Analytic Number Theory and Riemann Zeta Function – Winter 2018 Study Updated

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2018-05-17

Abstract. Through my study about Summation formula with Bernoulli function, I was intrigued to study analytic number theory, namely, the calculation of the zeta function. In this study, I learned the proof of prime number theorem.

The distribution of prime number is inconsistent from the local viewpoint, while consistent from the global viewpoint.

Contents

1 Analytic Functions over $\mathbb{C}$

1.1 Overview of Complex Analysis

Since the early age of mathematics history, the significance of complex number in mathematics are often indicated, since complex number can represent the symmetrical property of mathematical objects. One distinguishable example is cubic polynomial equation is solved by roots of unity, which takes advantage of the property of polynomial in complex number representation. Also, Euler points out the famous formula $e^{i\theta} = \cos\theta + i\sin\theta$, which indicates the symmetricity of the trigonometry. Though the geometrical representation of complex number is two dimensional vector space, one of our main missions is to reduce the multi-dimensional calculation with sophisticated technique of complex analysis.

We define a function f whose differentiation is independent from the limit of path, and there exists unique $\frac{df}{dz}$. Thereby we can treat it as a single variable differentiation.

$$\frac{df}{dz} = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \tag{1}$$

This function f is called as holomorphic. In this definition, the differentiation of the function has an exact form(note: complex surface has two dimension), and the closed path integral will be zero, if the domain is simply connected. This is called as Cauchy's integral theorem. Fortunately, the most important elementary function, such as polynomial and trigonometric functions are all holomorphic.

$$\oint_C f(z)dz = 0 \tag{2}$$

Therefore, all we have to care is the singular points of the function inside of the path. Assuming the function is C^∞ differentiable, the solution of the integral is provided by its residue[Residue Theorem]. To sum it up, I introduced the overview of Complex calculus.

1.1.1 Note

There are also myriad of non-holomorphic functions, but their calculation don't apply complex analysis method, but they are rather real analytic.

1.2 Gamma Function

The gamma function is a generalization of factorial $n!$, and if the variable is natural number, it is defined as:

$$\Gamma(n+1) = n! \tag{3}$$

where $n \in \mathbb{N}$

Euler and Riemann extended the domain of gamma function into the entire complex plane except for few singular points. One reason is to apply gamma function with the method of real/complex analysis. In my study, I use gamma function to study the properties of Riemann zeta function. The gamma function is defined as below:

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx \\ \text{where } s \in \mathbb{C} \tag{4}$$

Gamma function is one of the most important functions in analysis, and its application varies. For example, the transformation of the hyper-geometric function, which is a generalization of trigonometric, elliptic functions, which may contain factorials of complex number. Also, analytical approach is strong to the numerical approximation and to probability.

1.3 Riemann zeta function

Riemann zeta function is also the most famous function, especially in analytic number theory. It represents some arithmetic functions and, in particular, the distribution of prime number which I will discuss here. Below is the definition of Riemann zeta function.

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n^s} \\ \text{where } s \in \mathbb{C} \tag{5}$$

An interesting property of the function is that there is a prime-number-related representation of Riemann zeta function as below, and this property is the trigger to study the relation of the prime number and Riemann zeta function.

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \tag{6}$$

Also, some points out the symmetricity of Riemann zeta function with the line $s = \frac{1}{2} + it$.

$$\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}\zeta(s) = \Gamma(\frac{1-s}{2})\pi^{\frac{s-1}{2}}\zeta(1-s) \tag{7}$$

The whole property of Riemann zeta function has not yet been uncovered. For example, Riemann states all zero points of Riemann zeta function is over $s = \frac{1}{2} + it$ [Riemann Hypothesis(RH)], which is unsolved.

1.4 Theta function

The theta function is defined over

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z} \tag{8}$$

Theta function has a significant property, which is called Jacobi identity.

$$\begin{gathered}\vartheta(\frac{z}{\tau};\frac{-1}{\tau}) = \alpha\vartheta(z;\tau) \\ \text{where } \alpha = \sqrt{-i\tau}e^{\frac{\pi}{\tau}iz^2}\end{gathered} \tag{9}$$

1.4.1 Modular Form

We call a function has a modular form iff the function satisfies:

$$\begin{gathered}\exists k \text{ s.t.} \\ f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \\ \text{where } a,b,c,d \in \mathbb{Z},\ ad-bc = 1\end{gathered} \tag{10}$$

Let $f \in Aut(H,H)$ be

$$\begin{gathered}f: H \to H \\ z \mapsto \frac{az+b}{cz+d} \\ \text{where } a,b,c,d \in \mathbb{Z},\ ad-bc = 1\end{gathered} \tag{11}$$

