Algebraic and Geometry, not Algebraic Geometry

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Contents

1 Preface

We will mainly talk pivot of algebraic geometry, namely geometry from algebraic perspective, not algebraic geometrically. It could be Poisson(Sympectic), Lie, representation theoretic perspective, whose theory is based on linear algebra, so we need to introduce what linear algebra is, and we'll start from pure vector space, the standard form in linear algebra, and not considering grading vector space or anything algebraic.

What do we mean by "understanding" mathematics? It might be starting from a finger counting, and for the case of linear algebra, the only invariant of vector space is its dimension, which is typically the natural number or infinite, that can be evaluated by trace, and another question is volume, evaluated by determinant. Does the wedge product spontaneously arise from matrix? Yes, matrix is a tensor.

The generalization of vector space is module. R-module where R is a commuative ring corresponds to a vector bundle of an algebraic variety of R.

2 Linear Algebra

Generally speaking, linear algebra is an operation of vector spaces. What is vector spaces? For one thing, the only invariant of vector spaces is dimension, and we could define direct sum decomposition by choosing basis i.e. Peter-Weyl theorem for functional analysis, or for the case of finite dimension, it might be Jordan canonincal form, so this is spontaneously eigenvalue problem.

Definition 2.0.1 *(Hilbert Space)*

An infinite dimensioanl space $\mathfrak{H}$ is called Hilbert space if it has inner product $< \cdot, \cdot >$ i.e. $< f, g >= \int f\overline{g}dx$, and $\mathfrak{H}$ is often interpreted as a functional space.
Just be careful, however, the precise definition of the vector space $\mathfrak{H}$ is actually not unique, and it could be often interpreted as $\mathfrak{H} = L^2(\mathbb{R}^3)$ for wave

function of 3-dim diffyq or $\mathfrak{H} = L^2(\mathbb{R})$ if it's one-dimensioanl, and it's often depends on context.

2.1 Linear Transformation

Consider linear transformation in tensor product. Tensor is often used to study graph theory and braiding structure,

Definition 2.1.1 *(Tensor Network)*

subsection Trace (Characteristic Polynomial)

Example 2.2 *(Trace)*

We skip the definition of trace, it's obvious. A purpose of trace is evaluate the rank of a matrix. In particular, for an $n \times n$ matrix A such that $A^2 = A$, the trace corresponds to its rank as $tr(A) = rank(A)$.

Consider the trace commutes the matrix, $tr(AB) = tr(BA)$, so the diagonalization $A = P^{-1}\Lambda P$ makes $tr(A) = tr(\Lambda)$. Since the eigenvalue of A is always 1 or 0 (solve Cayley-Hamilton from $A^2 - A = 0$), the diagonal matrix Λ will be

$$\Lambda = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Definition 2.2.1 *(Characteristic Polynomial (Determinant))*

For an $n \times n$ matrix A, its characteristic polynomial is $p_A(t) = det(tI - A)$

There is an alternative version of characteristic polynomial using trace.

Definition 2.2.2 *(Characteristic Polynomial (Trace))*

For an $n \times n$ matrix A, its characteristic polynomial is

$$p_A(t) = \Sigma_{k=0}^{n} t^{n-k}(-1)^k tr(\wedge^k A)$$

where $tr(\wedge^k A)$ is

$$tr(\wedge^k A) = \frac{1}{k!} \begin{bmatrix} tr(A) & k-1 & 0 & \cdots & 0 \\ tr(A^2) & tr(A) & k-2 & \cdots & 0 \\ \cdots & \cdots & 0 & \cdots & \cdots \\ tr(A^{k-1}) & tr(A^{k-2}) & \cdots & \cdots & 1 \\ tr(A^k) & tr(A^{k-1}) & \cdots & \cdots & tr(A) \end{bmatrix}$$

2.3 Trace (Character Theory)

Now, what is representation? Representation is a morphism, and set theoretically, a morphism consists of domain, codomain and its mapping. So the famous result is that Lie algebra representation classifies the simple Lie algebra of finite dimension using Dynkin diagram, so to classify the domain, but equally imporant is to see how the functions behave over it.

Definition 2.3.1 *(Representation)*

A group representation (ϕ, V) is a group morphism $\phi : G \to GL(n, V)$, where G is a group and V is a vector space.

The purpose of representation is to reduce the abstract algebra problem to linear algebra.

Definition 2.3.2 *(Character)*

For a representation ϕ, a character $\chi(X) = tr(\phi(X))$ is

Example 2.4 *(Representation of Finite Group)*

Mashcke's theorem.

Representation of $\mathbb{Z}_2$ makes super graded vector space.

Representation of $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 2.5 *(Modular Representation)*

If the vector space V of $GL(V)$ is positive characteristic $char(V) = p$.

Example 2.6 *(Unitary Representation)*

A unitary representation of a group G on a Hilbert space H is a map

$\pi : G \to U(H)$

where $U(H)$ is a unitary group of H such that $\pi(g)$ is a unitary operator. This means, $\pi(g) : H \to H$ is a map such that $UU^* = U^*U = Id$

From here, we will introduce some trace describing some math/physics invariant.

Example 2.7 *(Parition Function (Trace))*

Parititition function in quantum mechanical discrete system is

$Z = tr(e^{-\beta \hat{H}})$

Example 2.8 *(Curvature (Trace))*

Let F be strength field tensor. Then the trace $Tr(F^2) = Tr(F \wedge \star F)$ is the curvature, where $\star$ is Hodge star.

Example 2.9 *(Equivariant map)*

Let $S : V_\delta \to V_\delta$ be a representation.

$\int_{SO(3)} \chi_\delta(g^{-1})\delta(g)vdg$

2.10 Wedge Product

Definition 2.10.1 *(Wedge Product)*

Definition 2.10.2 *(Volume Form)*

From geometric perspective, a vector is a one-dimensional arrow.
If we use wedge product,
In wedge product,

Definition 2.10.3 *(Determinant)*

For a matrix $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$
Determiant is given by wedge product

$v_1 \wedge v_2 \wedge \cdots \wedge v_n = det(v_1, v_2, \cdots, v_n)e_1 \wedge e_2 \wedge \cdots \wedge e_n$

where e_i is the standard basis.

Definition 2.10.4 *(Super Manifold)*

A supermanifold M of dimension $p|q$ is a sheaf of topological space M with a sheaf of superalgebras. Usually denoted $\mathscr{O}_M$ locally isormophic to $C^\infty(\mathbb{R}^p) \otimes \bigwedge^{\cdot}(\xi_1, \cdots, \xi_q)$. So far the coordinate is $(x_1, \cdots, x_p, \xi_1, \cdots, \xi_q)$ and the dimension is $n = p + q$. (Indeed, $\mathbb{R}^{n|m}$ is a super vector space (or supergraded) $\mathbb{R}^{n|m} = V_0 \oplus V_1$).

Definition 2.10.5 *(Odd/Even Symplectic Form)*

This is an analogy of symplectic form in a supermanifold.

Odd symplectic form is $\omega = \Sigma_i dx_i \wedge d\xi_i$

Even symplectic form is $\omega = \Sigma_i dp_i \wedge dq_i + \Sigma_j \frac{\epsilon_j}{2}(d\xi_j)^2$

where x_i*,* p_i*, and* q_i *are even coordinate and* ξ_j *is an odd coordinate, and* $\epsilon_j = \pm 1$ *is parity.*

Definition 2.10.6 *(Boson/Fermion)*

Boson = integer spin Fermion = half-integer spin Hadron = classifcation of particle in physics

3 Lie Group

We will introduce several basic real Lie group and its representation, because of studying harmonic properties of diffyq arising from physics problems. First of all, we'll briefly define Lie group representation and its properties, and let's think why this definition is important. We are interested in representation theory because we'll reduce complexity of math problem from group theory to linear algebra, because trace is an important invariant in representation, called character, in particular, also called Killing form.

Now, what kind of Lie groups we are intersted in. SO(3) is double cover of SU(2). Pauli metrices quotient by i generate $\mathfrak{su}(2)$. The Lie algebra of Lorentz group is equivalent to twice of su(2). Also, SU(2) is symplectic. So many lie group might be rooted in $\mathfrak{su}(2)$.

