbf{y}$ , and
2. an operation called multiplication by a scalar (a real number), which for each vector  $\mathbf{x} \in V$ , and each scalar  $c \in \mathbb{R}$  produced another vector in  $V$  denoted  $c\mathbf{x}$

Furthermore, the two operations must satisfy the following axioms:  $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \forall c, d \in \mathbb{R}$ ,

1.  $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
2.  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
3.  $\exists \mathbf{0} \in V$  s.t.  $\mathbf{x} + \mathbf{0} = \mathbf{x}$  (additive identity)
4.  $\exists -\mathbf{x} \in V$  s.t.  $\mathbf{x} + -\mathbf{x} = \mathbf{0}$  (additive inverse)
5.  $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
6.  $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$
7.  $(cd)\mathbf{x} = c(d\mathbf{x})$
8.  $1\mathbf{x} = \mathbf{x}$

**Smooth functions**  $C^\infty$

Most functions are not smooth.

## 1.2 Subspaces

**Example**  $C^\infty(\mathbb{R}) < C^k(\mathbb{R}) < \text{Differentiable functions} < C(\mathbb{R}) < F(\mathbb{R})$

**Definition** Let  $V$  be a vector space and Let  $W \subseteq V$  be a subset. Then  $W$  is a (vector) subspace of  $V$  if  $W$  is a vector space itself under the operations of vector sum and scalar multiplication from  $V$ .

**Theorem 1.2.8** Let  $V$  be a vector space and Let  $W \subseteq V$  be a **nonempty** subset of  $V$ . Then  $W$  is a subspace of  $V$  iff  $\forall \mathbf{x}, \mathbf{y} \in W$ , and all  $c \in \mathbb{R}$ , we have  $c\mathbf{x} + \mathbf{y} \in W$ .

proof:  $\rightarrow$ : If  $W$  is a subspace of  $V$ , then  $\forall \mathbf{x}, \mathbf{y} \in W$  and  $c \in \mathbb{R}$ ,  $c\mathbf{x} + \mathbf{y} \in W$  holds since  $W$  itself is a real vector space.

$\leftarrow$ : If  $\forall \mathbf{x}, \mathbf{y} \in W$ , and all  $c \in \mathbb{R}$ , we have  $c\mathbf{x} + \mathbf{y} \in W$

Can have  $c = 1$ , so  $\mathbf{x} + \mathbf{y} \in W$  (close under addition)

$c = -1$  and  $\mathbf{y} = \mathbf{x}$ , so  $-\mathbf{x} + \mathbf{x} = \mathbf{0} \in W$  (additive identity)

$\mathbf{y} = \mathbf{0}$ , so  $c\mathbf{x} \in W$  (close under scalar multiplication)

These implies all the axioms. ■

### Examples

1.  $W = \{f \in C(\mathbb{R}) | f(\pi) = 0\}$ .  $W$  subspace of  $C(\mathbb{R})$ ? -Yes
2.  $W = \{f \in C(\mathbb{R}) | f(e) = e\}$ .  $W$  subspace of  $C(\mathbb{R})$ ? -No, not close under addition
3.  $W = \{(x_1, \dots, x_n) | x_i \geq 0 \forall i\}$ .  $W$  subspace of  $C(\mathbb{R})$ ? -No, there is no additive inverse for each item in  $W$ .

**Theorem 1.2.13** Let  $V$  be a vector space. Then the intersection of any collection of subspaces of  $V$  is a subspace of  $V$ .

proof: Consider any collection of subspace of  $V$ . Note that the intersection of the subspaces is not empty since at least the zero vector from  $V$  is in it. Now let  $\mathbf{x}, \mathbf{y}$  be any two vectors in the intersection, so they are in every single subspace in the collection. Therefore  $c\mathbf{x} + \mathbf{y}$  is also in every single subspace in the collection, so that it is in the intersection as well. Hence the intersection is a subspace of  $V$ . ■

**Application** The set of all solutions of any simultaneous system of equations is a subspace of  $\mathbb{R}^n$ .

**Corollary 1.2.14** Let  $a_{ij} (1 \leq i \leq m, 1 \leq j \leq n)$  be any real numbers and let  $W = \{(x_1, \dots, x_n) \in \mathbb{R}^n | a_{i1}x_1 + \dots + a_{in}x_n = 0 \text{ for all } i, 1 \leq i \leq m\}$ . Then  $W$  is a subspace of  $\mathbb{R}^n$ .

## 1.3 Linear Combinations

**Definition 1.3.1** Let  $S$  be a subset of a vector space  $V$ .

1. A *linear combination* of vectors in  $S$  is any sum  $a_1\mathbf{x}_1 + \dots + a_n\mathbf{x}_n$  where the  $a_i \in \mathbb{R}$ , and the  $\mathbf{x}_i \in S$
2. If  $S \neq \emptyset$ , the set of all linear combinations of vectors in  $S$  is called the *span* of  $S$ , and denoted  $\text{Span}(S)$ . If  $S = \emptyset$ , we define  $\text{Span}(S) = \{\mathbf{0}\}$ . (Remark: It is a mathematician convention)
3. If  $W = \text{Span}(S)$ , we say  $S$  spans (or generates)  $W$ .

