Vector Bundle and Homology

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Contents

1 Preliminary

The major purpose of this pdf is to introduce homological technique to study geometry both algebraically and differentially. What is homological algebra? It depends, but homological algebra is a linearlization of geometry or its abelianization, which can measure some geometric invariants.

The alternating sum of dimension of homology classes of each degree is Euler characteristic. This is a topological invariant, but in sheaf cohomology, it helps us define Hilbert polynomial, the flatness condition of the algebraic variety.

If the given geometry is complex geometry, homological algebra has extra structures e.g. Hodge structures.

vector bundles. First, the question is how vector/fiber bundle of a geometry arises homological structures. I will introduce what is vector bundle and fibre bundle of a smooth real manifold. These defines global/local sections, where we can find de Rham cohomology. If it further looks carefully, the algebraic structure (complex or almost complex) generates extra data on de Rham cohomology group called Hodge decomposition. Other than vector bundle, Sheaf is a nice generalization of vector bundles, for at least, it has a functorial property, and it can generalize notion of algebraic variety to any moduli space. I'll introduce stack(orbifold) and groupoid theory.

What kind of structures does homological algebra has? As shown in de Rham cohomology, homological algebra is a linearlization of geometry. Or from a categorical viewpoint, it is derived category, having triangulated structures.

2 Homology and Cohomology

2.1 de Rham Cohomology

2.2 Hodge Theory

Definition 2.2.1 *()*

Definition 2.2.2 *(Hodge Decomposition)*

Definition 2.2.3 *(Mixed Hodge Structure)*

2.3 Lie Algebra Cohomology

I'd suppose that the purpose of Lie algebra cohomology is to compare cohomology of a geometry M and its quotient $M//G$ where G is a Lie group.

Definition 2.3.1 *(Lie Algebra Cohomology)*

Chevalley-Eilenberg complex.
$Hom(\bigwedge^\bullet \mathfrak{g}, M)$

Definition 2.3.2 *(Hamiltonian Vector Field)*

Moment map is similar to Hamiltonian vector field.

Definition 2.3.3 *(Symplectic Reduction)*

Let (M, ω) be a symplectic manifold where a Lie group G acts on it, and $\Phi : M \to \mathfrak{g}^*$ is a momentum map. Let $M_0 = \Phi^{-1}(0)$ and $0 \in \mathfrak{g}^*$ is a regular value. Then, there exists a symplectic quotient $\tilde{M} = M_0/G$ (often denoted as $M//G$), and this $\tilde{M}$ is also a symplectic manifold whose differential form is induced by the pullback as $\pi^*\tilde{\omega} = \iota^*\omega$ where $\pi : M_0 \to \tilde{M}$ and $\iota : M_0 \to M$.

Then the functional space is Lie algebra cohomology.
$C^\infty(\tilde{M}) = C^\infty(M_0)^{\mathfrak{g}} = H^0(\mathfrak{g}, C^\infty(M_0))$

where $C^\infty(M_0) = C^\infty(M)/$
since every smooth function on M_0 can be extended to M. coincides with the ideal generated by momentum map $I[\Phi]$.

Definition 2.3.4 *(Koszul Resolution)*

From symplectic reduction $\tilde{M}$ of a geometry M, we'll construct a Koszul resolution

$K^\bullet = \bigwedge^\bullet C^\infty(M) \to C^\infty(M_0)$.

For a derivation $\delta : K^q \to K^{q-1}$,

$\delta f = 0$ *and* $\delta X = \phi_X$. *Let X_i be a basis of $\mathfrak{g}$.*

Similarly, the higher degree $\delta : \mathfrak{g} \otimes C^\infty(M) \to C^\infty(M)$ is

$\delta(\Sigma_i X_i \otimes f_i) = \Sigma_i f_i \phi_i$

For $\bigwedge^2 \mathfrak{g} \otimes C^\infty(M) \to \mathfrak{g} \otimes C^\infty(M) \to C^\infty(M) \to 0$

$\delta(X \wedge Y \otimes f) = Y \otimes \phi_X f - X \otimes \phi_Y f$

where δ is extended to an odd derivation. Notice that $\delta^2 = 0$.

Definition 2.3.5 *(BRST Complex)*

BRST complex is a double complex $(C^{\bullet,\bullet}, D)$

where $C^{p,q} = C^p(\mathfrak{g}; K^q) = \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^p \mathfrak{g} \otimes C^\infty(M)$ and $D = \{Q, -\}$ is the differential.

Or explicitly, $Q = \alpha^i \phi_i - \frac{1}{2} f^i_{jk} \alpha^j \wedge \alpha^k \wedge X_i$

where X_i is a basis of $\mathfrak{g}$ and $[X_i, X_j] = f^i_{jk} X_k$ and α^j is dual basis of $\mathfrak{g}^*$.

2.4 Floer Homology

Definition 2.4.1 *(Hessian and Morse Index)*
For a geometry M, if $f : M \to \mathbb{R}$ is a smooth function, let $p \in M$ be a critical point $f'(p) = 0$, then Hessian $Hf_p(u, v)$ is

$Hf_p(u, v) = \Sigma^n_{i,j=1} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) a_i b_j$

where $u = \Sigma^n_{i=1} a_i \frac{\partial}{\partial x_i}_p$ and $v = \Sigma^n_{j=1} b_j \frac{\partial}{\partial x_j}_p$. If f is a Morse function (that is definable from any smooth functions), the Hessian Hf_p is nondegenerate.
Finally, we define Morse index $\mu(p)$ as the number of negative eigenvalues of the Hessian $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$.

Definition 2.4.2 *(Morse Homology)*

Let's define Morse chain complex.

For each k, the C_k is generated by critial points of degree k as $C_k := \oplus_{\mu(p)=k} \mathbb{Z}p$.

Let's define Morse differential.

For two critical points $p, q \in M$, the set

$M(p, q) = \{x : \mathbb{R} \to M | \dot{x} = -grad(f), \lim_{\tau \to -\infty} x(\tau) = p, \lim_{\tau \to \infty} x(\tau) = q, \}$

descibes path which follows the gradient. Further we take quotient as $\hat{M}(p, q) = M(p, q)/\mathbb{R}$ by translation action.

The derivative at critical point $p \in M$ is

$$\partial : C_k \to C_{k-1}$$
$$\partial p = \Sigma_{\mu(q)=k-1} \# \hat{M}(p,q) q$$

Indeed $\partial^2 = 0$, so the Morse chain complex is a chain complex.