Definition 3.0.1 *(Lie Group Representation)*

The formal defintion of Lie group representation is as follows: For G *is a Lie group and* V *is a* k*-vector space, Lie group representation is a morphism*

$\Pi : G \to GL(V)$

Definition 3.0.2 *(Unitary Representation)*

In particular, for the vector space is a Hilbert space $V = \mathfrak{H}$*, and* π *is a swew-adjoint operator,* $\pi : G \to U(\mathfrak{H})$ *is unitary representation if* $\forall g \in G$*,* $\pi(g)$ *is unitary, that is* $\pi(g)^* = \pi(g)^{-1}$*, or* $< \pi(g)v, w >=< v, \pi(g^{-1})w >$*. For* *example, by using Haar intgral, we could define inner product as*

$< v, w >= \int_G < \pi(x)v, \pi(x)w >_1 dx$

This is indeed unitary because

$$\begin{aligned}
< \pi(g)v, \pi(g)w > &= \int_G < \pi(x)\pi(g)v, \pi(x)\pi(g)w >_1 dx \\
&= \int_G < \pi(xg)v, \pi(xg)w >_1 dx \\
&= \int_G < \pi(y)v, \pi(y)w >_1 dy \\
&=< v, w >
\end{aligned} \tag{1}$$

Strongly continuous/smooth/analytic unitary representation is

$g \mapsto \pi(g)\xi$ *is a norm continuous/smooth/analytic function.*

Definition 3.0.3 *(Haar Measure)*

- *(Pontryagin duality)*
  For a group G and a circle group T, a set of continuous functions is denoted by hat
  $\hat{G} = Hom(G, T)$
  and the double hat is dual as $\hat{\hat{G}} \cong G$

- *(Peter-Weyl I)*
  The set of matrix coefficients of G is dense in the space of continuous complex functions $C(G)$ on G, equipped with the uniform norm.

- *(Peter-Weyl II)*
  Let ρ be a unitary representation of a compact group G on a complex Hilbert space H. Then H splits into an orthogonal direct sum of irreducible finite-dimensional unitary representations of G.

- *(Haar measure)*
  Haar measure is a measure of a topological group G, and let G be a locally compact Hausdorff group, and we have a σ algebra generated by all open subsets of G. Let's determine the structure of Haar measure μ.

  - *(Left/Right Translation)*
    $\mu(S) = \mu(gS) = \mu(Sg)$ *for $g \in G$ and $S \subset G$.*

- *(Compact)*
  $\mu(K) < \infty$ *for all* $K \subset G$ *compact.*
- *(Outer Regular)*
  $\mu(S) = inf\{\mu(U) : S \subset U, U \text{ is open}\}$
- *(Inner Regular)*
  $\mu(U) = sup\{\mu(K) : K \subset U, K \text{ is compact}\}$

Then, if G *is compact, there is a unique Haar measure with* $\mu(G) = 1$ *and* $\mu(U) > 0$ *for all* U*.*

- *(Spectrum of* C^**-algebra)*

  C^**-algebra is a Banach algebra over a field of complex numbers, together with a map* $x \mapsto x^*$ *for* $x \in A$*.*

Definition 3.0.4 *(Schur's Lemma)*

For a representation $\pi : G \to GL(V)$*,*

π *is irreducible implies* $End_G(V) = \mathbb{C}Id_V$

If π *is unitarizable, and* $End_G(V) = \mathbb{C}Id_V$*, then* π *is irreducible.*

Definition 3.0.5 *(Measure)*

$\int_G dx = 1$

$< f, g >= \int_G f(x)\overline{g(x)}dx,\ f, g \in L^2(G)$

Definition 3.0.6 *(Schur Orthogonality)*

The matrix coefficient of π *as a map* $m : G \to \mathbb{C}$ *is*

$m(g) = m_{v,w}(g) =< \pi(g)v, w >$

or alternatively this matrix is

$m_{v,w}(g) = Tr(\pi(x)L_{v,w})$ *where* $L_{v,w}(u) =< u, w > v$

$C(G)_\pi$ *is a linear span of the space of matrix coefficients* $m : G \to \mathbb{C}$*.*

Definition 3.0.7 *(Character)*

A character $\chi_\pi : G \to \mathbb{C}$ *is a morphism, and trace of a representation* $\chi_\pi(g) = Tr(\pi(g))$*.*

From orthogonality,

$\chi_\pi(x) = \Sigma < \pi(x)e_i, e_i >= \Sigma m_{e_i,e_i}(x)$

Also, if $\pi \sim \pi'$*, the inner product is*

$< \chi_\pi, \chi_{\pi'} >_{L^2} = 1$

or if $\pi \not\sim \pi'$,

$< \chi_\pi, \chi_{\pi'} >_{L^2} = 0$

Definition 3.0.8 *(Peter-Weyl)*

$L^2(G) = \hat{\bigoplus}_{\delta \in \hat{G}} C(G)_\delta$

3.1 U(1) (Electro-Magnetic)

Definition 3.1.1 *(U(1) representation)*

On $\mathbb{C}^n$*, We take Hermitian inner product*

$< u, v >_{inv} = \int_0^{2\pi} < R(e^{i\theta})u, R(e^{i\theta})v > \frac{d\theta}{2\pi}$

which we prove is invariant and $< R(e^{i\theta})u, R(e^{i\theta})v >_{inv} = < u, v >_{inv}$ *for all* $u, v \in \mathbb{C}^n$ *and* $e^{i\theta} \in U(1)$ *. Indeed,* $< u, v >_{inv}$ *is an Hermitian inner product.*

3.2 SO(3) (Spherical Harmonic)

Definition 3.2.1 *(SO(3))*
SO(3) is rotation.

$O(3) = \{A \in M_{3,3} | AA^T = I\}$

$SO(3) = \{A \in M_{3,3} | AA^T = I, det(A) = 1\}$

Definition 3.2.2 *(Harmonics Analysis)*

Spherical harmonics is a solution of Laplacian equation, and for example, Schrödinger equation is a diffyq of a Hamiltonian operator, which contains Laplacian.
$\hat{H}\psi = E\psi$ *where* $\hat{H} = \nabla + V$.

Definition 3.2.3 *(Spherical Harmonic Functions)*

Harmonic funciton is $f : \mathbb{R}^n \to \mathbb{C}$ harmonic iff $f \in C^\infty(\mathbb{R}^n)$ and $\nabla f = 0$. If $n = 3$,

$$\mathfrak{h}_l = \{p \in P_l(\mathbb{R}^3 | \nabla p = 0\}$$

where P_l is the space of homogeneous harmonic polynomials in $P_l(\mathbb{R}^3)$.

Now we define $SO(3)$-module $\mathfrak{h}_l$ as the representation $\rho_l : SO(3) \to GL(\mathfrak{h}_l)$.

Definition 3.2.4 *(Spherical Harmonics)*

$$Y_l^m = (-1)^m (\tfrac{(2l+1)(l+m)!}{4\pi(l-m)!})^{\frac{1}{2}} e^{im\phi} P_l^m(cos\theta)$$

where

$P_l^m(s) = \frac{(1-s^2)^{-\frac{m}{2}}}{2^l l!} \frac{d^{l-m}}{ds^{l-m}}(s^2 - 1)^l$, $|m| \leqq l$, $l \in \mathbb{N}$ constitute basis for $C(S^2)_l \subset L^2(S^2)$

3.3 Lorentz Group

As seen before, SO(3) representation for the Laplacian equation as Schrödiger equation was time-independent system, while time this the Lorentz group representation is for solving diffyq of time-dependent system e.g. Klein-Gordon equation. One of the difficulty is that Lorentz group is not compact, hence the representation might not be unitary.

Lorentz group is a Lie group structure of Minskowski space. The problem is Minkowski space is not symmetric, or it's not a metric space, since it contains imaginary number i. Conversely, we consider metrization starting from inner product (so metrization is symmetrization right?).

Definition 3.3.1 *(Indefinite Orthogonal Group)*

Indefinite orthogonal group is denoted as $O(p, q)$. Let $g \in M_{p+q}$ be a diagonal matrix

$g = diag(1, \cdots, 1, -1, \cdots - 1)$ of p of 1 and q of -1.

We definte symmetric bilinear form

$$[x, y]_{p,q} =< x, gy >= x_1y_1 + \cdots x_py_p - x_{p+1}y_{p+1} - \cdots + x_{p+q}y_{p+q}$$

where $< \bullet, \bullet >$ is an inner product on $\mathbb{R}^{p+q}$.

$$O(p, q) = \{A \in M_{p+q}(\mathbb{R}) | [Ax, Ay]_{p,q} = [x, y]_{p,q}, \forall p, q \in \mathbb{R}^{p+q}\}$$

Or more explicitly,

$gA^Tg = A^{-1}$.

