QE II – Cheat Sheet (except Toric Variety)

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1 Intro – A-Model & B-Model

- General
  Today I will introduce mirror symmetry, which is part of one of the six string theories. The 10-dimensional space is modeled by $\mathbb{R}^{1,3} \times X$, $\mathbb{R}^{1,3}$ is a composition of 4-dimensional Minkowski space and X is a 6-dimesional Calabi-Yau manifold. for it contains $SU(3)$ holonomy. In mirror symmetry, we define A-model and B-model in Calabi-Yau, and we compare these two relations. In particular, in this talk, the mirror symmetry also exists in the hypersurface of Calabi-Yau, namely a weighted projective space, and in particular, a projective space $\mathbb{P}^2$ is interesting.

A-model is a symplectic side of study and it is studied with Gromov-Witten invariants, which is intersection theoretic while B-model is a study of complex manifold and period integral. According to M. Gross, mirror symmetry can be defined by Barannikov map between the semi-infinite variation of Hodge struture of A-model and that of B-model.

- A-model
  A-model is an intersection theory for a Kontsevich moduli space $\overline{M}_{g,n}(X, \beta)$, where X is a scheme, $\beta \in H^*(X, \mathbb{Z})$ is an element of the cohomology ring, and this Kontsevich moduli space induces Gromov-Witten invariant. The evaluation morphism $ev_i : \overline{M}_{g,n}(X, \beta) \to X$ pull-backs morphism of cohomology, Gromov-Witten invariant, invariant over the moduli problem, can be a rational cohomology problem, but the problem can have furthermore translation to quantum cohomology, which is because Gromov-Witten potential works as an invariant of the cohomological computation.

  Recall that $X = \mathbb{P}^2$ has a singular homology so it naturally induces a cohomology ring $H^*(X, \mathbb{Q})$ where each element $[Y] \in H^*(X, \mathbb{Q})$ has a representation of a subscheme $Y \subset X$. A cohomology ring has intersection properties by cup product defined by $[Y] \cup [Z] = [Y \cap Z]$, where the latter is a set theoretical intersection of schemes. If Y and Z are codimension n and m, then $Y \cap Z$ is codimension $n+m$, and notice that the dimension of the scheme will be 0 if its codimension is $n+m = 4 = dim(\mathbb{P}^2)$, and since the given scheme is noetherian and zero dimensional, it is simply a finite set of isolated points, whose cardinality is the degree: this is intersection theory, how to count geometry by number.

  Similarly, a quantum cohomology ring $H^*(X, \mathbb{C}[[y_0, ..., y_m]])$ is induced from the formal completion of the coefficients ring (a ring is an abelian group), the change of coefficients are guaranteed by the universal coefficient theorem, and lim and Hom are functorial. This is the way we compute Gromov-Wittwn invariants, so afterall, the Moduli problem reduced to the quantum cohomology of a scheme. This quantum cohomology is difficult to compute, but it can be considered as a Frobenius manifold, and its Dubrouvin connection defines a quantum diffyq, which derives some nice value of Gromov-Witten invariants, which might help us to compute semi-infinite variation of its Hodge structure $\mathscr{H}_A$. This Hodge structure has a $N \times N$ matrix representation with Pressley-Segal Grassmannian, which is a generalized Grassmannian of infinite dimensional algebraic geometry.

- B-model
  In B-model, our goal is to compute semi-infinite variation of the Hodge structure, similar to A-model. We study the pair $(\check{X}, W)$ where $\check{X} = (\mathbb{C}^\times)^n$ is a dual of $X = \mathbb{P}^n$ and $W = x_0 + x_1 + ... + x_n$ is a potential, and this describes Landau-Ginzburg model. In Landau-Ginzburg model, our

concern is a computation of $\int g(x)e^{f(x)}dx$ for some $g(x)$ and $f(x)$, and this corresponds to the pair $(\check{X}, W)$.

Define a twisted de Rham complex $(\Omega^{\cdot}_X, d+dW\wedge)$, where Ω^p_X is a sheaf of p-form, and this chain complex of sheaves induces cohomology of sheaves. This cohomology is diffcult to compute, but it can be equivalent to say hypercohomology, that can be computed by the spectral sequence of the cohomology of abelian groups(modules). It turns out, in fact, each hypercohomology $\mathbb{H}^n(X, (\Omega^{\cdot}_X, dW\wedge))$ for $n \in \mathbb{N}$ is a Milnor ring. This hypercohomology helps compute an oscillatory integral $\int_{\Xi} e^{W/\hbar}\omega$ for some algebraic cycle. More generally, if we change the value of potential W, the integral can be generalized. Let a moduli space $\tilde{M}$ be a ringed space $(\mathbb{C}, \mathscr{O}_{\tilde{M}})$ that generalizes the potential W, then $\int_{\Xi} e^{W/\hbar} f\Omega \in \mathscr{O}_{\tilde{M}\times\mathbb{C}^{\times}}$ for some function f and differential form $\Omega = \frac{dx_0\wedge\ldots\wedge dx_n}{x_0\ldots x_n}$. Let we let the sheaf of the moduli space is $\mathscr{R} = R \otimes \mathscr{O}_{\tilde{M}\times\mathbb{C}^{\times}}$, then $[f\Omega]$ is a section of $\mathscr{R}$, and if we define a Gauss-Manin connection $\nabla^{GM}_X[f\Omega]$ over $\tilde{M}\times\mathbb{C}^{\times}$, which defines E, ∇, $(-,-)_E$ and Gr yield a semi-infinite variation of Hodge structure $\mathscr{H}_B$.

- Mirror Symmetry
  Our goal is to define a mirror map $m : \tilde{M}_A \to \tilde{M}_B$, that can be described by the semi-infinite variations of Hodge structure.

