Naïve Intro to Langlands Program

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Contents

We use this math

- Number theory
- Lie Algebra Rep & Algebraic Group
- Algebraic Geometry

Lie Algebra Representation

Definition

- $\rho : \mathfrak{g} \to gl(n, \mathbb{C})$
- $[X, Y] = XY - YX$: Lie bracket

Example

$$\begin{aligned} ad(Y) : \mathfrak{g} &\to gl(\mathfrak{g}) \\ X &\mapsto [X, Y] \end{aligned} \tag{1}$$

Root Space Decomposition

If $\mathfrak{g}$ is semi-simple, $\mathfrak{g}$ has a root space decomposition with a root system Φ:

Definition

- $\Delta \subset \Phi \subset H \subset H \otimes_{\mathbb{Z}} \mathbb{C}$: $|\Phi| < \infty$, Δ=basis of H.
- $\mathfrak{g} = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$
- $\Phi \subset \Lambda_\Phi \subset \Lambda$: root/weight lattice
- W is a Weyl group $|W| < \infty$.

Proposition

Semisimple Lie algebra can be classified to the following

- A_n, B_n, C_n, D_n
- E_6, E_7, E_8, F_4, G_2.

Character Formula

Definition

- $X^*(\mathbb{G}_m, \mathbb{C})$
- $\mathbb{C}[X^*(\mathbb{G}_m, \mathbb{C})] = Rep(G)$
- $\mathbb{G}_m$ is a multiplicative group.

Definition

For a Lie group G over F, the representation ring $Rep(G)$ is $Rep(G) = \mathbb{C}[X^*(T)]^W$

Classical Class Field Theory

Only abelian Galois extension!

- $Gal(\overline{\mathbb{Q}}/\mathbb{Q})^{ab} \leftrightarrow Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p)$
- $\mathbb{Z}_p = lim \mathbb{Z}/\mathbb{Z}_{p^n}$
- For ex, $\mathbb{F}_{p^n} = \mathbb{F}_p[x_1, x_2, ..., x_{n-1}]$ could be a field.

Example

- If $p = 7$
- $5 = (5)$.
- $8 = (1, 1)$.
- $25 = (3, 4)$.
- $57 = (1, 1, 1)$.

Example

- Roots of Polynomial f of $deg(f) = n$
- Shimura-Taniyama conjecture

Adele Ring

Definition

- $\mathbb{Q}_p \subset \mathbb{R}$
- $\mathbb{Z}_p \subset \mathbb{Q}_p$
- $\mathbb{F}_p = p$
- $ch(\mathbb{Z}_p) = 0$

Definition

- $\mathbb{A}_{\mathbb{Q}} = \prod \mathbb{Q}_p \times \mathbb{R}$
- $G(\mathbb{A}_{\mathbb{Q}}) = \coprod G(\mathbb{Q}_p) \coprod G(\mathbb{R})$

Algebraic Group

Definition

An algebraic group is a functor $\mathscr{G} : Alg_k \to Gp$.
For $\mathscr{G} = GL_n$,

- $G = GL_n(F)$ where $F = \mathbb{Q}_p$).
- $K^{\mathfrak{o}} = GL_n(\mathfrak{o})$ where $\mathfrak{o} = \mathbb{Z}_p$).
-
- G is a totally disconnected, profinite group.
- $K^{\mathfrak{o}} \subset G$ is a maximal compact subgroup.
-
- F is an non-archimedean field
- $\mathfrak{o} \subset F$ is a ring of integer.

Proposition

- $K^{\mathfrak{o}} = GL(n, \mathfrak{o})$ *be a maximal compact subgroup of* G.
- $\mathfrak{o} \subset F$ *is a complete discrete valuation ring,*

Neighborhood

Definition

- $K(N) = \{g \in K^o | g \equiv 1 \bmod \mathfrak{p}^N\} \subset G$, $\forall N \in \mathbb{N}$.
- $\{K(N)\}_{N \in \mathbb{N}}$ forms a basis of neighborhood of the identity.