Definition 3.3.2 *(Lorentz Group)*

Lorentz Group is an indefinite orthogonal group $O(1,3)$ preserving the quadratic form

$(t,x,y,z) \mapsto t^2 - x^2 - y^2 - z^2$

or let's consider $SO(1,3)$. This transformation is, in other words, an element $\Lambda \in SO(1,3)$ acts on $\mathbb{R}^{1,3}$ and

$x^\rho \mapsto x'^\rho = \Lambda^\rho_\mu x^\mu$

note Minskowski norm $\Lambda^T g \Lambda = g$

Definition 3.3.3 *(Lorentz Lie Algebra)*

J_i where $i = 1,2,3$ are generators of $SO(3)$, and we let K_i where $i = 1,2,3$ be

$$K_x = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; K_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; K_z = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

and

- $[J_i, J_j] = i\epsilon_{ijk}J_k$
- $[J_i, K_j] = i\epsilon_{ijk}K_k$
- $[K_i, K_j] = -i\epsilon_{ijk}J_k$

Definition 3.3.4 *($\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$)*

We'll claim $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$, and thus the irreducible representations of $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$ are the same.

If we use the complexification $\frac{so}{(}1,3)_\mathbb{C} = \frac{so}{(}1,3) \otimes \mathbb{C}$, the Lorentz algebra works nicely. So, if we use $M_i^\pm = \frac{J_i \pm iK_i}{2}$ (so that we can reconstrct original real Lie algebra $K_i = M_i^+ - M_i^-$ and $J_i = M_i^+ + M_i^-$ can be retrieved from $M_i\pm$)

Their relation is as follows

- $[M_i^\pm, M_j^\pm] = i\epsilon_{ijk}M_k^\pm$

- $[M_i^+, M_j^-] = 0$

Thus we hve following picture

$\mathfrak{so}(1,3) \hookrightarrow \mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$

Now, further as Lie group $SO^+(1,3) \cong PSL(2,\mathbb{C}) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$*, so* $SL(2,\mathbb{C})$ *is a double cover, that is,* $SL(2,\mathbb{C}) \cong Spin(1,3)$

$\mathfrak{so}(1,3) \hookrightarrow \mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \hookleftarrow \mathfrak{sl}(2,\mathbb{C})$

and finally $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$*. This means their irreducible reprensetation* $\mathfrak{so}(1,3)$ *and* $\mathfrak{sl}(2,\mathbb{C})$ *are the same.*

Definition 3.3.5 *(Representation of Lorentz Group For Klein-Gordon Equation)*

Let $\phi(x^\mu)$ *be a scalar field, and Loretz transformation be*

$\phi'(x^\mu) = phi(\Lambda^{-1}x^\mu)$

Now Lagrangian of Klein-Gordon equation $\partial_\mu\partial^\mu\phi + m^2\phi = 0$ *is*

$\underset{=}{L} \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$

The full group of the spacetime isometries is Poincare group.

3.4 AdS Group

Definition 3.4.1 *(de-Sitter Space)*

de-Sitter space dS_n *is maximally symmetric Lorentizian manifold with constant positive scalar curvature. It is the Lorentizian analogue of* n*-sphere.*

$ds^2 = -dx_0^2 + \Sigma_{i=1}^n dx_i^2$
dS_n *is submanifold of* $-dx_0^2 + \Sigma_{i=1}^n dx_i^2 = \alpha$.

Or by using Lie group, dS_n *is given by quotient*

$O(1,n)/O(1,n-1)$

Definition 3.4.2 *(Anti de-Sitter Space)*

Our universe is believed to be of positive cosmological constant, but we'll consider Anti de-Sitter space AdS_n*, which is negative curvature version of* dS_n*,* *and the cosmological constant Λ is negative.*

Or by using Lie group, AdS_n is given by quotient

$$AdS_n = O(2, n-1)/O(1, n-1)$$

Definition 3.4.3 *(Anti de-Sitter Algebra)*

We will constrct $\mathfrak{o}(2,n)$. Consider $\mathfrak{o}(1,n) \subset \mathfrak{o}(2,n)$ is a subalgebra, and explicitly, it is generated by

$$\mathfrak{H} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \leftarrow & v^t & \rightarrow \\ \cdots & \uparrow & & & \\ 0 & v & & B & \\ \cdots & \downarrow & & & \end{bmatrix}$$

where B is a skew-symemtry matrix. The complementary part of the generators are explicitly

$$\mathfrak{Q} = \begin{bmatrix} 0 & a & \leftarrow & w^t & \rightarrow \\ -a & 0 & \cdots & 0 & \cdots \\ \uparrow & \cdots & & & \\ w & 0 & & 0 & \\ \downarrow & \cdots & & & \end{bmatrix}$$

and thus generators of $\mathfrak{o}(2,n)$ are $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{Q}$. The Lie bracket structures are

- $[\mathfrak{H}, \mathfrak{Q}] \subset \mathfrak{Q}$
- $[\mathfrak{Q}, \mathfrak{Q}] \subset \mathfrak{H}$

Thus, anti de-Sitter space is reductive homogeneous space, and a non-Riemannian symmetric space.

Definition 3.4.4 *($\mathfrak{o}(3,2)$ Case)*

We consider the case of $n = 4$, then the group is $\mathfrak{G} = \mathfrak{so}(3,2)$, which is 10-dimensional. We will study the sturucture by complxification, actually identified as $\mathfrak{G}^{\mathbb{C}} = \mathfrak{so}(5,\mathbb{C})$, and we can use the same basis for both. we have standard triangular decomposition

$$\mathfrak{G}^{\mathbb{C}} = \mathfrak{G}_- \oplus \mathfrak{H} \oplus \mathfrak{G}_+$$

where $\mathfrak{H}$ is the diagonal two-dimensional Cartan subalgebra generated by H_1 and H_2, and $\mathfrak{G}_+$ and $\mathfrak{G}_-$ are generated by X_i^+ and X_i^- where $i = 1, 2, 3, 4$ resp. These matrices are explicitly given by

$$H_1 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} \; H_2 = \begin{bmatrix} e_2 & 0 \\ 0 & -e_1 \end{bmatrix}$$

$$X_1^+ = \begin{bmatrix} \sigma_+ & 0 \\ 0 & -\sigma_+ \end{bmatrix} \; X_1^- = \begin{bmatrix} \sigma_- & 0 \\ 0 & -\sigma_- \end{bmatrix}$$

$$X_2^+ = \begin{bmatrix} 0 & \sigma_- \\ 0 & 0 \end{bmatrix} \; X_2^- = \begin{bmatrix} 0 & 0 \\ \sigma_+ & 0 \end{bmatrix}$$

$$X_3^+ = \begin{bmatrix} 0 & 1_2 \\ 0 & 0 - \sigma_+ \end{bmatrix} \; X_3^- = \begin{bmatrix} 1_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X_4^+ = \begin{bmatrix} 0 & \sigma_+ \\ 0 & 0 \end{bmatrix} \; X_4^- = \begin{bmatrix} 0 & 0 \\ \sigma_- & 0 \end{bmatrix}$$

$$e_1 = \frac{1}{2}(1 + \sigma_3) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e_2 = \frac{1}{2}(1 - \sigma_3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\sigma_- =^t \sigma_+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

where σ_i are 2×2 Pauli matrices. Using this basis, the AdS algebra $\mathfrak{G}$ has the following Bruhat decomposition

$$\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{M} \oplus \mathfrak{A} \oplus \mathfrak{N}_+$$

in which the four subalgebras have physical meaning related to the fact that $\mathfrak{G}$ is also the conformal algebra of the 3-dimensional Minkowski space-time M^3.

Definition 3.4.5 *(Verma Module)*

Verma module is for classification of finite dimensional representation of $\mathfrak{g}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ is CSA, and R is the corresponding root system. λ is the highest weight and v is the corresponding vector.

$W_\lambda = \{Y_{\alpha_{i_1}} Y_{\alpha_{i_2}} \cdots Y_{\alpha_{i_M}} \cdot v\}$

Definition 3.4.6 *(Representation of AdS algebra)*

Using quantum numbers E_0 and s_0 called energy and spin, we characterize the explicit realization of positive energy UIR $D(E_0, s_0)$

$$(\partial_z)^{m_1} \hat{\phi} = 0$$

For example, if $E_0 = \frac{1}{2}$ and $s_0 = 0$. These are equations

$\partial_z \hat{\phi} = 0$ *where* $m_1 = 1$ $(\partial_x^2 - \partial_u \partial_v)\hat{\phi} = (\partial_0^2 - \partial_1^2 - \partial_2^2)\hat{\phi}\Box\hat{\phi}\hat{\phi}'$ *where* $m_3 = 2$

3.5 Supersymmetry Algebra(BPS State)

BPS state is a condition of a supersymmetry algebra used for supersymmetry physics, and it is an invariant which preserves mass in physics. Like Schrödinger with SO(3), consider Hilbert space $\mathfrak{H}$ and its self-adjoint operator $H : \mathfrak{H} \to \mathfrak{H}$.

