Intro to Langlands Program

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December 2, 2022

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1. This is for KU/KSU talk on Dec 3 2022, I will introduce Langlands program in this talk. Langlands program may be the unified theory of mathematics, which gives us a universal perspective of mathematics, which is interesting, but requires variety of subjects in mathematics: Number theory, Algebraic Geometry, Lie Theory, Differential Geometry, Étale Cohomology Theory, Mathematical Physics etc, and we need a thorough study.

   Langlands program is a vast generalization of Shimura-Taniyama conjecture proposed in 1967, much earlier than the Fermat's Last Theorem was proven in 1993, but many of the problem in this program is unknown, and we have a room of research. Nevertheless, it was the huge interest of ours for it helps us describe many, which is for example, quantum field theory, mirror symmetry of moduli space of Higgs bundle, Kazhdan Lusztig conjecture, Quantum Cohomology etc.

   So what is Langland program in few sentences? I must start from Shimura-Taniyama conjecture. Shimura-Taniyama conjecture claims a correspondence of Galois representations or class field theory that studies the abelian extensions of local/global fields on one side, and geometric symmetry of complex upper half plane quotiened by $SL_2(\mathbb{Z})$, called a moduli space of modular curves, and Langlands program considers the non-abelian Galois extension, which is why Langlands is a generalization.

   For the study of the Langlands program, we use Satake isomorphism and Shimura variety as the key players in the project, and it will tell us a theory that connects number theory and geometry and Lie algebra.

2. (Satake Isomorphism)
   We prove classical Satake isomorphism in an explicit way from a spherical Hecke algebra that helps construct the Langlands dual group ${}^LG$. Its technical heart is Hecke algebra and affine Grassmannian and few of the decomposition algorithm helping the construction of classical Satake isomorphism, identifying Spherical Hecke algebra or double coset to the representation ring.

   On the other side, Geometric Satake correspondence is its generalization from the classical Satake isomorphism, where for the geometry X open neighborhood of each point D_x where $x \in X$ constitutes an isomorphism of Hecke algebra and representation ring but with a flavor of D-module category. We care geometric Satake isomorphism because of quantum cohomology and mirror symmetry.

3. (Shimura Variety)
   Shimura variety describes multi dimensional symmetry on a Hermitian symmetric domain D on a complex manifold if Galois group is one dimensional symmetry. Hermitian symmetric domain D is a complex manifold. Let $Hol(D)$ be a set of all automorphism of D, then $Hol(D)$ is a Lie group. We study D from curvature(geometry) perspective and Lie algebra perspective assuming that D is a semisimple Lie group. Moreover, the manifold can be considered as an algebraic variety because there is a natural functor from a category of compact riemann surfaces to a category of projective algebraic curve, which is an equivalence of category.

4. A modular curve is a particular case of Shimura variety. Let complex upper-half plane be Hermitian symmetric domain where a Lie group $SL_2(\mathbb{Z})$ acts on it. This $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$, which is a naturally defined arithmetic subgroup. Here we define a group action of $SL_2(\mathbb{Z})$, and we have a geometric quotient of the complex upper half plane by its orbits. The quotient is a Riemann surface. Non-trivial example of Hermitian symmetric domain is a Siegel upper-half plane H_g, a complex space of dimension g and the set of its automorphism makes $Sp_n(R)$, whose dimension is $g(g+1)/2$.

1 Lie Theory and Algebraic Group

1.1 Reviewing Commutative Algebra

Definition 1.1.1. *(Local Ring)*
Let R be a ring. R is a local ring if it has a unique maximal ideal $m \subset R$.

Definition 1.1.2. *(Localization Of Ring)*
Let R be a ring, let S be a multiplicative subgroup $S \subset R$. We define localization of R as

$$\begin{aligned} R &\to S^{-1}R \\ 1 &\mapsto \frac{1}{1} \end{aligned} \tag{1}$$

where addition and multiplication are defined as $\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$, and $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$. In particular, for some prime ideal $p \subset R$, we can let $S = R \subset p$ to define $S^{-1}R$.

Definition 1.1.3. *(Discrete Valuation Ring)*
Let R be a ring. R is Discrete Valution Ring if it is principal domain and it is a local ring.

Definition 1.1.4. *(Dedekind Domain)*
Let A be a ring. All non proper ideals are written by some product of prime ideals, as for $I \subset A$, $I = \prod_i p_i^{e_i}$ for some $e_i \in \mathbb{N}$.

Proposition 1.1.5. *(Dedekind Domain)*
Localization of Dedekind rings by a prime ideal is a DVR.

Definition 1.1.6. *(Integrally Closed Domain)*
Let R be a domain. R is integrally closed domain if for polynomial $f(x) \in R[X]$, their roots exist in its field of fraction.

Proposition 1.1.7. *(ICD and DVR)*
Let R be an integral closed domain, and let $\mathfrak{p}$ be a minimal prime ideal. Then the localization $R_\mathfrak{p}$ is a discrete valuation ring. The intersection of the local ring $\bigcap R_\mathfrak{p}$ is R itself.

Definition 1.1.8. *(Krull Dimension)*
Let R be a commutative ring. The set of all the prime ideals contained in R creates an ordered lattice structure by inclusion relations as $\mathfrak{p}_1 \subset ... \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n \subset R$ for some $n \in \mathbb{N}$, where $\mathfrak{p}_n$ is called a maximal ideal and $\mathfrak{p}_1$ is called a minimal ideal. The length of resolution $n \in \mathbb{N}$ is called Krull Dimension.

1.2 Reviewing Galois Theory And Field

1.3 Galois Theory And Field

Definition 1.3.1. *(Finite Galois theory)*
For any field K, an field extension is a field L defined by $L = K[X]/(f(X))$ for some polynomial $f(X) \in K[X]$, and particularly Galois extension is field extension which is separable and normal.

Field extension is separable if the polynomial $f(X)$ does not have a multiplicity of roots. Also, field extension is normal if L splits over K, mean that all roots of $f(X)$ is contained in L. But why do we call it "normal"? For a field extension L/K, let there be an intermediate extension E/K. We call it Galois if the corresponding group is the normal subgroup $Gal(E/K) \trianglelefteq Gal(L/K)$.

If L is a Galois extension of K, then we define Galois group as $Gal(L/K) = Aut_K(L)$. If Galois extension is finite, then Galois extension and its Galois group has a 1-to-1 correspondence.

Definition 1.3.2. *(Infinite Galois theory)*
Let L be a field extension of K then we have TFAE:

1. $L = \bigcup_i L_i$ *where L_i/K is a finite Galois extension.*
2. *L is the splitting field over K of a set of separable polynomials in $K[X]$.*
3. $L^{Aut(L/K)} = K$
4. *L/K is both separable and normal.*

If we satisify the above condition, we call L is a Galois extension of K. Particularly if the Galois group $Gal(L/K)$ is an infinite group, then the extension if infinite Galois extension.

Example 1.3.3. *(Infinite Galois extension)*

1. $L = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, ...)$
   Then the corresponding group is $Gal(L/\mathbb{Q}) = \prod_i \mathbb{Z}/2\mathbb{Z}$.
2. $L = \mathbb{Q}(\zeta_{p^\infty}) = \bigcup \mathbb{Q}(\zeta_{p^n}) = \bigcup L_i$,
   so $Gal(L/\mathbb{Q}) = \bigcup Gal(L_i/\mathbb{Q}) = \lim \mathbb{Z}/p^i\mathbb{Z}$.
3. $Gal(\mathbb{F}_p^{sep}/\mathbb{F}_p)$ *for some finite field* $\mathbb{F}_p$.
   The Galois group is isomorphic to $\hat{\mathbb{Z}} \cong \prod \mathbb{Z}_n$ *where* $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ *is cyclic abelian group. We call the Galois extension to the separable closure is called absolute Galois group.*
4. $Gal(\mathbb{Q}^{sep}/\mathbb{Q})$ *in a similar way, but the group structure is still unknown. However, according to Shafarevich's conjecture, it is a believe to be profinite group.*

Definition 1.3.4. *(Profinite Group)*
All Finite/Infinite Galois Group is a topological space, which is profinite group. Profinite group is compact Hausdorff and totally disconnected topological space, and its name "profinite group" comes from projective limit of finite groups.

Proposition 1.3.5. *(Topology of Galois Group)*

1. *All infinite Galois group is countable as a set, so its base is at most second countable.*
2. *For all infinite Galois group, all finite sets are closed.*
3. *From the previous claim, all infinite Galois group is locally connected.*

Definition 1.3.6. *(Galois Representation)*

$$\rho : G_{\mathbb{Q}} \to GL_n(F)$$

is called Galois Representation of dimension n, which is is a linear map where $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is an absolute Galois group, and F is any topological field. Topological field is an ordinary field where each inverse, multiplication, addition operations are continuous.

Particularly if F is an (finite) extension of $\mathbb{Q}_l$, we call ρ an l-adic Galois representation.

Proposition 1.3.7. *(Character Formula)*
Define the cyclotomic character of l as χ_l
$\chi_l : G_{\mathbb{Q}} \twoheadrightarrow Gal(\mathbb{Q}(\mu_{l^\infty}/\mathbb{Q})) \cong \mathbb{Z}_l^\times$

$$G_{\mathbb{Q}} \twoheadrightarrow \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times = GL_1(\mathbb{Q}_l)$$

Definition 1.3.8. *(Ramification)*
Let p be any prime number, and $\mathfrak{p}$ be a prime ideal lying on some field extension K. Then, we have a morphism
$D_{\mathfrak{p}} \to Gal(\mathbb{F}_{\mathfrak{p}}/\mathbb{F}_p)$

where $D_{\mathfrak{p}} = \{\sigma \in Gal(K/\mathbb{Q}) : \sigma(\mathfrak{p}) = \mathfrak{p}\}$ and $\mathbb{F}_{\mathfrak{p}} = O_K/\mathfrak{p}$ is the residue field at $\mathfrak{p}$. If the kernel of the morphism is trivial, theni σ_p generates $D_{\mathfrak{p}}$, and K is unramified at p. Furthermore, this idea can be naturally extended to the absolute extensions.

The morphism $D_p \to Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p)$

Example 1.3.9. *(l-adic Cyclotomic Character)*
Consider a natural morphism induces a character χ_l
$G_{\mathbb{Q}} \twoheadrightarrow Gal(\mathbb{Q}(\mu_{l^\infty})/\mathbb{Q}) = limGal(\mathbb{Q}(\mu_{l^n})/\mathbb{Q})$
Then, $G_{\mathbb{Q}} \twoheadrightarrow \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times$

and each extension $\mathbb{Q}(\mu_l^n)/\mathbb{Q}$ is unramified at all$p \neq l$.

The notion might be generalized to Galois category.

Proposition 1.3.10. *(Cardinality of Field)*

1. *Cardinality of $\mathbb{Q}$ is $\aleph_0$.*
2. *Cardinality of $\mathbb{F}_p$ is p, which is finite.*
3. *Let k be a field. If k is a finite field, the cardinality of algebraic closure $card(\bar{k}) = \aleph_0$*
4. *If k is a field of countable elements, then the cardinality of algebraic closure $card(\bar{k}) = \aleph_0$*

1.4 Reviewing Algebraic Geometry

Remark 1.4.1. *(Open Sets In Euclidean Topology)*
For an Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$, topological basis is an open ball B_ϵ. The open ball is always n-dimensional. Considering from that fact, Zariski topology related each geometry to a closed subset, since they don't have to be n-dimensional. For example, maximal ideal is related to a closed point, but if it would be open, it is quite weird.

Definition 1.4.2. *(Algebraic Variety & Scheme)*
For affine variety, let R be a noetherian ring. Affine variety is a subset of an algebraic set $\mathbb{A}^n$ such that where $Spec(R)$ the spectrum of prime ideal of R corresponds to the closed subset of the corresponding affine variety. Mainly, Affine variety satisfies:

1. *If two ideals $I\_1 \subset I_2$, then the corresponding closed geometries $V(I_1) \supset V(I_2)$ have reverse inclusion.*
2. *If an ideal $I = \bigcap \mathfrak{q}_i$ is the primary decomposition, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then $V(I)$ is an union $V(I) = \bigcup V(\mathfrak{p}_i)$.*
3. *$SpecmR$ is set of all closed points of R.*

For general definition of abstract variety and scheme, more to be added.

Definition 1.4.3. *(Section)*
In Zariski topology, we define section as:
$\mathscr{O}_{X,x} = \lim \Gamma(U, \mathscr{O}_X)$
In Étale topology, we define section as:
$\mathscr{O}_{X,\bar{x}} = \lim \Gamma(U, \mathscr{O}_U)$

Definition 1.4.4. *(Generic Points, Closed Points, And Irreducible Varierty)*
Let X be a scheme. A generic point is a point $x \in X$ where for some irreducible subscheme $U \subset X$, closure of the point generates the subscheme U as $\bar{\{x\}} = U$.

For example, $\mathbb{R}[X]$ be an algebraic variety, and a prime ideal $(x^2+1) \subset \mathbb{R}[X]$ corresponds to an irreducible variety. Let $Specm(\mathbb{R}[X])$ be a set of maximal ideals of $\mathbb{R}[X]$, and this is equivalent to the set of all closed points, since each prime ideal corresponds to a closed subset of the variety particularly a maximal ideals to a point.

