Intro to Math Physics

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Contents

1 Preface

This pdf is for mathematician, and I wrote this pdf not for research but for exposition practice. I'll regret my terrible writing in advance. The purpose of this pdf titled "introduction to math physics" is to introduce the basic framework of modern physics from mathematical perspective, and this pdf might inspire further study of math algorithm, but I'll stop by introducing what each subject is. There is no unified method of describing physics mathematically, but the famous idea is standard model and string theory, and this modern frameworks of physics enabled to identify several symmetric properties or invariants in physics, and in QFT, in particular, we have CFT, TQFT, AQFT etc, while mirror symmetry in string theory, or more mathematically speaking, symmetry is a Lie group theory, for instance, and Lie group is a symmetry that can arise from infinitesimal modification of particle traveling path, or by using physics term, Lie group describes symmetric propery of Lagrangian invariant infinitesimal modification of path. On the other hand, string theory is mathematically mirror symmetry and SYZ conjecture. String theory and gauge theory are independent, but some people claims the correspondence between them.

2 Prelim

2.1 Differential Calculus

Covariant derivative is a way of specifying derivative in tangent bundle.

Definition 2.1.1 *(Covariant Derivative)*

Covariant derivative of f at p is a scalar at p, denoted $(\nabla_v f)_p$

$\phi : [-1,1] \to M$ *s.t.* $\phi(0) = p$ *and* $\phi'(0) = v$

$(\nabla_v f)_p = (f \circ \phi)'(0) = \lim_{t\to 0} \frac{f(\phi(t)) - f(p)}{t}$

Example 2.2 *(Covariant Derivative)*

In flat space, covariant derivative is same as

$\nabla_{\frac{\partial}{\partial \lambda}} v = \frac{\partial v}{\partial \lambda}$

but in general, covariatn derivative is subtracting normal vector

$\nabla_{\frac{\partial}{\partial \lambda}} v = \frac{\partial v}{\partial \lambda} - n$

where n is the normal vector. Or

$\nabla_{\frac{\partial}{\partial u^i}} v = \frac{\partial}{\partial u^i}(v^j e_j) = \frac{\partial v^j}{\partial u^i} e_j + v^j \frac{\partial e_j}{\partial u^i}$

where $\frac{\partial e_j}{\partial u^i} v = \Gamma^k_{ij} e_k$

If we use $g_{ij} = e_i \cdot e_j = e_j \cdot e_i = g_{ji}$, The Christoffel symbol Γ^i_{jk} is written concisely (ex: describe Γ^i_{jk} using only g_{ij}).

Definition 2.2.1 *(Parallel Transport)*

v is parallel transport if $\nabla_w v = 0$.

v is called geodesic if $\nabla_v v = 0$.

What is Levi-Civita connection? First of all, it is covariant derivative, and determining Christoffel symbol defines what covariant derivative is. What is Covariant derivative? It might be anything that satisfies product rule.

Definition 2.2.2 *(Levi-Civita Connection)*

On Riemannian manifold, there is a unqiue Levi-Civita connection. Levi-Civita connection is torsion free, and it has a metric compatibility.

Torsion freeness is that

$\nabla_{e_i} e_j = \nabla_{e_j} e_i$
$\Gamma^k_{ij} = \Gamma^k_{ji}$

Metric compatibility is that

$\nabla_w(v \cdot u) = (\nabla_w v) \cdot u + v \cdot (\nabla_w u)$
$\partial_k g_{ij} = \Gamma^l_{ik} g_{jl} + \Gamma^l_{jk} g_{il}$

$\Gamma^m_{jk} = \frac{1}{2} g^{im}(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk})$

Koszul connection is a connection which defines a directional derivative for sections of a vector bundle.

Definition 2.2.3 *(Exterior Covariant Derivative)*

$d^\nabla : \Omega^k(M, E) \to \Omega^k(M, E)$

Definition 2.2.4 *(Curvature Form)*

Curvature form is $\mathfrak{g}$*-valued* 2*-form*

$\Omega = d\omega + [\omega \wedge \omega] = D\omega$

where D *is exterior covariant derivative.*

Definition 2.2.5 *(Bianchi Identity)*

Bianchi identity is an identity arising from the exterior derivative (curvature form).

- *(First Identity)*
  $D\Theta = \Omega \wedge \theta$

- *(Second Identity)*
  $D\Omega = 0$

Definition 2.2.6 *(Exterior Covariant Derivative)*

For a principal G*-bundle* $P \to M$*, the tangent space* $T_uP = H_u \oplus V_u$ *has a direct sum decomposition, and* $h : T_uP \to H_u$ *be the projection.*

If ϕ *is a* k*-form on* P*, then its covariant exterior derivative* $D\phi$ *is a form defined by*

$D\phi(v_0, v_1, \cdots, v_k) = d\phi(hv_0, \cdots, hv_k)$ *where* $v_i \in T_uP$

2.3 Hamiltonian and Lagrangian

Definition 2.3.1 *(Lagrangian)*

Lagrangian L is

$L = T - V$

where T is kinematic energy and V is potential energy.

Definition 2.3.2 *(Euler-Lagrangian Equation)*

The action functional $S : P(a, b, x_a, x_b) \to \mathbb{R}$ is defined via

$S[q] = \int_a^b L(t, q(t), \dot{q}(t))dt$

where $q(t)$ is a path. This path is a stationary point if it satisfies EL equation, or EL equation is a path that minimizes action. Formally,

$\frac{\partial L}{\partial q} - \frac{d}{dx}\frac{\partial L}{\partial q'} = 0$

See wikipedia for derivation.

Definition 2.3.3 *(Legendre Transformation)*

Hamiltonian is derived from Lagrangian by Legendre transformation. Legendre transformation is an invertible maps defined by

$f(x) \to g(p)$

Legendre transformation transforms Lagrangian to Hamiltonian. In theory of manifold, let M be a smooth n-dimensioal real manifold with Lagrangian, then the Hamiltonian is generated and onto $2n$-dimensional symplectic manifold, whose coordinate maps

$(q, \dot{q}) \mapsto (p, q)$

Definition 2.3.4 *(Phase Space)*

For n-dimensional mechanics, phase space is a space parametrized by position and momentum. Namely, phase space is a space with a coordinate $(q_1, \cdots, q_n, p_1, \cdots, p_n)$

Definition 2.3.5 *(Hamiltonian)*

So far we have already introduced Lagrangian, that can describe the stanard physics model, and the obvious reason for using Hamiltonian is because Hamiltonian is mathematically convenient. Hamiltonian is mathematically derived from

Lagrangian, using Legendre transformation.

Hamiltonian H is

$H = T + V$

where T is kinematic energy and V is potential energy. Hamiltonian is derived from Legendre transformation of Lagrangian.

$H(p, q, t) = \Sigma_{i=1}^{n} p_i \dot{q}^i - L(q, \dot{q}, t)$

where $p = \frac{\partial L}{\partial \dot{q}}$

Then, to summarize the following Hamilton's equation is equivalent to EL equation

- $\frac{dq}{dt} = \frac{\partial H}{\partial p}$
- $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$

Example 2.4 *(Hamiltonian)*
Hamiltonian is sum of positional energy and momentum $H = L + K$. For example, Hamiltonian or a particle is

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} k \hat{x}^2$

Definition 2.4.1 *(Hamiltonian Operator)*

If Hamiltonian operator $\hat{H}$ act on wave function $|\psi\rangle$, the particle becomes observable.

Definition 2.4.2 *(Hamiltonian Vector Field)*

For a symplectic manifold (M, ω), using a smooth function $H : M \to \mathbb{R}$, Hamiltonian vector field is defined by

$dH(-) = \omega(X_H, -)$

Definition 2.4.3 *(Poisson Bracket)*

We define Poisson bracket from a symlectic manifold (M, ω). If we don't $\omega = \Sigma_{i=1}^{n} dp_i \wedge dq^i$ For a smooth function f and g, Poisson bracket is

$\{f, g\} = \omega(X_f, X_g)$

where X_f is a vector field defined from f. Or more explicitly,

$\{f, g\} = \Sigma_{i=1}^{n}(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i})$

Example 2.5 *(Canonical Poisson Bracket)*
In symplectic geometry, canonical coordinate is $2n$-dimensional basis $(p_1, \cdots, p_n, q_1, \cdots, q_n)$. The vector field is defined by

- $X_{p_i} = \frac{\partial}{\partial q_i}$

- $X_{q_i} = -\frac{\partial}{\partial p_i}$

the bracket will be

- $\{p_k, p_l\} = 0$
- $\{q_k, q_l\} = 0$
- $\{q_k, p_l\} = \delta_{kl}$
- $\frac{dq}{dt} = \frac{\partial H}{\partial p} = \{q, H\}$,
- $\frac{dp}{dt} = -\frac{\partial H}{\partial q} = \{p, H\}$
- $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$

2.6 Theory of Manifold

Definition 2.6.1 *(Symplectic Manifold)*

obvious.

Example 2.7 *(Cotangent Bundle)*

Cotangent bundle T^*M has a canonical symplectic structure.

Definition 2.7.1 *(Moment Map)*

Definition 2.7.2 *(Principal G-Bundle)*

Definition 2.7.3 *(Holonomy Group)*

Lie group valued parameter, and it calculates the gap arising from parallel transport.

Definition 2.7.4 *(Kähler Manifold)*

Kähler manifold is both symplectic, complex, and Riemannian manifold.

Definition 2.7.5 *(Calabi-Yau Manifold)*

Compact Kähler manifold, vanishing first Chern class, and Ricci flat metric.