**Theorem 1.3.4** Let  $V$  be a vector space and let  $S$  be any subset of  $V$ . Then  $\text{Span}(S)$  is a subspace of  $V$ .

*proof:*  $\text{Span}(S)$  is non-empty by definition. Let  $\mathbf{x}, \mathbf{y} \in \text{Span}(S)$ , then they are linear combinations of vectors in  $S$ . Check that  $c\mathbf{x} + \mathbf{y}$  is also a linear combination of vectors in  $S$ , so  $c\mathbf{x} + \mathbf{y} \in \text{Span}(S)$ . Hence  $\text{Span}(S)$  is a subspace of  $V$ . ■

**Definition** Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . The *sum* of  $W_1$  and  $W_2$  is the set

$$W_1 + W_2 = \{\mathbf{x} \in V \mid \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \text{ for some } \mathbf{x}_1 \in W_1 \text{ and } \mathbf{x}_2 \in W_2\}$$

**Proposition 1.3.8 The basis of sum is the union of two bases** Let  $W_1 = \text{Span}(S_1)$  and  $W_2 = \text{Span}(S_2)$  be subspaces of a vector space  $V$ . Then  $W_1 + W_2 = \text{Span}(S_1 \cup S_2)$

**Theorem 1.3.9** Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Then  $W_1 + W_2$  is also a subspace of  $V$ .

**Proposition 1.3.11**  $W_1 + W_2$  is the smallest subspace containing  $W_1 \cup W_2$ : Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$  and let  $W$  be a subspace of  $V$  such that  $W_1 \cup W_2 \subseteq W$ . Then  $W_1 + W_2 \subseteq W$

**Remark**  $W_1 \cup W_2$  is a subspace of  $V$  iff one is contained in another.

## 1.4 Linear Dependence and Linear Independence

**Definitions 1.4.2** Let  $V$  be a vector space, and let  $S$  be a subset of  $V$ .

1. A *linear dependence* among the vectors of  $S$  is an equation

$$a_1\mathbf{x}_1 + \dots + a_n\mathbf{x}_n = \mathbf{0}$$

where the  $x_i \in S$ , and the  $a_i \in \mathbb{R}$  are not all zero (i.e., at least one of the  $a_i \neq 0$ ).

2. the set  $S$  is said to be *linearly dependent* if there exists a linear dependence among the vectors in  $S$ .

**Fact** Let  $S$  be a set. If  $\mathbf{0} \in S$ , then  $S$  is dependent.

**Definition 1.4.4** A subset  $S$  of a vector space  $V$  is *linearly independent* if whenever we have  $a_i \in \mathbb{R}$  and  $\mathbf{x}_i \in S$  such that  $a_1\mathbf{x}_1 + \dots + a_n\mathbf{x}_n = \mathbf{0}$ , then  $a_i = 0$  for all  $i$ .

**Example** In any vector space the empty subset  $\emptyset$  is linearly independent.

**Proposition 1.4.7**

1. Let  $S$  be a linearly independent subset of a vector space  $V$ , and let  $S'$  be another subset of  $V$  that contains  $S$ . Then  $S'$  is also linearly dependent.
2. Let  $S$  be linearly independent subset of a vector space  $V$  and let  $S'$  be another subset of  $V$  that is contained in  $S$ . Then  $S'$  is also linearly independent.

## 1.5 Interlude on Solving Systems of Linear Equations (MAT223)

## 1.6 Bases And Dimension (Jan 17)

**Definition** A subset  $S$  of vector space  $V$  is called a *basis* of  $V$  if  $V = \text{Span}(S)$  and  $S$  is linearly independent.

**Remark** A basis is the maximal set of linearly independent vectors / minimal set of spanning vectors.

**Examples**

1. the standard basis  $S = \{e_1, \dots, e_n\}$  in  $\mathbb{R}^n$ , since every vector  $(a_1, \dots, a_n) \in \mathbb{R}^n$  may be written as the linear combination  $(a_1, \dots, a_n) = a_1e_1 + \dots + a_ne_n$
2. The vector space  $\mathbb{R}^n$  has many other bases as well. e.g., in  $\mathbb{R}^2$ , consider the set  $S = \{(1, 2), (1, -1)\}$ , which is l.i.
3. Let  $V = P_n(\mathbb{R})$  and consider  $S = \{1, x, x^2, \dots, x^n\}$ , which is a basis of  $V$ .

proof: It is clear that  $S$  spans  $V$ . For independence, consider

$$a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n = \mathbf{0}$$

Take the derivative of both sides,

$$\frac{d^n}{dx^n}(a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n) = \frac{d^n}{dx^n}(0)$$

$$n!a_n = 0 \implies a_n = 0$$

Similarly, we have  $a_i = 0$  for all  $i$ , as wanted.

4. The empty subset,  $\emptyset$ , is a basis of the vector space consisting only of a zero vector,  $\{\mathbf{0}\}$ .

**Theorem 1.6.3** Let  $V$  be a vector space, and let  $S$  be a nonempty subset of  $V$ . Then  $S$  is a basis of  $V$  iff every vector  $\mathbf{x} \in V$  may be written **uniquely** as a linear combination of the vectors in  $S$ .

