

From Complex Analysis to Algebraic Geometry

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1 Preliminary

1.1 Algebraic Curve

Complex Analysis:

We'll consider algebraic interpretation of complex analysis, where the holomorphic functions are calculated by their singularities, and the integral pertaining this has monodromy, or it has a Laurant expansion to calculate residue theorem, or in diffyq, singularities is more than power series solution, and we'll study algebraic geometry and representation theory for this.

The first question is the inverse problem: given the complex domain(plane), with some singularities, can we determine all the possible function whose domain is it? In 1-dimensional complex analysis, if the domain of the function $f(z)$ is $\mathbb{C} \setminus \{0\}$, then the function $f(z)$ could be $f(z) = \frac{1}{z}$ but also $f(z) = \log(z)$, This is called monodromy, and we use monodromy representation to determine all the possibilities. This problem is reduced to the combinational problem of

hypergeometric functions or KZ equation in general.

Moduli Problem:

Answering for what is moduli space is a very ambiguous problem, but we should approach from the concrete examples, that can be generated by GIT quotient of some geometries: for example, it might be M_g moduli space i.e. a set of stable algebraic curves of genus g , or in particular, $M_{g,n}$ a moduli space of genus g of n -marked points as M_g consists of their disjoint union $M_g = \coprod M_{g,n}$, and this idea is closely related to complex analysis, and our first goal is to describe the moduli spaces explicitly.

Another argument that sounds is why we should use the technique of algebraic geometry for complex analysis, and I believe it's because algebraic variety could be considered to be "hard" geometry, compared with smooth manifold in general, lacking diversity of their shapes, and especially the automorphism of the geometry could be a finite group, also called stability condition. For example, \mathbb{CP}^1 is the only algebraic curve, which is 1-dimensional compact projective geometries of genus 0, and similarly an elliptic curve is the only genus 1 compact algebraic curve.

The moduli problem can be further generalized by the language of stacks. Or perhaps derived algebraic geometry and so on.....

1.2 GIT Quotient

Definition 1.2.1 (*GIT Quotient*)

By context of moduli problem, GIT quotient is modularization. For an algebraic variety X , if a reductive group G acts on X algebraically, it will define quotient $X//G$ of good properties. For example,

$$M_{0,n} = \text{Conf}_n(C)/SL_2(C)$$

Why G needs to be reductive group because if a polynomial ring A is fixed by G , A^G is finitely generated (or in other words, if G is not reductive, GIT is undefined, or it doesn't have a good property).

Proposition 1.3 (*Keel-Mori Theorem*)

For any schemes, GIT quotient exist as Deligne-Mumford stack.

Remark 1.4 (*Compactness*)

Compactness is also a very important notion in the context of algebraic geometry, since compactness guarantees the bounray problem in geometry, and it the

geometry satisfies compactness, it'll enables us to discuss intersection theory, including Poincare duality. If the geometry is compact, then the automorphism is beautiful,

$$Aut(\mathbb{P}^1) = PGL_2(\mathbb{C})$$

while for non-compact space, the automorphism

$$Aut(\mathbb{C}) = \mathbb{C} \rtimes \mathbb{C}$$

is ∞ -dimensional, and it's very dirty.

2 Complex Analysis

The very question of complex analysis is to classify the possible class of functions depending on the domains.

2.1 Residue Theorem

Skipped since this is part of Qual.

2.2 Mondromy

Definition 2.2.1 (*Linear and Non-Linear Equations*)

- (Generalized Hypergeometric Function)

$${}_pF_q(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}$$

where $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{C}$ but $c \neq 0, -1, -2, \dots$, and $z \in \mathbb{C}$.