Definition 2.7.6 *(Lagrangian Foliation)*

2.8 Quantum State

Definition 2.8.1 *(Bra and Ket)*

- *(ket)*
  Ket $|\psi\rangle$ *is a state of some quantum system. Mathematically,* $|psi\rangle \in V$ *where* V *is a Hilbert space. In particular,* $|0\rangle$ *is a quantum vacuum state.*
- *(bra)*
  Bra $\langle f|$ *is* $\langle f| \in V^*$*, an element of dual space.*

Since inner product is defiend on Hilbert space, define the product of bra and ket as $\langle f|\psi\rangle \in \mathbb{C}$*.*

Definition 2.8.2 *(Quantum State)*

Let H *be the Hilbert space as quantum state space. Quantum state* $v \in H$ *is an element of the Hilbert space. This means that quantum state is given by a wave function or given as an element of Hilbert space.*

In particular, quantum vacuum state $|0\rangle \in H$ *is norm* 1*, so* $||\,|0\rangle\,|| = 1$*. The dual* $\langle 0|$ *is also norm* 1*, and* $||\,\langle 0|\,|| = 1$*, but dual to* $|0\rangle$*, so the paring as inner product is* $\langle 0|0\rangle = 1$*.*

Definition 2.8.3 *(Fock Space)*

Fock space $F = \oplus_{n\in\mathbb{N}} S_\nu H^{\otimes n}$ *is the generalization of the quantum state space. Of course the Hilbert space* H *describes quantum state in* 1*-particle, and the tensor product* $H \otimes H$ *describes* 2*-particle system, so in the higher degree.* S_ν *takes values* $S_\nu = \pm 1$ *and if* $S_\nu = 1$*, it's Boson, and* $S_\nu = -1$*, it's Fermion.*

3 Quantum Mechanics

3.1 Quantum Model

Quantum mechanics is to study physical behavior from miniscule perspective. In this chapter, we just pick up basic keywords of what modern physical idea is in quantum/classic level. We will introduce field theory in the next chapter for the mathematical formalization.

Definition 3.1.1 *(Standard Model)*

In particle physics, the physical system can be described by 17 particles and their interactions. They are largely classified by their spins: Ferminon and Boson. Fermion is a particle of spin half-integer $\frac{1}{2}, \frac{3}{2}, \cdots$*, and they're quarks and leptons. Boson is a particle of spin integer, and they're gauge particles whose spin is* 1*, and gauge particles describe interactions of particles of four forces. Higgs particle is spin* 0*, graviton is spin* 2*. In quantum mechanics, all physical objects are considered to be waves but also particles (it's not only photon), and objects, forces, waves are all particles!*

Is there any way to study the physical phenomena from math perspective? We consider field theory to define the physical space, and consider how a particle behaves on it, and thus we construct (or conjecture) standard model, the possible particles involving to the physics phenomena.

The most elementary symmetry is Maxwell equation: symmetry of electric force and magnetic force, but these could be described by one unified force called electromagnetic force, and they behave electric/magnetic by their perspective, and this symmetry has $U(1)$ *symmetry. And the weak force has* $SU(2)$ *and The strong force has* $SU(3)$*.*

Weinberg-Salam theory unifies weak $SU(2)$ *and electromagnetic* $U(1)$ *forces. The grand unified theory of physics is still conjectured, unifying strong* $SU(3)$*, weak* $SU(2)$*, electromagnetic* $U(1)$ *forces. Moreover, the absolute unified theory of the four forces in physics has not been achieved.*

Definition 3.1.2 *(Four Forces)*

Physics has four forces.

- *(Strong Forces)*
  QCD (chromodynamics), $SU(3)$*-symmetry.* 8*-dimensional gluon.*
  Connecting proton and neutron in atomic nucleus.
- *(Weak Forces)*
  QFD (flavordynamics) $SU(2)$*-symmetry.* 3*-dimensional gluon.*
  Forces for nuclear reaction. β*-collapse transforms proton to neutron.*
- *(Electroic Magnetic Forces)*
  QED (electrodanmics) $U(1)$*-symmetry.* 1*-dimensional photon.*
- *(Gravity Forces)*
  ? Existence of graviton is still conjectured.

Definition 3.1.3 *(Wave Function)*

In quantum mechanics, each particle has a wave function, multi-particle system can also be described by wave functions. The synthesis of the wave functions is multiplication. We cannot find the precise value of n-body particle system, but we have Hartree approximation, and synthesis of the wave function will be

$$\phi(x_1, x_2) = \phi(x_1)\phi(x_2)$$

If the particle is Fermion,

$$\phi(x_1, x_2) = \phi(x_1)\chi(x_2) - \phi(x_2)\chi(x_1)$$

or we can use determinant for easily generalizing the higher dimension.

$$\phi(x_1, x_2) = \begin{bmatrix} \phi(x_1) & \phi(x_2) \\ \chi(x_1) & \chi(x_2) \end{bmatrix}$$

Definition 3.1.4 *(Pauli Exclusion Principle)*

Pauli exclusion principle says that no two electrons on the atom have the same spin.

Definition 3.1.5 *(Quantum Number)*

Four different kind of quantum numbers can specify the quantum state of the particle.

- *Principal Quantum Number (n)*
  K,L,M orbit
  $n \in \mathbb{N}$
- *Azimuthal Quantum Number (l)*
  s, p, d, f orbits
  $0 \leqq l \leqq n - 1$
- *Magnetic Quantum Number (m_l)*
  $|m_l| \leqq l$
- *Spin Quantum Number (m_s)*
  spin is $\pm\frac{1}{2}$, so two possibilities.

These quantum numbers

3.2 Relativistic Quantum Mechanics

Definition 3.2.1 *(Klein-Gordon Equation)*

If a Klein-Gordon field $\phi(x,t)$ describes mass m free particle, then Klein-Gordon equation is

$$[\Box + \mu^2]\phi(x,t) = 0$$

where c is speed of light, $\hbar$ is reduced Plank constant, and $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ and $\mu = \frac{mc}{\hbar}$).

3.3 Radioactive Mechanics

Definition 3.3.1 *(Radioactive Decay)*

- *(α Decay)*
  α particle (helium nucleus) separated by atomic nucleus not by strong force but by tunnel effect. α wave is a radioactive wave of α particle.
- *(β Decay)*
  β decay is an effect of neutron changing to proton by weak force. α wave is a radioactive wave of an electron.
- *(γ Decay)*
  It doesn't change the mass and charge of an atom.

4 Field Theory

4.1 Classical Field Theory

The interest of math physics is to study symmetry, since according to Neother's theorem, Lagrangian invariant (conservation of energy) corresponds to mathematical symmetry, namely Lie group.

Definition 4.1.1 *(Lie Group)*

For a smooth manifold M, a Lie group G is a set of all infinitesimal automorphisms $G = \{\Gamma_\epsilon : M \to M\}$

Definition 4.1.2 *(Noether's Theorem)*

Noether's theorem claims the the group theoretical symmetry corresponds to
.

Imagine a particle traveling on the manifold M starting at time t_1 and ending t_2, and the particle is traveling along the path $x(t)$, $t_1 \leqq t \leqq t_2$, and for each point of manifold M, Lagrangian is defined. This path has an action $S = \int L(x(t))dt$. Consider another path infinitesimally modified $x'(t) = x(t) + \epsilon(t)$, so that the path is invariant action: $\int L(x'(t))dt = S + \delta S + O(\epsilon^2)$ where $\delta S = 0$.

The matehmatical statement of Neother's theorem is

$\frac{\partial \mathscr{L}}{\partial \dot{x_i}}\zeta_i + \mathscr{H}\tau = 0$

where $\mathscr{L} = \mathscr{L}(q_i, \dot{q}_i, t)$, *and* $\mathscr{H} = \dot{x}_i \frac{\partial \mathscr{L}}{\partial \dot{x_i}} - \mathscr{L}$
Also, Refer to Rund-Trautman identity. Rund-Trautman identity is a gap arising from the gauge transformation $\Gamma_\epsilon(\mathscr{L})$ *from* $\mathscr{L}$. *If Rund-Trautman identity is equal to* 0, *then it means the transformation is gauge invariant.*

Definition 4.1.3 *(Gauge Theory)*

Gauge theory is a branch of field theory, where we study global/local symmetry of the geometry, which is Lagrangian invariant by the change of coordinate by physics context. More generally, gauge theory is function equivariant wrt Lie group action.

In particular, the gauge theory is Yang-Mills theory, the Lie group is $SU(n)$ *or compact Lie groups. If the Lie group is compact,* $\int_X tr(F^2) dvol_g$ *becomes positive-definite.*

Definition 4.1.4 *(Ehresmann Connection (Gauge Field))*

Under constrction.

Proposition 4.2 *(Ehresmann Connection (Gauge Field))*

Ehresmann connection ∇ *defines (gauge) covariant derivative* $\nabla_\mu = \partial_\mu - igA_\mu$ *where* g *is a coupling constant.*
Field strength (curvature) is zero everywhere.

Example 4.3 *($U(1)$ Gauge Symmetry)*

$U(1)$ *gauge symmetry describes local coordinate symmetry of Lagrangian. Consider that* $U(1) = \{e^{i\theta} | \theta \in \mathbb{R}\}$, $U(1)$ *gauge transformation is*

$\psi(x) \mapsto e^{i\alpha(x)}\psi(x)$

where $\psi(x)$ *is wave function, and* $\alpha(x)$ *is local symmetry, and its value changes at every point in the coordinate.*

We'll consider gauge transformation (change of coordinate) to make Lagrangian invariant. Here we use covariant derivative $D_\mu = \partial_\mu - ieA_\mu$, *and if we use covariant derivative, we can costruct and gauge invariant transformation of gauge field* A_μ.

