# Complex Analysis Perspective on Algebraic Geometry

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## Contents



# 1 Preliminary

## 1.1 Algebraic Curve

Complex Analysis:

We'll consider algebraic interpretation of complex analysis, where the holomorphic functions are calculated by their singularities, and the integral pertaining this has monodromy, or it has a Laurant expansion to calculate residue theorem, or in diffyq, singularities is more than power series solution, and we'll study algebraic geometry and representation theory for this.

The first question is the inverse problem: given the complex domain(plane), with some singularities, can we determine all the possible function whose domain is it? In 1-dimensional complex analysis, if the domain of the funcion $f(z)$ is $\mathbb{C}\backslash\{0\}$, then the function $f(z)$ could be $f(z) = \frac{1}{z}$ but also $f(z) = log(z)$, This is called monodromy, and we use monodromy representation to determine all the possibilities. This problem is reduced to the combinational problem of hypergeometric functions or KZ equation in general.

Moduli Problem:

Answering for what is moduli space is a very ambiguous problem, but we should approach from the concrete examples, that can be generated by GIT quotient of some geometries: for example, it might be $M_g$ moduli space i.e. a set of stable algebraic curves of genus $g$, or in particular, $M_{g,n}$ a moduli space of genus $g$ of $n$-marked points as $M_g$ consists of their disjoint union $M_g = \coprod M_{g,n}$, and this idea is closely related to complex analysis, and our first goal is to describe the moduli spaces explicitly.

Another argument that sounds is why we should use the technique of algebraic geometry for complex analysis, and I believe it's because algebraic variety could be considered to be "hard" geometry, compared with smooth manifold in general, lacking diversity of their shapes, and especially the automorphism of the geometry could be a finite group, also called stability condition. For example, $\mathbb{CP}^1$ is the only algebraic curve, which is 1-dimensional compact projective geometries of genus 0, and similarly an elliptic curve is the only genus 1 compact algebraic curve.

The moduli problem can be further generalized by the language of stacks. Or perhaps derived algebraic geometry and so on......

## 1.2 GIT Quotient

**Definition 1.2.1** *(GIT Quotient)*

*By context of moduli problem, GIT quotient is modularization. For an algebraic variety $X$, if a reductive group $G$ acts on $X$ algebraically, it will define quotient $X//G$ of good properties. For example,*

$$M_{0,n} = Conf_n(C)/SL2(C)$$

*Why $G$ needs to be reductive group because if a polynomial ring $A$ is fixed by $G$, $A^G$ is finitely generated (or in other words, if $G$ is not reductive, GIT is undefiend, or it doesn't have a good property).*

**Proposition 1.3** *(Keel-Mori Theorem)*

*For any schemes, GIT quotient exist as Deligne-Mumford stack.*

**Remark 1.4** *(Compactness)*
*Compactness is also a very important notion in the context of algebraic geometry, since compactness guarantees the bounray problem in geometry, and it the geometry satisfies compactness, it'll enables us to discuss intersection theory, including Poincare duality. If the geometry if compact, then the automorphism is beautiful,*

$$Aut(\mathbb{P}^1) = PGL_2(\mathbb{C})$$

*while for non-compact space, the automorphism*

$$Aut(\mathbb{C}) = \mathbb{C} \rtimes \mathbb{C}$$

*is $\infty$-dimensional, and it's very dirty.*

## 1.5 Module Theory

**Definition 1.5.1** *(Tensor product as module)*

*Let $M$ and $N$ be $R$-module. The tensor product as $R$-module is tensor product as a vector space with quotient as follows:*

$$M \otimes_R N = M \otimes N/(m, r \cdot n) \sim (m \cdot r, n)$$

*ex:*

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/gcd(m,n)\mathbb{Z}$$

**Definition 1.5.2** *(Flat module)*

*$R$-module $M$ is flat if any injection $K \hookrightarrow L$ maps to $K \otimes_R M \to L \otimes_R M$ is injective.*