Definition 1.4.5. *(Base Of Scheme)*
Let X and Z be a scheme. We call X as a scheme over Z if X can be considered to be scheme $(X, \mathscr{O}_X)$ together with a morphism $X \to Z$.

Let .

Can I make a scheme which is over Z? Since category of scheme allows fiber product, we may induce a morphism $X \times_Y Z \to Z$ for some scheme Z, and this morphism has a base Z. This is called change of base.

Definition 1.4.6. *(Weil Divisor)*
Let X be regular in codimension one, namely, every local ring $\mathscr{O}_X$ of X of dimension one is regular. If X is moreover noetherian integral separated scheme, then we can define a prime divisor, which is a closed integral subscheme $Y \subset X$ of codimension one.

Let Set be a category of all prime divisors of X and let Ab be a category of abelian groups. Consider a functor $Free : Set \to Ab$ such that for $\{Y_i\} \in Ob(Set)$ and $\{Y_i\} \mapsto \{\Sigma n_i[Y_i]\}$, and the domain is free abelian group denoted $DivX$.

Weil divisor Y is an element of the free abelian group $DivX$.

Remark 1.4.7. *(What is the meaning of Weil Divisor After All?)*
Recall an affine variety has 1-to-1 correspondence between closed algebraic sets and prime divisors. Also, each prime divisor has an inverse inclusion relationships. If we take $\mathfrak{p}_1$, $\mathfrak{p}_2$ be prime ideals as $\mathfrak{p}_1 \subset \mathfrak{p}_2$, then the corresponding varieties are containing one another but in an opposite way $V(\mathfrak{p}_2) \subset V(\mathfrak{p}_1)$.

For example, maximal ideals $\mathfrak{m} \subset R$ correspond to a point, and by Nullstellensatz, the minimal prime ideals represents irreducible proper subvariety of X, namely a subvariety of codimension 1. Furthermore, the localization of minimal prime ideal $S^{-1}R$ where $S = R - \mathfrak{p}$ is a discreate valuation ring, quite number theoretical. Also, intersection of all the minimal ideals of noetherian ring R is nilradical, so it's identical to (0) ideal through Nullstellensatz.

Proposition 1.4.8. *(Weil-Divisor Sequence)*
$0 \to A^\times \to K^\times \to \bigotimes_{ht(\mathfrak{p})=1} \mathbb{Z} \to 0$
$0 \mapsto a \mapsto ord_{\mathfrak{p}}(a)$
for $A = \{a \in K | ord_{\mathfrak{p}}(a) \geqq 0 \text{ all } \mathfrak{p}\}$
for $A_{\mathfrak{p}} = \{a \in K | ord_{\mathfrak{p}}(a) \geqq 0\}$
for $A^\times = \{a \in K | ord_{\mathfrak{p}}(a) = 0 \text{ all } \mathfrak{p}\}$

Similarly, for any open subscheme $U \subset X$,
$0 \to \mathscr{O}_X^\times \to \Gamma(U, K^\times) \to Div(U) \to 0$
is a left exact, and it is short exact sequence if X is regular.

Definition 1.4.9. *(Ramification In Algebraic Geometry)*
Let $f : X \to Y$ be a morphism of schemes. The support of quasi-coherent sheaf $\Omega_{X/Y}$ is called the ramificatio of locus of f and the image of the ramification locus, $f(Supp(\Omega_{X/Y}))$ is called the branch locus of f. If $\Omega_{X/Y} = 0$, we say that f is formally unramified, and if f is also of locally finite presentation we say f is unramified.

Definition 1.4.10. *(Picard Group)*
For a ringed space X over k. Picard group is a group $Pix(X)$, where each element is an invertible sheaf of X.

Definition 1.4.11. *(Relative Picard Group)*
Let X be a curve over k. For any scheme T over k, we define $Pic^o(X \times T)$ be the subgroup of $Pic(X \times T)$ consisting of invertible sheaves whose restriction to each fibre X_t for $t \in T$ has degree 0.

Letp $: X \times T \to T$ *be the second projection. For any invertible sheaf* $\mathscr{N}$ *on* T, $p^*\mathscr{N} \in Pic^o(X \times T)$, *because it is in fact, trivial on each fibre. We define* $Pic^o(X/T) = Pic^o(X \times T)/p^*Pic(T)$, *and we regard its elements as families of invertible sheaves of degree* 0 *on* X, *parametrized by* T.

Justification for this is the fact that if T *is integral and of finite type over* k, *and if* $\mathscr{L}, \mathscr{M} \in Pic(X \times T)$, *then* $\mathscr{L}_t \cong \mathscr{M}_t$ *on* X_t *for all* $t \in T$ *iff* $\mathscr{L} \otimes \mathscr{M}^{-1} \in p^*Pic(T)$.

Definition 1.4.12. *(Jacobian Variety)*
Let X *be a curve over* k. *The Jacobian variety of* X *is a scheme* J *of finite type over* k, *together with an element* $\mathscr{L} \in Pic^o(X/J)$ *over* k *having the following universal property:*

For any scheme T *of finite type over* k, *and for any* $\mathscr{M} \in Pic^o(X/T)$, *there is a unique morphism* $f : T \to J$ *such that* $f^*\mathscr{L} \cong \mathscr{M}$ *in* $Pic^o(X/T)$.

Definition 1.4.13. *(Abelian Category)*

Definition 1.4.14. *(Derived Category)*
For an abelian category A, $Kom(A)$ *is a category of chain complexes,* $K(A)$ *is a homotopy category, and* $D(A)$ *is a derived category induced from a localization.*

Definition 1.4.15. *(Six Operations)*

1. *direct image* f_*
   $f_* : SH(X) \to SH(Y)$
2. *inverse image* f^*
   $f^*\mathscr{F} : f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$
   where $f^{-1}\mathscr{F}(U) = \mathscr{F}(f(U))$
3. *proper direct image* $f_!$
4. *proper inverse image* $f^!$
   If $f : X \to Y$ *is an immersion of locally closed subspace, then it is possible to define*
   $f^!(F) = f^*G$
   where G *is a subsheaf of* F *of which the sections on some open subset* U *of* Y *are the sections* $s \in F(U)$ *whose support is contained in* X. $f^!$ *is a left exact functor, and the exceptional image functor is also defined together:*
   $Rf^! : D(Y) \to D(X)$
5. *internal tensor product* $\bigotimes$
6. *internal hom* Hom

Definition 1.4.16. *(Local Systems)*

Definition 1.4.17. *(Grothendieck Groups)*

1.5 Local Field And Global Field

Definition 1.5.1. *(Global And Local Field)*
The formal definition of global field is a field which is either

1. *a field of finite extension of $\mathbb{Q}$.*
2. *The function field of an algebraic curve over a finite field. Or equivalently, finite extension of $F_q(T)$, the field of rational functions, called global function field.*

The formal definition of local field is a field if it is complete with respect to the topology induced by a discrete valuation ν, and residue field k is finite, and it is either

1. *Archimedian local fields: $\mathbb{R}$ and $\mathbb{C}$.*
2. *Non-archimedian local fields of characteristic 0: finite extension of $\mathbb{Q}_p$.*
3. *Non-archimedian local fields of characteristic p: field of Laurent series $\mathbb{F}_{p^n}((T))$*

In fact, p-adic number is characteristic 0 because $\mathbb{Z}_p \cong \mathbb{Z}[[t]]/(t-p)$.

Definition 1.5.2. *(p-adic Number)*
For each prime number $p \in \mathbb{Z}$, p-adic number is a ring, as a set of elements that can be written by the series $r = \Sigma_{n=-k}^{\infty} a_n p^{-n}$.

Alternatively, p-adic number is an inverse limit of finite abelian group $lim_p(\mathbb{Z}/p^n\mathbb{Z})$. Inverse limit on a small category is a product of sequences of objects $\{A_i\}$ where $A_i \in Ob(C)$, $i \in I$, which is quotioned by an equivalence class, and it is formally written as $lim_i(A_i) = \prod_i A_i/\sim$.

Also, p-adic field is $\mathbb{Q}_p$, which is a field of fraction of $\mathbb{Z}_p$.

Proposition 1.5.3. *(p-adic Topology)*

1. $\mathbb{Q} = \bigcap_p \mathbb{Q}_p$
2. $\bar{\mathbb{Q}} \subset \bigcup_p \mathbb{Q}_p$
3. $\bar{\mathbb{Q}_p} \neq \mathbb{Q}_p$
4. *Each p-adic field $\mathbb{Q}_p$ is Hausdorff.*
5. *Each p-adic field $\mathbb{Q}_p$ is complete.*
6. *cardinality of p-adic field $card(\mathbb{Q}_p)$ is $\aleph_0$*
7. *p-adic integer $\mathbb{Z}_p$ is homeomorphic to Cantor set. Also, Cantor set is compact because it is closed and bounded.*

Note that the union of $\mathbb{Q}_p$ contains algebraic closure of $\mathbb{Q}$, but each of them is not algebraic closure of itself.

Also, p-adic field $\mathbb{Q}_p$ is a complete metric space. First of all, Considered that the cardinality if $\aleph_0$, its topology is paracompact, and paracompact Hausdorff is a normal space, which is enough to say metrizable (Urysohn's metrizability theorem).

Proposition 1.5.4. *(p-adic Manifold)*
p-adic field $\mathbb{Q}_p$ is totally disconnected Hausdorff space whose topological dimension is 0. By these properties, $\mathbb{Q}_p$ can also be considered as a 0-dimensional manifold, since this is Hausdorff and disconnectedness means that each point is open. Furthermore, in fact, p-adic field $\mathbb{Q}_p$ is an infinite Galois group $\mathbb{Q}_p = \bigcup_n \mathbb{Q}(\zeta^{p^n})$, and if that is considered, $\mathbb{Q}_p$ is a Lie group.

Now this notion can extended to K-analytic manifold where K is some p-adic field, by adding some conditions that each open neighborhood is homeomorphic to K^n-vector space.

From now, we will introduce Adele ring.

Definition 1.5.5. *(Haar measure)*
Haar measure is namely a measure defined over a group.

Let G be a locally compact Hausdorff topological group, and let $\mu : G \to \mathbb{R}_{\geqq 0}$ be a measure. μ is Haar measure if it satisfies the following properties:

1. $\mu(S) = \mu(gS)$ *for Borel sets* $S \subset G$, $g \in G$.
2. $\mu(S) = \mu(Sg)$ *for Borel sets* $S \subset G$, $g \in G$.
3. $\mu(K) < \infty$ *for a compact set* $K \subset G$.
4. $\mu(G) = 1$
5. $\mu(S) = inf\{\mu(U) : S \subset U, U \ \text{open}\}$ *(outer regular)*
6. $\mu(S) = sup\{\mu(U) : K \subset U, K \ \text{compact}\}$ *(inner regular)*

For example, for Euclidean space is trivially an additive group $(\mathbb{R}, +)$, and this case, haar measure is Lebesgue measure.

Remark 1.5.6. *(Ostrowski)*
In fact, the only complete metric space over $\mathbb{Q}$ is $\mathbb{R}$ or $\mathbb{C}$ if archimedean absolute value is defined, or p-adic fields if non-archimedian absolute value is defined (Ostrowski).

Definition 1.5.7. *(Valuation Ring)*
An integral domain R is a valuation ring if for all $x \in F$ of fraction of field F, x or x^{-1} is contained in R. That says, $R \subset F$. This valuation ring R is, in fact, a local ring, and every ideal of R is totally orderd by inclusion.

The valuation of R is a map $\nu : F \to \Gamma \bigcup \{\infty\}$ where for $x \in R$, $\nu(x) \geqq 0$, and Γ is an abelian group.

Definition 1.5.8. *(Adele Ring)*
Let K be a global field, and $\mathbb{A}_K$ be an adel ring if $\mathbb{A}_K = \prod_\nu (K_\nu, O_\nu)$ where ν is all the possible valuations.

Particularly, if $K = \mathbb{Q}$, by Ostrowski's theorem, all the valuation rings are real field $\mathbb{R}$ or p-adic integers $\mathbb{Z}_p$. Namely, $\mathbb{A}_K = \mathbb{R} \times \prod_p (\mathbb{Q}_p, \mathbb{Z}_p)$.

2 Lie Algebra

2.1 Lie Algebra

Definition 2.1.1. *(Kac-Moody Lie Algebra)*
Kac-Moody Lie algebra is generated by a pair e_i, h_i, and f_i with a Lie bracket such that

1. $[h_i, h_j] = 0$
2. $[h_i, e_j] = c_{ij} e_j$
3. $[h_i, f_j] = -c_{ji} f_j$
4. $[e_i, f_j] = \delta_{ij} h_i$

where c_{ij} is an element of a generalized Cartan matrix. Generalized Cartan matrix is a matrix (c_{ij}) such that

1. *The diagonal $c_{ii} = 2$ is always 2*
2. *If $c_{ij} = 0$, then the symmetry side is $c_{ji} = 0$.*
3. *non diagonal entry $c_{ij} < 0$ for $i \neq j$.*
4. *can be written as DS where D is a diagonal matrix and S is a symmetric matrix.*

Definition 2.1.2. *(Universal Enveloping Algebra)*
Universal Enveloping algebra is an associative algebra whose representation corresponds to that representation of Lie algebra.