The $U(1)$ *gauge action makes* A_μ *and wave function* $D_m u\psi$ *as*

$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu \alpha(x)$

$D_\mu \psi \mapsto e^{i\alpha(x)} D_\mu \psi$

Then, the Lagrangian $L = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}$ *where* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ *is invariant under* $U(1)$ *transformation.*

Definition 4.3.1 *(Yang-Mill Functional)*

The Yang-Mills functional is

$YM(A) = \int_X ||F_A||^2 dvol_g$

where A *is a connection on a principal bundle, and* vol_g *is a volume form of* X*, typically* $dvol_g = dx^4$*.* F_A *is a strength tensor field (also called curvature) defined by* $F_A = \partial_\nu A_\mu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ *or alternatively,* $F_A = dA + A \wedge A$*. This is magnitude of field strengh. For example, in electrodynamics, its unit might be* (V/m)*.*

$L_{gf} = ||F_A||^2$ *is Lagrangian, so* $YM(A)$ *is an action.* $||F_A||^2$ *is often alternatively denoted by* $L_{gf} = ||F_A||^2 = -\frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2}Tr(F^2) = -frac12Tr(F \wedge \star F = -frac12Tr(F_{\mu\nu}F^{\mu\nu})$*, that's representation theory perspective.*

What is trace? Let F_A *be a second tensor, defined over the Lie algebra whose basis be* $\{T^a\}$*, and its trace is* $Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$*, so trace is functioning as alternative to inner product. Using basis of the Lie algebra* $\{T^a\}$*,* $F_{\mu\nu}$ *is given by* $F_{\mu\nu} = F^a_{\mu\nu}T^a$*. Thus,*

$$
\begin{aligned}
&Tr(F_{\mu\nu}F^{\mu\nu}) \\
&\qquad = Tr(F^a_{\mu\nu}T^aF^{\mu\nu}_bT^b) \\
&\qquad = Tr(T^aT^b)F^a_{\mu\nu}F^{\mu\nu}_b \\
&\qquad = \frac{1}{2}\delta^{ab}F^a_{\mu\nu}F^{\mu\nu}_b
\end{aligned}
\tag{1}
$$

Proposition 4.4 *(Yang-Mills Connection)*

Yang-Mills connection is a exterior covariant derivative that satisfies Yang-Mills equation $d_AF_A = 0$*. We will derive the Yang-Mills functional* $YM(A) = \int_X ||F_A||^2 dvol_g$ *makes the Yang-Mills equation* $d_A \star F_A = 0$*.*

$F_{A+ta} = F_A + td_Aa + t^2 a \wedge a$

$$\begin{aligned}\frac{d}{dt}(YM(A+ta))_{t=0} &= \frac{d}{dt}(\int_X < F_A + td_Aa + t^2 a\wedge a, F_A + td_Aa + t^2 a \wedge a > dvol_g)_{t=0} \\ &= \frac{d}{dt}(\int_X ||F_A||^2 + 2t < F_A, d_Aa > + 2t^2 < F_A, a\wedge a > + t^4 ||a\wedge a||^2 dvol_g)_{t=0} \\ &= 2\int_X < F_A, d_A a > dvol_g \\ &= 2\int_X < d_A^* F_A, a > dvol_g \end{aligned} \tag{2}$$

This connection A is a critical point of Yang-Mills functional if it vanishes for every a. Formally,

$$< d_A^* F_A, a >= 0$$

means that

$$d_A^* F_A = 0$$

Remark 4.5 *Irreducible unitary Yang-Mills connection corresponds to stable vector bundle. A vector bundle is stable vector bundle if it satisfies stability condition.*

Proposition 4.6 *(Self Dual/Anti-Self Dual Equations)*

If in particular X is 4-dimensional, the star operator becomes endomorphism

$$* : \Omega^2(X) \to \Omega^2(X)$$

Since the square of $*$-operator becomes $\pm Id$, the eigenvalue of $*$ is either 1 or -1, and we have decomposition

$$\Omega^2(X) = \Omega_+(X) \oplus \Omega_-(X)$$

Then the curvature $F_A = *F_A$ or $F_A = - * F_A$.

Proposition 4.7 *(What Yang-Mill Theory Derives Lie Structure)*

Let Lagrangian $L_{gf} = -\frac{1}{2}tr(F^2) = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}$
$tr(T^aT^b) = \frac{1}{2}\delta^{ab}$, $[T^a, T^b] = if^{abc}T^c$ where f^{abc} are structure constants.

We define covariant derivative $D_\mu = I\partial_\mu - igT^aA^a_\mu$

commutator $[D_\mu, D_\nu] = -igT^aF^a_{\mu\nu}$ can derive the relation

$F_{\mu\nu}^{a} = \partial_\mu A_\nu^a + \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$

By some calculation, a Bianchi identity $(D_\mu F_{\nu\kappa})^a + (D_\kappa F_{\mu\nu})^a + (D_\nu F_{\kappa\mu})^a = 0$ *holds, that's equivalent to Jacobi identity* $[D_\mu, [D_\nu, D_\kappa]] + [D_\kappa, [D_\mu, D_\nu]] + [D_\nu, [D_\kappa, D_\mu]] = 0$.

Definition 4.7.1 *(Standard Model)*

If gauge theory unifies all forces of particles (e.g. gravitational, electromagnetic, weak, strong), this will be called standard model.

Definition 4.7.2 *(Sigma Model)*

Specific Gauge theory. It depends on how we define symmetry, i.e. it could be typically $SU(N)$ *or in condensed matter physics, it could be* $O(N)$.

4.8 Quantum Field Theory

Similar to classical mechanics, quantum mechanics describes the state of a particle as given by position (x) mementum (p). The easiest qunatization is canonical quantization, but we also introduce several different generalizations.

Definition 4.8.1 *(Schödinger Equation)*
In quantum mechanics, let the quantum state state be $|\psi >$ *be an element in the Hilbert space, and observables are represented by a quantum state. Also, let* $\psi >= \Sigma_{n=0}^{\infty} a_n \psi_n >$

Time-independent Schödinger Equation is

$\hat{H}|\psi_n >= E_n|\psi_n >$
where E_n *is characteristic energy associated to* ψ_n *eigenstate. and*

where a_n *are constant coefficients.*

Definition 4.8.2 *(Canonical Quantization)*

For a one-particle system, canonical quantization is described by

$\{A, B\} \mapsto \frac{1}{i\hbar}[\hat{A}, \hat{B}]$

This Lie bracket encodes uncertainty principle ($\Delta x \Delta p \geqq \hbar/2$)*, and*

$[\hat{X}, \hat{P}] = \hat{X}\hat{P} - \hat{P}\hat{X} = i\hbar$

while Poisson bracket is defined in classical mechanics, and $\{x, p\} = 1$.

Definition 4.8.3 *(Weyl Quantization)*

Weyl transformation (Weyl quantization) is an operation in Hilbert space which is

$\Phi[f] = \frac{1}{(2\pi)^2} \int \int \int \int f(q,p)(e^{i(a(Q-q)+b(P-p))})dqdpdadb$

that is a linear transformation.

theorem:

If f is polynomial of $deg(f) \geqq 2$ and g is polynomial of any degree, then

$\Phi(\{f,g\}) = \frac{1}{i\hbar}[\Phi(f), \Phi(g)]$
that's similar to canonical quantization.

Definition 4.8.4 *(Moyal Bracket)*
We define Moyal bracket as

$\{\{f,g\}\} = \frac{1}{i\hbar}(f \star g - g \star f) = \{f,g\} + O(\hbar^2)$

where $\star$ is a star product in Moyal space.

Definition 4.8.5 *(Poisson Bivetor)*

$\Pi^0(f_1, f_2) = f_1 f_2$

$\Pi^1(f_1, f_2) = \{f_1, f_2\}$

$\Pi^n(f_1, f_2) = \Sigma_{k=0}^n \binom{n}{m}(\frac{\partial^k}{\partial p^k}\frac{\partial^{n-k}}{\partial q^{n-k}} f_1) \times (\frac{\partial^k}{\partial q^k}\frac{\partial^{n-k}}{\partial p^{n-k}} f_2)$

ex:

Gaussians compose hyperbolically e.g.

$\delta(p) \star \delta(q) = \frac{2}{\hbar} e^{2i\frac{qp}{\hbar}}$

Definition 4.8.6 *(Deformation Quantization)*

Given a Poisson algebra $(A, \{\cdot,\cdot\})$, its deformation quantization is an associative product $\star$ on the algebra of the formal power series $A[[\hbar]]$ with axioms

- $f \star g = fg + O(\hbar)$
- $[f,g] = f \star g - g \star f = i\hbar\{f,g\} + O(\hbar^2)$

That's called Kontsevich qunatization formula. On a Poisson manifold, we could add

$f \star g = fg + \Sigma_{k=1}^{\infty} \hbar^k B_k(f \otimes g)$

where B_k are linear bidifferential operators of degree at most k. We generalized to k-th degree because manifold is locally biholomorphic iff analytic.

In particular, for the Weyl-qunatization, the deformation quantization formula will be
$f \star g = fg + \Sigma_{k=1}^{\infty} \frac{1}{n!} (\frac{i\hbar}{2})^k \Pi^n (f \otimes g)$

Also, consider another generalization of quantization.

Definition 4.8.7 *(Prequantization)*

Let (M, ω) be a symplectic manifold, and symplectic potential θ be $d\theta = \omega$. The prequantization $Q(f)$ is defied as

$Q(f) = -i\hbar(X_f + \frac{1}{i\hbar}\theta(X_f)) + f$

where X_f is a Hamiltonian vector field associated to f. Or more concisely,

$Q(f) = -i\hbar \nabla_{X_f} + f$

with ∇ connection. For all smooth functions f and g, this prequantum operator satisfy
$[Q(f), Q(g)] = i\hbar Q(\{f, g\})$

By choice of polarization, prequantization becomes quantizaiton.

4.9 Path Integral

Definition 4.9.1 *(Green Function)*

Let L be a linear operator, and $\delta(x)$ is a Dirac delta functino. $G(x, y)$ is a green function if

$LG(x, y) = \delta(x - y)$

Example 4.10 *(Propagator)*

A notable example of Green function is a propagator in QFT. Propagator is a function of probability of the particle moving from x to x' in time t to t'.

- *(Non-Relativistic Propagator)*
  $(H_x - i\hbar\frac{\partial}{\partial t})K(x, x', t, t') = i\hbar\delta(x - x')\delta(t - t')$
  whose solution is
  $K(x, x', t, t') =< 0|\hat{U}(t, t')|0 >$

- *(Relativistic Propagator)*
  $(\Box_x + m^2)G(x, y) = -\delta(x - y)$
  where $\Box_x = \frac{\partial^2}{\partial t^2} - \nabla^2$

- *(Feynman Propagator)*
  $G_F(x, y) = \lim_{\epsilon\to 0} \frac{1}{(2\pi)^4} \int d^4p \frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon}$
  $G_F(x - y) = -i < 0|T\{\Phi(x)\Phi(y)\}|0 >$

So far we have a propagator for one particle system, by using the Green function. Or it's alternatively given by partition functions, so we can generalize to n-body system.

Definition 4.10.1 *(Correlation Function)*

Let $\phi(x)$ be a scalar field state, and $|\Omega >$ be a vacuum state at every point x. The n-point correlation function (also called Green function) is the vacuum expectation value of time-ordered products

$$G_n(x_1, \cdots, x_n) =< \Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega >$$

There $T\{\cdots\}$ is a time-ordering operator, just reordering each state by time-ordering

$T\{A, B\} = AB$ *if* $\tau_A > \tau_B$
$T\{A, B\} = BA$ *if* $\tau_A < \tau_B$

Or we can depict the correlation function as

$$G_n(x_1, \cdots, x_n) = \frac{<0|T\{\phi(x_1)\cdots\phi(x_n)\}e^{iS[\phi]}|0>}{<0|e^{iS[\phi]}|0>}$$

where $|0\rangle$ is a free ground state and $S[\phi]$ is the action.