*Especially if $- \otimes_R M$ is faithfully flat if and only if short exact sequence maps to short exact sequence.*

- *(non-ex I)*
  *$\mathbb{Z}/2\mathbb{Z}$ is not flat. For example an injection $\phi : \mathbb{Z} \to 2\mathbb{Z}$ doesn't make $\phi \otimes \mathbb{Z}/2\mathbb{Z}$ injective.*
- *(non-ex II)*
  *$R/I$ cannot be flat, since it's torsion, but $I$ could be flat.*
- *(ex I)*
  *If $M$ and $N$ are flat over $R$, then $M \otimes N$ is flat.*

- *(ex II)*
  *If $S \to S'$ is flat and $M$ is flat, then $M \otimes_S S'$ is flat.*
- *(ex III)*
  *If $M$ is $R$-module, its localization $M_{\mathfrak{p}}$ is flat $R_{\mathfrak{p}}$ module.*

*Another Characterization :*

*For all linear combinations $\Sigma_{i=1}^{m} r_i x_i = 0$ where $r_i \in R$ $x_i \in M$, we have change of basis*

*there exists an element $y_j \in M$ and $a_{i,j} \in R$ such that*

$\Sigma_{i=1}^{m} r_i a_{i,j} = 0$ *for* $0 \leqq j \leqq n$

$x_i = \Sigma_{j=1}^{n} a_{i,j} y_j = 0$ *for* $0 \leqq i \leqq m$

Why is this definition equivalent to $- \otimes_R M$ is an exact functor? Apparantly, $- \otimes_R M$ is right exact, so if the second morphism of $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is injective, it makes short exact sequence.

**Example 1.6** *(Flat which is not projective)*

*If $R = C^{\infty}(\mathbb{R})$, and $M = R_{\mathfrak{m}} = R/I$ where $\mathfrak{m} = \{f \in R | f(0) = 0\}$*
*and $I = \{f \in R | \exists \epsilon > 0 : f(x) = 0, \forall x, |x| < \epsilon\}$.*

*Note: $R$ is not integral domain.*

**Definition 1.6.1** *(Projective Module)*
*A module $P$ is projective iff every short exact sequence $0 \to A \to B \to P \to 0$ that splits.*

*ex:*

*$R$ is a Dedekind domain $R = \mathbb{Q}[\sqrt{-6}]$ and $P = (2, \sqrt{-6})$, then if $A = (3, \sqrt{-6})$, then $B \cong A \oplus P$.*

*Projecitve module is flat.*

**Definition 1.6.2** *(Flat Ring Extension)*

*A ring homomorphism is flat extension if it's a flat morphism.*

*ex:*

*$R \to R[x]$ is flat.*

## 2 Complex Analysis

The very question of complex analysis is to classify the possible class of functions depending on the domains.

### 2.1 Singularity

In complex analysis, the singular points defined on the region is the clue to understand the behavior of the functions. Conversely, if is it possible to reconstruct the function from the singular points is the monodromy problem. For the future convenience, in the lagunge of algebraic geometry, such singular points is called divisor.

**Definition 2.1.1** *(Divisor)*
*Divisor is the singlar regions of the function in the domain. Typically codimension* 1.

On extension of divisor argument, let's consider the intersection theory of the divisors in higher dimensions. Somehow, the elementary analogy of explanation is possible by using Euclidean topology.

**Definition 2.1.2** *(Intersection in Euclidean Topology) For an Euclidean space* $\mathbb{R}^n$*, the basis of topology is open balls* $B_\epsilon(x) = \{y|d(x,y) < \epsilon\}$ *for* $\epsilon > 0$ *and* $x \in \mathbb{R}^n$*, so any open subset is a union of some open balls. Consider the dimension of an open ball* $B_\epsilon(x)$ *is* $n$*, namely* $dim(B_\epsilon(x)) = n$*, and the dimension of any open subsets are* $n$*, and the dimension of an open subset is invariant by union and intersection. On the other hand, a close subset is not always dimension* $n$*. For example, hyperplane is, by defintion, codimension* 1*, which means dimension* $n-1$.