In fact, Morse homology is isomorphic to singular homology.

Definition 2.4.3 *(Singular Homology)*

Remark 2.5 *(Moment Map and Morse Theory)*

Definition 2.5.1 *(Lagrangian Intersection)*

If M is a symplectic manifold, a smooth function $H_t : M \times [0,1] \to \mathbb{R}$ has a vector field X_t such that $\omega(\cdot, X_t) = dH_t$ called Hamiltonian vector field. Further, we define a Hamiltonian isotopy ϕ_t such that

- $d\phi_t = X_t \circ \phi_t$
- $\phi_0 = id$

$\phi_t : M \to M$ is a symplectomorphism, and if $L \subset M$ is a Lagrangian submanifold, $\phi_t(L)$ is also a Lagrangian submanifold. For example, 0_X is a zero section of the cotangent bundle T^*X and its Lagrangian submanifold.

For a function $H = f \circ \pi : T^*M \to \mathbb{R}$ where $\pi : T^*M \to M$ is a projection, Its Hamiltonian isotopy ϕ_t is, in particular, $\phi_1(0_X)$ is the graph of df. If f is a Morse function, 0_X and $\phi_1(0_X)$ intersects transversally.

Floer homology is an infinite dimensional analogue of finite dimensional Morse theory.

Definition 2.5.2 *(Floer Homology)*

For ϕ_t, we define

$$\Omega = \{l : [0,1] \to M | l(0) \in L, l(1) \in \phi_1(L), l \textit{ is homotopic to } \phi_t(x_0)\}$$

where $x_0 \in L$ is a fixed point. The universal covering space $\tilde{\Omega}$ of Ω is

$$\tilde{\Omega} = \{u : [0,1] \times [0,1] \to M | u(\tau, 0) \in L, u(\tau, 1) \in \phi_1(L), u(0,t) = \phi_t(x_0)\} / \textit{homotopy}$$

We introduce a function $F : \tilde{\Omega} \to \mathbb{R}$

$F = \int_0^1 d\tau \int_0^1 dt\omega(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial \tau})$

Now taking two intersection points $p, q \in L \cap \phi_1(L)$*, the set*

$M(p,q) = \{u : \mathbb{R} \to \Omega | \frac{\partial u}{\partial \tau} = -grad(F), \lim_{\tau \to -\infty} u(\tau, [0,1]) = p, \lim_{\tau \to \infty} u(\tau, [0,1]) = q, \}$

Now Ω *is infinite dimensional smooth manifold with Riemannian metric, we are similarly allowed to define Morse theory.*

2.6 Simplicial Method

In this section, we define cotangent complex. Simplicial method generates a simplial resolution.

Definition 2.6.1 *(Simplex Category)*
A simplicial category Δ *is a category whose objects are* $[n] = \{0, 1, 2, \cdots, n\} \in \Delta$*, and its morphisms are non-strictly increasing map of finite order, which might be generated by coface and codegeneracy maps.*

- *(coface)*
  $\delta^{n,i} : [n-1] \to [n]$

  is a morphism that misses i namely, $\delta^{4,2} : [3] \to [4]$ *maps to* $(0, 1, 2, 3) \mapsto (0, 1, 3, 4)$.

- *(codegeneracy)*
  $\sigma^{n,i} : [n+1] \to [n]$

  is a morphism that counts i twice: namely, $\sigma^{4,2} : [5] \to [4]$ *maps to* $(0, 1, 2, 3, 4, 5) \mapsto (0, 1, 2, 2, 3, 4)$.

Definition 2.6.2 *(Simplicial Object)*

A simplicial object U *is a contravariant functor*

$U : \Delta \to \mathscr{C}$

esp if $\mathscr{C}$ *is a Set, then* U *is simplicial set. Or if* $\mathscr{C}$ *is a category of commutative ring, then* U *is simplicial commutative ring.*

Now, denote $U_n = U([n])$*,* $d_j^n = U(\delta_j^n) : U_n \to U_{n-1}$ *and* $s_j^n = U(\sigma_j^n) : U_n \to U_{n+1}$

Definition 2.6.3 *(Skeleton)*

$$sk_n : Simp(\mathscr{C}) \to Simp_n(\mathscr{C}) \ cosk_n : Simp_n(\mathscr{C}) \to Simp(\mathscr{C})$$

Definition 2.6.4 *(Simplicial object and fiber product)*

Let V and W be simplicial objects of a category $\mathscr{C}$, and morphism $a : V \to U$ and $b : W \to U$. Assume product $V_n \times_{U_n} W_n$ exists in $\mathscr{C}$, then

- $(V \times_U W)_n = V_n \times_{U_n} W_n$
- $d_i^n = (d_i^n, d_i^n)$
- $s_i^n = (s_i^n, s_i^n)$

In other words, $U \times V$ is a product of presheaves on U and V on Δ.

In fact, for any ring map $A \to B$, the simplicial resolution is of finite type, and in fact a trivial Kan fibration.

Example 2.7 *(Linear algebra on Fiber product)*

Consider the tensor product of modules

$$k[x, y] = k[x] \otimes_k k[x]$$

$$k[x, y, z] = k[x, y] \otimes_{k[x]} k[x, y]$$

and the dual of tensor product is fiber product. Consider $Spec(k) = pt$, $Spec(k[x]) = \mathbb{A}^1$, $Spec(k[x, y]) = \mathbb{A}^2$, and $Spec(k[x, y, z]) = \mathbb{A}^3$, and the tensor product of modules above will be

$$\mathbb{A}^2 = \mathbb{A}^1 \otimes_{pt} \mathbb{A}^1$$

$$\mathbb{A}^3 = \mathbb{A}^2 \otimes_{pt} \mathbb{A}^2$$

Now in general, Let $X_i = k[x_0, x_1, \cdots, x_i]$, then $X_{i+1} = X_i \otimes X_{i-1} X_i$.

Definition 2.7.1 *(Simplicial Resolution)*

Let A be a ring, and B be A-algebra. We'll construct a simplicial resolution $\epsilon : P^{\cdot} \to B$.

where $P_0 = B$, $P_1 = A[B]$, $P_2 = A[A[B]]$, and $P_n = A[A[A[\cdots A[B] \cdots]$ and so on.

This can be alternatively $P. = X.(B)$.

Here, X. is defined as follows.

