A_∞ and NC geom (Progress & Question)

Koji Sunami

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1 Intro

Definition 1.0.1 *(Hopf Structure)*
Let $\mathscr{C} = Vect_k^{\mathbb{Z}}$ and let $\mathscr{C}_f \subset \mathscr{C}$ be its subcategory, where each element is finite dimensional, accepting finite projective limit and fiber product, and $\mathscr{C}_f$ called an Artinian category.

Also, let $Alg_{\mathscr{C}_f}$ be a category of algebra which is derived from $\mathscr{C}_f$, where each element has an algebraic structure, and similarly $CoAlg_{\mathscr{C}_f}$ is a category of coalgebra, where eachelemetn has a coalgebra structure.

- *mult* $m : A \otimes A \to A$
- *unit* $\epsilon : k \to A$
- *comult* $\Delta : A \to A \otimes A$
- *counit* $\eta : A \to k$

Note that $CoAlg_{\mathscr{C}_f} = Alg_{\mathscr{C}_f}^{op}$

(remark: "algebra" is an attempt of how to install algebraic structure in vector space, but unlike classical ring or group theory, there is not guarantee that it is unital or at least most of the case.)

Definition 1.0.2 *(NC thin scheme)*
Consider that the functor

$$\begin{aligned} F : Alg_{\mathscr{C}_f} &\to Sets \\ A &\mapsto Hom_{Coalg}(A^*, B) \end{aligned} \tag{1}$$

it representable, and let $Fun(Alg_{\mathscr{C}_f,Sets}$ *as a category of functors. Next, we define non-commutative spectrum as*

$$\begin{aligned} Spc : CoAlg_{\mathscr{C}_f} &\to Fun(Alg_{\mathscr{C}_f,Sets}) \\ B &\mapsto Spc(B) \end{aligned} \tag{2}$$

so nc spectrum $Spc(B)$ *is a functor.*

Definition 1.0.3 *(Tensor Stuff)*
$T(V) = \oplus_{n\geq 0} V^{\otimes n}$
$V[1] = \oplus_{n\geq 1} V^{\otimes n}$

Definition 1.0.4 *(Pointed manifold)*

Definition 1.0.5 *(Formal, Smooth etc)*

- *(Def Formal Manifold)*
  A nc formal manifold X *is a nc thin scheme* $X \cong Spc(T(V))$ *for some* V*. Here* $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ *is a coalgebra where*

  $$\Delta(v_0 \otimes \cdots \otimes v_n) = \Sigma(v_0 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n) \tag{3}$$

  The dimension of X *is* $dim_k(V)$*.*

  The algebra $\mathscr{O}(X)$ *of function on* X *is isomorphic to* $k << x_0, \cdots, x_n >>$*, a topological algebra of formal power series of free graded variables* $x_1, \cdots, x_n$*.*

- *(Completion)*
  For nc affine thin scheme X*, we can define its completion* $\hat{X}_x$ *by defining functor*

  $$\begin{aligned} F_{\hat{X}_x} : Alg_C^f &\to Sets \\ A &\mapsto \left\{ f : A^* \to B_X | \exists A_i, \begin{matrix} f : A^* \to A_1^* \to B_X \\ x : k \to A_1^* \to B_X \end{matrix} \right\} \end{aligned} \tag{4}$$

  Its coalgebra $B_{\hat{X}_x} \subset B_X$ *is a preimage of* $B_X \to B_X/x(k)$*.*

- *(Smooth scheme)*
  A 2-sided ideal $J \subset A$ *is called nilpotent if* $J^{\otimes n} \to J$ *should be zero map for sufficiently large* n.

  Now a coalgebra B *is smooth if*

$$\begin{aligned} F_B : Alg_C^f &\to Sets \\ A &\mapsto Hom_{Coalg}(A^*, B) \end{aligned} \tag{5}$$

  For any $J \subset A$*, the natural projection* $A \to A/J$ *induces* $F_B(A) \to F_B(A/J)$ *is surjective. Also,* X *is smooth if* B_X *is smooth.*

- *(Inner Hom* $(Maps(X, Y))$*)*

  First of all, what is inner hom? Inner hom can be defined over a symmetric monoidal category $\mathscr{C}$ *with the following property: For a bifunctor*

  $[-, -] : \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{C}$

  where there exists a natural homomorphism

  $Hom(X, [Y, Z]) \cong Hom(X \otimes Y, Z)$

  For a nc thin scheme X *and* Y*, consider a functor*

$$\begin{aligned} F_{X,Y} : Alg_C^f &\to Sets \\ A &\mapsto Hom_{Coalg}(A^*, B) \end{aligned} \tag{6}$$

  are representable, and we denote this scheme as $Maps(X, Y)$.

Definition 1.0.6 *(dg manifold)*

- *(Homological Vector Field)*
  A homological vector field on X *is a morphism* $d : B_X \to B_X$. *This is degree* $+1$*, naturally making a graded structure. Also, consider the coalgebra* B_X *can have a Lie bracket, we define* d *as* $[d, d] = 0$.

  Also, d *vanishes at* x_0 *if the image of* k *is killed e.g.* $d(k) = 0$ *means* $d|_{x_0} = 0$

- *(DG Manifold)*
  Recall that A pointed thin scheme is a pair (X, x_0) *where* X *is a nc thin scheme, and* $x_0 : k \to B_X$ *is a counit of coalgebra.*

  $d|_{x_0} = 0$

  Finally, we will be able to define dg-manifold. dg-manifold (X, x_0, d_X) *derivative of coalgebra naturally defines a grading structure.*

Definition 1.0.7 *(Cofree Algebra)*
Given a vector space V, its cofreee coalgebra $C(V)$ is coalgebra such that for a forgetful functor $C(V) \to V$, any morphism $X \to V$ factors through $X \to C(V) \to V$.

Definition 1.0.8 *(Compact Objects)*
X is a compact object if direct limit commutes i.e.

$$colimHom(X, Y_i) = Hom(X, colimY_i)$$

is a bijetion for any filtered system of objects Y_i in $\mathscr{C}$.
Compactly Generated category:

A category $\mathscr{C}$ is compactly generated if any object can be expressed as a filtered colimit of compact objects in $\mathscr{C}$. For example, any vector space V is the filtered colimit of its finite-dimensioal subspaces. Hence, the category of vector spaces are compactly generated.

Especially if the category is compactly generated accepting colmits, then it's accessible category.

(Ind scheme)
$Ind(\mathscr{C})$
Let I be a small filtering category, and A be any category. For a covariant functor $\phi : I \to A$, the inductive limit "lim" is given by

$$\begin{aligned} "lim_{\to}"\phi(X) \in \hat{A} \text{ where} \\ "lim_{\to}"\phi(X) = Hom(X, \phi(i)) \end{aligned} \tag{7}$$

All the inductive forms create a category $Ind(A)$. Also, I abbreviate the case of proejctive limit since it's barely dual of it.

Now consider $NAff_{\mathscr{C}}$ is a category of nc affine scheme, and $\phi : I \to NAff_{\mathscr{C}}$ is a covariant functor. "$lim_{\to}$"F is a NC affine scheme if for $i \to j$, $F(i) \to F(j)$ is a closed embedding i.e. the corresponding morphism of the coalgebras $B_X \to B_Y$ are injective.

A NC ind affine scheme $\hat{X}$ is called formal if it can be written by $\hat{X} = lim_{\to} Spec(A_i)$ where $\{A_i\}$ is a projective system s.t. $A_i \to A_j$ is surjective, and have nilpotent kernel for all $i \to j$.

example:
Let $Y \subset X$ be a closed subscheme of X, that can be defined by $\mathscr{O}(Y) = \mathscr{O}(Y)/J$ for some 2-sided ideal J. Now the formal completion of Y is given by a projective limit $lim_{\leftarrow_n} \mathscr{O}_X/J^n$ denoted by $\hat{X}_Y$ or $Spf(\mathscr{O}_X/J)$.

A NC affine scheme over k form symmetric monoidal category. The tensor structure can be given by an ordinary tensor product.

(local nilpotency)

The category N_X is a nilpotent extension of X i.e. $\phi : X \to U$ where $U = Spc(D)$ is a nc thin scheme s.t. $D/f(B_X)$ is locally nilpotent. Here, $a \in D/f(B_X)$ is a local nilpotent element if $\exists n \geq 2$ s.t. $\Delta^{(n)}(a) = 0$.

For $f : X \to Y$, let $G_f : N_X^{op} \to Sets$ s.t. $G(X, \phi, U)$ is a set of all morphisms $\psi : U \to Y$ s.t. $\psi \circ \phi = f$. The functor G_f is represented by the triple $(X, \pi, \hat{Y}_X)$. Where $\hat{Y}_X$ is a formal nbhd of $f(X)$ in Y (or completion of Y along $f(X)$).

(tangent space)

The tangent space at x_0 $T_{x_0}X$ is canonically isomorphic to the graded vector space of primitive elements of the coalgebra B_X i.e. $\{a \in B_X | \Delta(a) = 1 \otimes a + a \otimes 1, B_X \ni 1 = x(1)\}$.

Definition 1.0.9 *(triangulated category)*

Definition 1.0.10 *(A_∞algebra)*

(unital A_∞ algebra)

An A_∞ algebra is called unital if $\exists 1 \in V$ of degree zero, and $m_2(v, 1) = m_2(1, v)$ and $m_n(\cdots, 1, \cdots) = 0$ for $n \neq 2$.

Or A_∞ algebra is weakly unital if its homology $H^0(V)$ has a unit $\exists 1 \in H^0(V)$.

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Definition 2.0.1 *(Ext and Tor)*
For simplicity, I only say Ext.

$Ext^i(A, B) = (RT^i)(B)$ where injective resolution

$$0 \to B \to I^0 \to I^1 \to I^2 \to \cdots$$

induces a chain complex

$$0 \to Hom(A, I^0) \to Hom(A, I^1) \to Hom(A, I^2) \to \cdots$$

whose homology is $Ker(\partial^{i+1})/Im(\partial^i)$.

Definition 2.0.2 *(Hochschild)*

- *(Hochschild homology)*
  Let $A^e = A \otimes A^o$ be an enveloping algebra where A is an associative k-algebra, and A^o is the opposite ring of it. Then, Hochschild cohomology is

defined by:

$$\begin{aligned} HH^n(A,M) &= Ext^n_{A^e}(A,M) \\ HH_n(A,M) &= Tor^n_{A^e}(A,M) \end{aligned} \tag{8}$$

$C_n(A,M) = M \otimes A^{\otimes n}$ *with a boundary operator* d_i *given by*

$$\begin{aligned} d_0(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ d_i(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ d_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned} \tag{9}$$

and if we let $b_n = \Sigma_{i=0}^n (-1)^i d_i$, $b_{n-1} \circ b_n = 0$ *will be a chain complex, so we can let* $(C_n(A,M), b_n)$ *as a Hochschild complex.*

- *(Standard Complex)*
  $\cdots \to A \otimes A \otimes A \to A \otimes A \to A \to 0$

  with the differential given by

  $d(a_0 \otimes a_{n+1}) = \Sigma_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$

  If A *is a unital* k*-algebra, then the standard complex is exact.*

- *(Hochschild and its Relation to Standard complex)*

  Consider the case if $M = A$ *and* A *is an associative* k*-algebra. Then we will denote* $HH(A/k) = HH(A,M)$.

  $HH(A/k) \cong A \otimes_{A^e} B(A/k)$

- *(As a derived intersection)*

  For a scheme (or derived scheme) X *over* S, *let* $\Delta : X \to X \times_S^L X$,

  $HH(X/S) := \Delta^*(\mathscr{O}_X \otimes^L_{\mathscr{O}_X \otimes^L_{\mathscr{O}_S} \mathscr{O}_X} \mathscr{O}_X)$

  In particular, if $S = Spec(k)$ *and* $X = Spec(A)$, *then*
  $HH(X/k) \cong_{qiso} A \otimes^L_{A \otimes^L_k A} A$

Definition 2.0.3 *(Operad)*

We study operad because it can manipulate Hochschild complex.

- *(Operad)*
  Let $\mathscr{C}$ be a closed symmetric monoidal category. A polynomial functor is a functor $F : \mathscr{C} \to \mathscr{C}$ such that $F(V) = \oplus(F_n \otimes V^{\otimes n})_{S_n}$ where F_n is a reprentation of finite symmetric group. These polynomial functors naturally generates a category PF whose objects are polynomial functors, $Hom(F,G) = \prod \underline{Hom}(F_n, G_n)$.

  An operad is an element $R \in PF$ such that it's also monoid with morphisms $R \circ R \to R$ and $1 \to R$ satisfying associativity and unit axioms.

  The particular case of a operad is a colored operad. For a set I, let $\mathscr{C}^I$ be a category with objects $(V_i)_{i\in I}$. Let a functor $F : \mathscr{C}^I \to \mathscr{C}^I$ be defined by $F((V_i))_j = \oplus_{a:I\to\mathbb{Z}_{\geq 0}} F_{a,j} \otimes_{\prod S_{a(i)}} \otimes_{i\in I}(V_i^{\otimes a(i)})$. F is a colored operad if it is monoid, and then it will be denoted as $\mathscr{O}\mathscr{P}$

- *(Operad and Tree)*

  For a polynomial functor $F : \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$, $F_{(m_i),n}$ is a representation of a group $S_{n,(m_k)} = S_n \times \prod_{k\geq 0}(S_{m_k} \rtimes S_k^{m_k})$.

  In particular, we take an operad $\mathscr{O}\mathscr{P} : \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$ with a forgetful functor operads $Operads \to \mathscr{P}\mathscr{F}$ It gives rise to I_0 colored operad $\mathscr{O}\mathscr{P} = \mathscr{O}\mathscr{P}_{(m_i),n}$, that can be described by using the notion of trees.

  Tree (T) is a graph with a set of vertices $V(T)$ and edges $E(T)$.

  A colored operad corresponds to a tree.

  Free operad P generated by a polynomial functor F that can be explicitly descibed as follows:
  $P_n = \oplus_{[T]\in|Tree(n)|}(\otimes_{v\in V_i(T)} F_{N^{-1}(v)})_{Aut(T)}$ where $[T]$ is a class of isomorphism of T.

- *(Tautological Operad)*

  Let M be a closed symmetric monoidal category, and $M(X,Y)$ be the inner hom. We define the tautological operad $Op(X)$ given by $Op(X)_n = M(X^{\otimes n}, X)$.

  The operad identity is a map $1_X : I \to M(X,X)$ and the operad multiplication is given by the composite

$$
\begin{aligned}
M(X^{\otimes k}, X) \otimes M(X^{\otimes n_1}, X) \otimes \cdots \otimes M(X^{\otimes n_k}, X) &\to M(X^{\otimes k}, X) \otimes M(X^{\otimes n_1 + \cdots + n_k}, X^{\otimes k}) \\
&\to M(X^{\otimes n_1 + \cdots + n_k}, X)
\end{aligned} \tag{10}
$$

- *(Algebra over Operad)*

  Let M be a closed symmetric monoidal category, and O be an operad. An algebra over the operad O is an object $X \in M$ s.t.

  O-algebra is given by a sequence of the map

  $O(k) \otimes X^{\otimes k} \to X$

  The data of an O

- *(dgla)*

  Let $\mathscr{C} = Vect_k^{\mathbb{Z}}$ be a closed symmetric monoidal category, esp category of $\mathbb{Z}$-graded k-vector spaces, and let F be a polynomial functor, and P be the generating free operad $P = Free_{\mathscr{OP}}(F)$. Let g_P be a Lie algebra $g_P = \prod_{n \geq 0} \underline{Hom}(F_n, P_n)^{S_n}$. Also, a structure of dg-operad on P will be given by $d_P \in g_P^1$ as $[d_P, d_P] = 0$

  Here we define a dg Lie Algebra as a pair $(g_P, [d_P, -])$.

- *(A_∞ case)*

  Let A be a A_∞ algebra together with morphisms $m_n : A^n \to A$, and for a dg Lie algebra $(g_P, [d_P, -])$, we define the differential structure d_P as follows:

  - $d_P(m_2) = 0$

  - $d_P(m_n)(v_1 \otimes \cdots \otimes v_n) = \Sigma_{k+l=n} \pm m_k(v_1 \otimes \cdots \otimes v_i \otimes m_l(v_{i+1} \otimes \cdots \otimes v_{i+l}) \otimes \cdots \otimes v_n)$ *where* $n > 2$

- *()*

  $A(n, m)$ and $B(n, m)$ are $\mathbb{Z}$-graded vector spaces, and .

  $A(n, 0) \otimes C(A, A)^{\otimes n} \to C(A, A)$

  Let M be a minimal operad. $A(n, 0)$ coincides with the n-section M_n.

  the basis of M_n is formed by a tree. The total degree of M is given by degree of the tree $deg(T)$

Definition 2.0.4 *(A_∞ algebra)*
A_∞ algebra over k is a $\mathbb{Z}$-graded vector space $A = \oplus A^p$ with a grading maps $m_n : A^{\otimes n} \to A$ $n \geq 1$ of degree $2-n$ with the following conditions.

- *$m_1 \circ m_1 = 0$ so (A, m_1) is a chain complex.*

- *$m_1 \circ m_2 = m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1)$ for $A^{\otimes 2} \to A$*

- *$m_2 \circ (1 \otimes m_2 - m_2 \otimes 1) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$ for $A^{\otimes 3} \to A$*

- *$\Sigma(-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$ where $n = r+s+t$ and $u = r+1+t$.*

Definition 2.0.5 *(Bar complex form agebra)*

We claim the construction of bar complex from the $\mathbb{Z}$-graded vector space $TV = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$. The coalgebra Δ of $\mathbb{Z}$-graded vector space TV generates A_∞ structure.

$$\begin{gathered} \Delta : TV \to TV \otimes TV \\ (v_1, \cdots, v_n) \mapsto \Sigma(v_1, \cdots, v_i) \otimes (v_{i+1}, \cdots, v_n) \end{gathered} \tag{11}$$

$1^{\otimes r} \otimes b_s \otimes 1^{\otimes t} : V^n \to V^u$ where $u = r+1+t$ and $n = r+s+t$. In particular $\Delta(v_1) = 0$, and $\Delta(v_1, v_2) = v_1 \otimes v_2$.

A graded coderivation $b : TV \to TV$

Definition 2.0.6 *(Bar complex)*
Now we define a $\mathbb{Z}$-graded algebra A generates A_∞ struture. Consider SA as suspension i.e. $(SA)^p = A^{p+1}$. Consider that $b_n : (SA)^{\otimes n} \to SA$ of degree 1, and maps $m_n : A^{\otimes n} \to A$ for $n \geq 1$ by the commutative square:

$$\begin{tikzcd} (SA)^{\otimes n} \arrow[r, "b_n"] & SA \\ A^{\otimes n} \arrow[u, "s^{\otimes n}"] \arrow[r, "m_n"] & A \arrow[u, "s"'] \end{tikzcd}$$

where s is a canonical map of degree -1, and m_n is of degree $2-n$.
TFAE:
1) $m_n : A^{\otimes n} \to A$ is an A_∞ structure on A.
2) the coderivation $b : TSA \to TSA$
3) $\Sigma b_u(1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) = 0$

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Definition 3.0.1 *()*

Definition 3.0.2 *(Limit)*

- *Cone*
  A diagram $F : \mathscr{J} \to \mathscr{C}$ *is a functor where* $\mathscr{J}$ *is an index category and* $\mathscr{C}$ *is a category. A cone* (N, ψ) *is a pair of an object* $N \in \mathscr{C}$ *together with* $\psi_X : N \to F(X)$ *s.t for a morphism* $f : X \to Y$ *for* $X, Y \in Ob(\mathscr{C})$, $F(f) \circ \psi_X = \psi_Y$.

  Note that the index category $\mathscr{J}$ *doesn't have to be a small or finite category.*

- *Limit*
  A limit of a the diagram is some cone (L, ϕ) *s.t. for every other cone* (N, ψ) *to* F *there exists a unique morphsim* $u : N \to L$ *s.t.* $\phi_X \circ u = \psi_X$ *for all* $X \in Ob(\mathscr{J})$.

  Note that in other words, (L, ϕ) *is a terminal object of the category of cones to* F*, or one says that the limit* (L, ϕ) *is a universal cone since it factors through it.*

- *Colimit*
  Colimit is the dual notion of limit. For a diagram $F : \mathscr{J} \to \mathscr{C}$, *. cocone is a pair* (N, ψ) *where* $N \in Ob(\mathscr{C})$ *and* $\psi_X : F(X) \to N$ *for all* $X \in \mathscr{J}$*, and for all* $f : X \to Y$, $\psi_Y \circ F(f) = \psi_X$.

  Colimit is a pair (L, ϕ) *s.t. for each cocone* (N, ψ) *there exists a morphism* $u : L \to N$ *s.t.* $u \circ \phi_X = \psi_Y$*, which is an initial object of category of cocones.*

Example 3.1 *(Examples of Limits)*

- *(Discrete Category)*
  For a category $\mathscr{J}$*, for* $X, Y \in \mathscr{J}$*, if all morphism* $Hom(X, X) = id$ *and* $Hom(X, Y) = \emptyset$ *for* $X \neq Y$,

  then the category $\mathscr{J}$ *is called discrete category.*

  Then for an diagram $F : \mathscr{J} \to \mathscr{C}$*, the categorical limit L is direct product of all objects of* $F(X)$*, and morphisms are projections.*

- *(Equalizer)*
  If $Ob(\mathscr{J}) = \{X, Y\}$ *is a category with two objects, then for any* $f, g : X \to Y$, *there exists respectively* $F(f) \circ \psi_X = \psi_Y$ *and* $F(g) \circ \psi_X = \psi_Y$. *Here, by definition, equalizer* $Eq(f, g)$ *is* $Eq(f, g) = \{x \in X | f(x) = g(x)\}$, *indeed by definition the limit* L *is an equializer of the diagram.*

- *(Pullback)*
  Trivial.

- *(Topological Limits)*
  See filters.

Note 3.2 *(Existence of Limits)*
Note that the limit of diagram does not always exists. In other words, from the previous examples in particular, the existence of product, equalizers, or pullback are not guaranteed to exist.

Product, equalizers, and pullback are all the same limit but differently defined with the different index categories.

Here a complete category $\mathscr{C}$ *is a category which the limit exists for any diagram* $F : \mathscr{J} \to \mathscr{C}$ *for any index categories* $\mathscr{J}$. *Dually, if colimit exists for any diagram* $F : \mathscr{J} \to \mathscr{C}$, *then the category* $\mathscr{C}$ *is called cocomplete.*

Definition 3.2.1 *(Universal Constructions)*
A further generalization of limit is a universal constrution. That is, for a category $\mathscr{C}$ *and its categorical functor* $\mathscr{C}^{\mathscr{J}} = Fun(\mathscr{J}, \mathscr{C})$, *let a diagonal functor* $\Delta : \mathscr{C} \to \mathscr{C}^{\mathscr{J}}$ *be* $\Delta(N) : \mathscr{J} \to \mathscr{C}$ *where* $\Delta(N)(X) = N$, *and* $\Delta(N)(f) = id_N$.

Now given a diagram $F : \mathscr{J} \to \mathscr{C}$, *a natural transformation* $\psi : \Delta(N) \to F$ *is the same thing as a cone from* N *to* F. *That is, for all* $X \in \mathscr{J}$, $\psi_{\mathscr{J}} : N \to F(X)$ *is a cone's diagram which commutes by functoriality.*

Therefore, a limit of F *is alternatively given by a universal morphism from* Δ *to* F, *and a colimit of* F *is alternatively given by a universal morphism from* F *to* Δ.

Definition 3.2.2 *(Adjunction of Limit)*
So far we have defined limit as defined over a categorical limit, but if all the diagram $F : \mathscr{C}^{\mathscr{J}} \to \mathscr{C}$ *has a shape* $\mathscr{J}$, *then limit can also be considered as a functor* $lim : \mathscr{C}^{\mathscr{J}} \to \mathscr{C}$.

This limit functor lim *is right adjoint to the diagonal functor* $\Delta : \mathscr{C} \to \mathscr{C}^{\mathscr{J}}$, *which is in other words, there exists an isomorphism* $Hom(N, \lim F) \cong Cone(N, F)$. *Similarly, a colimit functor* $colim : \mathscr{C}^{\mathscr{J}} \to \mathscr{C}$ *is left adjoint to* Δ, *and* $Hom(colimF, N) \cong Cocone(F, N)$.

There is respectively unique canonical morphisms:
$Hom(N, \lim F) \cong \lim Hom(N, F-)$
$Hom(colimF, N) \cong \lim Hom(F-, N)$

Definition 3.2.3 *(Inverse Limits)*
Inverse (or projective) limit is a reverse version of original limits of system. A diagram $F : \mathscr{J} \to \mathscr{C}$ is a contravariant functor, so functorially $i \to j$ makes $f_{ij} : A_j \to A_i$. Here, f_{ii} is an identity, and $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leqq j \leqq k$. Then $((A_i), (f_{ij}))$ is called an inverse system over $\mathscr{J}$.

Algebraically, we define the inverse system $\varprojlim_{i \in \mathscr{J}} A_i$ as a subgroup of direct product $\varprojlim_{i \in \mathscr{J}} A_i = \{a_i \in \prod A_i | a_i = f_{ij}(a_j)\}$.

More general perspective is a categorical perspective. Inverse limit of a system $((X_i), (f_{ij}))$ is an object $X = \varprojlim_{i \in \mathscr{J}} X_i$ such that for a projectition $\pi_i : X \to X_i$, $f_{ij} \circ \pi_i = \pi_j$ commutes, and the pair (X, π_i) is universal in a sense that for any other pair (Y, ψ_i), there is a unique morphism $u : Y \to X$ that commutes the diagram.

Example 3.3 *(Inverse limits)*

- *p-adic integers*
- *pro-finite groups*

Proposition 3.4 *(Mittag-Leffler Condition)*
Derived functor of inverse limits
If $\mathscr{C}$ is an abelian category, then $\varprojlim : \mathscr{C}^{\mathscr{J}} \to \mathscr{C}$ is left exact. Here, let's consider when does this lim *functor becomes exact, and this condition is called Mittag-Leffer condition.*

Derived functor of inverse limits $\varprojlim^1$ is a derived functor, so for any system (A_i, f_{ij}), (B_i, g_{ij}), and (C_i, h_{ij}), there is an exact sequence $0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i \to \varprojlim^1 A_i \to \cdots$ by the use of functors $\varprojlim^i$.

If in particular, the system (A_i, f_{ij}) are stationary, then for $f_{kj}(A_j) = f_{ki}(A_i)$, one says that the system satisfies Mittag-Leffler condition.

Some notable examples of ML condition is as follows:

- *A system in which the morphism f_{ij} is surjective.*
- *A system of finite dimensional vector spaces or finite abelian groups or modules of finite length or Artinian modules.*

Note that categorical dual of inverse limit is direct limit.

Definition 3.4.1 *(Filters)*

- *Filters*
  Filter was originally a topological notion that are used to define the convergence of sequences, working as a generalization of convergence in metric spaces, replacing direct system to natural numbers.

  Let a partial orded set $(P, \leqq)$ (which is a set with extra condition). A subset $F \subset P$ is called a filter if it satisfies the following axioms:

  - *(Non-Triviality)*
    $F \neq \emptyset$

  - *(Downward Directed)*
    For all $x, y \in F$ there exists $z \in F$ s.t. $z \leqq x$ and $z \leqq y$.

  - *(Upward Closure)*
    For every $x \in F$ and $p \in P$, the condition $x \leqq p$ implies $p \in F$.

  In particular, if $F \neq P$, then F is called a proper filter.

  Principal filter of $p \in P$ is $\{x | p \leqq x\}$ is the smallet filter containing p, and p is said to be the principal filters of F or generate F. It looks obvious, but Frechet filter is non-principal.

- *(Linear Filters)*
  For a vector space, a set of all vector subspaces makes an oreordered set, so it's a filter.

- *(Filter on a set)*
  A powerset $P(S)$ of some set S is partially ordered by inclusion. By abuse of notation, a filter F defined over this poset $P(S)$ is often called "a filter on S". The basic properties are $A, B \in F$ makes $A \cap B \in F$, and $A \in F$ and $A \subset B \subset S$ makes $B \in F$.

- *(Free Filters)*
  A filter is free iff it's a Frechet Filter (also called cofinite filter). A set of all cofinite subset of X is $F = \{A \subset X | X \backslash A \text{ is a finite set}\}$ makes a filter on a lattice $(P(X), \subset)$. If satisfies intersection and upper-set condition.

- *(Ultrafilter)*
  Ultrafilter U is the maximal filter on X. Explicitly, U is a filter s.t. $A \subset X$ implies $A \subset U$ or $A^c \subset U$.

- *Filter in Topology*
  $\mathbb{R}$ *is an example of partially ordered set. Consider a real-valued function* $f : \mathbb{R} \to \mathbb{R}$ *by a pointwise comparison, then the set of functions* $\{f : \lim_{x\to\pm\infty} f(x) = \infty\}$ *is a filter on* $\mathbb{R} \to \mathbb{R}$

  Here, for a given point $x \in X$, *a neighborhood filter* $V(x)$ *is a filter whose elements are a set contains* x *in the interior. Also, let* F *be a set of filters of* X.

  $F_{x,A} = \{G \in F | V(x) \cup A \subset G\}$ *the set of filters finer than* A *and that converges to* x. *The the set of filter* F *are given a small and thin category structure by adding and arrow* $A \to B$ *iff* $A \subset B$. *The injection* $I_{x,A} : F_{x,A} \to F$ *becomes a functor and* x *is a topological limit iff* A *is a categorical limit of* $I_{x,A}$, *where* $F_{x,A}$ *works as the index category,* F *as a category.*

- *Convergence*
  A filter F *can be interpreted as a sequence, and* F *converges to a point* x *iff* $N \leqq B$ *where* $N \in V(x)$ *is a nbhd filter.*

- *Thin category*
  Here, a thin category $\mathscr{C}$ *is a category s.t. for all* $f, g : x \to y$, $x, y \in \mathscr{C}$, $f = g$.

