The Kalman filter

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◦ State space model

(Linear Gaussian state space model)

y_t : $l \times 1$ vector, t is a time index.

$$\text{SSM}\begin{cases} \overset{(k\times1)}{x_t} = \overset{(k\times k)}{F}\,\overset{(k\times1)}{x_{t-1}} + \overset{(k\times n)}{E}\,\overset{(n\times1)}{u_t} + \overset{(k\times1)}{v_t} \\ \overset{(l\times1)}{y_t} = \overset{(l\times k)}{H}\,\overset{(k\times1)}{x_t} + \overset{(l\times1)}{w_t} \end{cases}$$

$$v_t \sim N(\overset{(k\times1)}{0}, \overset{(k\times k)}{V})$$

$$w_t \sim N(\overset{(l\times1)}{0}, \overset{(l\times l)}{W})$$

$Q \to V$

$R \to W$

$V \to P$

Determine $\overset{(k\times1)}{x\_0}$, $\overset{(k\times k)}{P\_0}$

$\underset{(k\times k)}{F}$, $\underset{(k\times n)}{E}$, $\underset{(l\times k)}{H}$, $\underset{(k\times k)}{V}$, $\underset{(l\times l)}{W}$.

◦ Kalman filter recursion

Kalman_recursion ($y_{1:T}$, $u_{1:T}=0$,

$x0$, $P0$, F, $E=0$, H, V, W)

$x_{0|0} = x\_0$ $(k\times1)$

$P_{0|0} = P\_0$ $(k\times k)$

$S = 0$ $(l\times l)$

$e = 0$ $(l\times1)$

I : $(k\times k)$

x_store : $(k\times1\times T)$

V_store : $(k\times k\times T)$

K : $(k\times l)$

for (i in 1:T) {

$(x_{t|t}, P_{t|t}) \leftarrow$ Kalman_filter (y_t, u_t, $x_{t-1|t-1}$,

$P_{t-1|t-1}$, S_{t-1}, e_{t-1}, F, H, V, W, I)

}

The Kalman filter $(y_t, u_t, x_{t-1|t-1}, P_{t-1|t-1}, S_{t-1}, e_{t-1}, F, H, E, V, W, I, K)$

[y_t : $l \times 1$ vector

$$\overset{k\times1}{x_t} = \overset{k\times k}{F}\,\overset{k\times1}{x_{t-1}} + \overset{k\times n}{E}\,\overset{n\times1}{u_t} + \overset{k\times1}{v_t}$$

$$\overset{l\times1}{y_t} = \overset{l\times k}{H}\,\overset{k\times1}{x_t} + \overset{l\times1}{w_t} \qquad x_0$$

[x_t : $k \times 1$ vector

u_t : $n \times 1$ vector, exogeneous

v_t : $k \times 1$, $v_t \sim N(0, V)$ — $k \times k$

w_t : $l \times 1$, $w_t \sim N(0, W)$ — $l \times l$

kalman_prediction$(x_{t-1|t-1}, P_{t-1|t-1}, u_t, F, E, V)$

$$\overset{k\times1}{x_{t|t-1}} = \overset{k\times k}{F}\,\overset{k\times1}{x_{t-1|t-1}} + \overset{k\times n}{E}\,\overset{n\times1}{u_t}$$

$$\overset{k\times k}{P_{t|t-1}} = \overset{k\times k}{F}\,\overset{k\times k}{P_{t-1|t-1}}\,\overset{k\times k}{F^t} + \overset{k\times k}{V}$$

$\rightarrow (x_{t|t-1}, P_{t|t-1})$

kalman_innovation$(y_t, x_{t|t-1}, P_{t|t-1}, H, W)$

$$e_t = \overset{l\times1}{y_t} - \overset{l\times k}{H}\,\overset{k\times1}{x_{t|t-1}}$$

$$\overset{l\times l}{S_t} = \overset{l\times k}{H}\,\overset{k\times k}{P_{t|t-1}}\,\overset{k\times l}{H^t} + \overset{l\times l}{W}$$

$\rightarrow (e_t, S_t)$

kalman_update$(x_{t|t-1}, P_{t|t-1}, H, e_t, S_t, I, K)$

$$\overset{k\times l}{K_t} = \overset{k\times k}{P_{t|t-1}}\,\overset{k\times l}{H^t}\,\overset{l\times l}{S_t^{-1}}$$

$$\overset{k\times1}{x_{t|t}} = \overset{k\times1}{x_{t|t-1}} + \overset{k\times l}{K_t}\,\overset{l\times1}{e_t}$$

$$\overset{k\times k}{P_{t|t}} = (\underbrace{I}_{k\times k} - \underbrace{\overset{k\times l}{K_t}\,\overset{l\times k}{H}}_{k\times k})\,\overset{k\times k}{P_{t|t-1}}$$

