Intro to Homology

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In this pdf, we'll just introduce language of homological algebra, not going to the applications e.g. geometry. In category theory, an abelian category generates a category of chain complexes, that contains homological and homotopical information, whose calculation is done by transformating category to homotopic/derived/triangulated/dg/∞-category, and all of them is distinct and very interesting, and the key idea of homological algebra is that homology is a linearization of homotopy, so generally speaking homology is easier to calculate, while homotopy is not, but we are missing many data in the process of linearization. On the other hand, ∞-algebra is another linearization of homotopy without losing data, but this is rather complicated. I'll also introduce several concrete homology including $A/L/E-\infty/BV$-algebra and coalgebra.

Contents

The major purpose of this pdf is to introduce homological technique to study geometry both algebraically and differentially. What is homological algebra? It depends, but homological algebra is a linearlization of geometry or its abelianization, which can measure some geometric invariants.

The alternating sum of dimension of homology classes of each degree is Euler characteristic. This is a topological invariant, but in sheaf cohomology, it helps us define Hilbert polynomial, the flatness condition of the algebraic variety.

If the given geometry is complex geometry, homological algebra has extra structures e.g. Hodge structures.

vector bundles. First, the question is how vector/fiber bundle of a geometry arises homological structures. I will introduce what is vector bundle and fibre bundle of a smooth real manifold. These defines global/local sections, where we can find de Rham cohomology. If it further looks carefully, the algebraic structure (complex or almost complex) generates extra data on de Rham cohomology group called Hodge decomposition. Other than vector bundle, Sheaf is a nice generalization of vector bundles, for at least, it has a functorial property, and it can generalize notion of algebraic variety to any moduli space. I'll introduce stack(orbifold) and groupoid theory.

What kind of structures does homological algebra has? As shown in de Rham cohomology, homological algebra is a linearlization of geometry. Or from a categorical viewpoint, it is derived category, having triangulated structures.

1 Preliminary

1.1 Linear Algebra

The very question in linear algebra is what is the specific applications in algebra and geometry. First of all, how trace is used in Lagrangian, or trace incharacter theory. Exterior algebra for volume form and Hodge star operator.

Definition 1.1.1 *(Inner Product)*
Inner product $< \cdot, \cdot >$ is a biliner form $V \times V \to \mathbb{R}$ for some vector space V.

Definition 1.1.2 *(Hermitian Inner Product)*
Hermitian inner product is

- *bilinear,*
- $< \alpha u, v >= \alpha < u, v >$
- $< u, \alpha v >= \overline{\alpha} < u, v >$
- $\overline{< u, v >} =< v, u >$
- $< u, u >\geqq 0$ *where* $< u, u >= 0$ *iff* $u = 0$.

A basic example if $h(z, w) = \Sigma z_i \overline{w_i}$

Definition 1.1.3 $< A^* u, v >=< u, Av >$

Definition 1.1.4 *(Trace)*

Example 1.2 *(Trace)*

- *(Character)*
  A character of ρ is $\chi(g) = Tr(\rho(g))$
- *(Hermitian metric)*
  $< A, B >= Tr(A^\dagger B)$
- *(Idempotent Matrix)*
  If $A = A^2$, $Tr(A) = dim(A)$ because eigenvalue of A is either 1 or 0. So if $A = Q^{-1}\Lambda Q$ where Λ is a diagonal matrix whose value is either 1 or 0.
- *(Exponential)*
  $det(exp(tA)) = 1 + t(Tr(A)) + O(t^2)$

1.3 Homology

Definition 1.3.1 *(Monomorphism and ker/coker)*
We generalize homological algebra to category theory from abstract algebra, so we can extend this notion to sheaf theory, while this generalization requires us reinterpretation of basic operations as subobject/quotient and kernel/image etc.

In a category $\mathscr{C}$, suppose $f \circ g_1 = f \circ g_2$ means $g_1 = g_2$ for all $g_1, g_2 \in Mor(\mathscr{C})$, then the morphism f is monomorphism. By using this monomorphism, we can generalize notion of kernel into abelian category. In the language *of category, kernel is defined $ker(f) = (K, i)$ as a pair where i is monomorphism $i : K \to X$ such that $f \circ i = 0$ (this is equivalent expression with $f \circ i = g \circ i$ in abelian category), and this $ker(f)$ is a subobject $ker(f) \subset X$, where subobject is defined by the pair (A, i) where $i : A \to X$ is a monomorphism. (just notice that on some cases monomorphism has the equal meaning with injectivity. For example, in category of Sets, monomorphism is equivalent to injectivity, thus $ker(f)$ is just equivalent to ordinary kernel in set theory). Similarly, for $A \subset X$, the quotient is defined by cokernel $X \xrightarrow{p} X/A = coker(i)$ so that $p \circ i = 0$, and in particular, $im(d_{n+1}) \subset ker(d_n) \subset X$ is a sequence of subobjects, thus $ker(d_n)/im(d_{n+1})$ is defined.*

Example 1.4 *(Abelian Category)*
$\mathcal{A}\lfloor$, $Coh(X)$ where X is noetherian, and $QCoh(X)$ where X is quasi-separated, not necessarily notherian.

Definition 1.4.1 *(Homology)*
In theory of homology, the chain complex is a sequence $\cdots C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$ defined over objects C_n of the abelian category. However, the object C_n doesn't need to be an abelian group e.g. coherent sheaf.

Definition 1.4.2 *()*

2 Concrete Homology

2.1 Simplicial Homology

Definition 2.1.1 *(Simplicial Homology)*
obvious.

Definition 2.1.2 *(Relative Homology)*
To step forad to Borel-Moore, we introduce relative homology. For $A \subset X$, the relative homology $H_i(X, A)$ is i-th homology of X ignoring A part, which is simply $C_i(X, A) := C_i(X)/C_i(A)$. This results the long exact sequence $\cdots H_{i+1}(X, A) \to H_i(A) \to H_i(X) \to H_i(X, A) \to \cdots$. In other words, using $A = X \backslash K$, the relative homology $H_i(X, X \backslash K)$ is called K-localized homology.

Definition 2.1.3 *(Borel-Moore Homology)*
Borel-Moore homology is basically the same as simplicial but generalization to noncompact case. It's defined by $H_i^{BM}(X) = \varinjlim_{K \subset X} H_i(X, X \backslash K)$ for K compact subset of X.

Proposition 2.2 *(Intuition of Homotopy)*
For a topology map $f : X \to Y$, its chain map $C.f : C.(X) \to C.(Y)$ has mapping $(\sigma : \Delta^n \to X) \to (f \circ \sigma : \Delta^n \to Y)$ for each n, whose homology is $H_n f$:

$H_n(X) \to H_n(Y)$. Consider the homotopy is a map $H : X \times [0,1] \to Y$, that can define homotopy relation $f \sim g$ where $f(x) = H(x,0)$ and $g(x) = H(x,1)$, then it naturally adds 1-dimension to the geometry, thus the homotopy H can be divided into $H_n : X^n \times [0,1] \to Y^{n+1}$, so $H = \Sigma_n H_n$. This is not surprising argument, as we can see in homotopy lifting, and now $\partial \circ H + H \circ \partial : X^n \to Y^n$ is a morphism between the same degrees. Here consider $H : X \times [0,1] \to Y$ in two dimensional, if $a \subset X$ is a 1-chain, thus $f(a)$ and $g(a)$ are also 1-chain, that can continuously changed in $[0,1]$, and $\partial(H(a)) = g(a) - f(a) - H(\partial(a))$, thus removing a, $f - g = \partial \circ H + H \circ \partial$.

2.3 de Rham Cohomology

Definition 2.3.1 *(Vector Field)*
A vector field X is a linear map $X : C^\infty(M) \to C^\infty(M)$ that satisfes product rule.

$$X(fg)p = fX(g) + X(f)g \text{ for all } f, g \in C^\infty(M)$$

Vector bundle is an abstraction of differential forms on a smooth manifold M. In particular, vector bundle is tangent bundle TM, whose tensor or direct sum $\bigwedge^* TM$ and $TM \oplus TM$ and dual TM^*, and trivially line bundle $M \to \mathbb{R}$. In differential form, differential of 1-form becomes 2-forms, and from 0-form to 1-form, so in analogy, we could define a morphism from the global section of a vector bundle to another global section of a vector bundle, which is called connection $\nabla : \Gamma(E \otimes \bigwedge^* TM) \to \Gamma(E \otimes \bigwedge^* TM)$ (what is differential? It is any morphism what satisfies product rule).

Definition 2.3.2 *(Vector Bundle)*
A vector bundle is a morphism $\pi : E \to B$ such that the inverse $\pi^{-1}(x) = E_x$ is a vector space. A section is an inverse map $\sigma : B \to E$ such that comosition $\pi \circ \sigma = id$ is identity. We will denote $\Gamma(E, U)$ for $U \subset B$ as section.

Remark 2.4 *(Zero Section)*
Zero section $\sigma : B \to E$ is local/global section of vector bundle, and the morphism maps to zero as $\sigma(x) = 0 \in E_x$, for $x \in B$.

Remark 2.5 *(Existence of Global Section)*
The global section of vector bundle always exists. At least zero section is a global section, and every vector bundle has it. On the other hand, global section of principal bundle exists iff it's trivial bundle. Of course, we are interested in any non-zero sections. There exists nowhere vanishing global section in a vector bundle if its Euler class is zero.

Definition 2.5.1 *(Connection Form)*
$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M) = \Gamma(E) \otimes \Gamma(T^*M)$

$\nabla(fv) = \nabla(v) \otimes df + f\nabla(v)$

where ∇ is the exterior derivative of f.

In general, the connection form is generalized to n-th power product

$\nabla : \Gamma(E) \to \Gamma(E \otimes \bigwedge^n T^*M) = \Gamma(E) \otimes \Gamma(\bigwedge^n T^*M)$

$\nabla(v \wedge \alpha) = \nabla(v) \wedge d\alpha + (-1)^{deg(v)} v \wedge d\alpha.$

Definition 2.5.2 *(Flat Connection)*
If the curvature derived from the connection vanishes, the connection ∇ is flat, and $\nabla^2 = 0$, so it can form the chain complex.

Definition 2.5.3 *(Curvature Form)*
The problem is curvature from derived from the connection ∇ has a two different way of representations.

First, the curvature form of connection is $F_\nabla = dA + A \wedge A$ or $A \wedge A = \frac{1}{2}[A, A]$. Or alternatively, $\Omega = d\omega + \omega \wedge \omega$. These two forms have actually the same Riemannian curvature tensor.

Explicitly, $F_\nabla(s) = \Sigma_{i,j=1}^{k} \Sigma_{p,q=1}^{n} R_{pqi}^{j} s^i dx^p \wedge dx^q \otimes e_j$

where R_{pqi}^{j} is a Riemann curvature tensor of F_∇.

Definition 2.5.4 *(Elliptic Operator)*
Elliptic operator is a generalization of Laplacian. Let L be a linear differential operator of order m in the domain $\Omega \subset \mathbb{R}^n$ given by

$Lu = \Sigma_{|\alpha| \leqq m} a_\alpha(x) \partial^\alpha u$

L is called Elliptic if for all $x \in \Omega$ and every non-zero $\xi \in \mathbb{R}^n$,
$\Sigma_{|\alpha| \leqq m} a_\alpha(x) \xi^\alpha \neq 0$

Definition 2.5.5 *(Elliptic Complex)*
Let $E_0, E_1, \cdots, E_k$ be vector bundles on a smooth manifold M.
For the sequence of vector bundles $0 \to E_0 \to E_1 \to \cdots \to E_k \to 0$.

For the sequence of vector bundles

$0 \to \Gamma(E_0) \to^{P_1} \Gamma(E_1) \to^{P_2} \cdots \to^{P_k} \Gamma(E_k) \to 0$

such that $P_{i+1} \circ P_i = 0$.

$$0 \to \pi^* E_0 \to^{\sigma(P_1)} \pi^* E_1 \to^{\sigma(P_2)} \cdots \to^{\sigma(P_k)} \pi^* E_k \to 0$$

is exact outside of zero section. Here π is the projection of cotangent bundle $T^*M \to M$ and π^* is the pullback of a vector bundle.

Proposition 2.6 *(Property of Elliptic Complex)*
In Elliptic complex, the kernel is of finite dimension, and analytically nice wrt smoothness and Fredholm. Therefore, the cohomology is of finite dimensional, and it can be idenfied with Laplacian.