Spherical Hecke Algebra

Definition

$H_{K^\circ} = \{\phi : G \to \mathbb{Z}\}$ where ϕ is K°-invariant, compactly supported, locally constant is a vector space.

Moreover, H_{K° has a ring structure.

Definition

A convolution $(\phi \star \psi)$ is a multiplication of H_{K°.
Also, the convolution $\phi \star \psi = \psi \star \phi$ is commutative.

Proof.

- $(\phi \star \psi)(g) = \int_G \phi(gx^{-1})\psi(x)dx$
- $\pi(\phi)v = \int_G \phi(g)\pi(g)vdv,$
- $\pi(\phi \star \psi) = \pi(\phi) \circ \pi(\psi).$

$\square$

Spherical Hecke Algebra

In conclusion,

Proposition

Spherical Hecke Algebra is a vector space H_{K^0} but also a commutative ring.

Philosophy of Torus and Borel Subalgebra

Proposition

- *T decides root system.*
- *B decides positive root.*

Definition

Positive Weyl chamber
$P^+ = \{\lambda \in X_*(T) | < \lambda, \chi > \geqq 0 \text{ for every } \chi \in \Phi^+\} = \{\lambda \in X_*(T) | < \lambda, \chi > \geqq 0 \text{ for every } \chi \in \Delta\}$
half-sum of positive roots ρ is defined by $\rho \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$, and $2\rho = \Sigma_{\chi \in \Phi^+} \chi$ in $X^*(T)$.

Character

Definition

Let $T \subset G$ be a torus as a subgroup of a reductive group G. We define the character and the cocharacter as

- $X^{\cdot}(T) = Hom(T, \mathbb{G}_m)$
- $X_{\cdot}(T) = Hom(\mathbb{G}_m, T)$

Definition

Dual Lie Group $\hat{G}$ be a complex dual of G, and $\hat{T}$ is the dual torus of T and $\hat{\Phi} = \Phi(\hat{G}, \hat{T})$. The their root data are

- $\hat{G} : (X_*(\hat{T}), X^*(\hat{T}), \hat{\Phi}, \hat{\Phi^\vee}),$
- $G : (X_*(T), X^*(T), \Phi, \Phi^\vee).$

Cartan decomposition

Definition

$G(F) = \coprod_{\lambda \in P^+} K\lambda(\omega)K$ where $\lambda \in X_*(T)$ where G is semisimple. Note that $(\lambda + \mu)(\omega) = \lambda(\omega) + \mu(\omega)$.

Hence basis of $C(G(F)//K)$ is $c_\lambda = 1_{K\lambda(\omega)K}$ for all $\lambda \in P^+$, and we have a following formula:

$c_\lambda \star c_\mu = \Sigma_{\nu \in P^+} d_{\lambda,\mu}(\nu) c_\nu = c_{\lambda+\mu} + \Sigma_{\nu < \lambda+\mu} d_{\lambda,\mu}(\nu) c_\nu$ where $d_{\lambda,\mu}(\nu) \in \mathbb{Z}$, and $d_{\lambda,\mu}(\nu) = \#\{(i,j) | \nu(\omega) \in x_i y_j K\} \in \mathbb{Z}$.
In particular, if $G = T$, then $c_\lambda \star c_m u = c_{\lambda+\mu}$.

Iwasawa decomposition

Definition

$T \lneqq B \lneqq G$, $N = R_v(B)$ be a unipotent radical of B.

- $G(F) = B(F)K$;
- $B(F) \bigcap K = (T(F) \bigcap K)(N(F) \bigcap K)$;
- $T(F) \bigcap K \lneqq T(F)$ is maximal compact.

As a result, $G(F) = T(F)N(F)K$

Definition

Haar measure dg on $G(F)$ is decomposed to $dg = \delta_B(t)dtdndk$ with $dk(K) = 1 = dn(N(F) \bigcap K)$.