Definition 3.4.2 *(Yoneda Lemma)*
Yoneda lemma claims $Nat(h_A, F) \cong F(A)$ *where* $h_A = Hom(A, -)$ *and* F *are covariant functors* $\mathscr{C} \to Sets$ *where* $\mathscr{C}$ *is a locally small category. A category is locally small if for all objects* $A, B \in Ob(\mathscr{C})$, *the hom* $Hom(A, B)$ *is a proper set.*

proof:
For a morphism $f : A \to X$ *in category* $\mathscr{C}$, *a natural transformation* Φ *makes a commutative diagram:*

$$\begin{tikzcd} Hom(A,A) \arrow[r, "Hom(A,f)"] \arrow[d, "\Phi_A"] & Hom(A,X) \arrow[d, "\Phi_X"] \\ F(A) \arrow[r, "F(f)"] & F(X) \end{tikzcd}$$

$$\begin{tikzcd} id_A \arrow[r] \arrow[d] & f \arrow[d] \\ u = \Phi_A(id_A) \arrow[r] & (Ff)u = \Phi_X(f) \end{tikzcd}$$

Hence, for any $\Phi \in Nat(h_A, F)$ *there exists* $u \in F(A)$. *On the other hand, consider* $(Ff)u = \Phi_X(f)$, *given* u, *we can canonically define morphism* Φ_X *for any* X, *so we have* Φ. *Therefore, a morphism* $\Phi \in Nat(h_A, F) \mapsto u \in F(A)$ *is*

1-to-1.

Universal element:
Here, for convenience, Here we let a pair (A, u) of functor F by means of above. The universal element of F is a pair (A, u) s.t. for any pair (X, v) factors through it. Namely, for any $X \in \mathscr{C}$ and the element $v \in F(X)$, there exists any morphism $f : A \to X$ that satisfies $(Ff)u = v$.

The contravariant version of Yoneda lemma is for a contravariant functors $h^A = Hom(-, A)$ and G, we have $Nat(h^A, G) \cong G(A)$.

Finally, a notable example of Yoneda lemma is Yoneda embedding. If in particular, $F = h_B$ for some $B \in \mathscr{C}$, then $Nat(h_A, h_B) \cong Hom(A, B)$.

Definition 3.4.3 *(Universal Morphism and Adjointness)*
For a functor $F : \mathscr{C} \to \mathscr{D}$, a universal morphism of $X \to F$ is a pair (A, u) where $u : X \to F(A)$. any morphism $X \to F(A')$ can be factored through some morphism $h : A \to A'$.

Now, let $G : \mathscr{D} \to \mathscr{C}$ and $X \in \mathscr{C}$. Then if (A, ϕ) is a universal morphism $X \to G$ iff (A, G) is a representation of functor $Hom(X, G-) : \mathscr{D} \to \mathscr{S}$. It follows G has a left adjoint F iff $Hom(X, G-)$ is representable for all $X \in \mathscr{C}$. The natural isomorphism $\Phi_X : Hom(FX, -) \to Hom(X, G-)$ yields adjointness. That is,

$$\Phi_{X,Y} : Hom(FX, Y) \cong Hom(X, GY)$$

is a bijection for all X and Y. The dual statement is also true.

Definition 3.4.4 *(Filtered Category)*
A category $\mathscr{D}$ is a filtered category if

- *$Ob(\mathscr{D})$ is not empty.*
- *for all $j, j' \in \mathscr{D}$, there exists $k \in \mathscr{D}$ and $f : j \to k$ and $f' : j' \to k$.*
- *for all $u, v : i \to j$ for $j \in Ob(\mathscr{D})$, there exists $k \in \mathscr{D}$ and $w : j \to k$ s.t. $wu = wv$.*

Note that for a Filtered category $\mathscr{D}$, the functor $F : \mathscr{D} \to \mathscr{C}$ is a diagram, because a filtered category is an index category, thus we can define the colimit of the diagram.

Similarly, a category $\mathscr{D}$ is a cofiltered category if the opposite $\mathscr{D}^{op}$ is filtered. In other words,

- *$Ob(\mathscr{D})$ is not empty.*
- *for all $j, j' \in \mathscr{D}$, there exists $k \in \mathscr{D}$ and $f : k \to j$ and $f' : k \to j'$.*

- *for all $u, v : j \to i$ for $j \in Ob(\mathscr{D})$, there exists $k \in \mathscr{D}$ and $w : k \to j$ s.t. $uw = vw$.*

A cofiltered limit is a limit of functor $F : \mathscr{D} \to \mathscr{C}$ where $\mathscr{D}$ is a cofiltered category.

Definition 3.4.5 *(Ind-Objects & Pro-Objects)*
For a small category $\mathscr{C}$, a presheaf of sets $F : \mathscr{C}^{op} \to Sets$ that is a small filtered colimit of representable presheaves, is called an ind-object of the category $\mathscr{C}$. The Ind-objects of a category $\mathscr{C}$ forms a full subcategory $Ind(\mathscr{C})$ in the category of functors $\mathscr{C}^{op} \to Sets$.

The category $Pro(\mathscr{C}) = Ind(\mathscr{C}^{op})^{op}$ is a category of Pro-objects of $\mathscr{C}$. In other words, there is a presheaf $\mathscr{C} \to Sets$ whose limit is .

Definition 3.4.6 *(Pro-Representable Functor)*
A functor $F : \mathscr{C} \to Sets$ is a pro-representable functor if it is a small filtered colimit of representables. In other words, it corresponds to a pro-objects.

Definition 3.4.7 *(Artinian and Noetherian Categories)*
An Artinian category is a category in which all descending sequence of objects have infinite sequences of subobjects. An Noetherian category is dual of it.
(Subobjects)

(locally noetherian category)
In particular, an abelian category is called locally noetherian if it satisfies axiom (AB5) and has small generating families of noetherian objects.
(Grothendieck AB condition)

- *(AB1)*
  existence of kernel and cokernel.
- *(AB2)*
  for any f the canonical morphism $coim(f) \to im(f)$ is an isomorphism.
- *(AB3)*
  abelian category possessing arbitrary coproducts.
- *(AB4)*
  AB3 category and direct sums preserves exactness, means that for short exact sequences $0 \to A_i \to B_i \to C_i \to 0$, the direct sum $0 \to \oplus A_i \to \oplus B_i \to \oplus C_i \to 0$ is also exact (In other words, the coproduct of monomorphisms are).
- *(AB5)*
  AB3 categry and has filtered colimit of exact sequences are exact. Namely, $0 \to colim A_i \to colim B_i \to colim C_i \to 0$ is also exact.

- *(AB6)*

Definition 3.4.8 *(Generator of category)*

For a category $\mathscr{C}$, a family of generators $\mathscr{G} \subset \mathscr{C}$ is a collection of objects such that for any $f, g : X \to Y$ such that $f \neq g$ there exists some $G \in \mathscr{G}$ and $h : G \to X$ such that $f \circ h \neq g \circ h$. If the collection consists of one objects, then G is called a generator. The dual notion is a cogenerator.

In particlar, for the category with zero objects, if G is a generator, for any nonzero H, there exists a nonzero morphism $f : G \to H$.

(Examples)

- *category of abelian group*
  The generator is $\mathbb{Z}$.
- *category of R-modules*
  The generator is R.
- *category of sets*
  The generator is one point set.

Definition 3.4.9 *(Grothendieck Category)*
Grothendieck categories are AB5 category with a generator.

(Examples)

- *category of abelian group*
- *category of Mod(R)*
- *For a topological space X, category of sheaves of abelian groups over X*
- *For a ringed space $(X, \mathscr{O}_X)$, sheaves of $\mathscr{O}_X$ modules.*
- *for an affine or projective variety V, category $QCoh(V)$.*
- *For a site $(\mathscr{C}, J)$, category of all sheaves of abelian groups.*

A divisible abelian group is an abelian group M such that for all $g \in M$ and $n \in \mathbb{N}$, there exists $y \in M$ such that $ny = g$. For example, $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group.

A divisble abelian group M is an injective $\mathbb{Z}$-module, and Baer's criterion says for any ideals $\mathfrak{a} \subset R$ and morphisms $\mathfrak{a} \to M$ and $\mathfrak{a} \to R$, there exists a unique morphism $R \to M$ that commutes the diagram.

(Properties)

- *Injective cogenerator*
  $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator.
- *(Baer's Theorem)*
  If M is free, then M^* is injective right R-module and M^* is a direct product of copies of the right R-modules R^*.
- *N is injective iff there exists a free M such that N is isomorphic to a direct summand of M^**
- *M is injective iff direct summand of character module of a free module F.*
- *For any right R-module N, there exists a free module M such that N is submodule of M^*.*

(Finitely Generated)
An object X is finitely generated if X is a sum of all subobjects.

(Locally Finitely Generated)
If a category $\mathscr{C}$ has a finitely generated generators (G_i), $\mathscr{C}$ is called locally finitely generated.

(Finitely Presented)
Finitely presented $W \to X$. finitely generated domain W has a finitely generated kernel.

(Finitely Generated)
Let $X \in \mathscr{A}$ be an object of Grothendieck category $\mathscr{A}$, and X is finitely presented iff for any direct system (A_i) in $\mathscr{A}$, there exists a natural isomorphism $\varinjlim Hom(X, A_i) \to Hom(X, \varinjlim A_i)$.

(Coherent)
X is coherent if it is finitely presented, and if each of its finitely generated subobjects is also finitely presented. The full subcategory of $\mathscr{A}$ of all coherent objects is abelian, and its inclusion functor is exact.

(Character Module)
For a module M, its character module M^* is $M^* = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. the elements in this group are called characters.

(Lambek's theorem)
left R-module M is flat iff the character module M^* is an injective right R-module.

Definition 3.4.10 *(Formal scheme)*
Formal scheme doesn't have to be defined over Noetherian case, but it can usually only be defined over a Noetherian or locally Neotherian.

(linear topology)
Let A be a ring, and A is a linear topology if base of neighborhood of 0 (topological base of A that contains 0) consisting of left ideals.

Note that it satisfies filter axioms:

- *if $a \subset b$ and a is open, then b is also open.*
- *if a and b are open ideals, then $a \cup b$ is also open.*
- *(uniform filter)*
  If a is open, then $(a : r) = \{x \in A | xr \in a\}$
  If A is commutative, then it satisfies the first axiom.

(ideal of definition)
For a linearly topologized ring A, ideal of definition $J \subset A$ is an open ideal such that evry open neighborhood V of 0, there exists $n \in \mathbb{N}$ such that $J^n \subset V$. A generally topologized ring is preadmissible if it admits an ideal of definition, and admissible it is also complete.

$O_{spf(A)}(D_f) = \hat{A}_f$ where $\hat{A}_f$ is a completion of localization of f.

The structure sheaf of $spf(A)$ is $\varinjlim \mathscr{O}_{spf(A)/J_\lambda}$.
Locally noetherian scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a topologically ringed space where for each point in $\mathfrak{X}$, there exists open neighborhood isomorphic to a formal spectrum of a noetherian ring.
A/J represents the formal spectrum.

Definition 3.4.11 *(Formal moduli space)*

We need to review the definition of scheme, since everything turns to be noncommutative, nonnoetherian, nonassociative etc, everything might look differently.

Definition 3.4.12 *(Prime Ideal for noncommutative rings)*
The notion of prime ideal can be extended to oncommutative ring. Let R be a ring. $P \subsetneqq R$ is a prime ideal if for all ideal $A, B \subset R$, $AB \subset P$ means $A \subset P$ or $B \subset P$.

Definition 3.4.13 *(Local Ring)*
A ring is a local ring if it only has one maximal left ideal.

If G is a p-group, then its group algebra $k[G]$ is local.

Definition 3.4.14 *(Artinian Local Ring)*
If in particular, the ring R is an Artinian local ring, then the maximal ideal $\mathfrak{m} \subset R$ is nilpotent.

Definition 3.4.15 *(what is nc affine scheme)*
Let category of noncommutative algebra. Category of nc affine scheme is just the opposite category of it.

Definition 3.4.16 *(Monoidal Category (Tensor Category))*
$\mathscr{C}$ be a category with a bifunctor

$$\otimes : \mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$$

Definition 3.4.17 *(Injective Hull)*
(Essential extension)
For a left R-module N, an essential extension M is an R-module such that for all submodules $H \subset M$, $H \cap N = \{0\}$ implies $H = 0$.

(properties)
For any $N \subset M$, there exists maximal $C \subset M$ such that $N \oplus C \subset M$.

If in particular, $C = 0$, then N is an injective module.
(Injective Hull)
For every module M, there exists the maximal essential extension $E(M)$, called the injective hull of M. Injective hull is necessarily an injective module unique up to isomorphism. This essential extension $E(M)$ can also be featurized by the smallest injective module containing it.
(Examples)

- *injective module itself.*
- *injective hull of integral domain is field of fraction*
- *injective hull of cyclic p-group is Prüfer group.*
- *injective hull of torsion free abelian group A is $\mathbb{Q} \otimes_{\mathbb{Z}} A$*
- *injective hull of $R/rad(R)$ is $Hom_k(R, k)$ where R is finite dimensional k-algebra*
- *injective hull of discrete valuation ring $(R, \mathfrak{m}, k)$ where $\mathfrak{m} = x \cdot R$ is R_x/R*
- *injective hull of $\mathbb{C}$ in $(\mathbb{C}[[t]], (t), \mathbb{C})$ is the module $\mathbb{C}((t))/\mathbb{C}[[t]]$*

Definition 3.4.18 *(Injective Module)*
There can be three equivalent definition of injective module a left module Q over R.

- *Q is injective if morphisms from any $0 \to X \to Y$ commutes.*
- *Any short exact sequence $0 \to Q \to M \to K \to 0$.*
- *If $Q \subset M$ for some M, then there exists another submodule $K \subset M$ such that M is an internal direct sum of Q and K i.e. $Q + K = M$ and $Q \cap K = \{0\}$.*
- *$Hom(-, Q)$ is an exact functor.*

Definition 3.4.19 *(Grothendieck Category)*
(property)

- *Every Grothendieck category contains an injective cogenerator.*
- *For example, the injetive cogenerator of the category of abelian group is $\mathbb{Q}/\mathbb{Z}$.*

- *Every objects in Grothendieck category $\mathscr{A}$ contains an injective hull in $\mathscr{A}$. This allows us to construct injective resolutions, to define a derived functor.*

- *Every Grothendieck category $\mathscr{A}$ is complete. Also, by definition, $\mathscr{A}$ is co-complete.*
- *For any objects $X \in \mathscr{A}$, the collection of subobjects (U_i) has supremum ΣU_i and infimum $\cap U_i$, both of which are again subobjects of X. If further (U_i) is directed and for $V \in \mathscr{A}$, we have $\Sigma(U_i \cap V) = (\Sigma U_i) \cap V$.*

(Gabriel-Popescu Theorem)
A Grothendieck category $\mathscr{A}$ is equivalent to a full subcategory of $Mod(R)$, a category of right R-module where R is a unital ring, and $\mathscr{A}$ can be obtained by a Gabriel quotient of $Mod(R)$ by some localizing subcategory.
(Localization of Category)

- *(Quotient Category)*
  For a category $\mathscr{C}$, a congruence relation R on $\mathscr{C}$ is given by equivalence relation $R_{X,Y}$ on $Hom(X, Y)$ for each $X, Y \in \mathscr{C}$, such that compositions of equivalences are also an equivalence as follows:

  For
  $f_1, f_2 \in Hom(X, Y)$
  $g_1, g_2 \in Hom(Y, Z)$
  such that $f_1 \sim f_2$ and $g_1 \sim g_2$ are equivalences, then the composition $g_1 \circ f_1 \sim g_2 \circ f_2$ are also an equivalences in $Hom(X, Z)$.

Now, we define quotient category $\mathscr{C}/R$ whose objects are those of $\mathscr{C}$, and whose morphisms are equivalence classes of morphisms of $\mathscr{C}$, namely:

$$Hom_{\mathscr{C}/R}(X,Y) = Hom_{\mathscr{C}}(X,Y)/R_{X,Y}$$

- *(Serre Subcategory)*
  For an abelian category $\mathscr{A}$, a subcategory $\mathscr{C}$ is caleld Serre subcategory if for any short exact sequence $0 \to A' \to A \to A'' \to 0$, the object A exists in $\mathscr{C}$ if A' and A'' belong in $\mathscr{C}$. In words, $\mathscr{C}$ is closed under subobjects, quotients objects ane extensions.

  Serre subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ is also abelian, and an inclusion functor $\mathscr{C} \to \mathscr{A}$ is exact.

- *(Quotient of Abelian Category)*
  The quotient $\mathscr{A}/\mathscr{C}$ is an abelian category with same objects of $\mathscr{A}$, where there is a canonical exact functor $\mathscr{A} \to \mathscr{C}$ whose kernel is $\mathscr{C}$. Or for all $X, Y \in \mathscr{A}/\mathscr{C}$, the morphism $X \to Y$ are given by

  $$Hom_{\mathscr{A}/\mathscr{C}}(X,Y) := \varinjlim Hom_{\mathscr{A}}(X', Y/Y')$$

  where the limit is subobjects $X' \subset X$, $Y' \subset Y$ such that $X/X', Y' \in \mathscr{C}$ and $(X',Y') \leqq (X", Y")$ iff $X" \subset X'$ and $Y" \subset Y'$

- *(Quotient Functor)*
  The canonical functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ is exact and surjective on objects. The kernel of Q is $\mathscr{C}$ i.e. $Q(X)$ is zero in $\mathscr{A}/\mathscr{C}$ iff X belongs to $\mathscr{C}$.

- *(Subobjects)*
  Let A be an object of some category and let $u : S \to A$ and $v : T \to A$ be monomorphisms of codomain A, then equivalence $u \equiv v$ iff existence of an isomorphism $\phi : S \to T$ which factors as $u = v \circ \phi$. A subobject of A is an equivalence class defined by it.

- *(Quotient objects)*
  An quotient object is a dual notion of a subobject. For any epimorphisms $u : A \to S$ and $v : A \to T$ with reverse arrows. A quotient object is an equivalence class of epimorphisms.

- *(Localizing Subcategory)*
  For an abelian category $\mathscr{A}$ and its Serre subcategory $\mathscr{C}$, the canonical functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ sends an object X to itself, and a morphism $f : X \to Y$ to the element of direct limit with $X' = X$ and $Y' = 0$.

  An alternative definition of localization is calculus of fraction: If a class of morphism S of $\mathscr{A}$ is a multiplicative system, we let $\mathscr{A}[S^{-1}]$ be a category *where all morphisms in S inverts. By Gabriel-Zisman the localization functor $Q : \mathscr{A} \to \mathscr{A}\mathscr{C}$ is additive.*

(Property of localizing subcategory)

- *(Isomorphism Theorem)*
  For three abelian categories $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, we have $\mathscr{A}/\mathscr{B} \cong \mathscr{C}$ iff there exists an essentially surjective functor $F : \mathscr{A} \to \mathscr{C}$ whose kernel is $\mathscr{B}$ such that every morphism $f : FX \to FY$, there exist morphism $\phi : W \to X$ and $\psi : W \to Y$ such that $F\phi$ is an isomorphism and $f = (F\psi) \circ (F\phi)^{-1}$.

- *(Coherent Sheaves on a Projective Scheme)*
  Let X be a projective scheme $X = Proj(R)$ over some commutative graded noetherian ring R. Then $Coh(X) \cong mod^{\mathbb{Z}}(R)/mod^{\mathbb{Z}}_{Tor}(R)$ where $mod^{\mathbb{Z}}(R)$ is a category of finitely generated graded modules over R, and $mod^{\mathbb{Z}}_{Tor}(R)$ is its Serre subcategory consisting of modules M which are trivial on its higher degree i.e. $M_n = 0$ for all $n \leqq n_0$ for some $n_0 \in \mathbb{N}$.

- *(Gabriel-Popescu Theorem)*

Definition 3.4.20 *(Preadditive Category)*

- *(Preadditive category)*
  A category $\mathscr{C}$ is a preadditive category if for any $X, Y \in \mathscr{C}$, $Hom(X, Y)$ is an abelian group. That is, for any $f, g \in Hom(X, Y)$, their addition is $f + g \in Hom(X, Y)$, also the composition of morphisms are bilinear:

  $(f + g) \circ h = f \circ h + g \circ h$ *and* $h \circ (f + g) = h \circ f + h \circ g$.

- *(Additive Functor)*
  For a functor $F : \mathscr{C} \to \mathscr{D}$ of preadditive categories, F is additive if the function $F : Hom(A, B) \to Hom(FA, FB)$ is a group homomorphism.

  Note that most functors studied in preadditive categories are additive functors.

- *(Biproducts)*
  B is a biproduct of $A_1, \cdots, A_n$ iff there are projection maps $p_j : B \to A_j$ and injection maps $i_j : A_j \to B$ such that $(i_1 \circ p_1) + \cdots + (i_n \circ p_n)$ is an identity morphism of B, $p_j \circ i_j$ is an identity morphism of A_j, $p_j \circ i_k$ is an zero morphism $A_k \to A_j$ if $k \neq j$.

  Biproducts are often written as $A_1 \oplus \cdots \oplus A_n$

- *(Kernel and Cokernel)*
  For a morphism in preadditive category $f : A \to B$, the kernel of f is the equalizer of f and the zero morphism $0 : A \to B$. The cokernel of $f : A \to B$ is a coequalizer of f and 0. More explicitly, the kernel of $f : X \to Y$ is an object K together with a morphism $k : K \to X$ such that the composition $f \circ k$ is zero, and any other pair $k' : K' \to X$ factors through $k : K \to X$.

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
 & K \arrow[u, "k"] \arrow[r, "0_{KY}"] & Y \\
K' \arrow[uur, "k'"] \arrow[ur, dashed, "u"] \arrow[urr, "0_{K'Y}"'] & &
\end{tikzcd}$$

  Cokernel is the dually given.

- *(Zero Objects)*
  A zero object is an object what is both initial and terminal object.

- *(Normal Morphism)*
  A monomorphism is normal if it's a kernel of some morphism. An epimorphism is conormal if it's a cokernel of some morphism.

(Examples)

- *(A Ring as a category)*
  A ring can be considered as a preadditive category with one object and vice versa.

- *(Module category)*
  Similarly, a module can be generalized categorically. For a preadditive category $\mathscr{C}$, $Mod(\mathscr{C}) = Add(\mathscr{C}, \mathscr{A})$ is a functor category, whose objects are additive functors.

- *(Additive category)*
  An additive category is a preadditive category with fiinite biproducts.

- *(R-linear category)*
  A R-linear category $\mathscr{C}$ is a monoidal category of modules over a commutative ring R, in other words, for all $A, B \in \mathscr{C}$, $Hom(A, B)$ is an R-module.

(Monomorphism and Injectivity)

- *(Divisible Abelian Groups: non-injective)*
  The quotient map $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is not an injective map, but monomorphism, meaning that for any $h : G \to \mathbb{Q}$, $q \circ h = 0$ implies $h = 0$. The proof is complicated.

(Epimorphism and Surjectivity)

- *(Compact Hausdorff : surjective)*
  In category of compact Hausdorff spaces, epimosphism is surjective. For proof, suppose $f : X \to Y$ is not surjective, then f is not an epimorphism. Consider that by Urysohn's lemma, there exists a morphism $g_1 : Y \to [0,1]$ such that 0 in fX, but 1 in $y \in Y - fX$, the composition $g_1 \circ f = 0 = 0 \circ f$ where $g_1 \neq 0$. Hence, f is not an epimorphism.

  Note more generally, in the category of Topological spaces, an epimorphism is always surjective.

- *(Monoid : non surjective)*
  In category of monoids, $\mathbb{N} \to \mathbb{Z}$ is non-surjective epimorphism.

- *(Hausdorff : non-surjective)*
  In category of Hausdorff, an epimorphism is precisely a continuous function with dense image.

Definition 3.4.21 *(Additive Category)*
For each object A, we denote $\Delta : A \to A \oplus A$ by $\Delta = i_1 + i_2$, and $\nabla A \oplus A \to A$ by $\nabla = p_1 + p_2$. Then $p_k \circ \Delta = 1_A$ and $\nabla \circ i_k = 1_A$.

Then for two morphisms $\alpha_1, \alpha_2 : A \to B$, the gluing $\alpha_1 \oplus \alpha_2 : A \oplus A \to B \oplus B$ such that $p_l \circ (\alpha_1 \oplus \alpha_2) \circ i_k$ equals to α_k iff $j = k$ and 0 otherwise. Therefore, $\alpha_1 + \alpha_2 = \nabla \circ (\alpha_1 \oplus \alpha_2) \circ \Delta$.

It means that for $a \in A$, $(\alpha_1 + \alpha_2)(a) = \alpha_1(a) + \alpha_2(a)$.

In general, we can define a matrix $f : A^{\oplus k} \to B^{\oplus l}$ for some $k, l \in \mathbb{N}$.

Definition 3.4.22 *(Abelian Category)*
An abelian category is a preadditive category with existence of

- *zero object*
- *binary products*
- *kernel and cokernel*
- *all monomorphisms and epimorphisms are normal*

Definition 3.4.23 *(Presheaf)*

Presheaf is a functor $F : \mathscr{C}^{op} \to Sets$. In particular, if $\mathscr{C}$ is the poset of open sets is a topological space, then this is the usual notion of presheaf.

The Yoneda emebedding $\mathscr{C} \to \hat{\mathscr{C}}$ is fully faithful where $\hat{\mathscr{C}} = Set^{\mathscr{C}^{op}}$.

Density theorem says every presheaf is a colimit of representable presheaves. In fact, $\hat{\mathscr{C}}$ is the colimit completion of $\mathscr{C}$.
$\hat{\mathscr{C}} = Set^{\mathscr{C}^{op}}$ admits small limits and small colimits.

- *Cartesian Closed Category*

  - *It has a terminal object.*
  - *For any $X, Y \in \mathscr{C}$, we have product $X \times Y \in \mathscr{C}$.*
  - *For any $Y, Z \in \mathscr{C}$, we have product $Z^Y \in \mathscr{C}$ where Z^Y is a set of all functions $Y \to Z$.*

- *Comma Category (Slice Category)*

  For a pair of categories $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ with two functors $S : \mathscr{A} \to \mathscr{C}$ and $T : \mathscr{B} \to \mathscr{C}$, A comma category $(S \downarrow T)$ is constructed as follows: an object (A, B, h) where $A \in \mathscr{A}$, $B \in \mathscr{B}$, and $h : S(A) \to T(B)$ a morphisn in $\mathscr{C}$. The morphism $(A, B, h) \to (A', B', h')$ is a pair (f, g) such that $f : A \to A'$ and $g : B \to B'$, and the following diagram commutes.

$$\begin{tikzcd}
S(A) \arrow[r, "S(f)"] \arrow[d, "h"] & S(A') \arrow[d, "h'"] \\
T(A) \arrow[r, "T(g)"] & T(A')
\end{tikzcd}$$

  Slice cateogry is a particular case of comma categeory, replacing $\mathscr{B} = 1$, a category with one object $*$ and functor $S = id_{\mathscr{A}}$.

$$\begin{tikzcd}
S(A) \arrow[r, "f"] \arrow[dr, "\pi_A"] & S(A') \arrow[d, "\pi_{A'}"] \\
 & *
\end{tikzcd}$$

  For a category, if all the slice category is Cartesian closed, then it's locally Cartesian closed.

- *Density Theorem*

All presheaves are colimit of some representable presheaves.

Let F be a presheaf of a category $\mathscr{C}$ i.e. $F \in \hat{\mathscr{C}}$, and let I be an index category with objects (U, x) where $U \in \mathscr{C}$ and $x \in F(U)$, and morphism $(U, x) \to (V, y)$ such that $u : U \to V$ in $\mathscr{C}$ such that $(Fu)(y) = x$. Also, it comes with a forgetful functor $p : I \to \mathscr{C}$. Now wht the forgetful functor the composition below creates a diagram

$I \to_p \mathscr{C} \to_U \hat{\mathscr{C}}$

where the second functor is Yoneda embedding. We claim F is colimit of the diagram.

- *Universal Property*

  Let $\mathscr{C}$ be a category and $\mathscr{D}$ be a category with small colimits, also denote $\hat{\mathscr{C}} = Fct(\mathscr{C}^{op}, Set)$. Then each functor $\eta : \mathscr{C} \to \mathscr{D}$ factorizes by $\eta = \hat{\eta} \circ y$ as follows:

  $\mathscr{C} \to_y \hat{\mathscr{C}} \to_{\hat{\eta}} \mathscr{D}$

  where y is an Yoneda embedding, and $\tilde{\eta}$ is Yoneda extension of η. Given a presheaf $F = \varinjlim yU_i$ where $U_i \in \mathscr{C}$. Let $\tilde{\eta}F = \varinjlim \eta U_i$, which exists by assumption. Since $\varinjlim$ is functorial, this determines $\tilde{\eta} : \hat{\mathscr{C}} \to \mathscr{D}$.

  Finally, the above results implies that a functor $\mathscr{C} \to \mathscr{D}$ induces another functor $\hat{\mathscr{C}} \to \hat{\mathscr{D}}$.

- *Kan Extension*

  For categories $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, and two functors $X : \mathscr{A} \to \mathscr{C}$ and $F : \mathscr{A} \to \mathscr{B}$, we will construct Kan extension of X along F.