**Definition 2.1.3** *(Residue Theorem)*
*Skipped*

### 2.2 Mondromy

**Definition 2.2.1** *( Linear and Non-Linear Equations )*

- *(Generalized Hypergeometric Function)*
  ${}_pF_q(a,b,c;z) = \Sigma_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n} \frac{z^n}{n!}$

  *where* $a_1, \cdots, a_p, b_1, \cdots, b_q \in \mathbb{C}$ *but* $b_i \neq 0, -1, -2, \cdots,$ *and* $z \in \mathbb{C}$.

- *(Hypergeometric Function)*

  *If* $p = 0$ *and* $q = 1$*, generalized hypergeometric function becomes ordinary hypergeometric function. Hypergeometric function is a solution of diffyq*

$z(1-z)\frac{d^2f}{dz^2} + [c-(a+b+1)z]\frac{df}{dz} - abf = 0$

- *(Bessel Function)*

  *If $p = 0$ and $q = 1$, generalized hypergeometric function becomes Bessel function. That is,*

  ${}_0F_1(a, b, c; z) = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{(b)_n} \frac{z^n}{n!}$

- *( Heun's Equation )*

  $\frac{d^2w}{dz^2} + [\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}]\frac{dw}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}w = 0$

- *(Theta Function)*

  *Used for solitons and abelian varieties.*

  $\vartheta(z;\tau) = \Sigma_{n=-\infty}^{\infty} exp(\pi i n^2 \tau + 2i\pi n z)$

- *(Painleve)*

  *Painleve equation is a transcendents are solution of non-linear second ODE, but it has an integrabl structure.*

  $\frac{d^2y}{dt^2} = 6y^2 + t$

**Definition 2.2.2** *(Monodromy Representation)*

*Monodromy representation is a finite dimensional representation, which means $dim(V) < \infty$, and $\pi_1(X)$ is a fundamental group. Monodromy representation is defined by*

$\rho : \pi_1(X) \rightarrow GL(V)$

*We could define the complex functions whose domain is $X$, and by choosing arbitrary dimension of $V$, we have several different solutions.*

**Example 2.3** *(Monodromy Representation $X = \mathbb{C}\backslash\{0\}$ and $dim(V) = 1$)*
*So there is only one singularity, and $f(z)$ could be either $f(z) = z^{\alpha}$ where $\alpha \in \mathbb{C}\backslash\mathbb{Z}$ or $f(z) = log(z)$.*

**Example 2.4** *(Monodromy Representation $X = \mathbb{C}\backslash\{0\}$ and $dim(V) = 2$)*
$f(z) = Az^{\alpha} + Bz^{\beta}$

**Example 2.5** *(Monodromy Representation $X = \mathbb{C}\backslash\{0,1\}$ and $dim(V) = 1$)*
$\pi_1(\mathbb{C}\backslash\{0,1\}) = F_2$ *is a free group of 2-generators, and Jordan-Canonical form (or could be diagonalized)*

$$f(z) = z^{\alpha}(1-z)^{\beta}$$

**Example 2.6** *(Monodromy Representation $X = \mathbb{C}\backslash\{0,1\}$ and $dim(V) = 1$)*
$\begin{bmatrix} e^{\alpha} & 1 \\ 0 & e^{\beta} \end{bmatrix}$, *and*

*All the possible functions are the base solution $f(z) = z^{\alpha}(1-z)^{\beta}$ and the matrix multiplication.*

**Example 2.7** *(Monodromy Representation $X = \mathbb{C}\backslash\{0,1\}$ and $dim(V) = 1$)*

#### 2.7.1 KZ Equation

Quasi-Hopf algebra for KZ equation.