2.7 Hodge Theory

Definition 2.7.1 *(Hodge Star Operator)*
A Hodge star is a linear operation in the exterior algebras $\star : \bigwedge^k(V) \to \bigwedge^{n-k}(V)$, and Hodge star maps an object $w \in V$ to its dual $\star w \in V$, and twice the operations is plus-minus identity as $\star \circ \star = (-1)^{k(n-k)} Id$. Hodge star operator is multi-linear map.

Let's see an explicit illustration. The domain $\bigwedge^k(V)$ consists of k-dim bivector space. If $k = 1$, an element is a vector, and vector is an arrow, vector is geometrically interpreted as 1-dim bounded line. For higher dimension, we let bivector given by wedge product $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ as a k-dimensional cube as a subset of V.

For example, if $n = 3$, the hodge dual of a vector $e_1 \in \mathbb{R} = V$ is $e_2 \wedge e^3 \in \mathbb{R} \wedge \mathbb{R}$, so the Hodge dual corresponds to the complement space.

Definition 2.7.2 *(Hermitian Metric)*
$h_p(\eta, \overline{\zeta}) = \overline{h_p(\zeta, \overline{\eta})}$ for all $\zeta, \eta \in E_p$ and

$h_p(\zeta, \zeta) > 0$ for all $\zeta \neq 0 \in E_p$

Hermitian metric is a complex valued function, but we can also define Riemannian metric if it we add conjugate, or we subtract to make (1,1)-form on M.

$g = \frac{1}{2}(h + \overline{h})$
$\omega = \frac{i}{2}(h - \overline{h})$

Definition 2.7.3 *(Hermitian Connection (or Dolbeault Operator))*
A Hermitian connection ∇ is a connection on a Hermitian vector bundle E over a smooth manifold M compatible with a Hermitian metric $< \cdot, \cdot >$ on E, meaning that

$$v < s, t > = < \nabla_v s, t > + < s, \nabla_v t >$$

for all smooth vector field $v \in C^\infty(M,E)$ and all smooth sections s,t of E. Then there is a unique Hermitian connection whose $(0,1)$ part is Dolbeault operator $\overline{\partial}_E$ on E associated to the holomorphic structure. Formally, Dolbeault operator is a projection $\overline{\partial}_E = \pi^{(0,1)}\nabla$. This ∇ is called Chern connection on E.

More explicitly, any section $s \in \Gamma(E)$ is written as $s = \Sigma s^i e_i$ for some function $s^i \in C^\infty(U_\alpha, \mathbb{C})$. Define an operator locally by

$\overline{\partial}_E(s) := \Sigma\overline{\partial}(s^i) \otimes e_i$ where $\overline{\partial}_E$ is a Cauchy-Riemann operator of the base manifold. That is,

$\overline{\partial}(\alpha) = \Sigma \frac{\partial f_{IJ}}{\partial \overline{z}^l} d\overline{z}^l \wedge d\overline{z}^I \wedge d\overline{z}^J$
(Curvature)
The curvature $\Omega = d\omega + \omega \wedge \omega$ be the curvature form of ∇. Since $\pi^{0,1}\nabla = \overline{\partial}_E$ is vanishing if squared, so $\Omega^{2,0}$ is trivial, so ∇ is skew-symmetric $\Omega^{0,2}$. Thus Ω is a $(1,1)$-form given by

$\Omega = \overline{\partial}_E \omega$

Definition 2.7.4 *(Holomorphic Vector Bundle)*
Over a complex manifold X, the projection map $E \to X$ is holomorphic, which means the local trivialization is a biholomorphic map $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$.

Example 2.8 *(Holomorphic Tangent Bundle)*
For a complex manifold M, the real $2n$-dimensional real vector bundle TM on M, and endomorphism $J : TM \to TM$ s.t. $J^2 = -Id$.

If complexified, $J : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ where $J(X + iY) = J(X) + iJ(Y)$ where $X, Y \in TM$

Since $J^2 = -Id$, eigenvalues are i and $-i$, and it splits

$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$

The holomorphic tangent bundle of M is vector bundle $T^{1,0}M$, and anti-holomorphic tangent bundle is $T^{0,1}M$.

Definition 2.8.1 *(Hermtian Yang-Mills Connection (or Instanton))*
For a principal bundle $P \to E$, a connection $A_\alpha \in \Omega^1(U_\alpha, ad(P))$. If the connection A satisfies $d_A \star F_A = 0$, then A is Yang-Mills connection (also called instanton). This is Hermitian YM connection, if in addition, the connection A is connection on a Hermitian vector bundle E over a Kähler manifold X of dimension n.

Then the Hermitian YM equaitons are:

$F_A^{0,2} = 0$
$F_A \cdot \omega = \lambda(E) Id$ *for some* $\lambda(E) \in \mathbb{C}$

Recall the curvature F_A *is skew-Hermitian, so* $F_A^{0,2} = 0$ *implies* $F_A^{2,0} = 0$*. If* X *is compact, then* $\lambda(E)$ *can be computed using Chern-Weil theory. Namely*

$$\begin{aligned} deg(E) &= \int_X c_1(E) \wedge \omega^{n-1} \\ &= \frac{i}{2\pi} \int_X Tr(F_A) \wedge \omega^{n-1} \\ &= \frac{i}{2\pi} \int_X Tr(F_A \wedge \omega) \omega^n \end{aligned} \tag{1}$$

Hence,

$\lambda(E) = -\frac{2\pi i}{n! Vol(X)} \mu(E)$

$\mu(E) = \frac{deg(E)}{rank(E)}$

Definition 2.8.2 *(Adjoint Operator)*
(Adjoint Operator)
Given an operator is a $T = \Sigma a_k(x) D^k$ *where* D *is differential, adjoint operator* T^* *is an operator that satisfies* $< Tu, v > = < u, T^* v >$ *where* $< \cdot, \cdot >$ *is an inner product.*

(Laplacian)
Given a de Rham cochain complex $(\Omega^* * (X), d)$*, let adjoint operator be* $\delta : \Omega^{k+1}(X) \to \Omega^k(X)$*, and the Laplacian* $\Delta = d\delta + \delta d : \Omega^k(X) \to \Omega^k(X)$

(Harmonic)
A form on X *is harmonic if* $H^k_\Delta(M) = \{\alpha \in \Omega^k(X) | \Delta\alpha = 0\}$

Proposition 2.9 *(Hodge Decomposition)*
There exists a unique decomposition

$\omega = d\alpha + \delta\beta + \gamma$

in which γ *is harmonic* $\Delta\gamma = 0$ *In terms of* L^2 *metric on differential forms, this gives an orthogonal direct sum decomposition*

$\Omega^k(M) \cong im(d_{k-1}) \oplus ker(d_{k+1}) \oplus H^k_\Delta(M)$

Hodge theorem states that $H^k_\Delta(M) \cong H^k(M, \mathbb{R})$

Proposition 2.10 *(Hodge Conjecture)*
Let X be a non-singular complex projective variety, and let Hodge classes be

$$Hdg^k(X) = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$$

We claim every Hodge class $Hdg^k(X)$ on X is a linear combination with rational coefficients of the cohomology clases of complex subvarieties of X.

Definition 2.10.1 *(Pullback Bundle)*
Let $\pi : E \to B$ be a fiber bundle and $B' \to B$ be a continuous map. Define a pullback bundle is

$$f^*E = \{(b', e) \in B' \times E | f(b') = \pi(e)\} \subset B' \times E$$

and the projection map $\pi' : f^*E \to B'$ given by the projection onto the first factor, i.e. $\pi'(b', e) = b'$ and another projection $h : f^*E \to E$. These maps creates a commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{h} & E \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

Definition 2.10.2 *(Hodge Theory on Elliptic Complex)*
Let elliptic complex be $L_i : \Gamma(E_i) \to \Gamma(E_{i+1})$ be linear differential operators, and denote it by

$$E^{\cdot} = \oplus \Gamma(E_i)$$

$$L = \oplus L_i : E^{\cdot} \to E^{\cdot}$$

and let $L^{\cdot}$ is the adjoint of L, and $\Delta = LL^{\cdot} + L^{\cdot}L$. As usual, this yields the harmonic sections

$$\mathfrak{H} = \{e \in E^{\cdot} | \Delta e = 0\}$$

Let $H : E^{\cdot} \to \mathfrak{H}$ be the orthogonal projection, and let G be the Green's operator for Δ. Then, Hodge theorem states

- *H and G are well-defined*
- *$Id = H + \Delta G = H + G\Delta$*
- *$LG = GL$ and $L^{\cdot}G = GL^{\cdot}$*
- *$H(E_j) \cong \mathfrak{H}(E_j)$ (cohomology correspondence)*

Definition 2.10.3 *(Dolbeault Complex)*
From the above definition of $\overline{\partial}_E$, Dolbeault complex is a complex if $\overline{\partial}_E^2 = 0$. Note in a complex manifold, $\overline{\partial}_E^2 = 0$ always exists, but in general (e.g. symplectic manifold), it doesn't always work.

First a warning that de Rham cohomology might be merely a real analytic notion, so in complex geometry, complexity has an extra symmetry in the homological algebra, and the adjoint operator of the differential operator defines Laplacian, thus Hodge decomposition. A generalization of de Rham cocomplex is elliptic complex, and "elliptic" sounds real analytic (indeed in complex geometry, we cannot distinguish elliptic and hyperbolic, or if we mention elliptic, it might be real analytic). In elliptic comlex, the differential operator is elliptic, meaning that the differential is non-vanishing (what do I mean by that).

Definition 2.10.4 *(Mixed Hodge Theory)*
Mixed Hodge theory generalizes pure Hodge theory to geometry of singularity or non-compact case. In mixed Hodge theory, the Hodge decomposition is not directly calculated, but there exists weight filtration and Hodge filtaration, and we can recover the Hodge decomposition by combining them. $0 = W_{-1} \subset W_0 \subset W_1 \subset \cdots W_n = H$ where W_k is a weight space that contains all weight k component or less than k, which simply means $W_k = \bigoplus_{p \leqq k} H^p$.

Definition 2.10.5 *(Variation of Hodge Theory)*
For a fibration $\pi : X \to S$ we define cohomology $H^k(X_s)$ for each fiber $X_s = \pi^{-1}(s)$ on $s \in S$, and Gauss-Manin connection is defined by $\nabla : H^k_{dR}(X_s) \to H^k_{dR}(X_s) \otimes \Omega^1_S$, and as usual it make a Hodge decomposition for each fiber $H^k_{dR}(X_s) = \bigoplus_{p+q=k} H^{p,q}(X_s)$ (assuming the fiber is always complex).

Hodge filtration is defined by $F^p H^k_{dR}(X) := \bigoplus_{r \geqq p} H^{r,k-r}(X)$, and indeed $H^k = F^0 \supset F^1 \supset F^2 \cdots F^k \supset F^{k+1} = 0$ is inclusion relation, and Gauss-Manin connection ∇ shifts Hodge filtration $\nabla F^p \subset F^{p-1} \otimes \Omega^1_S$ by no more than 1 by Griffith transversality.

Definition 2.10.6 *(The Case of Log Scheme)*
For non-compact scheme X, the divisor taing from compactification $D = \overline{X} \backslash X$ defines logarithmic complex $\Omega^{\cdot}_{\overline{X}}(log(D))$, and for example if $D = D_1 \cup D_2$, D_1 D_2 are weight 1, $D_1 \cap D_2$ are weight 2, more precisely W_1 consists of log differential form of either D_1 or D_2.

Definition 2.10.7 *(Perverse Sheaf)*
For an open embedding $j : U \hookrightarrow X$, there is an intermediaate extension $j_{*!}\mathbb{Q}_U[n]$ where $n = dim(X)$. $D = X \backslash U$. Log de Rham cohomology is the same as sheaf cohomology of perverse sheaf, and $H^i_{log}(X) \cong H^i(X, j_{!*}\mathbb{Q}_U[n])$. However, log scheme is defined by the pair (U, D), but it depends on the compactification.

2.11 Lie Algebra Cohomology

I'd suppose that the purpose of Lie algebra cohomology is to compare cohomology of a geometry M and its quotient $M//G$ where G is a Lie group.

Definition 2.11.1 *(Lie Algebra Cohomology)*

Let $\mathfrak{g}$ be a Lie algebra over R, and M be a representation of $\mathfrak{g}$. If M is trivial representation,

$H^n(\mathfrak{g}; M) = Ext^n_{U\mathfrak{g}}(R, M)$

Definition 2.11.2 *(Chevalley-Eilenberg Complex)*

Chevalley-Eilenberg complex is $Hom_k(\bigwedge^\bullet \mathfrak{g}, M) \cong M \otimes \bigwedge^\bullet \mathfrak{g}^*$

Definition 2.11.3 *(Hamiltonian Vector Field)*

Moment map is similar to Hamiltonian vector field.