We can say that

$$Aut^{al}_{\mathbb{Z}}(H,H) \cong PSL(2,\mathbb{Z}) \tag{12}$$

where

$$PSL(2,\mathbb{Z}) \cong SL(2,\mathbb{Z})/\{I,-I\}$$

PSL is a short for Projective Simplectic Group. Note that the domain is defined by the upper half plane. For example, scalar matrix $-I$ represents $\frac{-az+0}{0z-d} = \frac{az+0}{0z+d}$. Hence, $-I$ represents the same as the identity matrix I. Now, all matrix $A \in PSL(2,\mathbb{Z})$ will be generated from S and T iff

$$\begin{gathered}I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\end{gathered} \tag{13}$$

$$\begin{aligned} f(u+1) &= f(u) \\ f(\frac{-1}{u}) &= u^k f(u) \end{aligned} \tag{14}$$

Therefore, all the operations that are related to a modular group are a combination of S and T, namely, periodicity and inverse. Now, theta function has a modular form if $z = 0$.

$$\begin{aligned} \vartheta(0;\tau+2) &= \sum_{n=0}^{\infty} e^{\pi i n^2(\tau+2)} \\ &= \sum_{n=0}^{\infty} e^{\pi i n^2\tau + 2\pi i n^2} \\ &= \sum_{n=0}^{\infty} e^{\pi i n^2\tau} \\ &= \vartheta(0;\tau) \end{aligned} \tag{15}$$

Also,

$$\vartheta(0;\frac{-1}{\tau}) = \sqrt{-i\tau}\,\vartheta(0;\tau) \tag{16}$$

Hence, theta function has a modular form when if $z = 0$.

2 Riemann Zeta Function

2.1 Gamma Function and Symmetry of Zeta over 1/2

The half formula of gamma function can represent the relationship between the zeta functions with different values.

$$\Gamma(\frac{s}{2}) = \int_0^{\infty} u^{\frac{s}{2}-1} e^{-u} du \tag{17}$$

Let u be $u = \pi n^2 x$

$$\begin{aligned} \Gamma(\frac{s}{2}) &= \int_0^{\infty} (\pi n^2 x)^{\frac{s}{2}-1} e^{-\pi n^2 x} (\pi n^2 dx) \\ &= \pi^{\frac{s}{2}} n^s \int_0^{\infty} x^{\frac{s}{2}-1} e^{\pi n^2 x} dx \end{aligned} \tag{18}$$

$$\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}n^{-s} = \int_0^\infty x^{\frac{s}{2}-1}e^{\pi n^2 x}dx$$

If we sum up the equation for all n,

$$\begin{aligned}\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}\zeta(s) &= \Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}\sum_{n=1}^{\infty}n^{-s} \\ &= \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{\pi n^2 x}dx \\ &= \int_0^\infty x^{\frac{s}{2}-1}\omega(x)dx\end{aligned} \tag{19}$$

By using the modularity of modular form, $\omega(x^{-1})$ will be transformed with the inverse formula as:

$$\omega(x^{-1}) = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2} \tag{20}$$

we will show

$$\begin{aligned}\int_0^1 x^{\frac{s}{2}-1}\omega(x)dx &= \int_\infty^1 x^{\frac{s}{2}-1}(\sqrt{x}\omega(x^{-1}) + \frac{\sqrt{x}}{2} - \frac{1}{2})(-x^2dt) \\ &= \int_\infty^1 -t^{\frac{-s}{2}-\frac{1}{2}}\omega(t)dt + \frac{1}{2}\int_1^\infty (t^{\frac{-s}{2}-1} - t^{\frac{-s}{2}-\frac{-1}{2}})dx \\ &= \int_1^\infty t^{\frac{-s-1}{2}}\omega(t)dt + \frac{1}{s(s-1)}\end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned}\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}n^{-s} &= \int_0^1 x^{\frac{s}{2}-1}e^{\pi n^2 x}dx + \int_1^\infty x^{\frac{s}{2}-1}e^{\pi n^2 x}dx) \\ &= \int_1^\infty (t^{\frac{-s-1}{2}} + t^{\frac{s}{2}-1})\omega(t)dt + \frac{s(s-1)}{2}\end{aligned} \tag{22}$$

Let

$$\begin{aligned}\xi(s) &= \Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}n^{-s} \\ &= \int_1^\infty (t^{\frac{-s-1}{2}} + t^{\frac{s}{2}-1})\omega(t)dt + \frac{s(s-1)}{2} \\ &= \Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}n^{-s} \\ &= \xi(1-s)\end{aligned} \tag{23}$$

This function is symmetry to the line $s = \frac{1}{2} + it$ over the complex plane where $t \in \mathbb{R}$.