  Consider each of the Hodge structure H_A and H_B can be decomposed to $H_A = E_A \oplus H_A^-$ and $H_B = E_B \oplus H_B^-$. The quotients of the negative Hodge structure have isomorphisms $\hbar H_A^-/H_A^- \cong H^*[T_1]/(T_1)^{n+1}$ and $\hbar H_B^-/H_B^- \cong H^*[\alpha]/(\alpha)^{n+1}$ Hence we have an explicit way of mirror symmetry map

$$\begin{aligned} \hbar H_A^-/H_A^- &\to \hbar H_B^-/H_B^- \\ T_1^i &\to \hbar^{-(n+1)\alpha}(\hbar\alpha)^i \end{aligned} \tag{1}$$

  Hence, .

$$E^A \to (\hbar H_-^A/H_-^A)\otimes_{\mathbb{C}} \mathscr{O}_{\tilde{M}}\{\hbar\} \to (\hbar H_-^B/H_-^B)\otimes_{\mathbb{C}} \mathscr{O}_{\tilde{M}}\{\hbar\} \to E^B \tag{2}$$

  which is the mirror symmetry for $\mathbb{P}^n$.

2 Moduli Problem

Definition 2.0.1 *(What is moduli space?)*
The definion of Moduli space varies; it could be a set of isomorphism class of objects, or it could be a moduli functor, not space. Here, let's list Kontsevich moduli space we need for this argument, and several analogies.

- *M_g is a moduli space of curve, which is a set of smooth, proper, irreducible complex projective curves of genus g quotioned by its isomorphism classes.*

- *$\overline{M}_{g,n}$ – Fine moduli space of curve of genus g and n-marked points. This is a set where each element is $(\pi, C, p_1, ..., p_n)$ where $\pi : C \to S$ is a projection and $p_1, ..., p_n : S \to C$ are sections for some fixed scheme S.*

- *$\overline{M}_{g,n}$ is a scheme.*
  $S \mapsto \{$isom classes of flat familes $C \to S$ with sections $\sigma_1, ..., \sigma_n : S \to C$ such that $(C_{\overline{s}}, \sigma_1(\overline{s}), ..., \sigma_n(\overline{s}))$
  is a stable n-pointed genus g curve for every geometric point $\overline{s}$ of $S\}$
  The morphism is a functor $Hom(-, \overline{M}_{g,n}) : Sch^{op} \to Set$

- *$\overline{M}_{g,n}(X, \beta)$ – Kontsevich moduli space.*
  Consider a moduli functor
  $S \mapsto \{$isom classes of flat familes $C \to S$ with sections $\sigma_1, ..., \sigma_n : S \to C$and morphism$f : C \to X$ such that $f : (C_{\overline{s}}, \sigma_1(\overline{s}), ..., \sigma_n(\overline{s})) \to X$
  is a stable map of genus g representing β for every geometric point $\overline{s}$ of $S\}$
  where $\beta \in H_2(X, \mathbb{Z})$
  The moduli functor is representable by a proper DM-stack $Hom(S, \overline{M}_{g,n}(X, \beta))$, not a scheme anymore. We write the stack of n-pointed stable maps of genus g representing β as $\overline{M}_{g,n}(X, \beta)$.

Proposition 2.1 *(Properties of Moduli)*

- *Chow ring*
  Proper DM-stack is not a scheme, but we can define a Chow ring. All proper DM-stack X has an embedding to $[[A]^{ss}/GL_r]$ for some $r \in \mathbb{N}$ and $\mathbb{A}$, where a Chow ring is definable over a scheme $\mathbb{A}$, so naturally in the quotient $[\mathbb{A}^{ss}/GL_r]$, thus in the proper DM-stack X.

  The evaluation map $ev_i : \overline{M}_{g,n}(X, \beta) \to X$ induces a pull back ev^*, a morphism of Chow rings.

- *Compactification*
  we consider singular homology in the moduli space, and compactification is required to define Poincare duality. Poincare duality is used to define a "degree map", a way to numerize intersection property of the geometry. However, the trouble is that smoothness is lost by the compactification, that can be solved by log scheme.

$\overline{M}_{g,n}$ is a compactification of $M_{g,n}$, and in fact, it is a set of isomorphism class of stable maps.

- *Virtual Fundamental Class*
  In intersection theory in the Chow ring of a scheme X, the fundamental class $[X]$ is an identity of the ring. Similarly, we have a Chow ring in the Kontsevich moduli space, and it must have an identity, which we call virtual fundamental class .

Definition 2.1.1 *(Homology)*

- *Singular homology*
  Singular homology can compute a geometry with (countable?) amount of singlar points, because CW complex can contain singular points.
- *Rational Cohomology*
  Singular cohomology is a singular homology $H^*(X)$ applied cohomological functor $Hom(-, R)$ for some unital ring R.
- *Cohomology Ring*
  We define a cohomology ring as $H^*(\overline{M}_{g,n}) = H^*(\overline{M}_{g,n}(\mathbb{C}), \mathbb{Q})$

Note 2.2 *(Sheaf)*
A vector bundle is a sheaf!

Example 2.3
If $[l]$ is an isomorphism class of lines, then $n[l]$ for $n \in \mathbb{Z}$ is an isomorphism class of curves of degree n.

3 semi-infinite variation of Hodge structure

Definition 3.0.1 *(Frobenius Manifold)*
Smooth manifold is a generalization of calculus, and it may be used to write diffyq concisely. A connection is an analogy of the differential operator, and a vector(line) bundle is a generalization of function. For example, the cohomology ring is a Frobenius manifold. It is Riemannian manifold since it is a vector space.