Consider functors $\mathscr{C} \leftrightarrows F, U\mathscr{D}$, where $\mathscr{D} = Set$, $\mathscr{C} = k - Mod$ and forgetful functor U and some functor F which maps a set to generating object, that is left adjoint to for. We let composition $X_0 = F \circ U$, and $X_1 = U \circ F \circ U \circ F$, $X_n = U \circ F \circ \cdots \circ F$ of n times of iterations. Notice X_n is an endfunctor $X_n \in Fun(\mathscr{C}, \mathscr{C})$.

Write $L = FU$ and $\delta = F\eta U$, so that we define counit and comultiplication

- $\epsilon : L \to I$
- $\delta : L \to L^2$

The pair (L, ϵ, δ) is called comonad.

- $d_i = L^{n-i}\epsilon L^i : L^{n+1} \to L^n$
- $s_i = L^{n-1-i}\delta L^i : L^n \to L^{n+1}$

for $i = 0, 1, 2, \cdots, n-1$.

2.8 Cotangent Complex

Definition 2.8.1 *(Cotangent Complex)*
The cotangent complex is $L_{B/A} = \Omega_{P./A} \otimes P./\epsilon B$

whose homology is

$H^0(L_{B/A}) = \Omega_{B/A}$

Definition 2.8.2 *(Derived Lower Schriek)*

If category $\mathscr{C}$ is Derived category,

Definition 2.8.3 *(Chaotic/Indiscrete Topology)*

A category $\mathscr{C}$ has a canonically a site if $\{f : V \to U | f \text{ is an isomorphism}\}$ is a covering of U.

This corresponding topology is caleld chaotic or indiscrete topology.

Definition 2.8.4 *(Derived Lower Schriek)*
ex:

$\mathscr{C}^{op}_{B/A}$ is a category with object $\alpha : P \to B$ or $\alpha : P' \to B$ and morphism is $s : \alpha \to \alpha'$.

Let $A \to B$ be a ring map.

$L_{B/A} = L_{\pi!}(Li^*\Omega_{\mathscr{O}/A}) = L_{\pi!}(i^*\Omega_{\mathscr{O}/A}) = L_{\pi!}(\Omega_{\mathscr{O}/A} \otimes_{\mathscr{O}} \underline{B})$

in $D(B)$ where $Li^* : D(\mathscr{O}) \to D(\underline{B})$ and $L_{\pi!} : D(\underline{B}) \to D(B)$.

Definition 2.8.5 *(Polynomial Differential Form)*

$\Omega^{\cdot}_{poly}([n]) = \mathbb{Q}[t_1, \cdots, t_n, dt_1, \cdots, dt_n]/(\Sigma t_i - 1, \Sigma dt_i)$

and morphism $[n] \to [m]$ induces functorially $\Omega([m]) \to \Omega([n])$, and $t_i \mapsto \Sigma_{u(j)=i} t_j$.

2.9 Quantum Cohomology

Definition 2.9.1 *(Quantum Cohomology)*

Definition 2.9.2 *(Gromov-Witten Invariant)*

3 Vector Bundle

3.1 Linear Algebra

The very question in linear algebra is what is the specific applications in algebra and geometry. First of all, how trace is used in Lagrangian, or trace incharacter theory. Exterior algebra for volume form and Hodge star operator.

Definition 3.1.1 *(Hermitian Inner Product)*

Hermitian inner product is

- *bilinear,*
- $< \alpha u, v >= \alpha < u, v >$
- $< u, \alpha v >= \overline{\alpha} < u, v >$
- $\overline{< u, v >} =< v, u >$
- $< u, u >\geqq 0$ *where* $< u, u >= 0$ *iff* $u = 0$.

A basic example if $h(z, w) = \Sigma z_i \overline{w_i}$

Definition 3.1.2 $< A^*u, v >=< u, Av >$

Definition 3.1.3 *(Trace)*

Example 3.2 *(Trace)*

- *(Character)*
  A character of ρ *is* $\chi(g) = Tr(\rho(g))$

- *(Hermitian metric)*
  $< A, B >= Tr(A^\dagger B)$

- *(Idempotent Matrix)*
  If $A = A^2$, $Tr(A) = dim(A)$ *because eigenvalue of* A *is either* 1 *or* 0. *So if* $A = Q^{-1}\Lambda Q$ *where* Λ *is a diagonal matrix whose value is either* 1 *or* 0.

- *(Exponential)*
  $det(exp(tA)) = 1 + t(Tr(A)) + O(t^2)$

Definition 3.2.1 *(Hodge Star Operator)*

A Hodge star is a linear operation in the exterior algebras $\star : \bigwedge^k(V) \to \bigwedge^{n-k}(V)$, *and Hodge star maps an object* $w \in V$ *to its dual* $\star w \in V$, *and twice the operations is plus-minus identity as* $\star \circ \star = (-1)^{k(n-k)}Id$. *Hodge star operator is multi-linear map.*

Let's see an explicit illustration. The domain $\bigwedge^k(V)$ *consists of* k*-dim bivector space. If* $k = 1$, *an element is a vector, and vector is an arrow, vector is geometrically interpreted as* 1*-dim bounded line. For higher dimension, we let bivector given by wedge product* $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ *as a* k*-dimensional cube as a subset of* V.

For example, if $n = 3$, *the hodge dual of a vector* $e_1 \in \mathbb{R} = V$ *is* $e_2 \wedge e^3 \in \mathbb{R} \wedge \mathbb{R}$, *so the Hodge dual corresponds to the complement space.*

Definition 3.2.2 *(Vector Field)*

A vector field X *is a linear map* $X : C^\infty(M) \to C^\infty(M)$ *that satisfes product rule.*

$X(fg)p = fX(g) + X(f)g$ *for all* $f, g \in C^\infty(M)$

3.3 Vector Bundle and Homology

Vector bundle is an abstraction of differential forms on a smooth manifold M. In particular, vector bundle is tangent bundle TM, whose tensor or direct sum

$\bigwedge^* TM$ and $TM \oplus TM$ and dual TM^*, and trivially line bundle $M \to \mathbb{R}$. In differential form, differential of 1-form becomes 2-forms, and from 0-form to 1-form, so in analogy, we could define a morphism from the global section of a vector bundle to another global section of a vector bundle, which is called connection $\nabla : \Gamma(E \otimes \bigwedge^* TM) \to \Gamma(E \otimes \bigwedge^* TM)$ (what is differential? It is any morphism what satisfies product rule).