Definition 3.5.1 *(BPS State)*

Let M be the mass of the state, and Z is the linear combination of the central charges, and the inequality $M \leqq |Z|$ is called BPS bound. The supersymmetry algebra is caleld BPS state if $M = |Z|$.

Definition 3.5.2 *($N = 2$ supersymmetry in dimension $d = 1$)*

We define Lie superalgebra A by

$A = A^0 \oplus A^1$ *where*

$A^0 = \mathbb{C} \cdot H$

$A^1 = \mathbb{C} \cdot Q \oplus \mathbb{C} \cdot \overline{Q}$

- $[Q, \overline{Q}] = 2H$
- $[Q, Q] = 0$
- $[\overline{Q}, \overline{Q}] = 0$
- $[Q, H] = 0$
- $[\overline{Q}, H] = 0$

$\mathbb{Z}/2\mathbb{Z}$-graded representation of A means $A^i : H^j \to H^{j+i}$, and if the representation is unitary, H is formally self adjoint operator, and Q and $\overline{Q}$ are formally self adjoint with one another.

For example, the representation is unitary if we take $Q = d$, $\overline{Q} = d^*$, and $H = \frac{1}{2}\Delta$ where

$\mathfrak{H} = \mathfrak{H}^0 \oplus \mathfrak{H}^1$ *where* $\mathfrak{H}^0 = \oplus_k \Omega^{2k}_{L^2}(M)$ *and* $\mathfrak{H}^1 = \oplus_k \Omega^{2k+1}_{L^2}(M)$.

$2 < \psi, H\psi > = < \psi, Q\overline{Q}\psi > + < \psi, \overline{Q}Q\psi > = ||Q\psi||^2 + ||\overline{Q}\psi||^2 \leqq 0$, *and the norm is nondegenerate, so $H\psi = 0$ iff $Q\psi = 0$ and $\overline{Q}\psi = 0$.*

Definition 3.5.3 *($N = (2, 2)$ supersymmetry in dimension $d = 2$)*

4 odd generators $Q^{\pm}$ *and* $\overline{Q}^{\pm}$

and 6 *even generators* P^0*,* P^1*,* B*,* Z*,* $\overline{Z}$*,* F

where B *is generator of* $\mathfrak{so}(1,1)$ *and* Z *and* $\overline{Z}$ *are central generators. Their algebraic relation are defined as*

- $[Q^+, \overline{Q}^+] = P^+$
- $[Q^-, \overline{Q}^-] = P^-$
- $[Q^+, Q^+] = Z$
- $[Q^-, Q^-] = \overline{Z}$

where we defined $P^{\pm} = P^0 \pm P^1 = H \pm P^1$*.*

The generator F *as Fermion number obeys*

- $[F, Q^{\pm}] = -Q^{\pm}$
- $[F, \overline{Q}^{\pm}] = -\overline{Q}^{\pm}$

The application of $N = 2$ and $d = 4$ supersymmetry is supersymmetric blackhole entropies.

Definition 3.5.4 *($N = 2$ and $d = 4$ supersymmetry)*

For $N = 2$ *and* $d = 4$ *superalgebra, the odd part of generators* Q *have relations as*

$$\{Q^A_\alpha, \overline{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B$$

$$\{Q^A_\alpha, Q^B_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\overline{Z}$$

$$\{Q_{\dot{\alpha}A}, Q_{\dot{\beta}B}\} = 2\epsilon^{\alpha\beta}\epsilon_{AB} Z$$

where α *and* $\dot{\beta}$ *are Lorentz indices, and* A *and* B *are R-symmetry indices.*

Definition 3.5.5 *(R-matrices)*

$$R^A_\alpha = \xi^{-1} Q^A_\alpha + \xi \sigma^0_{\alpha\dot{\beta}} \overline{Q}^{\dot{\beta}B}$$

$$T^A_\alpha = \xi^{-1} Q^A_\alpha - \xi \sigma^0_{\alpha\dot{\beta}} \overline{Q}^{\dot{\beta}B}$$

Consider a state ψ which has a momentum $(M,0,0,0)$, and if we apply the following operator to the state

$$(R_1^1 + (R_1^1)^\dagger)^2\psi = 4(M + Re(Z\xi^2))\psi$$

Definition 3.5.6 *($N = 1$ supersymmetry)*

Definition 3.5.7 *(Isometry Group (ISO(p,q)))*

A isometry group of a metric space is the set of all bijective isometries (e.g. distance-preserving maps).

Definition 3.5.8 *(Poincare Group)*

Poincare group is Lie group extension of Lorentz group $O(1,3)$, and it contains "translation", "rotation", and "boosts". That is,

$$\mathbb{1},\mathbb{3} \rtimes O(1,3)$$

with group multiplications

$$(\alpha, f)\cdot(\beta, g) \mapsto (\alpha + f\cdot\beta, f\cdot g)$$

Definition 3.5.9 *(Poincare Algebra)*

Continuing from the previous definition, let P be the translation, and M be the generators of the Lorentz transformations, η is Minkowski metric $(+,-,-,-)$, then the bracket will be

- $[P_\mu, P_\nu] = 0$
- $\frac{1}{i}[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu$
- $\frac{1}{i}[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}$

Definition 3.5.10 *(Dirac Matrices (Gamma Matrices))*

Dirac matrices (or gamma matrices) $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ are 4 matrices existing in 4 dimensional space as

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Let's describe its mathematicall property. If we use anti-commutator $\{\}$, the matrices describe Minkowski metric $\eta^{\mu\nu}$.

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I_4$$

These matrices are called Dirac matrices because it's used to describe Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Also, the relativitstic spin matrices $\sigma^{\mu\nu}$ is given by Dirac matrices as

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Definition 3.5.11 *(Supercharge)*

Supercharge Q is an operator that transform Boson to Fermion and vice versa, so this operator itself is a spin. It's described by super Poincare algebra ,and it commutes with Hamlitonian as

$$[Q, H] = 0$$

The index Q_α is $\alpha = 1, 2, \cdots, N$ where the number of $N \in \mathbb{N}$ depends.

Definition 3.5.12 *(Super Poincare Algebra)*

$$\{Q_\alpha, \overline{Q_{\dot\beta}}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu$$

Q_α and $\overline{Q_{\dot\alpha}}$ are called super charges. σ^μ are Pauli matrices and P^μ are generators of translation. The dot $\dot\beta$ is to remind that this index transforms according to the inequivalent conjugate spinor representation.

Super Poincare algebra is a Poincare algebra in addition to the following structures

- $[M^{\mu\nu}, Q_\alpha] = \frac{1}{2}(\sigma^{\mu\nu})_\alpha^\beta Q_\beta$
- $[Q_\alpha, P^\mu] = 0$
- $\{Q_\alpha, \overline{Q_{\dot\beta}}\} = 2(\sigma)^\mu_{\alpha\dot\beta} P_\mu$

These condition will lead us to supergravity.

Definition 3.5.13 *(Representation of Poincare Group)*

3.6 Spinor (Clifford)

So, what is spin? $SU(2)$ is a double cover of $SO(3)$, and describes spin of $SO(3)$. In fact as Lie algebra, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ is isomorphic, means there is locally same property, so we need to think of it as Lie group to study global property.

Definition 3.6.1 *($SU(2)$ and $SO(3)$)*

$SO(3)$ is generated by M_θ and M_ϕ, where

$$M_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \}$$
$$M_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix} \}$$

and $0 \leqq \theta < 2\pi$ and $0 \leqq \phi < \pi$. Note that the domain of ψ is only π, but if we expand to 2π, it will be $SU(2)$. The group structure of $SU(2)$ is genearted by M_θ and M_ϕ where

$$M_\phi = \begin{bmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{bmatrix}$$
$$M_\theta = \begin{bmatrix} cos(\frac{\theta}{2}) & isin(\frac{\theta}{2}) \\ isin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{bmatrix}$$

and $0 \leqq \theta < 2\pi$ and $0 \leqq \phi < 2\pi$. So indeed, it's a double cover of $SO(3)$.

