

# STA261 Probability and Statistics II

## Lecture Notes

Yuchen Wang

June 1, 2019

### **Contents**

## 1 Converge in distribution

## 2 Normal Distribution Theory

**Theorem: Sum of independent normal random variables** Suppose  $X_i \sim N(\mu_i, \sigma_i^2)$  for  $i = 1, 2, \dots, n$  and that they are independent random variables. Let  $Y = (\sum_i a_i X_i) + b$  for some constants  $\{a_i\}$  and  $b$ . Then

$$Y \sim N((\sum_i a_i \mu_i) + b, \sum_i a_i^2 \sigma_i^2)$$

**Corollary: The distribution of the sample mean of normal random variables** Suppose  $X_i \sim N(\mu, \sigma^2)$  for  $i = 1, 2, \dots, n$  and that they are independent random variables, If  $\bar{X} = (X_1 + \dots + X_n)/n$ , then  $\bar{X} \sim N(\mu, \sigma^2/n)$

**Theorem: The covariance of sums of normal random variables** Suppose  $X_i \sim N(\mu_i, \sigma_i^2)$  for  $i = 1, 2, \dots, n$  and also that the  $\{X_i\}$  are independent. Let  $U = \sum_{i=1}^n a_i X_i$  and  $V = \sum_{i=1}^n b_i X_i$  for some constants  $\{a_i\}$  and  $\{b_i\}$ . Then  $Cov(U, V) = \sum_i a_i b_i \sigma_i^2$ . Furthermore,  $Cov(U, V) = 0$  if and only if  $U$  and  $V$  are independent.

## 3 Expectation and Covariance

### 3.1 Expectation -Discrete case

**Definition of expectation** Let  $X$  be a discrete random variable, taking on discrete values  $x_1, x_2, \dots$ , with  $p_i = P(X = x_i)$ . Then the *expected value* (or *mean* or *mean value*) of  $X$ , written  $E(X)$  (or  $\mu_x$ ), is defined by

$$E(X) = \sum_i x_i p_i$$

**Theorem: expectation involving nested functions**

1. Let  $X$  be a discrete random variable, and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be some function such that the expectation of the random variable  $g(X)$  exists. Then

$$E(g(X)) = \sum_x g(x) P(X = x)$$

2. Let  $X$  and  $Y$  be discrete random variables, and let  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  be some function such that the expectation of the random variable  $h(X, Y)$  exists. Then

$$E(h(X, Y)) = \sum_{x,y} h(x, y) P(X = x, Y = y)$$

**Theorem: Linearity of expected values** Let  $X$  and  $Y$  be discrete random variables, let  $a$  and  $b$  be real numbers, and put  $Z = aX + bY$ . Then

$$E(Z) = aE(X) + bE(Y)$$

**Theorem: Expectation of product of independent r.v** Let  $X$  and  $Y$  be discrete random variables that are independent. Then

$$E(XY) = E(X)E(Y)$$

**Monotonicity** Let  $X$  and  $Y$  be discrete random variables, and suppose that  $X \leq Y$  (Remember that this means  $X(s) \leq Y(s)$  for all  $s \in S$ ) Then  $E(X) \leq E(Y)$ .

### 3.2 Expectation - Continuous case

**Definition of expectation** Let  $X$  be an absolutely continuous random variable, with density function  $f_X$ . Then the *expected value* of  $X$  is given by

$$E(x) = \int_{-\infty}^{\infty} xf_X(x)dx$$

**Theorem: expectation involving nested functions**

1. Let  $X$  be a an absolutely continuous random variable with density function  $f_X$ , and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be some function such that the expectation of the random variable  $g(X)$  exists. Then

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$$

2. Let  $X$  and  $Y$  be discrete random variables, and let  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  be some function such that the expectation of the random variable  $h(X, Y)$  exists. Then

$$E(h(X, Y)) = \int_{-\infty}^{\infty} h(x, y)f_{X,Y}(x, y)dxdy$$

**Theorem: Linearity of expected values** Let  $X$  and  $Y$  be jointly absolutely continuous random variables, let  $a$  and  $b$  be real numbers. Then

$$E(aX + bY) = aE(X) + bE(Y)$$

**Monotonicity** Let  $X$  and  $Y$  be jointly continuous random variables, and suppose that  $X \leq Y$  (Remember that this means  $X(s) \leq Y(s)$  for all  $s \in S$ ) Then  $E(X) \leq E(Y)$ .