Definition 4.10.2 *(Vacuum State)*

Vacuum state (or quantum ground state) is the quantum state of the lowest energy, where quantum state is given by bra-ket notation, a vecotor in a Hilbert space thus denoted $|0\rangle$.

As noted , vacuum state is used to define a correlation function.

Definition 4.10.3 *(Partition Function)*

$Z[J] = \lim < 0|e^{-i\hat{H}T}|0 > = \int (D\phi)e^{i\int dx(L+J\phi)}$

where $J(x)$ *is the fictitious source current.*

Or if it's convenient we could also write

$W[J] = -iln(Z[J])$

In fact, this partition funciton describes one-particle system. Let's construct n*-body correlatio function.*

$$\begin{aligned}
\frac{\delta}{\delta J(x)}Z[J] &= \int (D\phi)\frac{\delta}{\delta J(x)}e^{i\int dx(L+J\phi)} \\
&= \int (D\phi)i(\int \delta(x-J)\phi(J)d^4J)e^{i\int dx(L+J\phi)} \\
&= \int (D\phi)i\phi(x)e^{i\int dx(L+J\phi)} \\
&=< 0|\phi(\hat{x})|0 > \\
&= \frac{-i}{Z[0]}\frac{\delta Z[J]}{\delta J(x)}|_{J=0}
\end{aligned} \tag{3}$$

n*-body system is*

$< 0|\phi(\hat{x_1})\phi(\hat{x_2})\cdots\phi(\hat{x_N})|0 >= \frac{(-i)^N}{Z[0]}\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_N)}$

Definition 4.10.4 *(Wick Rotation)*

Wick rotation is a method of finding a solution from a math problem in Minkowski space to a problem in Euclidean space, by transforming imaginary number to real number.

Definition 4.10.5 *(S-Matrix (Scattering Amplitude))*

Scattering Matrix is a matrix that describes network of n-inputs and n-outputs.

ex:

QFT

Electric circuit

Definition 4.10.6 *(Entropy Formula)*

In the closed system, the average temperature T of the closed system and heat fluctuation δQ, entropy is $\int \frac{\delta Q}{T} = 0$. Prove $S = \Sigma p_i ln(p_i)$.

$dS = \frac{\delta Q}{T} = (\frac{\partial S}{\partial T})_P dT + (\frac{\partial S}{\partial P})_T dP = \frac{C_P}{T} dT - V\alpha_V dP$

$S = C_P ln(T) - Rln(P)$

Or the sum is

$S = \Sigma C_P ln(T) - Rln(P)$

Definition 4.10.7 *(Path Integral)*

Path integral is the explicit description of correlation function. We define path integral as

$\int_A^B e^{iS}\phi(x_1)\cdots\phi(x_n)D\phi =< A|\phi(x_1)\cdots\phi(x_n)|B >$

Path integral is similar to expectation values of A and B. Or in particular,

$\frac{\int e^{iS}\phi(x_1)\cdots\phi(x_n)D\phi}{\int e^{iS}D\phi} =< 0|\phi(x_1)\cdots\phi(x_n)|0 >$

This could be written alternatively

$G_n(x_1,\cdots,x_n) = (-i)^n \frac{1}{Z[J]}\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}|_{J=0}$

or alternatively

$G_n^c(x_1,\cdots,x_n) = (-i)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}|_{J=0}$

where

$Z[J] = \int D\phi e^{iS[\phi]+i\int d^dxJ(x)\phi(x)}$

Definition 4.10.8 *(Schrödinger Picture)*

Definition 4.10.9 *(Heisenberg Picture)*

Heisenberg picture is used to calculate expectation value of an observable.

Definition 4.10.10 *(Interaction Picture)*

Definition 4.10.11 *(Feynman Diagram and Correlation Function)*

Feynman diagram describes what particles do we assume to exist in the system, and when and where they react. From the picture (coordinated by time t and position x, we can paraphrase to the correlation function)

For example, the two-point correlation function

$$\langle 0| \Phi(x_1)\Phi(x_2) |0\rangle$$

calculates the expectation value of a particle moving from x_1 to x_2 in time t_1 to t_2.

The four-point correlation function

$$\langle 0| \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4) |0\rangle$$

describes four particle system: e.g. beta collapse $n \to p + e^- + \nu_e$ makes neutron transforms to 3 particles, but totally four kind of particles in the system, so it's four-body problem.

Definition 4.10.12 *(Ward identity)*
A Ward identity is an identity between the correlation functions that follows global or gauge symmetries. More generally, Ward identity is a quantum version of classical current conservation of associated with a continuous symmetry by Noether's theorem.

In particular, in QED, $M(k) = \epsilon_\mu(k) M^\mu(k)$ be the amplitude for some QED with external photon of momentum k, where $\epsilon_\mu(k)$ is the polarization vector of the photon. Then,

$$k_\mu \epsilon_\mu(k) = 0$$

Definition 4.10.13 *(Polarization Vector)*

For example in QED, a photon is a wave osciallating in a direction through a polarization plane. The photon wave oscillates in 1-dimensional, but our living space is 3-dimensional, so there is the direction of oscillation called the polarization. It can be given in a vector form as follows

$$|\psi\rangle = \begin{bmatrix} cos(\theta)exp(i\alpha_x) \\ sin(\theta)exp(i\alpha_y) \end{bmatrix}$$

Definition 4.10.14 *(S-Matrix)*

S-matrix is a short of Scattering matrix, and S-matrix relates initial states and final states of physical system undergoing scattering process. Scattering process describes how particles change direction by collision. In particular in QFT, Feynman diagram describes how particles collide, which is exactly the scattering process.

Let S be a scattering matrix, and $\langle\Psi_{out}|$ is an outgoing wave function, $|\Psi_{in}\rangle$ in an incoming wave function. Then the relation is

$\langle\Psi_{out}| S |\Psi_{in}\rangle$

For detail, S-Matrix(video)

Definition 4.10.15 *(Faddeev-Popov Ghost)*

FP ghost field $c^a(x)$ is a notion in QFT, and works as a complementary field to make path integral consistent. For example in YM theory, its Lagrangian L_{ghost} is

$L_{ghost} = \partial_\mu \overline{c}^a \partial^\mu c^a + g f^{abc}(\partial^\mu \overline{c}^a) A^b_\mu c^c$

whose first term is kinetic term, and the second term is interaction with the gauge fields as well as the Higgs field.

4.11 Conformal Field Theory

The problem in CFT is that conformal symmetry of physical system is described by Virasoro algebra, and WZW model (or to put simply gauge theory) has an affine Lie algebra structure, and it has translated to Virasoro algebra by Sugawara construction. In this construction, we have the equations that calculate Lie algebra invariant, called KZ equation, and it's mathematically very intersting. The purpose of Sugawara construction is that by context Virasoro algebra is more enhanced than affine Lie algebra, because Virasoro algebra measures invariant of the correlation functions, and a correlation function is used to describe path integral of n-body system, and it's invariant with respect to the conformal transformation.

Definition 4.11.1 *(Central Charge)*

$Z(\mathscr{A}) : K(\mathscr{A}) \to \mathbb{C}$ is central charge for $\mathscr{A}$ is an abelian category.
In physics, charge usually refers to an invariant with respect to Hamiltonian. So, q is a charge such that

$\{q, H\} = 0 = \frac{dq}{dt}$

Definition 4.11.2 *(Virasoro Algebra)*

Virasoro algebra is defined by central extension of Witt algebra with central charge c. Witt algebra is

$Der(\mathbb{C}[z, z^{-1}]) = \{L_n = -z^{n+1}\frac{\partial}{\partial z} | n \in \mathbb{Z}\}$

and $[L_n, L_m] = (m-n)L_{m+n}$.

Now, Virasoro algebra is a Lie algebra with a bracket

$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$.

Definition 4.11.3 *(Chiral CFT)*

Let $V(z)$ *be a field, and*

$\frac{\partial}{\partial z}V(z) = L_{-1}V(z)$

It follows that OPE

$T(y)V(z) = \Sigma_{n\in\mathbb{Z}}\frac{L_nV(z)}{(y-z)^{n+2}}$

where (L_n) *is a generator of the Virasoro algebra. defines a locally holomorphic field* $T(y)$ *that doesn't depend on* z*. This* $T(y)$ *is called energy momentum tensor.*

Let $V_\Delta(z)$ *be a primary field, a field that is of lowest weight representation as*

- $L_{n>0}V_\Delta(z) = 0$
- $L_0V_\Delta(z) = \Delta V_\Delta(z)$

Theorem 4.11.1 *(Correlation Function and Conformal Identities)*

The correlation function is defined by using functional integral (path integral)

$< \phi(x_1)\cdots\phi(x_n) >= \frac{1}{Z}\int[d\phi]\phi(x_1)\cdots\phi(x_n)e^{iS[\phi]}$

where

$Z = \int[d\phi]e^{iS[\phi]}$ *and* ϕ *is a scalar field*

If we apply transformation R *to* ϕ *as* $\phi \mapsto R(\phi)$
$< R(\phi(x_1))\cdots R(\phi(x_n)) >= \frac{1}{Z}\int[d\phi]R(\phi(x_1))\cdots R(\phi(x_n))e^{iS[\phi]}$

Note that $[d\phi] = [dR(\phi)]$ and $S[\phi] = S[R(\phi)]$ are invariance, and

$< R(\phi(x_1)) \cdots R(\phi(x_n)) > = \frac{1}{Z} \int [dR(\phi)] R(\phi(x_1)) \cdots R(\phi(x_n)) e^{iS[R(\phi)]} = < \phi(R(x_1)) \cdots \phi(R(x_n)) >$

Now let using Ω, the transformation makes

$R(\phi(x)) = \phi'(x') = \Omega^{\frac{\Delta}{2}} \phi(x)$

in particular, $\Omega = 1$ if R is conformal.