Definition 3.6.2 *(Clifford Algebra)*

Let a vector space V and scalar field K, typically $\mathbb{R}$ or $\mathbb{C}$, and the quadratic form $Q : V \to K$. For all $v \in V$,

$v^2 = Q(v) \cdot 1$, or if we take $uv + vu = d(u, v)$ where $d(x, y)$ is a bilinear form and $u, v \in V$, then all the elements $v \in V$ generates $Cl(V)$ as

$$Cl(V) = Cl(V, d) = T(V)/(v \otimes v - Q(v))$$

Proposition 3.7 *(Grading Structure)*

$Cl(V)$ is naturally graded and given as

$$Cl(V) = Cl^0(V) \oplus Cl^1(V) \oplus Cl^2(V) \oplus \cdots \oplus Cl^n(V) \text{ where } n = dim(V)$$

$$Cl^0(V) = \mathbb{R}$$

$Cl^1(V) = V$

$Cl^2(V) = \mathfrak{spin}(n)$

In particular, $Cl^2(V)$ is called a spin algebra.

Definition 3.7.1 *(Spinor Algebra)*

The spin algebra $Cl^2(V) = \mathfrak{spin}(V)$ is, in fact, a Lie algebra, whose dimension is $n(n-1)$. It naturally has a short exact sequence as Lie algebra

$$0 \to \mu_2 \to \mathfrak{spin}(n) \to \mathfrak{so}(n) \to 0$$

Thise sequence as Lie algebra doesn't split. in fact $dim(Spin(n) = SO(n)) = n(n-1)$ is the same dimension. Also, μ_2 is an algebraic group of roots of unity.

Definition 3.7.2 *(Spinor Group)*
Geometrically, spinor as a group $Spin(V)$ is defined by $Spin(V) = Pin(V) \cap Cl^{even}$.

$Spin(V)$ is double cover of $SO(V)$, and it's simply connected, so it's universal cover of $SO(V)$.

Pin group $Pin(V) \subset Cl(V,Q)$ is a subset of $Cl(V,Q)$ where Q is non-degenerate, and consisting of the elements of the form $v_1 v_2 \cdots v_k$ where v_i is $Q(v_i) \pm 1$.

Proposition 3.8 *(Pauli Matrices)*

There are 3 Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \; \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

whose multiplication is

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

where $\epsilon_{ijk} = +1$*if (i,j,k) = (1,2,3),(2,3,1),(3,1,2)*-1*if (i,j,k) = (3,2,1),(1,3,2),(2,1,3)*0*if* $i = j$ *or* $j = k$ *or*

is 3-dimensional Levi-Civita symbol.

Also, imaginary times of Pauli matrices generate $\mathfrak{su}(2)$. Its basis is explicitly, $(i\sigma_x, i\sigma_y, i\sigma_z)$.

Also, Pauli matrices are involution as $\sigma_i^2 = \sigma_j^2 = \sigma_k^2 = I$

Also, $det(\sigma_i) = -1$, *and* $tr(\sigma_i) = 0$

Definition 3.8.1 *(Whitehead Tower)*

According to Postnikov tower construction, there is an exact sequnce

$Fivebrane(n) \to String(n) \to Spin(n) \to SO(n) \to O(n) \to 0$
Note that $Fivebrane(n)$ *and* $String(n)$ *are not necessarily Lie group.*

3.9 Heisenberg Group

Definition 3.9.1 *(Heisenberg Group)*

Heisenberg group H *is a 3x3 matrix group whose elements are*

$$\begin{bmatrix} 1 & a & c \\ 1 & 1 & b \\ 1 & 1 & 1 \end{bmatrix}$$

where $a, b, c \in R$ *where* R *is any commutative ring. Heisenberg group is often used in one-dimensional quantum mechanics by context of Stone-von Neumann theorem.*

More abstractly, Heisenberg group is created by central extension

$0 \to (\mathbb{R}, +) \to H \to (\mathbb{R}, +)$

Definition 3.9.2 *(Heisenberg Algebra)*

$$\begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$$

whose basis is X *and* Y *and* Z *and*

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \; Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $[X, Y] = Z$, $[X, Z] = 0$, *and* $[Y, Y] = 0$. *Consider* $X = \hat{x}$ *and* $Y = \hat{p}$ *and* $Z = i\hbar I$,

$[\hat{x}, \hat{p}] = i\hbar I$, $[\hat{x}, i\hbar I] = 0$, *and* $[\hat{p}, i\hbar I] = 0$.

Definition 3.9.3 *(Stone-von Neumann Theorem)*

Any unitary irresucible unitary representation can be transformed to a Schrödinger representation.

For any irreducible unitary representation π of H on a Hilbert space $\mathfrak{H}$, there is a unitary operator $\mathfrak{H} \to L^2(\mathbb{R})$ such that

$$U\pi U^{-1} = \pi_S$$

where π_S is a Schrödinger representation.

We consider reprensetation and how representation applies to integral.

Definition 3.9.4 *(Fock Space)*

Consider a Hilbert space with inner product as

$$< f(w), g(w) >= \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(w)} g(w) e^{-|w|^2}$$

Then, $< w^m, w^n >= n! \delta_{m,n}$

So $\frac{w^n}{\sqrt{n!}}$ *are orthonormal.*

Consider $a = \frac{d}{dw}$ *and* $a^\dagger = w$.

Definition 3.9.5 *(Schrödinger Representation)*
For Schrödinger representation $\pi_S : H \to L^2(\mathbb{R})$ of Heisenberg Lie algebra, explicitly given by

$$\pi_S(x, y, z)\psi(q) = e^{-iz} e^{i\frac{1}{2}xy} e^{-ixq} \psi(q - y)$$

$$\pi_S(x, y, z)\pi_S(x', y', z') = \pi_S(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)$$

- $\pi'_S(iX) = Q$
- $\pi'_S(iY) = P$
- $\pi'_S(iZ) = 1$

and so

$$\pi'_S(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$$

Definition 3.9.6 *(Bargmann-Fock Representation)*

Bargmann-Fock representation $\pi_{BF} : H \to \mathfrak{H}_F$ as

- $\pi'_{BF}(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{d}{dw}$

- $\pi'_{BF}(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{d}{dw}$

- $\pi'_{BF}(\frac{1}{\sqrt{2}}(iX - i(iY))) = a^\dagger = w$

- $\pi'_{BF}(iZ) = 1$

Now by Stone-von Neumann, we have $U : \mathfrak{H}_F \to L^2(\mathbb{R})$ *called Bargmann transform, and it's given by*

$(U^{-1}\psi)w = (\frac{1}{\pi})^{\frac{1}{4}} e^{-\frac{1}{2}w^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}q^2} e^{\sqrt{2}wq} \psi(q) dq$

Thus

$U \frac{d}{dw} U^{-1} = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$

/

$UwU^{-1} = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$

Definition 3.9.7 *(Heisenberg Group and Symplectic Geometry)*

Consider higher dimensional analogue of Heisenerg group with basis X_j, Y_j, *and* Z *where* $j = 1, 2, \cdots, n$, *and all Lie brackets are zero except*

$[X_i, Y_j] = \delta_{ij} Z$

Symmetric form S *on a vector space* V *is non-degenerate anti-symmetric bilinear form*

$V \times V \ni (v_1, v_2) \mapsto S(v_1, v_2) \in \mathbb{R}$

which is symmetric version of inner product.

Consider algebra $V \oplus \mathbb{R}$ *whose bracket is*

$[(v, z), (v', z')] = (0, S(v, v'))$

and one gets a corresponding Lie group

$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}S(v, v'))$

For Symplectic basis X_j *and* Y_j, $S(X_j, X_k) = S(Y_j, Y_k) = 0$ *and* $S(X_j, Y_k) = \delta_{jk}$

3.10 CCR and CAR

Let V be a real vector space with antisymmetric bilinear form $(\cdot,\cdot)$. The unital *-algebra is

Definition 3.10.1 *(*-algebra)*

$*$-ring is a ring with a map $* : A \to A$, an antiautomorphism and an involution. Namely,

- $(x+y)^* = x^* + y^*$
- $(xy)^* = y^* x^*$
- $1^* = 1$
- $(x^*)^* = x$

**-ring is *-algebra if further* $(rx)^* = r' x^* \forall r \in R x \in A$

$(\lambda x + \mu y)^* = \lambda' x^* + \mu' y^*$ *for* $\lambda, \mu \in R$ *and* $x, y \in A$.

