# The Kalman filter and the square root Kalman filter using only QR decompositions

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## Introduction

In this note, I outline the algorithms for the Kalman filter and the square root Kalman filter using only QR decompositions. The Kalman filter (Kalman (1960)) is a recursive algorithm that estimates the state vector of a linear Gaussian state space model. The square root Kalman filter using only QR decompositions is an alternative to the plain-vanilla Kalman filter, which is proposed by Tracy (2022). The square root Kalman filter is numerically more stable than the Kalman filter.

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### A linear Gaussian state space model

A (invariant) linear Gaussian state space model is defined by the following two equations.

$$x_t = Fx_{t-1} + Eu_t + w_t$$

and

$$y_t = Hx_t + v_t,$$

where $x_t$ is the $(k \times 1)$ state vector , $y_t$ is the $(l \times 1)$ observation vector, $u_t$ is the $(n \times 1)$ exogenous vector, $v_t$ is the $(k \times 1)$ state noise, $w(t)$ is the $(l \times 1)$ observation noise, $t$ is a time index. The matrices $F$, $E$, and $H$ are $(k \times k)$, $(k \times n)$, and $(l \times k)$, respectively. The system noise $w_t$ and the observation noise $v_t$ are sampled from $N(0, W)$ and $N(0, V)$, respectively, where $W$ is a $(k \times k)$ covariance matrix and $V$ is an $(l \times l)$ covariance matrix. The Kalman Filter is used to estimate the state vector $x_t$ given the observations $y_t$.

The aims of the problem obtain the estimate $x_{t|t}$ of the state vector $x_t$ and the estimate $P_t$ of the covariance matrix of the $x_t$ at time $t$. The two estimates are defined as follows:

$$x_{t|t} = E[x_t | y_1, y_2, \ldots, y_t]$$

$$P_{t|t} = \text{cov}[(x_t - x_{t|t})] = \text{E}[(x_t - x_{t|t})(x_t - x_{t|t})^t]$$

**The Kalman filter**

The Kalman filter is implemented using the following recursion:

1. Prediction step:

    1. $x_{t|t-1} = Fx_{t-1|t-1} + Eu_t$

    2. $P_{t|t-1} = FP_{t-1|t-1}F^t + V$

2. Innovation step:

    1. $e_t = y_t - Hx_{t|t-1}$

    2. $s_t = HP_{t-1|t-1}H^t + W$

3. Update step:

    1. $K_t = P_{t|t-1}H^t s_t^{-1}$

    2. $x_{t|t} = x_{t|t-1} + K_t e_t$

    3. $P_{t|t} = (I - K_t H)P_{t|t-1}$

where $x_{t|t}$ is the estimated state vector at, $P_{t|t}$ is the estimated state covariance matrix, and $K_t$ is the Kalman gain at time t.

**The outline of derivation of $P_{t|t}$ and $K_t$**

$$P_{t|t} = \text{cov}[x_t - x_{t|t}]$$
$$P_{t|t} = \text{cov}[x_t - (x_{t|t-1} + K_t e_t)]$$
$$P_{t|t} = \text{cov}[x_t - (x_{t|t-1} + K_t(y_t - Hx_{t|t-1}))]$$
$$P_{t|t} = \text{cov}[x_t - (x_{t|t-1} + K_t(Hx_t + v_t - Hx_{t|t-1}))]$$
$$P_{t|t} = \text{cov}[(I - K_tH)(x_t - x_{t|t-1}) - K_t v_t)]$$
$$P_{t|t} = (I - K_tH)\text{cov}[(x_t - x_{t|t-1})](I - K_tH)^t + K_t\text{cov}[v_t]K_t^t$$
$$P_{t|t} = (I - K_tH)P_{t|t-1}(I - K_tH)^t + K_t R_t K_t^t,$$

where $R_t = \text{cov}[v_t]$.

$$P_{t|t} = P_{t|t-1} - K_tHP_{t|t-1} - P_{t|t-1}H^t K_t^{\ t} + K_t R K_t^t$$
$$\frac{\partial tr(P_{t|t})}{\partial K_t} = -2(HP_{t|t-1})^t + 2K_t S_t = 0$$
$$K_t = P_{t|t-1}H^t S_t^{-1},$$

See Kitagawa, (2010) for more details.

**The square root Kalman filter using only QR decompositions**

The square root Kalman filter is an alternative to the Kalman filter that is numerically more stable. Tracy (2022) proposes the following recursion for The square root Kalman filter using only QR decompositions (hereafter the QR Kalman filter).

The QR Kalman filter is implemented using the following recursion

1. Prediction step

    1. $x_{t|t-1} = Fx_{t-1|t-1} + Eu_t$
    2. $\Sigma_{t|t-1} = gr_r(\Sigma_{t-1|t-1}F^t, \Gamma_v)$

2. Innovation step:

    1. $e_t = y_t - Hx_{t|t-1}$
    2. $G_t = gr_r(\Sigma_{t|t-1}H^t, \Gamma_w)$

3. Update step:

  1. $K_t = [G_t^{-1}(G_t^{-t}H)(\Sigma_{t|t-1})^t(\Sigma_{t|t-1})]^t$

  2. $x_{t|t} = x_{t|t-1} + K_t e_t$

  3. $\Sigma_{t|t} = gr_r(\Sigma_{t|t-1}(I - K_t H)^t, \Gamma_w K_t^{\ t})$,

where $\Sigma_{t|t} = \sqrt{P_{t|t}}$, the Cholesky decompositions of the covariance matrices $P_{t|t}$. The function $gr_r$ returns the matrix $R$ of the QR decomposition of the matrix. The matrices $\Gamma_v$ and $\Gamma_w$ are the Cholesky decompositions of $V$ and $W$, respectively. See Tracy (2022) for more details.

## Running Code

See "run_kf_qr_1.R" and "run_kf_qr_2.R" in "tools/".

**To be added:**

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1 + 1
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[1] 2
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