- (Hypergeometric Function)

If $p = 0$ and $q = 1$, generalized hypergeometric function becomes ordinary hypergeometric function. Hypergeometric function is a solution of diffyq

$$z(1-z)\frac{d^2f}{dz^2} + [c - (a+b+1)z]\frac{df}{dz} - abf = 0$$

- (Bessel Function)

If $p = 0$ and $q = 1$, generalized hypergeometric function becomes Bessel function. That is,

$${}_0F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(b)_n} \frac{z^n}{n!}$$

- (Heun's Equation)

$$\frac{d^2 w}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right] \frac{dw}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} w = 0$$

- (Theta Function)

Used for solitons and abelian varieties.

$$\vartheta(z; \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2i \pi n z)$$

- (Painleve)

Painleve equation is a transcendentals are solution of non-linear second ODE, but it has an integrabl structure.

$$\frac{d^2 y}{dt^2} = 6y^2 + t$$

Definition 2.2.2 (Monodromy Representation)

Monodromy representation is a finite dimensional representation, which means $\dim(V) < \infty$, and $\pi_1(X)$ is a fundamental group. Monodromy representation is defined by

$$\rho : \pi_1(X) \rightarrow GL(V)$$

We could define the complex functions whose domain is X , and by choosing arbitrary dimension of V , we have several different solutions.

Definition 2.2.3 (Riemann-Hilbert Correspondence)

RH correspondence is a 1-to-1 correspondence of flat connections and the monodromy representations. In other words, all the possible complex functions can be reconstructed from monodromy representations. Note that flat connection ∇ can define a system of diffyq $\nabla \phi = 0$, thus existence of ϕ , that can be derive from the representations.

Example 2.3 (Monodromy Representation $X = \mathbb{C} \setminus \{0\}$ and $\dim(V) = 1$)

So there is only one singularity, and $f(z)$ could be either $f(z) = z^\alpha$ where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ or $f(z) = \log(z)$.

Example 2.4 (Monodromy Representation $X = \mathbb{C} \setminus \{0\}$ and $\dim(V) = 2$)

$$f(z) = Az^\alpha + Bz^\beta$$

Example 2.5 (*Monodromy Representation $X = \mathbb{C} \setminus \{0, 1\}$ and $\dim(V) = 1$*)
 $\pi_1(\mathbb{C} \setminus \{0, 1\}) = F_2$ is a free group of 2-generators, and Jordan-Canonical form
(or could be diagonalized)

$$f(z) = z^\alpha(1 - z)^\beta$$

Example 2.6 (*Monodromy Representation $X = \mathbb{C} \setminus \{0, 1\}$ and $\dim(V) = 1$*)
 $\begin{bmatrix} e^\alpha & 1 \\ 0 & e^\beta \end{bmatrix}$, and

Example 2.7 (*Monodromy Representation $X = \mathbb{C} \setminus \{0, 1\}$ and $\dim(V) = 1$*)

Definition 2.7.1 (*KZ Equation*)

Definition 2.7.2 (*Degeneration*)

Formally speaking, a degeneration is a map

$$\pi : \mathfrak{X} \rightarrow S$$

where \mathfrak{X} is the total space, and S is the base i.e. \mathbb{A}^1 , and for each $s \in S$, $\mathfrak{X}_s = \pi^{-1}(s)$, together a degeneration at $s_0 \in S$ if the fiber \mathfrak{X}_{s_0} is of idiosyncratic shape. For instance,

$$\mathfrak{X} = \{(x, y, t) \in \mathbb{A}^3 | y^2 = x^3 + tx\}$$

where elliptic curve if $t \neq 0$, and singular if $t = 0$.

The reason why we need to study degeneration is that if we compactify moduli spaces, we might add some singular fibers, by which we mean degeneration.

Definition 2.7.3 (*Degeneration of Moduli Space*)

Degeneration is part of a moduli problem, and we'll particularly consider a moduli space arising from monodromy representation i.e. the monodromy representation defines complex functions, and we'll define moduli space of the domain of the complex functions. Here, by context, degeneration of moduli space is a moduli space of a domain where Painleve transcendence is defined.

Example 2.8 (*Degeneration*)

- ${}_pF_q(a, b, c; z)$ is a degeneration of ${}_{p+n}F_{q+m}(a, b, c; z)$
- Painleve transcendence is a degeneration of hypergeometric functions.