Definition 2.11.4 *(Symplectic Reduction)*

Let (M, ω) be a symplectic manifold where a Lie group G acts on it, and $\Phi : M \to \mathfrak{g}^*$ is a momentum map. Let $M_0 = \Phi^{-1}(0)$ and $0 \in \mathfrak{g}^*$ is a regular value. Then, there exists a symplectic quotient $\tilde{M} = M_0/G$ (often denoted as $M//G$), and this $\tilde{M}$ is also a symplectic manifold whose differential form is induced by the pullback as $\pi^*\tilde{\omega} = \iota^*\omega$ where $\pi : M_0 \to \tilde{M}$ and $\iota : M_0 \to M$.

Then the functional space is Lie algebra cohomology.
$C^\infty(\tilde{M}) = C^\infty(M_0)^{\mathfrak{g}} = H^0(\mathfrak{g}, C^\infty(M_0))$

where $C^\infty(M_0) = C^\infty(M)/$
since every smooth function on M_0 can be extended to M. coincides with the ideal generated by momentum map $I[\Phi]$.

Definition 2.11.5 *(Koszul Resolution)*
From symplectic reduction $\tilde{M}$ of a geometry M, we'll construct a Koszul resolution

$K^\bullet = \bigwedge^\bullet C^\infty(M) \to C^\infty(M_0).$

For a derivation $\delta : K^q \to K^{q-1}$,

$\delta f = 0$ *and* $\delta X = \phi_X$*. Let X_i be a basis of $\mathfrak{g}$.*

Similarly, the higher degree $\delta : \mathfrak{g} \otimes C^\infty(M) \to C^\infty(M)$ *is*

$\delta(\Sigma_i X_i \otimes f_i) = \Sigma_i f_i \phi_i$

For $\bigwedge^2 \mathfrak{g} \otimes C^\infty(M) \to \mathfrak{g} \otimes C^\infty(M) \to C^\infty(M) \to 0$

$\delta(X \wedge Y \otimes f) = Y \otimes \phi_X f - X \otimes \phi_Y f$

where δ *is extended to an odd derivation. Notice that* $\delta^2 = 0$.

Definition 2.11.6 *(BRST Cohomology)*
For a functional operator $Q : \mathfrak{H} \to \mathfrak{H}$ *is an endomorphism, and particularly if* $Q^2 = 0$, *it'll be a chain complex, called BRST cochain complex.* C^i *consists of* i*-th ghost,* 0*-th ghost is a wave function of the actual physics state.*

Definition 2.11.7 *(BRST Double Complex)*
Furthermore BRST has both gauge symmetry and anti-gauge symmetry structure, so we can have two distinct differential operators, and accordingly the wave function can be classified by the p-ghost and q-anti ghost, and we let $C^{p,q}$ *consists of it. To sum it up,* $(C^{\cdot,\cdot}, Q_{BRST}, d_{other})$ *is a double complex. Or by using the newly defined differential operator* $D = Q_{BRST} + d_{other}$, *BRST double complex becomes just a complex* $(C^{\bullet,\bullet}, D)$

Definition 2.11.8 *(BRST Cohomology With Lie Algebra)*
The BRST cochain complex has an explicit construction using Chevalley-Eilenberg cohomology with the gauge group $\mathfrak{g}$,

$C^{p,q} = C^p(\mathfrak{g}; K^q) = \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^p \mathfrak{g} \otimes C^\infty(M)$

where $D = \{Q, -\}$ *is the differential, and* $Q = \alpha^i \phi_i - \frac{1}{2} f^i_{jk} \alpha^j \wedge \alpha^k \wedge X_i$, *where* X_i *is a basis of* $\mathfrak{g}$ *and* $[X_i, X_j] = f^i_{jk} X_k$ *and* α^j *is dual basis of* $\mathfrak{g}^*$.

2.12 Floer Homology

Definition 2.12.1 *(Hessian and Morse Index) For a geometry* M, *if* $f : M \to \mathbb{R}$ *is a smooth function, let* $p \in M$ *be a critical point* $f'(p) = 0$, *then Hessian* $Hf_p(u, v)$ *is*

$Hf_p(u, v) = \Sigma^n_{i,j=1} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) a_i b_j$

where $u = \Sigma^n_{i=1} a_i \frac{\partial}{\partial x_i}_p$ *and* $v = \Sigma^n_{j=1} b_j \frac{\partial}{\partial x_j}_p$. *If* f *is a Morse function (that is definable from any smooth functions), the Hessian* Hf_p *is nondegenerate.*

Finally, we define Morse index $\mu(p)$ *as the number of negative eigenvalues of the Hessian* $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$.

Definition 2.12.2 *(Morse Homology)*
Let's define Morse chain complex. For each k, the C_k is generated by critial points of degree k as $C_k := \oplus_{\mu(p)=k} \mathbb{Z}p$.

Let's define Morse differential.

For two critical points $p, q \in M$, the set

$M(p,q) = \{x : \mathbb{R} \to M | \dot{x} = -grad(f), \lim_{\tau \to -\infty} x(\tau) = p, \lim_{\tau \to \infty} x(\tau) = q,\}$

descibes path which follows the gradient. Further we take quotient as $\hat{M}(p,q) = M(p,q)/\mathbb{R}$ by translation action.

The derivative at critical point $p \in M$ is

$\partial : C_k \to C_{k-1}$
$\partial p = \Sigma_{\mu(q)=k-1} \# \hat{M}(p,q) q$

Indeed $\partial^2 = 0$, so the Morse chain complex is a chain complex.

In fact, Morse homology is isomorphic to singular homology.

Definition 2.12.3 *(Singular Homology)*

Remark 2.13 *(Moment Map and Morse Theory)*

Definition 2.13.1 *(Lagrangian Intersection)*
If M is a symplectic manifold, a smooth function $H_t : M \times [0,1] \to \mathbb{R}$ has a vector field X_t such that $\omega(\cdot, X_t) = dH_t$ called Hamiltonian vector field. Further, we define a Hamiltonian isotopy ϕ_t such that

- $d\phi_t = X_t \circ \phi_t$
- $\phi_0 = id$

$\phi_t : M \to M$ is a symplectomorphism, and if $L \subset M$ is a Lagrangian submanifold, $\phi_t(L)$ is also a Lagrangian submanifold. For example, 0_X is a zero section of the cotangent bundle T^*X and its Lagrangian submanifold.

For a function $H = f \circ \pi : T^*M \to \mathbb{R}$ where $\pi : T^*M \to M$ is a projection, Its Hamiltonian isotopy ϕ_t is, in particular, $\phi_1(0_X)$ is the graph of df. If f is a Morse function, 0_X and $\phi_1(0_X)$ intersects transversally.

Floer homology is an infinite dimensional analogue of finite dimensional Morse theory.

Definition 2.13.2 *(Floer Homology)*
For ϕ_t, we define

$\Omega = \{l : [0,1] \to M | l(0) \in L, l(1) \in \phi_1(L), l \text{ is homotopic to } \phi_t(x_0)\}$

where $x_0 \in L$ is a fixed point. The universal covering space $\tilde{\Omega}$ of Ω is

$\tilde{\Omega} = \{u : [0,1]\times[0,1] \to M | u(\tau, 0) \in L, u(\tau, 1) \in \phi_1(L), u(0,t) = \phi_t(x_0)\}/homotopy$

We introduce a function $F : \tilde{\Omega} \to \mathbb{R}$

$F = \int_0^1 d\tau \int_0^1 dt\omega(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial \tau})$

Now taking two intersection points $p, q \in L \cap \phi_1(L)$, the set

$M(p,q) = \{u : \mathbb{R} \to \Omega | \frac{\partial u}{\partial \tau} = -grad(F), \lim_{\tau \to -\infty} u(\tau, [0,1]) = p, \lim_{\tau \to \infty} u(\tau, [0,1]) = q, \}$

Now Ω is infinite dimensional smooth manifold with Riemannian metric, we are similarly allowed to define Morse theory.

2.14 Hochschild Homology

Definition 2.14.1 *(Hochschild Complex)*
$C^*(A,A) = \bigoplus_{n \geqq 0} Hom(A^{\otimes n}, A)$ *with differential d and cup product. Its differential d is* $d\phi(a_1, \cdots, a_{n+1}) = a_1 + \phi(a_2, \cdots, a_{n+1}) + 2\sum_{n=1}^{n} (-1)^i \phi(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1}\phi(a_1, \cdots, a_n)a_{n+1}$*, and cup product is* $f \cup g(a_1, \cdots, a_{m+n}) = f(a_1, \cdots, a_m)g(a_{m+1}, \cdots, a_{n+m})$*, and* $d(f \cup g) = (df) \cup g + (-1)^m f \cup (dg) +$ *higher homotopy terms. $HH^1(A)$ shows derivation, $HH^2(A)$ shows deformation, and $HH^3(A)$ shows obstruction.*

Definition 2.14.2 *(Operad)*
An operad is a collection $\{O(n)\}_{n \in \mathbb{N}}$ with multi-array operation $O(k) \times O(l_1) \times O(l_2) \times \cdots \times O(l_k) \to O(l_1 + l_2 + \cdots + l_k)$, where $O(n)$ could be any objects, and if each $O(k)$ is a topological space, it's topological operad. An operad can describe algebra by acting on Hochschild cochain complex $C^*(A,A)$, and O-algebra is generated by an operad O together with how that acts on it by the action maps $\gamma_n : O(n) \otimes V^{\otimes n} \to V$. However, depending on the action γ_n, the O-algebra should be constructed differently, and if $O = Ass$, O-algebra can be strictly associative, DGA, or even A_∞-algebra.

One of the applications of operad is Deligne conjecture, which claims Hochschild cochain complex is Gerstenhaber algebra, and it's also E_2-algebra, or Koszul duality claims the duality of A_∞-L_∞ conversion. A_∞ can be understood without using operad, so as L_∞-algebra, since associative operad can generate A_∞ algebra as trivial case, and it doesn't need to be A_∞ operad.

For transformation of different operads, operad morphism $\phi : O_1 \to O_2$ where $O_i \to O_i - algebra$ maps functorially $\phi^* : O_1 - algebra\ to O_2 - algebra$.

Example 2.15 *(Operad Algebra)*
List of algebra operad acting on Hochschild cochain complex.

- *(A_∞)*
  A_∞-operad is a topological operad, since $A_\infty(n) = K_n$ for each $n \in \mathbb{N}$ is assigned to Stasheff polytope K_n, working as an associahedron. A_∞-algebra is generated by A_∞ operad acting on Hochschlild chain complex $A_\infty(n) \times C^*(A, A)^{\otimes n} \to C^*(A, A)$. If $n = 2$, homotopy means the existence of continuous transformation from $(fg)h$ to $f(gh)$. In particular, if $m_k = 0$ where $k > 2$, it's dg algebra.

- *(L_∞)*
  L_∞-operad is an algebraic operad, also called dg-operad. l_1 is differential, l_2 is Lie bracket, and l_n for $n \geqq 3$ is higher Jacobi identity. L_∞ algebra if L_∞ operad acts on Gerstenhaber algebra. In particular, if $l_k = 0$ where $k > 2$, it's dg Lie algebra.

- *(E_2)*
  E_2-operad (also called little disks operad) is a topological operad $\{E_2(n)\}$ where $E_2(n) = \{(D_1, \cdots, D_n) \subset D^2 | D_i \cap D_j = \emptyset\}$, that can be interpreted as a moduli space of circles inside of a unit circle D, which is in other words, a configuration space $Conf_n(D^2) = \{(x_1, x_2) | x_1 \neq x_2\}$. E_2-algebra is Gerstenhaber algebra. The Fulton-Mcpherson operad $FM_2(n) = Conf_n(\mathbb{R}^2)/\mathbb{R} \rtimes \mathbb{R}_{\leqq 0}$ is a compactification of E_2-operad, but it generates the same algebra.

- *(E_n, $n \geqq 3$)*
  E_n-algebra could be $Poisson_n$ algebra.

- *(Poisson Operad)*
  Poisson operad is just one colored operad, since there are two operations but it's defined on the same space.

- *(dg-Operad)*
  An operad $O(n)$ is a complex.