3.3.1 what is vector bundle

Definition 3.3.1 *(Vector Bundle)*
A vector bundle is a morphism $\pi : E \to B$ such that the inverse $\pi^{-1}(x) = E_x$ is a vector space.

A section is an inverse map

$$\sigma : B \to E$$

such that comosition $\pi \circ \sigma = id$ is identity.

We will denote $\Gamma(E, U)$ for $U \subset B$ as section.

Remark 3.4 *(Zero Section)*

Zero section $\sigma : B \to E$ is local/global section of vector bundle, and the morphism maps to zero as $\sigma(x) = 0 \in E_x$, for $x \in B$.

Remark 3.5 *(Existence of Global Section)*

The global section of vector bundle always exists. At least zero section is a global section, and every vector bundle has it. On the other hand, global section of principal bundle exists iff it's trivial bundle. Of course, we are interested in any non-zero sections. There exists nowhere vanishing global section in a vector bundle if its Euler class is zero.

3.5.1 Connection Form of Vector Bundle

Definition 3.5.1 *(Connection Form)*

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M) = \Gamma(E) \otimes \Gamma(T^*M)$$

$$\nabla(fv) = \nabla(v) \otimes df + f\nabla(v)$$

where ∇ is the exterior derivative of f.

In general, the connection form is generalized to n-th power product

$\nabla : \Gamma(E) \to \Gamma(E \otimes \bigwedge^n T^*M) = \Gamma(E) \otimes \Gamma(\bigwedge^n T^*M)$

$\nabla(v \wedge \alpha) = \nabla(v) \wedge d\alpha + (-1)^{deg(v)} v \wedge d\alpha.$

Definition 3.5.2 *(Flat Connection)*

If the curvature derived from the connection vanishes, the connection is flat.

Definition 3.5.3 *(Curvature Form)*

The problem is curvature from derived from the connection ∇ has a two different way of representations.

First, the curvature form of connection is $F_\nabla = dA + A \wedge A$ or $A \wedge A = \frac{1}{2}[A, A]$. Or alternatively, $\Omega = d\omega + \omega \wedge \omega$. These two forms have actually the same Riemannian curvature tensor.

Explicitly, $F_\nabla(s) = \Sigma_{i,j=1}^{k} \Sigma_{p,q=1}^{n} R_{pqi}^{j} s^i dx^p \wedge dx^q \otimes e_j$

where R_{pqi}^{j} is a Riemann curvature tensor of F_∇.

Definition 3.5.4 *(Hermitian Metric)*

$h_p(\eta, \overline{\zeta}) = \overline{h_p(\zeta, \overline{\eta})}$ *for all* $\zeta, \eta \in E_p$ *and*

$h_p(\zeta, \zeta) > 0$ *for all* $\zeta \neq 0 \in E_p$

Hermitian metric is a complex valued function, but we can also define Riemannian metric if it we add conjugate, or we subtract to make (1,1)-form on M.

$g = \frac{1}{2}(h + \overline{h})$
$\omega = \frac{i}{2}(h - \overline{h})$

Definition 3.5.5 *(Hermitian Connection (or Dolbeault Operator))*

A Hermitian connection ∇ is a connection on a Hermitian vector bundle E over a smooth manifold M compatible with a Hermitian metric $< \cdot, \cdot >$ on E, meaning that

$v < s, t > = < \nabla_v s, t > + < s, \nabla_v t >$

for all smooth vector field $v \in C^\infty(M, E)$ and all smooth sections s, t of E. Then there is a unique Hermitian connection whose $(0, 1)$ part is Dolbeault operator $\overline{\partial}_E$ on E associated to the holomorphic structure. Formally, Dolbeault operator is a projection $\overline{\partial}_E = \pi^{(0,1)}\nabla$. This ∇ is called Chern connection on E.

More explicitly, any section $s \in \Gamma(E)$ is written as $s = \Sigma s^i e_i$ for some function $s^i \in C^\infty(U_\alpha, \mathbb{C})$. Define an operator locally by

$\overline{\partial}_E(s) := \Sigma \overline{\partial}(s^i) \otimes e_i$ *where $\overline{\partial}_E$ is a Cauchy-Riemann operator of the base manifold. That is,*

$\overline{\partial}(\alpha) = \Sigma \frac{\partial f_{IJ}}{\partial \overline{z}^l} d\overline{z}^l \wedge d\overline{z}^I \wedge d\overline{z}^J$
(Curvature)
The curvature $\Omega = d\omega + \omega \wedge \omega$ be the curvature form of ∇. Since $\pi^{0,1}\nabla = \overline{\partial}_E$ is vanishing if squared, so $\Omega^{2,0}$ is trivial, so ∇ is skew-symmetric $\Omega^{0,2}$. Thus Ω is a $(1,1)$-form given by

$\Omega = \overline{\partial}_E \omega$

Definition 3.5.6 *(Dolbeault Complex)*

From the above definition of $\overline{\partial}_E$, Dolbeault complex is a complex if $\overline{\partial}_E^2 = 0$. Note in a complex manifold, $\overline{\partial}_E^2 = 0$ always exists, but in general (e.g. symplectic manifold), it doesn't always work.

Definition 3.5.7 *(Holomorphic Vector Bundle)*

Over a complex manifold X, the projection map $E \to X$ is holomorphic, which means the local trivialization is a biholomorphic map $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$.

Example 3.6 *(Holomorphic Tangent Bundle)*

For a complex manifold M, the real $2n$-dimensional real vector bundle TM on M, and endomorphism $J : TM \to TM$ s.t. $J^2 = -Id$.

If complexified, $J : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ where $J(X + iY) = J(X) + iJ(Y)$ where $X, Y \in TM$

Since $J^2 = -Id$, eigenvalues are i and $-i$, and it splits

$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$

The holomorphic tangent bundle of M is vector bundle $T^{1,0}M$, and anti-holomorphic tangent bundle is $T^{0,1}M$.

Definition 3.6.1 *(Hermtian Yang-Mills Connection (or Instanton))*

For a principal bundle $P \to E$, a connection $A_\alpha \in \Omega^1(U_\alpha, ad(P))$. If the connection A satisfies $d_A \star F_A = 0$, then A is Yang-Mills connection (also called instanton). This is Hermitian YM connection, if in addition, the connection

A is connection on a Hermitian vector bundle E over a Kähler manifold X of dimension n.