3 Prime Number Theorem

3.1 Background and Purpose

We want to count how many of prime numbers exist in a specified domain. Let $\pi(x)$ be representing the distribution of prime number, so that the $\pi(x)$th number will be approximately x. We can write it with formality as below:

$$\begin{aligned}\pi(x) &= \sum_p{}' 1 \\ &= \begin{cases} \sum_p 1 & (x \notin \{prime\}) \\ \left(\sum_p 1\right) + \frac{1}{2} & (x \in \{prime\}) \end{cases}\end{aligned} \tag{24}$$

Note that $\pi(x)$ uses a Greek letter similar to the ratio of circumference. However, there is no relation between the two. Since the value of $\pi(x)$ changes where x is integer, $\pi(x)$ is also called as a step function.

Gauss and Chebyshev discovered a simple integration that approximates $\pi(x)$ as below.

$$\pi(x) \sim Li(x) \tag{25}$$

where

$$\begin{aligned}Li(x) &= \int_2^x \frac{dt}{\ln t} \\ &= \frac{x}{\ln x} + O(x^{\frac{1}{2}})\end{aligned} \tag{26}$$

The slope of the function will gradually become gentle, but the value of function goes to infinity as $\lim_{x\to\infty} Li(x) = \infty$, because there are infinite amount of prime numbers. Now, because of gentle slope, the density of prime number becomes, on average, very thin, if we increase the domain.

For example, Miller-Rabin primality test is a probability method to judge if a chosen number is prime or not, and this algorithm is used for RSA encryption system, and it requires huge prime numbers for its security purpose. However, when a natural number is picked up randomly from the domain, the probability of picking up a prime number decreases, as we choose a larger number. So it is hard to estimate the prime number, and it becomes hard to generate an arbitrary prime number. Therefore, to cope with this issue, Miller-Rabin primality test's algorithm is based on Riemann Hypothesis, which is more advanced method.

Also, the precise distribution of prime number is required in pure mathematics. Twin prime conjecture states there are infinite amount of sets of prime number $[p, p+2]$. However, 2 is trivial if a huge number is considered.

4 Riemann's Prime Number Theorem

4.1 Logarithm of Zeta Function

Riemann proposed the fixed formula that explains the distribution of prime number. He used Riemann zeta function. First of all, we prove the following formula.

$$\frac{1}{s}\ln\zeta(s) = \int_1^\infty \Pi(x)x^{s-1}dx \tag{27}$$

Now, let us begin the transformation of Riemann zeta function. First of all, the $\ln|1+x|$ will be written by analytic function with Taylor exponential.

$$\ln|1+x| = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}x^k \tag{28}$$

if $x = p^{-s}$,

$$\ln|1-p^{-s}| = \sum_{k=1}^{\infty}\frac{1}{k}p^{-ks} \tag{29}$$

$$\begin{aligned}
\ln\zeta(s) &= \ln\prod_p\frac{1}{1-p^{-s}} \\
&= -\sum_p \ln(1-p^{-s}) \\
&= -\sum_p\sum_{k=1}^{\infty}\frac{1}{k}p^{-ks} \\
&= -\sum_{k=1}^{\infty}\frac{1}{k}\sum_p p^{-ks} \\
&= -\sum_{k=1}^{\infty}\frac{1}{k}\sum_p\int_{p^k}^{\infty}x^{-s}dx \\
&= \int_1^\infty \Pi(x)x^{-s-1}dx
\end{aligned} \tag{30}$$

Where, $\Pi(x)$ function is defined as:

$$\Pi(x) = \sum_{n=1}^{\infty}\mu(n)\pi(x^{\frac{1}{n}}) \tag{31}$$

Therefore, we get the following formula.

$$\frac{1}{s}\ln\zeta(s) = \int_1^\infty \Pi(x)x^{-s-1}dx \tag{32}$$

4.1.1 Möbius Transformation and $\Pi(x)$ function

For convenience, we define another function $\Pi(x)$ to relate $\pi(x)$ for the sake of calculation. $\Pi(x)$ is a Möbius transformation of $\pi(x)$, so we can say there also exists the inverse formula of $\Pi(x)$ function.