Definition 2.1.3. *(Verma Module)*

Definition 2.1.4. *(Harish-Chandra Module)*

2.2 Algebraic Group

Definition 2.2.1. *(Algebraic Group)*
Let Alg_k is a category of k-algebra. An algebraic group G is a functor $Alg_k \to Set$ with additional structures $m : G \times G \to G$, $e : * \to G$, and $inv : G \to G$. Namely, if we plugin some algebra $R \in Ob(Alg_k)$, $G(R)$ is a set with three operations $m(R) : G(R) \times G(R) \to G(R)$, $e(R) : * \to G(R)$, and $inv(R) : G(R) \to G(R)$.

Especially such G is called affine algebraic group if G is a representable functor, which means there is an k-algebra A such that $G \cong Hom(A, -)$. We call such A as a coordinate ring of G, denoted by $\mathscr{O}(G)$. Then $G \times G \to G$ makes $\Delta : \mathscr{O}(G) \to \mathscr{O}(G) \bigotimes \mathscr{O}(G)$ or $\Delta : A \to A \bigotimes A$. It is because $Hom(A, R) \times Hom(A, R) \to Hom(A, R)$ for all algebra R, and by tensor-hom adjointness property, $Hom(A, R) \times Hom(A, R) \cong Hom(A, Hom(A, R)) \cong Hom(A \bigotimes A, R)$ which is given as a morphism $Hom(-, R)(\Delta)$. Hereafter all the algebraic group is affine algebraic group.

Definition 2.2.2. *(Semisimple, Reductive)*
Let G be a connected algebraic group, and G is called semisimple if commutative normal subgroup can only be trivial, G is called reductive if commutative normal subgroup can only be tori.

Definition 2.2.3. *(Connectedness)*
For an algebraic group G, let $|G| = Spm(\mathscr{O}(G))$ be a group in the category of algebraic varieties over k. If H is an algebraic subgroup of G, then $|H|$ is a closed subvariety of $|G|$.

G is connected if $|G|$ is connected, or $\mathscr{O}(G)$ contains no étale k-algebra except k.

Definition 2.2.4. *(Borel Subgroup, Parabolic Subgroup)*
Lett G be an algebraic group. We call $B \subset G$ as a Borel subgroup if it is Zariski closed and connected algebraic solvable algebraic subgroup.

A subgroup $P \subset G$ is called parabolic if $P \supset B$.

Definition 2.2.5. *(Split/Quasi-Split Group)*
A reductive group G over k is split if it contains a split maximal torus T over k.

A reductive group G over k is quasi-split if it contains a Borel subgroiup over k.

Asplit reductive group G is quasi-split.

Note 2.2.6. *(Example)*
Redutive group is not always split or quasi-split. For example, $SO(p, q)$ over $\mathbb{R}$ is split iff $|p - q| \leqq 1$, and quasi-split iff $|p - q| \leqq 2$.

Definition 2.2.7. *(Character)*
A character of an algebraic group G is $\chi : G \to \mathbb{G}_m$. We write $X_k(G)$ for the group of characters of G over k, and $X^*(G)$ for the similar group over an algebraic closure of k.

Definition 2.2.8. *(Representation)*
Let V be a k-vector space, and R be a k-algebra, $V(R)$ be $V(R) = V \bigotimes_k R$. Then a group action $G(R) \times V(R) \to V(R)$ makes a representation $r(R) : G(R) \to Aut(V(R)) = GL_V(R)$, then r is a linear representation of G.

Such finite dimensional representations form a category $Rep(G)$.

Definition 2.2.9. *(Monoidal Category)*
Monoidal category C is a category equipped with a bi-functor $\bigotimes : C \times C \to C$ that is associative up to a natural isomorphism. Plus we have some other properties:

1. *a bifunctor $\bigotimes : C \times C \to C$*
2. *existence of a unit object $I \in Ob(C)$*
3. *pentagon and some other commutative diagrams etc*

Definition 2.2.10. *(Tannakian Duality)*
Let G be an algebraic group, $\lambda_V \in End(V(R))$ for some fixed k-algebra R, then if it satisfies

1. *$\lambda_{V \bigotimes W} = \lambda_V \bigotimes \lambda_W$ for all representations V and W.*
2. *$\lambda_1 = id_1$ where $k = 1$.*
3. *$\lambda_W \circ \alpha_R = \alpha_R \circ \lambda_V$ for all G-equivalent maps $\alpha : V \to W$.*

Then there exists a $g \in G(R)$ such that $\lambda_V = r_V(g)$ for all V. Hence it means that we can recover G from a category of finite-dimensional representation $Rep(G)$.

Let $G'(R)$ be a set of families (λ_V) satisfies that conditions, then we have a natural morphism $G \to G'$, which is an isomorphism, then we call it Tannakian duality holds for G.

Definition 2.2.11. *(Dual)*

Definition 2.2.12. *(Rigid)*
A tensor category is rigid if every object admits a dual. For example, the category Vec_k of finite-dimensional vector spaces over k and the category of finite-dimensional representations of a Lie algebra are rigid.

Definition 2.2.13. *(Neutral Tannakian Category)*
Neutral tannakian category is a pair (C, ω) where C is an abelian $k-$linear tensor category and ω is an exact tensor functor $\omega : C \to Vec_k$. Such a functor ω is called a fibre functor over k.

It is usual to write $Aut^{\bigotimes}(\omega)$ for the affine group G attached to the neutral tannakian category (C, ω), and we call it tannakian dual or tannakian group of C.

Definition 2.2.14. *(Tannnaka/Galois Correspondence)*
Let $S = \{X | X \subset C = Rep(G)\}$ be a set of objects of category $C = Rep(G)$, and denote $H(S)$ be a largest subgroup of G acting trivially on all V in S; thus,

$H(S) = \bigcap_{V \in S} Ker(r_V : G \to Aut(V))$

Then the maps $S \mapsto H(S)$ and $H \mapsto G^H$ form a Galois correspondence

$\{subsets\ of\ Ob(C)\} \leftrightarrow \{algebraic\ subgroup\ of\ G\}$

Furthermore, we induce 1-1 order reversing correspondence between tannakian subcategories of C and normal algebraic groups of G.

Definition 2.2.15. *(Lie Algebra of an Algebraic Group)*
Let k be a field of characteristic zero. Let G be an algebraic group, and the action of G on itself $G \times G \to G$ by conjugation, $(g, x) \mapsto gxg^{-1}$ fixes e, and so it defines a representation of G on the tangent space $\mathfrak{g}$ of G at e, $G \to GL_{\mathfrak{g}}$. Note that $\mathfrak{g}$ is a vector space.

Definition 2.2.16. *(Definition of $\mathfrak{g}(R)$)*
Let R be a k-algebra, and let $R[\epsilon] = R[X]/(X^2)$. Thus $R[\epsilon] = R \bigoplus R\epsilon$ as an R-module and $\epsilon^2 = 0$. Let $\pi : R[\epsilon] \to R$ where $\pi(a + b\epsilon) = a$. Then we define $\mathfrak{g}(R) = Ker\{\pi : G(R[\epsilon]) \to G(R)\}$.

2.3 Character Formula

Character formula is a symmetric function.

Definition 2.3.1. *(Schur Polynomial)*
Given a partition $\lambda = (\lambda_1 \geqq \lambda_2 \geqq ... \geqq \lambda_n)$ where each λ_k is a non-negative integer. We have a polynomial

$$a_{(\lambda_1+n-1, \lambda_2+n-2, ..., \lambda_n)}(x_1, ..., x_n) = det \begin{pmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & ... & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & ... & x_n^{\lambda_2+n-2} \\ & & ... & \\ x_1^{\lambda_n} & x_2^{\lambda_n} & ... & x_n^{\lambda_n} \end{pmatrix}$$

In particular, if $\lambda = 0$, then the polynomial $a_{(n-1, n-2, ..., 0)} = \prod_{i<j}(x_i - x_j)$ will be Van der Monde polynomial. Now we define a quotient

$s_\lambda(x_1, x_2, ..., x_n) = \frac{a_{(\lambda_1+n-1,\lambda_2+n-2,...,\lambda_n)}(x_1,...,x_n)}{a_{(n-1,n-2,...,0)}(x_1,...,x_n)}$

called a Schur polynomial. it is a special case of Weyl character formula.

Definition 2.3.2. *(Symmetric Polynomials)*
Elementary symmetric polynomial is

$e_k(X_1, ..., X_n) = \Sigma_{\Sigma k_i = k} X_1^{k_1} ... X_n^{k_n}$ *where* $k_i \in \mathbb{N}$.

Monomial symmetric polynomial is similar to elementary symmetric polynomial, but it is weighted:

$m_{(k_1,...,k_n)}(X_1, ..., X_n) = \Sigma_{\sigma \in W} X_1^{\sigma(k_1)} X_n^{\sigma(k_n)}$ *where* W *is the permutation of* $(k_1, ..., k_n)$.

2.4 Hecke Algebra

Definition 2.4.1. *(Coxeter Group)*
Coxeter group is a group generated by a Coxeter graph. Let Coxeter Graph G *be a graph, which consists of vertices* $\{s_i\}_{i \in I}$ *where* I *is an index, or number of generators, and edges* $\{m_i j\}_{i,j \in I}$ *where* m_{ij} *is the number of edges* $+2$ *between two vertices* s_i *and* s_j. *The Coxeter group generated by* G *is defined by* $W = s_i | {s_i}^2 = e, (s_i s_j)^{m_{ij}} = e$.

Definition 2.4.2. *(Hecke Algebra)*
Here, a Hecke algebra is a generalization of a Coxeter group. For a Coxeter group W, *Hecke algebra* $H_q(W)$ *is an algebra generated by some collection of elements* $\{T_i\}_{i \in I}$, *where the multiplication of any two product* $(T_i T_j)^{m_{ij}} = 1$ *is cyclic (could be infinite order) for* $i \neq j$, *and if* $i = j$, *it satisfies a quadratic relationship* ${T_i}^2 = (q-1)T_i + q$. *The ground field* F *is a field but order must be bigger than* q. *Notice that, in particular, when* $q = 1$, $H_1(W)$ *is just a Coxeter group algebra* $\mathbb{C}[W]$. *In general,* $H_q(W)$ *may be thought of as a deformation of* $\mathbb{C}[W]$.

The above definition is the simplest form of Hecke algebra, but I need some generalization.

Definition 2.4.3. *(Admissible Representation)*
Let $F = \mathbb{Q}_p$ *be a p-adic field and* $\mathfrak{o} = \mathbb{Z}_p$ *be a p-adic integer ring. Notice that* $\mathfrak{o}$ *is a discrete valuation ring, and its quotient by the maximal ideal* $\mathfrak{p}$ *is* $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_p$.

Let $G = GL(n, F)$ *be a reductive algebraic group and* $K^o = GL(n, \mathfrak{o})$ *be a maximal compact subgroup of* G. *Let for any integer* $N \in \mathbb{N}$, $K(N) = \{g \in K^o | g \equiv 1 \text{ mod } \mathfrak{p}^N\}$, *and notice that the family* $\{K(N)\}_{N \in \mathbb{N}}$ *forms a basis of neighborhood of the identity.*

A representation $\pi : G \to GL(V)$ *for a complex vector space* V *is called smooth if for all* $0 \neq v \in V$, *the stabilizer* $\{k \in G | \pi(k)v = v\}$ *is open.*

Also, π is admissible if it is smooth, and for any open subgroup $K \subset G$, V^K is finite dimensional.

Example 2.4.4. *(Unitary Representation)*
Unitary representation $\pi : G \to GL(H)$ is a unitary representation where H is a Hilbert space, Then H contains a dense subspace $V \subset H$ on which G acts, thus π is admissible.

Definition 2.4.5. *(Spherical Hecke Algebra)*
For G is a Lie group, and $K \subset G$ is a closed subgroup, Gelfand pair is a pair (G, K) such that the algebra of (K, K)-double invariant compactly supported continuous measures on G with multiplication defined by convolution is commutative.

Definition 2.4.6. *(Spherical Hecke Algebra)*
Recall $K \subset GL(n, \mathbb{Q}_p$ is a open subset, Recall $K^o = GL(n, \mathfrak{o})$, and Here we define a ring structure of K. Let $\phi : G \to \mathbb{Z}$ be a compactly supported function over G such that $\phi(kgk') = \phi(g)$ for all $k, k' \in K$, and ϕ is locally constant on each of its orbit. If we let $(\phi \star \psi)(g) = \inf_G \phi(gx^{-1})\psi(x)dx$ and $\pi(\phi)v = \inf_G \phi(g)\pi(g)vdv$, then $\pi(\phi \star \psi) = \pi(\phi) \circ \pi(\psi)$. Moreover since naturally $(\phi \star \psi)(g) = \psi \star \phi)(g)$, the Hecke algebra H_K is commutative. In particular, H_{K^o} is commutative, and we call it spherical Hecke algebra.