Construction of Classical Satake Isomorphism

Proposition

$0 \to T(\mathscr{O}_F) \to T(F) \xrightarrow{\gamma} X_*(T) \to 0$

Proposition

$T(\mathscr{O}_F)/T(F) \cong X_*(T)$

The SES naturally induces the isomorphism where the left hand side is Spherical Hecke algebra.

Modular Quasi-Character

Definition

Modular Quasi-Character

$$\begin{aligned} \delta_B : B(F) &\to \mathbb{R}^{>0} \\ b &\mapsto |det_{\mathfrak{b}}(b)|_{\mathfrak{p}} \end{aligned} \tag{2}$$

Example

If $t = \mu(\omega) \in T(F)$ for $\mu \in X_*(T)$,

$$\begin{aligned} \delta_B(t)^{1/2} &= |det(ad(t)|Lie(N))|_{\mathfrak{p}}^{1/2} \\ &= |2\rho(t)|_{\mathfrak{p}}^{1/2} \\ &= |\omega^{<\mu,2\rho>}|_{\mathfrak{p}}^{1/2} \\ &= q^{-<\mu,\rho>} \end{aligned} \tag{3}$$

where $q = |\mathscr{O}_F/\mathfrak{p}|$.

Description of Satake Morphism

Proposition

$$\begin{aligned} Sf : T(F) &\to \mathbb{C} \\ t &\mapsto \delta_B(t)^{1/2} \int_{N(F)} f(tn)dn \end{aligned} \tag{4}$$

for $f \in {C_c}^{\infty}(G(F)//K)$.

Hence

Proposition

$$\begin{aligned} S : {C_c}^{\infty}(T(F)/T(F) \bigcap K) &\cong \mathbb{C}[X_*(T)] \\ f &\mapsto Sf \end{aligned} \tag{5}$$

Classical Satake Isomorphism

Proposition

$H_T \cong R(\hat{G})$

where

- $H_T = {C_c}^{\infty}(G(F)//K)$
- $R(\hat{G}) = \mathbb{C}[X^*(T)]^{W(\hat{G},\hat{T})(\mathbb{C})}$

Classical Satake Isomorphism 2

There is a natural generalization of Satake isomorphism.

Proposition

$$H_T \cong R(\hat{G}) \tag{6}$$

$$\begin{aligned} H_T \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] &\cong R(\hat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \\ (H_T \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}])^W &= R(\hat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \\ S : H_G \to H_T \bigotimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \\ S : H_G \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] &\cong R(\hat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \\ \textit{For } \rho \in X^*(T), \textit{ then } S : H_G \otimes \mathbb{Z}[q^{-1}] &\cong R(\hat{G}) \otimes \mathbb{Z}[q^{-1}] \end{aligned} \tag{7}$$

Geometric Satake Isomorphism

Affine Grassmannian

If $G = GL_n$, then Affine Grassmannian $Gr = Gr_G$

Definition

$$Gr_{GL_n} : Alg_k \to Vect$$
$$R \mapsto \{\bigwedge \bigotimes_{k[[t]]} k((t))^n\} \quad n = dim(\bigwedge) \tag{8}$$

k-algebra R maps to the set of R-families of lattices in $k((t))^n$.

Definition

(Lattice Presheaf)
Let X be a presheaf over $\mathscr{O} = k[[t]]$

$$X : Alg_k \to R[[t]] - VSR \quad \mapsto \{\Lambda \bigotimes R((t))^n\} dn = dim(\Lambda) \tag{9}$$

Definition

Loop group

- $L^n X(R) = X(R[t]/t^n)$
- $LX(R) = X(R[[t]])$
- $L^+ X = lim_{\leftarrow}(L^n X)$

Affine Grassmannian and Loop Group

Proposition

Gr can be written by Loop Group! Let $\underline{G} = G \bigotimes \mathscr{O}$ *where* $\mathscr{O} = k[[t]]$*. The affine Grassmannian* $Gr_{\underline{G}}$ *can be identified with a fpqc quotient* $[L\underline{G}/L^{+}\underline{G}]$

Definition

(Weyl Algebra)

A Weyl algebra is a ring of diffenrential operators with polynomial coefficients, namely expressions of the form

$f_m(X)\partial_X^m + f_{m-1}(X)\partial_X^{m-1} + ... + f_0(X)$

where $f_k(X) \in F[X]$ is a polynomial over a field F for any k, and ∂_X is a derivative with respect to X, and this algebra is generated by X and ∂_X.