$$\Pi(x) = \sum_{n=1}^{\infty} \mu(n)\pi(x^{\frac{1}{n}}) \tag{33}$$

$$\pi(x) = \sum_{n=1}^{\infty} \mu(n)\Pi(x^{\frac{1}{n}}) \tag{34}$$

4.2 Mellin Transformation

Mellin Transformation is a multiplicative version of both side Laplace transformation, and the following relationship exists:

$$M(f(\ln t))(s) = B(f(t)(s) \tag{35}$$

where

$$B(f(t))(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

because if we define u as $u = e^t$,

$$\begin{aligned} B(f(t))(s) &= \int_{-\infty}^{\infty} f(t)e^{-st}dt \\ &= \int_{0}^{\infty} f(\ln u)u^{-s}(\frac{1}{u}du) \\ &= \int_{0}^{\infty} f(\ln u)u^{-s-1}du \\ &= M(f(\ln t))(s) \end{aligned} \tag{36}$$

Now, we can deduct $\Pi(x)$ with the inverse transformation of Mellin transformation. Though $\Pi(x)$ is a step function, we can apply the transformation.

$$\begin{aligned} \Pi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c-i\infty} \frac{1}{s} \ln\zeta(s)x^{-s-1}ds \\ = li(x) - \sum_{\rho} li(x^{\rho}) - \ln 2 - \int_{0}^{\infty} \frac{dt}{t(1-t^2)\ln t} \end{aligned} \tag{37}$$

4.3 Prime Distribution Formula

If we apply the inversion formula of Möbius transformation to the $\Pi(x)$, we will get the distribution of prime number.

$$
\begin{aligned}
\pi(x) &= \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \left(li(x^{\frac{1}{m}}) - \sum_{\rho} li(x^{\frac{\rho}{m}}) - \ln 2 - \int_0^{\infty} \frac{dt}{t(1-t^2)\ln t} \right) \\
&= \sum_{m=1}^{\infty} \left(\frac{\mu(m)}{m} Li(x^{\frac{1}{m}}) - \sum_{\rho} Li(x^{\frac{\rho}{m}}) \right)
\end{aligned} \tag{38}
$$

where

$$
li(x) = \int_0^{\infty} \frac{1}{\ln t} dt = \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} \frac{1}{\ln t} dt + \int_{1+\varepsilon}^{\infty} \frac{1}{\ln t} dt \right) \tag{39}
$$

5 Riemann Hypothesis

5.1 Statement

One of the most important properties of Riemann zeta function is a calculation of zero points, whose method has not yet been fully discovered. As I stated in Chapter 1, the significant property of holomorphic function in the complex analysis is its singular points. We usually calculate the zeta function with its logarithm, namely $\ln \zeta(s)$, and the singular points of the function corresponds to the zero point of Riemann zeta function. Thereby, we can identify the all singular points of the logarithmic function in the closed path to calculate the integral. However, mathematicians realized it is difficult to find all of the non-trivial zero points with pure calculus.

Riemann states that the non-trivial zero points can only exists over $s = \frac{1}{2} + it$ where $t \in \mathbb{R}$. It is based on one of his investigations in zeta calculation. The following formula describes the symetricity of Riemann zeta function over $s = \frac{1}{2} + it$ plane.

Since

$$
\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}\zeta(s) = \Gamma(\frac{1-s}{2})\pi^{\frac{s-1}{2}}\zeta(1-s)
$$

$$
\zeta(s) = \chi(s)\zeta(1-s) \tag{40}
$$

where

$$
\chi(s) = \frac{\Gamma(\frac{1-s}{2})\pi^{s-\frac{1}{2}}}{\Gamma(\frac{s}{2})} \tag{41}
$$

5.2 Conclusion

On the other hand, we began to use algebraic operation to the functions. For example, elliptic function and theta function are widely used, but it is unable to calculate the functions directly. Our attempt is to calculate the discrete points, such as half period point, third period points, or rational period points. Fortunately, Riemann zeta function is also transformed into the algebraic equation of elliptic function and theta function.

The application of the distribution of prime number are many in both of pure and applied mathematics. It is used to describe the spectra of atomic energy level in quantum mechanics. Also, an evidence is pointed out that it is also used for the solution of the differential equation of Navier Stokes. The theorem of Navier Stokes is used to solve the three-dimension fluid dynamics with viscosity is considered. Moreover, Miller- Rabin primality test is used to estimate a huge prime number, which is based on extended Riemann Hypothesis.

On the other hand, the Grand Riemann Hypothesis (GRH) is important to apply the similar result to study the property of L function. One of the most important L function is Hasse Weil L function. Also, extended Riemann Hypothesis(ERH) is important for Miller-Rabin primality test.

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