Definition 2.4.7. *(Iwahori Hecke Algebra)*
Let $J \subset K^o$, naturally the inclusion relation $H_J \supset H_{K^o}$ is contravariant.
$J = \{k \in K^o | \overline{k} \in GL(n, \mathbb{F}_q)\}$ where $\overline{k}$ is an image of homomorphism
$k \in GL(n, \mathfrak{o}) \to GL(n, \mathbb{F}_q)$

Here H_J is not abelian anymore, but it has a beautiful structure because it has a braid relationship, analogy to Coxeter group. Since Iwahori Hecke algebra H_J is an algebra, it has generators $T_0, ..., T_{n-1}$, and T_i and T_j commute unless $j \equiv i \pm 1 \mod n$ with a braid relations $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ (if $i = n-1$, then $i = 0$), and ${T_i}^2 = (q-1)T_i + q$. Thus $T_0, ..., T_{n-1}$ generate Iwahori Hecke algebra H_J. Its Coxeter group is affine Weyl group ${A_{n-1}}^{(1)}$, and the extra element t has the effect $tT_it^{-1} = T_{i+1}$.

Note 2.4.8.
Let's consider V^K as an H_K module with multiplication $\phi \cdot v = \pi(\phi)v$

3 Classical Satake Isomorphism

3.1 Construction of Classical Satake Isomorphism

With Hecke algbra in mind, in this chapter, we construct Satake isomorphism in an explicit way. Let F be a non-archimedean local field, and G be a reductive algebraic group over F. G is unramified if G is quasi-split (has a Borel subgroup) and is split over an unramified finite degree extension of F, and fix a hyperspecial subgroup $K \leqq G(F)$. Let (π, V) be an associated unramified irreducible representation of $G(F)$, so that $V^K \neq 0$. V^K is naturally a module over a spherical Hecke algebra ${C_c}^{\infty}(G(F)//K)$ with associated action $\pi(f)v := \int_{G(F)} f(g)\pi(g)vdg$ for dg a Haar measure on $G(F)$. We obtain a map ${C_c}^{\infty}(G(F)//K) \to End_{\mathbb{C}}(V^K) \cong \mathbb{C}$ where $f \mapsto tr(\pi(f))$ called Hecke character of π.

Let $T \leqq G$ be a maximal torus of a reductive algebra group G where G is split. We will assume K is $G(\mathscr{O})_F$. Then we have a short exact sequence $0 \to T(\mathscr{O}_F) \to T(F) \xrightarrow{\gamma} X_*(T) \to 0$ of locally compact groups. This is equivalent to say that the the quotient of group can be described by character formula $T(\mathscr{O}_F)/T(F) \cong X_*(T)$, but this quotient is the spherical Hecke algebra. Recall that $T(F)$ is abelian and the quotient $T(\mathscr{O}_F)/T(F) \cong X_*(T)$ is discrete,

Proposition 3.1.1. *(Grassmannian)*
For $G(k) \cong G(k[[t]])/G(k((t)))$, $G(k)$ is a Grassmannian.

Definition 3.1.2. *(Character)*
Let $T \subset G$ be a torus as a subgroup of a reductive group G. We define the character and the cocharacter as

1. $X^{\cdot}(T) = Hom(T, \mathbb{G}_m)$
2. $X_{\cdot}(T) = Hom(\mathbb{G}_m, T)$

Consider a character morphism $< \cdot, \cdot >$: $X^{\cdot}(T) \times X_{\cdot}(T) \to \mathbb{Z}$ $(\lambda, \chi) \mapsto [\chi \circ \lambda]$

which are free abelian groups of rank $l = dim(\mathbb{G}_m)$. Particularly if $T = \mathbb{G}_m$, then

$X_{\cdot}(\mathbb{G}_m) = Hom(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$

We care character because it defines a root system. From the character of the torus T, we have a Weyl chamber.

We choose a Borel subgroup $B \leqq G$ containing T, and the choice of T corresponds to the choice of the root system $\Phi(X, T)$, and the choice of B corresponds to the choice of positive roots $\Phi^+ \subset \Phi(X, T)$. This will naturally induce the positive Weyl chamber.

Definition 3.1.3. *(Positive Weyl chamber)*
$P^+ = \{\lambda \in X_*(T) | < \lambda, \chi > \geqq 0 \text{ for every } \chi \in \Phi^+\}$
$= \{\lambda \in X_*(T) | < \lambda, \chi > \geqq 0 \text{ for every } \chi \in \Delta\}$

half-sum of positive roots ρ is defined by $\rho \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$, and $2\rho = \Sigma_{\chi \in \Phi^+} \chi$ in $X^*(T)$.

Definition 3.1.4. *(Cartan decomposition)*
Cartan decomposition is a decomposition of semisimple Lie group and Lie algebra. Here we will only discuss semisimple group case.

$G(F) = \coprod_{\lambda \in P^+} K\lambda(\omega)K$ *where* $\lambda \in X_*(T)$.

Note that $(\lambda + \mu)(\omega) = \lambda(\omega) + \mu(\omega)$.

This induces a fact that the spherical Hecke algebra $C(G(F)//K)$ is a $\mathbb{C}$-vector space basis given by $c_\lambda = 1_{K\lambda(\omega)K}$ for all $\lambda \in P^+$, and we have a following formula:
$c_\lambda \star c_\mu = \Sigma_{\nu \in P^+} d_{\lambda,\mu}(\nu) c_\nu = c_{\lambda+\mu} + \Sigma_{\nu < \lambda+\mu} d_{\lambda,\mu}(\nu) c_\nu$ where $d_{\lambda,\mu}(\nu) \in \mathbb{Z}$, and $d_{\lambda,\mu}(\nu) = \#\{(i,j) | \nu(\omega) \in x_i y_j K\} \in \mathbb{Z}$.

In particular, if $G = T$, then $c_\lambda \star c_m u = c_{\lambda+\mu}$.

Definition 3.1.5. *(Iwasawa decomposition)*
For an algebraic group G*, let* $B \leqq G$ *be a Borel subgroup,* $T \leqq B \leqq G$ *be a maximal torus,* $N = R_v(B)$ *be a unipotent radical of* B*. Without loss of generality, we may assume* K*,* B*,* T *as*

$G(F) = B(F)K;$
$B(F) \bigcap K = (T(F) \bigcap K)(N(F) \bigcap K);$
$T(F) \bigcap K \leqq T(F)$ *is maximal compact.*

The above Iwasawa decomposition gives $G(F) = T(F)N(F)K$ *so we may decompose any choice of Haar measure* dg *on* $G(F)$ *via* $dg = \delta_B(t) dt dn dk$ *with* $dk(K) = 1 = dn(N(F) \bigcap K)$.

Also, we let $\delta_B : B(F) \to \mathbb{R}^{>0}$ *the modular quasicharacter characterized by* $d(bnb^{-1}) = \delta_B(n) dn$*, and which is trivial on* $N(F)$*, or we can say that*

$\delta_B : B(F) \to \mathbb{R}^{>0}$ *is* $b \mapsto |det_{\mathfrak{b}}(b)|_{\mathfrak{p}}$

Then given that $f \in C_c{}^\infty(G(F)//K)$*, define* $Sf : T(F) \to \mathbb{C}$ *by*
$Sf(t) := \delta_B(t)^{1/2} \int_{N(F)} f(tn) dn$

notice that Sf *is compactly suported, locally constant, and left* $T(F) \bigcap K$ *invariant, and* $C_c{}^\infty(T(F)/T(F) \bigcap K) \cong \mathbb{C}[X_*(T)]$ *so it may be an element of* $\mathbb{C}[X_*(T)]$*. Also,* S *is an algebra homomorphism.*

Here in particular if $t = \mu(\omega) \in T(F)$ *for* $\mu \in X_*(T)$*,* $\delta_B(t)^{1/2}$ *is explicitly computed as*

$\delta_B(t)^{1/2} = |det(ad(t)|Lie(N))|_{\mathfrak{p}}^{1/2} = |2\rho(t)|_{\mathfrak{p}}^{1/2} = |\omega^{<\mu, 2\rho>}|_{\mathfrak{p}}^{1/2} = q^{-<\mu,\rho>}$
where $q = |\mathscr{O}_F / \mathfrak{p}|$.

Definition 3.1.6. *(Dual Lie Group)*
Let $\hat{G}$ be a complex dual of G characterized by the root datum $(X_*(\hat{T}), X^*(\hat{T}), \hat{\Phi}, \hat{\Phi^\vee})$, and $\hat{T}$ is the dual torus of T and $\hat{\Phi} = \Phi(\hat{G}, \hat{T})$ is the dual to the root datum of $(X_*(T), X^*(T), \Phi, \Phi^\vee)$.

This finally leads us to the following theorem.

Proposition 3.1.7. *(Classical Satake Isomorphism)*
Let G be a reductive algebraic group and F is a non-archimedean field with $\mathfrak{o} \subset F$ the ring of integer. Let K be a maximal compact subgroup of $G(F)$. Then we have the following isomorphism called classical Satake isomorphism:

$$C_c{}^\infty(G(F)//K) \cong \mathbb{C}[X^*(T)]^{W(\hat{G},\hat{T})(\mathbb{C})}$$

The formula can be alternatively written by the simpler form:

$$H_T \cong R(\hat{G})$$

where the left hand side is the spherical Hecke algebra that can be denoted by $H_T = C_c{}^\infty(G(F)//K)$, while the right hand side is $\mathbb{Z}$-algebra of torus denoted by $R(\hat{G}) = \mathbb{Z}[X^*(T)]$.

The particular case of the classical Satake isomorphism is given by

$$C_c{}^\infty[G(\mathfrak{o}) \setminus G(F)/G(\mathfrak{o})] \cong \mathbb{C}[X_*(T(\mathbb{C}))]^W$$

Proposition 3.1.8. *()*
The Satake isomorphism in the previous statement $H_T \cong R(\hat{G})$ can be naturally extended to $H_T \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \cong R(\hat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$,
but in fact, the left hand side is equal to itself by the Weyl group invariant, so it may be rewritten as
$(H_T \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}])^W = R(\hat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Here, we will let another ring homomorphism $S : H_G \to H_T \bigotimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$ with image in the invariants for the Weyl group. This says,

$$S : H_G \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \cong R(\hat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

Especially if $\rho \in X^*(T)$, then $S : H_G \otimes \mathbb{Z}[q^{-1}] \cong R(\hat{G}) \otimes \mathbb{Z}[q^{-1}]$.

3.2 Langlands Program

Definition 3.2.1. *(Langlands Dual Group)*
We will construct Langlands dual group.

If G is split, then ${}^{L}G = \hat{G}(\mathbb{C}) \times Gal(F)$ where $Gal(F)$ is an absolute Galois group of F.

More generally, if G is quasi-split, then ${}^{L}G = \hat{G}(\mathbb{C}) \rtimes Gal(F)$ where $Gal(F)$ is an absolute Galois group of F.

Definition 3.2.2. *(L-Function)*
One of the applications of Satake isomorphism is computation of L-function.

Let (π, V) be an unramified representation such that $\pi = \pi(s)$ for some Satake parameter s, which is a semisimple class in $\hat{G}(\mathbb{C})$, then we let an L-function as

$$L(\pi, V, X) = det(1 - sX|V)^{-1} \in \mathbb{C}[[X]]$$

$$det(1 - sX|V) = \Sigma_{k=0}^{dimV}(-1)^k Tr(s|\bigwedge^k V)X^k.$$

Definition 3.2.3. *(The Trivial Representation)*
The trivial representation (π, V) of G is one of the unramified representation. Then c_λ acts by multiplication by

$$deg(c_\lambda) = \#\text{ of single } K\text{-cosets in } K\lambda(\pi)K$$

Let $P_\lambda \subset \underline{G}$ be the standard parabolic subgroup defined by the cocharacter λ. We have

$$Lie(P_\lambda) = Lie(T) + \bigoplus_{<\lambda,\alpha>\geqq 0} Lie(G)_\alpha$$

and

$$dim(G/P_\lambda) = \#\{\alpha \in \Phi| < \lambda, \alpha > \leqq 0\}$$

If $\lambda = 0$, then $P_\lambda = G$. If λ is regular we find $P_\lambda = B$. Let

$$l : \tilde{W}_a = X^*(\hat{T}) \rtimes W \to Z$$

be the length function on the extended affine Weyl group.

Proposition 3.2.4. *(Bruhat-Tits Decomposition)*
Let G be a reductive algebraic group, $B \subset G$ be a Borel subgroup, and W be a Weyl group of G corresponding to a maximal torus of B.

$$G = BWB = \coprod_{w \in W} BwB$$

Proposition 3.2.5. *()*
$deg(c_\lambda) = \Sigma_{W\lambda W} q^{l(y)} / \Sigma_W q^{l(w)} = \frac{\#(G/P_\lambda)(q)}{q^{dim(G/P_\lambda)} \cdot q^{<\lambda,2\rho>}}$

Moreover, λ is miniscule co-weight iff

$$deg(c_\lambda) = \#(G/P_\lambda)(q)$$

Satake parameter of the trivial representation is the conjugacy class $s = \rho(q) = 2\rho(q^{1/2})$ in $\hat{G}(\mathbb{C})$. Equivalently, if

$$s_0 = \begin{pmatrix} q^{1/2} & \\ & q^{-1/2} \end{pmatrix} \text{ in } SL_2(\mathbb{C})$$

is the satake parameter of the trivial representation of PGL_2, then s is the image in $\hat{G}(\mathbb{C})$ of s_0 in a principal SL_2. This gives a check on our various formulas. For example, if $G = G_2$, then we found

$$q^3\chi_1(s) = \alpha_1 + 1$$

$$q^5\chi_2(s) = \alpha_2 + \alpha_1 + 1 + q^4$$

Recall that on the trivial representation, we found

$$\alpha_1 = deg(c_{\lambda_1}) = q^6 + q^5 + q^4 + q^3 + q^2 + q$$

$$\alpha_2 = deg(c_{\lambda_2}) = q^{10} + q^9 + q^8 + q^7 + q^6 + q^5$$

and the characters $\chi_1(s)$ and $\chi_2(s)$ are computed.