**Definition 2.7.1** *(Quasi-Hopf Algebra)*
*Quasi-Hopf Algebra is a generalization of Hopf algebra. It's antipode with Quasi-bialgebra as*

$$\Sigma_i S(b_i)\alpha c_i = \epsilon(a)\alpha$$
$$\Sigma_i b_i \beta S(c_i) = \epsilon(a)\beta$$

*for all $a \in A$ and where*

$$\Delta(a) = \Sigma_i b_i \otimes c_i$$

*and*

$$\Sigma_i X_i \beta S(Y_i)\alpha Z_i = I$$
$$\Sigma_j S(P_j)\alpha Q_j \beta R_j = I$$

*where $\Phi = \Sigma_i X_i \otimes Y_i \otimes Z_i$ and $\Phi^{-1} = \Sigma_j P_j \otimes Q_j \otimes R_j$.*

**Definition 2.7.2** *(Knizhnik-Zamolodochikov Equation)*

*$\hat{frakg}_k$ be affine Lie algebra with level $k$ and dual Coxeter number $h$. $i, j = 1, 2, \cdots, N$, $t^a$ be basis of $\mathfrak{g}$*

$$((k+h)\partial_{z_i} + \Sigma_{j\neq i}\frac{\Sigma_{a,b}\eta_{ab}t_i^a \otimes t_j^b}{z_i - z_j}) < \Phi(v_N, z_N)\cdots\Phi(v_1, z_1) >= 0$$

We'd consider alternative approach of KZ equation.

**Definition 2.7.3** *(Monodromy)*
*An example of monodromy is a multi-valued function $f(z) = log(z)$, since this function doesn't have the same value if we rorate around the singular point $z = 0 \in \mathbb{C}$. In fact, consider $1 = e^{2\pi i}$, $log(z) = log(z * e^{2\pi i}) = log(z) + log(e^{2\pi i}) = log(z) + 2\pi i$, thus the function can take multiple value even though it takes the same inputs. This is the idea of monodromy.*

*These seemingly different functions $log(z)$ and $log(z) + 2\pi i$ can be joined together by analytic continuation. Consider the domain of $log(z)$ is entire of the plane except the singular point, means that $0 \leqq \theta < 2\pi$, while the other $log(z)+2\pi i$ for $2\pi \leqq \theta < 4\pi$. If we also consider intermediate function $log(z)+\pi i$, whose domain is $\pi \leqq \theta < 3\pi$, we have non-empty intersection with both of them, hence by identity theorem, $log(z)$ and $log(z)+2\pi i$ are identified to be the same function.*

## 2.8 Configuration Space

**Definition 2.8.1** *(Configuration Space)*
*The configuraiton space is the domain of the solution of KZ equation. The configuration space is defined as*

$$Conf_n(\mathbb{C}) = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$$

*$Conf_n(\mathbb{C})$ creates monodromy around the singular points. The monodromy is calculated by the monodromy representation of the fundamental group $\pi_1(Conf_n(\mathbb{C})) = B_n$, which is a braid group.*

Configuration space is a complex manifold.

**Definition 2.8.2** *(Configuration Space)*
*The surjective map of configuration spaces*

$$Conf_{n+N}(X) \to Conf_N(X)$$

*is a locally trivial fiber bundle with typical fiber $X \backslash \{x_1, \cdots, x_N\}$.*

**Definition 2.8.3** *(Horizontal Chord Diagrams)*
*The monoid of horizontal chord diagram is the free monoid*

$$D_n^{pb} = FreeMonoid((ij) | 1 \leqq i < j \leqq n)$$

*and each element $(ij)$ is a permutation of $i$ & $j$.*

*$2T$-relation*
*$(ij)$ and $(kl)$ commutes if i,j,k,l are all distinct, namely, $(ij)(kl) = (kl)(ij)$. So $2T$-relation is*