Definition 3.10.2 *(C*-algebra)*

Definition 3.10.3 *(CCR and CAR as *-algebra)*

- *(CCR)*
  $fg - gf = i(f,g)$
  $f^* = f$
  for any f, g *in* V *is called canonical commutation relation (CCR) algebra.*
- *(CAR)*
  $fg - gf = i(f,g)$
  $f^* = f$
  for any f, g *in* V *is called canonical anticommutation relation (CAR) algebra.*

Definition 3.10.4 *(CCR and CAR as C*-algebra)*

Let CCR algebra over H *is the unital* C^**-algebra generated by elements* $\{W(f) : f \in H\}$ *subject to*

$W(f)W(g) = e^{-i(f,g)} W(f+g)$
$W(f) = W(-f)$

If H is complex Hilbert space, CCR algebra is faithfully represented on the symmetric Fock space over H by setting

$$W(f)(1, g, \frac{g^{\otimes 2}}{2!}, \frac{g^{\otimes 3}}{3!}, \cdots) = e^{\frac{-1}{2}||f||^2 - <f,g>} W(f)(1, f+g, \frac{(f+g)^{\otimes 2}}{2!}, \frac{(f+g)^{\otimes 3}}{3!})$$

for any $f, g \in H$. How's CAR algebra? Omit.

4 Geometry

4.1 Parallel (Curvature)

4.1.1 Holonomy

Definition 4.1.1 *(Parallel Transport)*

Definition 4.1.2 *(Holonomy)*

Example 4.2 *(Sphere)*

4.2.1 Flatness Condition

Definition 4.2.1 *(Hilbert Polynomial)*

- *(Hilbert Function)*
  Let S be a graded commutative algebra $S = \oplus_{n \geqq 0} S_n$ and $S_0 = K$, then Hilbert function is
  $HF_S : n \mapsto dim_K S_n$
- *(Hilbert Series)*

  $HS_S(t) = \Sigma_{n=0}^{\infty} HF_n(n) t^n$

  that can also be written as

  $HS_S(t) = \frac{P(t)}{(1-t)^{\delta}}$

  where P is a polynomial of integer coefficients and δ is a Krull dimension of S.
- *(Hilbert Polynomial)*

  $HP_S(n) = \frac{P(1)}{(\delta-1)!} n^{\delta-1} +$ *terms of lower degree in n*

Example 4.3 *(Hilbert Polynomial)*

A flat morphism $\pi : X \to S$ has the same Hilbert polynomial.

Definition 4.3.1 *(Hilbert Scheme)*

Hilbert Scheme is a scheme but also moduli space.
$Hilb(n)$ of $\mathbb{P}^n$ is a moduli space of closed subschemes of projective spaces.

$Hilb(n, P)$ is a moduli space of Hilbert polynomial P.

Let $\underline{Hilb}_{X/S} : (Sch/S)^{op} \to Sets$ be the functor sending a relative scheme $T \to S$ to the set of isomorphism classes of the set

$$\underline{Hilb}_{X/S}(T) = \left\{ \begin{tikzcd} Z \arrow[r, "incl"] \arrow[d] & X \times_S T \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "="] & T \arrow[r] & S \end{tikzcd} : Z \to T \text{ is flat} \right\} / \sim$$

Universality:
$H = Hilb(n, P)$ has a universal subscheme $W \subset X \times H$ flat over H such that

- *the fiber W_x over closed points $x \in H$ are closed subschemes of X. For $Y \subset X$ denote this point x as $[Y] \in H$.*

- *H is universal wrt all flat families of X having Hilbert polynomial P. That is, given a scheme T and a flat family $W' \subset X \times T$, there is a unique morphism $\phi : T \to W$ such that $\phi^* W \cong W'$.*

  Tangent Space:
  The tangent space at $[Y] \in H$ is given by the global sections of the normal bundle $N_{Y/X}$. That is,
  $T_{[Y]}H = H^0(Y, N_{Y/X})$

  Unobstrctedness of complete intersections:
  For local complete intersections Y such that $H^0(Y, N_{Y/X}) = 0$, the point $[Y] \in H$ is smooth. This implies that the deformation of Y in X is unobstructed.

  (coalgebra is smooth, if it has a lifting property. morphism $F(A) \to F(A/J)$ is means surjective.)

  Complete Intersection:
  The ideal of V is generated by precisely codim V elements.

Example 4.4 *(Hilbert Schemes)*

- *(Fano Schemes of Hypersurfaces)*

- *(Hilbert schemes of n-points)*
  X^n/S_n is the nice geometric interpretation where the boundary loci $B \subset H$ describing the intersection of points can be tought of parametrizing points along with their tagent vectors.

  $X^{[n]} = Bl_{\Delta}(X \times X)/S_2$.

  Alternatively,

  $X^{[n]} = Hilb_X^P$ if the Hilbert polynomial P is constant $P(m) = n$ for all $m \in \mathbb{Z}$

- *(Degree d hypersurfaces)*

- *(Hilbert schemes of curves and moduli of curves)*

Definition 4.4.1 *(Blow up)*

For a scheme X, the blow up $\pi : \tilde{X} \to X$ with respect to a $\mathscr{I}$, as a coherent sheaf of ideals on X.

$\tilde{X} = Proj \oplus_{n=0}^{\infty} \mathscr{I}^n$

Example 4.5 *(Blow Up)*

- *($\mathbb{P}^2$)*
  The blowing of $\mathbb{P}^2$ at a point $P \in \mathbb{P}^2$ is $X = \{(Q, l)|Q \in \mathbb{P}l \in Gr(1, 2)\} \subset \mathbb{P}^2 \times Gr(1, 2)$.
  If $Q \neq P$, choosing Q can uniquely define a line that goes through both P and Q, while if $Q = P$, there could be any lines. namely, we only modify the origin.

  More concretely, for $P = [P_0 : P_1 : P_2]$, take $l = [L_0 : L_1 : L_2]$ is the set of all $[X_0 : X_1 : X_2]$ such that $X_0L_0 + X_1L_1 + X_2L_2 = 0$. Therefore, the blow up can be described as

  $X = \{([X_0 : X_1 : X_2], [L_0 : L_1 : L_2])|P \cdot L = 0, X \cdot L = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2$.

- *($\mathbb{A}^2$)*

  $$X = Proj \oplus Sym^r_{k[x,y]}\mathfrak{m}/\mathfrak{m}^2 = Projk[x,y][z,w]/(xz - yw)$$

  where x and y are degree 0 and z and w are degree 1.

  Is this just replacing projective tangent bundle the origin? and otherwise all the same?

- *($\mathbb{C}^2$)*

  In general, blow up of $\mathbb{C}^n$ is $\tilde{\mathbb{C}^n} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$ with equation $x_i y_j = x_j y_i$. Blow up is a morphism $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ induced by the projection $\pi : \mathbb{C}^n \times \mathbb{P}^{n-1} \to \mathbb{C}^n$ in

- *($\mathbb{R}^2$)*
  Real blow up of $\mathbb{R}^2$ at its origin is a Möbius strip. Similarly, real blow up of a 2-sphere S^2 is a real projective plane $\mathbb{RP}^2$. Indeed, Möbius strip is $\mathbb{RP}^2$ removed the disk D^2.

- *(Δ)*

  $\Delta \subset X \times X$ is the diagonal, whose blow up is you know what.

  $Bl_\Delta X$

Definition 4.5.1 *(Tautological Line Bundle)*

$\mathscr{O}_{\mathbb{P}^n}(-1)$ is dual of twisting sheaf of Serre.

Let $Gr(n,k)$ be Grassmannian, the total bundle of tautological line bundle is $Gr(n,k) \times \mathbb{R}^{n+k} = \{(V,v) | V \in Gr(n,k), v \in V\}$,

Definition 4.5.2 *(Hyperkähler Geometry)*

4.6 Dimension (Intersection Theory)

For the simplicity of argument, we'll start from Euclidean space, then Riemannian surface (Smooth manifold), and we'll later advance it to algebraic geometry.

4.6.1 Euclidean Topology

For an Euclidean space $\mathbb{R}^n$, the basis of topology is open balls $B_\epsilon(x) = \{y | d(x,y) < \epsilon\}$ for $\epsilon > 0$ and $x \in \mathbb{R}^n$, so any open subset is a union of some open balls. Consider the dimension of an open ball $B_\epsilon(x)$ is n, namely $dim(B_\epsilon(x)) = n$, and the dimension of any open subsets are n, and the dimension of an open subset is invariant by union and intersection.

On the other hand, a close subset is not always dimension n. For example, hyperplane is, by defintion, codimension 1, which means dimension $n-1$.

4.6.2 Riemannian Surface

Consider that a smooth manifold is . A complex manifold is locally analytic, Smooth manifold is

4.7 Symplectic Geometry

Definition 4.7.1 *(Symplectic manifold)*

Cotangent bundle is canonically a symplectic manifold. Cotangent bundle describes phase space of physics.
(volume form)

If symplectic manifold (M^n, ω) is $2n$-dimensional, volume form is
$vol = \omega^n$
(Cotangent bundle)

Cotangent bundle is

$$T^*M = \{(x, v^*) \in T^*\mathbb{R}^n | f(x) = 0, v \in T_x\mathbb{R}^n, df_x(v) = 0\}$$

or

$$\Gamma T^*M = \Delta^*(\mathscr{I}/\mathscr{I}^2).$$

Definition 4.7.2 *(Lagrangian submanifold)*

$L \subset M$ is a half dimensional isotropic submanifold.