Then the Hermitian YM equaitons are:

$F_A^{0,2} = 0$
$F_A \cdot \omega = \lambda(E) Id$ *for some* $\lambda(E) \in \mathbb{C}$

Recall the curvature F_A is skew-Hermitian, so $F_A^{0,2} = 0$ implies $F_A^{2,0} = 0$. If X is compact, then $\lambda(E)$ can be computed using Chern-Weil theory. Namely

$$\begin{aligned} deg(E) &= \int_X c_1(E) \wedge \omega^{n-1} \\ &= \frac{i}{2\pi} \int_X Tr(F_A) \wedge \omega^{n-1} \\ &= \frac{i}{2\pi} \int_X Tr(F_A \wedge \omega)\omega^n \end{aligned} \tag{1}$$

Hence,

$\lambda(E) = -\frac{2\pi i}{n! Vol(X)} \mu(E)$

$\mu(E) = \frac{deg(E)}{rank(E)}$

3.6.1 Elliptic Complex and Hodge Theory

First a warning that de Rham cohomology might be merely a real analytic notion, so in complex geometry, complexity has an extra symmetry in the homological algebra, and the adjoint operator of the differential operator defines Laplacian, thus Hodge decomposition. A generalization of de Rham cocomplex is elliptic complex, and "elliptic" sounds real analytic (indeed in complex geometry, we cannot distinguish elliptic and hyperbolic, or if we mention elliptic, it might be real analytic). In elliptic comlex, the differential operator is elliptic, meaning that the differential is non-vanishing (what do I mean by that).

Definition 3.6.2 *(Adjoint Operator)*

(Adjoint Operator)
Given an operator is a $T = \Sigma a_k(x) D^k$ where D is differential, adjoint operator T^* is an operator that satisfies $< Tu, v > = < u, T^* v >$ where $< \cdot, \cdot >$ is an inner product.

(Laplacian)
Given a de Rham cochain complex $(\Omega^* * (X), d)$, let adjoint operator be δ :

$\Omega^{k+1}(X) \to \Omega^k(X)$, *and the Laplacian* $\Delta = d\delta + \delta d : \Omega^k(X) \to \Omega^k(X)$

(Harmonic)
A form on X *is harmonic if* $H^k_\Delta(M) = \{\alpha \in \Omega^k(X) | \Delta\alpha = 0\}$

Proposition 3.7 *(Hodge Decomposition)*

There exists a unique decomposition

$\omega = d\alpha + \delta\beta + \gamma$

in which γ *is harmonic* $\Delta\gamma = 0$ *In terms of* L^2 *metric on differential forms, this gives an orthogonal direct sum decomposition*

$\Omega^k(M) \cong im(d_{k-1}) \oplus ker(d_{k+1}) \oplus H^k_\Delta(M)$

Hodge theorem states that $H^k_\Delta(M) \cong H^k(M, \mathbb{R})$

Proposition 3.8 *(Hodge Conjecture)*

Let X *be a non-singular complex projective variety, and let Hodge classes be*

$Hdg^k(X) = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$

We claim every Hodge class $Hdg^k(X)$ *on* X *is a linear combination with rational coefficients of the cohomology clases of complex subvarieties of* X*.*

Definition 3.8.1 *(Pullback Bundle)*

Let $\pi : E \to B$ *be a fiber bundle and* $B' \to B$ *be a continuous map. Define a pullback bundle is*

$f^*E = \{(b', e) \in B' \times E | f(b') = \pi(e)\} \subset B' \times E$

and the projection map $\pi' : f^*E \to B'$ *given by the projection onto the first factor, i.e.* $\pi'(b', e) = b'$ *and another projection* $h : f^*E \to E$*. These maps creates a commutative diagram*

$$\begin{tikzcd} f^*E \arrow[r, "h"] \arrow[d, "\pi^*"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Definition 3.8.2 *(Elliptic Operator)*

Elliptic operator is a generalization of Laplacian. Let L be a linear differential operator of order m in the domain $\Omega \subset \mathbb{R}^n$ given by

$$Lu = \Sigma_{|\alpha| \leqq m} a_\alpha(x) \partial^\alpha u$$

L is called Elliptic if for all $x \in \Omega$ and every non-zero $\xi \in \mathbb{R}^n$,
$\Sigma_{|\alpha| \leqq m} a_\alpha(x) \xi^\alpha \neq 0$

Definition 3.8.3 *(Elliptic Complex)*

Let $E_0, E_1, \cdots, E_k$ be vector bundles on a smooth manifold M.
For the sequence of vector bundles $0 \to E_0 \to E_1 \to \cdots \to E_k \to 0$.

For the sequence of vector bundles

$$0 \to \Gamma(E_0) \to^{P_1} \Gamma(E_1) \to^{P_2} \cdots \to^{P_k} \Gamma(E_k) \to 0$$

such that $P_{i+1} \circ P_i = 0$.

$$0 \to \pi^* E_0 \to^{\sigma(P_1)} \pi^* E_1 \to^{\sigma(P_2)} \cdots \to^{\sigma(P_k)} \pi^* E_k \to 0$$

is exact outside of zero section. Here π is the projection of cotangent bundle $T^*M \to M$ and π^* is the pullback of a vector bundle.

Definition 3.8.4 *(Hodge Theory on Elliptic Complex)*

Let elliptic complex be $L_i : \Gamma(E_i) \to \Gamma(E_{i+1})$ be linear differential operators, and denote it by

$$E^{\cdot} = \oplus \Gamma(E_i)$$

$$L = \oplus L_i : E^{\cdot} \to E^{\cdot}$$

and let $L^{\cdot}$ is the adjoint of L, and $\Delta = LL^{\cdot} + L^{\cdot}L$. As usual, this yields the harmonic sections

$$\mathfrak{H} = \{e \in E^{\cdot} | \Delta e = 0\}$$

Let $H : E^{\cdot} \to \mathfrak{H}$ be the orthogonal projection, and let G be the Green's operator for Δ. Then, Hodge theorem states

- *H and G are well-defined*
- *$Id = H + \Delta G = H + G\Delta$*
- *$LG = GL$ and $L^{\cdot}G = GL^{\cdot}$*

- $H(E_j) \cong \mathfrak{H}(E_j)$ *(cohomology correspondence)*

Definition 3.8.5 *(Cotangent Bundle)*

Cotangent bundle $T^*M \to M$ is a vector bundle dual to tangent bundle.

Definition 3.8.6 *(Almost)*

Definition 3.8.7 *(Integrability)*

A smooth real manifold is integrable if $\overline{\partial}_E^2 = 0$.