Definition 4.11.4 *(WZW Model)*

Σ is a Riemann Surface, G is a Lie group, $k \in \mathbb{N}$. We define G-WZW model on Σ at the level k. The model is a non-linear sigma model whose aciton is a functional $\gamma : \Sigma \to G$

$S_k(\gamma) = -\frac{k}{8\pi} \int_\Sigma d^2x K(\gamma^{-1}\partial^\mu\gamma, \gamma^{-1}\partial_\mu\gamma) + 2\pi k S^{WZ}(\gamma)$

where K is the Killing form on G. Its Wess-Zumino term of the action is

$S^{WZ}(\gamma) = 1\frac{1}{48\pi^2} \int_{B^3} d^3y \epsilon^{ijk} K(\gamma^{-1}\partial_i\gamma, [\gamma^{-1}\partial_j\gamma, \gamma^{-1}\partial_k\gamma])$

where B^3 is $\partial B^3 = \Sigma$.

This WZW model has a symmetric property, generating affine Lie structure. Let $\Omega(z)$ be any holomorhic G-valued function, $\bar{\Omega}(\bar{z})$ be any G-valued anti-holomorphic function.

This $S_k(\gamma) = S_k(\Omega\gamma\bar{\Omega}^{-1})$ is
$J(z) = -\frac{1}{2}k(\partial_z\gamma)\gamma^{-1}$ and $\bar{J}(\bar{z}) = -\frac{1}{2}k\gamma^{-1}\partial_{\bar{z}}\gamma$ are the conserved currents associated with this symmetry.

Or we denote by $J^a(z)$ where a is an adjoint indices where $\{t^a\}$ is the orthonomal basis of $\mathfrak{g}$ wrt Killing form.

Definition 4.11.5 *(Affine Lie Algebra (or sometimes current algebra))*
$\{t^a\}$ is orthnormal basis(wrt to the Killing form) of Lie algebra of G and $J^a(z)$ the quantization of the field $K(t^a, \partial_z gg^{-1})$ where $g : (B \coprod B)/\partial B \sim S^3 \to G$. Then

$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{if_c^{ab}j^c(w)}{z-w} + O(1)$

where f_c^{ab} is a coefficient of Lie bracket $[t^a, t^b] = \Sigma f_c^{ab} t^c$

$J^a(z) = \Sigma_{n\in\mathbb{Z}} J^a_n z^{-n-1}$

and the algebra generated by the $\{J^a_n\}$ *is called the affine Lie algebra associated to Lie algebra of* G*.*
$[J^a_n, J^b_m] = f^{ab}_c J^c_{m+n} + kn\delta^{ab}\delta_{n+m,0}$

We call $\hat{\mathfrak{g}} = \{J^a_n\}$ *is affine Lie algebra, or alternatively* $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$*.*

CFT is called In Chiral CFT, if the theory contains Virasoro algebra.

Definition 4.11.6 *(Energy-Momentum Tensor)*

The dependence of the field $V(z)$ *on its position is assumed to be determined by*

$\frac{\partial}{\partial z}V(z) = L_{-1}V(z)$

It follows that OPE

$T(y)V(z) = \Sigma_{n\in\mathbb{Z}} \frac{L_n V(z)}{(y-z)^{n+2}}$

defines a locally holomorphic field defined independent from z*. From here, its OPE(operator product expansion) will be*

$T(y)T(z) = \frac{\frac{c}{2}}{(y-z)^4} + \frac{T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z} + O(1)$

where c *is the central charge. OPE is a Laurent series expansion associated with a two operators.*

Sugawara construction is an embedding of Virasoro algebra to affine Lie algebra. If Virasoro algebra is for CFT, Sugawara construction shows that WZW model is actually CFT.

Definition 4.11.7 *(Sugawara Construction)*

Recall $\{J^a(t)\}$ *is Affine Lie algebra. the energy momentum tensor* $T(z)$ *for the Virasoro algebra.*

$T(z) = \frac{1}{2(k+h^\vee)}\Sigma_a : J^a J^a : (z)$

where ":" *denotes normal ordering,* $h^\vee$ *is the dual Coxter number. By using OPE and Wick's theorem,*

$T(y)T(z) = \frac{\frac{c}{2}}{(y-z)^4} + \frac{T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z} + O(1)$

which is equivalent to Virasoro algebra commutation relations. If we let central charge and its Virasoro algebra as

- $c = \frac{k dim \mathfrak{g}}{k+h^\vee}$
- $L_{n\neq 0} = \frac{1}{2(k+h^\vee)} \Sigma_a \Sigma_{m\in\mathbb{Z}} J^a_{n-m} J^a_m$
- $L_0 = \frac{1}{2(k+h^\vee)} 2(\Sigma_a \Sigma_{m=1}^{\infty} J^a_{-m} J^a_m + J^a_0 J^a_0$

then the momentum tensor will be

$T(z) = \Sigma_{m\in\mathbb{Z}} L_n z^{-n-2}$

Definition 4.11.8 *(Knizník-Zamolodchikov Equations)*

Let $\hat{frakg}_k$ *be an affine Lie algebra with level* k*, and dual Coxeter number* h*. Let* v *be a zero mode representation of* $\hat{\mathfrak{g}}_k$ *and* $\Phi(v, z)$ *the primary field associated with it.* t^a *is the basis of* $\mathfrak{g}$*, and* t^a_i *their representation on the primary field* $\Phi(v_i, z)$ *and* η *the killing form.*

$((k+h)\partial_{z_i} + \Sigma_{j\neq i} \frac{\Sigma_{a,b} \eta_{ab} t^a_i \otimes t^b_j}{z_i - z_j}) < \Phi(v_N, z_N) \cdots \Phi(v_1, z_1) >= 0$

The KZ equation appear from Sugawara construciton of Virasoro algebra.
$L_{-1} = \frac{1}{2(k+h)} \Sigma_{k\in\mathbb{Z}} \Sigma_{a,b} \eta_{ab} J^a_{-k} J^a_{k-1}$

4.12 Monstrous Moonshine

Monstrous Moonshine is a mathematical formulation derived from math physics using vertex algebra. Vertex algebra is a formal Laurent series where the coefficients are vectors.

Definition 4.12.1 *(Vertex Algebra/Vertex Operator Algebra)*

A vector space V*, called the space of states.*

- *(Identity)*
  $1 \in V$ *or denoted* $|0\rangle$ *or* $|\Omega\rangle$
- *(Translation)*
  $T : V \to V$
- *(Linear Multiplication)*
  $Y : V \otimes V \to V((z))$
  where $V((z))$ *is the space of all formal Laurent series with coefficients in* V*.*
  $\cdot_n : u \otimes v \mapsto u_n v$ *where* $u_n \in End(V)$
  such that $u_n v = 0$ *for all* $n < N$*. Then,*
  $u \otimes v \mapsto Y(u, z)v = \Sigma_{n\in\mathbb{Z}} u_n v z^{-n-1}$

Note that formal Laurent series is $f(z) = \Sigma_{n=N}^{\infty} a_n (z-c)^n$ *for some integer* $N \in \mathbb{Z}$. *In general, if* $N = -\infty$, $f(z)$ *is Laurent series.*

Axioms:

- *(Identity)*
  For any $u \in V$, $Y(1,z) = u$ *and* $Y(u,z) = 1 \in u + zV[[z]]$
- *(Translation)*
  $T(1) = 0$ *and for any* $u, v \in V$,
  $[T, Y(u,z)]v = TY(u,z)v - Y(u,z)Tv = \frac{d}{dz} Y(u,z)v$
- *(Locality (or Jacobi identity))*
  there exists a positive integer $N \in \mathbb{N}_{\geqq 0}$ $(z-x)^N Y(u,z)Y(v,x) = (z-x)^N Y(v,x)Y(u,z)$

(Vertex Operator Algebra)

Vertex operator algebra is a vertex algebra equipped with a conformal element $\omega \in V$, *such that the vertex operator* $Y(\omega, z)$ *is the weight two Virasoro field* $L(z)$

$$Y(\omega, z) = \Sigma_{n\in\mathbb{Z}} \omega_n z^{-n-1} = L(z) = \Sigma_{n\in\mathbb{Z}} L_n z^{-n-2}$$

and satisfies the following properties:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}\delta_{m_n,0}(m^3 - m)cId_V$$

L_0 *acts semisimply with integer eigenvalues that are bounded below.*

$$L_{-1} = T$$

Definition 4.12.2 *(Operator Product Expansion)*

$$Y(A,z)Y(B,w)C = \Sigma_{n\in\mathbb{Z}} \frac{Y(A_{(n)} \cdot B, w)}{(z-w)^{n+1}} \cdot C$$

Example 4.13 *(Monster Vertex Algebra)*

Monster vertex algebra (also called Moonshine Module) is a vertex algebra acted on Monster group.

Remark 4.14 *(AdS/CFT Correspondence)*

AdS/CFT correspondence relates Quantum gravity and QFT. AdS_5 *is a maximal symmetric solution of the Einstein equations in 5-dimension in cosmological constant, we are claiming that* AdS_5 *corresponds to the correlation function of CFT in 4-dimension.*

Definition 4.14.1 *(K3 Surface and CFT)*
symmetry is Mathieu group M24.

4.15 Lattice Field Theory

5 Condensed Matter Physics

5.1 Ising Model

describing classical mechanics behavior by accumlation of particle movements, i.e. the melting the ice, evapolation.