$2T = \{(ij)(kl) - (kl)(ij) | i, j, k, l$ *all distinct*$\}$

$4T$*-relation*

*Consider* $(ik)(ij) = (ij)(jk)$ *are the same permutation but by using different elements, and also* $(jk)(ij) = (ij)(ik)$ *is. We join them together and* $(ik)(ij) + (jk)(ij) = (ij)(jk) + (ij)(ik)$*.*

$4T = \{(ik)(ij) + (jk)(ij) - (ij)(jk) - (ij)(ik) | i, j, k$ *all distinct*$\}$

$A_n^{pb} = Span(D_n^{pb})/4T$

**Definition 2.8.4** *(KZ Connection)*
*(KZ Differential Form)*
$\omega_{KZ} \in \Omega(Conf(\mathbb{C}), A_n^{pb})$ *such that*

$$\omega_{KZ} = \Sigma d_{dR} log(z_i - z_j) \otimes t_{ij}$$

*and KZ connection is flat since*

$$d\omega_{KZ} + \omega_{kz} \wedge \omega_{KZ} = 0$$

*(KZ Connection)*
$\nabla_{KZ}\phi = d\phi + \omega_{kz} \wedge \phi$

*and KZ equation is to find* $\phi$ *such that*

*(KZ Equation)*
$\nabla_{KZ}\phi = 0$

KZ connection is flat, which means the result of integration doesn't change by the direction of path, so analytic continuation is well-defined.

**Definition 2.8.5** *(Degeneration)*
*Formally speaking, a degneration is a map*

$$\pi : \mathfrak{X} \to S$$

*where* $\mathfrak{X}$ *is the total space, and* $S$ *is the base i.e.* $\mathbb{A}^1$*, and for each* $s \in S$*,* $\mathfrak{X}_s = \pi^{-1}(s)$*, together a degeneration at* $s_0 \in S$ *if the fiber* $\mathfrak{X}_{s_0}$ *is of idiosyncratic shape. For instance,*

$$\mathfrak{X} = \{(x, y, t) \in \mathbb{A}^3 | y^2 = x^3 + tx\}$$

*where elliptic curve if* $t \neq 0$*, and singular if* $t = 0$*.*
*The reason why we need to study degeneration is that if we compactify moduli spaces, we might add some singular fibers, by which we mean degeneration.*

**Definition 2.8.6** *(Degeneration of Moduli Space)*
*Degeneration is part of a moduli problem, and we'll particularly cosider a moduli space arising from monodromy representation i.e. the monodromy representation defines complex functions, and we'll define moduli psace of the domain of the complex functions. Here, by context, degeneration of moduli space is a moduli space of a domain where Painleve trancendence is defined.*

**Example 2.9** *(Degeneration)*

- ${}_pF_q(a, b, c; z)$ *is a degeneration of* ${}_{p+n}F_{q+m}(a, b, c; z)$
- *Painleve trancendence is a degeneration of hypergeometric funcitons.*

## 2.10 Gauss-Manin Connection

**Definition 2.10.1** *(Eheresmann Fibration Theorem)*

*If a smooth mapping $X \to B$ where $X$ and $B$ are smooth manifolds is a submersion and a proper map, then it's a locally trivial fibration.*

*Note: locally trivial fibration means locally isomorphic to cartesian, which means locally each fiber is isomorphic (or diffeomorphic) to the same fibers.*

The idea of Gauss-Manin connection is to track the family of complex algebraic varieties by the change of their parameters. By considering its cohomology class, .