### 3.3 Variance, Covariance and Correlation

**Definition of variance** The *variance* of a random variable  $X$  is the quantity

$$\sigma_x^2 = \text{Var}(X) = E((X - \mu_X)^2)$$

where  $\sigma_X$  is the *standard deviation* of  $X$ .

**Theorem** Let  $X$  be any r.v. with  $\mu_X = E(X)$  and variance  $\text{Var}(X)$ . Then the following hold true:

1.  $\text{Var}(X) \geq 0$
2. If  $a$  and  $b$  are real numbers,  $\text{Var}(aX + b) = a^2 \text{Var}(X)$
3.  $\text{Var}(X) = E(X^2) - (\mu_X)^2 = E(X^2) - E(X)^2$
4.  $\text{Var}(X) \leq E(X^2)$

**Definition of covariance**

$$\text{Cov}(X, Y) = E((X - \mu_X)(Y - \mu_Y))$$

**Theorem: Linearity of covariance** Let  $X$ ,  $Y$  and  $z$  be three r.v.s. Let  $a$  and  $b$  be real numbers. Then

$$\text{Cov}(aX + bY, Z) = a\text{Cov}(X, Z) + b\text{Cov}(Y, Z)$$

**Theorem** Let  $X$  and  $Y$  be r.v.s. Then

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

**Theorem** If  $X$  and  $Y$  are independent, then

$$\text{Cov}(X, Y) = 0$$

**Theorem**

1. For any r.v.s  $X$  and  $Y$ ,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

2. More generally, for any r.v.s  $X_1, \dots, X_n$ ,

$$\text{Var}(\sum_i X_i) = \sum_i \text{Var}(X_i) + 2\sum_{i < j} \text{Cov}(X_i, X_j)$$

**Corollary**

1. If  $X$  and  $Y$  are independent, then  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
2. If  $X_1, \dots, X_n$  are independent, then  $\text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{Var}(X_i)$

**Definition** The *correlation* of two r.v.s  $X$  and  $Y$  is given by

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\text{Sd}(X)\text{Sd}(Y)}$$

provided  $\text{Var}(X) < \infty$  and  $\text{Var}(Y) < \infty$

## 4 Independent Random Variables

**Definition 1** Let  $X$  and  $Y$  be two continuous random variables. We say  $X$  and  $Y$  are independent if

$$f_{X,Y}(x, y) = f_X(x) \times f_Y(y)$$

$\forall x, y \in \mathbb{R}$

**Lemma 1**  $X$  and  $Y$  are two continuous random variables. If  $X$  and  $Y$  are independent, then

$$E[g(X)h(Y)] = E(g(X)) \times E[h(Y)]$$

for any two functions  $g()$  and  $h()$ .

proof:

$$\begin{aligned}
 E[g(X)h(Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f_{X,Y}(x,y) dx dy \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y) dx dy \\
 &= \int_{-\infty}^{\infty} f_Y(y)h(y) \int_{-\infty}^{\infty} g(x)f_X(x) dx dy \\
 &= \int_{-\infty}^{\infty} f_Y(y)h(y)E[g(X)] dy \\
 &= E[g(X)] \int_{-\infty}^{\infty} f_Y(y)h(y) dy \\
 &= E[g(X)]E[h(Y)]
 \end{aligned}$$

■

## 5 Types of Inferences

### Estimation:

1. Point estimation: Based on the sample observations, calculating a particular value as an estimate of the parameter  $\theta$
2. Interval estimation: Calculating a range of values that is likely to contain the parameter  $\theta$

**Hypothesis testing** Based on the sample, assess whether a hypothetical value  $\theta_0$  is a plausible value of the parameter  $\theta$  or not.

## 6 Different Types of Estimation

### 6.1 Method of Moments Estimation

Let  $X_1, X_2, \dots, X_n$  are independently and identically distributed (i.i.d.) random variables.

Let the  $k^{th}$  population moment be

$$\mu_k = E[X^k]$$

$k^{th}$  sample moment based on sample

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n X_i^k$$

We use  $\hat{\mu}_k$  as an estimator of  $\mu_k$

In other words, we use the sample moments as estimators of the population moments.