More generally, n-th Weyl algebra $A_n(X)$ is defined by n variables X_k and ∂_{X_k}, and each function $f_k(X_1, ..., X_n)$ is simply a n-variable polynomial.

In Weyl algebra, we have a Lie bracket $[x_i, \partial_{x_i}] = x_i\partial_{x_i} - \partial_{x_i}x_i = 1$, and for function f, $[\partial_{x_i}, f] = \partial f/\partial x_i$.

Definition

(D-Module)
D-module is simply a left-module M over a Weyl algebra $A_n(K)$ over a field K of characteristic zero, and it could be philosophically considered as a sheaf with a connection.
The slight generalization is the sheaf of differential oprators D_X, defined to be $\mathscr{O}_X$-algebra generated by the vector fields on X interpreted as derivations. Here, the left action $D_X \times M \to M$ is equivalent to specifying a K-linear map
$\nabla : D_X \to End_K(M)$ where $v \mapsto \nabla_v$ satisfying

- $\nabla_{fv}(m) = f\nabla_v(m)$
- $\nabla_v(fm) = v(f)m + f\nabla_v(m)$ (Leibniz rule)
- $\nabla_{[v,w]}(m) = [\nabla_v, \nabla_w](m)$

G-Bundle

Definition

- X – a smooth projective curve
- G – Lie group
- $Bun_G(X)$ – a moduli stack of G-bundles on X
- $QCoh(Bun_G(X))$ – a category of QCoh sheaves of $Bun_G(X)$.
- $D - Mod(Bun_G(X))$ – a category of D-modules on X.

Bundle

Example

$$\begin{tikzcd} Bun_G(X \cup \{x\}) \arrow[r] \arrow[d] & Bun_G(X) \arrow[d] \\ Bun_G(X) \arrow[r] & Bun_G(X - \{x\}) \end{tikzcd}$$

We consider an action

$$\begin{aligned} Bun_G(X \cup \{x\}) \times Bun_G(X) \to Bun_G(X) \\ (H, E) \mapsto H \cdot E = p_{2*}(H \otimes {p_1}^*(E)) \end{aligned} \tag{10}$$

for $H \in Bun_G(X \cup \{x\})$ and $E \in Bun_G(X)$.

Theory on Neighborhood Disk

Definition

- $D_x = Spec(\mathbb{C}[[t]])$ be a disk around x for some uniformizer t
- $D_x{}' = Spec(\mathbb{C}((t)))$ be a punctured disk.

Proposition

- $Bun_G(D_x{}') = G(O_x) \setminus G(K_x)/G(O_x)$
- $H_x = DMod(G(O_x) \setminus G(K_x)/G(O_x))$
- $H_x = DMod(Bun_G(D_x{}'))$

Global Theory

Proposition

$$DMod(Bun_G(X)) = DMod(G(\mathbb{C}(X)) \setminus \prod_{y \in X} G(K_y)/G(O_y))$$

Proposition

$$DMod(Bun_G(X)) = DMod(G(\mathbb{C}(X)) \setminus \prod_{y \neq X} G(K_y)/G(O_y) \times G(K_x))^{G(O_x)}.$$

Proposition

$$DMod(Bun_G(X)) = DMod(G(F) \setminus G(\mathbb{A}_F)/G(O_F))$$

Naïve Geometric Satake Correspondence

Proposition

$\mathbb{C}(G(k[[t]]) \setminus G(k((t)))/G(k[[t]])) \cong Rep(G^\vee)$
Notice that $\mathbb{Q}_p$ is replaced by $k((t))$.