Proof:  $\rightarrow$ : Assume  $S$  is a basis of  $V$ , then given  $\mathbf{x} \in V$ , there are scalars  $a_i \in \mathbb{R}$  and vectors  $x_i \in S$  s.t.  $\mathbf{x} = a_1x_1 + \dots + a_nx_n$ . To show this linear combination is unique, consider a possible second linear combination of vectors in  $S$  which also adds up to  $\mathbf{x}$ :  $\mathbf{x} = b_1x_1 + \dots + b_nx_n$ . Subtracting these two expressions for  $\mathbf{x}$ , we find that

$$\begin{aligned} \mathbf{0} &= a_1x_1 + \dots + a_nx_n - (b_1x_1 + \dots + b_nx_n) \\ &= (a_1 - b_1)x_1 + \dots + (a_n - b_n)x_n \end{aligned}$$

Since  $S$  is linearly independent, the equation implies that  $a_i = b_i$  for all  $i$ .

$\leftarrow$ : Assume every vector  $\mathbf{x} \in V$  may be written uniquely as a linear combination of the vectors in  $S$ . This implies  $\text{Span}(S) = V$ . We must show that  $S$  is l.i. Consider an equation

$$a_1x_1 + \dots + a_nx_n = \mathbf{0}$$

Note that it is also the case that

$$0\mathbf{x} = 0(x_1 + \dots + x_n) = \mathbf{0}$$

Since we assumed that every  $\mathbf{x}$  has a unique representation in  $S$ , then it must be true that  $a_i = 0$  for all  $i$ . Hence  $S$  is l.i.

**Theorem 1.6.6** Let  $V$  be a vector space that has a finite spanning set, and let  $S$  be a linearly independent subset of  $V$ . Then there exists a basis  $S'$  of  $V$ , with  $S \subset S'$

**Lemma 1.6.8** Let  $S$  be a linearly independent subset of  $V$  and let  $x \in V$ , but  $x \notin S$ . Then  $S \cup \{\mathbf{x}\}$  is l.i. iff  $\mathbf{x} \notin \text{Span}(S)$ .

**Insight** the number of vectors in a basis is, in a rough sense, a measure of “how big” the space is.

**Theorem 1.6.10 (Basis Theorem)** Let  $V$  be a vector space and let  $S$  be a spanning set for  $V$ , which has  $m$  elements. Then no linearly independent set in  $V$  can have more than  $m$  elements.

proof: It suffices to show that every set in  $V$  with more than  $m$  elements is linearly dependent. Write  $S = y_1, \dots, y_m$  and suppose  $S' = x_1, \dots, x_n$  is a subset of  $V$  with  $n > m$  vectors. Consider an equation

$$(1) a_1x_1 + \dots + a_nx_n = \mathbf{0}$$

Our goal is to show that  $a_i$  not all 0. Since  $S$  spans  $V$ , there are scalars  $b_{ij}$  s.t. for each  $i$ ,

$$x_i = b_{i1}y_1 + \dots + b_{im}y_m$$

Substituting these equations into (1), we get

$$a_1(b_{11}y_1 + \dots + b_{1m}y_m) + \dots + a_n(b_{n1}y_1 + \dots + b_{nm}y_m) = \mathbf{0}$$

Collecting terms and rearranging,

$$(a_1b_{11} + \dots + a_nb_{n1})y_1 + \dots + (a_1b_{1m} + \dots + a_nb_{nm})y_m = \mathbf{0}$$

Since S is l.i., this is equivalent to solving the system

$$b_{11}a_1 + \dots + b_{n1}a_n = 0$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$b_{1m}a_1 + \dots + b_{nm}a_n = 0$$

But this is a system with  $n$  unknowns and  $m$  equations and  $n > m$ , so there must exist a non-trivial solution  $\{a_1, \dots, a_n\}$ , which is what we wanted to show. ■

**Corollary 1.6.11** Let  $V$  be a vector space and let  $S$  and  $S'$  be two bases of  $V$ , with  $m$  and  $m'$  elements, respectively. Then  $m = m'$ .

proof:

Since  $S$  is a spanning set of  $V$  and  $S'$  is l.i., we have  $m' \leq m$ . Since  $S'$  is a spanning set of  $V$  and  $S$  is l.i.m we have  $m \leq m'$ . Hence  $m = m'$ . ■

### Definitions 1.6.12

1. If  $V$  is a vector space with some finite basis(possibly empty), we say  $V$  is finite-dimensional.
2. Let  $V$  be a finite-dimensional vector space. The dimension of  $V$ , denoted  $\dim(V)$ , is the number of vectors in a (hence any) basis of  $V$ .
3. If  $V = \{\mathbf{0}\}$ , we define  $\dim(V) = 0$ .

### Examples

1. For each  $n$ ,  $\dim(\mathbb{R}^n) = n$ , since the standard basis contains  $n$  vectors.
2.  $\dim(P_n(\mathbb{R})) = n + 1$ , since a basis for  $P_n(\mathbb{R})$  contains  $n + 1$  functions.
3. The vector spaces  $P(\mathbb{R})$ ,  $C^1(\mathbb{R})$  and  $C(\mathbb{R})$  are not finite-dimensional. We say that such spaces are infinite-dimensional.
4.  $\dim(\text{Span}\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}) = 2$



**Corollary 1.6.14** Let  $W$  be a subspace of a finite-dimensional vector space  $V$ . Then  $\dim(W) \leq \dim(V)$ . Furthermore,  $\dim(W) = \dim(V)$  iff  $W = V$ .

