CS204: Homework #5

Due on November 3, 2016 at 11:59pm

Prof. Sungwon Kang — Section A

20160051 Ohjun Kwon

Problem 1

(a)

$$
\begin{aligned}
H(1) &= 1 \\
H(2) &= 1 \\
H(3) &= H(2) + H(1) - H(0) \\
&= 1 + 1 - 0 = 2 \\
H(4) &= H(3) + H(2) - H(1) \\
&= 2 + 1 - 1 = 2 \\
H(5) &= H(4) + H(3) - H(2) \\
&= 2 + 2 - 1 = 3 \\
H(6) &= H(5) + H(4) - H(3) \\
&= 3 + 2 - 2 = 3 \\
H(7) &= H(6) + H(5) - H(4) \\
&= 3 + 3 - 2 = 4 \\
H(8) &= H(7) + H(6) - H(5) \\
&= 4 + 3 - 3 = 4 \\
H(9) &= H(8) + H(7) - H(6) \\
&= 4 + 4 - 3 = 5 \\
H(10) &= H(9) + H(8) - H(7) \\
&= 5 + 4 - 4 = 5
\end{aligned}
$$

(b)

For $n \geq 0$, we can guess that $H(n) = \left\lfloor \frac{n+1}{2} \right\rfloor$.
$\therefore H(100) = \left\lfloor \frac{100+1}{2} \right\rfloor = 50$

Problem 2

(a)

If we draw 3 points on the leftside and n points on the rightside, we can make a complete bipartite graph by connecting all points on the leftside with the points on the rightside. Then, we can make $K_{3,n}$ with $K_{3,n-1}$ by adding one point on the right and connect it with all the points on the left side (3 points). After doing this job, three more edges are added to the graph. So, we can make a recurrence relation based on this.

$$K_{3,n} = K_{3,n-1} + 3$$

(b)

We can think of the relation between $K_{n-1,n-1}$ and $K_{n,n}$. We can make $K_{n,n}$ from $K_{n-1,n-1}$ by drawing a point on the left side and another point at the right side. Then, we should connect them in line. First, we can draw n lines from the left point we have just drawed to the points at the right side. Next, we can draw $n-1$ lines from the right point to the points at the right side, because we have already drawn a line in between the new points. $\therefore K_{n,n} = K_{n-1,n-1} + 2n - 1$

Problem 3

Base case For $n \geq 0$,

$$\begin{aligned} C(0) &= \frac{3^{0+1} - 2 \cdot 0 - 3}{4} \\ &= 0 \end{aligned}$$

□

Inductive case

$$\begin{aligned} C(n) &= n + 3C(n-1) && (1) \\ &= n + 3 \cdot \frac{3^n - 2(n-1) - 3}{4} && (2) \\ &= n + 3 \cdot \frac{3^n - 2n - 1}{4} \\ &= n + \frac{3^{n+1} - 6n - 3}{4} \\ &= \frac{4n}{4} + \frac{3^{n+1} - 6n - 3}{4} \\ &= \frac{3^{n+1} - 2n - 3}{4} \end{aligned}$$

(1): recurrence relation
(2): inductive hypothesis
□

Problem 4

(a)

$$\begin{aligned} G(0) &= 1 \\ G(1) &= G(0) + 2 \cdot 1 - 1 \\ &= 1 + 2 - 1 = 2 \\ G(2) &= G(1) + 2 \cdot 2 - 1 \\ &= 2 + 4 - 1 = 5 \\ G(3) &= G(2) + 2 \cdot 3 - 1 \\ &= 5 + 6 - 1 = 10 \\ G(4) &= G(3) + 2 \cdot 4 - 1 \\ &= 10 + 8 - 1 = 17 \\ G(5) &= G(4) + 2 \cdot 5 - 1 \\ &= 17 + 10 - 1 = 26 \end{aligned}$$

(b)

We can make a sequence of differences from the six values. Figure 1 can be drawn from the sequence. The second sequence of differences is constant. This suggests that the sequence may have a formula of the form

Problem 4 continued on next page. . .