Definition 5.1.1 *(Entropy)*

Definition 5.1.2 *(Wick Transformation)*

$\frac{1}{k_B T} \mapsto \frac{it}{\hbar}$

Definition 5.1.3 *(Functional Derivative)*

A set of all function is $C^0(M, \mathbb{R})$ and an operator J is a morphism between them $J : C^0(M, \mathbb{R}) \to C^0(M, \mathbb{R})$

$J[f] = \int_a^b L(x, f(x), f'(x))dx$

$\delta J = \int_a^b (\frac{\partial L}{\partial f}\delta f(x) + \frac{\partial L}{\partial f'}\frac{d}{dx}\delta f(x))dx = (\frac{\partial L}{\partial f} - \frac{\partial L}{\partial f'}\frac{d}{dx})\delta f(x)dx + \frac{\partial L}{\partial f'}(b)\delta f(b) - \frac{\partial L}{\partial f'}(a)\delta f(a)$

Definition 5.1.4 *(Partition Function)*

- *(Classical Discrete System)*
  The canonical partition function is defined as

  $Z = \Sigma_i e^{-\beta E_i}$

  where $\beta = \frac{1}{k_B T}$ and E_i is the total energy of the system in the respective micro state. Also,

  $1 = \Sigma_i \rho_i$

- *(Classical Continuous System)*

  $Z = \frac{1}{h^3}\int exp(-\beta H(q,p))d^3q d^3p$

  $Z = \frac{1}{N! h^{3N}}\int exp(-\beta \Sigma_{i=1}^N H(q_i, p_i))d^3q_1 \cdots d^3q_N d^3p_1 \cdots d^3p_N = \frac{Z_{single}^N}{N!}$

- *(Quantum Mechanical Discrete System)*

  $Z = tr(e^{-\beta\hat{H}})$

- *(Quantum Mechanical Continuous System)*

  $Z = \frac{1}{h}\int <q,p|e^{\beta\hat{H}}|q,p> dqdp$

  Note that in addition

  $1 = \int |x,p><x,p|\frac{dxdp}{h}$

  Hence alternatively,

  $Z = \int tr(e^{-\beta\hat{H}}|x,p><x,p|)\frac{dxdp}{h} = \int <x,p|e^{-\beta\hat{H}}|x,p> \frac{dxdp}{h}$

Definition 5.1.5 *(Ising Model)*

Assume an physical object consists of atoms forming a lattice structure Λ, and each atom has a spin $\sigma_k \in \{\pm 1\}$.

The Hamiltonian of the system is

$H(\sigma) = \Sigma_{<ij>,i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - \mu\Sigma_j h_j\sigma_j$

picking i and j are adjacent each others where J_{ij} is interaction, μ is , h_j is an external magnetic field interacting with it.

$P_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta}$

where $\beta = \frac{1}{k_B T}$

The expectation value of f is given by

$<f>_\beta = \Sigma_\sigma f(\sigma)P_\beta(\sigma)$

Definition 5.1.6 *(Random Cluster Model)*

That's probability theory that generalizes and unifies Ising model.

5.2 Superconductivity

Definition 5.2.1 *(Free Energy)*

F is free energy

$F = U - TS$

where U is inner energy, T is temperatur, and S is an entropy.

Definition 5.2.2 *(Landau-Ginzburg Model)*

A physical model of superconductivity. Landau-Ginzburg equation is

$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m}(-i\hbar\nabla - 2e\mathbb{A})^2\psi = 0$ *(LG equation)*

which is derived from variational calculus of free energy

$F = F_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}|(-i\hbar\nabla - 2e\mathbb{A})\psi|^2 + \frac{|\mathbb{H}|^2}{2\mu_0}$ *(Free energy)*

where F_n is free energy of state of normal conductivity.

6 Chern-Simon Theory

We'll construct 3-dimensional example of Chern-Simon theory, namely if $\mathfrak{g} = \mathfrak{su}(2)$, since this might be easier, because knot is defined on 3-dimensional space, neither 2 or 4-dimension or any others. This is for weak theory in physics, but what else? For the higher dimensional case, knot is unable to be defined, but we can use quantum group and its braiding structure to knottify them.

6.1 For 3-Dim Case

The basic idea is that for the principal bundle $P \to M$ of fiber G, we'll consider traveling of particle on M, and track the change of Lie group value. This is alternatively a traveling of Lie group, and this is a geometric problem, since Lie group is a smooth manifold, and the holonomy exactly defines Wilson loop. If especially, for the sake of convenience, we choose the differential operator as flat connection, and the holonomy becomes trivial geometrically, and Wilson loop becomes purely topological invariant. That's how it's called TQFT.

Definition 6.1.1 *(Lie Algebra Valued Differential Form)*

If $\omega \wedge \omega$ is a 1-form, then

$\omega \wedge \omega(v_1, v_2) = [\omega(v_1), \omega(v_2)] - [\omega(v_2), \omega(v_1)]$
In general, if ω is p-form, η is q-form, then

$\omega \wedge \eta(v_1, \cdots, v_n) = \frac{1}{p!q!}\Sigma_\sigma sgn(\sigma)[\omega(v_{\sigma(1)}, \cdots, v_{\sigma(p)}), \eta(v_{\sigma(p_1)}, \cdots, v_{\sigma(p+q)})]$.

Definition 6.1.2 *(Chern-Weil Homomorphism)*

For a principla G-bundle $P \to M$, Chern-Weil homomorphism is

$\mathbb{C}[\mathfrak{g}]^G \to H^*(M, \mathbb{C})$.

Let $\Omega = D\omega = d\omega + \omega \wedge \omega$ be a curvature form, and $f \in \mathbb{C}[\mathfrak{g}]^G$ is a homogeneous polynomial function of degree k $f(ax) = a^k f(x)$ for $a \in \mathbb{C}$ and $x \in \mathfrak{g}$. Then, $f(\Omega)$ is a $2k$-form given by

$f(\Omega)(v_1, \cdots, v_{2k}) = \frac{1}{(2k)!} \Sigma \epsilon_\sigma f(\Omega(v_{\sigma(1)}, v_{\sigma(2)}), \cdots, \Omega(v_{\sigma(2k-1)}, v_{\sigma(2k)}))$

Thus we define a morphism

$$\begin{aligned} \mathbb{C}[\mathfrak{g}]^G &\to H^*(M, \mathbb{C}) \\ f &\mapsto [\overline{f}(\Omega)] \end{aligned} \tag{4}$$

An example of Chern-Weil homomorphism is Chern class.

Example 6.2 *(Chern Class)*

In differential gemetry, Chern polynomial is defined by the conneciton form. Let $E \to M$ be a vector bundle, and Ω is its conneciton form. Then

$c_t(E) = det(I - t\frac{\Omega}{2\pi i})$

and

$c_k(E) \in H^{2k}(M, \mathbb{Z})$

or

$c(E) = \Sigma c_k(E) \in H^*(M, \mathbb{Z})$

So, if $f(x) = det(I - t\frac{x}{2\pi i}) \in \mathbb{C}[\mathfrak{g}]^G$ is a character polynomial, plugging in $x = \Omega$ makes Chern-Weil homomorphism.

Definition 6.2.1 *(Chern-Simon Form)*

Let A be a connection 1-form.
$S = \frac{k}{4\pi} \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$

$F = dA + A \wedge A$

Definition 6.2.2 *(Wilson Loop)*

A Wilson loop is a gauge invariant operators. A Wilson loop is a holonomy around a loop in M, traced in given representation R of G.

$W[\gamma] = tr[Pexp(i \oint_\gamma A_\mu dx^\mu)]$

where $\gamma : [0,1] \to M$ is a loop. P is a path ordering operator.

The set of all Wilson lines is in one-to-one correspondence with the representations of the gauge group, which is:
Λ_w/W where W is Weyl group.

Definition 6.2.3 *(Knot Theory)*

A Knot is an embedding $S^1 \to M$ to a geometry M, but M needs to be compact (if we talk about $S^1 \to \mathbb{R}^3$, we spontaneously assume all the loops on $\mathbb{R}^3$ are bounded). Consider if M is compact Hausdorff, all closed subsets i.e $Im(S^1)$ are compact.

Example 6.3 *(G)*

Take $G = SU(2)$, and that's 3-dimensional real Lie group, and it's compact. Knot is definable.

6.4 For Higher Dimension

Now let's consider the case of higher dimensions, where we'll similarly consider knot theory, but considering by braiding structure. An algebraic way of describing braiding structure is based on noncommutativity. Swapping the neighboring path $\tau\sigma \neq \sigma\tau$ might be generally speaking different, usually makes a free group from the generators.

Definition 6.4.1 *(Braiding)*
Consider a map

$c_{V,W} : V \otimes W \to V \otimes W$

where R is R-matrix, and V is a representation space of $U_q(\mathfrak{su}(N))$. The braiding $c_{V,W}$ is determied by using $R \in U_q(\mathfrak{su}(N)) \otimes U_q(\mathfrak{su}(N))$ as,
$c_{V,W}(v \otimes w) = \tau \circ (\rho_V \otimes \rho_W)(R)(v \otimes w)$
where representation $\rho_1 : U_q(\mathfrak{su}(N)) \to V$, $\rho_2 : U_q(\mathfrak{su}(N)) \to W$ and $\tau(v \otimes w) = w \otimes v$

7 Higgs Bundle

7.1 Higgs Field

Higgs particle explains mass of physical system. Higgs particle doesn't have a mass itself, but it helps Boson and Ferminon particles obtain masses from spontaneous symmetry breaking. That is, Higgs field has a Mexican hat potential, it makes spontaneous symmetry breaking at the lowest energy, and this gap helps particles to have masses. Later Higgs field is more rigorously and mathematically studied by Higgs bundle.

Definition 7.1.1 *(Weinberg-Salam Theory)*

Weinberg-Salam theory unifies symmetry of weak $SU(2)$ and electromagnetic $U(1)$ forces. The total symmetry $SU(2)\times U(1)$ may be broken due to spontaneou symmetry breaking (that I explain later).

Definition 7.1.2 *(Higgs Field)*

Higgs Field Φ is defined by

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi^+\\ \phi^0\end{bmatrix}$$

THe vacuum expectation value of Φ is

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ v+h(x)\end{bmatrix}$$

Definition 7.1.3 *(Spontaneous Symmetry Breaking)*

Let Φ be a Higgs field, then the Higgs field's potential

$$V(\Phi) = \mu\Phi^\dagger\Phi + \lambda|\Phi|^4$$

whose shape is Mexican hat. At the origin, the particle can move any direction, meaning $U(1)$ symmetry, but the particle moved to lowest potential, the symmetry no longer exist. By analogy, if pencil is standing, the pencil can move to any direction, but once pencil falls and lying down, it no longer change the direction. That's symmetry breaking.

Proposition 7.2 *(Fermion Mass)*
The Higgs field defines interaction Lagrangian of Higgs field and Fermion

$$L_{int} = -y_f\overline{\psi}_L(\frac{v+h(x)}{\sqrt{2}})\psi_R + h.c.$$

If $v \neq 0$ is non-zero vacuum expectation value, fermion has a mass.

$$L_{mass} = -\frac{y_f v}{\sqrt{2}}\overline{\psi}\psi$$

$$m_f = \frac{y_f v}{\sqrt{2}}$$

Example 7.3 *(List of Masses of Particles)*

- *(W Boson)*
  $m_W = gv$

- *(Z Boson)*
  $m_Z = \frac{v}{2}\sqrt{g^2 + g'^2}$

- *(Photon)*
  Photon doesn't have mass.