**Corollary 1.6.15** Let  $W$  be a subspace of  $\mathbb{R}^n$  defined by a system of homogeneous linear equations. Then  $\dim(W)$  is equal to the number of free variables in the corresponding echelon form system.

**Theorem 1.6.18** Let  $W_1$  and  $W_2$  be finite-dimensional subspaces of a vector space  $V$ . Then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

**Remark** Analogous to the Principle of Inclusion-Exclusion

*proof:* Result obvious if either  $W_1$  or  $W_2$  is  $\{\mathbf{0}\}$ .

Therefore, we assume that neither  $W_1$  nor  $W_2$  is  $\{\mathbf{0}\}$ . Starting from a basis  $S$  of  $W_1 \cap W_2$ . We can always find sets  $T_1$  and  $T_2$  (disjoint from  $S$ ) such that  $S \cup T_1$  is a basis for  $W_1$  and  $S \cup T_2$  is a basis for  $W_2$ . We claim that  $U = S \cup T_1 \cup T_2$  is a basis for  $W_1 + W_2$ , since

$$U = S \cup T_1 \cup T_2 = (S \cup T_1) \cup (S \cup T_2)$$

$$\text{Span}(U) = \text{Span}((S \cup T_1) \cup (S \cup T_2)) = W_1 + W_2$$

Next, prove that  $U$  is linearly independent. Any potential linear dependence among the vectors in  $U$  must have the form

$$\mathbf{v} + \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}$$

where  $\mathbf{v} \in \text{Span}(S) = W_1 \cap W_2$ ,  $\mathbf{w}_1 \in \text{Span}(T_1) \subset W_1$ ,  $\mathbf{w}_2 \in \text{Span}(T_2) \subset W_2$ . (slice the linear combination into the sum of the vectors from 3 vector spaces). It suffices to prove that in any such potential linear dependence, we must have  $\mathbf{v} = \mathbf{w}_1 = \mathbf{w}_2 = \mathbf{0}$  (each vector is a lin comb, and equals  $\mathbf{0}$ ). Consider  $\mathbf{w}_2 = -\mathbf{v} - \mathbf{w}_1$ . Since  $-\mathbf{v} - \mathbf{w}_1 \in W_1$ ,  $\mathbf{w}_2 \in W_2$ , we must have  $\mathbf{w}_2 \in W_1 \cap W_2$ . By definition,  $\mathbf{w}_2 \in \text{Span}(T_2)$  But  $S \cap T_2 = \emptyset$ , hence  $\text{Span}(S) \cap \text{Span}(T_2) = \{\mathbf{0}\}$ . Therefore we must have  $\mathbf{w}_2 = \mathbf{0}$ . So then  $-\mathbf{v} = \mathbf{w}_1 \in W_1 \cap W_2$ . Since  $S \cap T_1 = \emptyset$ ,  $\text{Span}(S) \cap \text{Span}(T_1) = \{\mathbf{0}\}$  and we have  $\mathbf{w}_1 = \mathbf{0}$ , so  $\mathbf{v} = \mathbf{0}$  as well. So  $U$  is independent.

$$\begin{aligned}|U| &= |S| + |T_1| + |T_2| \\ &= \dim W_1 \cap W_2 + (\dim W_1 - \dim W_1 \cap W_2) + (\dim W_2 - \dim W_1 \cap W_2) \\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)\end{aligned}$$

**Exercises for 1.4** 1.(k), 7

**Exercises for 1.6** 1.(d)(e)(f), 3, 4, 16

## 2 Linear Transformations

### 2.1 Linear Transformations

A function  $T$  from  $V$  to  $W$  is denoted by  $T : V \rightarrow W$ . The vector  $\mathbf{w} = T(\mathbf{v})$  in  $W$  is called the *image* of  $\mathbf{v}$  under the function  $T$ . Loosely speaking, we want our functions to turn the algebraic operations of addition and scalar multiplication in  $V$  into addition and scalar multiplication in  $W$ .

**Definition 2.1.1** A function  $T : V \rightarrow W$  is called a *linear mapping* or a *linear transformation* if it satisfies

1.  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}$  and  $\mathbf{v} \in V$
2.  $T(a\mathbf{v}) = aT(\mathbf{v})$  for all  $a \in \mathbb{R}$  and  $\mathbf{v} \in V$

$V$  is called the *domain* of  $T$  and  $W$  is called the *target* of  $T$ .

We say that a linear transformation preserves the operations of addition and scalar multiplication.

**Property** A linear mapping always takes the zero vector in the domain vector space to the zero vector in the target vector space.