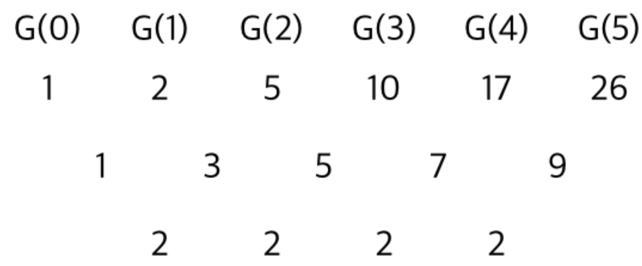


Figure 1: sequence of differences

$G(n) = An^2 + Bn + C$. If we substitute n with 0, 1, 2, and use $G(0)$, $G(1)$, $G(2)$; then we can make a system of equations like below.

$$\begin{aligned} 1 &= C \\ 2 &= A + B + C \\ 5 &= 4A + 2B + C \end{aligned}$$

We can get $A = 1$, $B = 0$, $C = 1$ from the system.
Therefore, we can assume that $G(n) = n^2 + 1$.

(c)

We want to show that the recurrence relation is equal with the hypothesized closed-form formula that we have obtained **4(b)**. Let $G(n)$: the given recurrence relation and $f(n)$: the hypothesized closed-form formula.

Base formula $G(0) = 1 = f(0)$ $\Box$
Inductive formula

$$\begin{aligned} G(n) &= G(n-1) + 2n - 1 && (3) \\ &= f(n-1) + 2n - 1 && (4) \\ &= (n-1)^2 + 1 + 2n - 1 \\ &= n^2 - 2n + 1 + 2n \\ &= n^2 + 1 \\ &= f(n) \end{aligned}$$

(3): recurrence relation
(4): inductive hypothesis
$\Box$

Problem 5

Applying the rule recursively, we can get a set $\mathbb{Z}$.

$$
\begin{aligned}
\mathbb{Z} = \{&(1,5),(1,7),(1,9),\\
&(2,4),(2,6),(2,8),(2,10),\\
&(3,5),(3,7),(3,9),\\
&(4,6),(4,8),(4,10),\\
&(5,7),(5,9),\\
&(6,8),(6,10),\\
&(7,9),\\
&(8,10)\}
\end{aligned}
$$

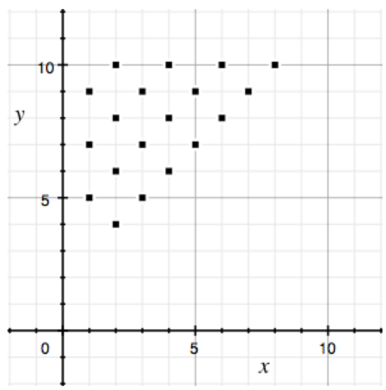


Figure 2: a plot of the set $\mathbb{Z}$

Problem 6

(a)

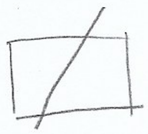
$\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$

(b)

If the set S has an element x, then S should contain a set which has the element x as an element, i.e. $\{x\} \in S$, since $\{x\}$ is a subset of S. However, this rule is applied to the all elements that are added to the S recursively. For example, if x is in S, $\{x\}, \{\{x\}\}, \dots$ should also be the elements of the set S. Therefore, the set S has infinitly many elements.

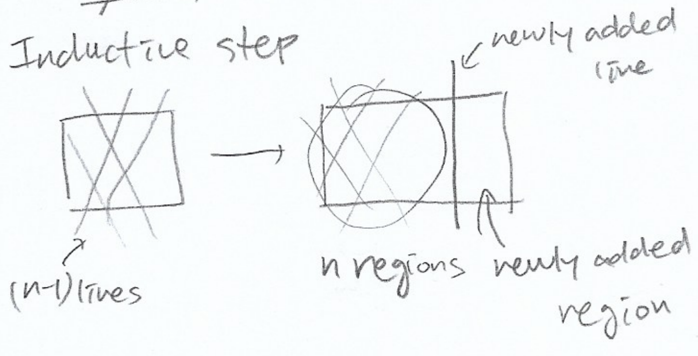
Problem 7

(a) Base step ($n=1$)



$\rightarrow$ 2 regions $1+1=2$ ✓

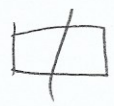
Inductive step



we can make only one region by adding a new line at the rightmost part of the rectangle. which is creating the most least regions.

$n+1 = \underline{n+1}$ ∎

(b) Base step ($n=1$)



$\rightarrow$ 2 regions. $2^1 = 2$ ✓

Inductive step.

We can make 2^{n-1} regions by drawing $(n-1)$ lines.

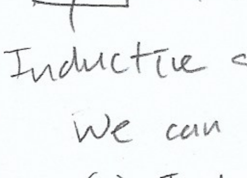
($\because$ Induction hypothesis)

From each region, we can make at most 2 regions by drawing line which passes through the region.