  For a right Kan extension, from the following diagram, find a functor R and it has natural transformation $\epsilon : RF \to X$ as in the following diagram:

$$\begin{tikzcd} & \mathscr{B} \arrow[d, "R"] \\ \mathscr{A} \arrow[ur, "F"] \arrow[r, "X"'] & \mathscr{C} \end{tikzcd}$$

  Formaly, the right Kan extension of X along F consists of a functor $R : \mathscr{B} \to \mathscr{C}$ and a natural transformation $\epsilon : RF \to X$ that is couniversal with respect to the specification, in the sense that the any functor $M : \mathscr{B} \to \mathscr{C}$ and natural transformation $\mu : MF \to X$, a unique natural transformation $\delta : M \to R$ is defined and fits into a commutative diagram:

$$
\begin{tikzcd}
 & RF \arrow[dl, "\epsilon"'] \\
X & MF \arrow[l, "\mu"] \arrow[u, "\delta_F"']
\end{tikzcd}
$$

where δ is a natural transformation with $\delta_F(a) = \delta(Fa) : MF(a) \to RF(a)$ for any objects $a \in A$. So in othe words, R is the terminal object of the class. This functor R is often dentoed by $Ran_F(X)$.

Applications:

- *Kan extension as colimits*
  Suppose $X : \mathscr{B} \to \mathscr{C}$ and $F : \mathscr{A} \to \mathscr{B}$ are two functors, and if $\mathscr{A}$ is small and $\mathscr{C}$ is cocomplete, then there exists a left Kan extension $Lan_F(X)$ of X along F, defined at each object b of $\mathscr{B}$ by

  $Lan_F(X) = \varinjlim_{f:Fa\to b} X(a)$

  where the colimit is taken over comma category, where $* \to \mathscr{B}$, $* \mapsto b$ is the constant functor.

- *Kan extension as coends*

  Right Kan extension can be computed with (co)end formula

  $(Ran_F X)b = \int_a Xa^{B(b,Fa)}$
- *Limit as Kan extension*

  The limit of a functor $F : \mathscr{C} \to \mathscr{D}$ can be expressed as a Kan extension by
  $limF = Ran_E F$

  where E is a unique functor $\mathscr{C} \to$. is a category with one object and one arrow, a terminal object in Cat.
- *Adjointness*
  A functor $\mathscr{C} \to \mathscr{D}$ possesses a left adjoint iff the right Kan extension of $Id : \mathscr{C} \to \mathscr{C}$ along F exists and preserved by F. In this case, the left adjoint is given by $Ran_F(Id)$.
- *Codensity Monad*
  The codensity monad of a functor $G : \mathscr{D} \to \mathscr{C}$ is a right Kan extension of G along itself.

- *Monad, Comonad*

  A monad is a pair (T, μ, η) of one endofunctor and two natural transformations, where $T : \mathscr{C} \to \mathscr{C}$ is an endofunctor and $\mu : T^2 \to T$ and $\eta : 1_{\mathscr{C}} \to T$ that satisfies the the following diagrams (sometimes called coherence conditions):

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \Big\downarrow{\scriptstyle \mu T} & & \Big\downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

$$\begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ \Big\downarrow{\scriptstyle T\eta} & & \Big\downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

  On the other hand, comonad is a monad for opposite category $\mathscr{C}^{op}$.

  - *Identity*
    An identity functor is a monad.
  - *Monad Arising from Adjointness*
    An example of monad comes from an adjointness: if F is left adjoint to G, then the composition $G \circ F$ becomes an endofunctor.
  - *Adjointness of Category of Vector Space*
    Consider an adjoint relation

    $(-)^* : Vect_k \leftrightarrow Vect_k^{op} : (-)^*$
  - *Closure Operators on Partially Ordered Sets*
  - *Free-Forgetful Adjunction*
  - *Codensity Monad*

    The codensity monad of $G : \mathscr{D} \to \mathscr{C}$ is defined by the right Kan extension of G along itself, provided Kan extension exists. We denote the right Kan extension as $T^G : \mathscr{C} \to \mathscr{C}$, and if $\mathscr{D}$ is small, the Kan extension exists.

- *Sheafification*

  Sheafification is a method of constructing sheaf from a presheaf on a topological space. Let $\mathscr{F} : Top/X \to Sets$ be a presheaf,

$\mathscr{F}^{\#}(U) = \{(s_u) \in \prod_{u \in U} \mathscr{F}_u$ *such that (*)*$\}$

where $(*)$ *is the property:*

For every $u \in U$*, there exists an open neighborhood* $u \in V \subset U$*, and a section* $\sigma \in \mathscr{F}(V)$ *such that for all* $v \in V$*, we have* $s_v = (V, \sigma)$ *in* $\mathscr{F}_v$*.*

Then the for $V \subset U \subset X$ *are open sets, the projection maps*

$\prod_{u \in U} \mathscr{F}_u \to \prod_{v \in V} \mathscr{F}_v$

maps elements of $\mathscr{F}^{\#}(U)$ *into* $\mathscr{F}^{\#}(V)$

Then $\mathscr{F}^{\#}$ *will turn into a presheaf of sets on* X*.*
Furthermore, the image $\mathscr{F}(U) \to \prod_{u \in U} \mathscr{F}$ *clearly has an image* $\mathscr{F}^{\#}$*, so*

$$\begin{array}{ccccc} \mathscr{F}(U) & \longrightarrow & \mathscr{F}^{\#}(U) & \longrightarrow & \prod_{u \in U} \mathscr{F}_u \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{F}(V) & \longrightarrow & \mathscr{F}^{\#}(V) & \longrightarrow & \prod_{u \in V} \mathscr{F}_v \end{array}$$

The product $\prod(\mathscr{F}) : U \mapsto \prod_{u \in U} \mathscr{F}_u$ *is a sheaf with an obvious restriction mappings, and by construction* $\mathscr{F}^{\#}$ *is a subsheaf of* $\prod(\mathscr{F})$*. In other words, we have a sequnce of presheaves* $\mathscr{F} \to \mathscr{F}^{\#} \to \prod(\mathscr{F})$*. From this, we will know that* $\mathscr{F}^{\#}$ *is a sheaf.*

applications:

- *For any morphism* $\mathscr{F}^{\#} \to \mathscr{G}^{\#}$ *factors by* $\mathscr{F}^{\#} \to \prod(\mathscr{F}) \to \mathscr{G}^{\#}$*.*

A support of function is given by

$supp(f) = \{x \in X | f(x) \neq 0\}$

especially if the support is compact set, it's called by a compact support.
Grothendieck Topos

Cosheaf

cosheaf with values in ∞*-category* $\mathscr{C}$ *that admits a colimit is a functor* F *from the category of open subsets of topological space* X *to* $\mathscr{C}$ *such that:*

- *1:*
  $F()$ *is the initial object.*

- *2:*
  For any increasing sequence of U_i *of open subsets with union* U*, the canonical map* $\varinjlim F(U_i) \to F(U)$ *is an equivalence.*

- *3:*
  $F(U \cup V)$ *is an pushout of* $F(U \cap V) \to F(U)$ *and* $F(U \cap V) \to F(V)$.

The basic example is $U \mapsto C_*(U; A)$ *where on the right is singular chain complex of* U *with coefficients in an abelian group* A.

example: If f *is a continuous map, then* $U \mapsto f^{-1}(U)$ *is a cosheaf.*

Definition 3.4.24 *(Grothendieck Six Operations)*
Grothendieck six operations are direct image f_**, inverse image* f^**, proper direct image* $f^!$*, proper inverse image* $f_!$*, internal tensor product, and internal Hom.*

f^* *and* f_* *are adjoint, also* $f^!$ *and* $f_!$ *are adjoint. Also, tensor and internal hom are adjoint.*

Direct Image with compact support $f^! : Sh(X) \to Sh(Y)$ *is a functor such that*

$$f_!(F)(U) = \{s \in F(f^{-1}(U)) | f|_{supp(s)} : supp(s) \to U \text{ is proper}\}$$

If f *is proper, then* $f^!$ *is equal to* f_*.

If f *is an open embedding, then* $f_!$ *identifies with the extension by zero functor.*

Definition 3.4.25 *(Category of Schemes)*

- *closed under limits*
- *base change*

Definition 3.4.26 *(Category of Rings)*

A category of rings is not a preadditive category because it doesn't contain zero morphisms. Instead, a category of rings is a symmetric monoidal category.

(Here preadditive category contains zero morphism, which is both constant and coconstant morphism. $f : X \to Y$ *is a constant morphism if for all* $W \in \mathscr{C}$ *and* $g, h : W \to X$, $fg = fh$*. A coconstant morphism is a dual notion.)*

However, a category of commutative ring is still a reflective subcategory of category of rings. Free commutative ring $\mathbb{Z}(E)$ generated by E gives a left adjoint functor to the forgetful functor $Comm \to Sets$

Reflective subcategory
$\mathscr{A} \subset \mathscr{B}$ is a reflective subcategory if the inclusion functor $\mathscr{A} \to \mathscr{B}$ has a left adjoint, and this adjoint is sometimes called a reflector or a localization. Also coreflective if the inclusion functor has right adjoint.

Definition 3.4.27 *(Category of Commutatve Rings)*

Eckmann-Hilton argument
For a monoid X, if $\circ$ and $\otimes$ are binary operations such that they are unital meaning that existence of identities $1_\circ$ and $1_\otimes$ such that $1_\circ \circ a = a = 1_\circ \circ a$ and $1_\otimes \otimes a = a = 1_\otimes \otimes a$ for all $a \in X$. If in addition, we suppose $(a \otimes b) \circ (c \otimes d) = (a \circ b) \otimes (c \circ d)$ for all $a, b, c, d \in X$, then $\circ$ and $\otimes$ are the same and in fact commutative and assocative.

Category of Monoids
A category $(\mathscr{C}, \otimes, I)$ where $\otimes$ is a binary product and I is a unit is a monoidal category. Each object (M, μ, η) is a monoid if $\mu : M \otimes M \to M$ is a multiplication and $\eta : I \to M$ is a unit, satisfying the pentagon axioms.

Now for two monoids (M, μ, η) and (M', μ', η') in a monoidal category $\mathscr{C}$, a morphism $f : M \to M'$ is a morphism of monoids when

- $f \circ \mu = \mu' \circ (f \otimes f)$
- $f \circ \eta = \eta'$

Now by Eckmann-hilton theorem, a monoid object of category of monoid is a commutative monoid.

Definition 3.4.28 *(Rng)*
For a category of rings without identity Rng, Ring is nonfull subcategory of Rng. But still the inclusion functor Ring $\to$ Rng has a left adjoint which formally looks adjoints an identity to any rng. The inclusion functor Rng $\to$ Ring respects limits but not colimits.

There exists a zero object in Rng, so Rng has zero morphisms. these are just rng homomorphism that maps everything to zero. Despite existence of zero morphisms, Rng is still not an preadditive category. The pointwise sum of two rng homomorphisms are not rng homomorphism.

However, there is a fully faithful functor Ab $\to$ Rng sending an abelian group to the associated rng of square zero.

Free rng generated by a set $\{x\}$ is ring of all integral polynomials over x without constant terms, while the free ring generated by $\{x\}$ is just polynomial ring $\mathbb{Z}[x]$.

Rng of square zero.
Rng of square zero. For any abelian group R, define multiplication as for all $x, y \in R$, the product $xy = 0$. Then it's a rng of square zero. Here, the only ring that's a rng of square zero is a trivial ring $\{0\}$.

Durroh extension
Durroh extension is a way of extending a rng R to a ring $R^\wedge$ by adding an identity element. If we let $R^\wedge = \mathbb{Z} \times R$ by

- $(n_1, r_1) + (n_2, r_2) = (n_1 + n_2, r_1 + r_2)$
- $(n_1, r_1) \cdot (n_2, r_2) = (n_1 n_2, n_1 r_2 + n_2 r_1 + r_1 r_2)$

where the multiplicative identity is $(1, 0)$, and $R^\wedge$ is indeed a ring. Now consider this operation is functorial, so it makes Rng $\to$ Ring, and this functor is left adjoint to the inclusion functor Ring $\to$ Rng.

Unital Homomorphism
For unital algebras A and B, an algebra homomorphism $f : A \to B$ is unital if sending the identity to the identity.

Definition 3.4.29 *(isomorphism of categoies)*
example

- *Group Representation and modules*
  category of k-linear group representation of G is isomorphic to the category of left modules over kG.
- *Boolean*
  Category of Boolean ring and category of Boolean lattice are isomorphic.

Definition 3.4.30 *(Monad)*

- *Closure Operator*
- *Galois Connection*

Definition 3.4.31 *(natural transformaiton)*

- *Dinatural Transformation*
  Dinatural transformation is a morphism between two functors $S, T : \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{D}$, written as $\alpha : S \to T$ carries every objects by $\alpha_c : S(c, c) \to T(c, c)$, and it satisfies the coherence property. For every morphism $f : c \to c'$, the following diagram

$$\begin{tikzcd}
 & S(c,c) \arrow[r, "\alpha_n"] & T(c,c) \arrow[dr, "T(1,f)"] & \\
S(c',c) \arrow[ur, "S(f,1)"] \arrow[dr, "S(1,f)"] & & & T(c,c') \\
 & S(c',c') \arrow[r, "\alpha_{c'}"] & T(c',c') \arrow[ur, "T(f,1)"] &
\end{tikzcd}$$

commutes. The composition of two natural transformations need not be dinatural.

- *Extranatural transformation*

  Let $F : A \times B^{op} \times B \to D$ *and* $G : A \times C^{op} \times C \to D$ *are functors of categories. A natural transformation* $\eta(a,b,c) : F(a,b,b) \to G(a,c,c)$ *is said to be natural in* a *and extranatural in* b *and* c *if the following holds.*

  - *(naturality)*
    $\eta(-,b,c)$ *is a natural transformation (in a usual sense).*
  - *(extranatuality in* b*)*
    For $\forall(g : b \to b') \in Mor(B)$, $\forall a \in A$, $\forall c \in C$, *the following diagram commutes.*

$$\begin{tikzcd}
F(a,b',b) \arrow[r, "F(1,1,g)"] \arrow[d, "F(1,g,1)"] & F(a,b',b') \arrow[d, "\eta(a,b',c)"] \\
F(a,b,b) \arrow[r, "\eta(a,b,c)"] & G(a,c,c)
\end{tikzcd}$$

  - *(extranatuality in* c*)*
    For $\forall(h : c \to c') \in Mor(C)$, $\forall a \in A$, $\forall b \in B$, *the following diagram commutes.*

$$\begin{tikzcd}
F(a,b,b) \arrow[r, "\eta(a,b,c')"] \arrow[d, "\eta(a,b,c)"] & G(a,c',c') \arrow[d, "G(1,h,1)"] \\
G(a,c,c) \arrow[r, "G(1,1,h)"] & G(a,c,c')
\end{tikzcd}$$

- *(End)*
  In category theory, an end of a functor $S : C \times C^{op} \to X$ *is a universal dinatural transformation from an object* e *of* X *to* S.

  More explicitly, this is a pair (e,ω) *where* e *is an object of* X, *and* $\omega : e \to S$ *is an extranatural transformation such that for every extranatural transformation* $\beta : x \to S$ *there exists a unique morphism* $h : x \to e$ *of* X *with* $\beta_a = \omega_a \circ h$ *for all* $a \in C$. *By abuse of language, if we let end of object as* e *such that*

$e = \int_c S(c,c)$ or just $\int_C S$.

Chracaterization as a limit: if X is complete, and $\mathscr{C}$ is small, the end can be described as an equalizer in the diagram.

$\int_c S(c,c) \to \prod_{c\in C} S(c,c) \rightrightarrows \prod_{c\to c'} S(c,c')$

Definition 3.4.32 *(Galois Connection and Closure Operator)*

- *Closure Operator*
  A closure operator on S is a function $cl : P(S) \to P(S)$ from the power set to itself with the following conditions

  - *$X \subset cl(X)$ (cl is extensive)*
  - *$X \subset Y \Rightarrow cl(X) \subset cl(Y)$ (cl is increasing)*
  - *$cl(cl(X)) = cl(X)$ (cl is idempotent)*

  Closure operator is alternatively called hull operator.

  An closure operator in algebra is, for example, that for all subsets of an algebra generates subalgebras, so we can define a closure operator. Also, linear span, and convex full in n-dimensional Euclidean space.

  An closure operator in lattice is, for example, Galois connection.

  In fact, every closure operator arises in this way is Galois connection.

- *Galois connection*

  Let $(A, \geqq)$ and $(B, \geqq)$ are two partial order sets. A monotone Galois connection between these posets consists of two monotone functions: $F : A \to B$, and $G : B \to A$ such that for all $a \in A$ and $b \in B$, we have

  $F(a) \geqq b$ iff $a \geqq G(b)$

  Then F is called lower(left) adjoint of G, and G is called upper(right) adjoint of F. monotone Galois conenction is a special case of pair of adjoint functors.

  An essential property of Galois connection is that

  $F(a)$ is the least element $\tilde{b}$ with $a \geqq G(\tilde{b})$ and
  $G(b)$ is the largest element $\tilde{a}$ with $F(\tilde{a}) \geqq b$.

If F or G is bijective, then one is inverse of the other $F = G^{-1}$.

Given adjointness of F and G, consider composition $GF : A \to A$ known as an associated closure operator, and $FG : B \to B$, known as a kernel operator. Both are monotone and idempotent, and we have $a \geqq GF(a)$ for all $a \in A$, and $FG(b) \geqq b$ for all $b \in B$.

Galois insertion of B into A is a Galois connection in which the kernel operator FG is the identity on B, and hence G is an order isomorphism of B onto the set of closed elements $GF[A]$ of A.

- *Galois connection and Closure operators*

  If f^* and f_* are Galois connection, then the composition $f_* \circ f^*$ are monotone, inflationary and idempotent, so this states that $_* \circ f^*$ is a closure operator. Dually, $f^* \circ f_*$ is monotone, deflationary, idempotent, so it's a kernel operator.

- *Categorical Interpretation*

  There is a paraphrase of the theory by category theory A category of partial ordered sets are where each object is element of partial ordered sets $(P, \geqq)$, and morphsims are uniquely existing $x \to y$ iff $x \geqq y$. Then Galois connection is nothing but a pair of adjoint functors between two categories of partial ordered sets.

Definition 3.4.33 *(Stone Space)*
Stonian space is an extremally disconnected, compact and Hausdorff.

A topological space is extremally disconnected if every open set.

Stone space is totally disconnected, compact Hausdorff.

In duality between Stone spaces and Boolean algebra, a Stonian space corresponds to a complete Boolean algebra. A Boolean algebra X is complete if every subset $U \subset X$, the supremum exists.

Definition 3.4.34 *(Stone Representation Theorem)*

For every Boolean algebra B, we can associate a Stone space $S(B)$ as follows:
Elements of $S(B)$ are ultrafilters on B, and the topology on $S(B)$, called Stone topology is generated by the form $\{F \in S(B) : b \in F\}$ where $b \in B$.

Stone representation theorem for Boolean algebra states that every Boolean algebra is isomorphic to the Boolean algebra of clopen sets of the Stone space $S(B)$.

Definition 3.4.35 *(Filter)*

For a partial order set $(P, \geqq)$, a filter F is a subset $F \subset P$ such that

- *Nontriviality*
  $F \neq \emptyset$
- *Downward Directed*
  For all $x, y \in F$, there is some $z \in F$ such that $z \geqq x$ and $z \geqq y$.
- *Upward Closure*
  For every $x \in F$ and $p \in P$ such that $x \geqq p$, then $p \in F$.

If additionally, $F \neq P$, then F is called proper filter.

Also, if a subset $S \subset F$ generates F, then S is called a basis of F. Especially, if a subset $B \subset P$ satisifies nontriviality and if it's downward directed, B generates all upward closure elements, so B generates a filter. Such basis B is called a prefilter.

A principal filter is that for a point $p \in P$, $\{x \in P | p \geqq x\}$, which is the smallest filter that contains p. It is often denoted by $\uparrow p$.

A filter $U \subset P$ is a ultrafilter if it's a proper filter, and there is no filter $F \subset P$ that properly extends U.

Principal ultrafilter is a filter containing a least element a. Free ultrafilter is a ultrafilter which is not principal.

U is a ultrafilter on a powerset of a set.

- $\emptyset \notin U$
- *If $A \in U$, then for all $B \supset A$, $B \in U$.*
- *If $A, B \in U$, then $A \cap B \in U$.*
- *If $A \subset X$, then A or $X \backslash A$ contained in U.*

Every filter converges to a point.
A Stone space is a compact totally disconnected Hausdorff space. X is homeomorphic to a projective limit of finite discrete spaces, or compact T0 and 0-dimensional. Or X is coherent and Hausdorff.

A Boolean algebra B has an associated topological space, which is a Stone space.

If its basis is $\{x \in S(B) | b \in x\}$, these sets are also closed set so clopen.

Boolean-Stone Correspondence
category theoretic duality between category of Boolean algebras and category of Stone spaces. Every Boolean algebra is isomorphic to a Boolean algebra of some clopen subset of Stone space $S(B)$.

Stone functor $S : Top^{op} \to Bool$. All the morphisms in Stone spaces are continuous because all objects are clopen.

Definition 3.4.36 *(Covering)*

For a lattice, an atom is an element a such that there exists no intermediate element between the least element 0 as $0 < x < a$.

A covering of x is an element y such that there is no intermediate element z such that $x < y < z$.

Geometric lattice is a finite atomistic semimodular lattice. Graded if $r(x) = r(y) + 1$ for x is a cover of y. Seminorm if $r(x) + r(y) \leqq r(x \wedge y) + r(x \vee y)$.

Geometric lattice is a finite version of Matroid. Matroid is atomistic semimodular lattice.

Domain Theory:

Definition 3.4.37 *(DCPO)*
We will define DCPO (category of Directed Complete Partial Order). Its objects are directed complete partial orders and its morphisms are D-continuous maps.

- *Informatic partial order:*
  $[a, b] \sqsubseteq [c, d] \in \mathbb{IR}$ *iff* $[c, d] \subset [a, b]$.

- *Directed Completeness:*
  $\emptyset \neq D \subset P$ *directed if* $x, y \in D \Rightarrow (\exists z \in D) x, y \geqq z$. *$P$ directed complete: D directed $\Rightarrow sup(D)$ exists.*

  $D \subset \mathbb{IR}$ *directed* $\Rightarrow sup(D) = \bigcap D$.

- *Approximation:*
  $x << y$ *iff* $y \geqq sup(D) \Rightarrow (\exists d \in D) x \geqq d$
  Domain: $\downarrow y = \{x | x << y\}$ *directed and* $y = sup \downarrow y$.

- *Morphisms:*
  $f : P \to Q$ *is D-continuous if*
  - f *is monotone*
  - D *is directed* $\Rightarrow f(sup(D)) = sup(f(D))$

- *:*
  $D \in DCPO$ is monotone with least element $\perp$, $f : D \to D$ is monotone. Then:

  - *Fix $f = sup_{\alpha \in Ord} f^{\alpha}(\perp)$ is the least fixed point of f.*
  - *f is D-continuous $\Rightarrow$ Fix $f = sup_{n \geqq 0} f^n(\perp)$*

- *Scott Topology:*
  U is Scott open if

  - *$U = \uparrow U = \{x \in P | (\exists u \in U) u \geqq x\}$*
  - *D directed, $sup(D) \in U \Rightarrow D \cap U \neq \emptyset$*

  Scott topology is always $T0$ and in fact, sober.

- *Sober space*
  For all closed sets $A \subset X$ are closure of some single point x as $A = \overline{\{x\}}$.

Definition 3.4.38 *(Pretopology)*

There are two different ways of defining pretopology.

- *filter*
  For a set X, $N(x)$ is a set of filters of a point x for all $x \in X$, interpreted as a neighborhood system of x. Pretopology is a set X with neighborhood system.

- *preclosure*
  For a set X, pretopology (X, cl) can be defined with some (Čech) preclosure operator cl. A preclosure operator is a closure operator without requiring to be idempotent.

  A pretopological space is topological space when its preclosure operator is idempotent.

  A map $f : (X, cl) \to (Y, cl')$ between two pretopological spaces is called continuous if it satisfies for all subsets $A \subset X$,

  $f(cl(A)) \subset cl'(f(A))$.

Definition 3.4.39 *()*

Closed immersion has TFAE

- *$i : Z \to X$ is a closed immersion.*

- *Every affine open* $Spec(R) = U \subset X$*, there exists an ideal* $I \subset R$ *such that* $i^{-1}(U) = Spec(R/I)$ *as schemes over* $U = spec(R)$*.*

- *There exists an open covering* $R = \cup_{j\in J} U_j$, $U_j = Spec(R_j)$ *and for every* $j \in J$ *there exists an ideal* $I_j \subset R_j$ *such that* $i^{-1}(U_j) = Spec(R_j/I_j)$ *as schemes over* $U_j = Spec(R_j)$*.*

- *The morphism* i *induces a homeomorphism of* Z *with a closed subset of* X *and* $i^{\#} : \mathscr{O}_Z \to i_*\mathscr{O}_X$ *is surjective, and the kernel* $ker(i^{\#})$ *is quasi-coherent sheaf of ideals.*

f is a closed immersion iff $\Delta_{X/S} \subset X \times_S X$ *is a closed subset.*
Quasi Compact Morphism has TEAE

- $f : X \to S$ *is quasi-compact*

- *For any open affine subscheme* $U \subset S$*, the inverse* $f^{-1}(U)$ *is quasi-compact.*

- *For any cover* $S = \cup U_i$, $f^{-1}(U_i)$ *is quasi-compact for all* i*.*

Noetherian
X is locally Noetherian if for all $x \in X$*, there exists open affine neighborhood* $Spec(R) = U \ni x$ *s.t.* R *is a Noetherian ring.*

X is Noetherian if it's locally Noetherian and quasi-compact.

Finite Type Morphism
$R \to A$ *is finite type if* A *is isomorphic to quotient of* $R[x_1, \cdots, x_n]$ *as* $R[x_1, \cdots, x_n]/J \cong A$ *as an* R*-algebra.*

f is of finite type at $x \in X$ *if there is an affine open neighborhood* $spec(A) = U \subset X$ *of* x *and an affine open* $spec(R) = V \subset S$ *with* $f(U) \subset V$ *such that the induced map* $R \to A$ *is finite type.*

f is of locally finite type if finite type at all $x \in X$*. Especially* f *is finite type if* f *is locally finite type and quasi-compact.*
Here is TFAE:

- f *is locally of finite type.*

- *For all affine open* $U \subset X$ *and* $V \subset S$ *with* $f(U) \subset V$ *the ring map* $\mathscr{O}_S(V) \to \mathscr{O}_X(U)$ *is of finite type.*

- *There exists an affine open covering* $S = \cup_{j\in J} V_j$ *and open coverings* $f^{-1}(V_j) = \cup_{i\in I_j} U_i$ *such that each of the morphisms* $U_i \to V_j$, $j \in J$ *and* $i \in I_j$ *is locally of finite type.*

- *There exists an affine open covering $S = \cup V_j$ and affine open coverings $f^{-1}(V_j) = \cup_{i \in I_j} U_i$ such that the ring map $\mathscr{O}_S(V_j) \to \mathscr{O}_X(U_i)$ is of finite type, for all $j \in J$ and $i \in I_j$.*

Separation Axiom

- *$f : X \to S$ is separated if $\Delta_{X/S}$ is a closed immersion.*
- *$f : X \to S$ is quasi-separated if $\Delta_{X/S}$ is a quasi-compact morphism.*
- *Y is separated if $Y \to Spec(\mathbb{Z})$ is separated.*
- *Y is quasi-separated if $Y \to Spec(\mathbb{Z})$ is quasi-separated.*

Ample
Given a global section s of $\mathscr{L}$, the set $X_s = \{x \in X | s \notin {}_x\mathscr{L}_x\}$ is open.
$X_s \cap X_{s'} = X_{ss'}$ where ss' denote the section $s \otimes s' \in \Gamma(X, \mathscr{L} \otimes \mathscr{L}')$
Let X be a scheme, and a sheaf $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. $\mathscr{L}$ is called ample if
X is quasi-compact, and for every $x \in X$, there exists $n \geqq 1$ and $s \in \Gamma(X, \mathscr{L}L^{\times n})$ such that $x \in X_s$ and X_s is affine.
$\mathscr{L}$ is ample iff $\mathscr{L}^{\times n}$ is ample.
Finite Presentation
$R \to A$ is of finite presentation if A is isomorphic to $R[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$. In Scheme, $f : X \to S$ if finite presentation at $x \in X$ if there exists $Spec(A) = U \subset X$ and $V \subset S$ such that $f(U) \subset V$ such that the induced morphism $\mathscr{O}_S(V) \to \mathscr{O}_X(U)$ is of finite presentation.

In other words, if $Z \to X$ is of finite presentation if $\mathscr{I}_Z \to \mathscr{O}_X$ is of finite type.
Projective
$f : X \to S$ is projective morphism if X is isomorphic to some subscheme of $\mathbb{P}(\mathscr{E})$ for some quasi-coherent finitely presented sheaf $\mathscr{E}$. H-projective locally-projective

open immersion if locally of finite presentation.

Definition 3.4.40 *()*

- *Ample Invertible sheaf*
  For preliminary, let $X_s = \{x \in X | s \in \mathfrak{m}_x\mathscr{L}_x\}$ and $X_s \cap X_{s'} = X_{ss'}$ where ss' denote section $s \otimes s' \in \Gamma(X, \mathscr{L} \otimes \mathscr{L}')$.

  Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$ module, then $\mathscr{L}$ is ample if

  - *X is quasi-compact.*

- *for every $x \in X$, there exists $n \geqq 1$ and $s \in \Gamma(X, \mathscr{L}^{\otimes n}$ such that $x \in X_s$ and X_s is affine.*

  The map $s : \mathscr{O}_{X_s} \to \mathscr{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathscr{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

- *Relatively Ample*

  $\mathscr{L}$ is an invertible sheaf on X, and $\mathscr{L}$ is relatively ample with $f : X \to S$ if $X \to S$ is quasi-compact, and if for every $V \subset S$ the restriction of $\mathscr{L}$ to the open subscheme $f^{-1}(V)$ on X is ample.