- *(Gluon)*
  Gluon doesn't have mass.

- *(Fermion)*
  $m_f = \frac{y_f v}{\sqrt{2}}$

where

- v *is vacuum expectation value of the Higgs field.*
- g *is coupling constant of* $SU(2)$
- g' *is coupling constant of* $U(1)$

7.4 Higgs Bundle

Higgs bundle (E, ϕ) is a holomorphic vector bundle E consisting Higgs field ϕ. Physically, Higgs particle, which defines Higgs field, experimentally proved its existence in 2012 in CERN, but it was originally conjectured to explain Weinberg-Salam theory, that unifies weak force and electromagnetic force in gauge theory.

Higgs field $\phi : M \to V$ is vector valued $\phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$,

Higgs bundle connects to Geometric Langlands correspondence.

From a constructive perspective, Hitchin system is a more general form of Higgs bundle.

Background of Higgs Bundle is to describe Higgs particle.

Definition 7.4.1 *(Integrable System)*

Definition 7.4.2 *(Hitchin System)*

Tangent space to the moduli space of G-bundles at F is

$T_{[F]}\mathfrak{M} = H^1(End(F))$

which by Serre duality is dual to

$\Phi \in H^0(End(F) \otimes K)$

where K is the canonical bundle. So a pair (F, Φ) is called Hitchin pair of Higgs bundle.

Definition 7.4.3 *(Higgs Bundle)*
Higgs bundle is a pair (E, ϕ) where E is holomorphic vector field and ϕ is a Higgs field ϕ, a holomorphic 1-form taking values in the bundle of endomorphisms of E such that $\phi \wedge \phi = 0$.

Higgs Field is a field of energy that is assumed to exist everywhere in the universe, and Higgs field can define mass of an element.

Theorem 7.4.1 *(Interface)*

By nonabelian Hodge correspondence, category of flat holomorphic connections on a smooth complex projective algebraic variety, category of representations of the fundamental group of that variety, and the category of Higgs bundles over this variety are actually quivalent.

Definition 7.4.4 *(Stability Condition of Higgs Bundle)*

Let $L \subset E$ be a rank 1 ϕ-invariant subbundle of E,
$deg(L) < \frac{1}{2} deg(\bigwedge^2 E)$

Definition 7.4.5 *(Hermitian Yang-Mill Connection)*

A hermtian metric h on a Higgs bundle (E, Φ) gives rise to a Chern connection ∇_A and a curvature F_A. The condition that Φ is holomorphic is phrased as $\overline{\partial_A}\Phi = 0$

$$\begin{gathered} F_A + [\Phi, \Phi^*] = \lambda Id_E \\ \overline{\partial_A}\Phi = 0 \end{gathered} \tag{5}$$

If we let a connection $D = \nabla_E + \Phi + \Phi^*$, D is called Hermitian Yang-Mill connection.

Definition 7.4.6 *(Non-Abelian Hodge Correspondence)*

claim 1:

A representation $\rho : \pi_1(X) \to GL(r, \mathbb{C})$ *is semisimple iff the flat vector bundle* $E = \hat{X} \times_\rho \mathbb{C}^r$ *admits a harmonic metric. Furthermore, the representation is irreducible iff the flat vector bundle is irreducible.*

claim 2:

A Higgs bundle (E, Φ) *has a Hermitian metric iff it's polystable. This metric is harmonic metric, and therefore arises semisimple representation of the fundamental group, iff Chern classes* $c_1(\Omega)$ *and* $c_2(\Omega)$ *vanishes. A Higgs bundle is stable iff it admits an irreducible Hermitian Yang-Mills connection, and therefore comes from an irreducible representation of the fundamental group.*

Moduli space version:

there are homeomorphism $M^{ss}_{Dol} \cong M_{dR} \cong M^{+}_{B}$ *of moduli spaces which restrict to homeomorphisms* $M^{s}_{Dol} \cong M^{*}_{dR} \cong M^{*}_{B}$

Definition 7.4.7 *(Geometric Langlands Correspondence)*

Might be already little mentioned in the previous exposition.

There is a conenction of Geometric Langlands correspondence and S-duality, a certain property of QFT.

Definition 7.4.8 *(S-duality)*

S-duality is short for strong-weak duality.

Connection to N=4 supersymmetry Yang-Mill theory.

Next, we will describe Yang-Mills existence and mass gap. The whole description is not fully constructed or proved yet.

Definition 7.4.9 *(Yang-Mills Existence and Mass Gap)*

It claims the construction of existence of mass of particles. For example, mass of weak bosons exist, and generated by the spontaneous symmetry breaking, and by analogy, we consider the existence of mass of gluon, that's QCD of pure Yang-Mills theory.

In other word, mass gap is the difference between second lowest energy and vacuum state. Indeed, roughly energy could be the paraphrase of mass, as $E = mc^2$.

The program claims that for any compact simple gauge group G, there exists a non-trivial quantum Yang-Mills theory on $\mathbb{R}^4$, and it has a mass gap $\Delta > 0$.

Definition 7.4.10 *(Example From Classical Theories)*

The mass gap already appears in classical mechanics.
$\Box\phi + \lambda\phi^3 = 0$

8 Super Theory

8.1 Super Symmetry

What is supersymmetry (or called SUSY)? Moreover, what is super? Super is related to $\mathbb{Z}/2\mathbb{Z}$ grading, which is in physics considered as a spin or anti-matter, making extra symmetry in gauge symmetry in physics. Spin is geometrically spinor. In particle physics, particles can be largely classified as boson and fermion, and fermion is spin $\frac{1}{2}$. Also, the question is mathematical construction of supersymmetry in Lie bracket and vertex operators. The particular example of supersymmetry algebra is BPS state. There are many applications of BPS states: magnetic monopoles, solitons and D-branes, SUNY gauge theory, external black hole etc.

Definition 8.1.1 *(Spinor)*

Under constrution. Here is youtube video.

https://www.youtube.com/watch?v=j5soqexrwqY

Definition 8.1.2 *(Λ-bracket)*

For an affine vertex algebra $V^k(\mathfrak{g})$ of level k associated with $\mathfrak{g}$, $V^k(\mathfrak{g})$ is generated by the parity reversed vector superspace $\overline{\mathfrak{g}}$ of $\mathfrak{g}$ and the Λ-bracket is given by

$[\overline{a}\Lambda\overline{b}] = (-1)^{p(a)}(\overline{[a,b]} + k\chi(a|b))$ *for* $a, b \in \mathfrak{g}$

Definition 8.1.3 *($\mathbb{C}[\nabla]$ Algebra)*
Let R be $\mathbb{C}[\nabla]$-module, where $\nabla = (\partial, D^1, D^2, \cdots, D^n)$ consists of an even operator ∂ and odd operators D^i such that

$[D^i, D^j] = \delta_{i,j}2\partial$ *for* $i, j = 1, 2, \cdots, n$

Note that the bracket is a supercommutator. In addition, consider a tuple $\Lambda = (\lambda, \chi^1, \chi^2, \cdots, \chi^n)$ of an even formal variable λ and odd formal variables χ^i subject to the relations

$[\chi^i, \chi^j] = -\delta_{i,j} 2\lambda$

bracket is again supercommutator.

Now a Λ*-bracket* $R \otimes R \to \mathbb{C}[\Lambda] \otimes R$ *is degree* $\overline{n}$ *map with sesquilinearity*

$[D^i a \Lambda b] = (-1)^{n+1} \chi^i [a \Lambda b]$
$[a \Lambda D^i b] = (-1)^{p(a)+n} (D^i + \chi^i)[a \Lambda b]$
$[D^i, \chi^j] = \delta_{i,j} 2\lambda \ [D^i, \lambda] = 0$

Definition 8.1.4 *(Supersymmetric(SUSY) Lie Superalgebra)*

An $N = n$ *SUNY Lie conformal algebra is* $[Z]/2[Z]$*]-graded* $\mathbb{C}[\nabla]$*-module* R *with a* Λ*-bracket which is a* $\mathbb{C}$*-linear map of degree* $\overline{n}$*:*

$[\cdot \Lambda \cdot] : R \otimes R \to \mathbb{C}[\Lambda] \otimes R$, $a \otimes b \mapsto [a \Lambda b]$

- *(Skew-Symmetry)*
  $[b \Lambda a] = (-1)^{p(a)p(b)+n+1} [a - \nabla - \Lambda b]$
- *(Jacobi Identity)*
  $[a \Lambda [b \Gamma c]] = (-1)^{(p(a)+1)n} [[a \Lambda b] \Lambda + \Gamma c] + (-1)^{(p(a)+n)(p(b)+n)} [b \Gamma [a \Lambda c]]$

for $a, b, c \in R$ *and* $\Gamma = (\gamma, \eta^1, \eta^2, \cdots, \eta^n)$*.*

Definition 8.1.5 *(Supersymmetric(SUSY) Vertex Algebra)*

An $N = n$ *SUSY vertex algebra is a pair* $(V, \nabla, [\cdot, \Lambda, \cdot], |0\rangle, ::)$ *such that*

- *(*$N = n$ *SUSY Lie conformal algebra)*
  $(V, \nabla, [\cdot \Lambda \cdot])$ *is an* $N = n$ *SUSY Lie conformal algebra.*
- *(Unital differential superalgebra)*
  $(V, \nabla, |0\rangle, ::)$ *is a unital differential superalgebra satisfying the following properties*
  - *(Quasi-commutativity)*
    $: ab : -(-1)^{p(a)p(b)} : ba := \int_{-\nabla}^{0} [a \Lambda b] d\Lambda$
  - *(Quasi-associativity)*
    $:: ab : c- : a : bc ::=: (\int_0^{\nabla} d\Lambda a)[b \Lambda c] : +(-1)^{p(a)p(b)} : (\int_0^{\nabla} d\Lambda b)[a \Lambda c] :$
- *(*Λ *and* $::$*)*
  Λ *and* $::$ *are related by*

– *(Non-commutative Wick Formula)*
$[a\Lambda : bc :] =: [a\Lambda b]c : +(-1)^{(p(a)+n)p(b)} : b[a\Lambda c] + \int_0^\Lambda [[a\Lambda b]\Gamma c]d\Gamma$
where the integral $\int_0^\Lambda d\Gamma = \partial_{\eta^1}\partial_{\eta^2}\cdots\partial_{\eta^n}\int_0^\lambda d\gamma$

Definition 8.1.6 *(R-Symmetry)*

Definition 8.1.7 *(BPS state)*

For $d = 4$ $N = 2$, extended supersymmetry algebra called BPS states have mass equal to the supersymmetry of central charge Z.