**Proposition 2.1.2** A function  $T : V \rightarrow W$  is a linear transformation if and only if for all  $a$  and  $b \in \mathbb{R}$  and all  $\mathbf{u}$  and  $\mathbf{v} \in V$

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

**Corollary 2.1.3** A function  $T : V \rightarrow W$  is a linear transformation if and only if for all  $a_1, \dots, a_k \in \mathbb{R}$  and for all  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ :

$$T\left(\sum_{i=1}^k a_i \mathbf{v}_i\right) = \sum_{i=1}^k a_i T(\mathbf{v}_i)$$

**Remark** prove this by induction!

### Examples

1. Let  $V$  be any vector space, and let  $W = V$ . The identity transformation  $I : V \rightarrow V$  is defined by  $I(\mathbf{v}) = \mathbf{v}$  for all  $\mathbf{v} \in V$ .

2. Let  $V$  and  $W$  be any vector spaces, and let  $T : V \rightarrow W$  be the mapping that takes every vector in  $V$  to the zero vector in  $W$ :

$$T(\mathbf{v}) = \mathbf{0}_W$$

for all  $\mathbf{v} \in V$ .  $T$  is called zero transformation.

3.  $T(\mathbf{x}) = a_1x_1 + \dots + a_nx_n$

4. *Differentiation, definite integration*

**Remark** The inner product plays a crucial role in linear algebra in that it provides a bridge between algebra and geometry, which is the heart of the more advanced material that appears later in the text.

**Proposition 2.1.14** If  $T : V \rightarrow W$  is a linear transformation and  $V$  is finite-dimensional, then  $T$  is uniquely determined by its values on the members of a basis of  $V$ .

proof: Show that if  $S$  and  $T$  are linear transformations that take the same values on each member of a basis for  $V$ , then in fact  $S = T$ .

$$\begin{aligned} T(v) &= T(a_1v_1 + \dots + a_kv_k) \\ &= a_1T(v_1) + \dots + a_kT(v_k) \\ &= a_1S(v_1) + \dots + a_kS(v_k) \\ &= S(a_1v_1 + \dots + a_kv_k) \\ &= S(v) \end{aligned}$$

Therefore,  $S$  and  $T$  are equal as mappings from  $V$  to  $W$ . ■

## 2.2 Linear Transformations Between Finite-Dimensional Vector Spaces

**Proposition 2.2.1** Let  $T : V \rightarrow W$  be a linear transformation between the finite-dimensional vector spaces  $V$  and  $W$ . If  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a basis for  $V$  and  $\{\mathbf{w}_1, \dots, \mathbf{w}_l\}$  is a basis for  $W$ , then  $T : V \rightarrow W$  is uniquely determined by the  $l \cdot k$  scalars used to express  $T(\mathbf{v}_j), j = 1, \dots, k$ , in terms of  $\mathbf{w}_1, \dots, \mathbf{w}_l$ .

**Definition 2.2.6** Let  $T : V \rightarrow W$  be a linear transformation between the finite-dimensional vector spaces  $V$  and  $W$ , and let  $\alpha = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  and  $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_l\}$ , respectively, be any bases for  $V$  and  $W$ . Let  $a_{ij}, 1 \leq i \leq l$  and  $1 \leq j \leq k$  be the  $l \cdot k$  scalars that determine  $T$  with respect to the bases  $\alpha$  and  $\beta$ . The matrix whose entries are the scalars  $a_{ij}, 1 \leq i \leq l$  and  $1 \leq j \leq k$ , is called the *matrix of the linear transformation  $T$  with respect to the bases  $\alpha$  for  $V$  and  $\beta$  for  $W$* . This matrix is denoted by  $[T]_{\alpha}^{\beta}$ .

**Remark** The basis vectors in the domain and target spaces are written in some particular order.

**Definition of coordinate vectors** If  $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_k\mathbf{v}_k$  and  $\mathbf{w} = b_1\mathbf{w}_1 + \dots + b_l\mathbf{w}_l$ , we can express  $\mathbf{v}$  and  $\mathbf{w}$  in coordinates, respectively, as a  $k \times 1$  matrix and as an  $l \times 1$  matrix, with respect to the chosen bases. These coordinate vectors will be denoted by  $[\mathbf{v}]_{\alpha}$  and  $[\mathbf{w}]_{\beta}$ , respectively. Thus

$$[\mathbf{v}]_{\alpha} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \text{ and } [\mathbf{w}]_{\beta} = \begin{bmatrix} b_1 \\ \vdots \\ b_l \end{bmatrix}$$

**Proposition 2.2.15** Let  $T : V \rightarrow W$  be a linear transformation between vector spaces  $V$  of dimension  $k$  and  $W$  of dimension  $l$ . Let  $\alpha = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a basis for  $V$  and  $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_l\}$  be a basis for  $W$ . Then for each  $\mathbf{v} \in V$ ,

$$[T(\mathbf{v})]_{\beta} = [T]_{\alpha}^{\beta}[\mathbf{v}]_{\alpha}$$

proof: Let  $\mathbf{v} = x_1\mathbf{v}_1 + \dots + x_k\mathbf{v}_k \in V$ . Then if  $T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + \dots + a_{lj}\mathbf{w}_l$

$$\begin{aligned} T(\mathbf{v}) &= \sum_{j=1}^k x_j T(\mathbf{v}_j) \\ &= \sum_{j=1}^k x_j \left( \sum_{i=1}^l a_{ij} \mathbf{w}_i \right) \\ &= \sum_{i=1}^l \left( \sum_{j=1}^k x_j a_{ij} \right) \mathbf{w}_i \end{aligned}$$