**Definition 2.10.2** *(Gauss-Manin Connection)*

*Let a smooth morphism of $X \to B$. The fiber $X_b$ is a smooth manifold and diffeomorphic to each other for all $b \in B$, and it's cohomology class $H^k(X_b)$ are all isomorphic, and consider that the cohomology classes are all vector spaces of the same dimension, which makes a vector bundle (note each $H^k(X_b)$ is isomorphic to each other as vector spaces, but it has different basis), and it naturally defines Gauss-Manin connection, which defines a parallel transport of this vector bundle.*

*The explicit construction of Gauss-Manin connection is as follows. For $[\omega] \in H^k(\Omega^{\cdot}_{X/S})$ where $d_{X/S}\omega = 0$, and $d_S$ is an exterior differential of $S$, we define conenction $\nabla$ is a morphism*

$\nabla : H^k \to H^k \otimes \Omega^1_S$ *(or* $\nabla : \Gamma(H^k) \to \Gamma(H^k) \otimes \Gamma(Omega^1_S)$ *),*

*which satisfies*

$\nabla[\omega] := [d_S\omega]$

**Definition 2.10.3** *(Parallel Transport)*

*Parallel transport is to move a vector on a geometry (the vector exists on the fiber of the point) along a curve, so that $\nabla_{\dot{\gamma}} v = 0$, and parallel transport is uniquely defined on a vector bundle using conneciton. Namely, if the cohomology group is finite dimension, the vector exists at least finitely many amount, but the connection which defines parallel transport is defined by the same connection.*

*For the path $\gamma : [0, 1] \to S$, parallel transport means to find $v(t)$ such that $\nabla_{ga\dot{m}ma(t)} v(t) = 0$. In other words, if we parallely transport $v(0) \in H^k(X_b)$, it'll be $v(t)$ for each $t$.*

**Proposition 2.11** *(Gauss-Manin Connection is Flat)*

*In order to prove flatness, the curvature needs to be zero, means $R_\nabla = \nabla^2 = \nabla \circ \nabla : H^k \to H^k \otimes \Omega^2_S$.*

*$\nabla \circ \nabla([\omega]) = [d_s(d_s\omega)] = 0$ for all $\omega$, since $d_S \circ d_S = 0$. Hence $\nabla^2 = 0$*

**Proposition 2.12** *(Gauss-Manin and Monodromy)*

*For a parallel transport along a curve $S$.*
*Monodromy is a holonomy defined on a flat connection, whose closed curve is defined as a loop around a singular point.*

Also, consider Riemann-Hilbert correspondence claims correspondence of flat connections and monodromy representations.

**Definition 2.12.1** *(Riemann-Hilbert Correspodence)*
*RH correspondence is a 1-to-1 correspondence of flat connections and the monodromy representations. In other words, all the possible complex functions can be reconstructed from monodromy representations. Note that flat connection $\nabla$ can define a system of diffyq $\nabla\phi = 0$, thus existence of $\phi$, that can be derive from the representations.*

**Remark 2.13** *(Determining the Representation)*

*The vector space $V$ of the monodromy representation of $\rho : \pi_1(S) \to GL(V)$ is the dimension of the vector bundle. In other words, automorphism of fiber $GL(V)$ corresponds to the fundamental group $\pi_1(S)$.*

# 3 Moduli Space

Moduli space is a space generated by geometric quotient of some objects, and each point corresponds to an isomorphism class of objects. For example, for a Mobius strip $M$ and a circle $S^1$, a natural projection $M \to S^1$ is a vector bundle with fiber $\mathbb{R}^1$, and the moduli space is $M/\mathbb{R}^1$.

**Remark 3.1** *Moduli space is more than a set of points.* $M/\mathbb{R}^1 \cong S^1$ *is an isomorphism, and* $M/\mathbb{R}^1$ *is a moduli space, but* $S^1$ *is not a moduli space, and these two are distinct only because each point corresponds to an isomorphism class.*

**Remark 3.2** *Moduli space is not a fiber bundle, but what is the difference from the fiber bundle exactly? The moduli space has each point as an isomorphism class, not as an actual fiber. There are few different varieties of vector bundles over circle* $S^1$*, which is annulus or mobius strip, whose fibers are both* $\mathbb{R}^1$*, but moduli space cannot distinguish them.*