- $\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\sigma^m_{\alpha\dot\beta}P_m\delta^A_B$
- $\{Q^A_\alpha, Q^B_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\overline{Z}$
- $\{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\beta B}\} = -2\epsilon_{al\dot{p}habe\dot{t}a}\epsilon_{AB}Z$

where α β are Lorentz group indices, and A and B are R-symmetry indices.

Example 8.2 *(BPS States)*

- *Magnetic Monopoles*
- *Solitons and D-branes*
- *SUNY gauge theory*
- *External black hole*

Definition 8.2.1 *(Magnetic Monopoles)*

Definition 8.2.2 *(External black hole)*

External black hole is a black hole with a minimum possible mass and that is compatible with its charge and angular momentum. This black hole is stable and no Hawking radiation. Their black hole entropy is calculated in string theory.

Definition 8.2.3 *(Solitons and D-branes)*

Definition 8.2.4 *(SUNY gauge theory)*

Definition 8.2.5 *(Super CFT)*

Definition 8.2.6 *(Wall Crossing Formula)*

Definition 8.2.7 *(Donaldson-Thomas Theory)*

Definition 8.2.8 *(Atiyah-Singer Index Theorem)*

Atiyah-Singer index theorem says that analytical index and topological index of the same elliptic differential operator on a compact manifold coincides.

There are many applications.

Definition 8.2.9 *(Fredholm Operator)*

Fredholm operator is an elliptic operator. More specifically, Let $T : X \to Y$ *be a bounded linear operator, where* X *and* Y *are Banach spaces with finite dimensional kernel* $ker(T)$ *and finite dimensional cokernel* $coker(T) = Y/ran(T)$*. We call this* T *Fredholm operator.*

The index of Fredholm operator is the integer $ind(T) = dim(ker(T)) - dim(coker(T))$

8.3 Super String Theory

Definition 8.3.1 *(Super String Theory)*

String theory is a theory in which each particle is consider to be a closed/open string.

String theory has an ability to describe gravity (Einstein equation), and it could be better than gauge theory.

Superstring theory is a enhancement of string theory by adding supersymmetry.

- *open string : spin 1. photon, weak boson, gluon.*
- *closed string : spin 2. graviton*

There are variety of different string theories.

- $SO(32)$
- $E_8 \times E_8$

- *TypeI*
- *TypeII*

Finally, M-theory is a theory that unifies all string theories. According to M-theory, our universe is 11*-dimension, consisting* 2 *or* 5*-dimensional branes.*

Definition 8.3.2 *(D-Branes)*

D-branes are a class of objects of open stirings can end with Dirichlet boundary conditions.

D-branes are classified by their dimension. DN-branes is N*-dimensional cube. D1-brane is a string, D2-brane is a plane.*

Definition 8.3.3 *(Dirichlet Boundary Condition)*

Dirichlet boundary condition is a boundary condition of PDE $\Delta y + y = 0$ *such that*

$y(x) = f(x),\ \forall x \in \partial\Omega$ *where* $\Omega \subset \mathbb{R}^n$

where f *is a known function defined on a boundary* $\partial\Omega$*.*

Definition 8.3.4 *(M-Theory)*

If we apply the 5*-dimensional Einstein equation, our universe is* 11*-dimensional instead of* 10*-dimensional.*
Mathematically, there are many evidences the our universe might be 5*-dimensional. AdS-CFT correspondence.*

Definition 8.3.5 *(Holographic Principle)*

Our Universe is actually encoded into holography, two-dimensional information space.

9 General Relativity

Definition 9.0.1 *(General Relativity)*

$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$

where Λ *is the gravitational constant.*

Definition 9.0.2 *(Einstein Instanton)*

Remark 9.1 *(General Relativity and Gauge Theory)*

Quantizing gravity into gauge theory.

Definition 9.1.1 *(de-Sitter Space)*

Definition 9.1.2 *(Anti de-Sitter Space)*

10 Soliton and Integrable System

Definition 10.0.1 *(Lax Pair)*

Let $L(t)$ and $P(t)$ be operators, acting on a Hilbert space, satisfying

$\frac{dL}{dt} = [P, L]$

where $[P, L] = PL - LP$ be a commutator.

Example 10.1 *(KdV Equation)*

KdV equation is

$u_t = 6uu_x - u_{xxx}$

can be reformulated as a Lax equation

$L_t = [P, L]$

with $L = -\partial_x^2 + u$ and $P = -4\partial_x^3 + 6u\partial_x + 3u_x$.

In fact, L is time-dependent Schrödinger operator with potential u.

Example 10.2 *(Heisenberg Picture)*

$\frac{d}{dt}A(t) = \frac{i}{\hbar}[H, A(t)]$

Definition 10.2.1 *(Toda Lattice)*

If we let Hamiltonian ;

$H(p, q) = \Sigma_{m\in\mathbb{Z}}(\frac{p(n,t)^2}{2} + V(q(n+1, t) - q(n, t))$

and the equation of motion

- $\frac{d}{dt}p(n,t) = -\frac{\partial H(p,q)}{\partial q(n,t)} = e^{-(q(n,t)-q(n-1,t))} - e^{-(q(n+1,t)-q(n,t))}$
- $\frac{d}{dt}q(n,t) = \frac{\partial H(p,q)}{\partial p(n,t)} = p(n,t)$

and Toda potential $V(r) = e^r + r - 1$.

Integrability of Toda Lattice depends on existence of Lax pair.

Example 10.3 *(Integrability of Toda Lattice)*

If we let $a(n,t) = \frac{1}{2}e^{-(q(n,t)-q(n-1,t))} - e^{-(q(n+1,t)-q(n,t))/2}$ *and* $b(n,t) = -\frac{1}{2}p(n,t)$, *Toda lattice reads* $\dot{a}(n,t) = a(n,t)(b(n+1,t) - b(n,t))$ *and* $\dot{b}(n,t) = 2(a(n,t)^2 - a(n-1,t)^2)$.

Now we let

$L(t)f(n) = a(n,t)f(n+1) + a(n-1,t)f(n-1) + b(n,t)f(n)$

$P(t)f(n) = a(n,t)f(n+1) - a(n-1,t)f(n-1)$

where $f(n)$ *are shift operators.*

11 String Theory

String theory is another formulation of physical model other than gauge theory, that successed to describe gravity. First of all, the dimension of string theory is 10-dimensional (geometrically principal bundle), since each 6-dimensional symmetry called Calabi-Yau manifold, existing as a fiber on each point of 4-dimensional Lorenzian space. Some others say that we could alternatively assume 5-dimensional space, which means totally 11-dimensional space according to AdS/CFT correspondence. The basic idea of string theory especially the difference from gauge theory is that each particle is considered to be a 1-dimensional (open/closed) string instead of a point, and the boundaries of the string are located in D-brane if it's open string. The D-brane is a Lagrangian submanifold of the Calabi-Yau manifold. The string as a whole is a subset of 10-dimensional total space. String theory has five different types IIA, IIB, I, Superstring, and Heterotic, but we aren't sure yet if there is the unified theory of five string theories, and if exists any, we call it M-theory.

Anyway, mathematical approach of string theory is Fukaya category to study intersection theory of D-branes.

Definition 11.0.1 *(D-Brane)*

Etymologically, D-brane stands for Dirichlet boundary condition for "D" and brane for membrane. D-brane is n-dimensional space.

An open string has two boundaries, and the boundaries connect to the D-brane.

Definition 11.0.2 *(Dirichlet Bondary Condition)*

Definition 11.0.3 *(Fukaya Category)*

Fukaya category is A_∞ category. Where each object is Lagrangian submanifold of the fixed symplectic geometry M, and the morphisms are Floer chain groups.

Definition 11.0.4 *(Homological Mirror Symmetry)*

The physical background is that A-model and B-model resp the physical model, typically both string theory, or A-model is string theory and B-model is Landau-Ginzburg model. Mathematically, a correspondence of Hodge structure (or equivalent) in both models.

11.1 SYZ Conjecture

Definition 11.1.1 *(SYZ Conjecture)*

11.2 Wall Crossing

Donaldson-Thomas invariant describes wall-crossing formulas.

Definition 11.2.1 *Donaldson Thomas Theory*

Definition 11.2.2 *Wall Crossing*
Wall crossing studies the index or space of BPS state.

12 Classical Mechanics

12.1 Navier Stokes Equation

Definition 12.1.1 *(Navier-Stokes Equation)*

Generalization formula of fluid dynamics. Its calculation contains velocity of flow, mass, viscosity, and compression. The general solution for 2.5 dimension has been given, using Ladyzhenskaya's inequality, but not 3 dimension yet. 3-dimensional NS equation is millenium prize problem by CMI.

- $\frac{\partial \rho}{\partial t} + div(\rho v) = 0$
- $\frac{\partial(\rho v)}{\partial t} + div(\rho vv) = div\sigma + \rho g$

where ρ is density, σ is stress, v is a velocity, g is a gracvity.

$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{1}{\rho}grad(p) + \frac{\mu}{\rho}\delta v + \frac{\chi + \frac{1}{3}\mu}{\rho}grad(\Theta) + \frac{1}{\rho}grad(v \cdot grad(\mu)) + \frac{1}{\rho}rot(v \times grad(\mu)) - \frac{1}{\rho}v\Delta\mu + g$

where $\Theta = div(v) = \frac{1}{2}tr(e)$, χ is volume viscosity, σ is shear viscosity.

Proposition 12.2 *(AdS/CFT/NSE correspondence)*

12.3 Diffusion

Definition 12.3.1 *(Fick's Law)*

Corresponding diffusion equation derived from Fick's law is

- *(First Fick's Law)*
  $J = -D(x)\nabla n(x,t)$
- *(Second Fick's Law)*
  $\frac{\partial n(x,t)}{\partial t} = \nabla \cdot (D(x)\nabla n(x,t))$

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