Thus, the  $i$ th coefficient of  $T(\mathbf{v})$  in terms of  $\beta$  is  $\sum_{j=1}^k x_j a_{ij}$  and  $[T(\mathbf{v})]_{\beta} =$

$$\begin{bmatrix} \sum_{j=1}^k x_j a_{1j} \\ \vdots \\ \sum_{j=1}^k x_j a_{lj} \end{bmatrix} \text{ which is precisely } [T(\mathbf{v})]_{\beta} = [T]_{\alpha}^{\beta}[\mathbf{v}]_{\alpha}. \quad \blacksquare$$

**Proposition 2.2.19** Let  $\alpha = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a basis for  $V$  and  $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_l\}$  be a basis for  $W$ , and let  $\mathbf{v} = x_1\mathbf{v}_1 + \dots + x_k\mathbf{v}_k \in V$

1. If  $A$  is an  $l \times k$  matrix, then the function

$$T(\mathbf{v}) = \mathbf{w}$$

where  $[\mathbf{w}]_\beta = A[\mathbf{v}]_\alpha$  is a linear transformation.

2. If  $A = [S]_\alpha^\beta$  is the matrix of a transformation  $S : V \rightarrow W$ , then the transformation  $T$  constructed from  $[S]_\alpha^\beta$  is equal to  $S$ .
3. If  $T$  is the transformation of (1) constructed from  $A$ , then  $[T]_\alpha^\beta = A$

**Proposition 2.2.20** Let  $V$  and  $W$  be finite-dimensional vector spaces. Let  $\alpha$  be a basis for  $V$  and  $\beta$  a basis for  $W$ . Then the assignment of a matrix to a linear transformation from  $V$  to  $W$  given by  $T$  goes to  $[T]_\alpha^\beta$  is injective and surjective.

### Notes

1. When proving a function  $T$  is not a linear transformation, can consider  $T(\mathbf{0}) \neq \mathbf{0}$ .

## 2.3 Kernel and Image

**Definition 2.3.1** The *kernel* of  $T$ , denoted  $\text{Ker}(T)$ , is the subset of  $V$  consisting of all vectors  $\mathbf{v} \in V$  such that  $T(\mathbf{v}) = \mathbf{0}$ .

**Remark** Kernel is different from null spaces. A null space is about a matrix, and it is something in  $\mathbb{R}^n$ .

**Proposition 2.3.2** Let  $T : V \rightarrow W$  be a linear transformation.  $\text{Ker}(T)$  is a subspace of  $V$ .

### Examples

1. Let  $V = P_3(\mathbb{R})$ . Define  $T : V \rightarrow V$  by  $T(p(x)) = \frac{d}{dx}p(x)$ .  $\text{Ker}(T)$  only consists constant polynomials.
2. Let  $V = W = \mathbb{R}^2$ . Let  $T$  be a rotation  $R_\theta$ . Then  $\text{Ker}(T) = \{\mathbf{0}\}$ .

**Proposition 2.3.7** Let  $T : V \rightarrow W$  be a linear transformation of finite-dimensional vector spaces, and let  $\alpha$  and  $\beta$  be bases for  $V$  and  $W$ , respectively. Then  $\mathbf{x} \in \text{Ker}(T)$  if and only if the coordinate vector of  $\mathbf{x}$ ,  $[\mathbf{x}]_\alpha$ , satisfies the system of equations

$$a_{11}x_1 + \dots + a_{1k}x_k = 0$$

$$\vdots$$

$$a_{l1}x_1 + \dots + a_{lk}x_k = 0$$

where the coefficients  $a_{ij}$  are the entries of the matrix  $[T]_\alpha^\beta$ .

**Remark** This says

$$x \in \ker(T) \iff [x]_\alpha \in \text{Nul}[T]_\alpha^\beta$$

**Proposition 2.3.8 Independence is Basis-Independent** Let  $V$  be a finite-dimensional vector space, and let  $\alpha = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a basis for  $V$ . Then the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_m \in V$  are linearly independent if and only if their corresponding coordinate vectors  $[\mathbf{x}_1]_\alpha, \dots, [\mathbf{x}_m]_\alpha$  are linearly independent.

**Definition 2.3.10** The subset of  $W$  consisting of all vectors  $\mathbf{w} \in W$  for which there exists a  $\mathbf{v} \in V$  such that  $T(\mathbf{v}) = \mathbf{w}$  is called the *image* of  $T$  and is denoted by  $\text{Im}(T)$ .

**Proposition 2.3.11** Let  $T : V \rightarrow W$  be a linear transformation. The image of  $T$  is a subspace of  $W$ .

**Proposition 2.3.12** If  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  is any set that spans  $V$  (in particular, it could be a basis of  $V$ ), then  $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_m)\}$  spans  $\text{Im}(T)$ .

**Corollary 2.3.13** If  $\alpha = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a basis for  $V$  and  $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_l\}$  is a basis for  $W$ , then the vectors in  $W$ , whose coordinate vectors (in terms of  $\beta$ ) are the columns of  $[T]_\alpha^\beta$ , span  $\text{Im}(T)$ .