## 6.2 Maximum Likelihood Estimation

**Definition of Likelihood Function** Suppose  $X_1, X_2, \dots, X_n$  has a joint density or mass function  $f(x_1, x_2, \dots, x_n | \theta)$

We observe sample,  $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$

Given the sample, the likelihood function of  $\theta$ , noted as  $L(\theta | x_1, x_2, \dots, x_n)$ , is defined as

$$L(\theta | x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n | \theta)$$

Often written as  $L(\theta)$ , is a function of  $\theta$ .

If  $X$  follows a discrete distribution, it gives the probability of observing the sample as a function of the parameter  $\theta$

If  $X_1, X_2, \dots, X_n$  are i.i.d. then their joint density is the product of marginal densities,  $f_\theta(x)$

Hence, in i.i.d. case we write

$$L(\theta) = \prod_{i=1}^n f_\theta(x_i)$$

### Comments

1.  $L(\theta)$  is NOT a pdf or pmf of  $\theta$
2. Likelihood introduces a belief ordering on parameter space,  $\Omega$
3. For  $\theta_1, \theta_2 \in \Omega$ , we believe in  $\theta_1$  as the true value of  $\theta$  over  $\theta_2$  whenever  $L(\theta_1) > L(\theta_2)$
4. Which means, the data is more likely to come from  $f_{\theta_1}$  than  $f_{\theta_2}$
5. The value  $L(\theta)$  is very small for every value of  $\theta$
6. So often, we are interested in the likelihood ratios:

$$\frac{L(\theta_1)}{L(\theta_2)}$$

**Maximum Likelihood Estimation**

1. Let's say we are interested in a point estimate of  $\theta$
2. A sensible choice will be to pick  $\hat{\theta}$  that maximizes  $L(\theta)$
3. So  $\hat{\theta}$  satisfies  $L(\hat{\theta}) \geq L(\theta)$  for all  $\theta \in \Omega$
4.  $\hat{\theta}$  is called the maximum likelihood estimate (MLE) of  $\theta$

**Computation of the MLE**

1. Define, log-likelihood function,  $l(\theta) = \ln L(\theta)$
2.  $\ln(x)$  is a 1-1 increasing function of  $x > 0 \implies L(\hat{\theta}) \geq L(\theta)$  for  $\theta \in \Omega$   
iff  $l(\hat{\theta}) \geq l(\theta)$
3. In other words, if  $L(\theta)$  is maximized at  $\hat{\theta}$  then  $l(\theta)$  will also be maximized at  $\hat{\theta}$
4. Therefore,

$$l(\theta) = \ln(\prod_{i=1}^n f_{\theta}(x_i)) = \sum_{i=1}^n \ln f_{\theta}(x_i)$$

5. The obvious benefit: It's much easier to differentiate a sum than a product
6. Solve the equation,  $\frac{\partial l(\theta)}{\partial \theta} = 0$  for  $\theta$
7. Say,  $\hat{\theta}$  is the solution. But it's still not the MLE
8. Need to check whether or not

$$\frac{\partial^2 l(\theta)}{\partial \theta^2} \Big|_{\theta=\hat{\theta}} < 0$$

**Properties of MLE**

1. MLE is not unique
2. MLE may not exist
3. The likelihood may not always be differentiable.



## 7 Sampling Distribution of an Estimator

1. Recall: An Estimator (T) is a random variable (infinite number of sample means)
2. If we repeat the sampling procedure and keep calculating T for each set of sample and finally draw a density histogram based on the T values we get the sampling distribution of T
3. **Standard error:** Standard deviation of an estimator is called the standard error (SE)

**Definition of Mean Squared Error** Let  $\psi(\theta)$  be any real valued function of  $\theta$ , suppose T is an estimator of  $\psi(\theta)$

$$MSE_{\theta}(T) = E_{\theta}[(T - \psi(\theta))^2]$$

**Corollary**

$$MSE_{\theta}(T) = Var_{\theta}(T) + (E_{\theta}(T) - \psi(\theta))^2$$

proof:

$$\begin{aligned} MST(T) &= E[(T - \psi(\theta))^2] \\ &= E[(T - E(T) + E(T) - \psi(\theta))^2] \\ &= E[(T - E(T))^2 + (E(T) - \psi(\theta))^2 + 2(T - E(T))(E(T) - \psi(\theta))] \\ &= E[(T - E(T))^2] + (E(T) - \psi(\theta))^2 + 2E[T - E(T)](E(T) - \psi(\theta)) \\ &= E[(T - E(T))^2] + (E(T) - \psi(\theta))^2 \\ &\quad \text{(Since } E[T - E(T)] = E(T) - E(T) = 0) \\ &= Var(T) + (E(T) - \psi(\theta))^2 \\ &= Var(T) + Bias^2(T) \end{aligned}$$