In particular, an almost complex manifold is integrable iff it's a complex manifold.

4 Fiber Bundle

4.1 Principal G-Bundle

Definition 4.1.1 *(Eheresmann Connection)*

4.2 Lagrangian Fibration

4.3 Elliptic Fibration

5 Sheaf and Cohomology

5.1 Sheaf by Examples

5.1.1 Sheaf by Examples

Example 5.2 *(Structure sheaf)*

Example 5.3 *(Constantt Sheaf)*

Example 5.4 *(Locally Constant Sheaf)*

Example 5.5 *(Perverse Sheaf)*

5.5.1 Basic Operation of Sheaf

Example 5.6 *(Six Oerations)*

Example 5.7 *(Six Oerations)*

5.8 Cotangent Sheaf

Definition 5.8.1 *(Tangent Sheaf)*

The sheaf θ_X *on a scheme* X *is for all* $U = Spec(A)$, $\theta_X(U) = Der_k(A, A)$, *or equivalently, it is a sheaf of the morphism* $Hom(\Omega^1_{X/k}, \mathscr{O}_X)$

The stalk is Zariski tangent space.

Definition 5.8.2 *(Cotangent Sheaf)*

Let $\mathscr{F}$ *be an* $\mathscr{O}_X$ *module.*

Cotangent sheaf is a sheaf of $\mathscr{O}_X$ *modules* $\Omega_{X/S}$ *that represents the* S*-derivation in the sense.*

$Hom(\Omega^1_{X/S}, F) = Der_S(\mathscr{O}_X, F)$

where $\Omega_{X/S}$ *is a Kähler differential.*

Definition 5.8.3 *(Kodaira-Spencer Map)*

Let M *be a complex manifold, and it has a transition maps*

$$\begin{tikzcd} M \arrow[r, "="] \arrow[d] & M \arrow[d] \\ U_i \subset \mathbb{R}^n \arrow[r, "f_{ij}"] & U_j \subset \mathbb{R}^n \end{tikzcd}$$

whose deformation is

$$\begin{tikzcd} \mathscr{M} \arrow[r, "="] \arrow[d] & sheaf M \arrow[d] \\ B \arrow[r, "\tilde{f}_{ij}"] & B \end{tikzcd}$$

where $B = U_i \times Spec(k[\epsilon])$, *and Kodaira-Spencer map is* $KS : T_0B \to H^1(M, T_m)$.

In scheme theory, deformation of smooth manifold can be

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ Spec(k) & \longrightarrow & Spec(k[\epsilon]) \end{array}$$

have a short exact sequence

$0 \to \pi^*\Omega^1_{Spec(k[\epsilon])} \to \Omega^1_{\mathscr{X}} \to \Omega^1_{\mathscr{X}/S} \to 0$

where $\pi : \mathscr{X} \to S = Spec(k[\epsilon])$. *If we tensored by the* $\mathscr{O}_{\mathscr{X}}$*-module* $\mathscr{O}_X$ *gives the short exact sequence*

$0 \to \mathscr{O}_X \to \Omega^1_{\mathscr{X}} \otimes \mathscr{O}_X \to \Omega^1_X \to 0$

Using derived categories, this can be

$RHom(\Omega^1_X, \mathscr{O}_X[+1]) \cong RHom(\mathscr{O}_X, T_X[+1]) \cong Ext^1(\mathscr{O}_X, T_X) \cong H^1(X, T_X)$.

Of Ringed Topoi, this can be described more abstractly. For the composition of maps of ringed topoi,

$X \to_f Y \to Z$

Then, associated to this composition is a distinguished triangle

$f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to^{+1}$

Definition 5.8.4 *(Flat topology)*

fppf cover of X *is a jointly surjective family of morphisms* $(\phi_a : X_a \to X)$ *where* X_a *affine* ϕ_a *flat, finitely presented. This generates pretopology.*

Definition 5.8.5 *(Flat Cohomology)*

Flat cohomology or etale cohomology??

Definition 5.8.6 *(Hilbert Scheme)*

If $f : X \to S$ *is flat, we have a canonically a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & Hilb_S \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \end{array}$$

where $Hilb_S$ *is a family of flat morphisms to* S. *Since* f *is flat, all fibers* $f_s : X_s \to s$ *for all* $s \in S$ *have the same Hilbert polynomial* Φ. *We could say* $Hilb^{\Phi}_S = Hilb_S$.

6 Stack and Geometry

6.1 Orbifold

Definition 6.1.1 *Groupoid Groupoid is a small category in which all morphisms are isomorphism (invertible).*

Example 6.2 *Groupoid*

- *(Symplectic Groupoid)*
  In Poisson geometry, a Sympletic groupoid is a Lie groupoid.

Definition 6.2.1 *Orbifold definition using Lie groupoid Let $G_1 \rightrightarrows G_0$ be a morphism where G_0 is a set of objects and G_1 is a set of arrows, and the structural maps $s, t : G_1 \rightrightarrows G_0$ and other structural maps i.e. unit, invert maps. It's called Lie groupoid if G_1 and G_0 are smooth manifolds, all maps are smooth, and s, t are submersions.*

The intersection of fibers $G_1(x) = s^{-1}(x) \cap t^{-1}(x)$ is a Lie group called isotropy group of G_1 at x.

A Lie groupoid is proper if $(s, t) : G_1 \to G_0 \times G_0$ is a proper map.

A Lie groupoid is Etale if both the source and target morphisms are local diffeomorphisms.

An orbifold groupoid is either

- *a proper etale Lie groupoid*
- *a proper Lie groupoid whose isotropies are discrete spaces*

Let $M \rightrightarrows G$ be a groupoid, and $|M/G|$ be the orbit space of the Lie groupoid G i.e. quotient of M by equivalence $x \sim y$ if there exists $g \in G$ such that $s(g) = x$ and $t(g) = y$. This shows that orbifolds are particular kind of differentiable stack.

An orbifold structure on Hausdorff space X is an Morita equivalence class of homeomorphism $|M/G| \cong X$.

