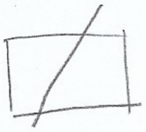
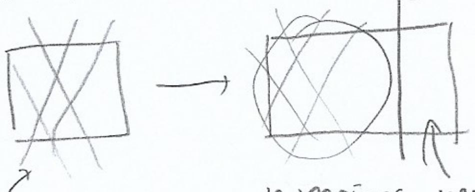
Problem 7

(a) Base step ($n=1$)



$\rightarrow$ 2 regions $1+1=2$ ✓

Inductive step



we can make only one region by adding a new line at the rightmost part of the rectangle. which is creating the most least regions.

$n+1=\underline{n+1}$ ∎

(b) Base step ($n=1$)

$\rightarrow$ 2 regions. $2^1=2$ ✓

Inductive step.

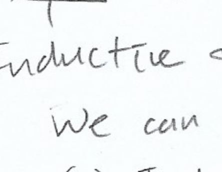
We can make 2^{n-1} regions by drawing $(n-1)$ lines.
($\because$ Induction hypothesis)

From each region, we can make at most 2 regions by drawing line which passes through the region.

Therefore, if we make a line that passes through all regions, we can double the number of the regions.

$\therefore 2\cdot 2^{n-1}=2^n$ ∎

(c)



(d) No, we cannot make a line map with three lines that have eight regions, because if we draw two lines, there will be four regions but we cannot draw a line that contains four points from the regions.

Problem 9

$\alpha = \frac{1+\sqrt{5}}{2} \quad \beta = \frac{1-\sqrt{5}}{2}$

- Base case

$n=1$

$L(1) = \alpha + \beta = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = 1$

$n=2$

$L(2) = \alpha^2 + \beta^2 = \frac{1+2\sqrt{5}+5}{4} + \frac{1-2\sqrt{5}+5}{4} = \frac{12}{4} = 3$ ✓

- Inductive case

I.H.: $L(k) = \alpha^k + \beta^k$ for $k \in (1, n-1)$ (strong induction)

For n

$$
\begin{aligned}
L(n) &= L(n-1) + L(n-2) \\
&= \alpha^{n-1} + \beta^{n-1} + \alpha^{n-2} + \beta^{n-2} \quad (\because \text{I.H.}) \\
&= \alpha^{n-2}(\alpha+1) + \beta^{n-2}(\beta+1) \\
&= \alpha^{n-2} \cdot \alpha^2 + \beta^{n-2} \cdot \beta^2 \\
&= \alpha^n + \beta^n
\end{aligned}
$$

$\blacksquare$

$(\because \alpha^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} + 1 = \alpha + 1$

$\beta^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} + 1 = \beta + 1)$