- *(PaB-Operad)*
  PaB consists of braid group B_n with projection $B_n \to S_n$, and $PaB(n) \cong E_2(n)$ is a weak equivalence by means of homotopy, and E_2 is a topological

operad, and PaB operad is a group operad not topological operad, but its classifying BB_n is a topological space, that's given by $\pi_1(BB_n) = B_n$, and $BB_n \cong Conf_n(\mathbb{R}^2)$.

- *(Swiss Cheese-operad)*

Definition 2.15.1 *(Algebras)*

- *(DGA)*
  For de Rham cochain $(\Omega^*(M), d)$, wedge product $\wedge : \Omega^p \times \Omega^q \to \Omega^{p+q}$ where $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$, being associative with unit $1 \in \Omega^0$. Its differential is $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ where $\alpha \in \Omega^p$. Other DGA could be Chevalley-Eilenberg and BRST.
- *(DGLA)*
  For de Rham cochain $(\Omega^*(M), d)$, DGLA is $\Omega^*(M, \mathfrak{g}) = \Omega^*(M) \otimes \mathfrak{g}$, whose bracket is $[\alpha \otimes X, \beta \otimes Y] = (\alpha \wedge \beta) \otimes [X, Y]_{\mathfrak{g}}$, and its differential is $d(\alpha \otimes X) = d\alpha \otimes X$, denoted by $(\Omega^*(M), d, [\cdot, \cdot])$.
- *(BV)*
- *(A_∞-algebra)*
  Loop space $\Omega X = \{\gamma : [0,1] \to X | \gamma(0) = \gamma(1) = x_0\}$, whose coposition $\gamma_1 \star \gamma_2(t) = \gamma_1(2t)$ if $(0 \leqq t \leqq 1/2)$ $\gamma_1 \star \gamma_2(t) = \gamma_2(2t-1)$ if $(1/2 \leqq t \leqq 1$. $(\gamma_1 \star \gamma_2) \star \gamma_3 \cong \gamma_1 \star (\gamma_2 \star \gamma_3)$ isn't the same, but equivalent up to homotopy.
- *()*

Definition 2.15.2 *(Colored Operad)*
An colored operad is a collection $\{O(c_1, \cdots, c_k; c)\}$ where c_i, c describe types of an operad, not number or variables with morphisms $f \circ_i g \in O(c_1, \cdots, c_i, d_1, \cdots, d_m, c_{i+1}, \cdots, c_m; d)$ for $f \in O(c_1, \cdots, c_n; c)$ and $g \in O(d_1, \cdots, d_m; d)$ An easy example is a colored operad with just one color, since the Jacobian bracket is 3-ary operation $O(2) \times O(2) \to O(3)$, whose colored operad version is simply replacing to dots $O(*, *; *) \times O(*, *; *) \to O(*, *, *; *)$.

Definition 2.15.3 *(PROP)*
PROP is a category with objects named $0, 1, 2, \cdots$ of countable cardinality, whose morphism $P(n, m)$ is an operad if $m = 1$ as $P(n, 1) = O(n)$, or more roughly PROP describes n inputs and m outputs, where an operad is just n inputs and one output, so PROP is simply a generalization of Operad. PROP is symmetric monoidal category. Furthermore, dg-PROP is a category, whose morphism $O(m, n) : \cdots \to O(m, n)^i \xrightarrow{d} O(m, n)^{i+1} \to \cdots$ itself is a chain complex. PROP can be used to define bialgebra and HOPF algebra, which aren't necessarily one-output.

Definition 2.15.4 *(Koszul Complex)*

Definition 2.15.5 *(Koszul Duality)*

Definition 2.15.6 *()*

Definition 2.15.7 *()*

Definition 2.15.8 *()*

Definition 2.15.9 *()*

Definition 2.15.10 *()*

Definition 2.15.11 *()*

2.16 Other Homologies

Definition 2.16.1 *(Quantum Cohomology)*
Skipped as already mentioned in Marc Gross.

Definition 2.16.2 *(Group Cohomology)*

Definition 2.16.3 *(L^2 Cohomology)*

Definition 2.16.4 *(Étale Homology)*

Definition 2.16.5 *(Cyclic Homology)*

Definition 2.16.6 *(Crystal Homology)*

Definition 2.16.7 *(Kolmogorov Homology)*

Definition 2.16.8 *(Borel Cohomology)*

3 Abstract Homology

We disucss the property of homology for itself and how homology connects to ∞-category problem. Or trig category, dg-category to name a few.

3.1 Simplicial Method

In this section, we define cotangent complex, which is a derived language of cotangent bundle T^*M, that can be defined by an extension of Kähler differential form $\Omega_{X/S}$ for morphism of scheme $X \to S$. Also, simplicial resolution.

Definition 3.1.1 *(Simplex Category)*
A simplicial category Δ is a category whose objects are $[n] = \{0, 1, 2, \cdots, n\} \in \Delta$, and its morphisms are non-strictly increasing map of finite order, which might be generated by coface and codegeneracy maps.

- *(coface)*
  $\delta^{n,i} : [n-1] \to [n]$

  is a morphism that misses i namely, $\delta^{4,2} : [3] \to [4]$ maps to $(0, 1, 2, 3) \mapsto (0, 1, 3, 4)$.
- *(codegeneracy)*
  $\sigma^{n,i} : [n+1] \to [n]$

  is a morphism that counts i twice: namely, $\sigma^{4,2} : [5] \to [4]$ maps to $(0, 1, 2, 3, 4, 5) \mapsto (0, 1, 2, 2, 3, 4)$.

Definition 3.1.2 *(Simplicial Object)*
A simplicial object U is a contravariant functor

$$U : \Delta \to \mathscr{C}$$

esp if $\mathscr{C}$ is a Set, then U is simplicial set. Or if $\mathscr{C}$ is a category of commutative ring, then U is simplicial commutative ring.

Now, denote $U_n = U([n])$, $d^n_j = U(\delta^n_j) : U_n \to U_{n-1}$ and $s^n_j = U(\sigma^n_j) : U_n \to U_{n+1}$

Definition 3.1.3 *(Skeleton)*
$sk_n : Simp(\mathscr{C}) \to Simp_n(\mathscr{C})$ $cosk_n : Simp_n(\mathscr{C}) \to Simp(\mathscr{C})$

Definition 3.1.4 *(Simplicial object and fiber product)*
Let V and W be simplicial objects of a category $\mathscr{C}$, and morphism $a : V \to U$ and $b : W \to U$. Assume product $V_n \times_{U_n} W_n$ exists in $\mathscr{C}$, then

- $(V \times_U W)_n = V_n \times_{U_n} W_n$
- $d_i^n = (d_i^n, d_i^n)$
- $s_i^n = (s_i^n, s_i^n)$

In other words, $U \times V$ is a product of presheaves on U and V on Δ.

In fact, for any ring map $A \to B$, the simplicial resolution is of finite type, and in fact a trivial Kan fibration.

Example 3.2 *(Linear algebra on Fiber product)*
Consider the tensor product of modules

$k[x, y] = k[x] \otimes_k k[x]$

$k[x, y, z] = k[x, y] \otimes_{k[x]} k[x, y]$

and the dual of tensor product is fiber product. Consider $Spec(k) = pt$, $Spec(k[x]) = \mathbb{A}^1$, $Spec(k[x, y]) = \mathbb{A}^2$, and $Spec(k[x, y, z]) = \mathbb{A}^3$, and the tensor product of modules above will be

$\mathbb{A}^2 = \mathbb{A}^1 \otimes_{pt} \mathbb{A}^1$

$\mathbb{A}^3 = \mathbb{A}^2 \otimes_{pt} \mathbb{A}^2$

Now in general, Let $X_i = k[x_0, x_1, \cdots, x_i]$, then $X_{i+1} = X_i \otimes X_{i-1} X_i$.

Definition 3.2.1 *(Simplicial Resolution)*

Let A be a ring, and B be A-algebra. We'll construct a simplicial resolution $\epsilon : P^{\cdot} \to B$.
where $P_0 = B$, $P_1 = A[B]$, $P_2 = A[A[B]]$, and $P_n = A[A[A[\cdots A[B] \cdots]$ and so on.
This can be alternatively $P. = X.(B)$.

Here, $X.$ is defined as follows.

Consider functors $\mathscr{C} \leftrightarrows F, U \mathscr{D}$, where $\mathscr{D} = Set$, $\mathscr{C} = k - Mod$ and forgetful functor U and some functor F which maps a set to generating object, that is left adjoint to for. We let composition $X_0 = F \circ U$, and $X_1 = U \circ F \circ U \circ F$, $X_n = U \circ F \circ \cdots \circ F$ of n times of iterations. Notice X_n is an endfunctor $X_n \in Fun(\mathscr{C}, \mathscr{C})$.

Write $L = FU$ and $\delta = F\eta U$, so that we define counit and comultiplication

- $\epsilon : L \to I$
- $\delta : L \to L^2$

The pair (L, ϵ, δ) is called comonad.

- $d_i = L^{n-i}\epsilon L^i : L^{n+1} \to L^n$
- $s_i = L^{n-1-i}\delta L^i : L^n \to L^{n+1}$

for $i = 0, 1, 2, \cdots, n-1$.

Definition 3.2.2 *(Kähler differential)*
Kähler differential form $\Omega_{S/R}$ on a commutative ring homomorphism $\phi : R \to S$ can be defined as $\Omega_{S/R} = I/I^2$ where $I = ker(S \otimes_R S \to S)$ generated by $ds = s \otimes_R 1 - 1 \otimes_R s$. Otherwise, $\Omega_{S/R}$ is defined that for $s, t \in S$ and $r \in R$

- $dr = 0$
- $d(s+t) = ds + dt$
- $d(st) = sdt + tds$ *(product rule)*

Remark 3.3 *()*
If R and S are local ring, this ideal I is the maximal ideal, and $\Omega_{S/R} = I/I^2$ is also called a cotangent space. Also, let's consider cotangent space is dual of tangent space, since cotangent space is generated by differential of geometry, while tangent space is differential of functions over the geometry, exactly dual.

Definition 3.3.1 *(Cotangent Complex)*
Given a ring morphism $B \to A$, we have resolution $P. \to A$, where $P_{n+2} = P_{n+1} \otimes_{P_n} P_{n+1}$, and the cotangent complex is $L^n_{B/A} = \Omega_{P_n/A} \otimes P_n/\epsilon B$ for $n \geqq 0$, so that its homology is $H^0(L_{B/A}) = \Omega_{B/A}$. This resolution $P.toA$ is not unique, but one of the easy interpretations is the standard resolution $\cdots P_3 = B \otimes_A B \otimes_A B \to P_2 = B \otimes_A B \to P1 = B \to A \to 0$.

Example 3.4 *(The Case $k[x]$)*
The cotanget complex has alredy a geometric property. If in particular the resolution $P. \to A$ has $A = k$ and $B = k[x]$ for some field k, each P_n corresponds to just an affine plane, since $P_2 = k[x, y] = k[x] \otimes_k k[x]$, thus inductively $P_n = k[x_1, \cdots, x_n] = k[x_1, \cdots, k_{n-1}] \otimes_{k[x_1, \cdots, k_{n-2}]} k[x_1, \cdots, k_{n-1}]$, whose geometry corresponds to $Spec(P_n) = \mathbb{A}^n$. This is just a single simplice, and Dold-Kan correspondence $[n] \mapsto P_n^{\otimes n}$ is applicable.

Definition 3.4.1 *(Koszul Complex)*
For a commutative ring A, and $s : A^r \to A$ is a linear map where $f_1, \cdots, f_n \in A$ are functions, so that the standard basis $e_i \in A^r$ becomes $s(e_i) = f_i$, Koszul complex is a sequence $\Lambda^i A^r \to \Lambda^{i-1} A^r \to \cdots \to \Lambda^1 A^r \to \Lambda^0 A^r \cong A$, or equivalently given by the graded structure $\Lambda^{\cdot} = \oplus_i \Lambda^i A^r$ whose endomorphism is defined by $\alpha^1 \wedge \cdots \wedge \alpha_k \mapsto \Sigma_{i=1}^k (-1)^{i+1} s(\alpha^i) \alpha^1 \wedge \cdots \hat{\alpha^i} \cdots \wedge \alpha_k$ for each $0 \geqq k \geqq r$.

Definition 3.4.2 *()*

Definition 3.4.3 *(Chaotic/Indiscrete Topology)*
A category $\mathscr{C}$ has a canonically a site if $\{f : V \to U | f$ is an isomorphism$\}$ is a covering of U.

This corresponding topology is called chaotic or indiscrete topology.