$\rightarrow (x_{t|t}, P_{t|t})$

The QR Kalman filter

∘ State space model

y_t : $l \times 1$ vector, t is a time index.

$$
\begin{cases}
\overset{(k\times1)}{x_t} = \overset{(k\times k)}{F}\,\overset{(k\times1)}{x_{t-1}} + \overset{(k\times n)}{E}\,\overset{(n\times1)}{u_t} + \overset{(k\times1)}{v_t} \\
\overset{(l\times1)}{y_t} = \overset{(l\times k)}{H}\,\overset{(k\times1)}{x_t} + \overset{(l\times1)}{w_t}
\end{cases}
$$

$$v_t \sim N(\overset{(k\times1)}{0}, \overset{(k\times k)}{V})$$

$$w_t \sim N(\overset{(l\times1)}{0}, \overset{(l\times l)}{W})$$

$$\overset{(k\times k)}{\Sigma_{t|t}} = \sqrt{P_{t|t}}$$

Determine $\overset{(k\times1)}{x0}$ $\overset{(k\times k)}{Sig0}$

$\overset{(k\times k)}{F}$, $\overset{(k\times n)}{E}$, $\overset{(l\times k)}{H}$, $\overset{(k\times k)}{V}$, $\overset{(l\times l)}{W}$

$\downarrow \sqrt{V}$ $\downarrow \sqrt{W}$

$\overset{(k\times k)}{P_v}$ $\overset{(l\times l)}{P_w}$

(Gmv) (Gmw)

∘ QR Kalman filter recursion

qr_kf_recursion ($y_{1:T}$, $u_{1:T}=0$, $x0$, $Sig0$, F, $E=0$, H, P_v, P_w, e_{t-1}, G_{t-1}, I)

$x_{0|0} = x0$, $(k\times1)$

$\Sigma_{0|0} = Sig0$ $(k\times k)$

$G = 0$ $(l\times l)$

$e = 0$ $(l\times1)$

I : $(k\times k)$ | K : $(k\times l)$

x_store : $(k\times 1\times T)$

Sig_store : $(k\times k\times T)$

for (i in 1:T){

$(x_{t|t}, \Sigma_{t|t}) \leftarrow$ qr_kalman_filter (y_t, u_t, $x_{t-1|t-1}$, $\Sigma_{t-1|t-1}$, F, E, H, P_v, P_w, e_{t-1}, G_{t-1}, I)

}

$qr\_kalman\_filter(\, y_t,\ u_t,\ x_{t-1|t-1},\ \Sigma_{t-1|t-1},\ F,\ E,\ H,\ P_v,\ P_w,\ e_{t-1},\ G_{t-1},\ I \,)$

- $qr\_kf\_prediction(\, x_{t-1|t-1},\ \Sigma_{t-1|t-1},\ u_t,\ F,\ E,\ P_v \,)$

$$\begin{cases} x_{t|t-1} = F\, x_{t-1|t-1} + E\, u_t \\ \Sigma_{t|t-1} = qr\_r(\Sigma_{t-1|t-1} F^t,\ P_v) \end{cases}$$

return $(x_{t|t-1},\ \Sigma_{t|t-1})$

- $qr\_kf\_innovation(\, y_t,\ x_{t|t-1},\ \Sigma_{t|t-1},\ H,\ P_w \,)$

$$\begin{cases} e_t = y_t - H\, x_{t|t-1} \\ G_t = qr\_r(\Sigma_{t|t-1} H^t,\ P_w) \end{cases}$$

return $(e_t,\ G_t)$

$qr\_kf\_update(\, x_{t|t-1},\ \Sigma_{t|t-1},\ H,\ P_w,\ e_t,\ G_t,\ I \,)$

$$\begin{cases} K_t = [\, G_t^{-1} (G_t^{-t} H) (\Sigma_{t|t-1})^t (\Sigma_{t|t-1}) \,]^t \\ x_{t|t} = x_{t|t-1} + K_t\, e_t \\ \Sigma_{t|t} = qr\_r(\Sigma_{t|t-1} (I - KH)^t,\ P_w K_t^t) \end{cases}$$

return $(x_{t|t},\ \Sigma_{t|t})$