- *The connection $\nabla : T_M \to T_M \otimes \Omega^1_M$ is a flat.*
- *metric $g : S^2(T_M) \to O_M$, and $d(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y)$*
- *$A : S^3(T_M) \to O_M$ is a symmetric tensor. The product $A(X, Y, Z) = g(X \circ Y, Z)$ is associative.*
- *$A(X, Y, Z) = XYZ\Phi$*

- *vector field* $X : M \to T_M \cong M$ *is an endomorphism.*

An Euler vector field E *on* M *if for all* Y *and* Z,

- $E(g(Y,Z)) - g([E,Y],Z) - g(Y,[E,Z]) = Dg(Y,Z)$
- $[E, Y \circ Z] - [E,Y] \circ Z - Y \circ [E,Z] = d_0 Y \circ Z$ *for some const* d_0.

Naturally Dubrouvin connection $\hat{\nabla}$ *is a connection such that*

- $\hat{\nabla}_X(Y) = \nabla_X(Y) + \hbar^{-1} X \circ Y$.
- $d_0 \hat{\nabla}_{\hbar\partial_\hbar}(Y) = \hbar \partial_\hbar Y - \hbar^{-1} E \circ Y + Gr_E(Y)$.
- $Gr_E : Y \mapsto [E,Y]$ *is a* O_M *linear map.*

and this connection defines the following two quantum diffyqs.

- $\hat{\nabla}_{\partial y_i} s = 0$
- $\hat{\nabla}_{\hbar\partial\hbar} s = 0$

and this quantum diffyq have a solution $s_i = T_i - \Sigma_{j=0}^{m} << \frac{T_i}{\hbar+\psi}, T_j >> T^j$.

Definition 3.0.2 *(Semi-Infinite Variation of Hodge Structure)*

- $\nabla : \mathscr{E} \to \Omega^1_M \otimes \hbar^{-1}\mathscr{E}$ *where* $\mathscr{E}$ *is a* $O_M(\hbar)$*-module.*
- $(\cdot,\cdot)_{\mathscr{E}} : \mathscr{E} \times \mathscr{E} \to O_M(\hbar)$
- $\mathbb{C}\{\hbar, \hbar^{-1}\} = \mathbb{C}\{\hbar\} \oplus \hbar^{-1} O(\mathbb{P} \setminus \{0\})$
- $O_{\tilde{M}}\{\hbar, \hbar^{-1}\}$ *is a sheaf which accepts a formal power series.*
- $H = \{s \in \Gamma(M, \mathscr{E} \otimes_{O_M(\hbar)} O_M\{\hbar, \hbar^{-1}\}) | \nabla s = 0\}$
  Since ∇ *is flat,* H *is a free* $\mathbb{C}(\hbar, \hbar^{-1})$*-module of the same rank as* $\mathscr{E}$.
- *semi-infinite Hodge structure induces a morphism* $M \to LGL(\mathbb{C})/L^+GL(\mathbb{C})$ *which is* $x \mapsto M^{-1}(x)$
- *Consider that* $\mathscr{E}_0 \to H$ *is a morphism as* $\mathbb{C}\{\hbar\}$*-submodule, we have a natural isomorphism* $H_- \oplus \mathscr{E}_0 \cong H$ *where* H_- *is a* $O(\mathbb{P} \setminus \{0\})$*-submodule of* H.
- $\mathscr{E}_0 \cap \hbar H_- \cong \mathscr{E}_0 / \hbar \mathscr{E}_0$
- $\mathscr{E}_0 \cap \hbar H_- \cong \hbar H_- / H_-$
- $\mathscr{E}_0 \cong (\mathscr{E}_0 \cap \hbar H_-) \otimes_{\mathbb{C}} \mathbb{C}\{\hbar\} \cong \frac{\mathscr{E}_0}{\hbar \mathscr{E}_0} \otimes_{\mathbb{C}} \mathbb{C}\{\hbar\} \cong \frac{\hbar H_-}{H_-} \otimes_{\mathbb{C}} \mathbb{C}\{\hbar\}$

4 Defining B-model

Definition 4.0.1 *The pair (X, W) defines a sheafcohomology, so hypercohomology, and it's isomorphic to Milnor ring. Now, two homology groups $H^n(X, Re\ W/\hbar, \mathbb{C})$ define period integral $\int_\Xi e^{W/\hbar}\omega$. In other words, B-model is a study of period integral from homological perspective.*

$$\begin{aligned} H^n(X, Re\ W/\hbar, \mathbb{C}) \times H^n(X, Re\ W/\hbar, \mathbb{C}) &\to \mathbb{C} \\ (\Xi,\omega) &\mapsto \int_\Xi e^{W/\hbar}\omega \end{aligned} \tag{3}$$

Definition 4.0.2 *(Gauss Manin Connection)*

5 Deformation Theory

- A-model
  A-model arises from a moduli problem. If a Riemann surface can be given through graph theory, then the problem is reduced to the counting problem, and this is equivalent to say the counting of Gromov-Witten invariant, that can be computed by quantum cohomology.

- B-model
  Milnor number $\mathbb{C}[y_0, ..., y_m]/J$ where $J = (\frac{\partial W}{\partial x_0}, ..., \frac{\partial W}{\partial x_m})$ is a Jacobian ideal is used to describe deformation theory in B-model.

- Log scheme
  Let $O_k = Spec(R_k)$ where $R_k = \mathbb{C}[T]/[T^{n+1}]$, and we have a natural sequence of embedding morphism $O_{k+1} \to O_k$, then a fiber bundle $X \to \mathbb{A}^1$ has family of morphisms $f_k : C_k \to X$, and this operation is called deformation for a smooth variety X. In general, in log scheme ${X_k}^\dagger$, ${O_k}^\dagger$ makes a deformation.