Definition 3.4.4 *(Derived Lower Schriek)*
For category $\mathscr{C}$ is Derived category, ex:

$\mathscr{C}^{op}_{B/A}$ is a category with object $\alpha : P \to B$ or $\alpha : P' \to B$ and morphism is $s : \alpha \to \alpha'$.

Let $A \to B$ be a ring map.

$L_{B/A} = L_{\pi!}(Li^*\Omega_{\mathscr{O}/A}) = L_{\pi!}(i^*\Omega_{\mathscr{O}/A}) = L_{\pi!}(\Omega_{\mathscr{O}/A} \otimes_{\mathscr{O}} \underline{B})$

in $D(B)$ where $Li^* : D(\mathscr{O}) \to D(\underline{B})$ and $L_{\pi!} : D(\underline{B}) \to D(B)$.

Definition 3.4.5 *(Polynomial Differential Form)*
$\Omega^{\cdot}_{poly}([n]) = \mathbb{Q}[t_1, \cdots, t_n, dt_1, \cdots, dt_n]/(\Sigma t_i - 1, \Sigma dt_i)$

and morphism $[n] \to [m]$ induces functorially $\Omega([m]) \to \Omega([n])$, and $t_i \mapsto \Sigma_{u(j)=i} t_j$.

3.5 Categorification of Chain Complexes

Definition 3.5.1 *(Category of Chain Complexes)*
For defining the category of chain complexes $Ch(\mathscr{A})$, its objects $C. \in Ob(Ch(\mathscr{A}))$ can be a chain complex $\cdots C_{n+1} \overset{d_{n+1}}{\to} \cdots C_n \overset{d_n}{\to} \cdots C_{n-1} \cdots$, where $d_n \in Mor(Ch(\mathscr{A}))$ is a differential operator such that $d_n \circ d_{n+1} = 0$ for all n, and dually, the category of cochain complex $Ch(\mathscr{A})^{op}$. The abelian category $\mathscr{A}$ could be Ab, $Coh(X)$, or $FinVect_{\Bbbk}$.

I'll introduce many distinct definitions derived from category of chain complexes e.g. homotopy category, derived category, triangulated category, dg category, ∞-category etc, and they are all seems to be used for different purposes, while equally important question is how these distinct definitions are related. The diagram below shows the life span of homotopy category ends by $infty$-theory, and it roughly shows all kinds of categories can be related.

$(HomotopyCategory) \overset{localization}{\Rightarrow} (DerivedCategory) \overset{\equiv}{\Rightarrow} (TriangulatedCategory) \overset{dg-enhancement}{\Leftarrow} (DGCategory) \overset{dg-nerve}{\Rightarrow} (\text{infty}-Category) \Rightarrow (HomotopyCategory)$

Definition 3.5.2 *(Variety of Consequences of Category of Chain Complex)*

- *(Homotopy Category)*
  The homotopy category induced by the category of chain complexes is $K(\mathscr{A}) = Ch(\mathscr{A})/\sim$ *where* $f \sim g$ *if* $f - g = hd + dh$.

- *(Derived Category)*
  Derived category is genenrated from either homotopy category or category of chain complexes, hence $D(\mathscr{A}) = K(\mathscr{A})[qis^{-1}]$ *where making all the quasi-isomorphism inversible, while similarly* $D(\mathscr{A}) = Ch(\mathscr{A})[qis^{-1}]$, *which makes the same result.*

- *(Triangulated Categeory)*
  Triangulated category consists of 4 axioms. First, (TR1) says that the existence of distinguished triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ *for any* $f : A \to B$ *where* $C = Cone(f)$ *is a mapping cone, where* $[1]$ *is the shifting or translation functor just adding suspension denoted* $[1]X = \Sigma X$. *In (TR2) if* $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ *is distinguished triangle, then* $B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{u[1]} B[1]$ *is also distinguished triangle. In (TR3), For a morphism of two triangles* $X \to Y \to Z$ *and* $X' \to Y' \to Z'$, *if existence of two morphism* $f : X \to X'$ *and* $g : Y \to Y'$ *makes the complement* $h : Z \to Z'$, *makeing the whole diagram commutes. (TR4) as octahedral axiom says the* 3 *distinguished triangles generate another distinguished triangle.*

  Its obvious consequence is that short exact sequence can be defined in the category of chain complexes, since it's an abelian category, while in category of chain complexes, we can also define distinguished triangle, namely derived version of short exact sequence, and triangulated category never becomes an abelian category, since it's already taken its quotient by the homotopy equivalence, and the kernel/cokernel is undefined.

- *(Model Category)*
  Model categories are, for example, category of simplicial sets $sSet$ *and category of topological spaces* Top, *and it consists of fibration, cofibration, and weak equivalences, which can be constructible anywhere where homotopy theory is definable. In* $sSet$, *fibration is Kan fibration, cofibration is injection, weak equivalence if its geometric realization* $|X| \to |Y|$ *is weak equivalence in topological space, which means its homotopy* $\pi_n(f) : \pi_n(|X|) \to \pi_n(|Y|)$ *is isomorphism for all* $n \in \mathbb{Z}_{\geqq 0}$.

- *(DG Category)*
  In short, DG category is a category whose $Hom(X, Y)$ *itself is a chain complex. DG category* $\mathscr{C}$ *consists of data* $Ob(\mathscr{C})$ *with morphisms* $Hom_{\mathscr{C}}(X, Y) = \oplus_{i \in \mathbb{Z}} Hom^i_{\mathscr{C}}(X, Y)$ *as* $\mathbb{Z}$*-graded* k*-vector space, and the differential is* $d : Hom^i_{\mathscr{C}}(X, Y) \to Hom^{i+1}_{\mathscr{C}}(X, Y)$ *such that* $d \circ d = 0$. *The composition* $\circ : Hom_{\mathscr{C}}(Y, Z) \otimes Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{C}}(X, Z)$ *needs to be the*

chain map, which means $d(f \circ g) = (df) \circ g + (-1)^{|f|} f \circ (dg)$. Also, id_X $inHom^0_{\mathscr{C}}(X, X)$. Listing few examples of DG category as a category of chain complexes $Ch(k)$ and category of elliptic complexes, and category of de Rham complexes etc.

- *(∞-Category)*
  ∞-category is a quasi-category consists of objects, 1-morphisms, 2-morphisms, and n-morphisms where $n \in \mathbb{Z}$ in general. Strictly this is not a category, since associativity failed but equivalent up to homotopy $(f \circ g) \circ h \cong f \circ (g \circ h)$.

Definition 3.5.3 *(Why Derived Categry is Triangulated category)*
What do you think?

Definition 3.5.4 *(Model Category the Same as Triangulated Category?)*
No, but it's somewhat common.

Definition 3.5.5 *(DG-Enhancement)*
Intuitively, if localization makes a category of chain complexes to a derived category(triangulated category), DG enhancement is the opposite operation. DG-enhancement is a process of finding a DG-category $\mathcal{A}$ equivalent to the triangulated category $\mathcal{T}$, so that $H^0((A)) \cong \mathcal{T}$.

Definition 3.5.6 *((∞,1)-Category)*
(∞,1)-Category is ∞-Category but all n-morphism of higher degree $n \geqq 2$ can be invertible.

Definition 3.5.7 *(dg-Nerve)*
dg-Nerve is a tool to generate (∞,1)-Category from DG category ((∞,1)-Category is already ∞-Category), and its idea is to take the positive part $Hom_{\mathcal{C}}(X, Y)^{\geqq 0}$, that can be transformed to simplicial complex. By Lurie's definition, for dg-category $\mathcal{C}$, the nerve $N_{dg}(\mathcal{C})$ is a simplicial set.

For $N_{dg}(\mathcal{C})$, the object is $Ob(\mathcal{C})$, 1-morphism is $Hom^0(X, Y)$ in some dg-category, 2-morphism is H such that $d(H) = g \circ f - h$, and $g \circ f \sim h$ needs to be homotopy, and $f, g, h \in Hom^0$ and $H \in Hom^1$, and n-morphism is just repeating the homotopy.

Definition 3.5.8 *(Pretriangulated Category)*
$\mathcal{C}$ is pretriangulated category is if shift [1] is defined, and it's closed under this shift operation, and the cone $Cone(f)$ exists for any morphisms $f : X \to Y$, and any twisted complex is closed under $\mathcal{C}$.

Twisted complex C is $C = \bigoplus_i X_i[k_i]$ whose differential is $d_C = \begin{bmatrix} d_{X_1} & q_{12} & \cdots \\ 0 & d_{X_2} & \cdots \\ \cdots & & \cdots \end{bmatrix}$, so that $d_C^2 = 0$.

Definition 3.5.9 *(Pretriangulated Hull)*
DG category $\mathcal{C}$ is not pretriangulated but pretriangularizable. DG category already defines the shifting operator and cones, and they are closed under operations, and the only remaining problem is the existence of the twisting complexes. If we accept twisting complex generated from the category, so that the newly generated category pretriangulated category, then it's called pretriangulization.

Definition 3.5.10 *(Triangulization)*
The triangulization of pretriangulated category $\mathbb{C}$ is possible, and it's simply taken to the categorical homotopy $H^0(\mathcal{C})$. If the morphism of $\mathcal{C}$ $Hom_{\mathcal{C}}(X,Y)$ is a chain complex, then its homology $H^0(Hom_{\mathcal{C}}(X,Y))$ is the triangulted category.

Definition 3.5.11 *(DG-Enhancement)*
dg-enhancement is the opposite operation from the triangulization, means to find a category $\mathcal{D}$ such that $H^0(\mathcal{D}) \cong \mathcal{T}$. In particular, if the triangulated category $\mathcal{T} = D(\mathcal{A})$ is derived category, then its dg-enhancement is $C(\mathcal{A})$. Or if $\mathcal{T}$ is small category, we can use dg-Yoneda embedding to make dg-enhancement $\mathcal{T} \hookrightarrow Fun(\mathcal{T}^{op}, Ch(\Bbbk))$.

3.6 ∞-World

Definition 3.6.1 *(A-∞ Algebra)*

Definition 3.6.2 *(A-∞ Category)*

Definition 3.6.3 *(Fukaya Category)*

Definition 3.6.4 *()*

3.7 Equivalence of Categories

Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalence of categories if the functor $F : \mathcal{C} \to \mathcal{D}$ has an inverse $G : \mathcal{D} \to \mathcal{C}$ such that the composisions $F \circ G \cong Id_{\mathcal{D}}$ and $G \circ F \cong Id_{\mathcal{C}}$ are naturally isomorphic to the identities. At one sight, equivalence of categories looks the same as isomorphism of categories, but they are actually many variety of difference, as we can see in Fourier-Mukai and Morita equivalence, so it's very rich in applications.

Example 3.8 *(Derived Categories)*
Equivalence of categories can be used for derived category by means of homological algebra, and $D(\mathcal{A}) \cong D(\mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, which in particular, $D^b(A - mod) \cong D^b(B - mod)$ where $A = k[x]/(x^2)$ and $B = k \oplus k$ for both satisfy SES $0 \to S \to A \to S \to 0$ where $S = k$.

Definition 3.8.1 *(Fourier-Mukai Transform)*
Fourier-Mukai is a paradigm to interpret algebraic variety by language of derived categories.

$$\Phi^{\mathcal{P}} : D^b(Coh(X)) \to D^b(Coh(Y))$$

Definition 3.8.2 *(Morita Equivalence)*
Morita equivalence is an equivalence of categories $A - Mod \cong B - Mod$ of two rings A and B. For example,

- *(Product)*
  $A - Mod \cong A \times A - Mod$
- *(Matrix)*
  $A - Mod \cong M_n(A) - Mod$
- *(Scheme)*
  $QCoh(X) \cong A{-}Mod$ *for an affine scheme* $X = Spec(A)$*, thus* $QCoh(X) \cong QCoh(Y)$.
- *(Quantum Group)*
  $A - Mod \cong U_q(\mathfrak{g}) - Mod$. *In particular,* $k[x]/(x^2) - Mod \cong U_q(\mathfrak{sl}_2) - Mod$
- *(Progenerator Module)*
  $RHom_A(T, -) : D^b(A - mod) \to D^b(End_A(T) - mod)$ T *is a tilting module. Tilting module if finite projective dimension and* $Ext^{>0}(T, T) = 0$.
- *(Quasi-Frobenius Algebra)*
  For $A - Mod \cong B - Mod$*, if* A *is QF, then* B *is also QF.*

3.9 Resolution Problem

Definition 3.9.1 *(Perfect Complex)*
$E^{\cdot} \in D(QCoh(X))$ *is perfect iff* $\forall x \in X$, $\exists U \ni x$, $E^{\cdot}|_U \cong$ *free complex of finite length where* $E^{\cdot}$ *is generated by some finite amount of free* $\mathscr{O}_U$*-module.* $E^{\cdot}$ *is perfect iff* $E^{\cdot} \in D(QCoh(X))$ *is compact object. The subcategory* $Perf(X) \subset D(QCoh(X))$ *is a category of perfect complexes.*

Example 3.10 *(Perfect Complex)*
Cotangent complex of moduli stack is perfect.

