Intro to Deformation Theory

Koji Sunami

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1 What is Deformation

1.1 integrability

Definition 1.1.1 *(Frobenius Theorem)*
Frobenius theorem claims the integrability of the distribution $N \subset M$ over a smooth manifold M, and for all the global section $X, Y \in X(TN)$ is closed under the distribution $[X, Y] \in X(TN)$, then the distribution is integrable. In other words, vector field is differential operator, which means that it's an element of the dgla.

Definition 1.1.2 *(Newlander-Nirenberg Theorem)*
Newlander-Nirenberg theorem claims the condition of the integrability of a smooth manifold M with an almost complex structure $J^2 = -Id$, and the theorem says the manifold has a vector bundle $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$, and each $T^{1,0}, T^{0,1} \subset TM \otimes \mathbb{C}$ As a distribution is integrable by means of Frobenius theorem, if Newlander-Nirenberg tensor $N_J(X, Y)$ vanishes, and this condition is equivalent to solving Maurer-Cartan equation but in a different language.

First, Newlander-Nirenberg tensor $N_J(X, Y)$ is defined by $N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$ for $X, Y \in TM$, and if this tensor is $N_J(X, Y) = 0$, the manifold is integrable, because this makes $[T^{1,0}, T^{1,0}] \subset T^{1,0}$, where

$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$. What is the problem?? In real smooth manifold, distribution $N = M$ satisfies Frobenius theorem trivially, thus naturally the manifold with almost complex structure is a real manifold, and thus $[X, Y] \in X(TM)$ for $X, Y \in X(TM)$. However, we don't guarantee that $[X, Y] \in X(T^{1,0} \cap TM)$ for $X, Y \in X(T^{1,0} \cap TM)$. However, it satisfies only if M is a complex manifold.

Definition 1.1.3 *(Frölicher-Nijenhuis Bracket)*
It makes graded Lie algebra. Similarly, Nijenhuis–Richardson bracket and the Schouten–Nijenhuis bracket or whatever.

Definition 1.1.4 *(Maurer-Cartan Equation)*
For a 1-form $\omega : TG \to \mathfrak{g}$, Maurer-Cartan equation is $d\omega + \omega \wedge \omega = 0$. By using $X, Y \in TG$, the equation is derived as follows. $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$ where $\omega([X, Y]) = [\omega(X), \omega(Y)]$, and $X(\omega(Y)) = Y(\omega(X)) = 0$ for left invariant X and Y.

Conversely, let's consider what properties Maruer-Cartan equation shows. Recall Lie group G is by definition a smooth manifold, and we discuss integrability condition of distribution of vector bundles TG. Recall Frobenius theorem says integrability condition is $d\omega \wedge \omega = 0$. If $X, Y \in ker(\omega) = D$, $\omega(X) = \omega(Y) = 0$, and $d\omega(X, Y) = 0$, which means $d\omega|_D = 0$. This $D \subset TG$ is a distribution, which satisfies Frobenius condition.

In general, a smooth manifold M has a global section of tangent bundle $\Gamma(TM)$ where we can define the Lie bracket to define Lie algebra, and its corresponding Lie group exists, for which we discuss Maurer-Cartan equation. This claim only applies locally.

2 Formal Deformation

2.1 Hopf Algebra

Definition 2.1.1 *(Algebra)*
We mean algebra by unitary associatve algebra (this means, linear algebra or Lie algebra is not algebra).

Definition 2.1.2 *(Coalgebra)*
Coalgebra C over K is a vector space with two structure morphisms

- $\Delta : C \to C \otimes C$
- $\epsilon : C \to K$

with the following identities

- $(id_C \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta$ *(coassociativity)*
- $(id_C \otimes \epsilon) \circ \Delta = id_C = (\epsilon \otimes id_C) \circ \Delta$

Definition 2.1.3 *(Hopf Algebra)*
Hopf algebra H is bialgebra with antipode, and bialgebra is coalgebra but also algebra. Antipode $S : H \to H$ is a k-linear map that commutes the diagram and

$S_{c_{(1)}} c_{(2)} = c_{(1)} S_{c_{(2)}} = \epsilon(c)1$ *for all* $c \in H$.

Definition 2.1.4 *(Representation of Hopf Algebra)*
Let A be Hopf algebra, and M and N are A-modules. Then $M \otimes N$ is also A-module, with

$a(m \otimes n) = \Delta(a)(m \otimes n) = (a_1 \otimes a_2)(m \otimes n) = (a_1 m \otimes a_2 n)$ *where* $m \in M$, $n \in N$, *and* $\Delta(a) = (a_1, a_2)$.

$a(m) = \epsilon(a)m$

$(af)(m) = f(S(a)m)$ *where* $f \in M^*$ *and* $m \in M$.