3 Moduli Space

Moduli space is a space generated by geometric quotient of some objects, and each point corresponds to an isomorphism class of objects. For example, for a Mobius strip M and a circle S^1 , a natural projection $M \rightarrow S^1$ is a vector bundle with fiber \mathbb{R}^1 , and the moduli space is M/\mathbb{R}^1 .

Remark 3.1 *Moduli space is more than a set of points. $M/\mathbb{R}^1 \cong S^1$ is an isomorphism, and M/\mathbb{R}^1 is a moduli space, but S^1 is not a moduli space, and these two are distinct only because each point corresponds to an isomorphism class.*

Remark 3.2 *Moduli space is not a fiber bundle, but what is the difference from the fiber bundle exactly? The moduli space has each point as an isomorphism class, not as an actual fiber. There are few different varieties of vector bundles over circle S^1 , which is annulus or mobius strip, whose fibers are both \mathbb{R}^1 , but moduli space cannot distinguish them.*

Remark 3.3 *Etymologically, Moduli stands for "mod"uli.*

3.4 $M_{g,n}$ Moduli Space

Definition 3.4.1 *(Configuration Space)*

Let X be a topological space, then the configuration space $Conf_n(X)$ is defined as

$$Conf_n(X) = \{(z_1, z_2, \dots, z_n) \in X^n | z_i \neq z_j \text{ for } i \neq j\}$$

Definition 3.4.2 *($M_{g,n}$ Moduli Space)*

The $M_{g,n}$ moduli space is GIT quotient of configuration space of \mathbb{P}^1 by $SL^2(\mathbb{C})$, which is a geometry of genus g with n -marked points of the base space \mathbb{P}^1 . In particular,

$$M_{0,n} = Conf_n(\mathbb{P}^1)/SL^2(\mathbb{C})$$

Remark 3.5 *($M_{0,n}$ Calculation)*

- $(n = 3)$
Consider that Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ can put arbitrary distinct 3-points z_1, z_2 , and z_3 mapped to arbitray distinct 3-points w_1, w_2 , and w_3 .

Since configuration space of 3-dimensions is a space of 3 independent distinct elements, each points could be arbitrarily mapped to any points of the

space, which means by $SL^2(\mathbb{C})$ quotient, it'll be all identified. Hence,

$$M_{0,3} = \{pt\}$$

- ($n = 4$)

Recall $SL^2\mathbb{C}$ fixed the 3-points, then the last 4th-point has some freedom, because if we let $\lambda = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$, Cross ratio λ could take any values of $\mathbb{P}\setminus\{0, 1, \infty\}$. $\lambda \neq 0$ or $\lambda \neq \infty$ because all points z_i are distinct, and none of the numerator or denominator could be zero, and if $\lambda \neq 1$, $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = 1$, then $(z_1 - z_3)(z_2 - z_4) = (z_1 - z_4)(z_2 - z_3)$, and by simple college algebra, it will be $(z_1 - z_2)(z_3 - z_4) = 0$, but it never happens since all points are distinct. Otherwise, λ could take any values. Hence,

$$M_{0,4} = \mathbb{P}\setminus\{0, 1, \infty\}$$

- ($n = 5$)

If we fix the mapped points w_1, w_2 , and w_3 depending on z_1, z_2 , and z_3 , w_4 is cross ratio depending on z_4 , while w_5 is another cross ratio by z_5 , which means

$$\begin{aligned} - \lambda &= \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} \\ - \mu &= \frac{(z_1-z_3)(z_2-z_5)}{(z_1-z_5)(z_2-z_3)} \end{aligned}$$

so we have two cross ratios λ and μ , so $M_{0,5}$ is two parameters. Hence,

$$M_{0,5} = \mathbb{P}\setminus\{0, 1, \infty\}$$

Finally, notice indeed $\dim(M_{0,n}) = n - 3$

Definition 3.5.1 ($M_{1,1}$ Moduli Space)