**Rank-Nullity Theorem 2.3.17** If  $V$  is finite-dimensional vector space and  $T : V \rightarrow W$  is a linear transformation, then

$$\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(V)$$

Equivalently,

$$\text{Nullity}(T) + \text{Rank}(T) = \dim(V)$$

## 2.4 Applications of the Dimension Theorem

**Proposition 2.4.2** A linear transformation  $T : V \rightarrow W$  is injective iff  $\dim(\text{Ker}(T)) = 0$ , or  $\dim(\text{Im}(T)) = \dim(V)$ .

**Remark** Analogously, in MAT223 we said that **a matrix is one-to-one if all the columns are l.i.**

**Corollary 2.4.3** A linear mapping  $T : V \rightarrow W$  on a finite-dimensional vector space  $V$  is injective iff  $\dim(\text{Im}(T)) = \dim(V)$ .

**Corollary 2.4.4** If  $\dim(W) < \dim(V)$  and  $T : V \rightarrow W$  is a linear mapping, then  $T$  is not injective.

proof:

$$\begin{aligned}\dim(\text{Im}(T)) &\leq \dim(W) < \dim(V) \\ \implies \dim(\text{Ker}(T)) &> 0\end{aligned}$$

**Proposition 2.4.7** If  $W$  is finite-dimensional, then a linear mapping  $T : V \rightarrow W$  is surjective iff  $\dim(\text{Im}(T)) = \dim(W)$

**Remark** Analogously, in MAT223 we said that **a matrix  $A \in M_{m \times n}(\mathbb{R})$  is onto if there is a pivot in every row, or the columns of  $A$  spans  $\mathbb{R}^m$ .**

**Corollary 2.4.8** If  $V$  and  $W$  are finite-dimensional, with  $\dim(V) < \dim(W)$ , then there is no surjective linear mapping  $T : V \rightarrow W$

proof:  $\dim(\text{Im}(T)) \leq \dim(V) < \dim(W) \implies T$  is not surjective

**Corollary 2.4.9** A linear mapping  $T : V \rightarrow W$  can be surjective iff

$$\dim(V) \geq \dim(W)$$

**Proposition 2.4.10** Let  $\dim(V) = \dim(W)$ . A linear transformation  $T : V \rightarrow W$  is injective iff it is surjective.

**Proposition 2.4.11** Let  $T : V \rightarrow W$  be a linear transformation, and let  $w \in \text{Im}(T)$ . Let  $v_1$  be any fixed vector with  $T(v_1) = w$ . Then every vector  $v_2 \in T^{-1}(\{w\})$  can be written uniquely as  $v_2 = v_1 + u$ , where  $u \in \text{Ker}(T)$



**Remark** In this situation  $T^{-1}(\{w\})$  is a subspace of  $V$  iff  $w = 0$ .

**Corollary 2.4.15** Let  $T : V \rightarrow W$  be a linear transformation of finite-dimensional vector spaces, and let  $w \in W$ . Then there is a unique vector  $v \in V$  such that  $T(v) = w$  iff

1.  $w \in \text{Im}(T)$ , and
2.  $\dim(\text{Ker}(T)) = 0$

**Proposition 2.4.16** With notation as before

1. The set of solutions of the system of linear equations  $A\mathbf{x} = \mathbf{b}$  is the subset  $T^{-1}(\{\mathbf{b}\})$  of  $V = \mathbb{R}^n$
2. The set of solutions of the system of linear equations  $A\mathbf{x} = \mathbf{b}$  is a subspace of  $V$  iff the system is homogeneous, in which case the set of solutions is  $\text{Ker}(T)$ .

**Corollary 2.4.17**

1. The number of free variables in the homogeneous system  $A\mathbf{x} = \mathbf{0}$  (or its echelon form equivalent) is equal to  $\dim(\text{Ker}(T))$
2. The number of basic variables of the system is equal to  $\dim(\text{Im}(T))$

**Definition 2.4.18** Given an inhomogeneous system of equations,  $A\mathbf{x} = \mathbf{b}$ , any single vector  $\mathbf{x}$  satisfying the system (necessarily  $\mathbf{x} \neq \mathbf{0}$ ) is called a particular solution of the system of equations.

**Proposition 2.4.19** Let  $\mathbf{x}_p$  be a particular solution of the system  $A\mathbf{x} = \mathbf{b}$ . Then every other solution to  $A\mathbf{x} = \mathbf{b}$  is of the form  $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ , where  $\mathbf{x}_h$  is a solution of the corresponding homogeneous system of equations  $A\mathbf{x} = \mathbf{0}$ . Furthermore, given  $\mathbf{x}$  and  $\mathbf{x}_p$ , there is a unique  $\mathbf{x}_h$  such that  $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ .

**Corollary 2.4.20** The system  $A\mathbf{x} = \mathbf{b}$  has a unique solution iff  $\mathbf{b} \in \text{Im}(T)$  and the only solution to  $A\mathbf{x} = \mathbf{0}$  is the zero vector.

## 2.5 Composition of Linear Transformations

**Definition** Let  $U$ ,  $V$ , and  $W$  be vector spaces, and let  $S : U \rightarrow V$  and  $T : V \rightarrow W$  be linear transformations. The *composition* of  $S$  and  $T$  is denoted  $TS : U \rightarrow W$  and is defined by

$$TS(\mathbf{v}) = T(S(\mathbf{v}))$$

Notice that this is well defined since the image of  $S$  is contained in  $V$ , which is the domain of  $T$ .