Definition 3.2.6. *(Cuspidal Representation)*
We let a complex-valued measurable function $f : G(\mathbb{A}) \to \mathbb{C}$ as :

1. $f(\gamma g) = f(g)$, $\forall \gamma \in G(K)$
2. $f(gz) = f(g)\omega(z)$, $\forall z \in G(Z_{\mathbb{A}})$
3. $\int_{Z(\mathbb{A})G(K)\backslash G(\mathbb{A})} |f(g)|^2 dg < \infty$
4. $\int_{U(K)\backslash U(\mathbb{A})} f(ug)du = 0$ *for* $g \in G(\mathbb{A})$, $U \subset G$ *is a parabolic subgroup of* G.

A cuspidal function generates a unitary representation of the group $G(\mathbb{A})$ on a complex Hilbert space V_f generated by the right translate of f. Here the action of $g \in G(\mathbb{A})$ on V_f is given by

$$(g \cdot u)(x) = u(xg)$$

$$u(x) = \Sigma_j c_j f(xg_j) \in V_f$$

A cuspidal representation of $G(\mathbb{A})$ is a pair (π, V_π) for some ω. ${L_0}^2(G(K) \backslash G(\mathbb{A}), \omega) = \hat{\bigoplus}_{\pi, V_\pi} m_\pi V_\pi \;\; m_\pi \in \mathbb{N}$

Proposition 3.2.7. *(Langlands-Hecke Correspondence)*

$$\begin{aligned} \text{Unramified representation} &\leftrightarrow \text{Hecke module category} \\ \text{representations of } (\pi, V_\pi) \text{ of } G \text{ generated } V^K &\leftrightarrow {C_c}^{\infty}(G//K)\text{-modules} \end{aligned} \tag{2}$$

3.3 Kazhdan-Lusztig

Definition 3.3.1. *(Kazhdan-Lusztig Polynomial)*
We let $\hat{P}(\mu) = \Sigma_{\mu=\Sigma n(\alpha^\vee)\alpha^\vee} q^{-\Sigma n(\alpha^\vee)}$ be a polynomial in q^{-1} which counts the number of expressions of μ as a non-negative sum of positive coroots. If μ cannot be expressed by such a sum, then $\hat{P}(\mu) = 0$. Since we include the empty sum, when $\mu = 0$, we have $\hat{P}(0) = 1$. In all cases, $q^{<\mu,\rho>} \cdot \hat{P}(\mu)$ is a polynomial in q with integral coefficients. If μ is in P^+ and $\mu \geqq 0$, the constant coefficient of $q^{<\mu,\rho>}\hat{P}(\mu)$ is equal to 1.

The coefficient of $d_\lambda(\mu)$ appearing in Satake isomorphism can be explicitly computed as
$d_\lambda(\mu) = P_\lambda, \mu(q) = q^{<\mu,\rho>} \Sigma_{\sigma \in W} \epsilon(\sigma) \hat{P}(\sigma(\lambda + \rho^\vee) - (\mu + \rho^\vee))$
where $\epsilon(\sigma) = det(\sigma|X_*(T))$ is the sign character of the Weyl group W. If $q = 1$, then $\hat{P}(\mu)$ becomes a partition function, and $P_{\mu,\lambda}(1) = dim(V_\lambda(\mu))$ by a formula of Kostant.

Note 3.3.2. *(Kazhdan-Lusztig Formula for E8)*
For $G = E8$, Kazhdan Luszig polynomial will be

$152q^{22}+3472q^{21}+38791q^{20}+293021q^{19}+1370892q^{18}+4067059q^{17}+7964012q^{16}$
$+11159003q^{15}+11808808q^{14}+9859915q^{13}+6778956q^{12}+3964369q^{11}+2015441q^{10}$
$+906567q^{9}+363611q^{8}+129820q^{7}+41239q^{6}+11426q^{5}+2677q^{4}+492q^{3}+61q^{2}+3q$

Definition 3.3.3. *(Kazhdan-Lusztig Conjecture)*
Let W be a Weyl group of finite degree, and for each $w \in W$ we denote M_w as Verman module of highest weight $-w(\rho) - \rho$ where ρ is the half-sum of positive roots, and let L_w be its irreducible quotient, the simple highest weight module of heighest weight $-w(\rho) - \rho$. If both M_w and L_w are locally-finite weight modules over the complex semisimple Lie algebra $\mathfrak{g}$ with Weyl group W, then it admits an algebraic character. We call $ch(X)$ be character of $\mathfrak{g}$-module X. Then the Kazhdan-Lusztig conjectures state:

$ch(L_w) = \sum_{y \le w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) ch(M_y)$

$ch(M_w) = \sum_{y \le w} P_{w_0 w, w_0 y}(1) ch(L_y)$

where w_0 is the element of maximal length of Weyl group. The statement is still a conjecture, but it was partially proven for the case of characteristic 0.

4 Geometric Satake Isomorphism

Geometric Satake isomorphism is a vast generalization of the classical Satake isomorphism. Recall that the double coset space appears in the local neighborhood of moduli of G-bundles, which might be related to character polynomial. Namely, this is the geometric version of Satake correspondence. This notion can be generalized globally through category of D-modules. In a sentence, the whole argument connects arithmetic theory and geometric theory.

First of all, we start from the construction of affine grassmannian and loop groups.

Definition 4.0.1. *(Lattice)*
We usually denote lattice by $\Lambda \subset \mathbb{Z}^n$, as a discrete subset of Euclidean space where it makes an $\mathbb{Z}$-module by an additional binary operation. However, here we define lattice in a general sense: For a k-algebra R,

R-family of lattices in $k((t))^n$ for some $n \in \mathbb{N}$ is a finitely generated projective $R[[t]]$-submodule of $R((t))^n$ such that $\Lambda \bigotimes_{R[[t]]} R((t)) = R((t))^n$.

Definition 4.0.2. *(Affine Grassmannian and Loop Groups)*
We will consider an affine Grassmannian Gr_G where G is an algebraic group. If, in particular, $G = GL_n$, then Gr_{GL_n} is a presheaf that assigns every k-algebra R the set of R-families of lattices in $k((t))^n$. Hereafter we will use $Gr = Gr_{GL_n}$ for abbreviation.

Proposition 4.0.3. *(Affine Grassmannian)*
The affine Grassmannian is represented by ind-projective scheme.
$\Lambda_0 = k[[t]]^n$ denote the standard lattice. $Gr^{(N)} \subset Gr$ denote the subspace classifying lattices Λ in $((t))^n$ that land in between $t^N\Lambda_0 \subset \Lambda \subset t^{-N}\Lambda_0$, then Gr is an increasing union of these $Gr^(N)$. $Gr^{(N)}$ can be represented by a projective scheme. For a lattice $\Lambda \in Gr^{(N)}$, the quotient $\Lambda/t^N\Lambda_0$ can be regarded as a subspace of $k^{2nN} \cong t^{-N}\Lambda_0/t^N\Lambda_0$ stable under action of t. In this way, $Gr^{(N)}$ is realized as a closed subscheme of the usual Grassmannian variety classifying finite dimensioal subspaces in k^{2nN}, and therefore it is a projective scheme.

Definition 4.0.4. *(Lattice Functor)*
Let X be a presheaf over $\mathscr{O} = k[[t]]$ given by $X : Alg_k \to R[[t]] - VS$ such that k-algebra R maps to a family of $R[[t]]$-modules where each of them is generated by a lattice Λ, meaning that $R \mapsto \{\Lambda \bigotimes R((t))^n\}$ where $n = dim(\Lambda)$.

The space of n-jets of X is the presheaf that assigns every R the set $L^nX(R) = X(R[t]/t^n)$. Similarly, Let L^+X be a presheaf such that $LX(R) = X(R[[t]])$ called a loop group, which is, in fact, $L^+X = lim(L^nX)$. Also, let LX be a presheaf such that $LX(R) = X(R((t)))$ called a loop group, which is, in fact, $LX = lim(L^nX)$.

Proposition 4.0.5. *()*
Let's see the affine Grassmannian $Gr_{\underline{G}}$ from an alternative perspective. Let

$\underline{G} = G \bigotimes \mathscr{O}$ *where* $\mathscr{O} = k[[t]]$*. The affine Grassmannian* $Gr_{\underline{G}}$ *can be identified with a fpqc quotient* $[L\underline{G}/L^{+}\underline{G}]$

Definition 4.0.6. *(Weyl Algebra)*
A Weyl algebra is a ring of diffenrential operators with polynomial coefficients, namely expressions of the form

$$f_m(X)\partial_X^m + f_{m-1}(X)\partial_X^{m-1} + ... + f_0(X)$$

where $f_k(X) \in F[X]$ *is a polynomial over a field* F *for any* k*, and* ∂_X *is a derivative with respect to* X*, and this algebra is generated by* X *and* ∂_X*.*

More generally, n*-th Weyl algebra* $A_n(X)$ *is defined by* n *variables* X_k *and* ∂_{X_k}*, and each function* $f_k(X_1, ..., X_n)$ *is simply a* n*-variable polynomial.*

In Weyl algebra, we have a Lie bracket $[x_i, \partial_{x_i}] = x_i\partial_{x_i} - \partial_{x_i}x_i = 1$*, and for function* f*,* $[\partial_{x_i}, f] = \partial f/\partial x_i$*.*

Definition 4.0.7. *(D-Module)*
D-module is simply a left-module M *over a Weyl algebra* $A_n(K)$ *over a field* K *of characteristic zero, and it could be philosophically considered as a sheaf with a connection.*

The slight generalization is the sheaf of differential oprators D_X*, defined to be* $\mathscr{O}_X$*-algebra generated by the vector fields on* X *interpreted as derivations. Here, the left action* $D_X \times M \to M$ *is equivalent to specifying a* K*-linear map*

$\nabla : D_X \to End_K(M)$ *where* $v \mapsto \nabla_v$ *satisfying*

1. $\nabla_{fv}(m) = f\nabla_v(m)$
2. $\nabla_v(fm) = v(f)m + f\nabla_v(m)$ *(Leibniz rule)*
3. $\nabla_{[v,w]}(m) = [\nabla_v, \nabla_w](m)$

Definition 4.0.8. *(G-Bundle)*
For a smooth projective curve X *and a Lie group* G*, we denote* $Bun_G(X)$ *by a moduli stack of fiber bungles (or called G-bundles) on* X*, and naturally let* $QCoh(Bun_G(X))$ *be a category of quasi-coherent sheaves of* $Bun_G(X)$*.* $D-Mod(Bun_G(X))$ *is a category of D-modules on* X*.*

Example 4.0.9. *()*
Let $X' = X \bigcup \{x\}$ *be a plane but a double point at* $x \in X$*. Then its G-bundle* $Bun_G(X')$ *will be given by a pull-back* $Bun_G(X') \cong Bun_G(X) \times_{Bun_G(X-x) Bun_G(X)}$*.*

We consider an action $Bun_G(X') \times Bun_G(X) \to Bun_G(X)$ *given by* $H \cdot E = p_{2*}(H \otimes p_1{}^*(E))$ *for* $H \in Bun_G(X')$ *and* $E \in Bun_G(X)$*.*

In particular, $D_x = Spec(\mathbb{C}[[t]])$ *be a disk around* x *for some uniformizer* t*, and similarly,* ${D_x}' = Spec(\mathbb{C}((t)))$ *be a punctured disk. We call* $H_x = D-Mod(Bun_G({D_x}'))$ *be a Hecke category for* $x \in X$*,* $X' = X \bigcup \{x\}$*.*

Proposition 4.0.10. *()*
A bundle $Bun_G({D_x}')$ *is given by a double coset space as* $Bun_G({D_x}') = G(O_x) \setminus G(K_x)/G(O_x)$*, thus the Hecke Category is* $H_x = DMod(G(O_x)\backslash G(K_x)/G(O_x))$*.*

In general, the idea can be extended globally. $DMod(Bun_G(X)) = DMod(G(\mathbb{C}(X))\setminus \prod_{y\in X} G(K_y)/G(O_y)) = DMod(G(\mathbb{C}(X))\backslash\prod_{y\neq X} G(K_y)/G(O_y)\times G(K_x))^{G(O_x)}$*. So* $DMod(Bun_G(X))$ *is indeed,* $G(O_x)$*-invariants of a* $G(K_x)$*-representation. The formula can be simplified by the further symbolization.* $DMod(Bun_G(X)) = DMod(G(F) \setminus G(\mathbb{A}_F)/G(O_F)$ *where* F *is a number field, and* $\mathbb{A}_F$ *is a ring of adeles.*