**Remark 3.3** *Etymologically, Moduli stands for "mod"uli.*

## 3.4 $M_{g,n}$ Moduli Space

**Definition 3.4.1** *(Configuration Space)*

*Let* $X$ *be a topological space, then the configuration space* $Conf_n(X)$ *is defined as*

$$Conf_n(X) = \{(z_1, z_2, \cdots, z_n) \in X^n | z_i \neq z_j \text{ for } i \neq j\}$$

**Definition 3.4.2** *($M_{g,n}$ Moduli Space)*

*The* $M_{g,n}$ *moduli space is GIT quotient of configuration space of* $\mathbb{P}^1$ *by* $SL^2(\mathbb{C})$*, which is a geometry of genus* $g$ *with* $n$*-marked points of the base space* $\mathbb{P}^1$*. In particular,*

$$M_{0,n} = Conf_n(\mathbb{P}^1)/SL^2(\mathbb{C})$$

**Remark 3.5** *($M_{0,n}$ Calculation)*

- *($n = 3$)*
  *Consider that Möbius transformation* $z \mapsto \frac{az+b}{cz+d}$ *can put arbitary distinct* 3*-points* $z_1$*,* $z_2$*, and* $z_3$ *mapped to arbitray distinct* 3*-points* $w_1$*,* $w_2$*, and* $w_3$*.*

  *Since configuration space of* 3*-dimensions is a space of* 3 *independent distinct elements, each points could be arbitrarily mapped to any points of the space, which means by* $SL^2(\mathbb{C})$ *quotient, it'll be all identified. Hence,*

  $$M_{0,3} = \{pt\}$$

- *($n = 4$)*

*Recall $SL^2\mathbb{C}$ fixed the 3-points, then the last 4th-point has some freedom, because kf we let $\lambda = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$, Cross ratio $\lambda$ could take any values of $\mathbb{P}\backslash\{0,1,\infty\}$. $\lambda \neq 0$ or $\lambda \neq \infty$ because all points $z_i$ are distinct, and none of the numerator or denominator could be zero, and if $\lambda \neq 1$, $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = 1$, then $(z_1 - z_3)(z_2 - z_4) = (z_1 - z_4)(z_2 - z_3)$, and by simple college algebra, it will be $(z_1-z_2)(z_3-z_4) = 0$, but it never happens since all points are distinct. Otherwise, $\lambda$ could take any values. Hence,*

$M_{0,4} = \mathbb{P}\backslash\{0,1,\infty\}$

- *($n = 5$)*

  *If we fix the mapped points $w_1$, $w_2$, and $w_3$ depending on $z_1$, $z_2$, and $z_3$, $w_4$ is cross ratio depending on $z_4$, while $w_5$ is another cross ration by $z_5$, which means*

  - $\lambda = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$
  - $\mu = \frac{(z_1-z_3)(z_2-z_5)}{(z_1-z_5)(z_2-z_3)}$

  *so we have two cross ratios $\lambda$ and $\mu$, so $M_{0,5}$ is two parameters. Hence,*

  $M_{0,5} = \mathbb{P}\backslash\{0,1,\infty\}$

*Finally, notice indeed $dim(M_{0,n}) = n - 3$*

**Definition 3.5.1** *($M_{1,1}$ Moduli Space)*
*The very definition of $M_{1,1}$ is a moduli space of algebraic curves of genus 1 of 1-marked points, but the question is that are all the algebraic curves of genus 1 of 1-marked points elliptic curves and vice versa? Yes, and this is, in fact, equivalently to that it's a moduli space of elliptic curves quotient by $j$-invariants. Another question is how to explicitly classify all the elliptic curves, and this could be done by moduli space of torus, and only to prove that torus is equivalent to some elliptic curve.*

*$M_{1,1} = \mathbb{H}/SL_2(\mathbb{Z}) \cong \Lambda$ where $\mathbb{H}$ is the upper-half plane.*

*where $dim(M_{1,1}) = 1$*

*Or equivalently, $M_{1,1} = \{(E,p)|$ genus 1 smooth curve, $p \in E\ \}/\sim$*

**Remark 3.6** *($M_{1,1}$ Lattice and Curve Correspondence)*

*In order to verify the previous definition. we'll start from the lattice correspondence with the elliptic curve.*