- *relatively Very Ample*
  For a quasi-coherent sheaf $\mathscr{E}$, we let the projective bundle associated to $\mathscr{E}$ is the morphism $\mathbb{P}(\mathscr{E}) \to S$ where $\mathbb{P}(\mathscr{E}) = \underline{Proj}_X(Sym(\mathscr{E}))$.

  An invertible sheaf $\mathscr{L}$ over X is relatively very ample on X/S if there is a $\mathscr{O}_S$-module $\mathscr{E}$ and an immersion $i : X \to \mathbb{P}(\mathscr{E})$ over S such that $\mathscr{L} \cong i^*\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

- *Integral*
  A scheme X is integral if it's non empty and for all open affine subscheme $U \subset X$ such that $U = Spec(R)$, R is integral domain.

- *Reduced*
  A ring is reduced if it has no non-zero nilpotent elements. In other words, there exists no $x \in R$ such that $x^2 = 0$. A scheme X is reduced if the local ring $\mathscr{O}_{X,x}$ is reduced for all $x \in X$.

- *irreducible*
  A scheme X is irreducible if there exists an open covering $X = \cup_{i \in I} U_i$ such that I is non-empty, U_i is irreducible for all $i \in I$ and $U_i \cup U_j \neq$ for all $i, j \in I$. An affine scheme U_i is irreducible if it is a closure of a generic point.

- *A sheaf of Ideals*
  A sheaf of ideals on X is a sheaf of modules which is a subsheaf of $\mathscr{O}_X$. In other words, for every open set U, (U) is an ideal in $\mathscr{O}_X(U)$.

Properties
A scheme X is integral iff X is reduced and irreducible.

Definition 3.4.41 *(Blow Up)*
Let X be a scheme, and $\mathscr{I} \subset \mathscr{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be a closed subscheme that corresponds to the ideal sheaf $\mathscr{I}$. The blowing up of X along Z (or along ideal sheaf $\mathscr{I}$ is the morphism

$$b : \underline{Proj}_X(\oplus_{n \geqq 0} \mathscr{I}^n) \to X$$

We let the exceptional divisor as $b^{-1}(Z)$, sometimes Z is called the center of blowup.

Here is the list of properties of blow up:

- *The exceptional divisor is an effective Cartier divisor.*
- $b|_{b^{-1}(X\backslash Z)} : b^{-1}(X\backslash Z) \to X\backslash Z$ *is an isomorphism.*
- *There is a canonical isomorphism* $\mathscr{O}_{X'}(-1) \cong \mathscr{O}_{X'}(E)$
- b' *is a projective morphism.*
- $\mathscr{O}_{X'}(1)$ *is a b-relatively ample invertible sheaf.*

Here is property for composition of blow ups.

$X" \to X' \to X$

and let $Z \subset X$, $Z' \subset X'$ be closed subschemes of finitely presented. Then, $Y = Z' \cup b(Z) \subset X$ is also a closed subscheme of finitely presented. Then, $X" \to X$ is isomorphism to blow up of X in Y.

Admissible Blow up
$X' \to X$ along $Z \subset X$ is U-admissible blow up if Z is of finite presentation and $Z \cap U = \emptyset$ and .
Closed Immersion

- *$f : Z \to X$ is a closed immersion*
- *For every open affine $U = spec(R) \subset X$, there exists an ideal $I \subset R$ such that $f^{-1}(U) = Spec(R/I)$ as schemes over U.*
- *There exists an open covering $X = \cup U_j$, $U_j = Spec(R_j)$ and for each j there exists an ideal $I_j \subset R_j$ such that $f^{-1}(U_j) = Spec(R_j/I_j)$ as schemes over U_j.*
- *there is a quasi-coherent sheaf of ideal $\mathscr{I}$ on X such that $f_*\mathscr{O}_Z \cong \mathscr{O}_X/\mathscr{I}$ and f is an isomorphism of Z onto the global spec of $\mathscr{O}_X/\mathscr{I}$ over X.*

Definition 3.4.42 *(Nagata Compactification)*

Let $X = U \cup V$ be a scheme and its open covering, and morphism of schemes $X \to S$. $\Delta : X \to X \times_S X$ is closed by using the open covering of $X \times_S X$, and if $U \to S$ and $V \to S$ are separated, and if $U \cap V \to U \times_S V$ is closed, then $X \to S$ is also separated.

$Z_1, Z_2 \subset X$ closed subschemes of finite presentation such that $Z_1 \cap Z_2 \cap U = \emptyset$, then there exists a U-admissible blowing up $X' \to X$ such that the strict transform of Z_1 and Z_2 are disjoint.

If $T_1, T_2 \subset U$ are disjoint constructive closed subsets, then there is a U-admissible blowing up $X' \to X$ such that the closures of T_1 and T_2 are disjoint.

Nataga Compactification

Let S is quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated, finite type morphism. Then X has a compactification over S.

Moreover, the compactification morphism $X \to S$ can be factored into an open immersion followed by a proper morphism, namely, $f = p \circ j$ for p a proper morphism and j an open immersion, and setting that $Rf^! = Rp_* \circ j_\#$. where $j_\#$ is the extension by zero functor.

Definition 3.4.43 *()*

- *Category derived from the functor*
  Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor, and $A \in \mathscr{C}'$.

  $\mathscr{C}_A = \{(X, s) | X \in \mathscr{C}, s : F(X) \to A\}$

  $Hom((X, s), (Y, t)) = \{f \in Hom_{\mathscr{C}}(X, Y) | s = t \circ F(f)\}$

- *Cofinal*
  $\phi : \mathscr{J} \to \mathscr{I}$ is cofinal if $\mathscr{J}^i$ is connected for any $i \in \mathscr{I}$. For cofinality, we have TFAE:

  - *ϕ is cofinal.*
  - *For any $\beta : \mathscr{I}^{op} \to Set$, the natural map $\varinjlim \beta \to \varinjlim(\beta \circ \phi^{op})$ is bijective.*
  - *For any $\beta : \mathscr{I}^{op} \to \mathscr{C}$ where $\mathscr{C}$ is any category, the natural map $\varinjlim \beta \to \varinjlim(\beta \circ \phi^{op})$ is an isomorphism in $\mathscr{C}^\wedge$.*
  - *For any $i \in \mathscr{I}$, $\varprojlim_{j \in \mathscr{J}} Hom_{\mathscr{I}}(i, \phi(j)) \cong \{pt\}$*

- *Filtrant*
  A category $\mathscr{I}$ is filtrant if

  - *$Ob(\mathscr{I})$ is non-empty*
  - *for any $i, j \in \mathscr{I}$, there exists $k \in \mathscr{I}$ such that $i \to k$ and $j \to k$ exist.*

- *for any morphisms $f, g : i \rightrightarrows j$, there exists a morphism $h : i \to k$ such that $h \circ f = h \circ g$.*

For any finite category $\mathscr{J}$ and $\phi : \mathscr{J} \to \mathscr{I}$, there exists $i \in \mathscr{I}$ such that $\varinjlim_{j \in \mathscr{J}} Hom_{\mathscr{I}}(\phi(i), j) \neq \emptyset$.

Let $\alpha : \mathscr{I} \to Set$ be a functor with $\mathscr{I}$ small and filtrant. Define a relation as $\alpha(i) \ni x \sim y \in \alpha(j)$ if there exist $s : i \to k$ and $t : j \to k$ such that $\alpha(s)(x) = \alpha(t)(y)$. Then $\sim$ is an equivalence relation, and $\varprojlim \alpha = \coprod_i \alpha(i)/\sim$

If $\mathscr{I}$ is a small category, then $\mathscr{I}$ is filtrant iff for any finite category $\mathscr{J}$ and any functor $\alpha : \mathscr{I} \times \mathscr{J}^{op} \to Set$, the natural morphism:

$$\varprojlim_i \varinjlim_j \alpha(i,j) \to \varinjlim_j \varprojlim_i \alpha(i,j)$$

is an isomorphism.

- *IPC Property*
  IPC is short for "inductive limit product commutation property".
- *Filtrant Categories*

  - *If a category has a terminal object, then it's filtrant.*
  - *If a category admits finite inductive limits then it's filtrant.*
  - *A product of filtrant category is filtrant.*
  - *If a category is filtrant, then it's connected.*

  Here is TFAE:

  - *ϕ is cofinal.*
  - *$\mathscr{J}^i$ is filtrant for all $i \in \mathscr{I}$.*
  - *For each $i \in \mathscr{I}$, there exists $j \in \mathscr{J}$ and a morphism $s : i \to \phi(j)$. Also, for any $i \in \mathscr{I}$ and $j \in \mathscr{J}$ and any pair of parallel morphisms $s, s' : i \rightrightarrows \phi(j)$ in $\mathscr{I}$, there exists a morphism $t : i \to k$ in $\mathscr{J}$ such that $\phi(t) \circ s = \phi(t) \circ s'$.*

  Further, if this conditions are satisfied, then $\mathscr{I}$ is filtrant.

Definition 3.4.44 *(Exact functor)*

Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor.

- *We say F is right exact if the category $\mathscr{C}_U$ is filtrant for any $U \in \mathscr{C}'$.*

- *We say F is left exact if $F^{op} : \mathscr{C}^{op} \to \mathscr{C}'^{op}$ is left exact, or equivalently $\mathscr{C}^U$ is cofiltrant for any $U \in \mathscr{C}'$.*
- *We say F is exact iff F is both right/left exact.*

Properties:

- *Let $F : \mathscr{C} \to \mathscr{C}'$ be an exact functor, and let $\mathscr{J}$ be a finite category and $\beta : \mathscr{J}^{op} \to \mathscr{C}$ be a functor. Assume that $\varinjlim \beta$ exists in $\mathscr{C}$. Then $\varinjlim (F \circ \beta)$ exists in $\mathscr{C}'$ and is isomorphic to $F(\varinjlim \beta)$. In particular, the left exact functor commutes with finite projective limits if $\mathscr{C}$ admits limits.*
- *If $\mathscr{C}$ admits finite projective limits, then F is left exact iff it commutes with such limits. Or if $\mathscr{C}$ admits finite projective limits, then it has TFAE:*

  - *F sends the terminal object to the terminal object.*
  - *For any $X, Y \in \mathscr{C}$, $F(X) \times F(Y)$ exists in $\mathscr{C}$, and $F(X) \times F(Y) \cong F(X \times Y)$.*
  - *F commutes with kernels i.e. for any parallel arrows $f, g : X \rightrightarrows Y$ in $\mathscr{C}$. $F(ker(f, g))$ is a kernel of the parallel arrows $(F(f), F(g))$.*

  Moreover, assuming all of the above,

  F commutes with fiber products i.e. $F(X \times_Z Y) \cong F(X) \times_{F(Z)} F(Y)$ for any pair of morphisms $X \to Z$ and $Y \to Z$ in $\mathscr{C}$.

- *If F admits right(left) adjoint, then F is right(left) exact.*
- *If $\mathscr{C}$ admits finite inductive limits and finite projective limits, then $Hom : \mathscr{C} \times \mathscr{C}^{op} \to Sets$ is left exact in each argument.*

Definition 3.4.45 *(Regular Category)*

A regular category can be a generalization of an abelian category, and a regular functor might be a left exact functor.

Let $\mathscr{C}$ be a category. $\mathscr{C}$ is a regular category if all finite limits exist and if coequalizers of a pair of morphisms called kernel pairs, satisfying the certain exactness conditions. It is an analogy of abelian categories (existence of images without additivity).

- *$\mathscr{C}$ is finitely complete (all finite limits exist).*

- *If a morphism $f : X \to Y$ in $\mathscr{C}$, and*

$$\begin{array}{ccc} Z & \xrightarrow{p_0} & Y \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & Y \end{array}$$

  is a pullback, then the coequalizer p_0 and p_1 exist. The pair (p_0, p_1) is called kernel pair of f. Being a pullback, the kernel pair is unique up to a unique isomorphism.

- *If a morphism $f : X \to Y$ in $\mathscr{C}$,*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & Y \end{array}$$

  is a pullback, and if f is a regular epimorphism, then g is also a regular epimorphism as well. A regular epimorphism is an epimorphism that appears as a coequalizer of some pair of morphisms.

  $R \rightrightarrows X \to Y$

  where $r, s : R \to X$ and $f : X \to Y$ is called exact sequence if it's both coequalizer and kernel pairs, and this is exact if $0 \to R \to^{(r,s)} X \oplus X \to^{(f,-f)} Y \to 0$ is a short exact sequence in the usual sense.

  A functor between regular categories is called regular if it preserves finite limits and coequalizers and kernel pairs. A functor is regular iff it preserves finite limits and exact sequences.

  Epi-mono factorization
  for a regular category, $f : X \to Y$ has a factorization with monomorphism $E \to Y$ and regular epimorphism $X \to E$ such that $f = me$. This factorization is unique up to isomorphism. A monomorphism m is called image of f.

Definition 3.4.46 *(Grothendieck Topology)*

Sieve
Let $U \in Ob(\mathscr{C})$. A sieve S over U is a subset $S \subset Ob(\mathscr{C}_U)$ such that composition $W \to V \to U \in S$ belongs to $V \to U \in S$.

A sieve is, for example, S_A is a sieve such that $V \to U \in S_A$ iff $V \to U$ decomposes as $V \to A \to U$.

Grothendieck Topology
A Grothendieck topology is a set of $\{SCov_U\}_{U\in Ob(\mathscr{C})}$ *such that the following conditions:*

- *GT1*
  $Ob(\mathscr{C}_U) \subset SCov_U$
- *GT2*
  If $S_1 \subset S_2 \subset Ob(\mathscr{C}_U)$ *are sieves in* $\mathscr{C}$*. If* $S_1 \in SCov_U$ *then* $S_2 \in SCov_U$*.*
- *GT3*
  Let $U \to V$ *be a morphism in* $\mathscr{C}$*. If* $S \in SCov_V$*, then* $S \times_V U \in SCov_U$*. Here*

  $S \times_V U := \{W \to U | W \to V \to U \text{ belongs to } SCov_V\}$
- *GT4*
  If S, S' *are sieves over* U*, and if* $S' \in SCov_U$ *and* $S \times_U V \in SCov_V$ *for all* $V \to U \in S'$*, then* $S \in SCov_U$*.*

Local Epimorphism

- *A morphism* $A \to U$ *in* $\mathscr{C}^\wedge$ *with* $U \in \mathscr{C}$ *is local epimorphism if the sieve* S_A *is a covering sieve over* U*.*
- *A morphism* $A \to B$ *in* $\mathscr{C}^\wedge$ *with* $U \in \mathscr{C}$ *is local epimorphism if for any morphisms* $V \to B$*,* $A \times_B V \to V$ *is a local epimorphism.*
- *LE1*
  For any $U \in \mathscr{C}$*,* $id_U : U \to U$ *is a local epimorphism.*
- *LE2*
  Let $A_1 \to_u A_2 \to_v A_3$ *be a morphism in* $\mathscr{C}^\wedge$*. If* u, v *are local epimorphisms, then* $v \circ u$ *is a local epimorphism.*
- *LE3*
  Let $A_1 \to_u A_2 \to_v A_3$ *be a morphism in* $\mathscr{C}^\wedge$*. If* $v \circ u$ *is a local epimorphism, then* v *is a local epimorphisms,*
- *LE4*
  $u : A \to B$ *is a local epimorhism iff for any* $U \in \mathscr{C}$ *and any morphisms* $U \to B$*, the morphism* $A \times_B U \to U$ *is a local epimorphism.*
- *LE1'*
  $u : A \to B$ *is an epimorhism, then* u *is a local epimorphism.*

Refinement:
Let $U \in \mathscr{C}$, and Let $S_1 = \{U_i\}_{i\in I}$ and $S_2 = \{V_j\}_{j\in J}$. S_1 is a refinement of S_2 if for any $i \in I$, there exist $j \in J$ and a morphism $U_i \to V_j$ in $\mathscr{C}_U$. We denote it by $S_1 \preceq S_2$.

Cover
Let $\mathscr{C}$ be a category with a Grothendieck topology. A small family $S = \{U_i\}_{i\in I}$ of objects of $\mathscr{C}_U$ is a covering of U if the morphism $\coprod_i U_i \to U$ is a local epimorphism.

Here we denote Cov_U the family of coverings of U. The family of coverings will satisfy the following axioms:

- *COV1*
  $\{U\}$ belongs to Cov_U.
- *COV2*
  If $S_1 \preceq S_2$ and $S_1 \in Cov_U$ and $S_2 \subset Ob(\mathscr{C}_U)$, then $S_2 \in Cov_U$.
- *COV3*
  If $S = \{U_i\}_{i\in I}$ belongs to Cov_U, then $S \times_U V = \{U_i \times_U V\}_{i\in I}$ belongs to Cov_V for a morphism $V \to U$ in $\mathscr{C}$.
- *COV4*
  If $S_1 = \{U_i\}_{i\in I}$ belongs to Cov_U, S_2 is a small family of objects of $\mathscr{C}_U$, and $S_2 \times_U U_i$ belongs to Cov_{U_i}, then S_2 belongs to Cov_U.

For local epimorphism, we have TFAE:

- *u is a local epimorphism*
- *for any $t : U \to B$ with $U \in \mathscr{C}$, there exist a local epimorphism $u : C \to U$ and a morphism $s : C \to A$ such that $u \circ s = t \circ u$.*
- *$Im(u) \to B$ is a local epimorphism.*

Example 1 (Op_X):
Let $\mathscr{C}_X = Op_X$ be a category of open sets of X. Note that $\mathscr{C}_X$ admits a terminal object X, and product of two objects U and V in $\mathscr{C}_X$ is $U \cap V$. Also note if $U \subset X$ is an open subset, then $(\mathscr{C}_X)_U \cong Op_U$. We define a Grothendieck topology by deciding that if a small family $S = \{U_i\}_{i\in I}$ of objects of Op_U belongs to Cov_U if $\cup U_i = U$.

Equivalently, we may also define Grothendieck topology as follows. A morphism $u : A \to B$ in $(\mathscr{C}_X)^\wedge$ is a local epimorphism if for any $U \in Op_X$ and any $t \in B(U)$, there exists a covering $U = \cup_i U_i$ and for each i an $s_i \in A(U_i)$ with $u(s_i) = t|_{U_i}$ (here $t|_{U_i}$ is an image of $B(U) \to B(U_i)$). Hence, a morphism

$A \to U$ is a local epimorphism if there exists an open covering $U = \cup U_i$ factroizes through A for every $i \in I$.

These two definitions give the same topology on $\mathscr{C}$.
Example 2 (Initial/Final/Pt Topology):

- *Final Topology:*
  Let $\mathscr{C}_X$ be a category with a Grothendieck topology by deciding the local epimorphisms in $\mathscr{C}^\wedge$ are epimorphisms.

- *Initial Topology:*
  Let $\mathscr{C}_X$ be a category with a Grothendieck topology by deciding all the morphisms are local epimorphisms.

- *Pt Topology:*
  Pt denotes a category with one object c and one morphism. We endow this topology with the final topology. Note that this topology is different from the initial one. Indeed, a morphism $\emptyset_{Pt^\wedge} \to c$ in $Pt^\wedge$ is a local epimorphism for the initial topology, not for the final one. Empty covering of pt is a covering for the initial topology, not for the final one.

Example 2 (Étale):
Let G be a finite group, and denote G-Top the category of small G topological spaces. An object is a small topological space X with a continuous action of G, and such an f is said to be G-equivariant. We define Category $\mathscr{E}t_G$. Its objects are those of G-Top and its morphisms are $f : V \to U$ as G-equivariant. such that f is local homeomorphism. Note that $f(V)$ is open in U. The category $\mathscr{E}t_G$ admits fiber products. If $U \in$ G-Top, then the category $\mathscr{E}t_G(U) := (\mathscr{E}t_G)_U$ admits finite projective limits.

- *Étale*
  The étale topology on $\mathscr{E}t_G$ is defined as follows. A sieve S over $U \in \mathscr{E}t_G$ is a covering sieve if for any x, there exists a morphism $V \to U$ in S such that $x \in f(V)$.

- *Nisnevich*
  The Nisnevich topology on $\mathscr{E}t_G$ is defined as follows. A sieve S over $U \in \mathscr{E}t_G$ is a covering sieve if for any $x \in U$, there exists a morphism $V \to U$ in S and $y \in V$ such that $f(y) = x$, and y has the same isotopy group as x (isotopy group G_y of y is the subgroup of G consisting of $g \in G$ such that $g \cdot y = y$).

- *Zariski*
  The Zariski topology on $\mathscr{E}t_G$ is defined as follows. A sieve S over $U \in \mathscr{E}t_G$ is a covering sieve if for any $x \in U$, there exists an open embedding $f : V \to U$ in S such that $x \in f(V)$.

Definition 3.4.47 *(Local Isomorhism)*

Def:
$u : A \to B$ *is a local monomorphism if* $A \to A \times_B A$ *is a local epimorphism. In particular,* $u : A \to B$ *is a local isomorphism if it's both local epimorphism and local monomorphism.*

Category of Local Isomorphisms:
Being derived from $\mathscr{C}$ *with the Grothendieck topology, we let* $\mathscr{L}I$ *be a set of local isomorphisms. In particular, let* $\mathscr{L}I_A$ *for* $A \in \mathscr{C}^\wedge$ *be the category given by*

- $Ob(\mathscr{L}I_A) = \{\text{the local isomorphisms } B \to A\}$
- $Hom_{\mathscr{L}I_A}((B \to_u A), (C \to_v A)) = \{w : B \to C; u = v \circ w\}$

Property 1:
The family $\mathscr{L}I$ *of local isomorphisms in* $\mathscr{C}^\wedge$ *is a left saturated multiplicative system.*

Property 2:
The category $\mathscr{L}I_A$ *admits finite projective limits. In particular,* $\mathscr{L}I_A$ *is filtrant.*

Property 3:
$\mathscr{C}$ *is small. Then for any* $A \in \mathscr{C}^\wedge$*, the category* $(\mathscr{L}I_A)^{op}$ *is cofinally small.*

Example (local isomorphism):
A functor $(-)^a$ *induces a local isomorphism.*

Localization by local isomorphisms:

Definition 3.4.48 *(Sheaf)*

From an abstract point of view, a sheaf is a site with the extra conditions. A site is a presite with some grothendieck topology.

Presite:

- *A presite* X *is nothing but a category* $\mathscr{C}_X$*.*
- *A morphism of presite* $f : X \to Y$ *is a functor* $f^t : \mathscr{C}_Y \to \mathscr{C}_X$*.*
- *A presite* X *is small if* $\mathscr{C}_X$ *is small. More generally, if* X *has a property* P *if* $\mathscr{C}_X$ *has a property* P*.*

Note:

- *We denote presite* $\hat{X}$ *associating to* $(\mathscr{C}_X)^\wedge$*, and morphism of presites* $h_X : \hat{X} \to X$ *corresponds to* $h_X^t : \mathscr{C}_X \to (\mathscr{C}_X)^\wedge$*.*
- *For a morphism of presites* $f : X \to Y$*, we denote* $\hat{f} : \hat{X} \to \hat{Y}$ *the associated morphism*

$$
\begin{aligned}
(f^t A)(U) &\cong \varinjlim_{(V \to A) \in (\mathscr{C}_Y)^\wedge} Hom_{(\mathscr{C}_Y)^\wedge_A}(f^t(V), U) \\
&\cong \varinjlim_{(U \to f^t(V)) \in (\mathscr{C}_Y)^U} A(U)
\end{aligned} \tag{12}
$$

for any $A \in (\mathscr{C}_Y)^\wedge$ and $U \in \mathscr{C}_X$. Note that $f^t : \mathscr{C}_{\hat{Y}} \to \mathscr{C}_{\hat{X}}$ commutes with a small inductive limits.

- *For a presite X, we denote by pt_X the terminal object of $(\mathscr{C}_X)^\wedge$.*

Presheaf:
$Psh(X, \mathscr{A}) = Fun((\mathscr{C}_X)^{op}, \mathscr{A})$
In particular,
$Psh(X) = Fun((\mathscr{C}_X)^{op}, \mathscr{U} - Set) = \mathscr{C}_{\hat{X}}$

examples:

- *Categories Op_X are presites.*
- *Let $\mathscr{C}^0(U)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued continuous functions on $U \in Op_X$.*
- *Locally constant sheaf $\mathbb{Z}_X$ of $\mathbb{Z}$-valued functions.*

Adjointness of Presheaves:
Given a presite $f : X \to Y$ with a functor $f^t : \mathscr{C}_Y \to \mathscr{C}_X$, we have the following functors:

$f_* : PSh(X) \to PSh(Y)$
$f^\dagger : PSh(Y) \to PSh(X)$
$f^\ddagger : PSh(Y) \to PSh(X)$

We construct the adjointness of these functors: so $f^\dagger \dashv f_*$ and $f_\dashv f^\ddagger$. These functors are defined as:

- $f_* F(V) = F(f^t(V))$
- $f^\dagger G(U) = \varinjlim_{(U \to f^t(V)) \in (\mathscr{C}_Y)^U} G(V)$
- $f^\ddagger G(U) = \varprojlim_{(f^t(V) \to U) \in (\mathscr{C}_Y)_U} G(V)$

Furthermore, we extend presheaves on X to presheaves on $\hat{X}$. We associate $h_X^\ddagger$ to the Yoneda functor $h_X^t = h_{\mathscr{C}_X}$. Hence, we have

$h_X^\ddagger F(A) \cong \varprojlim_{(U \to A) \in (\mathscr{C}_X)_A} F(U).$

Notice that $h_X^{\ddagger} : PSh(X) \to PSh(\hat{X})$ makes an equivalence of categories.

$$\begin{aligned} f_*F(B) &= \varprojlim_{V \in (\mathscr{C}_Y)_B} f_*F(V) \\ &\cong \varprojlim_{V \in (\mathscr{C}_Y)_B} F(f^tV) \\ &\cong F(\varinjlim_{V \in (\mathscr{C}_Y)_B} (f^tV)) \\ &= F(\hat{f}^t(B)) \end{aligned} \tag{13}$$

Consider $j_{A\to X} : A \to X$.

- $j_{A\to X *} : PSh(X, \mathscr{A}) \to PSh(A, \mathscr{A})$
- $j_{A\to X \dagger} : PSh(A, \mathscr{A}) \to PSh(X, \mathscr{A})$
- $j_{A\to X \ddagger}{}^* : PSh(A, \mathscr{A}) \to PSh(X, \mathscr{A})$

Let $G \in PSh(A, \mathscr{A})$ and $F \in PSh(X, \mathscr{A})$.
$j_{A\to X *}(F)(B \to A) \cong F(B)$ *for* $(B \to A) \in \mathscr{C}_A^{\wedge}$
${j_{A\to X}}^{\dagger}(G)(U) \cong \coprod_{s\in A(U)} G(U \to^s A)$ *for* $U \in \mathscr{C}_X$
${j_{A\to X}}^{\ddagger}(G)(B) \cong G(B\times_A \to A)$ *for* $B \in \mathscr{C}_X^{\wedge}$

The presheaf is called as an internal hom of (F, G):
$\mathscr{H}om_{PSh(X,A)}(F, G)(U) = Hom_{PSh(U,A)}(j_{U\to X *}F, j_{U\to X *}G)$

In other words

$Hom_{PSh(X,A)}(F, G) = \varprojlim_{U\in\mathscr{C}_X} \mathscr{H}om_{PSh(X,A)}(F, G)(U)$

Also we have isomorphisms

$j_{A\to X *}\mathscr{H}om_{PSh(X,A)}(F, G) \cong \mathscr{H}om_{PSh(A,A)}(j_{A\to X *}F, j_{A\to X *}G)$

$\mathscr{H}om_{PSh(X,A)}(F, G)(A) \cong Hom_{PSh(A,A)}(j_{A\to X *}F, j_{A\to X *}G)$

Site:
A site is a presite with a Grothendieck topology. A morphism of sites $f : X \to Y$ is a functor $f^t : \mathscr{C}_Y \to \mathscr{C}_X$ such that for any local isomorphisms $B \to A \in \mathscr{C}_{\hat{Y}}$, $f^t(B) \to f^t(A)$ is a local isomorphism in $\mathscr{C}_{\hat{X}}$. This makes a category of sites.
We consider if X and Y be sites, and $f : X \to Y$ is a morphism of presites.

- *If f is a morphism of sites, then if $\hat{f}$ sends a local epimorphism in $\mathscr{C}_{\hat{Y}}$ to a local epimorphism in $\mathscr{C}_{\hat{X}}$.*

- f is a morphism of sites iff for any $V \in \mathscr{C}_Y$ and any morphism $B \to V \in \mathscr{C}_{\hat{Y}}$ which is both monomorphism and local isomorphism, then $f^t(B) \to f^t(A)$ is a local isomorphism.

Exactness:

- f is left exact if f^t is left exact.
- f is weakly left exact if $\mathscr{C}_Y \to (\mathscr{C}_X)_{\hat{f}^t(pt_Y)}$ is left exact.
- If f is left exact, then it's weakly left exact.
- If f is weakly left exact, then $\hat{f}^t : \mathscr{C}_{\hat{Y}} \to \mathscr{C}_{\hat{X}}$ commutes with fiber products and sends the monomorphisms to monomorphisms.
- For any $A \in \mathscr{C}_{\hat{X}}$, the morphism $j_{A \to X} : X \to A$ is weakly left exact.

If assume that

- f is weakly left exact
- $\hat{f}^t$ sends local epimorphisms $B \to V$ with $V \in \mathscr{C}_Y$ to local epimorphisms.

Then, f is a morphism of sites.
Sheaf:

We define $F(A) = \varprojlim_{U \to A \in \mathscr{C}_A} F(U)$ for $A \in \mathscr{C}_{\hat{X}}$, and we obtain a functor $F : \mathscr{C}_{\hat{X}}{}^{op} \to \mathscr{A}$ which commutes with a small projective limits. In particular, if $A \to B$ is an epimorphism, then $F(A) \to F(B)$ is a monomorphism.