3 Misc

3.1 GIT

Definition 3.1.1 *(GIT and Reductive Group)*
In GIT, the question is if the quotient space has a nice property as a scheme, and it's known that a stable point $G \cdot x$ is not only a set theoretical point, but also a closed point in the Zariski topology of the scheme, but the problem is whether semistable point becomes a closed point in the Zariski topology of the scheme.

We claims if the quotient group G is reductive, we can relate semistale point to stable by relating $X//G$ and $Proj(R^G)$, since Hilbert theorem says R^G is finitely generated k-algebra if G is reductive, which makes $Proj(k[X]^G)$ scheme.

3.2 Hilbert Scheme

Definition 3.2.1 *(Hilbert Scheme)*

Hilbert Scheme is a scheme but also moduli space.
$Hilb(n)$ of $\mathbb{P}^n$ is a moduli space of closed subschemes of projective spaces.

$Hilb(n, P)$ is a moduli space of Hilbert polynomial P.

Let $\underline{Hilb}_{X/S} : (Sch/S)^{op} \to Sets$ be the functor sending a relative scheme $T \to S$ to the set of isomorphism classes of the set

$$\underline{Hilb}_{X/S}(T) = \left\{ \begin{array}{ccccc} Z & \xrightarrow{incl} & X \times_S T & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{=} & T & \longrightarrow & S \end{array} : Z \to T \text{ is flat} \right\} / \sim$$

Universality:
$H = Hilb(n, P)$ *has a universal subscheme* $W \subset X \times H$ *flat over* H *such that*

- *the fiber W_x over closed points $x \in H$ are closed subschemes of X. For $Y \subset X$ denote this point x as $[Y] \in H$.*
- *H is universal wrt all flat families of X having Hilbert polynomial P. That is, given a scheme T and a flat family $W' \subset X \times T$, there is a unique morphism $\phi : T \to W$ such that $\phi^* W \cong W'$.*

  Tangent Space:
  The tangent space at $[Y] \in H$ is given by the global sections of the normal bundle $N_{Y/X}$. That is,
  $T_{[Y]} H = H^0(Y, N_{Y/X})$

  Unobstrctedness of complete intersections:
  For local complete intersections Y such that $H^0(Y, N_{Y/X}) = 0$, the point $[Y] \in H$ is smooth. This implies that the deformation of Y in X is unobstructed.

  (coalgebra is smooth, if it has a lifting property. morphism $F(A) \to F(A/J)$ is means surjective.)

  Complete Intersection:
  The ideal of V is generated by precisely codim V elements.

Example 3.3 *(Hilbert Schemes)*

- *(Fano Schemes of Hypersurfaces)*
- *(Hilbert schemes of n-points)*
  X^n/S_n is the nice geometric interpretation where the boundary loci $B \subset H$ describing the intersection of points can be tought of parametrizing points along with their tagent vectors.

$X^{[n]} = Bl_{\Delta}(X \times X)/S_2.$

Alternatively,

$X^{[n]} = Hilb_X^P$ *if the Hilbert polynomial* P *is constant* $P(m) = n$ *for all* $m \in \mathbb{Z}$

- *(Degree d hypersurfaces)*

- *(Hilbert schemes of curves and moduli of curves)*

3.4 Symplectic Geometry

Definition 3.4.1 *(diff p-form and vector field)*

- *ex1:*

  $dx_i(\partial_j) = \delta_{ij}$

- *ex2:*
  $dx_{i_1} \wedge \cdots \wedge dx_{i_p}(\partial X_{j_1}, \cdots, \partial X_{j_p}) =$
  1 *if (*i_p *and* j_p *are even permutation)*
  -1 *if (*i_p *and* j_p *are odd permutation)*
  0 *(otherwise)*

- *ex3:*
  $\omega = d_x \wedge d_y$, $X = f(x)\partial_X$, $Y = g(x)\partial_X$,
  $\omega = f(x)g(x)(d_x(\partial_X) \wedge d_y(\partial_X) - d_y(\partial_X) \wedge d_x(\partial_X))$

Definition 3.4.2 *(interior product)*
$\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$

$$\iota_X \omega(X_1, \cdots, X_{p-1}) = \omega(X, X_1, \cdots, X_{p-1})$$

for any vector fields X *and* X_i.

Definition 3.4.3 *(Kähler manifold)*

- *(Symplectic Viewpoint)*
  $g(u, v) = \omega(u, Jv)$

- *(Complex Viewpoint)*
  Let h *be the Hermitian metric.*

$\omega(u,v) = Re(h(iu,v)) = Im(h(u,v))$

$g(u,v) = Re(h(u,v))$

- *(Riemannian Viewpoint)*

  Kähler manifold is Riemannian manifold X of dimension $2n$, whose holonomy is contained in $U(n)$.

  J preserves metric i.e. $g(Ju, Jv) = g(u,v)$, and J is preserved by parallel transport.

(Kähler potential)
Let ρ be a smooth real valued function.

$\omega = \frac{i}{2}\partial\bar{\partial}\rho$

is a Kähler potential.

References