Proposition

- $H_x = Rep(G^\vee)$
- $H = \bigotimes_{x \in X} H_x$
- $DMod(Bun_G(X)) = QCoh(Spec(H))$

Question: Structure of $Spec(H)$?

Six Operations

Definition

For $f : X \to Y$

- direct image $f_* : SH(X) \to SH(Y)$
- inverse image $f^*\mathscr{F} : f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ where $f^{-1}\mathscr{F}(U) = \mathscr{F}(f(U))$
- proper direct image $f_!$
- proper inverse image $f^!(F) = f^*G$ where $G \subset F$.
- internal tensor product $\bigotimes$
- internal hom Hom

Perverse Sheaf

Definition

Let X be a scheme, and for $x \in X$, let $j_x : \{x\} \to X$ be an inclusion morphism. Then, by the Grothendieck six operators induce

- $j_x{}^* : SH(\{x\}) \to SH(X)$
- $j_x{}^! : SH(\{x\}) \to SH(X)$

Here, We define a subscheme $Y \subset X$ such that $x \in Y$ if $H^{-i}(j_x{}^* C) \neq 0$ or $H^i(j_x{}^! C) \neq 0$, and they have real dimensions at most $2i$ for all i.

- $Perv(Gr) \subset D^b(Gr)$
  - abelian subcategory, category of perverse sheaves

Geometric Satake Correspondence

Proposition

$$K(Perv(Gr))\bigotimes_{\mathbb{Z}}\mathbb{C} \cong K(Rep(^L G))\bigotimes_{\mathbb{Z}}\mathbb{C} \tag{11}$$

where K is a Grothendieck Group.

Then this equivalence induces the equivalence by Tannakian duality.

$$Perv(Gr) \cong Rep(^L G) \tag{12}$$

Summary

Classical Satake Isomorphism

- Langlands Program
- Kazhdan-Lusztig formula

Geometric Satake Isomorphism

- Moduli of Higgs bundle
- Quantum Satake
- Shimura Variety

Langlands Program

- Shimura-Taniyama
  - want to study rep of $\mathbb{A}_F$
  - want to study $Gal(\overline{\mathbb{Q}}/\mathbb{Q})^{ab}$
- Langlands
  - want to study rep of $G(\mathbb{A}_F)$
  - want to study $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

Admissible Representation

Definition

$\pi : G \to GL(V)$ for $\mathbb{C}$-vector space V is called smooth if for all $0 \neq v \in V$, the stabilizer $\{k \in G | \pi(k)v = v\}$ is open.

Definition

π is admissible if π is smooth, and for any open subgroup $K \subset G$, V^K is finite dimensional.

Unramified Representation

Definition

G is unramified if G is quasi-split, and is split over an unramified finite degree extension of F,

Remark

- *G is quasi-split if* $\exists B \subset G$
- *G is split if* $\exists T \subset G$*, where* $T = \prod \mathbb{G}_m$.

Unramified Representation

Now, fix a hyperspecial subgroup $K \leqq G(F)$.

Definition

(π, V) is an unramified irreducible representation of $G(F)$ if

- $V^K \neq 0$.
- V^K is naturally a module over $C_c{}^\infty(G(F)//K)$ with associated action $\pi(f)v := \int_{G(F)} f(g)\pi(g)vdg$

Q: Is $G(F_v)$ unramified?

Cuspidal Representation

Definition

We let a complex-valued measurable function $f : G(\mathbb{A}) \to \mathbb{C}$ as :

- $f(\gamma g) = f(g)$, $\forall \gamma \in G(K)$
- $f(gz) = f(g)\omega(z)$, $\forall z \in G(Z_{\mathbb{A}})$
- $\int_{Z(\mathbb{A})G(K)\backslash G(\mathbb{A})} |f(g)|^2 dg < \infty$
- $\int_{U(K)\backslash U(\mathbb{A})} f(ug)du = 0$ for $g \in G(\mathbb{A})$, $U \subset G$ is a parabolic subgroup of G.