A presheaf is a sheaf if for any local isomorphism $A \to U$ with $U \in \mathscr{C}_X$ and $A \in \mathscr{C}_{\hat{X}}$, the morphism $F(U) \to F(A)$ is an isomorphism.
F is
Sheaf associated with Presheaf:
$\mathscr{A}$ admits projective and inductive limits.
small filtrant inductive limits are exact.
$\mathscr{A}$ satifities IPC-property.

Let $u : A \to A'$ be a morphism.
$\lambda_u : \mathscr{L}I_A \to \mathscr{L}I_{A'} : (B \to A) \mapsto (B \times_A A' \to A')$

Moreover if u is a local isomorphism, then

$\mu_u : \mathscr{L}I_{A'} \to \mathscr{L}I_A : (B \to A') \mapsto (B \to A' \to^u A)$

Then (λ_u, μ_u) is a pair of adjoint functors.
Let $F \in PSh(\hat{X}, \mathscr{A})$ and let $A \in \mathscr{C}_{\hat{X}}$. We set

$F^b(A) = \varinjlim_{(B \to A) \in \mathscr{L}I_A} F(B)$

a presheaf F defines a functor $\alpha : (\mathscr{L}I_A)^{op} \to \mathscr{A}$ and

$F^b(A) \cong \varinjlim \alpha$

For a morphism $u : A' \to A$, we define the morphism $F^b(A) \to F^b(A')$ by the chain of morphisms

$$\begin{aligned} F^b(A) &= \varinjlim F(B) \\ &\cong F(B \times_A A') \\ &\cong \varinjlim F(B') \\ &= F^b(A') \end{aligned} \tag{14}$$

We denote $(-)^b : PSh(\hat{X}, \mathscr{A}) \to PSh(\hat{X}, \mathscr{A})$ the functor given by above. Also, we define a natural transfomation $\epsilon_b : id \to (-)^b$.

Let $F \in PSh(X, \mathscr{A})$. If F commutes with small projective limits, then so does $F^b \in PSh(X, \mathscr{A})$.

Also, we define a functor $(-)^a : PSh(X, \mathscr{A}) \ni F \mapsto h_{X*}((h_X^{\ddagger} F)^b) \in PSh(X, \mathscr{A})$

Hence, for $F \in PSh(X, \mathscr{A})$ and $U \in \mathscr{C}_X$, we have

$F^a(U) \cong \varinjlim_{A \in \mathscr{L}I_U} F(A)$

This implies that

$(h_X^{\ddagger} F)^b \cong h_X^{\ddagger}(F^a)$

and hence we have

$F^a(A) \cong \varinjlim_{(B \to A) \in \mathscr{L}I_A} F(B)$ *for any $A \in \mathscr{C}_{\hat{X}}$.*

Here we let

$\epsilon : id_{PSh(X, \mathscr{A})} \to (-)^a$

The condition of sheaf:

Let $F \in PSh(X, \mathscr{A})$.

- *If F is separated, then $F \to F^a$ is a monomorphism.*

- *If F is a sheaf, then $F \to F^a$ is an isomorphism.*

Restriction and Extension of sheaf:

Recall ${j_{A\to X}}^t : \mathscr{C}_A := (\mathscr{C}_X)_A \to \mathscr{C}_X$ gives a morphism of presites $j_{A\to X} : X \to A$.

prop 1:
$C \to B$ is a local epimorphism iff $\hat{j}^t_{A\to X}(C) \to \hat{j}^t_{A\to X}(B)$ is a local epimorphism in $\mathscr{C}_{\hat{X}}$.

prop 2:
Let $G \in Sh(X, \mathscr{A})$. Then, $\hat{j}^{\ddagger}_{A\to X} G \in Sh(X, \mathscr{A})$ is a sheaf. Also, $\hat{j}^{\ddagger}_{A\to X}$ is right adjoint to $\hat{j}_{A\to X}*$.

prop 3:
If $\mathscr{A}$ is additive, which satisfies . Let $f : X \to Y$ weakly left exact morphism of sites, then f^{-1} is exact. In particular, ${j_{A\to X}}^{-1}$ is exact.

prop 4:
$j_{A\to X}*$ is exact. Moreover, it commutes with a small inductive limits and small projective limits.

prop 5:
If $A \to pt_X$ is local epimorphism, then $j_{A\to X}*$ is conservative and faithful.

def:

- $F_A = j^{-1}_{A\to X} j^{-1}_{A\to X} * (F)$
- $\Gamma_A(F) = j^{\ddagger}_{A\to X} j^{-1}_{A\to X} * (F)$

These are adjoint functors.

prop 1:

- *F_A is a sheaf associated with the presheaf $U \mapsto F(U)^{\coprod A(U)}$*
- *$\Gamma_A(F)(B) \cong F(A \times B)$ for $B \in \mathscr{C}_{\hat{X}}$*

prop 2:

- $(F_A)_B \cong F_{A\times B}$
- $\Gamma_B(\Gamma_A(F)) \cong \Gamma_{A\times B}(F)$
- $\Gamma(X, \Gamma_A(F)) \cong \Gamma(A; F)$

Examples of Sheaf:

- *constant sheaf*
  Also, locally constant function is section of constant sheaf.
  Also, locally constant sheaf is section of constant stack.

- *locally constant sheaf*
  For example, an orientation sheaf is locally constant.
  Also, for $P : \mathscr{O}_X \to \mathscr{O}_X$*, let* $P = z\frac{\partial}{\partial z} - \frac{1}{2}$*. Then the kernel of* P *is locally constant sheaf on* $X - \{0\}$ *but not constant there.*

- *skyscraper sheaf*
  The idea is similar to Dirac δ*-function.*

- *perverse sheaf*

- *Orientation Sheaf*
  An orientation sheaf on a manifold X *of dimension* n *is a locally constant sheaf* $\mathfrak{o}_X$ *on* X *such that the stalk of* $\mathfrak{o}_X$ *at a point* x *is*

  $$\mathfrak{o}_{X,x} = H_n(X, X - \{x\})$$

A presheaf is a sheaf if .

Definition 3.4.49 *(sheaf examples)*

- *Twisted sheaf:*
  A twisted sheaf is a variant of coherent sheaf. Precisely, it is specified by: an open covering in the étale topology U_i*, coherent sheaf* F_i *over* U_i*, a Čech 2-cocycle* θ *for* $\mathbb{G}_m$ *on the covering* U_i *as well as the isomorphisms*

  $$g_{ij} : F_i|_{U_{ij}} \cong F_j|_{U_{ij}}$$

  satisifying

  - $g_{ii} = id_{F_i}$
  - $g_{ij} = g_{ji}^{-1}$
  - $g_{ij} \circ g_{jk} \circ g_{ki} = \theta_{ijk} id_{F_i}$

- *Torsion sheaf:*
  A torsion sheaf is a sheaf of abelian groups $\mathscr{F}$ *on a site for which every object* U*, the space of sections* $\Gamma(U, \mathscr{F})$ *is a torsion abelian group. Similarly, for a prime* p*, a sheaf* $\mathscr{F}$ *is* p*-torsion if every section over any object is killed by a power of* p*.*

  A torsion sheaf on an étale site is the union of its constructible subsheaves.

- *Local system:*
  A local system is a locally constant sheaf on X. In other words, a sheaf $\mathscr{L}$ is a local system if every points has an open neighborhood U such that the restricted sheaf $\mathscr{L}|_U$ is isomorphic to the sheafification of some constant presheaf.
- *Perverse sheaf:*
- *Constructive sheaf:*
  A sheaf $\mathscr{F}$ is called constructive if X can be written as a finite union of locally closed subschemes $i_Y : Y \to X$ such that for each subscheme Y of the covering, the sheaf $\mathscr{F}|_Y = (i_Y)^*\mathscr{F}$ is a finite locally constant sheaf. In particular, there is a covering $\{U_i \to Y | i \in I\}$ such that for all étale subschemes in the cover of Y, the sheaf $(i_Y)^*\mathscr{F}|_{U_i}$ is constant and represented by a finite set.
- *Reflexive sheaf:*
- *Dualizing sheaf:*

Definition 3.4.50 *(Perverse sheaves)*

we define $DR(V, \nabla) = ker\nabla$ is a locally constant sheaf. a vector bundle $V = \mathscr{V} \otimes \mathscr{O}_Y$ carries an integrable connection ∇ such that $ker\nabla$ is $\mathscr{V}$.

Holonomic modules:
A finitely presented D-module M is holonomic if it restricts to vector bundle with connection on $\Delta^* = \Delta - \{0\}$.

M restricts to $\mathscr{O}_{\Delta^*}$ with connection $d + r\frac{dz}{z}$, $r \in \mathbb{C}$. If $r = 0$, then $M = \mathscr{O}$ with the standard D-module structure. If $r \neq 0$, then $z^{-1} \in M$, and by repeated differentiation we get all negative powers, so $M = \mathscr{O}[z^{-1}]$. The trivial connection is not simple for it properly contains $\mathscr{O}$, so we can take a quotient by $\mathscr{O}$.

Full subcategory of holonomic modules are abelian and artinian.

The dual of regular holonomic modules are also holonomic.

Perverse sheaves on disk:
Given a D-module, we now redefine $DR(M)$ as the complex

$M \to^{\partial} M$

shifted so that the first M starts in degree -1. By restricting to Δ^*, Poincaré lemma gives an exact sequence

$0 \to ker\nabla \to M|_{\Delta^*} \to^{\partial} M_{\Delta^*}$

thus $DR(M)|_{\Delta^*} \cong ker\nabla$ *up to shift.*

The provisional definition of perverse sheaf is that it is a complex of $\mathbb{C}$*-vector spaces on* Δ *quasi-isomorphic to* $DR(M)$ *for a regular holonomic D-module* M*. From this, the collection of perverse sheaves should form an artinian abelian category because* M*'s do.*

The actual definition of perverse sheaves is that $K = DR(M)$ *and dually* $DR(M^*)$ *have cohomology in exactly degree* 0 *and* -1*. The dual* $DR(M^*)$ *can be understood directly in terms of* K*. It is so called Verdier dual* $DK = R\mathscr{H}om(K, \mathbb{C}[-1])$*.*

Definition of perverse sheaves:
A perverse sheaf on Δ *is a bounded complex of sheaves* K *such that*

- *The cohomology sheaves* $H^i(K)$ *are zero unless* $i = 0, -1$*.* H^0 *supported at* 0*, and* H^{-1} *gives a local system on* Δ^**.*
- *The same conditions hold for* DK*.*

Examples of perverse sheaves:

- $\mathbb{C}_\Delta[-1]$ *is perverse. We can realize this as* $DR(\mathscr{O}_\Delta)$
- *Let* $M = \mathscr{O}_\Delta[z^{-1}]$ *with* $\partial \cdot 1 = \frac{r}{z}$*. Then* $DR(M)$ *consists of* $< z^{-r} >$ *in degree* -1*. Writing* $z = e^{-2\pi i t}$*, we can see that* $z^{-r} \mapsto az^{-r}$ *under* $t \mapsto t+1$*, where* $a = e^{-2\pi i r}$*. In other words,* $DR(M) = j_* L[-1]$*, where* L *is the rank one local system on* Δ^* *with monodromy* a*.*
- $DR(\mathscr{O}[z^{-1}]/\mathscr{O}) = \mathbb{C}_0$*, the skyscraper sheaf at* 0*.*

Definition 3.4.51 *(D-modules)*
Def:
A D-module is a left module over the Weyl algebra $A_n(K)$ *over a field* K *of characteristic zero.* $A_n(K)$ *is an algebra consists of* $x_1, \cdots, x_n, \partial_1, \cdots, \partial_n$*.*

$[\partial_i, x_j] = \delta_{ij}$

and this implies the relation

$[\partial_i, f] = \frac{\partial f}{\partial x_i}$

D-modules in algebraic variety:
The sheaf of differential operators D_X *is defined to be the* $\mathscr{O}_X$*-algebra by the vector fields on* X *interpreted as derivations.*

A left D_X-modules M is an $\mathscr{O}_X$ module with a left action of D_X on it. Giving such an action is equivalent to specifiying a K-linear map

$\nabla : D_X \to End_K(M), v \mapsto \nabla_v$
satisfying

- $\nabla_{fv}(m) = f\nabla_v(m)$
- $\nabla_v(fm) = v(f)m + f\nabla_v(m)$ *(Leibniz rule)*
- $\nabla_{[v,w]}(m) = [\nabla_v, \nabla_w](m)$

Here f is a regular function on X, v and w are vector fields, m a local section of M, $[-,-]$ a commutator. If in addition, M is a locally free $\mathscr{O}_X$ module, giving M a D-module structure is nothing else but equipping the vector bundle associated to M a flat connection.

Definition 3.4.52 *(Stack)*

Prestack:

- *A prestack $\mathfrak{S}$ is a 2-category*
- *for each $U \in \mathscr{C}_X$, $\mathfrak{S}(U)$ is a category.*
- *for any morphism $u : U_1 \to U_2$, a functor $r_u : \mathfrak{S}(U_2) \to \mathfrak{S}(U_1)$.*
- *for $u : U_1 \to U_2$ and $v : U_2 \to U_3$, a functor $c_{u,v} : r_u \circ r_v \cong r_{v\circ u}$ is a compostiion morphism.*
- $r_{id_U} = id_{\mathfrak{S}(U)}$ *and* $c_{id_U, id_U} = id_{id_{\mathfrak{S}(U)}}$ *for any* $U \in \mathscr{C}_X$.
- *thediagram*$r_u \circ r_v \circ r_w \to r_{w\circ v\circ u}$ *commutes.*

Examples:

- *An additive prestack is a prestack s.t $\mathfrak{S}(U)$ is an additive category, and r_u is an additive functor.*
- *An additive prestack is an abelian prestack if $\mathfrak{S}(U)$ is an abelian category and r_u is an exact functor.*
- *If $\mathfrak{S}$ is a prestack, $\mathfrak{S}^{op}$ is given by $\mathfrak{S}^{op}(U) = (\mathfrak{S}(U))^{op}$ with a natural restriction functors and the natural composition isomorphisms of such functors.*

direct image of stack:
$(f_*\mathfrak{S})(V) = \mathfrak{S}(f^t(V))$ *for* $u : V_1 \to V_2$, *and set* $r_u = r_{f^t(u)} : \mathfrak{S}(f^t(V_2)) \to \mathfrak{S}(f^t(V_1))$ *and for* $u : V_1 \to V_2$ *and* $v : V_2 \to V_3$, $c_{u,v} = c_{f^t(u),f^t(v)}$.
For $A \in \mathscr{C}_{\hat{X}}$, *we write* $\mathfrak{S}|_A$ *instead of* $j_{A\to X} * \mathfrak{S}$.

$\mathfrak{S}_\nu$ *where* $\nu = 1, 2$ *is a prestack on* X *with restriction functor* r_u^ν *and the composition morphisms* $c_{u,v}^\nu$. *A functor of prestacks* $\Phi : \mathfrak{S}_1 \to \mathfrak{S}_2$ *is a data of:*

- *For any* $U \in \mathscr{C}_X$, *a functor* $\Phi(U) : \mathfrak{S}_1(U) \to \mathfrak{S}_2(U)$
- *for any* $u : U_1 \to U_2$, *an isomorphism* Φ_u *of functors from* $\mathfrak{S}_1(U_2)$ *to* $\mathfrak{S}_2(U_1)$*:*
  $\Phi_u : \Phi(U_1) \circ r_u^1 \cong r_u^2 \circ \Phi(U_2)$.

$\mathscr{H}om_{\mathfrak{S}}(F_1, F_2)(A) \cong Hom_{\mathfrak{S}(A)}(F_1|_A, F_2|_A)$.

Since $A \cong \varinjlim_{(U\to A)\in\mathscr{C}_A} U$ *and both right hand side is isomorphic to* $\varprojlim_{(U\to A)\in\mathscr{C}_A} Hom_{\mathfrak{S}(U)}(F_1|_U, F_2|_U)$.

Separation and Stacks:
A prestack $\mathfrak{S}$ *is separated if for any* $U \in \mathscr{C}_X$, *and any* $F_1, F_2 \in \mathfrak{S}(U)$, $\mathscr{H}om_{\mathfrak{S}|_U}(F_1, F_2)$ *is a sheaf on* U.

Let $\mathfrak{S}$ *be a separated prestack. For* $A \in \mathscr{C}_{\hat{X}}$, *and* $F_1, F_2 \in \mathfrak{S}(A)$, *the presheaf on A:*
$\mathscr{H}om_{\mathfrak{S}|_A}(F_1, F_2) : (U \to A) \mapsto Hom_{\mathfrak{S}(U)}(F_1|_U, F_2|_U)$
is a sheaf on A.

$\mathfrak{S}$ *is separated iff for any local isomorphism* $A \to A'$, $\mathfrak{S}(A') \to \mathfrak{S}(A)$ *is fully faithful.*

Prestack is a stack if for any $U \in \mathscr{C}_X$ *and any local isomorphisms* $A \to U$, $\mathfrak{S}(U) \to \mathfrak{S}(A)$ *is an equivalence of categories.*

A stack is a separated prestack.

If $\mathfrak{S}$ *is a stack on* X, *then for any* $A \in \mathscr{C}_{\hat{X}}$, $\mathfrak{S}|_A$ *is a stack on* A.

- $\mathfrak{S}$ *is a stack.*
- $\mathfrak{S}(U) \to \mathfrak{S}(A) \rightrightarrows \mathfrak{S}(A \times_U A) \Rrightarrow \mathfrak{S}(A \times_U A \times_U A)$.
- *for any local isomorphism* $A \to B$, $\mathfrak{S}(B) \to \mathfrak{S}(A)$ *is an equivalence.*

Stack and category of Sheaves:

denote $\mathfrak{S}$ *by a prestack* $U \mapsto Sh(U, \mathscr{A})$. *Then,*

- $\mathfrak{S}(A)$ *is an equivalent of categories* $Sh(A, \mathscr{A})$
- $\mathfrak{S}$ *is a stack.*

Let $\mathscr{R}$ be a sheaf of rings on X, then $U \mapsto Mod(\mathfrak{R}|_U)$ is a stack.

Let $\mathfrak{S}$ is a stack on X. If there exists a local epimorphism $A \to pt_X$ such that $\Phi|_U : (\mathfrak{S}|_U)^{op} \to \mathscr{S}h_U$ is representable for any $U \in \mathscr{C}_X$ and $U \to A$, then we say that Φ is locally representable. If a functor $\Phi : \mathfrak{S}^{op} \to Sh_X$ is locally representable, then Φ is representable.

Definition 3.4.53 *(Morita Equivalence)*

$- \otimes_{\mathscr{R}} M : Mod(\mathscr{R}^{op}) \to Mod(\mathbb{Z}_X)$
is exact. If this functor is exact and faithful, then M is faithfully flat.

Let P be a flat $\mathscr{R}$-module of locally of finite presentation. Then

- *P is locally a direct summand of $\mathscr{R}^{\oplus n}$ for some n.*
- *For any $\mathscr{R}$-module M,*
  $\mathscr{H}om_{\mathscr{R}}(P, \mathscr{R}) \otimes_{\mathscr{R}} M \to \mathscr{H}om_{\mathscr{R}}(P, M)$
  is an isomorphism.

Morita Equivalence:
$\Phi : \mathfrak{Mod}(\mathscr{R}_2) \to \mathfrak{Mod}(\mathscr{R}_1)$ be an equivalence of $\mathscr{O}_X$ stacks. Then there exists an invertible $\mathscr{R}_1 \otimes \mathscr{R}_2^{op}$-module P such that.

Definition 3.4.54 *(Twisted Sheaves)*

An $\mathscr{O}_X$ stack $\mathfrak{S}$ is called an $\mathscr{O}_X$ stacko of twisted sheaves. If $\mathfrak{S}$ is locally equivalent to $\mathfrak{Mod}(\mathscr{O}_X)$, that is, if there is a local epimorphism $A \to pt_X$ such that for any $U \to A$ with $U \in \mathscr{C}_X$, there is an equivalence $\Phi_U : \mathfrak{S}|_U \cong \mathfrak{Mod}(\mathscr{O}_X)|_U$. Or equivalently, for any local epimorphism $A \to pt_X$ such that $\Phi_A : \mathfrak{S}|_A \cong \mathfrak{Mod}(\mathscr{O}_X)|_A$.

An object of $\mathfrak{S}(X)$ is called twisted $\mathscr{O}_X$-module.

For a stack $\mathfrak{S}$ of twisted $\mathscr{O}_X$ modules, an object of $\mathfrak{S}(X)$ is called a twisted $\mathscr{O}_X$ module.

Let $\mathfrak{S}$ be an $\mathscr{O}_X$ stack of twisted sheaves. Then for any $M \in Mod(\mathscr{O}_X)$ and $F \in \mathfrak{S}(X)$, the functor
$\mathfrak{S}(X) \ni L \mapsto Hom_{\mathscr{O}_X}(M, \mathscr{H}om_{\mathfrak{S}}(F, L))$
is representable. Indeed, functor $\mathfrak{S} \to \mathscr{S}h_X : U \mapsto \mathscr{H}om_{\mathscr{O}_U}(M|_U, \mathscr{H}om_{\mathfrak{S}|_U}(F, -))$ is locally representable, thus representable.

Any equivalence of $\mathscr{O}_X$ stack $\Phi : \mathfrak{S} \to \mathfrak{S}$ is isomorphic to $P \otimes -$ for some invertible $\mathscr{O}_X$-module P.

$\mathfrak{S}$ is an $\mathscr{O}_X$ stack of twisted sheaves.

Let $\mathfrak{S}$ be an $\mathscr{O}_X$ stack of twisted sheaves. Then, any equivalence of $\mathscr{O}_X$ stacks $\Phi : \mathfrak{S} \to \mathfrak{S}$ is isomorphic to $P \otimes -$ for some invertible $\mathscr{O}_X$-module P.

Definition 3.4.55 *(Quotient Stack)*
Let G be an affine smooth group scheme over a scheme S and X be a S-scheme on which G acts. Let the quotient stack $[X/G]$ be the category over the category of S-schemes.

Consider the projection $[X/G] \to X/G$ as a bundle, where each object $(P \to T) \in [X/G] \mapsto T$ be a principal G-bundle over $T \in X/G$. $(P \to T)$ is considered to be a fiber of a point T on the geometry X/G.

Note that the space is usually coarser, means that the geometric point and fibers are not one-to-one.

Coarse Moduli space
$\tau \cong Hom(-, M)$ is a natural transformation.
Artin's Criterion

Definition 3.4.56 *(Quotients)*
Categorical Quotients:
Given a category $\mathscr{C}$, a categorical quotient of X with an action of G is a morphism $\pi : X \to Y$ that

- *is invariant i.e. $\pi \circ \sigma = \pi \circ p_2$ where $\sigma : G \times X \to X$ is the given group action and p_2 is the projection.*
- *satisifies universal propery: any morphism $X \to Z$ satisfying the previous condition uniquely factors through π.*

Note that π need not be surjective. A categorical quotient π is universal categorical quotient if it's stable under base change: for any $Y' \to Y$, $\pi' : X' = X \times_Y Y' \to Y'$ is a categorical quotient.

Geometric quotients and GIT quotients are categorical quotients.
Geometric Quotients:
A geometric quotient is a morphism $\pi : X \to Y$ s.t.

- *for each $y \in Y$, the fiber $\pi^{-1}(y)$ is an orbit of G.*
- *the topology of Y is the quotient topology,*
- *For any open subsets $U \subset Y$, $\pi^{\#} : k[U] \to k[\pi^{-1}(U)]^G$ is an isomorphism.*

For example, if $H \subset G$ is a closed subgroup, then G/H is a geometric quotient.
GIT Quotients:

Definition 3.4.57 *(Algebraic Spaces)*

Groupoid:

Groupoid Object:
A groupoid object in a category $\mathscr{C}$ admitting finite fiber products consists of a pair of objects R and U together with five morphisms
$s, t : R \to U$, $e : U \to R$, $m : R \times_{U,t,s} R \to R$, $i : R \to R$
satisfying the following groipoid axioms:

- $s \circ e = t \circ e = 1_U$,$s \circ m = s \circ p_1$,$t \circ m = t \circ p_2$ *where* $p_1 : R \times_{U,s,t} R \to R$ *is projections.*
- $m \circ (1_R \times m) = m \circ (m \times 1_R)$
- $m \circ (e \circ s, 1_R) = m \circ (1_R, e \circ t) = 1_R$
- $i \circ i = 1_R$, $s \circ i = t$, $t \circ i = s$, $m \circ (1_R, i) = e \circ s$, $m \circ (i, 1_R) = e \circ t$

Groupoid Scheme:
A groupoid S-scheme (also called algebraic groupoid) is a groupoid object in the category of schemes over some fixed base scheme S. If $U = S$, then the groupoid scheme is the same as group scheme.

- *Scheme*

- *Algebraic Spaces*
  An algebraic space comprises a scheme U and a closed subscheme $R \subseteq U \times U$ satisfying the following two conditions:

  - *R is an equivalence relation as a subset of $U \times U$.*
  - *The projections $p_j : R \to U$ onto each factor are étale maps.*

  If R is trivial, then X is a scheme.

  $X = \{R \subseteq U \times U\}$

  The morphism of algebraic spaces is then defined by the descent sequence

  $Hom(Y, X) \to Hom(V, X) \rightrightarrows Hom(S, X)$

  which is exact.
  An alternative definition of algebraic space is a categorical perspective:

$\mathfrak{X} : (Sch/S)_{et}^{op} \to Sets$

such that

  - *There is étale surjective map* $h_X \to \mathfrak{X}$.
  - *the diagonal morphism* $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ *is representable.*

- *Deligne Mumford Stack*
  A Deligne Mumford stack F *is a stack* F *such that*

  - $\Delta : F \to F \times F$ *is representable, quasi-compact and separated.*
  - *There is a scheme* U *and étale surjective map* $U \to F$ *(called an atlas).*

  If "étale" is weakened to "smooth", then such a stack is called an algebraic stack.

  A DM stack F *is that any* X *in* $F(B)$*, where* X *is quasi-compact, has only finitely many automorphisms. A DM stack admits a presentation by a groupoid.*

- *Algebraic Stack*
  One way to view an algebraic stack:
  A groupoid scheme (R, U, s, t, m) *where* $R = \mu_n \times_S \mathbb{A}_S^n$, $U = \mathbb{A}_S^n$, $s = pr_U$, *and* t *is a group action* $\zeta_n \cdot (x_1, \cdots, x_n) = (\zeta_n x_1, \cdots, \zeta_n x_n)$*, and* m *is a multiplicatio map. Then given an* S*-scheme morphism* $\pi : X \to S$*, the groupoid scheme* $(R(X), U(S), s, t, m)$ *forms a groupoid.*

  Another way to view an algebraic stack:
  An algebraic stack is a fibred category
  $p : \mathcal{X} \to (Sch/S)_{fppf}$ *such that*

  - $\mathcal{X}$ *is a category fibred in groupoids. meaning overcategory for some* $\pi : X \to S$ *is a groupoid.*
  - $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ *of fibered categories is representable as algebraic spaces.*
  - *There exists an fppf scheme* $U \to S$ *and an associated 1-morphism of fibred categories* $\mathcal{U} \to \mathcal{X}$*, which is surjective and smooth called an atlas.*

- *Stack*

Pretopology:
For a set X, a pretopology is a set of neighborhoods $\{N(x)\}_{x\in X}$. A pair (X, cl) is a pretopology if cl is a $\check{C}$ech closure operator, and a subset $S \subset X$ is a neighborhood of a point x if $x \notin cl(X \backslash S)$.

Fppf topology:
Let X be an affine scheme. We define fppf cover of X to be a finite and jointly surjective family of morphisms

$$(\phi_a : X_a \to X)$$

with each X_a affine scheme and each ϕ_a is flat, finitely presented. This generates a pretopology. For X arbitrary, we define an fppf cover of X to be a family $(\phi_a : X_a \to X)$. Note that this is pretology but doesn't have to be topology, but can generate topology from it.

Smooth morphisms:
$f : X \to S$ is smooth morphisms if If f is flat, finitely reprented, and for any geometric point $\bar{s} \to S$, the fiber $X_{\bar{s}} = X \times_S \bar{s}$ is regular. This means for each geometric fiber of f is a nonsingular variety. Thus intuitively speaking, a smooth morphism gives a flat family of nonsingular varieties.

Equivalently, this can be defined as: Let $f : X \to S$ be a locally of finite presentation, then TFAE:

- *f is smooth.*
- *f is formally smooth.*
- *f is flat, and the sheaf of relative differential $\Omega_{X/S}$ is locally free of rank equal to the relative dimension of X/S.*
- *for any x there exists a nbhd $Spec(B)$ of x and nbhd $Spec(A)$ of $f(x)$ s.t. $B = A[t_1, \cdots, t_n]/(P_1, \cdots, P_m)$ and the ideal generated by the m-by-m minors of $(\partial \mathfrak{P}_i \partial t_j)$ is B.*
- *locally f factors into $X \to^g \mathbb{A}^n_S \to S$ where g is étale.*

examples:

Affine Stacks:
For defining a quotient of $\mathbb{C}^2$ by a cyclic group $C_n =< a|a^n = 1 >$, consider an action

$$a : (x, y) \mapsto (\zeta_n x, \zeta_n y)$$

acting on $\mathbb{C}^2$. Then the stack quotient $[\mathbb{C}^2/C_n]$ is an affine smooth Deligne-Mumford stack with a non-trivial stabilizer at the origin. If we wish to think about this as a category, fibered in groupoids over $(Sch/\mathbb{C})_{fpff}$ then given a scheme $S \to \mathbb{C}$ over the category is given by

$$Spec(\mathbb{C}[s]/(s^n - 1)) \times Spec(\mathbb{C}[x, y])(S) \rightrightarrows Spec(\mathbb{C}[x, y])(S)$$

Weighted Projective Line:

A non affine example is $\mathbb{P}(2, 3)$, that can be constructed by $[\mathbb{C} - \{0\}/\mathbb{C}^*]$ where $\mathbb{C}^*$-action is given by

$$\lambda \cdot (x, y) = (\lambda^2 x, \lambda^3 y)$$

The only stabilizers are finite, so the stack is Deligne-Mumford.

