From Complex Analysis to Algebraic Geometry

Koji Sunami

May 29, 2025

Contents

1 Preliminary

2 Complex Analysis

Definition 2.0.1 *(Hypergeometric Function)*

Definition 2.0.2 *(Configuraiton Space)*

3 Moduli Space

Moduli space is a space generated by geometric quotient of some objects, and each point corresponds to an isomorphism class of objects. For example, for a Mobius strip M and a circle S^1, a natural projection $M \to S^1$ is a vector bundle with fiber $\mathbb{R}^1$, and the moduli space is $M/\mathbb{R}^1$.

Remark 3.1 *Moduli space is more than a set of points. $M/\mathbb{R}^1 \cong S^1$ is an isomorphism, and $M/\mathbb{R}^1$ is a moduli space, but S^1 is not a moduli space, and these two are distinct only because each point corresponds to an isomorphism class.*

Remark 3.2 *Moduli space is not a fiber bundle, but what is the difference from the fiber bundle exactly? The moduli space has each point as an isomorphism class, not as an actual fiber. There are few different varieties of vector bundles over circle S^1, which is annulus or mobius strip, whose fibers are both $\mathbb{R}^1$, but moduli space cannot distinguish them.*

Remark 3.3 *Etymologically, Moduli stands for "mod"uli.*

3.4 $M_{g,n}$ Moduli Space

Definition 3.4.1 *(Configuration Space)*

Let X be a topological space, then the configuration space $Conf_n(X)$ is defined as

$$Conf_n(X) = \{(z_1, z_2, \cdots, z_n) \in X^n | z_i \neq z_j \text{ for } i \neq j\}$$

Definition 3.4.2 *($M_{g,n}$ Moduli Space)*

The $M_{g,n}$ moduli space is GIT quotient of configuration space of $\mathbb{P}^1$ by $SL^2(\mathbb{C})$, which is a geometry of genus g with n-marked points of the base space $\mathbb{P}^1$. In particular,

$$M_{0,n} = Conf_n(\mathbb{P}^1)/SL^2(\mathbb{C})$$

Remark 3.5 *($M_{0,n}$ Calculation)*

- *($n = 3$)*
  Consider that Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ can put arbitary distinct 3-points z_1, z_2, and z_3 mapped to arbitray distinct 3-points w_1, w_2, and w_3.

  Since configuration space of 3-dimensions is a space of 3 independent distinct elements, each points could be arbitrarily mapped to any points of the space, which means by $SL^2(\mathbb{C})$ quotient, it'll be all identified. Hence,

  $$M_{0,3} = \{pt\}$$

- *($n = 4$)*

  Recall $SL^2\mathbb{C}$ fixed the 3-points, then the last 4th-point has some freedom, because kf we let $\lambda = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$, Cross ratio λ could take any values of $\mathbb{P}\backslash\{0, 1, \infty\}$. $\lambda \neq 0$ or $\lambda \neq \infty$ because all points z_i are distinct, and none of the numerator or denominator could be zero, and if $\lambda \neq 1$, $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = 1$, then $(z_1 - z_3)(z_2 - z_4) = (z_1 - z_4)(z_2 - z_3)$, and by *simple college algebra, it will be* $(z_1 - z_2)(z_3 - z_4) = 0$*, but it never happens since all points are distinct. Otherwise,* λ *could take any values. Hence,*

$M_{0,4} = \mathbb{P} \backslash \{0, 1, \infty\}$

- *(*$n = 5$*)*

  If we fix the mapped points w_1*,* w_2*, and* w_3 *depending on* z_1*,* z_2*, and* z_3*,* w_4 *is cross ratio depending on* z_4*, while* w_5 *is another cross ration by* z_5*, which means*

  - $\lambda = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$
  - $\mu = \frac{(z_1 - z_3)(z_2 - z_5)}{(z_1 - z_5)(z_2 - z_3)}$

  so we have two cross ratios λ *and* μ*, so* $M_{0,5}$ *is two parameters. Hence,*

  $M_{0,5} = \mathbb{P} \backslash \{0, 1, \infty\}$

Finally, notice indeed $dim(M_{0,n}) = n - 3$

Definition 3.5.1 *(*$M_{1,1}$ *Moduli Space)*
The very definition of $M_{1,1}$ *is a moduli space of algebraic curves of genus* 1 *of* 1*-marked points, but the question is that are all the algebraic curves of genus* 1 *of* 1*-marked points elliptic curves and vice versa? Yes, and this is, in fact, equivalently to that it's a moduli space of elliptic curves quotient by* j*-invariants. Another question is how to explicitly classify all the elliptic curves, and this could be done by moduli space of torus, and only to prove that torus is equivalent to some elliptic curve.*