- *(Step 1:)*
  *Each lattice can be described by the standard form, which is*

  $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$

  *for some* $\tau \in \mathbb{C}$

- *(Step 2:)*
  *If* $E_\tau$ *is a lattice, this* $\mathbb{C}/E_\tau$ *can be a domain of the Weierstrass* $\wp$ *function, which is in other words period, and Weierstrass* $\wp$ *function defines an elliptic function. To formulate, if we let* $Y = \wp'(z, \mathbb{C}/E_\tau)$ *adn* $X = \wp(z, \mathbb{C}/E_\tau)$, *we have the following identities:*

  $Y^2 = 4X^3 - g_2 X - g_3$

  *where* $g_2$ *and* $g_3$ *are Eisenstein series.*

- *(Step 3: Legendre Transform)*

  *What else do we have? This correspondence is actually not one-to-one, but it will be, if we take quotient of the set of elliptic curves by isomorphism classes by* $j$*-invariant, that is equivalent to say that if elliptic curves have the same* $j$*-invariant, then they have isomorphism as algebraic varieties, and Legendre transform is the step to defien* $j$*-invariant. The previous elliptic function can be canonically transformed to the Legendre form*

  $y^2 = x(x-1)(x-\lambda)$ *where* $\lambda \in \mathbb{C}\backslash\{0, 1\}$

- *(Step 4:j-Invariant)*

  *By using* $\lambda$ *arised from Legendre transform, if we let* $j$*-invariant as*

  $j(\lambda) = 256 \cdot \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}$

  $j(\tau) = j(\lambda(\tau))$

*Note if two Weierstrass functions are distinct up to* $SL_2(\mathbb{Z})$ *transformation,* $g_2$ *and* $g_3$ *are always distinct.*

**Definition 3.6.1** *($M_{1,0}$ Moduli Space)*

*Also,* $M_{1,0}$ *is* $n = 0$, *and it's also moduli space of elliptic curves* $y^2 = x^3 + ax + b$,

*For* $n = 0$, $M_{1,0}$ *is* $M_{1,0} = M_{1,1}/Aut(C)$

**Remark 3.7** *(Group Structure)*
*Why the moduli space of elliptic curves are $M_{1,1}$ instead of $M_{1,0}$ is because each elliptic curve has a group structure, or in other words, if the point of the elliptic curve is depending on $X$ and $Y$ coordinates, whose value depends on the position of the lattice $E_\tau$. Each element of the group is a point of the elliptic curve, and of course the unit element $O$ is. The unit element $O$ is point at infinity.*

**Remark 3.8** *(Elliptic Curve Is Not Torus)*

**Definition 3.8.1** *($M_{2,0}$ Moduli Space)*

$$M_{2,0} = \mathbb{H}_2 / Sp_4(\mathbb{Z})$$

*where* $dim(M_{2,0}) = 3$

**Remark 3.9** *(Tree Structure)*

*$M_{0,n}$ can be combinatorially constructed by the tree structure of multiple $M_{0,4}$.*

### 3.10 Compactification of Moduli Space

**Definition 3.10.1** *()*

*The compactification $\overline{M_{0,4}}$ is $\overline{M_{0,4}} = \mathbb{P}^1$.*
$\overline{M_{0,4}}^{GIT} \cong (\mathbb{P}^1)^4 // PGL_2$

## 4 Unsorted

**Definition 4.0.1** *(Deligne-Mumford Space)*

**Definition 4.0.2** *(Compactification of Moduli Space)*

**Definition 4.0.3** *( )*

## References

[1] Akaho Manabu(2003), *A Crash Course of Floer Homology for Lagrangian Intersections*, https://pseudoholomorphic.fpark.tmu.ac.jp/akaho_a_crash_course_of_floer_homology_for_lagrangian_