Definition 3.4.58 *(Stack properties)*

Artin's criterion:

Artin's criterion claims the deformation functors.

A stack F is limit preserving if it's compatible with filtered direct limit Sch/S, means that for a family $\{X_i\}$, there is an equivalence of categories

$$\varinjlim F(X_i) \to F(\varinjlim X_i)$$

An element $x \in F(X)$ is called algebraic element if it's henselization of of $\mathscr{O}_S$-algebra of finite type.

A limit preserving stack F over Sch/S is called an algebraic stack if

- *For any pair of elements $x \in F(X)$, $y \in F(Y)$, the fiber product $X \times_F Y$ is represented as an algebraic space.*
- *There is a scheme $X \to S$ locally of finite type, and an element $x \in F(X)$ which is smooth and surjective such that for any $y \in F(Y)$ the induced map $X \times_F Y \to Y$ is smooth and surjective.*

Schlessinger's conditions:

Let $\mathscr{F}$ be a category cofibered with a groupoid over $\mathscr{C}_\Lambda$. We define conditions as follows:

- *(S1): every diagram in $\mathscr{F}$*

$$\begin{array}{ccc} & & x_2 \\ & & \downarrow \\ x_1 & \longrightarrow & x \end{array}$$

lying over

$$\begin{array}{ccc} & & A_2 \\ & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

in $\mathscr{C}_\Lambda$ *with* $A_2 \to A$ *surjective can be completed to a commutative diagram*

$$\begin{array}{ccc} z & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array}$$

lying over

$$\begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

- *(S2): The condition (S1) can be applied to*

$$\begin{array}{ccc} & & k[\epsilon] \\ & & \downarrow \\ A & \longrightarrow & k \end{array}$$

Moreover if we have two commutative diagrams in $\mathscr{F}$*,*

$$\begin{array}{ccc} y & \xrightarrow{c} & x_\epsilon \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ x & \xrightarrow{d} & x_0 \end{array}$$

and

$$\begin{array}{ccc} y' & \xrightarrow{c'} & x_\epsilon \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle b} \\ x & \xrightarrow{d} & x_0 \end{array}$$

lying over

$$\begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

then there exists a morphism $b : y \to y'$ *in* $\mathscr{F}(A \times_k k[\epsilon])$ *such that* $a = a' \circ b$

The condition (S1) requires that this functor be essentially surjective.
The condition (S1) requires that this functor be essentially surjective if f_2 equals the map $k[\epsilon] \to k$

remark:
(S1): If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathscr{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is surjective.
(S2): If $A \to k$ maps to $\mathscr{C}_\Lambda$, then the induced map $F(A_1 \times_k k[\epsilon]) \to F(A_1) \times_{F(k)} F(k[\epsilon])$ is bijective.

Schlessinger's theorem:

Keel-Mori Theorem:
Keel-Mori theorem claims the existence of the quotient of algebraic space by a group. As a consequence, the existence of coarse moduli space of a separated algebraic stack, which is roughly a best possible approximation to the stack by a separated algebraic space.

All algebraic spaces are assumed to be finite type over a locally Noetherian base. Suppose that $j : R \to X \times X$ is a flat groupoid whose stabilizer $f^{-1}\Delta$ is finite over X. The Keel-Mori theorem states that there is an algebraic space that is geometric and uniform categorical quotient of X by j, which is separated if j is finite.

Definition 3.4.59 *(Moduli Stack)*

- *Moduli of curves*
- *Kontsevich moduli spaces*
- *Picard stack*
- *inductive scheme*

Glueing sheaves:

If a family of morphisms $\phi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$, and ϕ_i and ϕ_j restrict the same map, then there exists $\phi : \mathscr{F} \to \mathscr{G}$.
cohen structure theorem:

Definition 3.4.60 *(base category)*

Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\mathscr{C}_\Lambda$ to be the category with

- *objects are a pair* (A, ϕ) *where* A *is an Artinian local* Λ *algebra where* $\phi : A/\mathfrak{m}_A \to k$ *is an isomorphism.*
- *morphisms* $f : (B, \psi) \to (A, \phi)$ *are local* Λ *algebra homomorphism such that* $\phi \circ (f \ mod\ \mathfrak{m}) = \psi$.

small extension:
$f : B \to A$ *is a small extension if it is surjective, and* $Ker(f)$ *is a nonzero principal ideal which is annihilated by* $\mathfrak{m}_B$.

lemma 1:
If f *is a surjective ring map in* $\mathscr{C}_\Lambda$*, then it can be factored as a composition of small extensions.*

lemma 2:
Let $f : A \to B$ *be a ring map in* $\mathscr{C}_\Lambda$*, then TFAE:*

- f *is surjective.*
- $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ *is surjective.*
- $\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda + \mathfrak{m}_B^2)$ *is surjective.*

Here the third condition uses the notion of relative cotangent space, which is: For a local homomorphism $R \to S$ *of local rings. The relative cotangent space of* R *over* S *is the* $S/\mathfrak{m}_S$*-vector space* $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$.

lemma :
Let $f_1 : A_1 \to A$ *and* $f_2 : A_2 \to A$ *be ring maps in* $\mathscr{C}_\Lambda$*. Then:*

- *If* f_1 *or* f_2 *surjective, then* $A_1 \times_A A_2$ *is in* $\mathscr{C}_\Lambda$.
- *If* f_2 *is a small extension, then so is* $A_1 \times_A A_2 \to A_1$.
- *If the field extension* k/k' *is separable, then* $A_1 \times_A A_2$ *is in* $\mathscr{C}_\Lambda$.

essential surjection:
$f : B \to A$ *is called essential surjection if*

- f *is surjective.*
- *if for* $g : C \to B$ *is a ring map in* $\mathscr{C}_\Lambda$*, and if* $f \circ g$ *is surjective, then* g *is surjective.*

Categories cofibered in groupoids:
Let $\mathscr{C}$ *be a category, and a category cofibered in groupoid over* $\mathscr{C}$ *is a category* $\mathscr{F}$ *equipped with a functor* $p : \mathscr{F} \to \mathscr{C}$ *such that* $\mathscr{F}^{opp}$ *is a category fibered in groupoids over* $\mathscr{C}^{opp}$ *via* $p^{opp} : \mathscr{F}^{opp} \to \mathscr{C}^{opp}$.

$p : \mathscr{F} \to \mathscr{C}$ *is cofibered in groupoids if:*

- *for any morphism $f : U \to V$ in $\mathscr{C}$ and every object x lying over U, there is a morphism $x \to y$ of $\mathscr{F}$ lying over f.*
- *every pair of morphisms $a : x \to y$ and $b : x \to z$ of $\mathscr{F}$ and any morphism $f : p(y) \to p(z)$ such that $p(b) = f \circ p(a)$, there exists a unique morphism $c : y \to z$ of $\mathscr{F}$ lying over f such that $b = c \circ a$.*

Categories fibered in groupoids:

Let $p : \mathscr{S} \to \mathscr{C}$ be a functor. We say $\mathscr{S}$ is fibered in groupoids over $\mathscr{C}$ if:

- *for morphism $f : V \to U$ in $\mathscr{C}$ and every lift x of U there is a lift $\phi : y \to x$ of f with target x.*
- *for every pair of morphisms $\phi : y\ tox$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique life $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.*

Quasi-Frobenius ring:

Definition 3.4.61 *(deformation functor)*
Let Λ be a complete Noetherian ring, and let ${}_\Lambda Art_k$ be a category of local Λ Artinian algebras with a residue field k.

A thickening is a sujection $A' \to^f A \to 0$ such that $I = Ker(f)$ satisfies $m_{A'} I = 0$. If, in paritcular, I is principal, then it's a small thickening.

A predeformation functor is a functor $F : {}_\Lambda Art_k \to Sets$ such that $F(k) = \{*\}$ is a singleton.

A predeformation functor is prorepresentable if $\hat{F}$ is representable.

A hull for F is a pair (R, η) where $R \in {}_\Lambda Noeth_k$ and $\eta \in \hat{F}(R)$ such that $h_R \to F$ is formally smooth and we have a bijection on tangent spaces $T_{h_R} \to T_F$.

By convention, $k[\epsilon] = k[\epsilon]/(\epsilon^2)$ with the trivial Λ-algebra structure.

The conditions:
For any predeformation functor F and two maps of rings $A' \to A$ and $A" \to A$, we get an induced map $F(A' \times_A A") \to F(A') \times_{F(A)} F(A")$ by functoriality of F and the universal property of a pullback. Call this map (*).

- *(H1): glue:*
  If $A" \to A$ is a small thickening, then (*) is surjective.

- *(H2): gluing being unique over infinitesimal nbhds:*
  If $A = k$ and $A" = k[\epsilon]$, then (*) is bijective.

- *(H3): having finite dimensional tangent space:*
  T_F is finite dimensional.

- *(H4): gluing being unique on a small thickening over itself:*
  If $A" = A'$ and $A' \to A$ is a small thickening, then (*) is bijective.

deformation functor:
If a predeformation functor F satisifies (H1) and (H2), then it's called a deformation functor.

Here we make precise what was meant in the motivation section. Suppose you have a moduli functor $\overline{F} : Sch \to Sets$ and you would like to know what the moduli space looks like in a nbhd of some point say $\eta_0 \in \overline{F}(Spec(k))$, Then if we consider the functor of deformations of η_0, we get $F : {}_{\Lambda}Art_k \to Set$ by the following prescription $F(A) = \{\eta \in \overline{F}(Spec(A)) : \eta|_{spec(k)} = \eta_0\}$. This F can be seen as a deformation functor.

Schlessinger's Criterion:
Schlessinger's criterion says (H1) and (H2) and (H3) are satisfied iff F has a hull. Furthurmore, F is prorepresentable if (H4) is satisfied.

Prorepresentability:
Let $F : \mathscr{C}_\Lambda \to Sets$ be a functor. We say F is prorepresentable if there exists an isomorphism $F \cong \underline{R}|_{\mathscr{C}_\Lambda}$.

Predeformation category:
$\mathscr{F}$ is a cateogry cofibered in grouopids over $\mathscr{C}_\Lambda$.

$\mathscr{F}(k)$ contains at least one object and there is a unique morphism between any two objects.

$\mathscr{F}(k)$ is equivalent to a category with a single object and a single morphism i.e. $\mathscr{F}(k)$ contains at least object and there is a unique morphism between any two objects. A morphism of predeformation category is a morphism of categories cofibered in groupoids over $\mathscr{C}_\Lambda$.

A feature of predeformation category is the following. Let $x_0 \in Ob(\mathscr{F}(k))$. Then every object of $\mathscr{F}$ comes with a single object and a single morphism. If x is an object of $\mathscr{F}$ over A, then we can choose a pushforward $x \to q_*x$ for a quotient map $q : A \to k$, then there is a unique morphism $q_*x \to x_0$ such that the composition $x \to q_*x \to x_0$ is the desired morphism.

Grothendieck's theorem:
Grothendieck's theorem says that a functor from category $\mathscr{C}$ of Artinian algebras to Sets is pro-representable iff it preserves finite limits. This condition is equivalent to asking that the functor preserves pullbacks and the final object. This in general applies to any category $\mathscr{C}$ with finite limits whose objects are Artinian.

Schlessinger's theorem:

Let $F, G : \mathscr{C}_\Lambda \to Sets$ be deformation functors. Let $\phi : F \to G$ be a smooth morphism which induces an isomorphism $d\phi : TF \to TG$ of tangent spaces. Then ϕ is an isomorphism.

Let $F : \mathscr{C}_\Lambda \to Sets$ be a deformation functor. Then F is prorepresentable iff

- *F is a deformation functor*
- *$dim_k TF$ is finite.*
- *$\gamma : Der_\Lambda(k, k) \to TF$ is injective.*

Artin Approximation Theorem:

Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ denote n-collection of indeterminates, $k[[x]]$ be a ring of formal power series with indeterminates x over a field k. Let $f(x, y) = 0$ be a system of polynomial equation in $k[x, y]$ and $c \in \mathbb{N}$, Then given a formal power series solution $\hat{y}(x) \in k[[x]]$, there is an algebraic solution $y(x)$ consisting of algebraic functions such that

$\hat{y}(x) \equiv y(x) \ mod \ (x)^c$.

Skeleton of Category:
In category theory, an skeleton of $\mathscr{C}$ is a smallest subcategory $\mathscr{D} \subset \mathscr{C}$ that does not contain any extraneous isomorphisms. A category is called skeletal if isomorphic objects are necessarily identical, and $\mathscr{D}$ equivalent to $\mathscr{C}$.

Any category is equivalent to its skeleton.

Definition 3.4.62 *(Dualizing Sheaf)*

Definition 3.4.63 *(Some Commutative Algebras)*
The Cohen structure theorem:

Let R be a complete local ring. Then, $R \to \lim_n R/\mathfrak{m}^n$, the completion of R with respect to $\mathfrak{m}$ is an isomorphism.

prop 1:
any quotient of noetherian local ring is also a noetherian local ring. Given a finite ring map $R \to S$, then S is a product of Noetherian complete local rings.

prop 2:
Let $(R, \mathfrak{m})$ be a complete local ring. If $\mathfrak{m}$ is finitely generated ideal then R is Noetherian.

def 1:
A subring $\Lambda \subset R$ is called a coefficient ring if the following conditions hold:

- *Λ is a complete local ring with a maximal ideal $\Lambda \cap \mathfrak{m}$.*
- *the residue field of Λ maps isomorphically to the residue field of R, and*
- *$\Lambda \cap \mathfrak{m} = p\Lambda$, where p is the characteristic of the residue field of R.*

def 2:
A Cohen ring is a complete discrete valuation ring with uniformizer p a prime number.

prop 3:
Let p be a prime. $ch(k) = p$. There exists a Cohen ring $\Lambda/p\Lambda \cong k$.

Definition 3.4.64 *(Cotangent Complex)*
Cotangent Sheaf:

For a morphism of schemes $f : X \to S$, a sheaf of $\mathscr{O}_X$-modules $\Omega_{X/S}$ that represents S-derivations in the sense: for any $\mathscr{O}_X$-modules F, there is an isomorphism

$$Hom_{\mathscr{O}_X}(\Omega_{X/S}, F) = Der_S(\mathscr{O}_X, F)$$

that depends naturally on F. In other words, the cotangent sheaf is characterized by the univer sal property: there is the differential $d : \mathscr{O}_X \to \Omega_{X/S}$ such that any S-derivation $D : \mathscr{O}_X \to F$ factors as $D = \alpha \circ d$ with some $\alpha : \Omega_{X/S} \to F$.

There are two important exact sequences:

- *If $S \to T$ is a morphism of schemes, then*
  $f^*\Omega_{S/T} \to \Omega_{X/T} \to \Omega_{X/S} \to 0$
- *If Z is a closed subscheme of X with the ideal sheaf I, then*
  $I/I^2 \to \Omega_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_Z \to \Omega_{Z/S} \to 0$

The cotangent sheaf is closely related to smoothness of variety or scheme. For exmample, an algebraic variety is smooth of dimension n iff Ω_X is locally free sheaf of rank n.

Construction through a diagonal morphism:

Let $f : X \to S$ be a morphism of schemes and let $\Delta : X \to X \times_S X$ be a diagonal morphism. Then the image of Δ is locally closed; i.e. closed in some open subset W of $X \times_S X$. Let I be the ideal sheaf of $\Delta(X)$ in W. One then puts:

$$\Omega_{X/S} = \Delta^*(I/I^2)$$

and checks this sheaf of modules satisfies the required universal property of a cotangent sheaf. The construction shows in particular that the cotangent sheaf is quasi-coherent. It is coherent if S is Noetherian and f is of finite type.

The above definition means that the cotangent sheaf on X is the restriction to X of the conormal sheaf to the diagonal embedding of X over S.

Relation to the tautological line bundle:

The cotangent sheaf on a projective sapce is related to the tautological line bundle $\mathscr{O}(-1)$ by the following exact sequence: writing $\mathbb{P}^n_R$ for the projective space over a ring R,

$$0 \to \Omega_{\mathbb{P}^n_R/R} \to \mathscr{O}_{\mathbb{P}^n_R}(-1)^{\oplus(n+1)} \to \mathscr{O}_{\mathbb{P}^n_R} \to 0$$

Derivation:

Let A be an algebra over a field k. A k-derivation $d : A \to A$ is a k-linear morphism of algebras with a Leibniz law. We denote collection of all k-derivations is $Der_k(A)$. For derivations $A \to M$, M an A-bimodule, $Der_k(A, M)$.

Cotangent Complex:

cotangent complex is a derived Kahler differentials.
Syzygies:

Model category:

The Naive Cotangent Complex:

Definition 3.4.65 *(Cotangent Complex)*

Derived lower shriek functor.

$$\pi : Sh(\mathscr{C}) \to Sh(*)$$

$$\pi_! : Sh(\mathscr{C}) \to Sh(*)$$

The derived lower shriek functor is

$$L\pi_! : D(\mathscr{O}_{\mathscr{C}}) \to D(\mathscr{O}_{\mathscr{D}})$$

which is left adjoint to π^*. Moveover, for $U \in Ob(\mathscr{C})$ we have

$$L\pi_!(j_{U!}\mathscr{O}_U) = g_! j_{U!}\mathscr{O}_U = j_{u(U)!}\mathscr{O}_{u(U)}$$

where $j_U!$ and $j_{u(U)!}$ are extension by zero associated to the localization morphisms $j_U : \mathscr{C}/U \to \mathscr{C}$ and $j_{u(U)} : \mathscr{D}/u(U) \to \mathscr{D}$.

Extension by zero:
Let $j : U \to X$ be an étale morphism of schemes, then the restriction functor $j^{-1} : Ab(X_{étale}) \to Ab(U_{étale})$ has a left adjoint. $j_! : Ab(U_{étale}) \to Ab(X_{étale})$ and called extension by zero.

Also, the restriction functor $j^{-1} : Sh(X_{étale}) \to Sh(U_{étale})$ has a left adjoint. $j_!^{Sh} : Sh(U_{étale}) \to Sh(X_{étale})$.

prop:
Let $u : \mathscr{C} \to \mathscr{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathscr{C}) \to Sh(\mathscr{D})$ be the corresponding morphism of topoi. Let $\mathscr{O}_{\mathscr{D}}$ be a sheaf of rings, and $\mathscr{I}$ is an injective $\mathscr{O}_{\mathscr{D}}$ module. If $g_!^{Sh} : Sh(\mathscr{C}) \to Sh(\mathscr{D})$ commutes with fibre products, then $g^{-1}\mathscr{I}$ is totally acyclic.
prop:
Let $u : \mathscr{C} \to \mathscr{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathscr{C}) \to Sh(\mathscr{D})$ be the corresponding morphism of topoi. Let $U \in Ob(\mathscr{C})$,

- *For M in $D(\mathscr{D})$, we have $R\Gamma(U, g^{-1}M) = R\Gamma(u(U), M)$.*
- *If $\mathscr{O}_{\mathscr{D}}$ is a sheaf of rings and $\mathscr{O}_{\mathscr{C}} = g^{-1}\mathscr{O}_{\mathscr{D}}$, then for M in $D(\mathscr{O}_{\mathscr{D}})$ we have $R\Gamma(U, g^*M) = R\Gamma(u(U), M)$*

cocontinuous functor:
Let $\mathscr{C}$ and $\mathscr{D}$ be sites, and $u : \mathscr{C} \to \mathscr{D}$ be a functor. u is called cocontinuous if for every $U \in Ob(\mathscr{C})$ and every covering $\{V_j \to u(U)\}_{j\in J}$ of $\mathscr{D}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathscr{C}$ such that the family of maps $\{u(U_i) \to u(U)\}_{i\in I}$ refines the covering $\{V_j \to u(U)\}_{j\in J}$
continuous functor:
Let $\mathscr{C}$ and $\mathscr{D}$ be sites, and $u : \mathscr{C} \to \mathscr{D}$ be a functor. u is called continuous if for every $\{V_j \to V\}_{i\in I} \in Cov(\mathscr{C})$, we have the following

- *$\{u(V_i) \to u(V)\}_{i\in I}$ is in $Cov(\mathscr{D})$, and*
- *For any morphism $T \to V$ in $\mathscr{C}$ the morphism $u(T_i \times_V V_i) \to u(T_i) \times_{u(V)} u(V_i)$ is an isomorphism.*

Recall that given a functor u as above, and a presheaf of sets $\mathscr{F}$ on $\mathscr{D}$ we have defined $u^p\mathscr{F}$ to be simply the presheaf $\mathscr{F} \circ u$. In other words, $u^p\mathscr{F}(V) =$

$\mathscr{F}(u(V))$ *for every object* $V \in \mathscr{C}$.

prop:
If $\mathscr{F}$ *is a sheaf on* $\mathscr{D}$, *then* $u^{\mathrm{p}}\mathscr{F}$ *is a sheaf as well.*
Finally,

The cotangent complex $L_{B/A}$ *of a ring map* $A \to B$ *is the complex of* B*-modules associated to the simplicial* B*-module*

$\Omega_{P./A} \otimes_{P.,\epsilon} B$

where $\epsilon : P. \to B$ *is a standard resolution of* B *over* A.

Definition 3.4.66 *(Simplicial Argument)*

- *Simplicial complex:*

  A simplicial complex K *is a set of simplices that satisfies the following conditions:*

  - *Every face of a simplex from* K *is also in* K.
  - *The non-empty intersection of two simplices* $\sigma_1, \sigma_2 \in K$ *is a face of both* σ_1 *and* σ_2.

  A n*-simplex is a representable functor*

  $\Delta^n(i) = Hom_{\Delta}([i], [n])$

- *skeleton and coskeleton:*

  The n*-skeleton of a topological space* X *is a simplicial complex (or CW complex) referred to as a simplicial subspace* X_n *that is a union of simplices of dimension* $n \leqq n$.

  Or we can construct skeleton and coskeleton in a functorial way. The restriction functor $i_* : \Delta^{op}Sets \to \Delta^{op}_{\leqq n}Sets$ *has the left adjoint* i^* *and the right adjoint* $i^!$. *So, the* n*-skeleton is*

  $sk_n(K) = i^* i_* K$

  and the n*-coskeleton is given by*

$cosk_n(K) = i^! i_* K$

By using adjointness, we have an intuitive interpretation of coskeleton as :

Let X be a Kan complex. Then, $\{\Delta^k \to cosk_n(X)\}$ is bijective to $\{sk_n(\Delta^k) \to X\}$.

- *Kan fibration and Kan Complex:*
  Kan fibration and Kan complex is a part of theory of simplicial sets. A morphism $f : X \to Y$ is a Kan fibration if for any $n \geqq 1$ and any $0 \leqq k \leqq n$ and for any maps $s : \Lambda^n_k \to X$ and $y : \Delta^n \to X$ such that $s = x \circ i$ and $y = f \circ x$.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "s"] \arrow[d, "i"] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "y"] \arrow[ur, "\exists x"] & Y \end{tikzcd}$$

  If a morphism $X \to \{*\}$ from a simplicial set X to a point $\{*\}$ is a Kan fibration, then we call X a Kan complex.

  For example, n-simplex Δ^n is not a Kan fibration.

- *Kan fibration example (Function complex)*

  For simplicial sets X, Y, there is an associated simplicial set called the function complex $Hom(X,Y)$, where the simplices are defined as

  $Hom_n(X,Y) = Hom)_{sSets}(X \times \Delta^n, Y)$

  and an ordinal map $\theta : [m] \to [n]$, there is an induced map

  $\theta^* : Hom(X,Y)_n \to Hom(X,Y)_m$

  defined by sending a map $f : X \times \Delta^n$ to the composition $X \times \Delta^m \to^{1\times\theta} X \times \Delta^n \to^f Y$.

  This complex has an exponential law of simplicial sets.

  $ev_* : Hom_{sSets}(K, Hom(X,Y)) \to Hom_{sSets}(X \times K, Y)$

  which sends a map $f : K \to Hom(X,Y)$ to the composite map

$X \times K \to^{1\times g} \to X \times Hom(X,Y) \to^{ev} Y$

where $ev(x,f) = f(x, \iota_n)$ *where* $\iota \in Hom_{\Delta}([n],[n])$ *lifted to the n-simplex* Δ^n.

- *Kan fibration example (Kan fibration and pull backs)*

  Given a Kan fibration $p : X \to Y$ *and an inclusion of simplicial sets* $i : K \hookrightarrow L$, *there is a fibration*

  $Hom(L,X) \to^{(i^*,p_*)} Hom(K,X) \times_{Hom(K,Y)} Hom(L,Y)$

  induced from a commutative diagram

$$\begin{array}{ccc} Hom(L,X) & \xrightarrow{p_*} & Hom(L,Y) \\ \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle i^*} \\ Hom(K,X) & \xrightarrow{p_*} & Hom(K,Y) \end{array}$$

- *Simplicial Lie algebra:*

  Simplicial Lie algebra is a simplicial object of category of Lie algebras. This is a simplicial abelian group, so it is a topic of Dold-Kan correspondence.

- *Simplicial abelian group:*

  Simplicial gropp is a simplicial object of category of groups. Simplicial abelian group is a simplicial object of category of abelian groups.

  Any simplicial abelian group A *is non-canonically equivalent to the product of Eilenberg-Maclane spaces* $\prod_{i \geqq 0} K(\pi_i A, i)$.

Definition 3.4.67 *(Prelim Standard Resolution)*

- *order perserving morphisms:*

  - $\delta^n_j : [n-1] \to [n]$ *is injective order preserving map skipping* j.
  - $\sigma^n_j : [n+1] \to [n]$ *is sujective order preserving map with* $(\sigma^n_j)^{-1}(\{j\}) = \{j, j+1\}$.

- *Simplicial objects:*
  For a category $\mathscr{C}$*, a simplicial object is a functor* $U : \Delta^{op} \to \mathscr{C}$*. If especially* $\mathscr{C} = Set$*, then* U *is just a simplicial set. If* $\mathscr{C} = Ab$*, then* U *is a simplicial abelian group.*

- *Skeleton functors:*

  $sk_n : Simp(\mathscr{C}) \to Simp_n(\mathscr{C})$

- *Simplicial Object in abelian category:*

  - *If* $\mathscr{A}$ *is an abelian category, the category* $Simp(\mathscr{A})$ *and* $Cosimp(\mathscr{A})$ *are also abelian.*
  - $f : A \to B$ *is inj/surjective if* $f_n : A_n \to B_n$ *is inj/surjective for all* n.
  - *The sequence* $A \to B \to C$ *is exact iff* $A_i \to B_i \to C_i$ *is exact for all* i.
  - *An Eilenberg-Maclane object is* $K(A, k)$ *is given by* $K(A, k) = i_{k!}U$.
  - *Simplicial objects in chain complex is a sequence* $\to U_{n+1} \to U_n \to U_{n-1} \to \cdots U_1 \to 0$*, and the boundary map* $d_n : U_n \to U_{n-1}$.

- *Augumentation:*

  An augmentation $\epsilon : U \to X$ *of* U *towards an object* X *of* $\mathscr{C}$ *is a morphism from* U *into the constant simplicial object* X.

  Let U *be a simplicial obejct of* $\mathscr{C}$*, and* $X \in \mathscr{C}$*. To give an augmentation of* U *towards* X *is the same as giving a morphism* $\epsilon_0 : U_0 \to X$ *such that* $\epsilon_0 \circ d_0^1 = \epsilon_0 \circ d_1^1$.

- *Dold-Kan:*

- *Homotopies:*

- *Homotopies in Abelian category:*

- *Trivial Kan Fibration:*

  Let $\partial\Delta[n] = i_{(n-1)!}sk_{n-1}\Delta[n]$ *be a boundary of* $\Delta[n]$*. Then we have a following diagram commutes.*

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, "\exists"] & Y \end{tikzcd}$$

More generally, for a trivial Kan fibration $f : X \to Y$ of simplicial sets, any simplicial complex Z and W with an (termwise) injective dotted arrow $Z \to W$, the diagram below commutes.

$$\begin{tikzcd} Z \arrow[r, "b"] \arrow[d] & X \arrow[d, "f"] \\ W \arrow[r, "a"] \arrow[ur, "\exists"] & Y \end{tikzcd}$$

prop 1:
Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism simplicial sets. Then $X \times_Y Y' \to Y'$ is a trivial Kan fibration.

prop 2:
A composition of trivial Kan fibrations is also a trivial Kan fibration.

Definition 3.4.68 *(preliminary to Standard resolution)*

Let $Y : sheafC \to \mathscr{C}$ be endofunctor, and let natural transformation for functors $d : Y \to Id$ and $s : Y \to Y \circ Y$. Using these, we define

$X_n = Y \circ Y \circ Y \circ \cdots \circ Y$ (n+1 compositions).

Observe that $X_{n+m+1} = X_n \circ X_m$. Next,

$d^n_j = 1_{X_{j-1}} \star d \star 1_{X_{n-j-1}} : X_n \to X_{n-1}$
$s^n_j = 1_{X_{j-1}} \star s \star 1_{X_{n-j-1}} : X_n \to X_{n+1}$

Simplicial Object:
If $1_Y = (d \star 1_Y) \circ s = (1_Y \star d) \circ s$ and $(s \star 1) \circ s = (1 \star s) \circ s$, then $X = (X_n, d^n_j, s^n_j)$ is a simplicial object in the category of endofunctors of $\mathscr{C}$, and $d = X_0 = Y \to id_{\mathscr{C}}$ defines an augmentation.

lemma:
Given a functor $F : \mathscr{A} \to \mathscr{C}$ and $G : \mathscr{C} \to \mathscr{B}$, we obtain a simplicial object $G \circ X \circ F$ in the category of functors from $\mathscr{A}$ to $\mathscr{B}$, which comes with an augmentation to $G \circ F$.