3.11 Algebraization

Definition 3.11.1 *(Ext and Tor)*

Definition 3.11.2 *(Yoneda Algebra)*

Definition 3.11.3 *()*

Definition 3.11.4 *()*

Definition 3.11.5 *()*

Definition 3.11.6 *()*

4 Sheaf and Cohomology

4.1 Sheaf

The idea of sheaf is to systematically classify the functions defined on the domain X, and for the structure, sheaf is defined to be a presheaf with extra structures on it. Presheaf is merely a contravariant functor, but it's already possible to construct simplicial category, while sheaf is a necessary step to define sheaf cohomology. But sheaf cohomology could be different from de Rham cohomology(or whatever non-sheaf theoretic), since sheaf can grasp the geometry from global perspective, while de Rham is just cohomology of differential forms, requiring smoothness, but most of algebraic geometry including moduli space contain singularity.

Example 4.2 *(Presheaf)*
For example, consider a functor $F : Top \to Ab$ where $Top = \mathscr{O}(X)$ is a category of topological space over X whose objects are open subsets $U \subset X$ by some topology, and their morphisms are inclusion morphisms, and Ab is a category of abelian groups, which in other words is to correspond each open neighborhood $U \subset X$ by set of functions $Hom(U, \mathbb{C})$ (of course it could be anything like $Hom(U, \mathbb{C}^n)$)

Why do we need to do this? Because not all the functions share the same domains. For example, if $X = \mathbb{C}$ and $f(z) = \frac{1}{z}$, then the function f has a singular point at $\{0\}$, which means the domain is $U = \mathbb{C} - \{0\}$, and $f \in Hom(U, \mathbb{C})$ but $f \notin Hom(X, \mathbb{C})$. This set of functions defines a vector bundle. Just for the sake of terminology, in algebraic geometry, the subscheme of codimension 1 is often called divisor, and in this example, singular points are divisors. Divisors in algebraic geometry implicitly means vector bundles. Alright, we have already talked many in presheaf, but we haven't reached sheaf yet.

My interpretation is that sheaf is a classification tool of function of some topological space, not algebraic geometry itself. In deed, sheaf is a broader subject, and for example, not all the functions needs to be analytic/holomorphic, it could be smooth functions or else.

Definition 4.2.1 *(Sheaf)*
obvious.

Definition 4.2.2 *(Cosheaf)*
Cosheaf is a copresheaf $G : Open(X) \to \mathscr{C}$ wtih a gluing condition with a pushout. The gluing condition can be coequalizer given by

$$G(U_1) \cap G(U_2) \rightrightarrows G(U_1) \oplus G(U_2) \to G(U) \text{ for all } U = U_1 \cup U_2$$

If sheaf makes sheaf cohomology, cosheaf makes sheaf homology

Definition 4.2.3 *(A-∞ Category)*
A-∞ category $\mathscr{A}$ consists of objects $Ob(\mathscr{A})$ such that the composition maps m_n satisfies the A-∞ conditions $\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes s}) = 0$.

Here is the list of A-∞ properties. A-∞ functor $F^n : A^{\otimes n} \to B$ whose composition satisfies the higher homotopy. It's A-∞ equivalent, if F_1 is quasi-isomorphism, in other words, $Hom_A(X,Y) \cong Hom_B(F_1X, F_1Y)$, and A-$\infty$ enhancement of T is an existence of A-∞ category s.t $H^0(A) \cong T$.

Example 4.3 *(Fukaya Category)*
Fukaya category $Fuk(Y)$ is an A-∞ category with symplectic structure with Floer homology. $Fuk(Y)$ only applies if Y is compact symplectic, since if it's non-compact, the Floer homology could be infinite-dimensional, while wrapped Fukaya category $W(Y)$ is a generalization of $Fuk(Y)$ to the non-compact case. The given symplectic geometry Y can be decomposed to each local data, and if its Fukaya category of each of the local data can be pushed out and glued, it will be the whole wrapped Fukaya category. Considered $Y \mapsto W(Y)$ is a morphism mapping geometry to its wrapped Fukaya category $Symp \to Cat_\infty$, which is atually a cosheaf.

Definition 4.3.1 *(Homological Mirror Symmetry)*
HMS Conjecture claims A-∞ equivalence of $D^b(Coh(X)) \cong Fuk(Y)$, as convention $D^b(Coh(X))$ is interpreted as A-∞ enhancement of itself. In addition, Verdier duality on A-model and Grothendieck duality on B-model correspond each other. Grothendieck duality on B-model side $RHom(-, \omega_X[dim(X)])$ corresponds to Verdier duality on A-model sides, which is a mirror of Fukaya categories, and Verdier duality is just inverse of Lagrangian as $SS(D\mathscr{F}) = -SS(\mathscr{F})$.

Definition 4.3.2 *(Derived Stack and MC Equation)*
For the moduli space M, L_∞ structure consists of $L_x = RHom(F,F)[1]$, $\frac{L_\infty}{gauge} \cong$ formal nbhd of $x \in M = \hat{M}_x$ from perspective of derived stack.

Example 4.4 *(Structure sheaf)*
Structure sheaf in algebraic geometry is a contravariant functor from Top to Ab

as $F : Top \to Ab$. In particular, Top is a category whose objects are all open subset $U \subset X$ with Zariski topology, if scheme theory is considered (and it could be alternately étale or fppf or fpqc etc).

It's clear the definition is, but for pedagogistic purpose, a sheaf doesn't define a scheme, but the sheaf is a structure defined over the scheme.

Example 4.5 *(Constant Sheaf)*
Another famous example of sheaf is constant sheaf, which means that the stalk is always having the same object, which means for all the point $p \in X$, the stalk $F_p = L$ constant.

The stalked value doesn't seem to be interesting, but in general for the open sets, a constant sheaf is a sheaf that maps $U \subset X$ to a set of functions, which each function is constant on U.

There are some variations of constant sheaf. For example, if X is connected and the codomain is category of abelian groups, then the locally constant sheaf is constant sheaf. This is called local system. Also, perverse sheaf is created by simply shifting a constant sheaf.

Definition 4.5.1 *(Local System)*
Now here is the tip. Constant in constant sheaf is that it's geometrically trivial but the topology might not be trivial, and local system as a constant sheaf bijectively corresponds to monodromy representations (which is group homomorphisms) $\rho : \pi_1(X, x) \to Aut(L)$.

Example 4.6 *(Local System)*
From horizontal sections of vector bundle with a flat connection ∇, we can define local system.

$$E_U^{\nabla} := \{s \in \Gamma(U, E) \text{ which are horizontal: } \nabla s = 0\}$$

List of flat connections

- *Gauss-Manin connection*
- *KZ connection*

Example 4.7 *(Perverse Sheaf)*
If X is a smooth complex algebraic variety and everywhere of dimension d, and F is a local system on X, then $F[d]$ is a perverse sheaf.

Let a category of sheaves $Sh_c(X)$ over X, which is an abelian category, and the local system $F \in Sh_c(X)$ could be an object, and we consider an embedding to its derived category $D_c^b(X)$, and indentify the local system as a chain complex $[\cdots \to 0 \to F \to 0 \to \cdots]$

$$
\begin{aligned}
Sh_c(X) &\to D^b_c(X) \\
F &\mapsto [\cdots \to 0 \to F \to 0 \to \cdots]
\end{aligned} \tag{2}
$$

and this chain complex as the mapped local system is $F[0]$ as an object of the derived category, where by abuse of notation we let $F = [\cdots \to 0 \to F \to 0 \to \cdots]$, and $F[d]$ is simply shifting of $F = F[0]$.

- *$dim(supp(H^{-i}(F))) \leqq i$ (support condition)*
- *$dim(supp(H^{i}(\mathbb{D}F))) \leqq i$ (cosupport condition)*

where $\mathbb{D}$ is Verdier duality. Verdier duality is a generalization of Poincare duality, which is $H^i_c(X,\mathbb{Z}) \cong H_{n-i}(X,\mathbb{Z})^*$ where H^i_c is a cohomology with a compact support and H_{n-i} is a homology, while Verdier duality consists of Verdier duality functor $\mathbb{D} : D^b(Sh(X)) \to D^b(Sh(X))$, that can be defined by $\mathbb{D}_X(\mathscr{F}) := RHom(\mathscr{F}, \omega^{\cdot}_X)$, where $\mathscr{F}$ is a sheaf over X and $\omega^{\cdot}_x$ is a dualizaing complex. It satisfies $\cong \mathbb{D}(\mathbb{D}(\mathscr{F}))$ and $H^i_c(X,\mathscr{F}) \cong H^{-i}(X, D_X(\mathscr{F}))^{\cdot}$, and $D_Y(Rf_!F) \cong Rf_*D_X(\mathscr{F})$.

4.8 Microlocal Analysis

Definition 4.8.1 *(Grothendieck Six Operations)*
For a scheme morphism $f : X \to Y$, the six operations f^*, f_*, $f_!$, $f^!$, ${}^L\otimes$, and $RHom(-,-)$ are the only peculiar problems in sheaf cohomology, otherwise sheaf cohomology is the same as module cohomology. For example, pullback $f^* : D(Y) \to D(X)$ and pushforward $f_* : D(X) \to D(Y)$ are the invariants of the derived geometry as $f^* = Lf^* = L(f^*)$ and $f^* = Rf^*$, which are adjoint $f^* \dashv f_*$, and the derived tensor $-{}^L\otimes- : D(X) \times D(X) \to D(X)$ and the derived hom $RHom(-,-) : D^{op}(X) \times D(X) \to D(X)$ are another adjoint ${}^L\otimes G \dashv RHom(G,-)$. The remaining two operators are complicated, but as an extension from the usual pullback and pushforward by adding a compact support.

For shriek pushforward $f_! : D(X) \to D(Y)$, to minimize the complexity of $f_!$, let's say the $f : X \to Y$ is the composition of $X \xrightarrow{j} \overline{X} \xrightarrow{\overline{f}} Y$, where $j : U \hookrightarrow X$ is an inclusion, and $j_!F(V) = \begin{cases} F(V) \text{ if } V \subset U \\ 0 \text{ if } V \not\subset U \end{cases}$, or $f_!$ is simply $Rf_!F = R\overline{f_*}(j_!F)$ if we use it.

For shriek pullback $f^! : D(Y) \to D(X)$, $f^!G \cong f^*G \otimes \omega_{X/Y}[d]$ where $\omega_{X/Y}[d] := \wedge^d\Omega^1_{X/Y}[d]$. Here $\Omega_{X/Y}$ is a sheaf of relative differential form, and $[1]$ is a cohomological shfit, which is shifting the degree of chain complex by 1 with differential times -1, and $[d]$ is doing $[1]$ d-times, and its adjointness relations is $f_! \dashv f^!$, but the order of the upper/lower case are the opposite.

Definition 4.8.2 *(Support)*
Support of sheaf $\mathscr{F}$ is defined by $supp(\mathscr{F}) = \overline{\{x \in X | F_x \neq 0\}} \subset X$, and it's

compact support if supp($\mathscr{F}$) is compact, whose reason to take the closure is the same as all closed subset of an algebraic variety is an algebraic set. Combining the six operators, $supp(f^*G) = f^{-1}(supp(G))$, $supp(f_*F) \subset f(supp(F))$ where its subset is equal if f is proper, and $supp(f^!G) = f^{-1}(supp(G))$, $supp(f_!F) \subset f(supp(F))$ if proper, and $supp(F \otimes G) = supp(F) \cap supp(G)$, and $supp(Hom(F,G)) \subset supp(F) \cap supp(G)$.

Definition 4.8.3 *(Microlocal Support)*
In addition, the sheaf theory version of microlocal support is $SS(\mathscr{F}) \subset T^*X \backslash 0$ such that $SS(\mathscr{F}) = \{(x,\xi) \in T^*X\backslash 0 |\ \mathscr{F}$ is not micro-locally flat in direction ξ at $x\}$, where micro-locally flat means $\exists$ functions ϕ such that $d\phi(x_0) = \xi_0$ and $H^*(\mathscr{F}_{\phi \geqq 0}) = 0$. $\mathscr{F}_{\phi \geqq 0} = R\Gamma_{\phi \geqq 0}(\mathscr{F})$ where $R\Gamma_Z(\mathscr{F}) \in D^b(\mathbb{Q} - mod)$, which is particularly $\Gamma_Z(\mathscr{F}) = \{s \in \Gamma(X, \mathscr{F}) | supp(s) \subset Z\}$.