6.3 Lie Gropoid/Algebroid

Definition 6.3.1 *(Fibered Category)*
For a functor $p : \mathscr{S} \to \mathscr{C}$, the category $\mathscr{S}$ is a fibered category if the morphism $f : V \to U \in Mor(\mathscr{C})$ induces $f^*x \to x$ for some $x \in \mathscr{S}$, and they make the strongly cartesian product.

$$
\begin{tikzcd}
f^*x \arrow[r, "p"] \arrow[d] & V \arrow[d, "f"] \\
x \arrow[r, "p"] & U
\end{tikzcd}
$$

A strong cartesian product is .

$$
\begin{tikzcd}
z \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d, "p"] & x \arrow[d, "p"] \\
p(z) \arrow[r] & V \arrow[r, "f"] & U
\end{tikzcd}
$$

If such a morphism $V \times_U x \to x$ exists, then it's called a strongly cartesian morphism.

Lie group/algebra are tools to study geometry of smooth manifold, but Lie group/algebra only define local property of the manifold, so I'm interested in Lie groupoid/algebroid generalization to discuss global situation. We will use category theoretic language for this.

Definition 6.3.2 *(Groupoid)*

Definition 6.3.3 *(A category fibered in groupoid)*

Let $p : \mathscr{F} \to \mathscr{C}$ be a functor of categories $\mathscr{F}$ and $\mathscr{C}$. For $x \in \mathscr{C}$, $p^{-1}(x) \in \mathscr{F}_c \subset \mathscr{F}$ be a subcategory, which we name by fiber category.

Definition 6.3.4 *(Lie Groupoid)*

Let G and M be smooth manifolds. Lie groupoid $G \rightrightarrows M$ consists of several morphisms.

- *(Two arrows)*
  $s, t : G \rightrightarrows M$ *be smooth submersions.*
- *(Multiplication map)*
  m: $G^{(2)} := \{(g,h) | s(g) = t(h)\} \subset G \times G \to G$
- *(Unit map)*
  $u : M \to G$
  Using unit map, identity map 1_x is defined as
  $1_x = u(x)$
- *(Inverse map)*
  $i : G \to G$
  $g^{-1} := i(g)$

Example 6.4 *(Lie Subgroupoid)*
For a Lie groupoid $G \rightrightarrows M$, its Lie subgroupoid is $H \rightrightarrows N$ where $H \subset G$ is an immersed submanifold. Some examples of Lie subgroupoid might be below

- *(Unit Lie subgroupoid)*
  $u(M)$ is a sub Lie groupoid
- *(Inner subgroupoid)*
  $IG = \{g \in G | s(g) = t(g)\}$

Example 6.5 *(Notable Example of Lie Groupoid – Trivial and Extreme)*

- *(Group)*
  A Lie groupoid $G \rightrightarrows *$ is the same as Lie group.
- *(Pair groipoid)*
  $M \times M \rightrightarrows M$ with precisely one morphism from one another (i.e. $s = t$).
- *(Trivial Groupoid)*
  $M \times G \times M \rightrightarrows M$ with structure maps
  - $s(x, g, y) = y$
  - $t(x, g, y) = x$
  - $m((x, g, y), (y, h, z)) = (x, gh, z)$
  - $u(x) = (x, 1, x)$
  - $i(x, g, y) = (y, g^{-1}, x)$
- *(Unit Groupoid)*
  $u(M) \rightrightarrows M$

Definition 6.5.1 *(a)*

Lie algebroid is an analogy of Lie algebra.

Definition 6.5.2 *(Lie Algebroid)*

Lie algebroid $(A, [\cdot, \cdot], \rho)$ consisting of

- *A vector bundle A over a manifold M.*
- *A Lie bracket $[\cdot, \cdot]$ on its space of sections $\Gamma(A)$*
- *A morphism of vector bundles $\rho : A \to TM$, called an anchor, where TM is the tangent bundle of M.*

with product rule

$[X, fY] = \rho(X)f \cdot Y + f[X, Y]$

where $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$, and $\rho(X)f$ is an image of f via derivation $\rho(X)$.

Example 6.6 *(Example of Lie Algebroid)*

- *(Tangent Lie Algebroid)*
  $TM \to M$ *and* $\rho = id_{TM}$ *is an identity.*
- *(Lie algebroid to a point)*
  $TM \to *$ *is just a Lie algebra.*
- *(Any bundle of Lie algebra)*
  Yes, and $\rho = id$.

6.7 Differentiable Stacks

Differentiable stack is an analogue of algebraic stack, and it is always Morita equivalent to Lie groupoid.

Definition 6.7.1 *(Morita equivalence)*

Definition 6.7.2 *(Differentiable Stacks)*

Definition 6.7.3 *(Quotient Stack)*
Let G be an affine group scheme over S, and X be an S-scheme on which G acts. A quotient stack $[X/G]$ is a stack where for each T defines a category $[X/G](T)$, and an object $P \to T \in Ob([X/G](T))$ is a principal G-bundle together with an equivariant map $P \to X$.

Definition 6.7.4 *(Categorical Quotient)*
X/G A categorical quotient X/G is a morphism $\pi : X \to Y$ that coequalizes $\sigma, pr_2 : G \otimes X \to X$, and that satisfies universal property: the other equalizer can be factored by this. Note π need not to be surjective.

Notable example of categorical quotients are geometric quotient G/H and GIT $X//G$.

symplectic reduction (symplectic quotient)
phase space cotangent bundle

6.8 Category Fibred in Gropoids

Definition 6.8.1 *(Categorical Quotient)*

Let X be an object of a category $\mathscr{C}$, G be a group. A categorical quotient X/G is a morphism $\pi : X \to Y$ such that the group action and projection $\sigma, p_2 : G \times X \to X$ is coequalized by $\pi \circ \sigma = \pi \circ p_2$, and any other morphisms $X \to Z$ factors by it.