Proposition 4.0.11. *(Geometric Langlands Correspondence)*
First, we recall a Satake isomorphism for the case of $k((t))$*, then:*

$\mathbb{C}(G(k[[t]]) \setminus G(k((t)))/G(k[[t]]))$ *is isomorphic to the complexified representation ring of* $G^\vee$*.*

This will be generalized to $H_x = Rep(G^\vee)$

Let Hecke category $H = \bigotimes_{x\in X} H_x$*, and* $DMod(Bun_G(X)) = QCoh(Spec(H))$*,*

and the remaining problem is to identify $Spec(H)$*. Geomeric Langlands program is to consider the spectral decomposition of* $DMod(Bun_G(X))$*.*

Definition 4.0.12. *(Perverse Sheaf)*
Let X *be a scheme, and for* $x \in X$*, let* $j_x : \{x\} \to X$ *be an inclusion morphism. Then, by the Grothendieck six operators induce*

${j_x}^* : SH(\{x\}) \to SH(X)$*.*

${j_x}^! : SH(\{x\}) \to SH(X)$*.*

Here, We define a subscheme $Y \subset X$ *such that* $x \in Y$ *if*

$H^{-i}({j_x}^* C) \neq 0$ *or* $H^i({j_x}^! C) \neq 0$*, and they have real dimensions at most* $2i$ *for all* i*.*

Definition 4.0.13. *(Perverse Sheaves)*
Intersection cohomology is, in particular, a Chow ring and rational cohomology ring with cup product. $IC(O)$ *be an intersection cohomology complex of the closure of* L^+G*-orbit* O *extended by zero to all* Gr*, and it is a well defined object of* $D^b(Gr)$*.*

$IC\lambda = IC(O_\lambda)$ $O_\lambda \subset Gr$ is an orbit of $G(O) \times Gr \to Gr$

$Heck_X = \{(P_1, P_2, x, \phi) | P_1, P_2 \in Bun_G, x \in X, \phi : P_1|_{X\setminus\{x\}} \cong P_2|_{X\setminus\{x\}}\}$

The stack $Heck_X$ is not algebraic stack.

$pr_i : Heck_X \to Bun_G$ where $i = 1, 2$

$D^b(Heck_X) \times D^b(Bun_G) \to D^b(X \times Bun_G)$ $(N, A) \mapsto pr_{2*}(N \otimes pr_1{}^! A)$

Definition 4.0.14. *(Perverse sheaves)*
$P(Gr)$ be a abelian full subcategory of $D^b(Gr)$ whose objects are perverse sheaves of on Gr isomorphic to finite direct sums of complexes $IC(O)$. Any object $L \in P(Gr)$ has finite-dimensional hyper-cohomology $H^*(L)$.

This $P(Gr)$ acts by convolution:

$$\begin{aligned} * : P(Gr) \times D^b(Bun_G) &\to D^b(Bun_G) \\ (M,A) &\mapsto M \star A \end{aligned} \tag{3}$$

Definition 4.0.15. *(Geometric Satake Equivalence)*
$K(Perv(Gr)) \bigotimes_{\mathbb{Z}} \mathbb{C} \cong K(Rep(^L G)) \bigotimes_{\mathbb{Z}} \mathbb{C}$ where K is a Grothendieck Group.

Then this equivalence induces the equivalence by Tannakian duality. $Perv(Gr) \cong Rep(^L G)$

4.1 Schubert Variety

Definition 4.1.1. *(Grassmannian Variety)*
Let $\mathbb{G}(k, n)$ be a set of k-dimensional vector subspaces of the n-dimensional vector space. Then $\mathbb{G}(k, n)$ is naturally an algebraic variety, and it is called a Grassmannian variety.

Definition 4.1.2. *(Plücker Coordinate)*
Define $f_{j_1, j_2, \ldots, j_n}$ as a homogeneous polynomial for a flag j given by determinant of the following matrix

$$\begin{pmatrix} x_{1,j_1} & x_{1,j_2} & \ldots & x_{1,j_n} \\ x_{2,j_1} & x_{2,j_2} & \ldots & x_{2,j_n} \\ & & \ldots & \\ x_{n,j_1} & x_{n,j_2} & \ldots & x_{n,j_n} \end{pmatrix}$$

$G(k, n)$ is an open set of the Zariski topology on $k \times n$ matrices defined as the union over all k-subsets of $\{1, 2, \ldots, n\}$ of complements of the varieties $V(f_{(j_1, j_2, \ldots, j_n)})$, and it could be embedded in $\mathbb{P}^{\binom{n}{k}}$.

Definition 4.1.3. *(Schubert Cell)*
Let $j = (j_1 < j_2 < j_3 < \ldots < j_k) \in [n]$ be a flag. A schubert cell is a subset of the Grassmannian variety, and each cell is identified with a flag, which is defined by:

$C_j = \{U \in G(k, n) | position(U) = \{j_1, \ldots, j_k\}\}$

and the original Grassmannian variety is given by $\mathbb{G}(k,n) = \bigcup C_j$ over all possible k-subsets of $[n]$.

Schubert variety X_λ is the Zariski closure of Schubert cell C_λ.

Example 4.1.4. *(Grassmannian Variety)*
For a Grassmannian variety $G(3,10)$, a Schubert cell $C_{\{3,7,9\}}$ is given by a matrix

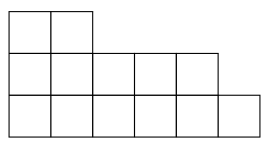
$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 1 & 0 & 0 \end{pmatrix}$$

and the dimension $dim(C_{\{3,7,9\}}) = 2+5+6 = 13$

In general, $dim(C_j) = \Sigma j_i - i$.

Definition 4.1.5. *(Partition and Young Diagram)*

Let a partition $\lambda = (\lambda_1 \geqq \lambda_2 \geqq ... \geqq \lambda_k)$ be a weakly increasing sequence of non-negative integers such that $n = \Sigma\lambda_i = |\lambda|$. Then the partition λ makes a block diagram called Ferrer diagram

$(2,5,6) \mapsto$ 

1. *$X_j = \bigcup_{i \subset j} C_i$*
2. *$dim(X_j) = |shape(j)|$*
3. *Grassmannian $G(k,n) = X_{\{n-k+1,...,n-1,n\}}$ is a Schubert variety.*

Definition 4.1.6. *(Schubert Class and Homology)*
Let $H^*(G(k,n))$ be a homology ring of the $G(k,n)$. Recall from the previous arguments, the given Grassmannian variety generates a family of Schubert varieties X_j inducing canonical basis elements of the cohoology ring called Schubert classes $[X_j]$. This hoology ring is a graded ring, where the multiplication is defined by the intersection:

$$[X_i][X_j] = [X_i(B^1) \cap X_j(B^2)]$$

Proposition 4.1.7. *(Pieri/Giambelli formula)*
Pieri/Giambelli formula are determined for Schubert variety and Schur functions.

1. *Giambelli formula: $[X_i] = det(e_{\lambda'_i - i + j})_{1 \leqq i,j \leqq k}$*

2. *Pieri formula :* $[X_i]e_r = \Sigma[X_i]$

On the other hand, there is an analogy of the formulae in Schur polynomials, and we will realize the comparison of Schubert classes and Schur polynomials.

1. *Giambelli formula:* $S_\lambda = det(e_{\lambda'_i - i + j})_{1 \leqq i,j \leqq k}$
2. *Pieri formula :* $S_\lambda e_r = \Sigma S_\mu$

Thus as rings $H^*(G(k,n)) \sim \mathbb{Z}[x_1, ..., x_n]^{S_n} / < S_n : \lambda \not\subset k \times n >$

$S_\lambda S_\mu = \Sigma c^\nu_{\lambda,\mu} S_\nu$ *where* $c^\nu_{\lambda,\mu}$ *is non-negative integer, and it is called a Littlewood-Richardson coefficient. This formula is an outcome of general young diagram, and it is inducedd by Giambelli formula. The special case of the formula is Pieri formula when one of the multipliers is young diagram of width* 1.

Consider that $s_p \cdot s_\lambda = \Sigma s_\nu$ *is a special case of adding* p *boxes.*

In general, using a order $\mu = (\mu_1 \geqq \mu_2 \geqq ... \geqq \mu_k)$,

$s_{\mu_1} \cdot s_{\mu_2} \cdot ... \cdot s_{\mu_r} \cdot s_\lambda = \Sigma K^\nu_{\lambda,\mu} s_\nu$ *where* $K^\nu_{\lambda,\mu} \in \mathbb{Q}$ *is a coefficient called Kostka number, and this can be computed by mathematical induction:*
the number of tableaux on ν/λ *with content* μ.

Definition 4.1.8. *(Quantum Homology Ring)*
$H^*(X) = \mathbb{Z}[\sigma_1, ..., \sigma_k]/(Y_{l+1}, ..., Y_n)$

Small quantum cohomology ring is given by:

$QH^*(X) = \mathbb{Z}[q, \sigma_1, ..., \sigma_k]/(Y_{l+1}, ..., Y_{n-1}, Y_n + (-1)^k q)$

where $Y_r = det(\sigma_{1+j-i})_{1 \geqq i,j \geqq r}$

4.2 Quantum Cohomology

Definition 4.2.1. *(Deligne-Mumford Stack)*
Deligne-Mumford stack is a stack F *such that*

1. *The diagonal morphism* $F \to F \times F$ *is representable, quasi-compact and separated*
2. *There is a scheme* U *and étale surjective map* $U \to F$ *(called an atlas)*

Definition 4.2.2. *(Moduli Space)*
We denote $M_{g,n}$ *by a moduli space of geometry of genus* g *and* n*-marked points. Moduli space is an algebraic stack. marked points means that when we consider an automorphism* $X \to X$, *the specified* n *points are fixed.*

Especially $M_{g,n}(X,\beta)$ for X a variety and for $\beta \in H_2(X,\mathbb{Z})$ is a moduli subspace.

Definition 4.2.3. *(Homology Ring)*
For a geometric object X, we have a singular homology group $H^i(X,\mathbb{Z})$ for each i. We consider a cup product $\cup : H^i(X,\mathbb{Z}) \times H^j(X,\mathbb{Z}) \to H^{i+j}(X,\mathbb{Z})$, so that the direct sum $H(X) = \bigoplus H^i(X,\mathbb{Z})$ will be a naturally graded ring.

Intuitively, the notion is in particular, a Chow ring $A(X)$.

Definition 4.2.4. *(Gromov-Witten Invariant)*
Let $M_{g,n}(X,\beta)$ be a moduli space. Consider a morphism $ev_i : M_{g,n}(X,\beta) \to X$ for $1 \leqq i \leqq n$, which naturally induces the product $ev = \prod ev_i : M_{g,n}(X,\beta) \to X^n$ for $\alpha_1, ..., \alpha_n \in H^*(X,\mathbb{Q})$ and $\beta \in H_2(X,\mathbb{Z})$, we define a Gromov-Witten invariant:

$$< \alpha_1, ..., \alpha_n >_{g,\beta} = \int_{[\overline{M_{g,n}}(X,\beta)]^{vir}} ev^*(\alpha_1 \times ... \times \alpha_n) \in \mathbb{Q}$$

for a virtual fundamental class $[\overline{M_{g,n}}(X,\beta)]^{vir} \in A_d(\overline{M_{g,n}}(X,\beta) \bigotimes \mathbb{Q}$, and the integral means that we are evaluating a cohomology class on a homology class.

Definition 4.2.5. *(Quantum Cohomology)*
Let $T_0, ...T_n \in H^*(X,\mathbb{Z})$ be elements of the homology ring, but $T_0 \in H^0(X,\mathbb{Z})$. Moreover, let $\gamma = \Sigma y_k T_k$, then we define Gromov-Witten potential as

$$\Phi = \Sigma_{n=0}^{\infty} \Sigma_{\beta \in H_2(X,\mathbb{Z}} \frac{1}{n!} < \gamma^n >_{0,\beta} \in \mathbb{C}[[y_0, ..., y_n]]$$

Using Φ, we define a quantum cohomology of X as the ring

$$H^*(X, \mathbb{C}[[y_0, ...y_n]])$$

with a product given by generators by

$T_i \star T_j = \Sigma_k(\partial_{y_i}\partial_{y_j}\partial_{y_k}\Phi)T^k$ *where $T^0, ..., T^n$ are Poincare dual basis of $T_0, ..., T_n$. It is often useful to define g^{ij} to be $g^{ij} = \int_X T_i \cup T_j$ so that $T_i = \Sigma g^{ij} T_k$. $T_i \star T_j = \Sigma_{a,k}(\partial_{y_i}\partial_{y_j}\partial_{y_k}\Phi)g^{ak}T_k$.*

Definition 4.2.6. *(WDVV Equation)*
Notice that the previous $\star$ operation has an associative property, so $(T_i \star T_j) \star T_k = T_i \star (T_j \star T_k)$, it directly deduces a PDE called WDVV equation.

$$\Sigma_{a,b}(\partial_{y_i}\partial_{y_j}\partial_{y_a}\Phi)g^{ab}(\partial_{y_b}\partial_{y_k}\partial_{y_l}\Phi) = (-1)^{deg(T_i)(deg(T_j)+deg(T_k))}\Sigma_{a,b}(\partial_{y_j}\partial_{y_k}\partial_{y_a}\Phi)g^{ab}(\partial_{y_b}\partial_{y_i}\partial_{y_l}\Phi)$$

4.3 Quantum Satake

We apply the Geometric Satake correspondence to Quantum Satake isomorphism to compute minuscule Grassmannians particularly Dynkin diagram of type D.