Now, together with the six operators, $SS(f^*G) \subset f_d^{-1}(SS(G))$ where $f_d : T^*M \to T^*N$ $SS(f_*F) \subset f_\pi(SS(F))$ where $f_\pi : T^*M \to T^*N$ is a fiber-wise map, and $f_!$ and $f^!$ are similar. Also, $SS(F \otimes G) \subset SS(F) \hat{+} SS(G)$ and $SS(Hom(F,G)) \subset SS(G) \hat{-} SS(F)$ where $\hat{+}$ and $\hat{-}$ are microlocal sum/difference, that's given by the direction of synthesis over the cotangent bundle, meaning $A \hat{+} B = \{(x, \xi + \eta) |\ \exists x_n \to x,\ \xi_n \in A\ ,\ \eta_n \in B \xi_n + \eta_n \ \to\ \xi + \eta\}$, and $A \hat{-} B = A \hat{+} (-B)$.

Example 4.9 *()*
For skyscraper sheaf k_x, $SS(k_x) = T_x^*X$ where X is a smooth complex algebraic variety and $\mathscr{F}$ is either D-module or constructible sheaf.

Definition 4.9.1 *(Perverse t-Structure)*
Perverse t-structure is t-structure (truncation-structucture) but extra dimensional conditions. First of all, the t-structure consists of a pair $(D^{\leqq 0}, D^{\geqq 0})$, where shift condition $D^{\leqq 0}[1] \subset D^{\leqq 0}$ and $D^{\geqq 0}[-1] \subset D^{\geqq 0}$, and $Hom_D(X,Y)$ $\forall X \in D^{\leqq 0}, Y \in D^{\geqq 1}$. For all $X \in D$, a triangle $A \to X \to B \to A[1]$ exists such that $A \in D^{\leqq 0}, B \in D^{\geqq 1}$. Perverse t-structure is as follows.

$D_{perv}^{\leqq 0} = \{K \in D_c^b(X) | dim(supp(H^i(K))) \leqq -i \text{ for all } i\}$
$D_{perv}^{\geqq 0} = \{K \in D_c^b(X) | dim(supp(H^i(DK))) \leqq -i \text{ for all } i\}$
$Perv(X) = D_{perv}^{\geqq 0} \cap D_{perv}^{\leqq 0}$ The category of perverse sheaves $Perv(X)$ is an abelian category.

Definition 4.9.2 *(Perverse t-Exact)*
A functor $F : D \to D'$ is called t-exact if it's both right/left t-exact, and it's right t-exact, if $F(D^{\geqq 0}) \subset D'^{\geqq 0}$ and left t-exact if $F(D^{\leqq 0}) \subset D'^{\leqq 0}$, and its main property is that it preserves t-structure. Moreover, a functor F is perverse t-exact, if in particular $D = Perv(X) = D^b(X)_{perv}^{\heartsuit}$.

Definition 4.9.3 *(Beilinson-Bernstein-Deligne)*
$D^b(Perv(X)) \cong D_c^b(X)$ where D_c means derived category of constructible sheaves.

Construcible sheaf is a sheaf, that can be decomposable to finitely many locally constant sheaves.

Definition 4.9.4 *(Nadler-Zaslow)*
The microlocal support SS(F) is always conic Lagrangian if X is a complex variety, and conic means $(x, \xi) \in SS(F)$ means $(x, \lambda\xi)$, $\lambda > 0$, and there exists Fukaya objects that coresponds to it: constructible sheaves on $X \leftrightarrow$ Fukaya category of T^*X, or by formula $Sh_c(X) \cong Fuk_{conic}(T^*X)$ is an equivalence as A-∞ category.

Definition 4.9.5 *()*
Use microlocal analysis to relate perverse sheaf and Fukaya category.

Definition 4.9.6 *(DT Sheaf)*
DT sheaf is an improvement of classical DT invariant, since classical DT invariant is fragile, lacking BPS data and rich information, and computation is complicated. If we use Behrend function for weighted Euler characteristic, it'll make DT invariant as $DT(M) = \chi(M, \nu_M) = \sum\limits_{x \in M} \nu(x)$, where Behrend function $\nu_M : M \to \mathbb{Z}$ takes value $\nu_M(x) = (-1)^{dim(M)}$ if it's smooth, or $|\nu_M(x)| > 1$ if it's singular. DT sheaf is just a perverse sheaf ϕ_f, called vanishing cycle perserve sheaf, which is locally defined by Behrend function ν_M, where $M \cong Crit(f) \subset U$ where $f : U \to \mathbb{C}$ and U is smooth ambient space, but it can be extended globally, so $DT(M) = \chi(M, \mathscr{F}_{DT})$, or using this, $\chi(\phi_f(\mathbb{Q}_U[dim(U)])_x) = \nu_M(x)$ and DT Sheaf$|_{crit(f)} := \phi_f(\mathbb{Q}_U[dim(U)])$.

Definition 4.9.7 *(Cohomological Integrality)*
For DT invariant $DT(\gamma) \in \mathbb{Z}$, $\sum\limits_{\gamma} DT(\gamma)q^\gamma \in \mathbb{Z} = \prod\limits_{\gamma}(1 - q^\gamma)^{-\Omega(\gamma)}$ where $\Omega(\gamma)$ is a BPS invaraints.

Definition 4.9.8 *(Kontsevich-Soibelman)*
DT is a new type of quantum KZ.

Definition 4.9.9 *(Cotangent Complex)*
Cotangent complex $L_{\mathfrak{M}}$ is an object of derived category, given by $Hom_{D(A)}(L_{\mathfrak{M}} \otimes_A M, A) \cong$ Derived Deformation Group$Der_A(A, M)$, and $\mathfrak{M} : Alg_k \to$ Groupoids. Alternatively, cotangent complex $L_X \in D(M)$ is $H^0(L_X) = \Omega^1_X$ for a scheme X, or if Deligne-Mumford stack $\mathfrak{M} = [X/G]$, cotangent complex $L_{\mathfrak{M}} \in D(\mathfrak{M})$, $L_{[X/G]} \cong [L_X \to \mathfrak{g}^* \otimes \mathscr{O}_X]$.

For the case of closed embedding $i : X \hookrightarrow Y$, $L_{X/Y} \cong [I/I^2 \to i^*\Omega^1_Y]$.

Also, for $X \xrightarrow{f} Y \xrightarrow{g} Z$, $f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to$ where $L_{X/Y}$ is a cotangent complex of $X \to Y$.

4.10 Cotangent Sheaf

Definition 4.10.1 *(Tangent Sheaf)*
The sheaf θ_X on a scheme X is for all $U = Spec(A)$, $\theta_X(U) = Der_k(A, A)$, or

equivalently, it is a sheaf of the morphism $Hom(\Omega^1_{X/k}, \mathscr{O}_X)$

The stalk is Zariski tangent space.

Definition 4.10.2 *(Cotangent Sheaf)*
Let $\mathscr{F}$ *be an* $\mathscr{O}_X$ *module.*

Cotangent sheaf is a sheaf of $\mathscr{O}_X$ *modules* $\Omega_{X/S}$ *that represents the* S*-derivation in the sense.*

$$Hom(\Omega^1_{X/S}, F) = Der_S(\mathscr{O}_X, F)$$

where $\Omega_{X/S}$ *is a Kähler differential.*

Definition 4.10.3 *(Kodaira-Spencer Map)*
Let M *be a complex manifold, and it has a transition maps*

$$\begin{tikzcd} M \arrow[r, "="] \arrow[d] & M \arrow[d] \\ U_i \subset \mathbb{R}^n \arrow[r, "f_{ij}"] & U_j \subset \mathbb{R}^n \end{tikzcd}$$

whose deformation is

$$\begin{tikzcd} \mathscr{M} \arrow[r, "="] \arrow[d] & sheaf M \arrow[d] \\ B \arrow[r, "\tilde{f}_{ij}"] & B \end{tikzcd}$$

where $B = U_i \times Spec(k[\epsilon])$*, and Kodaira-Spencer map is* $KS : T_0B \to H^1(M, T_m)$.

In scheme theory, deformation of smooth manifold can be

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ Spec(k) \arrow[r] & Spec(k[\epsilon]) \end{tikzcd}$$

have a short exact sequence

$$0 \to \pi^*\Omega^1_{Spec(k[\epsilon])} \to \Omega^1_{\mathscr{X}} \to \Omega^1_{\mathscr{X}/S} \to 0$$

where $\pi : \mathscr{X} \to S = Spec(k[\epsilon])$*. If we tensored by the* $\mathscr{O}_{\mathscr{X}}$*-module* $\mathscr{O}_X$ *gives the short exact sequence*

$$0 \to \mathscr{O}_X \to \Omega^1_{\mathscr{X}} \otimes \mathscr{O}_X \to \Omega^1_X \to 0$$

Using derived categories, this can be

$$RHom(\Omega^1_X, \mathscr{O}_X[+1]) \cong RHom(\mathscr{O}_X, T_X[+1]) \cong Ext^1(\mathscr{O}_X, T_X) \cong H^1(X, T_X).$$

Of Ringed Topoi, this can be described more abstractly. For the composition of maps of ringed topoi,

$$X \to_f Y \to Z$$

Then, associated to this composition is a distinguished triangle

$$f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to^{+1}$$

Definition 4.10.4 *(Flat topology)*
fppf cover of X is a jointly surjective family of morphisms $(\phi_a : X_a \to X)$ where X_a affine ϕ_a flat, finitely presented. This generates pretopology.

Definition 4.10.5 *(Flat Cohomology)*
Flat cohomology or etale cohomology??

Definition 4.10.6 *(Hilbert Scheme)*

If $f : X \to S$ is flat, we have a canonically a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Hilb_S \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \end{array}$$

where $Hilb_S$ is a family of flat morphisms to S. Since f is flat, all fibers $f_s : X_s \to s$ for all $s \in S$ have the same Hilbert polynomial Φ. We could say $Hilb^{\Phi}_S = Hilb_S$.

4.11 Derived Geometry

Category of coherent sheaves $Coh(X)$ is inconvenient in the derived language, since $A[1] \notin Coh(X)$ or $Cone(f) \notin Coh(X)$ in general case when $A \in Coh(X)$ or $f : A \to B$ where $A, B \in Coh(X)$.

5 Stack and Geometry

5.1 Orbifold

Definition 5.1.1 *Groupoid Groupoid is a small category in which all morphisms are isomorphism (invertible).*

Example 5.2 *Groupoid*

- *(Symplectic Groupoid)*
  In Poisson geometry, a Sympletic groupoid is a Lie groupoid.

Definition 5.2.1 *Orbifold definition using Lie groupoid Let $G_1 \rightrightarrows G_0$ be a morphism where G_0 is a set of objects and G_1 is a set of arrows, and the structural maps $s, t : G_1 \rightrightarrows G_0$ and other structural maps i.e. unit, invert maps. It's called Lie groupoid if G_1 and G_0 are smooth manifolds, all maps are smooth, and s, t are submersions.*

The intersection of fibers $G_1(x) = s^{-1}(x) \cap t^{-1}(x)$ is a Lie group called isotropy group of G_1 at x.

A Lie groupoid is proper if $(s, t) : G_1 \to G_0 \times G_0$ is a proper map.

A Lie groupoid is Etale if both the source and target morphisms are local diffeomorphisms.

An orbifold groupoid is either

- *a proper etale Lie groupoid*
- *a proper Lie groupoid whose isotropies are discrete spaces*

Let $M \rightrightarrows G$ be a groupoid, and $|M/G|$ be the orbit space of the Lie groupoid G i.e. quotient of M by equivalence $x \sim y$ if there exists $g \in G$ such that $s(g) = x$ and $t(g) = y$. This shows that orbifolds are particular kind of differentiable stack.

An orbifold structure on Hausdorff space X is an Morita equivalence class of homeomorphism $|M/G| \cong X$.