Definition 6.8.2 *(Quotient Stack)*

Let X be a S-scheme and G be an affine group S-scheme, and G acts on X. A quotient stack $[X/G]$ is a category over the category of S-schemes such that

- *an object is a pricipal G-bundle $P \to T$ together with an equivariant map $P \to X$.*
- *a morphism is a bundle map from $P \to T$ to $P' \to T'$ compatible with equivariant maps $P \to X$ and $P' \to X$*

A quotient stack is always an algebraic stack.

ex:

- *(trivial case)*
  $[pt/pt] = Sch_S$
- *()*
  For $[X/pt]$, the affine group pt is trivial, so the principal G-bundle is trivial bundle $T \to T$, that can be identified to T itself. Hence, the object is identified with a morphism $T \to X$, but since the group G is trivial, equivarint map could be any morphism $T \to X$.
- *()*
  For $[pt/G]$, an equivariant map $X \to pt$ is always trivial, which means unique, so it's naturally 1-to-1 to category of pricipal G-bundle of any basement S-scheme T.
- *(inclusion relation)*
  If $Y \subset X$ is a subscheme and $H \subset G$ is a subgroup scheme, then we have an inclusion relation

  $[Y/H] \subset [Y/G] \subset [Y/G] \subset [X/G]$

Definition 6.8.3 *(Algebraic Stack)*

An algebraic stack is a fibred category

$p : \mathscr{X} \to (Sch/S)_{fppf}$

over $(Sch/S)_{fppf}$*, which a category of scheme added with Grothethendieck topology, and*

- $\mathscr{X}$ *is a stack in groupoid over* $(Sch/S)_{fppf}$*.*
- $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ *is representable*
- *there exist a scheme* $U \in (Sch/S)_{fppf}$ *and a* 1*-morphism* $(Sch/S)_{fppf} \to \mathscr{X}$ *which is surjective and smooth.*

Diagonal morphism is representable ??
flat and smooth morphism ??

Definition 6.8.4 *(Stack)*

Prestack $\mathscr{S}$ *over a presite* X *is a* 2*-category. For each object* $U \in \mathscr{C}_X$, $\mathscr{S}(U)$ *is a category.*

Stack is a separated prestack.

Definition 6.8.5 *(Morita Equivalence)*

Let $G \rightrightarrows M$ *and* $H \rightrightarrows N$ *be Lie groupoids. They are Morita equivalent if there exists* P *such that*

$P \to M$ *is a principal* H*-bundle while* $P \to N$ *is a principal* G*-bundle such that two actions on* P *commutes.*

Definition 6.8.6 *(Differentiable Stack)*

Definition 6.8.7 *(Normal Bundle)*

- *(Normal Bundle)*
- *(Tangent Bundle)*
- *(Cotangent Bundle)*

Definition 6.8.8 *(Cotangent Complex)*

7 Derive Geometry

Definition 7.0.1 *(Koszul Complex)*

Originally used to calculate cohomology of Lie algebra.

For a commutative algebra A, its Koszul complex K_s is

$\wedge^r A^r \to \wedge^{r-1} A^r \to \cdots \to \wedge^1 A^r \to \wedge^0 A^r \cong A^r$

where the maps send

$\alpha_1 \wedge \cdots \wedge \alpha_k \mapsto \Sigma_{i=1}^k (-1)^{i+1} s(\alpha_i) \alpha_1 \wedge \cdots \wedge \hat{\alpha_i} \wedge \cdots \wedge \alpha_k.$

Or we can change A^r to any A-module.

Definition 7.0.2 *(Derived Scheme)*

Derived scheme is a homotopy theoretic generalization of a scheme in which commutative rings are replaced by dga An affine derived algebraic geometry is equivalent to the theory of commutative dg-rings.

$(X, \mathscr{O}.) = RSpec(R/(f_1) \otimes_R^L \cdots \otimes_R^L R/(f_k))$

where $f_i \in \mathbb{C}[x_1, \cdots, x_n] = R$

Then we get a derived scheme that

$RSpec : (dga_{\mathbb{C}})^{op} \to DerSch$

is the étale spectrum. Since we construct a resolution

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\cdot f_i"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & R/(f_i) \arrow[r] & 0
\end{tikzcd}$$

a derived ring $R/(f_1) \otimes_R^L \cdots \otimes_R^L R/(f_k)$, a derived tensor product, is a Koszul complex $K_R(f_1, \cdots, f_n)$. The truncation of this derived scheme to amplitude $[-1, 0]$ provides a classical model motivating derived algebraic geometry.

Notice that if we have a projective scheme

$Proj(\frac{\mathbb{Z}[x_1, \cdots, x_n]}{(f_1, \cdots, f_k)})$

where $deg(f_i) = d_i$ we can construct the derived scheme $(\mathbb{P}^n, \mathscr{E}^{\cdot}, (f_1, \cdots, f_k))$ where

$\mathscr{E}^{\cdot} = [\mathscr{O}(-d_1) \oplus \cdots \oplus \mathscr{O}(-d_k) \to^{(\cdot f_1, \cdots, \cdot f_k)} \mathscr{O}]$

with amplitude $[-1, 0]$.

Definition 7.0.3 *(Cotangent Complex)*

Let $(A., d)$ *be a fixed dga over char* 0. *Then* $A.$*-dga* $(R., d_R)$ *is called semi-free if*

- $R.$ *is a polynomial algebra over* $A.$.
- *there exists a filtration* $= I_0 \subset I_1 \subset \cdots$ *such that* $\cup I_n = I$, *and* $s(x_i) \in A.[(x_j)_{j \in I_n}]$ *for any* $x_i \in I_{n+1}$.

The relative cotangent complex of an $(A., d)$*-dga* $(B., d_B)$ *can be constructed using semi-free resolution* $(R., d_R) \to (B., d_B)$, *which is defined as*

$\mathbb{L}_{B./A.} = \Omega_{R./A.} \otimes_{R.} B.$

Example 7.1 *(Cotangent Complexes)*

For the cotangnet complex of hypersurface $X = \mathbb{V}(f) \subset \mathbb{A}^n_{\mathbb{C}}$,

dga $K_R(f)$ *representing the derived enhancement of* X. *Its cotangent complex is*

$0 \to R \cdot ds \to^{\Phi} \to \oplus_i R \cdot dx_i to 0$

where $\Phi(gds) = g \cdot df$ *and* d *is the universal derivation. If we take complete intersection, then the Koszul complex*

$R^{\cdot} = \frac{\mathbb{C}[x_1, \cdots, x_n]}{(f_1)} \otimes^L_{\mathbb{C}[x_1, \cdots, x_n]} \cdots \otimes^L_{\mathbb{C}[x_1, \cdots, x_n]} \frac{\mathbb{C}[x_1, \cdots, x_n]}{(f_k)}$

is quasi-isomorphic to

$\frac{\mathbb{C}[x_1, \cdots, x_n]}{(f_1, \cdots, f_k)}[+0]$

Example 7.2 *(Tangent Complex)*

Definition 7.2.1 *(Abelian Variety)*

References

[1] Akaho Manabu(2003), *A Crash Course of Floer Homology for Lagrangian Intersections*, https://pseudoholomorphic.fpark.tmu.ac.jp/akaho_a_crash_course_of_floer_homology_for_lagrangian_