Definition 5.0.1 *(Terminologies)*

- *what is sheaf?*
  The Hartshorne's definition says sheaf is a functor $F : Top \to Ab$, but in general, the codomain doesn't always need to be Ab. For example, a sheaf of monoid $Top \to Mon$.

- *Log scheme*
  Log Scheme $\underline{X}$ is a scheme $(X, \mathscr{O}_X)$ with a sheaf of monoid (M_X, α_X). A log scheme is fine if M_X is coherent and integral: saturated if M_X is

saturated. A monoid is integral if chancellation holds: if $x + y = x' + y$ then $x = x'$, while if for any choice of $mx \in M$, $x \in \mathbb{Z}$, $m \in M$, then M is saturated. $f : X \to Y$ is strict if the induced map $f M_Y \to M_X$ is an isomorphism.

- *Log Chart*
  P, M are monoids. A morphism $\theta : P \to M$ is called a chart if $\theta^a : P^a \to M$ is an isomorphism. α is quasi-coherent(or coherent) if locally on X it admits a chart (or chart of f.g. monoid).

- *Differential Form*
  A log form is ${\Omega_{\overline{X}}}^q(log\ D) \subset i_* {\Omega_X}^q$ where $i : X \to \overline{X}$ locally generated by $\{\frac{dx_1}{x_1}, ..., \frac{dx_p}{x_p}, dx_{p+1}, ..., dx_n\}$. $D = \overline{X} \setminus X$ is a normal crossing, locally given by $x_1 \cdots x_p = 0$.

  $\mathbb{H}^q(\overline{X}, \Omega^q_{\overline{X}}(log\ D)) \cong H^q(X, \mathbb{C})$

- *étale open cover*
  Étale topology is an extension of Zariski topology on X. In algebraic geometry, if X is a scheme, each stalk $\mathscr{O}_{X,x}$ for each point x has a strict henselization $\mathscr{O}_{X,\overline{x}}$, and its etale scheme $X_{\acute{e}t}$ is a scheme with each stalk is $\mathscr{O}_{X,\overline{x}}$

  Grothendieck topology is an alternative view. Etale topology on C is for all $U \subset X$, a sieve is a covering sieve if for all $x \in U$, there exists $f : V \to U$ in S_U such that $x \in f(V)$.

  In case of comparision, Zariski topology on C is for all $U \subset X$, a sieve is a covering sieve if for all $x \in U$, there exists an open immersion $f : V \to U$ in S_U such that $x \in f(V)$.

- *flat morphism*
  $f : X \to Y$ is flat morphism of scheme at $x \in X$ if $f_x : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat ring homomorphism. f_x is flat if it makes $\mathscr{O}_{X,x}$ a flat $\mathscr{O}_{Y,y}$-module, that is, for all ideals $\mathfrak{a} \subset \mathscr{O}_{Y,y}$, $\mathfrak{a} \otimes \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}$ is injective.

- *Log Smooth*
  (X, M) is log smooth over Spec k with a trivial log structure iff there exists an étale open covering $U = \{U_i\}$ of X and a divisor $D \subset X$ such that

  - *there exists a smooth morphism $h_i : U_i \to V_i$ where V_i is affine toric variety over k for each $i \in I$.*
  - *the divisor $D \cap U_i$ of U_i is the pull back of is the union of the closure of codimension 1 torus orbits of V_i by h_i for each $i \in I$.*

– *log structure* $M \to X$ *is equivalent to log structure* $O_X \cap j_* P^{\times}_{X-D} \to O_X$ *where* $j : X - D \to X$ *is the inclusion.*

Definition 5.0.2 *(Log Deformation)*

- *Deformation on a smooth scheme*
  Let $O_k = Spec\ R_k$ *where* $R_k = \Bbbk[x]/(x)^k$*.* $O_l \to O_k$ *is closed embedding for* $l < k$*, in particular,* $O_0 = \Bbbk$*. Let* $X \to Spec\ \Bbbk$*, there is a lifting* $X_k = X \times O_k \to O_k$ *whose restriction to* X *is* $X \to Spec\ \Bbbk$*.*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Spec\ \Bbbk \arrow[d] \\ X \times_{\Bbbk} O_k \arrow[r] & O_k \end{tikzcd}$$

- *Deformation on a log scheme*
  The slightly updated is deformation of a rational curve as a log scheme. It is a lifting of the following commutative diagram or abbreviated by $[f_k : C_k^\dagger / O_k^\dagger \to X^\dagger]$*. For all* $k \geqq 0$*,* $[f_k : C_k^\dagger / O_k^\dagger \to X^\dagger]$ *is inductively defined starting from* $k = 0$*. Lifting exists uniquely if there's no obstrcution, or* $H^1(U_{ij}, f_{0*}\Theta_{X^\dagger/\mathbb{A}^{1\dagger}_{\Bbbk}}) = 0$*.*

$$\begin{tikzcd} C_k^\dagger \arrow[r, "f_k"] \arrow[d] & X^\dagger \arrow[d, "\pi"] \\ O_k^\dagger \arrow[r, "\alpha_k"] & \mathbb{A}^{1\dagger}_{\Bbbk} \end{tikzcd}$$

- *Deformation on a log scheme with s-marked points and sections*
  Let's note the deformation by $[f_k : C_k^\dagger / O_k^\dagger \to X^\dagger, x^k]$ *for all* k *where* $x^k = \{x_1^k, ..., x_s^k\}$ *are s-marked points, where each marked point is a log smooth point.*

$$0 \to \Theta_{C_k^\dagger/\Bbbk^\dagger}(-x^0) \to \Theta_{C_k^\dagger/\Bbbk^\dagger} \to \bigoplus_{l=0}^{s} \Theta_{C_k^\dagger/\Bbbk^\dagger} \otimes k(x_l^0) \to 0$$

where $\Theta_{C_k^\dagger/\Bbbk^\dagger}(-x^0)$ *denotes the sheaf* $\Theta_{C_k^\dagger/\Bbbk^\dagger}$ *twisted by the line bundle* $\mathscr{O}_{C_0}(-\Sigma_{l=1}^{s} x_l^0)$ *where* $k(x_l^0)$ *is the residue field of* C_0 *at* x_l^0*.*