Also, $\omega|_L = 0$.

Definition 4.7.3 *(symplectic reduction and moment map)*

Let (M, ω) be a symplectic manifold, and a momentum map $\mu : M \to \mathfrak{g}^*$ is a map that satisfies $d(<\mu, \xi>) = \iota_{\rho(\xi)}\omega$ for all $\xi \in \mathfrak{g}$.

An G-action on a symplectic manifold (M, ω) is called Hamiltonian if it has a momentum map μ.

(Symplectic reduction)

Let $\mu : M \to \mathfrak{g}^*$

the symplectic reduction $M//G$ *is* $M//G = \mu^{-1}(0)/G$, *which is a symplectic manifold.*

Its dimension is $dim(M//G) = dim(M) - 2 * dim(G)$.

Definition 4.7.4 *(diff p-form and vector field)*

- *ex1:*

  $$dx_i(\partial_j) = \delta_{ij}$$

- *ex2:*
  $dx_{i_1} \wedge \cdots \wedge dx_{i_p}(\partial X_{j_1}, \cdots, \partial X_{j_p}) =$
  1 *if (*i_p *and* j_p *are even permutation)*
  -1 *if (*i_p *and* j_p *are odd permutation)*
  0 *(otherwise)*

- *ex3:*
  $\omega = d_x \wedge d_y$, $X = f(x)\partial_X$, $Y = g(x)\partial_X$,
  $\omega = f(x)g(x)(d_x(\partial_X) \wedge d_y(\partial_X) - d_y(\partial_X) \wedge d_x(\partial_X))$

Definition 4.7.5 *(interior product)*
$\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$

$$\iota_X \omega(X_1, \cdots, X_{p-1}) = \omega(X, X_1, \cdots, X_{p-1})$$

for any vector fields X *and* X_i.

Definition 4.7.6 *(Kähler manifold)*

- *(Symplectic Viewpoint)*
  $g(u, v) = \omega(u, Jv)$

- *(Complex Viewpoint)*
  Let h *be the Hermitian metric.*

  $$\omega(u, v) = Re(h(iu, v)) = Im(h(u, v))$$

  $$g(u, v) = Re(h(u, v))$$

- *(Riemannian Viewpoint)*

Kähler manifold is Riemannian manifold X of dimension $2n$, whose holonomy is contained in $U(n)$.

J preserves metric i.e. $g(Ju, Jv) = g(u, v)$, and J is preserved by parallel transport.

(Kähler potential)
Let ρ be a smooth real valued function.

$\omega = \frac{i}{2}\partial\bar{\partial}\rho$

is a Kähler potential.

Definition 4.7.7 *(Kähler differential)*

(Definition using derivations)
Let $\phi : R \to S$ be a commutative ring homomorphism, and $R \to M$
Kähler differential form $\Omega_{S/R}$ is

- $dr = 0$
- $d(s + t) = ds + dt$
- $d(st) = sdt + tds$

where $s, t \in S$ and $r \in R$.
(Definition using an augmentation ideal)

let $I = ker(S \otimes_R S \to S)$

then, the module of Kähler differential is defined by

$\Omega_{S/R} = I/I^2$

where

$ds = s \otimes 1 - 1 \otimes s$

4.8 Singualrty (Orbifold)

Definition 4.8.1 *(Orbifold)*

Example 4.9 *(Billiard)*

Example 4.10 *(Pillow Case)*

Definition 4.10.1 *(Tensor product as module)*

Let M and N be R-module. The tensor product as R-module is tensor product as a vector space with quotient as follows:

$M \otimes_R N = M \otimes N/(m, r \cdot n) \sim (m \cdot r, n)$

ex:

$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/gcd(m,n)\mathbb{Z}$

Definition 4.10.2 *(Flat module)*

R-module M is flat if any injection $K \hookrightarrow L$ maps to $K \otimes_R M \to L \otimes_R M$ is injective.

Especially if $- \otimes_R M$ is faithfully flat if and only if short exact sequence maps to short exact sequence.

- *(non-ex I)*
  $\mathbb{Z}/2\mathbb{Z}$ is not flat. For example an injection $\phi : \mathbb{Z} \to 2\mathbb{Z}$ doesn't make $\phi \otimes \mathbb{Z}/2\mathbb{Z}$ injective.
- *(non-ex II)*
  R/I cannot be flat, since it's torsion, but I could be flat.
- *(ex I)*
  If M and N are flat over R, then $M \otimes N$ is flat.
- *(ex II)*
  If $S \to S'$ is flat and M is flat, then $M \otimes_S S'$ is flat.
- *(ex III)*
  If M is R-module, its localization $M_{\mathfrak{p}}$ is flat $R_{\mathfrak{p}}$ module.

Another Characterization :

For all linear combinations $\Sigma_{i=1}^m r_i x_i = 0$ where $r_i \in R$ $x_i \in M$, we have change of basis

there exists an element $y_j \in M$ and $a_{i,j} \in R$ such that

$\Sigma_{i=1}^m r_i a_{i,j} = 0$ *for* $0 \leqq j \leqq n$

$x_i = \Sigma_{j=1}^n a_{i,j} y_j = 0$ *for* $0 \leqq i \leqq m$

Why is this definition equivalent to $-\otimes_R M$ is an exact functor? Apparently, $-\otimes_R M$ is right exact, so if the second morphism of $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is injective, it makes short exact sequence.

Example 4.11 *(Flat which is not projective)*

If $R = C^\infty(\mathbb{R})$, and $M = R_{\mathfrak{m}} = R/I$ where $\mathfrak{m} = \{f \in R | f(0) = 0\}$ and $I = \{f \in R | \exists \epsilon > 0 : f(x) = 0, \forall x, |x| < \epsilon\}$.

Note: R is not integral domain.

Definition 4.11.1 *(Projective Module)*

A module P is projective iff every short exact sequence $0 \to A \to B \to P \to 0$ that splits.

ex:

R is a Dedekind domain $R = \mathbb{Q}[\sqrt{-6}]$ and $P = (2, \sqrt{-6})$, then if $A = (3, \sqrt{-6})$, then $B \cong A \oplus P$.

Projecitve module is flat.

Definition 4.11.2 *(Flat Ring Extension)*

A ring homomorphism is flat extension if it's a flat morphism.

ex:

$R \to R[x]$ is flat.

5 Algebra

5.1 Quiver

5.1.1 ADE Classification

The Lie algebra of type ADE is the synthesis of $\mathfrak{sl}_2$ of type A1, which is well-knowned, and the quiver structure of type ADE, also called simply laced dynkin diagram, can determine the bigger simple Lie algebra of type ADE, and each vertex in the dynkin diagrams corresponds to $\mathfrak{sl}_2$. The famous application is to construct the quantum group $U_q(\mathfrak{g})$ of $\mathfrak{g}$ is type ADE.

First of all, $\mathfrak{sl}_2(\mathbb{C})$ is very clear mathematically, but physics is often real geometric. In fact, the gauge theory uses $\mathfrak{su}_2(\mathbb{R})$ symmetry, which is real geometric, but algebraically it might be replaced to $\mathfrak{sl}_2(\mathbb{C})$ by complexification.

$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2(\mathbb{R}) \otimes \mathbb{C}$

First, Dynkin diagram generated by Cartan matrix. Discrete Laplace operator generates Cartan matrices.

Definition 5.1.1 *(Discrete Laplace Operator)*

Discrete Laplace operator is a discrete version of continuous Laplace operator. Discrete Laplace operator is used in graph theory, and simply take the difference of functional value of neighboring points. For a given graph $\Gamma = (V, E)$ and a function $\phi : \Gamma \to \mathbb{R}$,

$\Delta\phi(v) = \Sigma_{w\in\Gamma, d(v,w)=1}(\phi(v) - \phi(w))$

where $v \in \Gamma$ is a vertex, and $d(v, w)$ is a distance function of two points.

Here, it's possible to recognize ϕ as a vector and Δ as a matrix to think this problem linear algebraically.