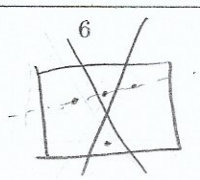
Therefore, if we make a line that passes through all regions, we can double the number of the regions.

$\therefore 2 \cdot 2^{n-1} = 2^n$ ∎

(c)



(d) No, we cannot make a line map with three lines that have eight regions, because if we draw two lines, there will be four regions but we cannot draw a line that contains four points from the regions.



Problem 8

Base case For $n = 1$ (odds' base case),

$$\begin{aligned} H(2 \cdot 1) &= H(2) = 1 \\ H(2 \cdot 1 - 1) &= H(1) = 1 \\ n &= 1 \end{aligned}$$

$\therefore H(2n) = H(2n-1) = n$ for $n = 1$ □

For $n = 2$ (evens' base case),

$$\begin{aligned} H(2 \cdot 2 - 1) &= H(3) = H(2) + H(1) - H(0) \\ &= 1 + 1 - 0 = 2 \\ H(2 \cdot 2) &= H(4) = H(3) + H(2) - H(1) \\ &= 2 + 1 - 1 = 4 \\ n &= 2 \end{aligned}$$

$\therefore H(2n) = H(2n-1) = n$ for $n = 2$ □

Inductive case
i) n is even Let $n = 2k$,

$$\begin{aligned} H(n) &= H(2k) \\ &= H(2k-1) + H(2k-2) - H(2k-3) && (5) \\ &= H(2k-1) + H(2(k-1)) - H(2(k-1)-1) \\ &= k + (k-1) - (k-1) && (6) \\ &= k \end{aligned}$$

ii) n is odd Let $n = 2k - 1$,

$$\begin{aligned} H(n) &= H(2k-1) \\ &= H(2k-2) + H(2k-3) - H(2k-4) && (7) \\ &= H(2(k-1)) + H(2(k-1)-1) - H(2(k-2)) && (8) \\ &= (k-1) + (k-1) - (k-2) \\ &= k \end{aligned}$$

(5), (7): recurrence relation
(6), (8): inductive hypothesis

Problem 9

$\alpha = \frac{1+\sqrt{5}}{2}$ $\beta = \frac{1-\sqrt{5}}{2}$

- Base case

$n=1$

$L(1) = \alpha + \beta = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = 1$

$n=2$

$L(2) = \alpha^2 + \beta^2 = \frac{1+2\sqrt{5}+5}{4} + \frac{1-2\sqrt{5}+5}{4} = \frac{12}{4} = 3$ ✓

- Inductive case

I.H: $L(k) = \alpha^k + \beta^k$ for $k \in (1, n-1)$ (strong induction)

For n

$$L(n) = L(n-1) + L(n-2)$$
$$= \alpha^{n-1} + \beta^{n-1} + \alpha^{n-2} + \beta^{n-2} \quad (\because \text{I.H})$$
$$= \alpha^{n-2}(\alpha+1) + \beta^{n-2}(\beta+1)$$
$$= \alpha^{n-2} \cdot \alpha^2 + \beta^{n-2} \cdot \beta^2 \quad \left(\because \alpha^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} + 1 = \alpha + 1,\ \beta^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} + 1 = \beta + 1\right)$$
$$= \alpha^n + \beta^n$$

▣.

Problem 10

Base case For a element from Q-sequence $< x, 4-x >$, $x + (4-x) = 4$. Therefore, the condition holds.
Inductive case Suppose $< x_1, x_2, \ldots, x_j >$ and $< y_1, y_2, \ldots, y_k >$ is a Q-sequence.
Then, $x_1 + x_2 + \cdots + x_j = 4$ and $y_1 + y_2 + \cdots + y_k = 4$ by the induction hypothesis.
For $< x_1 - 1, x_2, \ldots, x_j, y_1, y_2, \ldots, y_{k-1}, y_k - 3 >$, the sum of the number is

$$\begin{aligned}
&x_1 - 1 + x_2 + \cdots + x_j + y_1 + \cdots + y_{k-1} + y_k - 3 \\
&= x_1 + x_2 + \cdots + x_j + y_1 + y_2 + \cdots + y_{k-1} + y_k - 4 \\
&= 4 + 4 - 4 = 4
\end{aligned}$$

$\therefore$ the proposition holds. $\square$