Moreover, define sections $\sigma_1, ..., \sigma_n : \mathbb{A}^1 \to X$ *by* $f_0(x_i^0) = \sigma_i(0)$ *for* $1 \leqq i \leqq s$*, or alternatively* $f_0 \circ x_i^0 = \sigma_i \circ \alpha_0$*. By the use of deformation, it can be lifted to arbitrary* k *as* $f_k \circ x_i^k = \sigma_i \circ \alpha_k$*.*

6 Singularities

- A-model
  Singular homology accepts singularities, so it has a cohomology ring.

- B-model
  Lefschetz thimbles are algebraic cycles defined over the finite amount of isolated singular points of X. Therefore, the period integral of singularity varieties is defined.

- Log Scheme
  The compactified moduli space has curves of several singularities, and this is how we care log scheme.

7 Prelim for Calabi-Yau

- Pricipal G-bundle and Holonomy
- Complex differential form
- Canonical bundle
- Kähler manifold
- Calabi-Yau manifold

Definition 7.0.1 *(Pricipal* G*-bundle and Holonomy)*

- *Riemannian manifold*
  Why Riemannian manifold? Because it makes itself simple.
  - *Connection* ∇ *is a generalization of a differential operator*
  - *Line bundle is a paraphrase of a function.*

- *Parallel transport*
  For $\gamma : [0,1] \to M$, if a diffyq $\nabla_{\gamma(t)'} X = 0$, then the section X is called parallel. Now we define parallel transport as a map $\Gamma(\gamma)_t^s : E_{\gamma(s)} \to E_{\gamma(t)}$

  If, in particular, γ is a loop based at $x \in M$ and $s = t$, then $P_\gamma = \Gamma(\gamma)_t^s$ will be an automorphism $P_\gamma : E_x \to E_x$.

- *Holonomy*
  Holonomy group of ∇ at $x \in M$ is a set of parallel transport $G = \{P_\gamma \in GL(E_x)\}$, and P_γ is a liner map, and it is invertible, where γ is a loop. By this group G, we can naturally define G-bundle.

- *Structure Group*
  Let M be a Riemannian manifold, and $\{\phi_i, U_i\}$ be an atlas.

$$\begin{aligned} \phi_i \circ {\phi_j}^{-1} : (U_i \cap U_j) \times F \quad &\to (U_i \cap U_j) \times F \\ (x,\xi) \quad &\mapsto (x, t_{ij}(x)\xi) \end{aligned} \tag{4}$$

  where $t_{ij} : U_i \cap U_j \to G$ is a morphism for some topological group G, having a group structure:

  - $t_{ii} = 1$
  - $t_{ij} = {t_{ji}}^{-1}$
  - $t_{ik} = t_{ij} \circ t_{jk}$

  where the third condition might be applied on triple overlaps $U_i \cap U_j \cap U_k$, and it is called cocycle condition in Čech cohomology. This group G is called a structure group.

- *Principal G-bundle*
  Principal G-bundle P is a fiber bundle $\pi : P \to M$ with a continuous action $G \times P \to P$, where each fiber F maps to itself by the action of G. This each of the fibers is called G-torsor.

Definition 7.0.2 *(Complex Differential Form)*
We define complex differential form and its Hodge structure with de Rham complex

- *Review of de Rham cohomology*
  De Rham cohomology is defined by a family of n-forms $\{\Omega_X^n\}$ with a chain $d : \Omega_X^n \to \Omega_X^{n+1}$.

- *(p,q)-form*
  Let $(1,0)$-form be generated by $\Omega_X^{1,0} = \{dz\}$ for some basis element z, while $(0,1)$-form by $\Omega_X^{0,1} = \{d\bar{z}\}$ for its complex conjugate. Then we define (p,q)-form as $\Omega_X^{p,q} = \wedge_p \Omega_X^{1,0} \wedge \wedge_q \Omega_X^{0,1}$.

- *n-form*
  The n-form Ω_X^n can be decomposed to the sum of (p,q)-forms as $\Omega_X^n = \bigoplus_{p+q=k} \Omega_X^{p,q}$.
- *Doubeault operator*
  Let $\pi^{p,q} : \Omega_X^k = \bigoplus \Omega_X^{p,q} \to \Omega^{p,q}$ be a projection, and we can define the Doubeault operators with it. One of the Doubeault operators is ∂ be $\partial = \pi^{p+1,q} \circ d : \Omega_X^{p,q} \to \Omega_X^{p+1,q}$, while the other Doubeault operator is $\overline{\partial}$ be $\overline{partial} = \pi^{p,q+1} \circ d : \Omega_X^{p,q} \to \Omega_X^{p,q+1}$.
- *Relation between the operators*

  - $d = \partial + \overline{\partial}$
  - $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$
- *Ricci Curvature*
  $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$

Definition 7.0.3 *(Canonical Bundle)*

- *sheaf of kähler differential form*
  For a scheme X, Ω_X is a sheaf of 1-form. Namely, each element of the local ring $O_X(U)$ is differentiated:

$$\begin{aligned} O_X : Top &\to Ab \\ U &\mapsto R = \{f\} \\ \Omega_X : Top &\to Ab \\ U &\mapsto S = \{df\} \end{aligned} \tag{5}$$

- *canonical bundle*
  Canonical bundle $\omega_x = \wedge \Omega_X$ is n-form where $n = dim(X)$, so it is also called a determinant matrix.