**Proposition 2.5.1** Let  $S : U \rightarrow V$  and  $T : V \rightarrow W$  be linear transformations, then  $TS$  is a linear transformation.

**Remark** In general,  $ST$  is not equal to  $TS$ . We emphasize that the composition is well defined only if the image of the first transformation is contained in the domain of the second.

### Proposition 2.5.4

1. Let  $R : U \rightarrow V$ ,  $S : V \rightarrow W$  and  $T : W \rightarrow X$  be linear transformations of the vector space  $U, V, W$  and  $X$  as indicated. Then

$$T(SR) = (TS)R \text{ (associativity)}$$

2. Let  $R : U \rightarrow V$ ,  $S : V \rightarrow W$  and  $T : W \rightarrow X$  be linear transformations of the vector space  $U, V, W$  and  $X$  as indicated. Then

$$T(R + S) = TR + TS \text{ (distributivity)}$$

3. Let  $R : U \rightarrow V$ ,  $S : V \rightarrow W$  and  $T : W \rightarrow X$  be linear transformations of the vector space  $U, V, W$  and  $X$  as indicated. Then

$$(T + S)R = TR + SR \text{ (distributivity)}$$

**Proposition 2.5.6** Let  $S : U \rightarrow V$  and  $T : V \rightarrow W$  be linear transformations. Then

1.  $\text{Ker}(S) \subset \text{Ker}(TS)$
2.  $\text{Im}(TS) \subset \text{Im}(T)$

proof:

1. If  $\mathbf{u} \in \text{Ker}(S)$ ,  $S(\mathbf{u}) = \mathbf{0}$ . Then  $TS(\mathbf{u}) = T(\mathbf{0}) = \mathbf{0}$ . Therefore  $\mathbf{u} \in \text{Ker}(TS)$ .
2. If  $\mathbf{x} \in \text{Im}(TS)$ , then  $\exists \mathbf{u} \in U$  s.t.  $TS(\mathbf{u}) = T(S(\mathbf{u})) = \mathbf{x}$ , then  $\exists \mathbf{v} = S(\mathbf{u}) \in V$  s.t.  $T(\mathbf{v}) = \mathbf{x}$ . Therefore  $\mathbf{x} \in \text{Im}(T)$  ■

**Corollary 2.5.7** Let  $S : U \rightarrow V$  and  $T : V \rightarrow W$  be linear transformations of finite-dimensional vector spaces. Then

1.  $\dim(\text{Ker}(S)) \leq \dim(\text{Ker}(TS))$
2.  $\dim(\text{Im}(TS)) \leq \dim(\text{Im}(T))$

**Proposition 2.5.9** If  $[S]_{\alpha}^{\beta}$  has entries  $a_{ij}$ ,  $i = 1, \dots, n$  and  $j = 1, \dots, m$  and  $[T]_{\beta}^{\gamma}$  has entries  $b_{kl}$ ,  $k = 1, \dots, p$  and  $l = 1, \dots, n$ , then the entries of  $[TS]_{\alpha}^{\gamma}$  are  $\sum_{l=1}^n b_{kl}a_{lj}$

**Definition 2.5.10** Let  $A$  be an  $n \times m$  matrix and  $B$  a  $p \times n$  matrix, then the *matrix product*  $BA$  is defined to be the  $p \times m$  matrix whose entries are  $\sum_{l=1}^n b_{kl}a_{lj}$  for  $k = 1, \dots, p$  and  $j = 1, \dots, m$ .

**Proposition 2.5.13** Let  $S : U \rightarrow V$  and  $T : V \rightarrow W$  be linear transformations between finite-dimensional vector spaces. Let  $\alpha, \beta$  and  $\gamma$  be bases for  $U, V$  and  $W$ , respectively. Then

$$[TS]_{\alpha}^{\gamma} = [T]_{\beta}^{\gamma}[S]_{\alpha}^{\beta}$$

In words, the matrix of the composition of two linear transformations is the product of the matrices of the transformations

**Proposition 2.5.14**

1. Let  $A, B$  and  $C$  be  $m \times n, n \times p$  and  $p \times r$  matrices, then

$$(AB)C = A(BC) \text{ (associativity)}$$

2. Let  $A, B$  and  $C$  be  $m \times n, n \times p$  and  $p \times r$  matrices, then

$$A(B + C) = AB + AC \text{ (distributivity)}$$

3. Let  $A, B$  and  $C$  be  $m \times n, n \times p$  and  $p \times r$  matrices, then

$$(A + B)C = AC + BC \text{ (distributivity)}$$

### 3 Problem Notes

**1**  $S = \{\mathbf{a}\} \subseteq \mathbb{R}^2$ , then we cannot determine whether S is dependent (when  $\mathbf{a} = \mathbf{0}$ ) or independent (when  $\mathbf{a} \neq \mathbf{0}$ )

**2** If a set in a vector space contains the zero vector, then it is linearly dependent.

**3** Multiple Choice questions: check the hypothesis of theorems!