MAT224 Linear Algebra II
Lecture Notes

Yuchen Wang

January 17, 2019

Contents

1 Vector Spaces

1.1 Bases And Dimension (Jan 17)

Definition A subset S of vector space V is called a *basis* of V if V = Span(S) and S is linearly independent.

Examples

1. the standard basis S = $\{e_1,...,e_n\}$ in $\mathbb{R}^n$, since every vector $(a_1, ..., a_n) \in \mathbb{R}^n$ may be written as the linear combination $(a_1, ..., a_n) = a_1e_1 + ... + a_ne_n$
2. The vector space $\mathbb{R}^n$ has many other bases as well. e.g., in $\mathbb{R}^2$, consider the set $S = \{(1, 2), (1, -1)\}$, which is l.i.
3. Let $V = P_n(\mathbb{R})$ and consider $S = \{1, x, x^2, ..., x^n\}$, which is a basis of V.
4. The empty subset, $\emptyset$, is a basis of the vector space consisting only of a zero vector, $\{\mathbf{0}\}$.

Theorem 1.6.3 Let V be a vector space, and let S be a nonempty subset of V. Then S is a basis of V iff every vector $\mathbf{x} \in V$ may be written uniquely as a linear combination of the vectors in S.
Proof:

Theorem 1.6.6 Let V be a vector space that has a finite spanning set, and let S be a linearly independent subset of V. Then there exists a basis S' of V, with $S \subset S'$

Lemma 1.6.8 Let S be a linearly independent subset of V and let $x \in V$, but $x \notin S$. Then $S \cup \{\mathbf{x}\}$ is l.i. iff $\mathbf{x} \notin Span(S)$.

Insight the number of vectors in a basis is, in a rough sense, a measure of "how big" the space is.

Theorem 1.6.10 Let V be a vector space and let S be a spanning set for V, which has m elements. Then no linearly independent set in V can have more than m elements.

Corollary 1.6.11 Let V be a vector space and let S and S' be two bases of V, with m and m' elements, respectively. Then m = m'.

Definitions 1.6.12

1. If V is a vector space with some finite basis(possibly empty), we say V is *finite-dimentional*.
2. Let V be a finite-dimensional vector space. The dimension of V, denoted dim(V), is the number of vectors in a (hence any) basis of V.
3. If $V = \{\mathbf{0}\}$, we define dim(V) = 0.

Examples

1. For each n, $\dim(\mathbb{R}^n)$ = n, since the standard basis contains n vectors.
2. $\dim(P_n(\mathbb{R})) = n+1$, since a basis for $P_n(\mathbb{R})$ contains n + 1 functions.
3. The vector spaces $P(\mathbb{R}), C^1(\mathbb{R})$ and $C(\mathbb{R})$ are not finite-dimensional. We say that such spaces are *infinite-dimentional*.

Corollary 1.6.14 Let W be a subspace of a finite-dimensional vector space V. Then $dim(W) \leq dim(V)$. Furthermore, $dim(W) = dim(V)$ iff $W = V$.

Corollary 1.6.15 Let W be a subspace of $\mathbb{R}^n$ defined by a system of homogeneous linear equations. Then dim(W) is equal to the number of free variables in the corresponding echelon form system.

Theorem 1.6.18 Let W_1 and W_2 be finite-dimensional subspaces of a vector space V. Then

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$$

2 Linear Transformations

2.1 Linear Tranformations