Definition 7.0.4 *(Kähler Manifold)*

Kähler manifold is a manifold with Sympletic, complex, Riemannian structures:

- *Symplectic View*

  $$g(u,v) = \omega(u, Jv)$$

  where

  - $\omega : TM \to TM$ *is a symplectic form*
  - J *is an integrable almost complex structure, a smooth tensor field with* $J^2 = -I$

- J is compactible with ω.
- g is a Riemannian metric.

- *Almost complex View* Hermitian metric h constitute ω and g.

$$\omega(u, v) = Reh(iu, v) = Imh(u, v)$$
$$g(u, v) = Reh(u, v)$$

- *Riemannian View* Let X be a Riemannian manifold of dimension $2n$ whose holonomy group is contained in the unitary group $U(n)$.

In conclusion, as long as we will know at least one structure of the three, and if we know the manifold X is Kähler, then we will know the other two.

Definition 7.0.5 *(Calabi-Yau Manifold)*
The definition of Calabi-Yau manifold varies. One of the classical definition by Yauis

- *Compact Kähler*
- *Ricci flat*
- *vainishing first Chern class*

Alternative definition can be given by:

- *Canonical bundle of M is trivial*
- *M has a holomorphic n-form that vanishes nowhere*
- *The structure group of the tangent bundle of M can be reduced from $U(n)$ to $SU(n)$.*
- *M has a Kähler metric with a global holonomy contained in $SU(n)$*

8 Intersection Theory

Definition 8.0.1 *(Divisor)*

- *Prime Divisor*
  Prime divisor of X is $Z \subset X$ regular of codimension 1.
- *Weil Divisor*
  Weil divisor is an element $\Sigma n_i[Z_i]$ where n_i is an integer, Z_i is a prime divisor.

- *Principal Divisor*
  A principal divisor is $(f/g) = \{\textit{set of zeros of } f/g\} - \{\textit{set of poles of } f/g\}$ *where* f/g *is a rational function where both sets are counted with multiplicities. Alternatively, principal divisor is given by a rational function* $(f/g) = \Sigma_Z ord_Z(f/g)Z$ *where* Z *is the prime divisors,* $ord_Z(f/g)$ *is the multiplicities, and* f *and* g *are holomorphic functions.*

  Note that set of all principal divisors (f/g) *forms a group: invertible* $(g/f) = (f/g)^{-1}$*, identity* $(0) = id$*, and* $(f/g)(g/h) = (f/h)$*. The free abelian group can be naturally quotioned by the group, then we call* $Cl(X) = Div(X)/\{principal\}$ *as a class group.*

- *Cartier Divisor*
  Similar stuff to Weil divisor. Skip.

- *Cartier-Picard Relations*

  Cartier divisor corresponds to a Line bundle

  - $[D] \mapsto \mathscr{L}(D)$ *is 1-to-1. Or,* $Cl(X) \cong Pic(X)$
  - $\mathscr{O}_X(D - D') \cong \mathscr{O}_X(D) \otimes \mathscr{O}_X(D)^{-1}$
  - $\mathscr{O}_X(D) \cong \mathscr{O}_X(D')$ *if* $D \sim D'$.

Definition 8.0.2 *(Chern Class)*

- *First Chern class*
  For any line bundle $\mathscr{O}_X(D)$ *with a Cartier divisor* D*, the first Chern class is* $c_1(\mathscr{O}_X(D)) = [D]$.

- *Chern Polynomial*
  $c_t(\mathscr{L}) = 1 + c_1(\mathscr{L}t + c_2(\mathscr{L}t^2 + ... + c_n(\mathscr{L}t^n$

- *Chern Character*
  $c(E) = \Sigma_{k=0}^{n} c_k(E)$

- *Chern Class Formulae*

  - $c_0(E) = 1$
  - $c_1(\mathscr{O}_X) = [X]$
  - $c_1(E \otimes E') = c_1(E) + c_1(E')$
  - $c_t(E \oplus E') = c_t(E)c_t(E')$
  - $c_1(E) = 0$ *if* E *is a trivial vector bundle.*

–

$$\begin{aligned} c(\mathbb{CP}^n) &= c(T\mathbb{CP}^n) \\ &= c(\mathscr{O}_{\mathbb{CP}^n}(1))^{n+1} \\ &= (1+a)^n \textit{ for some } a \in H^2(\mathbb{CP}^n, \mathbb{Z}) \end{aligned} \tag{6}$$

– $c_k(\mathbb{CP}^n) = \binom{n}{k} a^k$.

Definition 8.0.3 *(Tangent Sheaf)*

- *sheaf of Kähler differential :* $\Omega_{X/S}$
- *Tangent sheaf:* $\mathscr{T}_X = \mathscr{H}_{\mathscr{O}_X}(\Omega_{X/k}, \mathscr{O}_X)$*, a locally free sheaf of rank* $n = dim(X)$
- *Cotangent sheaf : dual of tangent sheaf*

For all scheme $X \subset \mathbb{P}^n$*, There is a short exact sequence*

$$0 \to \mathscr{T}_X \to \mathscr{T}_{\mathbb{P}^n}|_X \to \mathscr{N}_{X/\mathbb{P}^n} \to 0 \tag{7}$$

Definition 8.0.4 *(Grothendieck Riemann Roch theorem)*

9 Sheaf Cohomology & Spectral Sequence

Definition 9.0.1 *(Hypercohomology and Period Integral)*

- *hypercohomology*

  $\mathbb{H}^n(X, \mathscr{F}) \cong H^n(\Gamma(X, I^{\cdot}), d)$
  For a toric Fano variety X*, there can be defined* $X^\vee$ *is mirror dual to* X*.*
  For example, if $X = \mathbb{P}^n$*,* $X^\vee = (\mathbb{C}^\times)^n \subset \mathbb{C}^{n+1}$

  The hypercohomology is naturally dual to a homology group $H^n(X, Re(W/\hbar) << 0; \Omega_X^n)$

  By Poincare lemma, a locally constant sheaf $\mathscr{O}_X = \mathbb{C}_X$ *has an injective resolution:*

  $0 \to \mathscr{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \cdots$

  which is a de Rham co-chain complex.