Definition 4.3.1. *(Pfaffian)*
Pfaffian is a polynomial $pf(A)$ given that A is a $n \times n$ skew-symmetric matrix, and $pf(A)^2 = det(A)$.

Definition 4.3.2. *(Clifford algebra)*
Clifford algebra for spinor variety.

Definition 4.3.3. *(Minuscule Representation)*
Minuscule representation of a semi-simple Lie algebra is an irreducible representation such that Weyl group acts transitively on the weights. Also, highest weight of minuscule representation is called minuscule weight.

Definition 4.3.4. *(Orthogonal Grassmannian)*
Orthogonal Grassmannian $OG(k, n)$ is defined by a geometric quotient of an algebraic group G by parabolic subgroup $P \subset G$,

One of the application of Quantum Satake isomorphism for spinor variety $OG(5, 10)$ is Apery's differential equation.

Example 4.3.5. *(Apery's Diffyq)*
Assuming that $t \neq 0, 1, (\sqrt{2}+1)^2, \infty$, then Apery's differential equation is given by

$$S_t = 1 - (1 - XY)Z - tXYZ(1 - X)(1 - Y)(1 - Z) = 0.$$

and S_t is birationally equivalent to K3 surface X_t!

Definition 4.3.6. *(Quadric)*
In algebraic geometry, quadric (or quadric hypersurface) $\mathbb{Q}^{2n-2}$ is the subspace of N-dimensional projective space defined by a polynomial equation of degree 2 over a field. For n-dimensional quadric, we have n variables $\{x_i\}$ makes a quadric relation

$$x_i x_j = x_k x_l$$

4.4 Higgs Bundle

Definition 4.4.1. *(Jacobian Variety)*
Let Σ_g be a Riemann surface of genus g. Also, let $Jac_0(\Sigma_g)$ be a Jacobian variety of a Riemann surface Σ_g.

We care the classification problem of $Jac_0(\Sigma_g)$.

Definition 4.4.2. *(Higgs Bundle)*
A Higgs bundle is a pair (E, Φ) *such that* $\Phi : E \to E \bigotimes K$ *is an End E-valued holomorphic* $(1,0)$*-form on* Σ_g*. That is,*

$$\Phi \in H^0(\Sigma_g, End(E \bigotimes K))$$

This Φ *is called a Higgs field of the Higgs bundle.*

Definition 4.4.3. *(Moduli of Higgs Bundle)*
Define a moduli space of stable Higgs bundle of rank n*, degree* d *on a Riemann surface* Σ_g *by*

$$M^g_{n,d} := B^s / \mathscr{G}_{\mathbb{C}}$$

where B^s *denotes the subset of* B *consisting of stable Higgs bundles.*

Definition 4.4.4. *(Narasimhan-Seshadri)*

Definition 4.4.5. *(Hitchin Integrable System)*
A map $M^g_{n,d} \to \bigoplus_{k=1}^n H^0(\Sigma_g, K^k) =: A$

Theorem of Hitchin claims

$$dim(A) = \frac{1}{2} dim(M^g_{n,d}) = 2 + 2n^2(g-1)$$

and h *is completely integrable Hamiltonian system.*

Example 4.4.6. *(Physics)*
$N^g_{n,d}$ *and* $M^g_{n,d}$ *are the configuration spaces of Chern-Simon gauge theory for* $U(n)$*,* $GL(n, \mathbb{C})$ *in* $2+1$ *dimensions, for the three-manifolds* $\Sigma_g \times [0,1]$*.*

$M^g_{n,d}$ *is used to describe S-duality in string theory, and mathematically this translates to a geometric Langlands correspondence.*

Example 4.4.7. *(Mirror Symmetry)*
Moduli space of G-Higgs bundle of arbitrary Lie groups G *is denoted by* $M(G)$*. Then, the two Hitchin systems* $h_1 : M(G) \to A$ *and* $h_2 : M({}^LG) \to A$ *generates SYZ-fibration in mirror symmetry, and it is expected to work for any pair of groups* G *and* ${}^LG$*.*

Example 4.4.8. *(Non-Abelian Hodge Theorem)*
The Hodge theorem states that

$$H^1(\Sigma_g, \mathbb{C}) = H^{1,0}(\Sigma_g) \bigoplus H^{0,1}(\Sigma_g)$$

which is straight-forward to see that

$H^1(\Sigma_g, \mathbb{C}) \cong H^1(\pi_1(\Sigma_g), \mathbb{C}) = Hom(\pi_1(\Sigma_g), \mathbb{C})$

Here, the non-abelian Hodge decomposition is exactly the same but the replacement by $GL(n, \mathbb{C})$
$H^1(\Sigma_g, GL(n, \mathbb{C})) \cong H^1(\pi_1(\Sigma_g), GL(n, \mathbb{C})) = Hom(\pi_1(\Sigma_g), GL(n, \mathbb{C}))/GL(n, \mathbb{C})$

and produces a holomorphic vector bundle and a Higgs field, i.e. an element of

$H^1(\Sigma_g, \mathscr{GL}(n, \mathbb{C}) \bigotimes H^0(\Sigma_g, \mathscr{GL}(n, \mathbb{C} \bigotimes K)$

4.5 Shimura Variety

Geometric satake isomorphism is used to define excursion operator of Shimura variety.

5 Shimura Variety

5.1 Modular and Cusp

Definition 5.1.1. *(Modular Form)*
We call a function f is called a weakly modular of weight k if for the entire upper half plane $H = \{z | Im(z) \geqq 0\}$ and for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\tau \in H$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(\tau) = \frac{1}{(cx+d)^k} f(\tau)$.

In particular, f is called a modular function if f is a weak modular form of weight k, and it is holomorphic in the entire of H and at ∞.

The set of all such functions is $M(SL_2(\mathbb{Z})) = \bigoplus M_k(SL_2(\mathbb{Z}))$ where M_k is a set of all modular functions of weight k. Notice that these M is naturally graded.

Definition 5.1.2. *(Congruence Group)*
We define some important subgroup of $SL_2(\mathbb{Z})$

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{(mod N)}\}$$
$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{(mod N)}\}$$
$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{(mod N)}\}$$

The above three makes an inclusion $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$, and moreover, they are normal subgroups $\Gamma(N) \trianglelefteq \Gamma_1(N) \trianglelefteq \Gamma_0(N) \trianglelefteq SL_2(\mathbb{Z})$.

Definition 5.1.3. *(Weil Pairing)*
Let $[N] : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ *such that* $z + \Lambda \mapsto Nz + \Lambda$*, which is an isogeny since* $N\Lambda \subset \Lambda$*, and its kernel is* $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$*, and this is denoted by* $E[N]$*. More precisely,* $E[N] = \{P \in \mathbb{C}/\Lambda | [N]P = 0\} =< \omega_1/N + \Lambda > \times < \omega_2/N + \Lambda >$*.*

Definition 5.1.4. *(Modular Function wrt* Γ*)*
A function f *is modular with respect to* Γ *if* f *is holomorphic,* f *is weakly modular of weight* k *by an action of* $\Gamma \subset SL_2(\mathbb{Z})$*, and* $f[\alpha]_k$ *is holomorphic at* ∞ *for all* $\alpha \in SL_2(\mathbb{Z})$*.*

Definition 5.1.5. *(Moduli Space of Elliptic Curve/ Moduli Space of Modular Curve)*
An enhanced elliptic curve for $\Gamma_0(N)$ *is an ordered pair* (E, C) *where* E *is a complex elliptic curve and* C *is a cyclic subgroup of* E *or order* N*. Take* (E, C) *and* (E', C') *are equivalent if there exists an isomorphism* $E \cong E'$ *that takes* C *to* C'*. We let* $S_0(N)$ *as a given by a set of enhanced elliptic curves for* $\Gamma_0(N)$ *quotioned by the equivalence class.*

Similarly, let (E, Q) *be a set of an elliptic curve* E *and a point* $Q \in E$ *of order* N*.* $S_1(N)$ *is given by the set of all* (E, Q) *quotioned by its equivalence class.*

Similarly, let (E, P, Q) *be a set of* E *and points* $P, Q \in E$ *that generates a torsion subgroup* $E[N]$ *with Weil pairing* $e_N(P, Q) = e^{e\pi i/N}$*.* $S(N)$ *is given by the set of all* (E, Q) *quotioned by its equivalence class.*

Definition 5.1.6. *(Modular Curve)*
Modular curve $Y(\Gamma) = \Gamma \backslash H = \{\Gamma\tau | \tau \in H\}$ *is a quotient space of orbits under* Γ*, and similarly,* $Y_0(N) = \Gamma_0(N) \backslash H$*,* $Y_1(N) = \Gamma_1(N) \backslash H$*, and* $Y(N) = \Gamma(N) \backslash H$*.*

Moreover, its compactification is $X(\Gamma) = \Gamma \setminus H^* = Y(\Gamma) \bigcup \Gamma \setminus (\mathbb{Q} \bigcup \{\infty\})$

Proposition 5.1.7. *()*
$S_0(N) \cong Y_0(N)$
$S_1(N) \cong Y_1(N)$
$S(N) \cong Y(N)$*.*

Definition 5.1.8. *(Riemann Surface)*

Proposition 5.1.9. *()*
The upper half complex plane is a Riemann surface, and the quotient is also a Riemann surface.

Definition 5.1.10. *(Elliptic Points)*
Let there be an isotropy subgroup of τ *as* $\Gamma_\tau = \{\gamma \in \Gamma | \gamma(\tau) = \tau\}$*, a point* $\tau \in H$ *is an elliptic point for* Γ*.*

Proposition 5.1.11. *(Shimura Taniyama Conjecture)*
There exists an isomorphism between $X_0(N)$ to the elliptic curve E. The morphism $X_0(N) \to E$ is called a modular parametrization of E.

Definition 5.1.12. *(Hecke Operators)*
We have two kind of Hecke operators, where one is called a diamond operator denoted by $< d >$.

$< d >: M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$

so that $< d > f = f[\alpha]_k$ *for any* $\alpha = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(N)$ *with* $\delta \cong d (mod\ N\ )$.

The other is T_n. For the case of n is prime,

$T_p : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$

given by $T_p f = f[\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N)]_k$.

The operator can be generalized for any natural number $n \in \mathbb{N}$.
$T_{p^r} = T_p T_{p^{r-1}} - p^{r-1} < p > T_{p^{r-2}}$ *for* $r \geqq 2$.
Consider that all natural numbers have a prime decomposition $n = \prod p_i^{e_i}$, so defining $T_n = \prod T_{p_i{}^{e_i}}$, we have following relationship:

$T_n T_m = T_{nm}$ *if* $gcd(n, m) = 1$

$< n >< m >=< nm >$ *if* $gcd(n, m) = 1$

5.2 Hermitian Symmetric Domain

Definition 5.2.1. *(Hermitian Form)*
Let V be a vector space, and $J : V \to V$ be an endomorphism such that $J^2 = -1$. An Hermitian form on (V, J) is an operation $(|) : V \times V \to \mathbb{C}$ such that $(Ju|v) = i(u|v)$ and $(u|v) = \overline{(v|u)}$.

In particular, $(u|u)$ is a real value because $(u|u) = \overline{(u|u)}$.

Definition 5.2.2. *(Analytic and Holomorphic)*
A function f is holomorphic at $p \in M$ if f is complex differentiable at the point. , particularly function f is holomorphic if it is holomorphic at all $p \in M$.

A function f is analytic at $p \in M$ if f is given by a power series at some neighborhood of p.

For a one-variable complex function $f : U \to \mathbb{C}^n$, f is holomorphic iff it is analytic. However, it is not the case for the case of higher dimensions.

Definition 5.2.3. *(Complex Strucutre of Manifold)*
A complex manifold is a manifold M with a complex structure i.e. a sheaf of $\mathscr{O}_M$ of $\mathbb{C}$-valued functions such that $(M, \mathscr{O}_M)$ is locally isomorphic to $\mathbb{C}^n$ with its sheaf of analytic functions, and for any covering $\{U_\alpha\}$, namely $M = \bigcup U_\alpha$, and morphisms $u_\alpha : U_\alpha \to \mathbb{C}^n$, $u_\alpha \circ {u_\beta}^{-1}$ is analytic.

Almost complex structure on a smooth manifold M is a smooth tensor field $J = (J_p)_{p \in M}$ where $J_p : T_pM \to T_pM$ such that $J_p^2 = -1$ for all p.

Definition 5.2.4. *(Hermitian Symmetric Space)*
A manifold is said to be homogeneous if for each point $p, q \in M$, there exists an automorphism $f \in Aut(M)$ such that $f(p) = q$.