■

**Bias** The bias of an estimator T of  $\psi(\theta)$  is given by

$$E_{\theta}(T) - \psi(\theta)$$

**Unbiased estimator:** When the bias of an estimator is zero, it's called unbiased

**Remark**

1. For unbiased estimators,

$$MSE_{\theta}(T) = Var_{\theta}(T)$$

2. If all the other properties are similar, then an unbiased estimator is preferred over a biased estimator.
3. In practice, often an biased estimator with lower variance is preferred over an unbiased estimator with really high variance. **We minimize MSE.**

**8 Population Variance ( $\sigma^2$ )**

**Definition**  $\sigma^2 = E[(X - \mu)^2]$  where  $\mu = E[X]$ .

If we have equally likely  $N$  data points in our population, this is equivalent of

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^N (X_i - \mu)^2$$

**In words:** It's the average squared difference of each of the data points ( $X_i$ ) from the mean ( $\mu$ )

**Estimate  $\sigma^2$  based on a sample of size  $n$**  When we are estimating based on the sample of size  $n$ , we replace  $\mu$  by  $\bar{X}$ , so the numerator is  $\sum_{i=1}^n (X_i - \bar{X})^2$ . We can divide it by both  $n$  or  $n - 1$ . The latter one is unbiased!

The fraction,  $\frac{n-1}{n} \rightarrow 1$  as  $n \rightarrow \infty$ . So for large  $n$ , both estimator will produce similar estimate. In statistical literature, whenever we say *sample variance* we refer to the *unbiased* one. Hence, from now on,

**Definition of sample variance**

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

## 9 Sampling distribution of $S^2$ (under Normal Distribution)

**Theorem** Suppose  $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$  iid,  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$  and  $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$ .  
Then

1.  $\bar{X}$  and  $S^2$  are independent, and
2.  $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(df=n-1)}$

proof:

**Part 1** Let

$$\begin{aligned} U = \bar{X} &= \frac{1}{n}X_1 + \dots + \frac{1}{n}X_n \\ V = X_1 - \bar{X} &= X_1 - \left(\frac{1}{n}X_1 + \dots + \frac{1}{n}X_n\right) \\ &= \left(1 - \frac{1}{n}\right)X_1 - \left(\frac{1}{n}X_2 + \dots + \frac{1}{n}X_n\right) \end{aligned}$$

$$\begin{aligned} Cov(\bar{X}, X_1 - \bar{X}) &= Cov(\bar{X}, X_1) - Cov(\bar{X}, \bar{X}) \\ &= Cov\left(\frac{1}{n}X_1 + \dots + \frac{1}{n}X_n, X_1\right) - \frac{\sigma^2}{n} \\ &= \frac{1}{n}Cov(X_1, X_1) - \frac{\sigma^2}{n} \\ &= \frac{\sigma^2}{n} - \frac{\sigma^2}{n} \\ &= 0 \end{aligned}$$

Hence by E&R theorem, U and V are independent. Similarly, we can show  $\bar{X}$  is independent to each  $X_i - \bar{X}$  for  $i = 1, \dots, n$

Therefore,  $\bar{X}$  is independent to  $\sum_{i=1}^n (X_i - \bar{X})^2$

Therefore,  $\bar{X}$  is independent to  $\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1} = S^2$

**Part 2**

$$\begin{aligned}
 \sum_i (X_i - \mu)^2 &= \sum_i (X_i - \bar{X} + \bar{X} - \mu)^2 \\
 &= \sum_i (X_i - \bar{X})^2 + \sum_i (\bar{X} - \mu)^2 + 2 \sum_i (X_i - \bar{X})(\bar{X} - \mu) \\
 &= \sum_i (X_i - \bar{X})^2 + \sum_i (\bar{X} - \mu)^2 + 2(\bar{X} - \mu) \sum_i (X_i - \bar{X}) \\
 &= \sum_i (X_i - \bar{X})^2 + \sum_i (\bar{X} - \mu)^2 + 2(\bar{X} - \mu) \left( \sum_i X_i - n\bar{X} \right) \\
 &= \sum_i (X_i - \bar{X})^2 + \sum_i (\bar{X} - \mu)^2 + 2(\bar{X} - \mu)(n\bar{X} - n\bar{X}) \\
 &= \sum_i (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 \\
 \implies \sum_i (X_i - \bar{X})^2 &= \sum_i (X_i - \mu)^2 - n(\bar{X} - \mu)^2
 \end{aligned}$$