- *What is* $H^k(X, Re(W/\hbar) << 0, \mathbb{C})$

- *Period Integral*
  Our goal is to define a period integral $\int_{\Xi_i} e^{W_0/\hbar} f\Omega$. *For a real* $2n$*-dimensional manifold* X*, its homology ring has a cup product* $H_k(X, Re(W/\hbar) << 0; \mathbb{C}) \times X_{2n-k}(X, \Omega_X^k) \to \mathbb{C}$. *With a poincare lemma* $H_k(X, Re(W/\hbar) << 0; \mathbb{C}) \cong H^{2n-k}(X)$*, the product naturally lifts to*

$$\begin{aligned} H_k(X, Re(W/\hbar) << 0; \mathbb{C}) \times H^k(X, \Omega_X^k) \to \mathbb{C} \\ (\Xi_i, \omega) \mapsto \int_{\Xi_i} e^{W/\hbar}\omega \end{aligned} \tag{8}$$

  for some fixed potential W*. Hence, the integral problem is an intersection problem. Let's consider the basis of* $H_k(X, Re(W/\hbar) << 0; \mathbb{C})$ *denoted by* Ξ_k*, and the basis of* $H^k(X, \Omega_X^k)$ *denoted by* $[f\Omega]$.

- *Lefschetz Thimbles*
  If Δ_p^+ *is a Lefschetz thimble, which is a stable manifold, and its dimension is* n*, because the vector field potential* W *is a complex holomorphic, and at each critical point its Hessian matrix always has the same amount of positive/negative eigenvalues.* Δ_p^+ *is generated by the positive elements, so totally* $n = 2n/2$*-dimensional. Similarly, for* Δ_p^- *and it's unstable manifold.*

- *a*
  $\Omega = \frac{dx_0 \wedge \cdots \wedge dx_n}{x_0 \cdots x_n}$

Definition 9.0.2 *(Local System R)*

- *moduli space*
  $\tilde{M} = (\mathbb{C}, \mathscr{O}_{\tilde{M}})$ *where* $\mathbb{C}$ *is viewed as a complex manifold with coordinate* t_1*, and* $\mathscr{O}_{\tilde{M}}(U) = \{\Sigma f t_0^{i_0} t_2^{i_2} \cdots t_n^{i_n}\}$ *where* f *is a formal power series and it's holomorphic on* U*, and* t_i *is of finite degree, also* $e^{t_1} = x_0 \cdots x_n$. $\tilde{M}$ *is a fine moduli space, and it is also a* $n+2$*-dimensional scheme*

  $M = Spec\ \mathbb{C}[t_0, t_1][[t_2, \cdots, t_n]]$.

  Also $\check{X} \subset \tilde{M} \times \mathbb{C}^{n+1}$ *with* $\pi : \tilde{M} \times \mathbb{C}^{n+1} \to \check{X}$ *is a projection.*

- *(Local System R)*
  In general, we could assume a universal unfolding $W : M \times \check{X} \to \mathbb{C}$ *so that* $W|_{0 \times \check{X}} = W_0$*. as* $W = t_0 + W_0 + \sigma t_i W_0^i$*, but it has more critical points than* W_0. $\{W_0^i\}$ *is a basis of the milnor ring.*

  The fibre of local system R over $(t_1, \hbar) \in \tilde{M} \times \mathbb{C}^\times$ *is* $H_n(\pi^{-1}(t_1), Re(W|_{\pi^{-1}(t_1)}/\hbar) << 0; \mathbb{C})$ *where* $W|_{\pi^{-1}(t_1)} = x_0 + \cdots + x_n$

and $x_0 \cdots x_n = e^{t_1}$, ignoring t_0,$t_2 \cdots t_n$, and this function has $n+1$ critical points.

Definition 9.0.3 *(Gauss-Manin Connection)*

- *(Gauss-Manin connection)*
  Gauss-Manin connection is a connection which defines a diffyq similar to quantum diffyq in A-model. what is GM connection explicitly?

$$\begin{aligned} \nabla^{GM} &: R \to R \otimes \Omega_S^1 \\ \frac{d}{dx} \mapsto \frac{d}{dx} &+ N \textit{ (where } N \textit{ is a } n \textit{ by } n \textit{ matrix)} \end{aligned} \tag{9}$$

$b_n \frac{d^n f}{dx^n} + \cdots + b_1 \frac{d^1 f}{dx^1} + b_0 f = 0.$
$\frac{\partial}{\partial x} \circ \phi = \phi \circ \nabla^{GM}_{\frac{\partial}{\partial x}}$

Ξ_k *is a local basis of local section of* R.

$\Delta_p^{\pm} \in H_n(X, Re(\pm W/\hbar) << 0, \mathbb{C})$ *is a Lefschetz thimbles.*

$\nabla^{GM}_X[f\Omega] = [(X(f) + \hbar^{-1}X(W)f)\Omega]$
$\nabla^{GM}_{\hbar\partial_\hbar}[f\Omega] = [(\hbar\partial_\hbar f - \hbar^{-1}Wf)\Omega]$

$[f\Omega] = \Sigma_{i=0}^n \alpha^i \int_{\Xi_i} e^{W_0/\hbar}\Omega$ *or* $[f\Omega] \in R$.