$M_{1,1} = \mathbb{H}/SL_2(\mathbb{Z}) \cong \Lambda$ *where* $\mathbb{H}$ *is the upper-half plane.*

where $dim(M_{1,1}) = 1$

Or equivalently, $M_{1,1} = \{(E, p) |$ *genus 1 smooth curve,* $p \in E$ $\}/\sim$

Remark 3.6 *(*$M_{1,1}$ *Lattice and Curve Correspondence)*

In order to verify the previous definition. we'll start from the lattice correspondence with the elliptic curve.

- *(Step 1:)*
  Each lattice can be described by the standard form, which is

$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$

for some $\tau \in \mathbb{C}$

- *(Step 2:)*
  If E_τ *is a lattice, this* $\mathbb{C}/E_\tau$ *can be a domain of the Weierstrass* $\wp$ *function, which is in other words period, and Weierstrass* $\wp$ *function defines an elliptic function. To formulate, if we let* $Y = \wp'(z, \mathbb{C}/E_\tau)$ *adn* $X = \wp(z, \mathbb{C}/E_\tau)$*, we have the following identities:*

  $Y^2 = 4X^3 - g_2X - g_3$

  where g_2 *and* g_3 *are Eisenstein series.*

- *(Step 3: Legendre Transform)*

  What else do we have? This correspondence is actually not one-to-one, but it will be, if we take quotient of the set of elliptic curves by isomorphism classes by j*-invariant, that is equivalent to say that if elliptic curves have the same* j*-invariant, then they have isomorphism as algebraic varieties, and Legendre transform is the step to defien* j*-invariant. The previous elliptic function can be canonically transformed to the Legendre form*

  $y^2 = x(x-1)(x-\lambda)$ *where* $\lambda \in \mathbb{C}\backslash\{0, 1\}$

- *(Step 4:j-Invariant)*

  By using λ *arised from Legendre transform, if we let* j*-invariant as*

  $j(\lambda) = 256 \cdot \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}$

  $j(\tau) = j(\lambda(\tau))$

Note if two Weierstrass functions are distinct up to $SL_2(\mathbb{Z})$ *transformation,* g_2 *and* g_3 *are always distinct.*

Definition 3.6.1 *($M_{1,0}$ Moduli Space)*

Also, $M_{1,0}$ *is* $n = 0$*, and it's also moduli space of elliptic curves* $y^2 = x^3 + ax + b$*,*

For $n = 0$*,* $M_{1,0}$ *is* $M_{1,0} = M_{1,1}/Aut(C)$

Remark 3.7 *(Group Structure)*
Why the moduli space of elliptic curves are $M_{1,1}$ *instead of* $M_{1,0}$ *is because each elliptic curve has a group structure, or in other words, if the point of the elliptic*

curve is depending on X and Y coordinates, whose value depends on the position of the lattice E_τ. Each element of the group is a point of the elliptic curve, and of course the unit element O is. The unit element O is point at infinity.

Remark 3.8 *(Elliptic Curve Is Not Torus)*

Definition 3.8.1 *($M_{2,0}$ Moduli Space)*

$M_{2,0} = \mathbb{H}_2/Sp_4(\mathbb{Z})$

where $dim(M_{2,0}) = 3$

Remark 3.9 *(Tree Structure)*

$M_{0,n}$ can be combinatorially constructed by the tree structure of multiple $M_{0,4}$.

3.10 Compactification of Moduli Space

Definition 3.10.1 *()*

The compactification $\overline{M_{0,4}}$ *is* $\overline{M_{0,4}} = \mathbb{P}^1$.
$\overline{M_{0,4}}^{GIT} \cong (\mathbb{P}^1)^4//PGL_2$

4 Unsorted

Definition 4.0.1 *(Deligne-Mumford Space)*

Definition 4.0.2 *(Compactification of Moduli Space)*

Definition 4.0.3 *( )*

References

[1] Akaho Manabu(2003), *A Crash Course of Floer Homology for Lagrangian Intersections*, https://pseudoholomorphic.fpark.tmu.ac.jp/akaho_a_crash_course_of_floer_homology_for_lagrangian_