$$\begin{aligned}
 \frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} &= \frac{\sum_i (X_i - \bar{X})^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2} \\
 \implies \sum_i \left( \frac{X_i - \mu}{\sigma} \right)^2 &= \frac{\sum_i (X_i - \bar{X})^2}{\sigma^2} + \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2 \\
 &= \frac{(n-1)S^2}{\sigma^2} + \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2 \\
 \implies \chi_{(n)}^2 &= \frac{(n-1)S^2}{\sigma^2} + \chi_{(1)}^2 \\
 \implies MGF(\chi_{(n)}^2) &= MGF\left(\frac{(n-1)S^2}{\sigma^2} + \chi_{(1)}^2\right) \\
 &= MGF\left(\frac{(n-1)S^2}{\sigma^2}\right) * MGF(\chi_{(1)}^2) \\
 \implies MGF\left(\frac{(n-1)S^2}{\sigma^2}\right) &= \frac{MGF(\chi_{(n)}^2)}{MGF(\chi_{(1)}^2)} \\
 &= \frac{(1-2t)^{-\frac{n}{2}}}{(1-2t)^{-\frac{1}{2}}} \\
 &= (1-2t)^{-\frac{n-1}{2}}
 \end{aligned}$$

which is the MGF of  $\chi_{(n-1)}^2$  ■

**E&R theorem 4.6.2**  $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$  i.i.d., U and V are two different linear combinations of the  $X_i$ 's, then  
 $Cov(U, V) = 0 \iff$  U and V are independent.

**Note** In general, zero covariance doesn't imply independent  
 Example:  $X \sim N(0, 1)$ ,  $Y = X^2$ , clearly X and Y are dependent, but

$$\begin{aligned} Cov(X, Y) &= E[XY] - E[X]E[Y] \\ &= E[X^3] - 0 \cdot E[Y] \\ &= E[X^3] \\ &= \int x^3 f(x) dx \\ &= 0 \quad (\text{since } x^3 f(x) \text{ is centro-symmetric}) \end{aligned}$$

**Unbiasedness of  $S^2$  using the Chi-sq distribution**

$$\begin{aligned} E\left[\frac{(n-1)S^2}{\sigma^2}\right] &= n-1 \\ \implies E[S^2] &= \sigma^2 \end{aligned}$$

This proves  $S^2$  is an unbiased estimator for  $\sigma^2$  under Normal distribution  
 There's another way to prove it under any arbitrary distribution with the assumption that  $X_i$ 's are i.i.d. and  $\mu, \sigma^2$  exists.

## 10 Some relationships among distributions

1.  $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{(n-1)}$
2.  $\frac{\chi^2_{(m)}}{m} \xrightarrow{P} 1$

## 11 Difference between sample variance and variance of sample mean

**variance of sample mean:** Expectation of squared difference of sample mean from the true mean

**sample variance:** average squared difference of each data points in the sample from the sample mean

## 12 Consistent Estimator

**Definition** Let  $T_n$  be an estimator of parameter  $\theta$ ,  $T_n$  is said to be consistent (in probability) if

$$T_n \xrightarrow{P} \theta$$

In words,  $T_n$  converges to  $\theta$  in probability.

**Note** If  $T_n \xrightarrow{a.s.} \theta$  then  $T_n$  is called consistent (almost surely). In this course we will only talk about consistent (in probability)

**Proving consistency using LLN** LLN tells us,  $\bar{X} = \frac{1}{n} \sum X_i \xrightarrow{P} E[X_i]$  for any distribution. Immediately that tells us:

1. If  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$  then  $\bar{X}$  is a consistent estimator of  $\mu$
2. If  $X_i \stackrel{iid}{\sim} Poisson(\lambda)$  then  $\bar{X}$  is a consistent estimator of  $\lambda$
3. And we can say this for few other known distributions