A cuspidal function generates a unitary representation of the group $G(\mathbb{A})$ on a complex Hilbert space V_f generated by the right translate of f. Here the action of $g \in G(\mathbb{A})$ on V_f is given by
$(g \cdot u)(x) = u(xg)$
$u(x) = \Sigma_j c_j f(xg_j) \in V_f$
A cuspidal representation of $G(\mathbb{A})$ is a pair (π, V_π) for some ω.
${L_0}^2(G(K) \setminus G(\mathbb{A}), \omega) = \hat{\bigoplus}_{\pi, V_\pi} m_\pi V_\pi \;\; m_\pi \in \mathbb{N}$

Langlands-Hecke Correspondence

Proposition

$$\{\textit{Unramified Rep}\} \leftrightarrow \{\textit{Hecke module category}\}$$
$$\{\textit{Rep of } (\pi, V_\pi) \textit{ of } G \textit{ generated } V^K\} \leftrightarrow \{{C_c}^\infty(G//K)\textit{-modules}\} \tag{13}$$

Hecke Character

${C_c}^\infty(G(F)//K) \to End_{\mathbb{C}}(V^K) \cong \mathbb{C}$ where $f \mapsto tr(\pi(f))$ called Hecke character of π.

Non-Abelian Hodge Theorem

Example

The Hodge theorem states that
$H^1(\Sigma_g, \mathbb{C}) = H^{1,0}(\Sigma_g) \bigoplus H^{0,1}(\Sigma_g)$
which is straight-forward to see that
$H^1(\Sigma_g, \mathbb{C}) \cong H^1(\pi_1(\Sigma_g), \mathbb{C}) = Hom(\pi_1(\Sigma_g), \mathbb{C})$
Here, the non-abelian Hodge decomposition is exactly the same but the replacement by $GL(n, \mathbb{C})$
$H^1(\Sigma_g, GL(n, \mathbb{C})) \cong H^1(\pi_1(\Sigma_g), GL(n, \mathbb{C})) =$
$Hom(\pi_1(\Sigma_g), GL(n, \mathbb{C}))/GL(n, \mathbb{C})$
and produces a holomorphic vector bundle and a Higgs field, i.e. an element of
$H^1(\Sigma_g, \mathscr{G}\mathscr{L}(n, \mathbb{C}) \bigotimes H^0(\Sigma_g, \mathscr{G}\mathscr{L}(n, \mathbb{C} \bigotimes K)$

Kazhdan-Lusztig Formula

- Springer Correspondence
- KL polynomial relates the failure of local Poincare duality and Schubert varieties.

Moduli Of Higgs Bundle

Definition

- Chern-Simon Gauge theory
- S-duality in string theory
- SYZ fibration in mirror symmetry
- Non-abelian Hodge theory

Non-Abelian Hodge Theorem

The abelian version states that

Proposition

$$
\begin{aligned}
H^1(\Sigma_g, \mathbb{C}) &\cong H^1(\pi_1(\Sigma_g), \mathbb{C}) \\
&= Hom(\pi_1(\Sigma_g), \mathbb{C})
\end{aligned}
\tag{14}
$$

Proposition

$$
\begin{aligned}
H^1(\Sigma_g, GL(n, \mathbb{C})) &\cong H^1(\pi_1(\Sigma_g), GL(n, \mathbb{C})) \\
&= Hom(\pi_1(\Sigma_g), GL(n, \mathbb{C}))/GL(n, \mathbb{C})
\end{aligned}
\tag{15}
$$

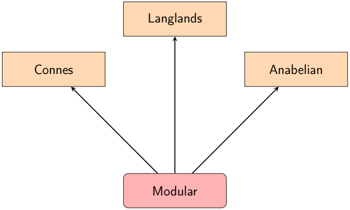
Quantum Satake Isomorphism

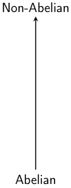
- used to compute minuscule Grassmannian
- Spinor variety $n = 5$ is idential to Apery' diffyq.
- Quadric

Shimura Variety

Geometric Satake isomorphism is used to construct Shimura variety.

My View of 21th Century Math





That's It!