5.3 Lie Gropoid/Algebroid

Definition 5.3.1 *(Fibered Category)*
For a functor $p : \mathscr{S} \to \mathscr{C}$, the category $\mathscr{S}$ is a fibered category if the morphism $f : V \to U \in Mor(\mathscr{C})$ induces $f^*x \to x$ for some $x \in \mathscr{S}$, and they make the strongly cartesian product.

$$\begin{tikzcd} f^*x \arrow[r, "p"] \arrow[d] & V \arrow[d, "f"] \\ x \arrow[r, "p"] & U \end{tikzcd}$$

A strong cartesian product is .

$$\begin{tikzcd} z \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d, "p"] & x \arrow[d, "p"] \\ p(z) \arrow[r] & V \arrow[r, "f"] & U \end{tikzcd}$$

If such a morphism $V \times_U x \to x$ exists, then it's called a strongly cartesian morphism.

Lie group/algebra are tools to study geometry of smooth manifold, but Lie group/algebra only define local property of the manifold, so I'm interested in Lie groupoid/algebroid generalization to discuss global situation. We will use category theoretic language for this.

Definition 5.3.2 *(Groupoid)*

Definition 5.3.3 *(A category fibered in groupoid)*

Let $p : \mathscr{F} \to \mathscr{C}$ be a functor of categories $\mathscr{F}$ and $\mathscr{C}$. For $x \in \mathscr{C}$, $p^{-1}(x) \in \mathscr{F}_c \subset \mathscr{F}$ be a subcategory, which we name by fiber category.

Definition 5.3.4 *(Lie Groupoid)*

Let G and M be smooth manifolds. Lie groupoid $G \rightrightarrows M$ consists of several morphisms.

- *(Two arrows)*
  $s, t : G \rightrightarrows M$ *be smooth submersions.*
- *(Multiplication map)*
  m: $G^{(2)} := \{(g,h)|s(g) = t(h)\} \subset G \times G \to G$
- *(Unit map)*
  $u : M \to G$
  Using unit map, identity map 1_x is defined as
  $1_x = u(x)$
- *(Inverse map)*
  $i : G \to G$
  $g^{-1} := i(g)$

Example 5.4 *(Lie Subgroupoid)*
For a Lie groupoid $G \rightrightarrows M$, its Lie subgroupoid is $H \rightrightarrows N$ where $H \subset G$ is an immersed submanifold. Some examples of Lie subgroupoid might be below

- *(Unit Lie subgroupoid)*
  $u(M)$ *is a sub Lie groupoid*
- *(Inner subgroupoid)*
  $IG = \{g \in G|s(g) = t(g)\}$

Example 5.5 *(Notable Example of Lie Groupoid – Trivial and Extreme)*

- *(Group)*
  A Lie groupoid $G \rightrightarrows *$ is the same as Lie group.
- *(Pair groipoid)*
  $M \times M \rightrightarrows M$ with precisely one morphism from one another (i.e. $s = t$).
- *(Trivial Groupoid)*
  $M \times G \times M \rightrightarrows M$ with structure maps

  - $s(x, g, y) = y$
  - $t(x, g, y) = x$
  - $m((x, g, y), (y, h, z)) = (x, gh, z)$
  - $u(x) = (x, 1, x)$
  - $i(x, g, y) = (y, g^{-1}, x)$
- *(Unit Groupoid)*
  $u(M) \rightrightarrows M$

Definition 5.5.1 *(a)*

Lie algebroid is an analogy of Lie algebra.

Definition 5.5.2 *(Lie Algebroid)*

Lie algebroid $(A, [\cdot, \cdot], \rho)$ consisting of

- *A vector bundle A over a manifold M.*
- *A Lie bracket $[\cdot, \cdot]$ on its space of sections $\Gamma(A)$*
- *A morphism of vector bundles $\rho : A \to TM$, called an anchor, where TM is the tangent bundle of M.*

with product rule

$$[X, fY] = \rho(X)f \cdot Y + f[X, Y]$$

where $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$, and $\rho(X)f$ is an image of f via derivation $\rho(X)$.

Example 5.6 *(Example of Lie Algebroid)*

- *(Tangent Lie Algebroid)*
  $TM \to M$ and $\rho = id_{TM}$ is an identity.
- *(Lie algebroid to a point)*
  $TM \to *$ is just a Lie algebra.
- *(Any bundle of Lie algebra)*
  Yes, and $\rho = id$.

5.7 Differentiable Stacks

Differentiable stack is an analogue of algebraic stack, and it is always Morita equivalent to Lie groupoid.

Definition 5.7.1 *(Differentiable Stacks)*

Definition 5.7.2 *(Quotient Stack)*
Let G be an affine group scheme over S, and X be an S-scheme on which G acts. A quotient stack $[X/G]$ is a stack where for each T defines a category $[X/G](T)$, and an object $P \to T \in Ob([X/G](T))$ is a principal G-bundle together with an equivariant map $P \to X$.

Definition 5.7.3 *(Categorical Quotient)*
X/G A categorical quotient X/G is a morphism $\pi : X \to Y$ that coequalizes $\sigma, pr_2 : G \otimes X \to X$, and that satisfies universal property: the other equalizer can be factored by this. Note π need not to be surjective.

Notable example of categorical quotients are geometric quotient G/H and GIT $X//G$.

symplectic reduction (symplectic quotient)
phase space cotangent bundle

5.8 Category Fibred in Gropoids

Definition 5.8.1 *(Categorical Quotient)*

Let X be an object of a category $\mathscr{C}$, G be a group. A categorical quotient X/G is a morphism $\pi : X \to Y$ such that the group action and projection $\sigma, p_2 : G \times X \to X$ is coequalized by $\pi \circ \sigma = \pi \circ p_2$, and any other morphisms $X \to Z$ factors by it.

Definition 5.8.2 *(Quotient Stack)*
Let X be a S-scheme and G be an affine group S-scheme, and G acts on X. A quotient stack $[X/G]$ is a category over the category of S-schemes such that

- *an object is a pricipal G-bundle $P \to T$ together with an equivariant map $P \to X$.*
- *a morphism is a bundle map from $P \to T$ to $P' \to T'$ compatible with equivariant maps $P \to X$ and $P' \to X$*

A quotient stack is always an algebraic stack.

ex:

- *(trivial case)*
  $[pt/pt] = Sch_S$

- *()*
  For $[X/pt]$*, the affine group* pt *is trivial, so the principal* G*-bundle is trivial bundle* $T \to T$*, that can be identified to* T *itself. Hence, the object is identified with a morphism* $T \to X$*, but since the group* G *is trivial, equivarint map could be any morphism* $T \to X$*.*

- *()*
  For $[pt/G]$*, an equivariant map* $X \to pt$ *is always trivial, which means unique, so it's naturally 1-to-1 to category of pricipal* G*-bundle of any basement* S*-scheme* T*.*

- *(inclusion relation)*
  If $Y \subset X$ *is a subscheme and* $H \subset G$ *is a subgroup scheme, then we have an inclusion relation*

  $$[Y/H] \subset [Y/G] \subset [Y/G] \subset [X/G]$$

Definition 5.8.3 *(Algebraic Stack)*
An algebraic stack is a fibred category

$$p : \mathscr{X} \to (Sch/S)_{fppf}$$

over $(Sch/S)_{fppf}$*, which a category of scheme added with Grothethendieck topology, and*

- $\mathscr{X}$ *is a stack in groupoid over* $(Sch/S)_{fppf}$*.*
- $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ *is representable*
- *there exist a scheme* $U \in (Sch/S)_{fppf}$ *and a* 1*-morphism* $(Sch/S)_{fppf} \to \mathscr{X}$ *which is surjective and smooth.*

Diagonal morphism is representable ??
flat and smooth morphism ??

Definition 5.8.4 *(Stack)*
Prestack $\mathscr{S}$ *over a presite* X *is a* 2*-category. For each object* $U \in \mathscr{C}_X$*,* $\mathscr{S}(U)$ *is a category.*

Stack is a separated prestack.

Definition 5.8.5 *(Morita Equivalence)*
Let $G \rightrightarrows M$ *and* $H \rightrightarrows N$ *be Lie groupoids. They are Morita equivalent if there exists* P *such that*

$P \to M$ is a principal H-bundle while $P \to N$ is a principal G-bundle such that two actions on P commutes.

Definition 5.8.6 *(Differentiable Stack)*

Definition 5.8.7 *(Normal Bundle)*

- *(Normal Bundle)*

- *(Tangent Bundle)*

- *(Cotangent Bundle)*

Definition 5.8.8 *(Cotangent Complex)*

6 Derive Geometry

Definition 6.0.1 *(Koszul Complex)*

Originally used to calculate cohomology of Lie algebra.

For a commutative algebra A, its Koszul complex K_s is

$\wedge^r A^r \to \wedge^{r-1} A^r \to \cdots \to \wedge^1 A^r \to \wedge^0 A^r \cong A^r$

where the maps send

$\alpha_1 \wedge \cdots \wedge \alpha_k \mapsto \Sigma_{i=1}^{k} (-1)^{i+1} s(\alpha_i) \alpha_1 \wedge \cdots \wedge \hat{\alpha_i} \wedge \cdots \wedge \alpha_k.$

Or we can change A^r to any A-module.

Definition 6.0.2 *(Derived Scheme)*

Derived scheme is a homotopy theoretic generalization of a scheme in which commutative rings are replaced by dga An affine derived algebraic geometry is equivalent to the theory of commutative dg-rings.

$(X, \mathscr{O}.) = RSpec(R/(f_1) \otimes^L_R \cdots \otimes^L_R R/(f_k))$

where $f_i \in \mathbb{C}[x_1, \cdots, x_n] = R$

Then we get a derived scheme that

$RSpec : (dga_{\mathbb{C}})^{op} \to DerSch$

is the étale spectrum. Since we construct a resolution

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{\cdot f_i} & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & R/(f_i) & \longrightarrow & 0 \end{array}$$

a derived ring $R/(f_1)\otimes^L_R \cdots \otimes^L_R R/(f_k)$*, a derived tensor product, is a Koszul complex* $K_R(f_1, \cdots, f_n)$*. The truncation of this derived scheme to amplitude* $[-1, 0]$ *provides a classical model motivating derived algebraic geometry.*

Notice that if we have a projective scheme

$Proj(\frac{\mathbb{Z}[x_1,\cdots,x_n]}{(f_1,\cdots,f_k)})$

where $deg(f_i) = d_i$ *we can construct the derived scheme* $(\mathbb{P}^n, \mathscr{E}^{\cdot}, (f_1, \cdots, f_k))$ *where*

$\mathscr{E}^{\cdot} = [\mathscr{O}(-d_1) \oplus \cdots \oplus \mathscr{O}(-d_k) \to^{(\cdot f_1,\cdots,\cdot f_k)} \mathscr{O}]$

with amplitude $[-1, 0]$.

Definition 6.0.3 *(Cotangent Complex)*

Let $(A., d)$ *be a fixed dga over char* 0*. Then* $A.$*-dga* $(R., d_R)$ *is called semi-free if*

- $R.$ *is a polynomial algebra over* $A.$*.*
- *there exists a filtration* $= I_0 \subset I_1 \subset \cdots$ *such that* $\cup I_n = I$*, and* $s(x_i) \in A.[(x_j)_{j\in I_n}]$ *for any* $x_i \in I_{n+1}$*.*

The relative cotangent complex of an $(A., d)$*-dga* $(B., d_B)$ *can be constructed using semi-free resolution* $(R., d_R) \to (B., d_B)$*, which is defined as*

$\mathbb{L}_{B./A.} = \Omega_{R./A.} \otimes_{R.} B.$

Example 6.1 *(Cotangent Complexes)*

For the cotangnet complex of hypersurface $X = \mathbb{V}(f) \subset \mathbb{A}^n_{\mathbb{C}}$ *,* *dga $K_R(f)$ representing the derived enhancement of X. Its cotangent complex is*

$$0 \to R \cdot ds \to^{\Phi} \to \oplus_i R \cdot dx_i to 0$$

where $\Phi(gds) = g \cdot df$ and d is the universal derivation. If we take complete intersection, then the Koszul complex

$$R^{\cdot} = \frac{\mathbb{C}[x_1, \cdots, x_n]}{(f_1)} \otimes^L_{\mathbb{C}[x_1,\cdots,x_n]} \cdots \otimes^L_{\mathbb{C}[x_1,\cdots,x_n]} \frac{\mathbb{C}[x_1,\cdots,x_n]}{(f_k)}$$

is quasi-isomorphic to

$$\frac{\mathbb{C}[x_1,\cdots,x_n]}{(f_1,\cdots,f_k)}[+0]$$

Example 6.2 *(Tangent Complex)*

Definition 6.2.1 *(Abelian Variety)*

References

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[2] *Chain Homotopies Geometrically*, https://grossack.site/2022/06/24/chain-homotopies.html