Goal: prove consistency when the estimator is not simply  $\bar{X}$  Still use LLN but with the help of a well known Lemma and the continuous mapping theorem

**Slutsky's Lemma** We have two different sequence  $X_n$  and  $Y_n$

If  $X_n \xrightarrow{P} X$  and  $Y_n \xrightarrow{P} Y$ , then  $X_n + Y_n \xrightarrow{P} X + Y$

If  $X_n \xrightarrow{P} X$  and  $Y_n \xrightarrow{P} Y$ , then  $X_n Y_n \xrightarrow{P} XY$

**Continuous mapping theorem** Let  $X_n \xrightarrow{P} X$  and  $g(\cdot)$  be a continuous function, then  $g(X_n) \xrightarrow{P} g(X)$

**Proving  $S^2$  is a consistent estimator of  $\sigma^2$  ...**

**MSE consistent** An estimator  $T_n$  is called MSE consistent if

$$MSE(T_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

Example: for  $N(\mu, \sigma^2)$   $MSE(\bar{X}) = \sigma^2/n \rightarrow 0$  as  $n \rightarrow \infty$  Therefore  $\bar{X}$  is a MSE consistent estimator of  $\mu$

In naive words, after you have calculated the MSE of an estimator, just check if it goes to zero for large  $n$

**Note** MSE consistent  $\implies$  consistent (in probability)

## 13 Efficient Estimator

**Definition of Efficiency** Let  $T_1$  and  $T_2$  be two different estimators of  $\theta$ , Efficiency of  $T_1$  relative to  $T_2$  is defined as

$$eff(T_1, T_2) = \frac{var[T_2]}{var[T_1]}$$

**Remark**

1.  $eff(T_1, T_2) > 1 \implies T_1$  has smaller variance  $\implies T_1$  is more efficient
2. This comparison is meaningful when  $T_1$  and  $T_2$  are both unbiased or both have the same bias.

**Lower bound of the variance of an unbiased estimator** This famous inequality provides a lower bound for the variance of all the unbiased estimators. In other words it gives a lower bound of the MSE (since Bias = 0). The estimator whose variance achieves this lower bound is said to be efficient. Before we state the inequality let's define few terms...

**Score function,  $S(\theta)$**  The derivative of the log-likelihood

$$S(\theta) = \frac{\partial l(\theta)}{\partial \theta}$$

For the random variable  $X$ ,  $S(\theta|X = x) = \frac{\partial}{\partial \theta} \ln f_{\theta}(x)$ . For an observed i.i.d sample, it's written as  $S(\theta|x_1, x_2, \dots, x_n)$  with

$$S(\theta|x_1, x_2, \dots, x_n) = \frac{\partial}{\partial \theta} \sum_i \ln f_{\theta}(x_i) = \sum_i \frac{\partial}{\partial \theta} \ln f_{\theta}(x_i) = \sum_i S(\theta|x_i)$$

**Fisher Information,  $I(\theta)$**  The function

$$I(\theta) = var_{\theta}[S(\theta|X)]$$

It's the amount of information that each observable random variable  $X$  contains about  $\theta$ .

Information of a sample of size  $n = var[S(\theta|x_1, x_2, \dots, x_n)] = nI(\theta)$

**A plot showing the randomness of  $S(\theta)$**  The likelihood function looks different for different data!

**One important property of  $S(\theta)$**  Under some assumptions,

$$E[S(\theta|X = x)] = 0$$

Which implies

$$E[S(\theta|x_1, x_2, \dots, x_n)] = \sum_i E[S(\theta|x_i)] = 0$$

**Cramer-Rao Inequality** Let  $X_1, X_2, \dots, X_n$  be i.i.d. with density  $f_\theta(x)$ ,  $T(X_1, X_2, \dots, X_n)$  be an unbiased estimator of  $\theta$ , Then under some assumptions on  $f_\theta(x)$ ,

$$\text{var}[T] \geq \frac{1}{nI(\theta)}$$

$\frac{1}{nI(\theta)}$  is also known as the Cramer-Rao lower bound (CRLB)

**Proof of Cramer-Rao Inequality** ...

**Definition of sufficient statistic** A statistic  $T(X_1, X_2, \dots, X_n)$  is said to be sufficient for  $\theta$  if the conditional distribution of  $X_1, X_2, \dots, X_n$ , given  $T = t$ , does not depend on  $\theta$