The very definition of $M_{1,1}$ is a moduli space of algebraic curves of genus 1 of 1-marked points, but the question is that are all the algebraic curves of genus 1 of 1-marked points elliptic curves and vice versa? Yes, and this is, in fact, equivalently to that it's a moduli space of elliptic curves quotient by j -invariants. Another question is how to explicitly classify all the elliptic curves, and this could be done by moduli space of torus, and only to prove that torus is equivalent to some elliptic curve.

$$M_{1,1} = \mathbb{H}/SL_2(\mathbb{Z}) \cong \Lambda \text{ where } \mathbb{H} \text{ is the upper-half plane.}$$

where $\dim(M_{1,1}) = 1$

Or equivalently, $M_{1,1} = \{(E, p) \mid \text{genus 1 smooth curve, } p \in E\} / \sim$

Remark 3.6 ($M_{1,1}$ Lattice and Curve Correspondence)

In order to verify the previous definition. we'll start from the lattice correspondence with the elliptic curve.

- (Step 1:)
Each lattice can be described by the standard form, which is

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$$

for some $\tau \in \mathbb{C}$

- (Step 2:)
If E_τ is a lattice, this \mathbb{C}/E_τ can be a domain of the Weierstrass \wp function, which is in other words period, and Weierstrass \wp function defines an elliptic function. To formulate, if we let $Y = \wp'(z, \mathbb{C}/E_\tau)$ and $X = \wp(z, \mathbb{C}/E_\tau)$, we have the following identities:

$$Y^2 = 4X^3 - g_2X - g_3$$

where g_2 and g_3 are Eisenstein series.

- (Step 3: Legendre Transform)

What else do we have? This correspondence is actually not one-to-one, but it will be, if we take quotient of the set of elliptic curves by isomorphism classes by j -invariant, that is equivalent to say that if elliptic curves have the same j -invariant, then they have isomorphism as algebraic varieties, and Legendre transform is the step to define j -invariant. The previous elliptic function can be canonically transformed to the Legendre form

$$y^2 = x(x-1)(x-\lambda) \text{ where } \lambda \in \mathbb{C} \setminus \{0, 1\}$$

- (Step 4: j -Invariant)

By using λ arising from Legendre transform, if we let j -invariant as

$$j(\lambda) = 256 \cdot \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}$$

$$j(\tau) = j(\lambda(\tau))$$

Note if two Weierstrass functions are distinct up to $SL_2(\mathbb{Z})$ transformation, g_2 and g_3 are always distinct.

Definition 3.6.1 ($M_{1,0}$ Moduli Space)

Also, $M_{1,0}$ is $n = 0$, and it's also moduli space of elliptic curves $y^2 = x^3 + ax + b$,

For $n = 0$, $M_{1,0}$ is $M_{1,0} = M_{1,1}/Aut(C)$

Remark 3.7 (Group Structure)

Why the moduli space of elliptic curves are $M_{1,1}$ instead of $M_{1,0}$ is because each elliptic curve has a group structure, or in other words, if the point of the elliptic curve is depending on X and Y coordinates, whose value depends on the position of the lattice E_τ . Each element of the group is a point of the elliptic curve, and of course the unit element O is. The unit element O is point at infinity.

Remark 3.8 (Elliptic Curve Is Not Torus)

Definition 3.8.1 ($M_{2,0}$ Moduli Space)

$$M_{2,0} = \mathbb{H}_2/Sp_4(\mathbb{Z})$$

where $\dim(M_{2,0}) = 3$

Remark 3.9 (Tree Structure)

$M_{0,n}$ can be combinatorially constructed by the tree structure of multiple $M_{0,4}$.

3.10 Compactification of Moduli Space

Definition 3.10.1 ()

The compactification $\overline{M}_{0,4}$ is $\overline{M}_{0,4} = \mathbb{P}^1$.
 $\overline{M}_{0,4}^{GIT} \cong (\mathbb{P}^1)^4//PGL_2$

4 Unsorted

Definition 4.0.1 (Deligne-Mumford Space)

Definition 4.0.2 (Compactification of Moduli Space)

Definition 4.0.3 ()

References

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https://pseudoholomorphic.fpark.tmu.ac.jp/akaho_a_crash_course_of_floer_homology_for_lagrangian