Given a transformation of functors $h_0 : G \circ F \to G \circ Y \circ F$ such that

$1_{G\circ F} = (1_G \star d \star 1_F) \circ h_0$

Then there is a morphism $h : G \circ F \to G \circ X \circ F$ of simplicial object such that $\epsilon \circ h = id$ where $\epsilon : G \circ X \circ F \to G \circ F$ is the augmentation.

lemma:

Let $F' : \mathscr{A} \to \mathscr{C}$ and $G' : \mathscr{C} \to \mathscr{B}$ be two functors. Let $(a_n) : G \circ X \to G' \circ X$ be a morphism of simplicial objects compatible via augmentation with $a : G \to G'$ and $(b_n) : X \circ F \to X \circ F'$ via augmentation with $b : F \to F'$. Then the two maps

$a \star (b_n), (a_n) \star b : G \circ X \circ F \to G' \circ X \circ F'$

are homotopic.

lemma:

Let $f : id_{\mathscr{C}} \to id_{\mathscr{C}}$, then $f \star 1_X, 1_X \star f : X \to X$ are maps of simplicial objects compatible with f via augmentation $\epsilon : X \to id_{\mathscr{C}}$. Moreover, $f \star 1_X$ and $1_X \star f$ are homotopic.

Definition 3.4.69 *(Standard Resolution)*

We have a further generalization of simplicial objects. Let $\mathscr{A}$ and $\mathscr{S}$ be categories, and let $V : \mathscr{A} \to \mathscr{S}$ be a functor with a left adjoint $U : \mathscr{S} \to \mathscr{A}$. From this, we define a simplicial object from $\mathscr{A}$ to $\mathscr{A}$.

Let unit be $d : U \circ V \to id_{\mathscr{A}}$ and counit be $\eta : id_{\mathscr{S}} \to V \circ U$. In categories, $V \to V \circ U \circ V \to V$ and $U \to U \circ V \circ U \to U$ are identity morphisms. Also, we set $X_n = (U \circ V)^{\circ(n+1}$, and $X_{n+m+1} = X_n \circ X_m$ for all $n, m \geqq -1$. We endow this sequence of functors $d_j^n : X_n \to X_{n-1}$, and $s_j^n : X_n \to X_{n+1}$ satisfying $d_j^n = 1_{X_{j-1}} \star d \star 1_{X_{n-j-1}}$ and $s_j^n = 1_{X_{j-1}} \star \eta \star 1_{X_{n-j-1}}$. Finally, we write $\epsilon_0 = d : X_0 \to X_{-1}$.

Then, $X = (X_n, d_j^n, s_j^n)$ is a simplicial object of $Fun(\mathscr{A}, \mathscr{A})$ and ϵ_0 defines an augmentation ϵ from X to the constant simplicial object with value $X_{-1} = id_{\mathscr{A}}$.

lemma:

In this situation, the maps $1_V \star \epsilon : V \circ X \to V$ and $\epsilon \star 1_V : X \circ U \to U$ are homotopy equivalence.

example:

Let R be a ring. We can take $i : Mod_R \to Sets$ to be the forgetful functor and $F : Sets \to Mod_R$ to be the functor that associates to a set E the free R-module $R[E]$ on E. For an R-module M, the simplicial R-module $X(M)$ will have the following shape

$X(M) = (R[R[R[M]]] \to R[R[M]] \to R[M])$

which comes with an augmentation with M.

Definition 3.4.70 *()*

- *Dold-Kan correspondence:*

  There is an equivalence of category of chain complexes and category $Ch_{\geqq 0}(Ab)$ of simplicial abelian groups sAb. Under the equivalence, the n-th homology group of a chain complex is the n-th homotopy group of the corresponding simplicial abelian group, and a chain homotopy corresponds to a simplicial homotopy (and the correspondence preserves model structures).

- *Model category:*

  - *Retracts:*

    If g is a morphism belonging to one of the distinguished classes, and f is a retract of g, then g belongs to the same distinguished class. Explicitly, the requirement that f is a retract of g means there is i, j, r, s such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\
A' \arrow[r, "j"] & B' \arrow[r, "s"] & A'
\end{tikzcd}$$

  - *2 of 3:*

    If f and g are maps in $\mathscr{C}$ such that gf is defined and any two of these are weak equivalences then so is the third.

  - *Lifting:*

    Acyclic cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to acyclic fibrations. Explicitly, if the outer square of the following diagram commutes, where i is a cofibration and p is a fibration, and i and p are acyclic, then there exists h completing the diagram:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \\
A' \arrow[r, "j"] \arrow[ur, "\exists h"] & B'
\end{tikzcd}$$

- *Factorization:*

  Every morphism f in $\mathscr{C}$ can be written as $f = p \circ i$ for p a fibration, i an acyclic cofibration.

  Every morphism f in $\mathscr{C}$ can be written as $f = p \circ i$ for p an ayclic fibration, i a cofibration.

- *Weak Equivalence:*

  $f : X \to Y$ is called weak equivalence if the induced functions $f_* : \pi_0(X) \to \pi_0(Y)$ is bijective, and for every points $x \in X$, and for all $n \geqq 1$, $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is bijective. Especially, in a simply connected topological space, the singular homology $f_* : H_n(Y, \mathbb{Z}) \to H_n(X, \mathbb{Z})$.

- *cofibration*

  A continuous mapping $i : A \to X$ is a cofibration if the homotopy lifting property wrt all topological spaces S. That is, i is a cofibration if for each topological space S, and for any continuous mapping $f, f' : A \to S$ and $g : X \to S$ with $g \circ i = f$ for any homotopy $h : A \times I \to S$ from f to f', there is a continuous map $g' : X \to S$ and a homotopy $h' : X \times I \to S$ from g to g' such that $h(i(a), t) = h(a, t)$ for all $a \in A$ and $t \in I$. Here, $I = [0, 1]$.

  This definition is dual of fibration, which requires homotopy lifting property wrt all spaces S. This is one instance of the broader Eckmann-Hilton duality.

  We can encode the condition of cofibration in the duality.

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, "i"'] & S^I \arrow[d, "p_0"] \\ X \arrow[r, "f'"] \arrow[ur, "\exists H'"] & S \end{tikzcd}$$

  where S^I is the path space of S equipped with compact open topology. A path space S^I is $S^I = \{f : I \to X | f \text{ cts } f(0) = *\}$ equipped with compact open topology.

- *Fibration:*

  Fibration is a generalization of fiber bundle. Fibration can be used in Postnikov system or obstruction theory.

Homotopy lifting property:
$p : E \to B$ satisfies homotopy lifting property for a space X if for every homotopy $h : X \times I \to B$ and for every mappig $\tilde{h}_0 : X \to E$ lifting $h|_{X\times 0} = h_0$, there exists a homotopy $\tilde{h} = X \times I \to E$ lifting E with $\tilde{h}_0 = \tilde{h}|_{X\times 0}$.

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d, "incl"] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] \arrow[ur, "\exists\tilde{h}"] & B \end{tikzcd}$$

– *Eckmann-Hilton duality*

Definition 3.4.71 *(CW complex)*

$\emptyset X_{-1} \subset X_0 \subset X_1 \subset \cdots$

such tht X_k is obtained from X_{k-1} by gluing copies of k-cells $(e^k_\alpha)_\alpha$, each homeomorphic to k-ball D^k, to X_{k-1} by continuous gluing maps $g^k_\alpha : \partial e^k_\alpha \to X_{k-1}$. The maps are called attaching maps. Each X_k is called the k-skeleton of the complex.

Definition 3.4.72 *(Eckmann-Hilton Duality)*

Eckmann-Hilton duality dicusses the fibration is dual to cofibration, by making all the arrows in the diagram reversed.
a map $X \times I \to Y$ is the same as a map $X \to Y^I$ where $Y^I = \{f : I \to Y\}$, which gives a duality between the reduced suspension ΣX, which is the quotient of $X \times I$, and the loop space ΩY, which is the subspace of Y^I. This then leads to the adjoint relation $< \Sigma X, Y > = < X, \Omega Y >$, which allows the study of spectra, which gives rise to cohomology theories.

Definition 3.4.73 *(Spectra)*

A spectrum is an object representing a generalized cohomology theory.

$\mathscr{E}^* : CW^{op} \to Ab$

There exists a space E^k such that evaluating the cohomology theory in degree k on a space X is equivalent to computing homotopy classes of maps to the space E^k, that is:
$\mathscr{E}^k(X) \cong [X, E^k]$

A spectrum is any sequence X_n of pointed topological spaces or pointed simplicial sets together with a structure maps $S^1 \wedge X_n \to X_{n+1}$. $S^1 \wedge X_n$ is homeomorphic to ΣX.

prop:
$\pi_n(E) = \varinjlim_k \pi_{n+k}(E_k)$

Definition 3.4.74 *(Currying)*

Given a function $f : (X \times Y) \to Z$, currying constructs a new function $g : X \to (Y \to Z)$, which is $g(x)(y) = f(x, y)$ for x of type X and y of type Y. We then also write $curry(f) = g$. Uncurrying is the reverse transformation.

Definition 3.4.75 *(Compact Open Topology)*

Let X and Y be two topological spaces, and $C(X, Y)$ be set of all continuous functions between X and Y. Given a compact subset K of X and an open subset U of Y, let $V(K, U)$ denote the set of all functions $f \in C(X, Y)$ such that $f(K) \subseteq U$. In other words, $V(K, U) = C(K, X) \times_{C(K,Y)} C(X, Y)$. Then the condition of all such $V(K, U)$ is a subbase for the compact open topology on $C(X, Y)$.

Definition 3.4.76 *(Grothendieck Connection)*

Grothendieck connection is a generalization of Gauss-Manin connection.

Definition 3.4.77 *(Leray Spectral Sequence)*

Acyclic:
An abelian category $\mathscr{C}$ with enough injectives, and given an additive functor $F : \mathscr{C} \to \mathscr{D}$, an acyclic object with respect to F, or simply an F-acyclic object, is an object A in $\mathscr{C}$ such that

$R^i F(A) = 0$ *for all* $i > 0$.

where $R^i F$ are the right derived functor of F.

Leray Spectral Sequence:

$Sh_{Ab}(X) \to^{f_*} Sh_{Ab}(Y) \to^{\Gamma} Ab$

Thus the derived functors of $\Gamma \circ f_*$ compute the sheaf cohomology for X:

$R^i(\Gamma \cdot f_*)(\mathscr{F}) = H^i(X, \mathscr{F})$

But becuase f_* and Γ send injective objects $Sh_{Ab}(X)$ to Γ-acyclic objects in $Sh_{Ab}(Y)$, there is a spectral sequence whose second page is

$$E_2^{pq} = (R^p\Gamma \cdot R^q f_*)(\mathscr{F}) = H^p(Y, R^q f_*(\mathscr{F}))$$

and which converges to

$$E^{p+q} = R^{p+q}(\Gamma \circ f_*)(\mathscr{F}) = H^{p+q}(X, \mathscr{F}).$$

This is called Leray spectral sequence.

Definition 3.4.78 *(Grothedieck Spectral Sequence)*

Definition 3.4.79 *(Beilinson Spectral Sequence)*

Definition 3.4.80 *(Hilbert Scheme)*

Definition 3.4.81 *(Quiver Variety)*

Definition 3.4.82 *(Hilbert Chow map)*

Definition 3.4.83 *(Derived Noncommutative Algebraic Geometry)*

Definition 3.4.84 *(Degeneration)*

$\pi : \mathfrak{X} \to C$ *of a variety to a curve* C *with origin* 0*, the fibers* $\pi^{-1}(t)$ *form a family of varieties over* C*, then the fiber* $\pi^{-1}(0)$ *may be thought of as a limit of* $\pi^{-1}(t)$ *as* $t \to 0$*. One says the family* $\pi^{-1}(t)$ *degenerates to a special fiber* $\pi^{-1}(0)$*.*

The limiting process behaves nicely when π *is a flat morphism, and in that case, the degeneration is called flat degeneration.*

In the study of moduli of curves, the important point is to understand the boundaries of moduli, which amounts to understand the degeneration of curves.
Infinitesimal Deformation:
Let $D = k[\epsilon]$ *be ring of dual numbers, and* Y *a finite type scheme over* k*. Let* $X \subset Y$ *be a close subscheme, then an embedded first order infinitesimal deformation of* X *is a closed subscheme* X' *of* $Y \times_{Spec(k)} Spec(D)$ *such that the projection* $X' \to Spec(D)$ *is flat and has* X *has the special fiber.*

If $Y = Spec(A)$ *and* $X = Spec(A/I)$ *are affine, then an embedded infinitesimal deformation amounts to an ideal* I' *of* $A[\epsilon]$ *such that* $A[\epsilon]/I'$ *is flat over* D

and the image of I' in $A = A[\epsilon]/\epsilon$ is I.

In general, given a pointed scheme $(S, 0)$ and a scheme X, a morphism of schemes $\pi : X' \to S$ is called the deformation of a scheme X if it is flat and the fiber of it over the distinguished point 0 of S is X. Thus, the above notion is a special case when $S = Spec(D)$ and there is some choice of embedding.

Mumford Degeneration:

Definition 3.4.85 *(Category of Schemes)*

For a scheme over a commutative ring R, an R-point of X means a section of a morphism $X \to Spec(R)$. One writes $X(R)$ for the set of R-points of X. In examples, this definition reconstructs the old notion of set of solutions of the defining equations of X with values in R. When R is a field k, $X(k)$ is called called the set of k-rational points of X.

More generally, for a commutative ring R and a commutative R-algebra S, an S-point of X means a morphism $Spec(S) \to X$ over R. One writes $X(S)$ for the set of S-points of X.

Definition 3.4.86 *(Fiber Bundle)*

def fiber bundle:
The formal definition of a fiber bundle is a pair (E, B, π, F) where E,B,F are topological spaces, and π is a continuous surjection satisfying the local triviality condition. Alternatively, a fiber bundle is denoted by the sequence $F \to E \to^{\pi} B$. It's a smooth fiber bundle if E, B, F are all smooth manifolds, and all the functions above are smooth maps.

Definition 3.4.87 *(Is fibration always fiber bundle?)*

If the fibration is smooth and the spaces involved are compact, then it is a fiber bundle.

A smooth math $p : E \to B$ is said to satisfy the homotopy lifting property in the smooth category if given the following commutative diagram where all maps are smooth, and there exists a morphism $\tilde{F}$ which makes the following diagram smooth:

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, "incl"'] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "F"] \arrow[ur, "\exists \tilde{F}"] & B
\end{tikzcd}$$

- *smooth (Hurewicz) fibration:*
  A smooth map is said to be smooth (Hurewicz) fibration if it satisfies the homotopy lifting property in the smooth category for all manifolds Y.

- *Smooth Serre fibration*
  A smooth map is said to be smooth Serre fibration if it satisfies the homotopy lifting property in the smooth category for all disks I^n, $n \geqq 0$.

- *Eheresmann's lemma:*
  If a smooth mapping $f : M \to N$ where M and N are smooth manifolds, and f is surjective submersion and a proper map. Then it's a locally trivial fibration.

Now here is the proof:
For a smooth Serre fibration $p : E \to B$, p is a surjective submersion, and if the spaces involved are compact, then by Eheresmann's lemma, the fibration p is fiber bundle.

Definition 3.4.88 *(Obstruction theory)*

Obstruction theory studies the cohomological invariant with a simplicial/CW method. Let $p : E \to B$ be a fibration of simplicial objects E and B.

Definition 3.4.89 *(augmentation)*

An augmentation of an associative algebra A over a commutative ring k is a k-algebra homomorphism $A \to k$, typically denoted by ϵ. An algebra together with an augmentation is called an augmentation algebra. The kernel of the augmentation is a two-sided ideal called the augmentation ideal A.
example:
If $A = k[G]$ is a group algebra with a finite group G, then

$$A \to k,\ \Sigma a_i x_i \mapsto \Sigma a_i$$

is an augmentation.

If A is a graded algebra which is connected, i.e. $A_0 = k$, then the homomorphism $A \to k$ which maps an element to its homogeneous component of 0 is an augmentation. For example,

$$k[x] \to k,\ \Sigma a_i x^i \mapsto a_0$$

is an augmentation on the polynomial $k[x]$.

Definition 3.4.90 *(dg-category)*

DG-category is a category whose morphisms are endowed with an additional structure of differential $\mathbb{Z}$-graded modules. This means that,

$$Hom(A, B) = \oplus_{n \in \mathbb{Z}} Hom_n(A, B)$$

and there is a differential d on this graded group, i.e. for each n,

$d : Hom_n(A, B) \to Hom_{n+1}(A, B)$

which has to satisfy $d \circ d = 0$. This is equivalent to saying that $Hom(A, B)$ is a cochain complex. Furthermore, the composition of morphisms $Hom(A, B) \otimes Hom(B, C) \to Hom(A, C)$ is required to be a map of complexes, and for all objects A of the category, one requires $d(id_A) = 0$.

Examples:

- *Any additive category may be considered to be a DG-category by imposing a trivial grading $Hom_n(A, B) = 0$ for all $n \neq 0$ and trivial differential $d = 0$.*
- *A little bit sophisticated is the category of complexes $C(\mathscr{A})$ over an additive category $\mathscr{A}$. By definition, $Hom_{C(\mathscr{A}),n}(A, B)$ is the group of maps $A \to B[n]$ which do no need to respect the differential of the complexes A and B, i.e.*

  $Hom_{C(\mathscr{A}),n}(A, B) = \prod_{l \in \mathbb{Z}} Hom(A_l, B_{l+n})$

  The differential of such a morphism $f = (f_l : A_l \to B_{l+n})$ of degree n is defined to be

  $f_{l+1} \circ d_A + (-1)^{n+1} d_B \circ f_l$

  This applies to the category of quasi-coherent sheaves on a scheme over a ring.
- *DG-category with one object is the same as DG-ring. A DG-ring over a field k is called DG-algebra.*

prop1:
The category of small dg-categories can be endowed with a model category structure such that weak equivalences are those functors that induce an equivalence of derived categories.

prop2:
Give an dg-category $\mathscr{C}$ over some ring R, there is a notion of smoothness and properness of $\mathscr{C}$ that reduces to the usual notions of smooth and proper morphisms in case $\mathscr{C}$ is the category of quasi-coherent sheaves on some scheme X over R.

Relation to triangulated categories:
namely dg-enhancement.

Definition 3.4.91 *(Koszul Complex)*

Definition 3.4.92 *(dg-Lie algebra)*

dg-Lie algebra is a graded vector space $L = \oplus L_i$ *over a field* k, $ch(k) = 0$ *with a bilinear map* $[\cdot, \cdot] : L_i \otimes L_j \to L_{i+j}$ *and differential* $d : L_i \to L_{i-1}$ *satisfying*

- $[x, y] = (-1)^{|x||y|}[y, x]$
- $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ *(Jacobi identity)*
- $d[x, y] = [dx, y] + (-1)^{|x||y|}[x, dy]$ *(product rule)*

Any reasonable formal deformation problem in characteristic zero can be described by Maurer–Cartan elements of an appropriate dgla. A Maurer-Cartan element $x \in L_{-1}$ *is a solution of Maurer-Cartan equation:*

$dx + \frac{1}{2}[x, x] = 0$

Definition 3.4.93 *(dg-algebra)*

dg-algebra is a graded algebra A *equipped with a map* $d : A \to A$ *that has either degree* 1 *or* -1 *that satisfies two conditions:*

- $d \circ d = 0$
- $d(a \cdot b) = (da) \cdot b + (-1)^{deg(a)} a \cdot b$ *where deg is a degree of homonegeneous elements.*

example:

- *(Tensor Algebra):*
  $T(V) = \oplus_{i \geqq 0} T^i(V) = \oplus_{i \geqq 0} V^{\otimes i}$
  where $V^{\otimes 0} = K$.

  If $e_1, \cdots, e_n$ *is a basis for* V *there is a differential* d *on the tensor algebra defined component-wise*

  $d : T^k(V) \to T^{k-1}(V)$

  sending basis elements to

  $d(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \Sigma e_{i_1} \otimes \cdots \otimes d(e_{i_j}) \otimes \cdots \otimes e_{i_k}$

In particular, we have $d(e_i) = (-1)^i$ *and so*

$$d(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \Sigma(-1)^{i_j} e_{i_1} \otimes \cdots \otimes \hat{e_{i_j}} \otimes \cdots \otimes e_{i_k}$$

- *Koszul Complex:*

  One of the foundational examples of dga, widely used in commutative algebra and algebraic geometry, is the Koszul complex.

- *De-Rham complex:*

- *singular cohomology:*

A dg-augmented algebra is a dga equipped with a dg morphism to the ground ring.

properties of dgla:

- *model structure*
- *Relation with* L_∞ *algebra*
- *Relation with dg-coalgebras*
- *Relation with simplicial Lie algebras*

Definition 3.4.94 *(dg-coalgebra)*

A dgca is a comonoid in the category of chain complexes. Equivalently, this is a graded coalgebra C *equipped with a coderivation*

$$D : C \to C$$

that is of degree -1 *and squares to* 0 *i.e.* $D^2 = 0$.

Detailed component definition:

- *Pre-coalgebra*
  A pre-graded coalgebra (C, Δ, ϵ) *is a pre-gvs* C *together with linear maps of degree* 0,

  $$\Delta : C \to C \otimes C,\ \epsilon C \to k$$

  such that the obvious diagrams commute.

- *Coaugmentation*

A coaugmentation of a pre-gc is a morphism $\eta : k \to C$. We will write 1 *for* $\eta(1)$.

The cokernel $\overline{C}$ *of* η *can be identified with* $ker(\epsilon)$ *and so can be considered as a subspace of* C.

The reduced diagonal $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$, *induced by* Δ *is defined by* $\Delta x = 1 \otimes x + x \otimes 1 + \overline{\Delta} x$. *The vector space of primitives of* C, *denoted* $P(C)$, *is the kernel of the reduced diagonal.*

A morphism of coaugmenteds pre-gcs, $\psi : (C, \eta) \to (C', \eta')$ *is a morphism of the pre-gcs which satisfies* $\eta' = \psi \circ \eta$. *It preserves primitives.*

- *the commutation morphism*

  $\tau : V \otimes V' \to V' \otimes V$

  is defined by $\tau(v \otimes v') = (-1)^{|v||v'|} v' \otimes v$, *on homogeneous elements.*

- *Coderivations of pre-graded coalgebras*

  If C *is a pre-gc, a coderivation of degree* $p \in \mathbb{Z}$, *is a linear map* $\theta \in Hom_p(C, C)$ *such that*

  $\Delta \circ \theta = (\theta \otimes id_C + \tau \circ (\theta \otimes id_C) \circ \tau) \circ \Delta$, *and* $\epsilon \circ \theta = 0$

  A coderivation θ *of a coaugmented pre-gc* (C, η) *is a coderivation of* C *such that* $\theta \circ \eta = 0$.

Finally, we have definition of dgca.

A differential ∂ *on a (coaugmented) pre-gc,* C, *is a coderivation of degree* -1 *such that* $\partial \circ \partial = 0$.

The pair (C, ∂) *is called a differential (coaugmented) pre-graded coalgebra (pre-dgc). Its homology* $H(C, \partial)$ *will be pre-gc.*

Definition 3.4.95 *(L_∞-algebra)*

definition using operad:
An L_∞ *algebra is an algebra over an operad in the category of chain complexes over the* L_∞ *operad.*
defnition in terms of higher brackets:

An L_∞ algebra is a $\mathbb{Z}$ graded vector space $\mathfrak{g}$, and there is a multi-linear map of n-ary bracket ($n \geqq 1$) of the form

$$l_n(\cdots) := [-,-,\cdots,-]_n : \mathfrak{g}\otimes\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}\to\mathfrak{g}$$

and of degree $n-2$ with the following conditions hold:

- *graded skew symmetry*

  Each l_n is graded anti-symmetric, in that for every permutation σ of n elements and for every n-tuple $(v_1,\cdots,v_n)$ of homogeneously graded elements $v_i \in \mathfrak{g}_{|v_i|}$ then

  $$l_n(v_{\sigma(1)}, v_{\sigma(2)}\cdots, v_{\sigma(n)}) = \chi(\sigma, v_1,\cdots,v_n)\cdot l_n(v_1,v_2,\cdots,v_n)$$

  where $\chi(\sigma, v_1,\cdots,v_n)\in\{-1,+1\}$ is a graded signature sign.

- *strong homotopy Jacobi identity*

  For all $n\in\mathbb{N}$, and for all n-tuples $(v_1,\cdots,v_n)$ of homogeneously graded elements $v_i\in\mathfrak{g}_{|v_i|}$ the following equation holds

  $$\Sigma_{i,j\in\mathbb{N},i+j=n+1}\Sigma_{\sigma\in UnShuff(i,j-1)}\chi(\sigma,v_1,\cdots,v_n)(-1)^{i(j-1)}l_j(l_i(v_{\sigma(1)},\cdots,v_{\sigma(i)}),v_{\sigma(i+1),\cdots,v_{\sigma(n)}}) = 0$$

definition in terms of semifree dgca:

It was pointed out that the higher brackets of an L_∞ algebra induce on the graded co-commutative cofree coalgebra $\vee^{\cdot}\mathfrak{g}$ over the underlying graded vector space $\mathfrak{g}$ the structure of dgca, with the differential $D = [-]+[-,-]+[-,-,-]+\cdots$. The higher brackets are extended coderivations. The higher Jacobi identity is equivalently the condition that $D^2=0$. The semifree dgca are an equivalent incarnation of L_∞ algebras.

definition in terms of semifree dga:

A Grassmannian algebra $\wedge^{\cdot}\mathfrak{g}^*$ is then naturally equipped with an ordinary differential $d = D^*$ which acts on $\omega\in\wedge^{\cdot}\mathfrak{g}^*$ as

$$(d\omega)(t_1\vee\cdots\vee t_n) = \pm\omega(D(t_1\vee\cdots\vee t_n))$$

When the grading-dust has settled one finds that with

$$\wedge^{\cdot}\mathfrak{g}^* = k \oplus \mathfrak{g}_1^* \oplus (\mathfrak{g}_1^* \wedge \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*) \oplus \cdots$$

with the ground field in degree 0*, the degree* 1*-elements of* $\mathfrak{g}^*$ *in degree* 1*, that* d *is of degree* $+1$ *and of course sequares to* 0 *i.e.* $d^2 = 0$*. This means that we have a semifree dga* $CE(\mathfrak{g}) := (\wedge^{\cdot}\mathfrak{g}^*, d)$*.*

properties:

- *Ind-nilpotency*

  For $\mathfrak{g}$ *an* L_∞ *algebra, then its CE chain dgc-coalgebra* $CE.(\mathfrak{g})$ *is ind-conilpotent.*

  This means that $CE.(\mathfrak{g})$ *is a filtered colimit of sub-dg-coalgebras which are conilpotent, in that for each of them there is* $n \in \mathbb{N}$ *such that their* n*-fold coproduct vanishes. As such these are like co-local Artin algebras.*

  Moreover, since every dg-coalgebra is the union of its finite-dimensional subalgebras, this means that $CE.(\mathfrak{g})$ *is a filtered colimit of finite dimensional conilpotent coalgebras.*

  This implies that the dual Chevalley-Eilenberg cochain algebra $CE.(\mathfrak{g})$ *is a filtered limit of finite-dimensional nilpotent dgc-algebras (actual local Artin algebras).*

- *Model category structure*

- *Relation to DG-Lie algebras*

  Every dg-lie algebra is evidently L_∞ *algebra.*

  Here there is a proproposition:

  Let k *be a field of* $ch(k) = 0$*. Write* $L_\infty Alg_k$ *as a category of* L_∞ *algebras over* k*. Then every object of* $L_\infty Alg_k$ *is quasi-isomorphic to a dg-Lie algebra. Moreover, there is a functorial replacement. There is a functor*

  $$W : L_\infty Alg_k \to L_\infty Alg_k$$

  such that for each $\mathfrak{g} \in L_\infty Alg_k$

– $W(\mathfrak{t})$ *is a dg-Lie algebra*

– *there is a quasi-isomorphism* $\mathfrak{g} \to W(\mathfrak{g})$

Definition 3.4.96 *(Chevalley-Eilenberg algebra)*

Chevalley-Eilenberg algebra $CE(\mathfrak{g})$ *of a Lie algebra* $\mathfrak{g}$ *is a dga of elements dual to* $\mathfrak{g}$ *whose differential encodes the Lie bracket on* $\mathfrak{g}$.

The cochain cohomology of the underlying cochain complex is the Lie algebra cohomology of $\mathfrak{g}$.

This generalizes to a notion of Chevalely-Eilenberg algebra for $\mathfrak{g}$ *and* L_∞ *algebra, a Lie algbroid and generally an* ∞*-Lie algebroid.*

Grading Convention:
The $\mathbb{N}$*-graded vector space we write*

$$\wedge^{\cdot} V := Sym(V[1]) = k \oplus (V_0) \oplus (V_1 \oplus V_0 \wedge V_0) \oplus (V_2 \oplus V_1 \wedge V_0 \oplus V_0 \wedge V_0 \wedge V_0) \oplus \cdots$$

for the free-graded commutative algebra on the graded vector space obtained by shifting V *up in degree by one.*

Def (CE algebra of Lie algebra) :

$CE(\mathfrak{g})$ *of a finite dimenaional Lie algebra* $\mathfrak{g}$ *is the semifree graded commutative dg-algebra whose underlying graded algebra is the Grassmannian algebra*

$$\wedge^{\cdot\cdot\cdot} \mathfrak{g}^* = k \oplus \mathfrak{g}^* \oplus (\mathfrak{g}^* \wedge \mathfrak{g}^*) \oplus \cdots$$

whose differential d *(of degree +1) is defined on* $\mathfrak{g}^*$ *as the dual of the Lie bracket*

$$d|_{\mathfrak{g}^*} := [-,-]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$$

and extended uniquely as a graded derivation on $\wedge^{\cdot} \mathfrak{g}^*$.