$f\Omega : R \to \mathscr{O}_{\tilde{M}\times\mathbb{C}^\times}$ *or* $f\Omega \in R^\vee$

Definition 9.0.4 *(Semi-infinite variation of Hodge structure)*

- *(Semi-infinite variation of Hodge structure)*
  Semi-infinite variation of Hodge sturcutre is a pair of $(M, \mathscr{E}, \nabla, (-,-)_{\mathscr{E}}, Gr)$, where $\mathscr{E}$ is $\mathscr{O}_M\{\hbar\} - module$, $\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega_M^1$ is a flat connection, and $Gr : \mathscr{E} \to \mathscr{E}$ is a morphism.
  - $\nabla^{GM}_X[f\Omega] = [(X(f) + \hbar^{-1}X(W)f)\Omega]$
  - $\nabla^{GM}_{\hbar\partial_\hbar}[f\Omega] = [(\hbar\partial_\hbar f - \hbar^{-1}Wf)\Omega]$
  - $E = (n+1)\partial_{t_1} + \Sigma_{i=0}^n (1-i)t_i\partial_{t_i}$
  - $Gr(s) = \nabla^{GM}_{\hbar\partial_\hbar + E}(s) - s$
  - $(s_1, s_2)_{\mathscr{E}} = \frac{(-1)^{n(n+1)/2}}{(2\pi i\hbar)^n}\Sigma_p(\int_{\Delta_p^-} s_1(-\hbar))(\int_{\Delta_p^+} s_2(\hbar))$
  - $\overline{\Omega}(s_1, s_2) = Res|_{\hbar=0}(s_1, s_2)_{\mathscr{E}} d\hbar$
  - $E(W) = W$
  - $Gr([f\Omega]) = [(\hbar\partial_\hbar f + E(f) - f)\Omega]$

– $([f\Omega],[g\Omega])_{\mathscr{E}} = \pm\Sigma_p \frac{f(p,0)g(p,0)}{Hess(W)(p)} + O(\hbar) \in \mathscr{O}_{\tilde{M}}\{\hbar\}$

– $\nabla(\hbar^{-(n+1)\alpha}\alpha^i) = 0$ *for all* $0 \leqq i \leqq n$*. So, they are flat sections forming a basis of* H *as free* $\mathbb{C}\{\hbar,\hbar^{-1}\}$*-module.*

– $H = \{s \in \Gamma(M, \mathscr{E} \otimes \mathscr{O}_{\tilde{M}}\{\hbar,\hbar^{-1}\}) | \nabla s = 0\}$

– H_- *is* $\mathscr{O}(\mathbb{P}^1 \backslash 0)$*-submodule of* H *and generated by* $\{(\hbar\alpha)^k \hbar^{-1}\hbar^{-(n+1)\alpha} | 0 \leqq k \leqq n\}$.
Conversely, $H = \hbar H_-/H_- \otimes \mathscr{O}_M\{\hbar,\hbar^{-1}\}$

– $H \cong H_- \oplus \mathscr{E}_0$

– $\hbar H_-/H_- \cong \mathbb{C}[\alpha]/(\alpha^{n+1})$

– $[f\Omega] = \Sigma_{i=0}^n \alpha^i \int_{\Xi_i} f e^{W/\hbar}\Omega$ *is a local section of* $R^\vee$*.*

– Ξ_i *is a local section of* R*.*

- *miniversality*
  Semi-infinite variation of Hodge structure (M, E, ∇) *is miniversal if*

$$\begin{aligned} T_M &\to \mathscr{E}/\hbar\mathscr{E} \\ X &\mapsto \hbar\nabla_X s_0 \end{aligned} \tag{10}$$

  is isomorphic.

Definition 9.0.5 *(Barannikov's period map)*

$$\begin{aligned} \hbar H_-^A/H_-^A &\cong \hbar H_-^B/H_-^B \\ T_1^i &\mapsto \hbar^{-(n+1)\alpha}(\hbar\alpha)^i \end{aligned} \tag{11}$$

$\psi^A : M^A \to \hbar H_-^A/H_-^A$
$\psi^B : M^B \to \hbar H_-^B/H_-^B$
are local isomorphism by miniversality. Now we want to show correspondence of semi-infinite variation of Hodge structures of A/B-models. Namely, we show the isomorphism $m : (M^A, 0) \to (M^B, 0)$*, and their Frobenius structure coincide: coincidence of* $\mathscr{E}^A$ *and* $\mathscr{E}^B$*, also* Gr^A *and* Gr^B*, also* $(-,-)^A_{\mathscr{E}}$ *and* $(-,-)^B_{\mathscr{E}}$*.*

$\mathscr{E}^A \subset H^A_{\tilde{M}}$ *and* $\mathscr{E}^B \subset H^B_{\tilde{M}}$ *are identified, and .*

10 Gromov-Witten Invariant

- Fundamental Class Axiom
  If $n + 2g \geqq 4$ or $\beta \neq 0$ and $n \geqq 1$ and $[X] \in H^0(X, \mathbb{Q})$, then
  $< \alpha_1, \cdots, \alpha_{n-1}, [X] >_{g,\beta} = 0$

- Divisor Axiom
  If $n + 2g \geqq 4$ or $\beta \neq 0$ and $n \geqq 1$ and $\alpha_n \in H^2(X, \mathbb{Q})$, then
  $< \alpha_1, \cdots, \alpha_n >_{g,\beta} = (\int_X \alpha_n) < \alpha_1, \cdots, \alpha_{n-1} >_{g,\beta}$

- Pointing Mapping Axiom
  if $g = 0$, and $\beta = 0$, then
  $< \alpha_1, \cdots, \alpha_n >_{0,0} = \int_X \alpha_1 \cup \alpha_2 \cup \alpha_3$ (if $n = 3$ otherwise the invariant is equal to 0.)