A manifold is said to be symmetric if it is homogeneous at some point p and the point p is a fix point $s_p(p) = p$, where ${s_p}^2 = 1$, and p is only the fix point of the neighborhood of p, so it is an isolated fix point.

Let $Hol(M)$ denote the set of all automorphism of M as a complex manifold, and $Hol(M)$ is a group. Automorphism of Hermitian manifold (M, g) is denoted by $Is(M, g)$ as a holomorphic isometries.

$$Is(M, g) = Is(M^\infty, g) \bigcap Hol(M)$$

A connected symmetric Hermitian form is called a Hermitian symmetric space.

Proposition 5.2.5. *(Classification and Decomposition by Curvature)*
Considered that a Hermitian symmetric manifold (M, g) as a Riemannian manifold, it has a curvature, and it classifies itself by the sign of curvature.

1. *H_1, noncompact, simply connected, negative curvature, and $Is(M, g)$ is adjoint and compact.*
2. *$\mathbb{P}^1(\mathbb{C})$ is compact, simply connected, positive curvature $Is(M, g)$ is adjoint and compact.*
3. *$\mathbb{C}/\Lambda$, zero curvature*

A Hermitian symmetric domain is called irreductible if there is no subdomain of lower dimension. In fact, all Hermitian symmetric domain M can be decomposed into a product of $M \cong M^0 \times M^- \times M^+$ where each of them is a Hermitian symmetric subdomain of zero, negative, positive curvatures.

Example 5.2.6. *()*
For a Hermitian symmetric domain (M, g), let's decomposed to M^0, M^-, and M^+ by its curvature, so denote the group of automorphism $Hol(M)^+$. Since it is a Lie group, expotential map defines a Lie algebra $\mathfrak{h}$. Then, there is a unique

connected algbraic subgroup G of $GL(\mathfrak{h})$ such that $G(\mathbb{R})^+ = Hol(M)^+$.

For such a G, $G(\mathbb{R})^+ = G(\mathbb{R}) \bigcap Hol(M)$

Therefore, $G(\mathbb{R})^+$ is a stabilizer in $G(\mathbb{R})$ of M.

Note that $G(\mathbb{R})^+$ is real Lie group, but there is a one-to-one correspondence with its complexification, and it may be classified by Dynkin diagram.

Definition 5.2.7. *(Cartan Involution)*
Let G be a connected algebraic group over $\mathbb{R}$, an involution θ of G is said to be Cartan if

$$G^\theta(\mathbb{R}) = \{g \in GL(\mathbb{C}) | g = \theta(\overline{g})\}$$

Proposition 5.2.8. *(Classification of Hermitian Symmetric Domain)*
For a real Lie group $Hol(D)^+$ induced from a Hermitian symmetric domain, there is a complexification with some representation $u : U_1 \to G(\mathbb{R}$, which is classified by Dynkin diagram if it is semisimpmle.

Definition 5.2.9. *(Quotient of Hermitian Symmetric Domain)*
Let D be a Hermitian symmetric domain, and Γ be a discrete subgroup of $Hol(D)^+$. If Γ is torsion-free, then Γ acts freely on D, and there is a unique complex structure $\Gamma \setminus D$ for which the quotient map $\pi : D \to \Gamma \setminus D$ is a local isomorphism. A morphism $\phi : \Gamma \setminus D \to M$ for some complex manifold M is holomorphic iff $\phi \circ \pi$ is holomorphic.

Let S_1 and S_2 be commensurable if $S_1 \bigcap S_2$ is a subgroup of S_1 and S_2 and its finite index respectively. Let G be an algebraic group over $\mathbb{Q}$, and a subgroup Γ of $G(\mathbb{Q})$ is arithemetic if it is commensurable with $G(\mathbb{Q}) \bigcap GL_n(\mathbb{Z})$ for some embedding $G \to GL_n$. It is then commensurable with $G(\mathbb{Q}) \bigcap GL_{n'}(\mathbb{Z})$ for every embedding $G \to GL_{n'}$.

Definition 5.2.10. *()*
There is a unique functor $(V, \mathscr{O}_V) \to (V^{an}, \mathscr{O}_{V^{an}}$ from nonsingular varieties over $\mathbb{C}$ to complex manifolds. We consider the following conditions:

$V = V^{an}$ as a sets, and a Zariski-open subsets are open in V^{an}, and every regular function is holomorphic.

If $V = \mathbb{A}^n$, then $V^{an}k = \mathbb{C}^n$ with its natural structure as a complex manifold

If $\phi : V \to W$ is étale, then $\phi^{an} : V^{an} \to W^{an}$ is an local isomorphism.

Note 5.2.11. *()*
Every compact Riemann surface of genus $g \geqq 2$ is a quotient of H_1 by a discrete subgroup of $PGL_2(\mathbb{R})^+$ acting freely on H_1.

If Γ is torsion free, then D is a universal covering space of $\Gamma \setminus D$.

Definition 5.2.12. *(Algebraic Variety)*
Let A be an affine k-algebra, and define $spm(A)$ to be a set of maximal ideals of A with a topology whose basis is given by $D(f0 = \{m | f \notin m\}$ and $f \in A$. There is a unique sheaf of k-algebras $\mathscr{O}$ on $spm(A)$ such that $\mathscr{O}(D(f)) = A_f$ for all f where A_f is an algebra obtained by inverting f. Thus an affine algebraic variety is given by $Spm(A) = (spm(A), \mathscr{O})$, and stalk at m is a local ring A_m, so it is a locally ringed space.

Let V be an algebraic variety over k, and let R be a k-algebra. We let $V(R)$ denote the set of points of V with coordinates in R, and when A is affine, $V(R) = Hom_{k-alg}(A, R)$, and in general, $V(R) = Hom_k(Spm(R), V)$.

Let $D(\Gamma) = \Gamma \setminus H_1$ Then it is an Zariski open subset of a projective algebraic variety $D(\Gamma)^*$ which is given by $D(\Gamma)^* = \Gamma \setminus {H_1}^* = \Gamma \setminus (H_1 \bigcup \mathbb{P}^1(\mathbb{Q}) \bigcup \{\infty\}) = \Gamma \setminus H_1 \bigcup \Gamma \setminus (\mathbb{P}^1(\mathbb{Q}) \bigcup \{\infty\})$.

Definition 5.2.13. *(Adele)*
A ring of finite adeles is $\mathbb{A}_f = \prod_l (\mathbb{Q}_l, \mathbb{Z}_l)$ where l runs over the finite primes of $\mathbb{Q}$. Note that $\mathbb{A}_f = \hat{\mathbb{Z}}j \bigoplus_{\mathbb{Z}} \mathbb{Q}$. Then,

$V(\mathbb{Z}_l) = Hom_{\mathbb{Z}}(\mathbb{Z}[x_1, ..., x_m], \mathbb{Z}_l) = V(\mathbb{Q}_l) \bigcap {\mathbb{Z}_l}^m$.

$V(\mathbb{A}_f) = \prod (V(\mathbb{Q}_l), V(\mathbb{Q}_l))$

In particular,

$G(\mathbb{A}_f) = \prod (G(\mathbb{Q}_l), G(\mathbb{Q}_l))$, *and*

$\mathbb{G}_m(\mathbb{A}_f) = \prod (\mathbb{G}_m(\mathbb{Q}_l), \mathbb{G}_m(\mathbb{Q}_l))$.

Definition 5.2.14. *(Connected Shimura Datum)*
A connected Shimura datum is a pair (G, D) consisting of a semisimple algebraic group G over $\mathbb{Q}$ and $G^{ad}(\mathbb{R})^+$- conjugacy class D, of homomorphisms $u : U_1 \to {G^{ad}}_{\mathbb{R}}$ satisfying the following conditions:

SU1: for all $u \in D$, only the characters 1, z, z^{-1} occur in the representation of U_1 on $Lie(G^{ad})_{\mathbb{C}}$ defined by $Ad \circ u$.

SU2: for all $u \in D$, $ad(u(-1))$ is a Cartan involution on ${G^{ad}}_{\mathbb{R}}$.

SU3: G^{ad} has no $\mathbb{Q}$-factor H such that $H(\mathbb{R}$ is compact.

Definition 5.2.15. *(Connected Shimura Variety)*
Let (G, D) be a connected Shimura datum. A connected Shimura variety relative to (G, D) is an algebraic variety of the form $D(\Gamma)$ with Γ an arithmetic subgroup of $G^{ad}(\mathbb{Q})^+$ containing the image of congruence subgroup of $G(\mathbb{Q})^+$ and such

that $\overline{\Gamma}$ is torsion free. The inverse system of such algebraic varieties, denoted $Sh^o(G,D)$ is called connected Shimura variety attached to (G,D).

5.3 Some Properties

Proposition 5.3.1. *(Shimura-Taniyama Formula)*

Proposition 5.3.2. *(Frobenius map π_V)*

6 Summary and Future

1. (What I have learned from this project)
   From this experience, I could successfully broaden my perspective in mathematics that connects to mirror symmetry. For example, I found Langlands program can connect arithemtic property to geometric property through D-module and Satake correspondence, and I learned how number theory and Lie algebra connect to geometry.

2. (What I have not done in this project)
   I need more study of basic category theory, Tannakian formalism, Hopf algebra, and its connection to Galois theory and foundational study of reductive algebraic groups. Also, I wanted to know the application of Langlands program for Higgs bundle, excursion operator of Shimura variety, computation of L-functions, D-module, Drinfeld module etc, but I gave them up, I am out of time.

3. (About my Former Project)
   In my prior project, I studied étale cohomology with a view toward Weil conjecture, but I have not master in étale cohomology yet, so I should continue studying this project. In addition, I recently founded there is an alternative proof of Weil conjecture with non-commutative geometry with a field with one element. Also, I need to study more about perverse sheaves for Geometric Langlands program.

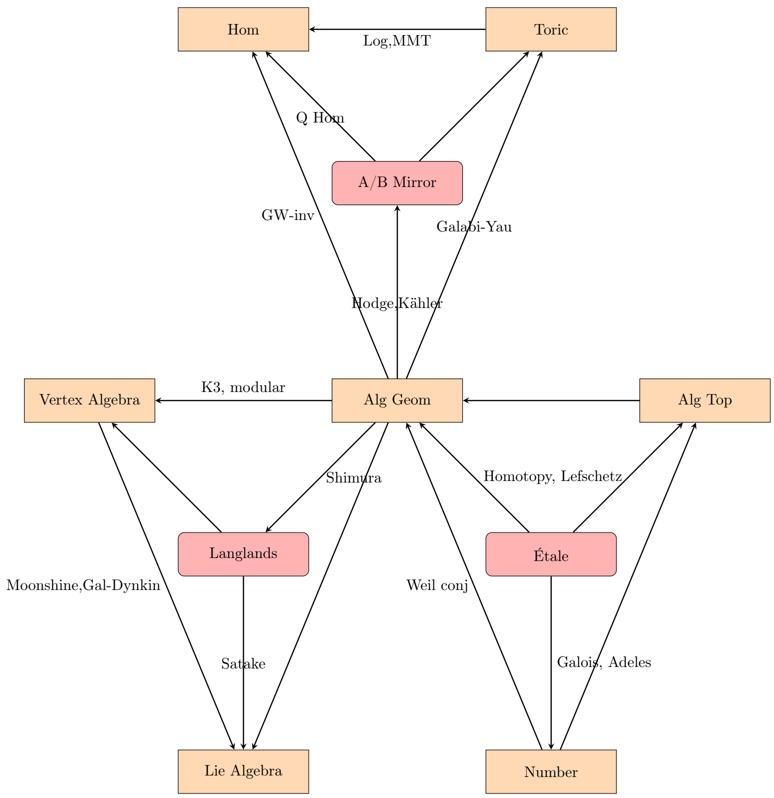
4. (What I will do in the future)
   In Spring 2023, I will study something else, but continue several of mini-projects. One of my interests is non-commutative geometry, which requires K-theory, C^*-algebra, Ergodic theory from the non-algebraic side, and algebraic side is non-commutative algebraic geometry, differential graded algebra. Another important component is field with one element $\mathbb{F}_1$,

which makes $\mathbb{F}_1$-geometry, that originates to the proof of Weil conjecture and Arakelov theory. Arakelov theory is a theory of compactification of $Spec(\mathbb{Z})$.

Another mini-project is to study more about the relationship between Lie theory and algebraic geometry. These two subjects were traditionally considered to be different, but we founded several common properties since 21st century, which is, for example, quantization of Poisson Lie group with Frobenius manifold. Calabi-Yau manifold is studied by Lie algebra or moonshine. Hilbert scheme studies ADHM construction.

Finally, I will study derived algebraic geometry, which is a prepration for summer school (Jun 24 2023), but also has an interesting connection to geometric Langlands program and Gauge theory. For example, "geometric Langlands twists of $N = 4$ Gauge theory from derived algebraic geometry" looks intersting. For derived algebraic geomety, many basic in algebraic geometry and category theory should be required.

Finally, if we complete the above three mini-projects, we could be familiar with Arakelov theory, p-adic Hodge theory and anabelian theory immediately or sooner, and we are on the entrance of Interuniversal Teichmüller theory, that is also interesting.



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