If we choose a dual basis $\{t^a\}$ *of* $\mathfrak{g}^*$ *and let* $\{C^a_{bc}\}$ *be the structure constants of the Lie bracket in that basis, then the action of the differetial on that basisi generator is*

$$d^a = -\frac{1}{2} C^a_{bc} t^b \wedge t^c$$

prop:
there is an equivalence of categories

$$CE : LieAlg \to (dgAlg_{sf,1})^{op}$$

Hence, Chevalley-Eilenberg algebra is just another way of looking at Lie algebra.

Def (CE algebra of L_∞ algebra) :

If $\mathfrak{g}$ is a graded vector space, then a differental D of degree -1 squaring 0 on $\vee^\cdot \mathfrak{g}$ is precisely the same as eqipping $\mathfrak{g}$ with the structure of an L_∞ algebra.

Dually, this corresponds to a general semifree dga,

$CE(\mathfrak{g}) := (\wedge^\cdot \mathfrak{g}^*, d = D^*)$

This is a Chevalley-Eilenberg algebra of the L_∞ algebra $\mathfrak{g}$.

So every commutative semifree dga is the Chevalley Eilenberg algebra of some L_∞-algebra of finite type.

This means that many constructions involving dg-algebra are secretly about ∞-Lie theory. For instance, the Sullivan construction in rational homotopy theory may be interpreted in terms of Lie interpretation of L_∞ algebras.
(semi-free dga):
A dga is semi-free if the underlying graded algebra is free.
Def (CE algebra of Lie algebroid) :

Def (CE algebra of ∞ Lie algebroid) :

Definition 3.4.97 *(Lie Algebroid)*

Definition 3.4.98 *(∞-Lie Algebroid)*

Definition 3.4.99 *(Properties of DG-algebras)*

- *(Model Category and dgla)*

  There exists a model category structure $(dgLie_k)_{proj}$ on the category $dgLie_k$ of dg-Lie algebras over a commutative ring $k \supset \mathbb{Q}$ such that

  - *fibrations the surjection maps*
  - *weak equivalences the quasi-isomorphisms on the underlying chain complexes.*

  This becomes a simplicial category with simplicial mapping spaces given by

  $dgLie(\mathfrak{g}, \mathfrak{h}) := ([k] \mapsto Hom_{dgLie}(\mathfrak{g}, \Omega^\cdot(Delta^k) \otimes \mathfrak{h}))$

where

- $\Omega^{\cdot}(\Delta^k)$ *is the dg-algebra of polynomial differential forms on the k-simplex*

- $\Omega^{\cdot}(\Delta^k) \otimes \mathfrak{h}$ *is the canonical dg-Lie algebra structure on the tensor product.*

- *(Relation to dg-coalgebras)*

- *(dg Lie-algebra and deformation)*

Definition 3.4.100 *(Model structure for dgla)*

Consider there is the Quillen adjunction

$(\mathscr{R} \dashv i) dgLie_k \to^i dgCocAlg_k$

where $\mathscr{R}$ *is a rectification functor,* i *is a inclusion functor. Therefore, the composition* $i \circ \mathscr{R}$ *is a resolution functor.*
Another functor

$CE : dgLie_k \to^i dgCocAlg_k$

$CE(\mathfrak{g}, \partial, [-,-]) = (\vee^{\cdot} \mathfrak{g}[1], D = \partial + [-,-])$

For $(X, D) \in dgCocAlg)_k$*, write*

$\mathscr{L}(X, D) := (F(\tilde{X}[-1]), \partial := D + (\Delta - 1 \otimes id - id \otimes 1)) \in dgLieAlg_k$

where

- $\tilde{X} = ker(\epsilon)$ *is the kernel of the counit, regarded as a chain complex*
- F *is the free Lie algebra functor*
- *on the right we are extending* $(\Delta - 1 \otimes id - id \otimes 1) : \tilde{X} \to \tilde{X} \times \tilde{X}$ *as a Lie algebra derivation.*

hence, we have an adjunction relation (Quillen functors)

$(CE \dashv \mathscr{L}) dgLie_k \to^i dgCocAlg_k$

and hence we have natural hom isomorphism

$Hom(\mathscr{L}(X), \mathfrak{g}) \cong Hom(X, CE(\mathfrak{g})) \cong MC(Hom(\tilde{X}, \mathfrak{g}))$

Definition 3.4.101 *(Model structure for L_∞ algebras)*

Consider that L_∞ algebras are the ∞-algebras on the category of chain complexes over the Lie operad. As such they carry a model structure on algebras over an operad. There is also a strictification to model structure of dgla.

More geometric way is to understand L_∞ algebras as being the tangent spaces to connected smooth ∞-groupoids, hence to the delooping/moduli ∞-stacks BG of smooth ∞-groups, hence as the first order infinitesimal neighborhood
$B\mathfrak{g} \hookrightarrow BG$

prop:
An L_∞ algebra on a graded vector space $\mathfrak{g}$ is equivalently a dg-coalgebra structure on a graded cofree coalgebra over $\mathfrak{g}$. Conversely, a category of L_∞ algebras is a full subcategory of category of dgca.
prop:

Let $k \supset \mathbb{Q}$ and $Ch.(k)$ category of chain complexes, the free-forgetful adjunction

$$(U \vdash F) : Alg(\mathscr{O}) \to_U Ch.(k)$$

where $\mathscr{O}$ is a symmetric operad.

Definition 3.4.102 *(Model structure for Simplicial Lie algebras)*

A simplicial Lie algebra is an object of the category of Lie algebras $LieAlg_k$. A category of simplicial Lie algebra is written by $(LieAlg_k)^{\Delta^{op}}$.

Relation to dg-Lie algebra

$$[-,-]_{N\mathfrak{g}} : (N\mathfrak{g}) \otimes_k (N\mathfrak{g}) \to^{\nabla} N(\mathfrak{g} \otimes_k \mathfrak{g}) \to^{N([-,-])} N(\mathfrak{g})$$

where the first morphism is the Eilenberg-Zilber map. This construction extends to a functor

$$N : LieAlg_k^{\Delta^{op}} \to dgLieAlg_k$$

and N has a left adjoint N^*, so $N^* \dashv N$.

Definition 3.4.103 *(Weakly homotopic equivalence)*
Let X, Y be a topological spaces or simplicial sets and $f : X \to Y$ be a continuous funciton or a simplicial map, f is called weakly homotopic equivalent if

- *f induces an isomorphism of connected components*
  $\Pi_0(f) : \Pi_0(X) \to \Pi_0(Y)$

- *for all points $x \in X$ and for all $1 < n$, $n \in \mathbb{N}$, f induces an isomorphism on homotopy groups*
  $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$

Equivalent characterization:

An equivalent characterization of weakly homotopic equivalence is precisely if for all $n \in \mathbb{N}$ and all commuting diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ D^n \arrow[r] \arrow[ur, "\exists\sigma"] & Y \end{tikzcd}$$

there exists $\sigma : D^n \to X$ which makes both the triangle commute.

Definition 3.4.104 *(A_∞ algebra)*
An A_∞ algebra in Top is called A_∞ space.

Every loop space is canonically A_∞ space.

Every A_∞ space is weakly homotopic equivalent to a topological monoid.

An A_∞ space is a homotopy type X that is equippeed with the structure of a monoid up to coherent higher homotopy. that means it is equipped with:

- *a binary product operation $\cdot : X \times X \to X$*

- *a choice of associativity homotopy $\eta_{x,y,z} : (x \cdot y) \cdot z \to x \cdot (y \cdot z)$*

- *a choice of pentagon law homotopy between five such ηs*

- *and so ever on controlled by the associahedra*

Definition 3.4.105 *(A_∞-algebra)*

A_∞ algebra is alternatively called a homotopy associative algebra, because in A_∞ algebra, associativity holds only up to a homotopy.
def:

For a fixed field k, an A_∞-algebra is a $\mathbb{Z}$ graded vector space

$A = \oplus_{p \in \mathbb{Z}} A^p$

such that for $d \geqq 1$ *there exists degree* $2-d$, *k-linear maps*

$m_d : (A^\cdot)^{\otimes d} \to A^\cdot$

which satisfy a coherence condition:

$\Sigma_{1 \leqq p \leqq d 0 \leqq q \leqq d-p} (-1)^{\alpha} m_{d-p+1}(a_d, \cdots, a_{p+q+1}, m_p(a_{p+q}, \cdots, a_{q+1}), a_q, \cdots, a_1) = 0$

where $\alpha = (-1)^{deg(a_1)+\cdots+deg(a_q)-q}$.

examples:

- *associative algebra*
  This is trivial
- *dga*
  if $m_1 = d$ *and* m_2 *is just a multiplication, and* $m_n = 0$ *for all* $n \geqq 3$, *then the* A_∞ *algebra will be dga.*
  Using the structure theorem of minimal models, there is a canonical A_∞ *structure on the graded cohomology algebra* $HA^\cdot$ *which preserves the quasi-isomorphism structure of the original dga. One common example of such dga comes from the Koszul algebra arising from a regular sequence. This is an important result because it helps pave the way for the equivalence of homotopy categories*
  $Ho(dga) \cong Ho(A_\infty alg)$
- *Cochain algebras and H-spaces*
- *Examples with infinitely many non-trivial* m_i

de Rham algebra $(\Omega^\cdot, d, \wedge)$ *and*
Hochschild cohomology algebra can be equipped with an A_∞ *structure.*

Definition 3.4.106 *(Minimal model for* A_∞ *algebra)*

One of the most important structure theorem for A_∞ *algebras is the existence and uniqueness of minimal models – which is* A_∞*-algebra where the differential* $m_1 = 0$ *is zero. taking a cohomological algebra* $HA^\cdot$ *of* $A^\cdot$ *from differential* m_1, *so as a graded algebra*

$HA^\cdot = \frac{ker(m_1)}{m_1(A^\cdot)}$

with multiplication map $[m_2]$. *It turns out this graded algebra can then canonically be induced with an* A_∞ *structure,*

$$(HA^{\cdot}, 0, [m_2], m_3, m_4, \cdots)$$

which is unique up to quasi-isomorphisms of A_∞ algebras. In fact, the statement is even stronger

$$(HA^{\cdot}, 0, [m_2], m_3, m_4, \cdots) \to A^{\cdot},$$

which lifts the identity map on $A^{\cdot}$. Note these higher products are given by the Massey product.

Definition 3.4.107 *(Massey product)*

Definition 3.4.108 *(Graded algebra from Ext structure: structure theorem of A_∞ algebra)*
Another structure theorem of A_∞ algebra is the reconstruction of an algebra from its ext algebra. Given a connected graded algebra

$$A = k_A \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$$

it is canonically an associative algebra, called its Ext algebra, defined as

$$Ext^{\cdot}_k(k_A, k_A)$$

where the multiplication is given by the Yoneda product. Then, there is an A_∞ quasi isomorphism between $(A, 0, m_2, 0, 0, \cdots)$ and $Ext^{\cdot}_k(k_A, k_A)$. This identification is important because it gives a way to show that all derived categories are derived affine, meaning they are isomophic the derived category of some algebra.

Definition 3.4.109 *(Homotopy Lie algebra)*

L_∞ is alternatively called a homotopy Lie algebra.

Formal manifold $\hat{S}\Sigma V^*$ where $\hat{S}$ is the completed symmetric algebra, Σ is the suspension of a graded vector space, and V^* denotes the linear dual.

Typically, one describes (V, m) as a homotopy Lie algebra and $\hat{S}\Sigma V^*$ with the differential m as its representing commutative differential graded algebra.

Let V and W be homotopy Lie algebras, and $f : V \to W$ is quasi-isomorphism if their differentials are just m_V and m_W. An important class of homotopy Lie algebras are called minimal homotopy Lie algebras, characterized by vanishing component of l_1.

Any homotopy Lie algebra is quasi-isomorphic to a minimal one, which is unique up to isomorphism therefore call its minimal model.

Definition 3.4.110 *(Suspension in topology and algebra)*
In topology, Suspension of homotopy X is $X \times [0,1]$ identified with $X \times \{0\}$ by a point, and $X \times \{1\}$ resp.

For a graded vector space, a r-suspension of V is given by $(s^r V)_n = V_{n-r}$ for $r \in \mathbb{Z}$.

Definition 3.4.111 *(Chevalley-Eilenberg Algebra)*

- *Of Lie algebra*
  $CE(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a dga of elements dual to $\mathfrak{g}$ whose differential encodes to the Lie bracket on $\mathfrak{g}$.

  The cochain cohomology of the underlying cochain complex is the Lie algebra cohomology of $\mathfrak{g}$.

  $\wedge^{\cdot} V := Sym(V[1]) = k \oplus (V_0) \oplus (V_1 \oplus V_0 \wedge V_0) \oplus (V_2 \oplus V_1 \wedge V_0 \oplus V_0 \wedge V_0 \wedge V_0) \oplus \cdots$

  for the free graded commutative algebra on the graded vector space obtained by shifting V up in degree by one.

  CE alg over Lie alg:

  A CE algebra $CE(\mathfrak{g})$ is $\wedge \mathfrak{g}^*$ as an algebra with a differential

  $d|_{\mathfrak{g}^*} := [-,-]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$

  and this naturally extends to d. Indeed, $d \circ d = 0$.

  $dt^a = -\frac{1}{2} C^a_{bc} t^b \wedge t^c$

- *Of L_∞ Algebra*

  CE algebra over L_∞ algebra $CE(\mathfrak{g}) := (\wedge^{\cdot} \mathfrak{g}^*, d = D^*)$ where $\mathfrak{g}$ is an L_∞ algebra is simply generalizing from the Grassmannian over $\mathfrak{g}^*$ to Grassmannian algebra over a graded vector space.

Definition 3.4.112 *(Operad)*

An operad is a gadget describing algebraic structures in symmetric monoidal categories. It is

- *a bunch of abstract operations of arbitrarily many arguments*
- *equipped with a notion of how to compose these*

- *subject to evindent associativity and unitarity conditions*

Just like a monoid can be seen as a single-object category, an operad is equivalently a single-object multicategory. Multicategories with multiple objects are also called colored operads.

An algebra over an operad is a concrete realization of these abstract operations; an object A equipped with n-ary operations $A \otimes A \otimes \cdots \otimes A \to A$ as specified by the operad, subject to the composition relation as specified by the operad.

Symmetric Operads:
Let V be a symmetric monoidal category. A permutative or symmetry operad in V consists of objects $F(n)$ of V indexed over the natural numbers $n = 0, 1, 2, \cdots$ equipped with the following extra structure

- *Right actions of symmetric groups $\rho_n : S_n \to Hom(F(n), F(n))$*
- *A unit $e : I \to F(1)$*
- *A composition operations*
  $F(k) \otimes F(n_1) \otimes F(n_2) \otimes \cdots \otimes F(n_k) \to F(n_1 + n_2 + \cdots + n_k)$.

These data are subject to obvious identities such as associativity and unitarity of composition, and compatibility of composition with symmetric group actions. For example, the unit laws say that the evident composite

$$F(n) \cong I \otimes F(n) \to^{e\otimes 1} F(1) \otimes F(n) \to^{comp} F(n)$$

is the identity map, as is

$$F(n) \cong F(n) \otimes I^{\otimes n} \to^{1\otimes e^{\otimes n}} F(n) \otimes F(1)^{\otimes n} \to^{comp} F(n)$$

Compatibility with symmetric group acitons means that for each element $\sigma \in S_k$, the composition operation

$$F(k) \otimes \bigotimes_{i=1}^{k} F(n_i) \to F(n_1 + \cdots + n_k)$$

coequalizes a pair of automorphisms

$$\rho(\sigma) \otimes 1, 1 \otimes \lambda(\sigma) : F(k) \otimes \bigotimes_{i=1}^{k} F(n_i) \rightrightarrows F(k) \otimes \bigotimes_{i=1}^{k} F(n_i)$$

where σ acts on the big tensor product on the left by permuting tensor factors in the obvious way. If V has suitable colimits, this condition could be expressed in terms of tensor products over S_n.

Colored Operads:

There is an evident generalization of the above where we allow the operad to have several objects called colors in operad theory. This models algebraic structures where elements of different types may be fed into n-ary operations.

Let C be a set, called the set of colors. Then a colored operad is

- *for each $n \in \mathbb{N}$, and each $(n+1)$-tuple $(c_1, \cdots, c_n, c)$ an object $P(c_1, \cdots, c_n; c) \in V$*
- *for each $c \in C$ a morphism $1_c : I \to P(c; c)$ in V - the identity on c;*
- *for each $n+1$-tuple $(c_1, \cdots, c_n, c)$ and n other tuples $(d_{1,1}, \cdots, d_{1,k_1}), \cdots, (d_{n,1}, \cdots, d_{n,k_n})$ a morphism*
  $P(c_1, \cdots, c_n; c) \otimes P(d_{1,1}, \cdots, d_{1,k_1}; c_1) \otimes P(d_{n,1}, \cdots, d_{n,k_n}; c_n) \to P(d_{n,1}, \cdots, d_{n,k_n}; c)$
  the composition operations.
- *for all n, all tuples, and each permutations σ in the symmetry group Σ_n a morphism $\sigma^* : P(c_1, \cdots, c_n; c) \to P(c_{\sigma(1)}, \cdots, c_{\sigma(n)}; c)$*
- *subject to the condition that*
  - *the σs form a representation of Σ_n.*
  - *the composition operation satisfies associativty and unitality in the obvious way*
  - *and is Σ_n equivariant in the evident way.*

  An equivalent term for a colored operad in V is a symmetric multicategory which is enriched over V.

  Algebras:

  An algebra over an operad F in V is just a semantics for interpreting the $F(n)$ as objects of actual n-ary operations on an object v. That is, an F-algebra structure on an object v in V consists of a collection of maps:

  $F(n) \otimes v^{\otimes n} \to v$

  which intuitively is a mapping like this:

  $\theta \otimes x_1 \otimes \cdots \otimes x_n \mapsto \theta(x_1, \cdots, x_n)$

so that elements of $F(n)$ are interpreted as n-ary operations on v. These data are subject to some natural conditions which implement this idea.

Perhaps the quickest way to define it is to suppose that V is just a semantics for interpreting the $F(n)$ as objects of actual n-ary operations on an object v. That is, an F-algebra structure on an object v in V consists of a collection of maps

$F(n) \otimes v^{\otimes n} \to v$

which intuitively a mapping like this

$\theta \otimes x_1 \otimes \cdots \otimes x_n \mapsto \theta(x_1, \cdots, x_n)$

so that "elements" of $F(n)$ are interpreted as an n-ary operations on v. These data are subject to some natural conditions which implement this idea.

Perhaps the quickest way to define it is to suppose that V is symmetric monoidal closed, and work by way of parellel to how representations or module work. Just as an R-module (or a ring over R) can be defined as a ring homomorphism

$R \to hom(A, A)$

where the hom is an internal hom of abelian groups, called an endomorphism ring, so there is such a thing as an endomorphism operad attached to any object v in a symmetric monoidal closed category, and an F-algebra over an operad F is the same thing as an operad morphism

$F \to hom(v^{\otimes =}, v)$

to an endomorphism operad.

Now the components of the endomorphism operad is given by $End(v)(n) = hom(v^{\otimes n}, v)$

Certainly S_n acts on the right on the hom object $hom(v^{\otimes n}, v)$. And clearly there is a map $e : I \to hom(v, v)$ to play the role of the unit. The operad composition involves an instance of enriched functoriality of iterated tensor products: there is a map

$$hom(v^{\otimes n_1}, v) \otimes \cdots \otimes hom(v^{\otimes n_k}, v) \to hom(v^{\otimes n_1 + \cdots + n_k}, v^{\otimes k})$$

The endomorphism operad composition is obtained by tensoring this last arrow with $hom(v^{\otimes n}, v)$ on the left, and composing the result with ordinary internal hom-composition

$$hom(v^{\otimes k}, v) \otimes hom(v^{\otimes n_1 + \cdots + n_k}, v^{\otimes k}) \to hom(v^{\otimes n_1 + \cdots + n_k}, v)$$

A closely related way of defining an F-algebra is via the monad attached to an operad, which we will describe below.

Definition 3.4.113 *(A detailed conceptual treatment of Operad)*

A (set-based) operad is a monoid in the monoidal category $(Psh(\mathbb{P}), \circ, I)$.

The monad attached to an operad.

There is a functor $i : Set \to Psh(\mathbb{P})$

which sends a set X to the functor

$\hat{X} : \mathbb{P}^{op} \to Set$

taking $[n]$ to X if $n = 0$, else to 0. This functor is full and faithful. Conceptually, it treats a set X as giving a set of 0-ary operations or constants indexed by itself. Notice that the composite

$Psh(\mathbb{P}) \times Set \to^{1 \times i} Psh(\mathbb{P}) \times Psh(\mathbb{P}) \to^{\circ} Psh(\mathbb{P})$

factors through an inclusion $i : Set \to Psh(\mathbb{P})$. This gives an action

$Psh(\mathbb{P}) \times Set \to Set$

for an acategory structure; as it is the restriction of the substitution product $\circ$ along the inclusion i in the second argument, we again denote it by $\circ$, by abuse of notation. Given $F : \mathbb{P}^{op} \to Set$ and a set X, we have

$F \circ X \cong \Sigma_{k \geqq 0} F(k) \otimes_{S_k} X^k$

and given $G : \mathbb{P}^{op} \to Set$, we also have coherent natural morphisms $(F \circ G) \circ X \cong F \circ (G \circ X)$, $I \circ X \cong X$.

Def: (monad associated with an operad)

The monad associated with an operad $(M, m : M \circ M \to M, u : I \to M)$ *is the functor* $\hat{M} : Set \to Set$ *taking* X *to* $M \circ X$*, equipped with a natural transformations*

$$\hat{M}\hat{M}X = M \circ (M \circ X) \cong (M \circ M) \circ X \to^{m \circ X} M \circ X = \hat{M}X$$

$$X \cong I \circ X \to^{u \circ X} M \circ X = \hat{M}X$$

which provide $\hat{M}$ *with the structure of a monad.*

The definition of the assocaited monad carries over with ease to the enriched case, and to variants such as nonpermutative operads, braided operads, and cartesian operads.

Notice that an algebra for the operad $\hat{M}$ *is a set* X *equipped with a structure map* $\alpha : M \circ X \to X$ *which makes* $i(X)$ *a module over the monoid* M *in the monoidal category* $Psh(\mathbb{P})$*.*
Properties (Change of Color):

Smooth differential forms

Definition 3.4.114 *(Smooth n-simplex)*

Smooth n-differential forms

Smooth n-simplex
The smooth n-simplex Δ^n_{smth} *is the smooth manifold with boundary and corners defined, up to isomorphism, as the following locus inside* $\mathbb{R}^{n+1}$*:*

$$\Delta^n_{smth} := \{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} | 0 \leqq x_i \leqq 1 \text{ and } \Sigma^n_{i=0} x_i = 1\} \hookrightarrow \mathbb{R}^{n+1}$$

For $0 \leqq i \leqq n$ *the function*

$$x_i : \Delta^n_{smth} \hookrightarrow \mathbb{R}$$

is the i-th component of the function.

For $f : [n_1] \to [n_2]$

a morphism of linear orders $[n] := \{0 < 1 < \cdots < n\}$*, let*

$$\Delta^n_{smth}(f) = \Delta^{n_1}_{smth} \to \Delta^{n_2}_{smth}$$

be the smooth function defined by $x_i \mapsto x_{f(i)}$

Definition 3.4.115 *(smooth differential form on the smooth n-simplex)*

$F^n := \{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} | \Sigma_{i=0}^n x_i = 1\} \hookrightarrow \mathbb{R}^{n+1}$

A smooth differential form on Δ^n_{smth} *of degree* k *is a collection of linear functions*

$\wedge^k T_x F^n \hookrightarrow \mathbb{R}$

Write $\Omega^{\cdot}(\Delta^n_{smth})$ *for the graded real vector space defiend this way. By definition then, a canonical linear map*

$\Omega^{\cdot}(F^n) \to \Omega^{\cdot}(\Delta^n_{smth})$

from the de Rham complex of F^n *and there is a unique structure of dg comm algebra on* $\Omega^{\cdot}(\Delta^n_{smth})$ *that makes is a homomorphism of dga from the de Rham algebra of* F^n*. This is the de Rham algebra of smooth differential forms on the smooth n-simplex.*

For $f : [n_1] \to [n_2]$*, the pullback form on* F^n *is*

$(\Delta_{smth}(f))^* : \Omega(\Delta^{n_2}_{smth}) \to \Omega(\Delta^{n_2}_{smth})$

$\Omega(\Delta^{-}_{smth}) : \Delta^{op} \to dgcAlg_{\mathbb{R}}$

Definition 3.4.116 *(Polynomial Differential Form)*

For $n \in \mathbb{N}$*, we write*

$\Omega^{\cdot}_{poly}(\Delta^n) := Sym^{\cdot}_{\mathbb{Q}} < t_0, \cdots, t_n, dt_0, \cdots, dt_n > /(\Sigma t_i - 1, \Sigma dt_i)$

for the quotient of the $\mathbb{Z}$*-graded symmetric algebra over the rational numbers. In particular, for degree* 0,

$\Omega^0_{poly}(\Delta^n) := \mathbb{Q}[t_0, \cdots, t_n]/(\Sigma_i t_i = 0)$

where a symmetric algebra $Sym(x_1, \cdots, x_n)$ *is an algebra generated by* x_i *for all* $1 \leqq i \leqq n$ *quotiented by* $xy - (-1)^{|x||y|} yx$.

Prop:

$\Omega^{\cdot}(\Delta^n_{smth}) \cong C^\infty(\Delta^n_{smth}) \otimes_{\Omega^0_{poly}(\Delta^n)} \Omega^{\cdot}_{poly}(\Delta^n)$

and this defines a canonical inclusion

$\Omega^{\cdot}_{poly}(\Delta^n) \hookrightarrow \Omega^{\cdot}(\Delta^n_{smth})$

Definition 3.4.117 *()*

Definition 3.4.118 *(PROP)*

PROP is a symmetric strict monoidal category whose objects are natural numbers n identified with the sets $\{0, 1, \cdots, n-1\}$. In other words, every object is given by $x^{\otimes n}$ for some object x.

Symmetric group of n is a subgroup of automorphism group of n. The name PROP is an abbreviation of PROduct and Permutation category.

Note that Operad is a perticular kind of PROP. In other words, Operad $\subset \frac{1}{2}PROP \subset PROP$, where the first category is a category of operads.

Examples:
An important elementary class of PROPs are the sets $R^{\cdot\times\cdot}$ of all matrices over some fixed ring R. More concretely, these matrices are the morphisms of PROP; the objects can be taken as either $\{R\}^{\infty}_{n=0}$. or just as a plan natural numbers. In this example,

- *composition of morphism $\circ$ is just a multiplication*
- *The identity morphism of an object n (or R^n) is the identity matrix with side n.*
- *The product $\otimes$ acts on objects like an addition $m \otimes n = m + n$ (or $R^m \otimes R^n = R^{m+n}$), so $\alpha \otimes \beta = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$*
- *The permutation in PROP is just a permutation matrices, both left/right actions.*

Other examples of PROP:

- *a discrete category of natural numbers $\mathbb{N}$.*
- *Augumented simplex category Δ_+ is NOT a PROP.*

Algebra of PRO:

An algebra of a PRO P in a monoidal category C is a strict monoidal functor from P to C. Every PRO P and a category C give rise to a category Alg^C_P of algebras whose objects are the algebras of P in C and whose morphisms are natural transformations between them.

- *An algebra of FinSet is a commutative monoid object of C.*
- *an algebra of Δ is a monoid object in C.*

Definition 3.4.119 *(Deligne's Conjecture)*

In deformation theory, Deligne's conjecture is about the operadic strucutre on Hochschild cochain complex.

Definition 3.4.120 *(Topological Hochschild Homology)*

We consider Topological Hochschil homology, as the generalization of Hochschild homology. We replace a category of k-modules by ∞-category $\mathscr{C}$, and A by an associative algebra in this category.

Applying this to a category of spectra $\mathscr{C} = Spectra$ and A being the Eilenberg-MacLane spectrum associated to an ordinary ring R yields topological Hochschild homology denoted by $THH(R)$. For exampdle, $\mathscr{C} = D(mathbbZ)$ derived category of $\mathbb{Z}$-modules.

Definition 3.4.121 *(dg-manifolds)*

Definition 3.4.122 *(Example: Algebraic Deformation)*

Definition 3.4.123 *(Example: Geometric Deformation)*

Definition 3.4.124 *(Integral Curve)*

Definition 3.4.125 *(Integral manifold)*

Definition 3.4.126 *(Phase Space)*

Definition 3.4.127 *(Foliation)*

Definition 3.4.128 *()*

Definition 3.4.129 *(Integrability Condition)*

Integrability condition is equivalent to the condition $\overline{\partial}^2 = 0$.
Integrability condition for the deformed structure is $(\overline{\partial} + \Sigma_i \gamma_i \partial\partial z_i)^2 = 0$.

Definition 3.4.130 *(q-cocycle condition (1-cocycle condition))*

Definition 3.4.131 *(Maurer-Cartan equation)*

Definition 3.4.132 *(Almost complex structure)*

$J : TM \to TM$ *such that* $J^2 = -I$

Definition 3.4.133 *(Cartan Formula)*

Definition 3.4.134 *(Lie derivative)*

Let M *be a smooth manifold and* $X : M \to TM$ *be a vector field, and let there be an integral curve* $\Phi_X : [0,1] \to M$ *derived from* X*, where* $\Phi_X(0) = p \in M$*. Then, the Lie derivative will be:*

$\mathscr{L}_X(f) = \lim \frac{f(\Phi_X(t)) - f(p)}{t}$

Definition 3.4.135 *(Lie algebra Differential Form)*

Definition 3.4.136 *()*

- *Commutative monoid operad*
- *Associative operad*
- *Lie operad*
- A_∞ *operad*
- L_∞ *operad*

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