Definition 5.1.2 *(Laplacian Matrix)*
We let the Laplacian matrix $L = \{L_{i,j}\}$ as

$$L_{i,j} = \begin{cases} deg(v_i) & \textit{if } i = j \\ -1 & \textit{if } i \neq j \textit{ and } v_j \textit{ is adjacent to } v_j \\ 0 & \textit{otherwise} \end{cases}$$

deg is the degree function, in which each vertex is counted by how many edges connect with it, and we could write simply $L = D - A$ where D is degree matrix and A is adjacency matrix.

Definition 5.1.3 *(Gabriel Theorem)*

A simple finite dimensional Lie algebra has a connected Dynkin diagram (I'll omit how to derive it), and Dynkin diagram is a graph. and by Gabriel theorem, dynkin diagram is classified by A-G type. Type A,D,E is called simply-laced graph.

Definition 5.1.4 *(McKay Correspondence)*

McKay correspondence claims the one-to-one correspndence of the Mckay graph to extended Dynkin diagram of ADE type Lie algebras. For the group represntation of a finite subgroup $G \subset SL(2, \mathbb{C})$, the Mckay graph Γ_G is a graph whose vertecies correspond each character of irreducible representations $\{\chi_1, \cdots, \chi_d\}$ (indeed, we only have finite possible variety of irreducible representations), and the arrows n_{ij} are calculated by

$n_{ij} =< V \otimes \chi_i, \chi_j >= \frac{1}{|G|}\Sigma_{g\in G}V(g)\chi_i(g)\overline{chi_j(g)}.$

and Γ_G is undirected because $n_{ij} = n_{ji}$. McKay graph of some finite subgroup of $SO(3)$ corresponds Dynkin diagram of ADE type. as follows.

- $A_n = \mathbb{Z}_{n+1}$
- $D_n = D_{2(n-2)}$ *(Dihedral Group if $n \geqq 4$)*
- $E_6 = T$ *(Tetrahedral)*
- $E_7 = O$ *(Octahedral)*
- $E_8 = I$ *(Icosahedral)*

Definition 5.1.5 *(Du Val Singularity)*

The geometric quotient of the finite group corresponding to the dynkin diagram is orbifold, called du Val singularity.

- $A_n = w^2 + x^2 + y^{n+1} = 0$
- $D_n = w^2 + y(x^2 + y^{n-2}) = 0$
- $E_6 = w^2 + x^3 + y^4 = 0$
- $E_7 = w^2 + x(x^2 + y^3) = 0$
- $E_8 = w^2 + x^3 + y^5 = 0$

5.1.2 Quiver Diagram

Definition 5.1.6 *(Quiver Diagram)*

Let Q be a finite graph, and each $v \in Q$ corresponds to a gauge group G_v. The gauge group is $\prod G_v$. Each edge of Q $e : u \to v$ corresponds to the defining representation $\overline{N_u} \otimes N_v$. There is a corresponding superfield Φ_e.

A super field is a morphism $\Phi : \mathbb{R}^{m|n} \to \mathbb{R}^{m|n}$

Definition 5.1.7 *(Quantum Group of Quiver)*

Quantum group of ADE type is

$U_q(\mathfrak{g})$ where $\mathfrak{g}$ is simple Lie group of type ADE.

5.1.3 ADHM Construction

Definition 5.1.8 *(Moment Map)*

(M, ω) is a symplectic manifold, and a Lie group G acts on M via symplectomorphism (each action of $g \in G$ preserves ω).

$< -, - >: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$

Any $\xi \in \mathfrak{g}$ induces a vector field $\rho(\xi)$ on M, defined locally as

$\rho(\xi))_x = \frac{d}{dt}|_{t=0} exp^{t\xi} \cdot x$

Since $\iota_{\rho(\xi)}\omega$ is closed and if moreover exact, then there exists $H_\xi : M \to \mathbb{R}$.
Now momentum map $\mu : M \to \mathfrak{g}^*$ is a map

$d(< \mu, \xi >) = \iota_{\rho(\xi)}\omega$

for all $x \in \mathfrak{g}$ and $< \mu, \xi > (x) =< \mu(x), \xi >$.

Definition 5.1.9 *(ADHM Data)*

Let V and W be vector spaces where $dim(V) = k$ and $dim(W) = N$, and $B_1, B_2 \in M_{n,n}$, and $I \in M_{n,N}$, and $J \in M_{N,n}$.
Let a real momentum map

$\mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger + J^\dagger J$

$\mu_{\mathbb{C}} = [B_1, B_2] + IJ$

We can construct ASD instanton in $SU(N)$ gauge theory given $\mu_{\mathbb{R}} = 0$ and $\mu_{\mathbb{C}} = 0$.

5.2 Hopf Algebra

Definition 5.2.1 *(Algebra)*

We mean algebra by unitary associatve algebra (this means, linear algebra or Lie algebra is not algebra).

Definition 5.2.2 *(Coalgebra)*

Coalgebra C over K is a vector space with two structure morphisms

- $\Delta : C \to C \otimes C$

- $\epsilon : C \to K$

with the following identities

- $(id_C \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta$ *(coassociativity)*
- $(id_C \otimes \epsilon) \circ \Delta = id_C = (\epsilon \otimes id_C) \circ \Delta$

Definition 5.2.3 *(Hopf Algebra)*

Hopf algebra H is bialgebra with antipode, and bialgebra is coalgebra but also algebra. Antipode $S : H \to H$ is a k-linear map that commutes the diagram and

$S_{c_{(1)}} c_{(2)} = c_{(1)} S_{c_{(2)}} = \epsilon(c)1$ *for all* $c \in H$.

Definition 5.2.4 *(Representation of Hopf Algebra)*

Let A be Hopf algebra, and M and N are A-modules. Then $M \otimes N$ is also A-module, with

$a(m \otimes n) = \Delta(a)(m \otimes n) = (a_1 \otimes a_2)(m \otimes n) = (a_1 m \otimes a_2 n)$ *where* $m \in M$, $n \in N$, *and* $\Delta(a) = (a_1, a_2)$.

$a(m) = \epsilon(a)m$

$(af)(m) = f(S(a)m)$ *where* $f \in M^*$ *and* $m \in M$.

5.2.1 Yang-Baxter Equation

One major purpose of Yang-Baxter equation is to describe statistical mechanics of the lattice structure of ice melting or magnetism, and YB equation calculates partition functions, that measures entropy, and another is that it's also used to describe quantum mechanics, so it's to connect between them. YB equation is mathematically, the identity of R-matrices, that's used to define an operator of the transpose $V \otimes W \to W \otimes V$, where V and W are representation spaces, hence YB generates braiding structure.

The physics interpretation of YB equation is six-vertex model. Suppose the ice structure is two dimensional, where each position is a water molecule (H2O), that's electrically unevenly charged, namely the position of hydrogen wrt oxygen, by the four directions (up/down/right/left), but as a total, it's neutrally charged. Our problem is to determine the direction of charge (up/down/right/left) of each molecule, that's YB equation, and we have six different combination of charges. Now, in physics, ice melting is entropy problem, thus here we define a partition function.

On the other hand, YB equation describes quantum mechanics if R is scattering matrix.

Definition 5.2.5 *(Partition Function)*

Of partition function, Boltzmann weight is $e^{\frac{\epsilon(a,b,c,d)}{k_B T}}$ *where* $\epsilon(a,b,c,d)$ *is the energy of the state. A particle has four legs, and each of them has Boltzmann weights, and If few particles are connected, which is called tensor network, the weight is given by adding each weights. The partition function is simply the sum*

$$\Sigma e^{\frac{\epsilon(a,b,c,d)}{k_B T}}$$

YB equation describes the 2d lattice structure, so supposely define a ice molecule on aligned in a row, and then determining the column. The the position of H2O molecule in the second row is somewhat determined by the first row, but not absolutely, and we have some freedom of positions. We define transfer matrix for systematically defines the entropy structure.

Definition 5.2.6 *(Transfer Matrix)*
Consider $M \times N$ *2d lattice, and if we determine the order of* 1*-st row, we will automatically determine the* 2*-nd row, that's precisely determined by transfer matrix. Transfer matrix is given by*

$$T_{\sigma,\sigma'} = \Sigma \prod w(a,b,c,d).$$

and the partition function is $Z = Tr(T^N)$ *where* N *is piling up* N *times.*

Proposition 5.3 *(R-Matrix and T-Matrix)*
$T = \prod_i R(u_i)$

R-matrix describes how one particle relates to the neighboring molecules. Pixel by pixel, not row by row.

Definition 5.3.1 *(R-Matrix)*

$$\Sigma w(a_1,\alpha|a_2,\gamma)w(\gamma,\beta|a_3,b_3)w(\alpha,b_1|\beta,b_2)\Sigma w(a_2,\beta|a_3\alpha)w(a_1,b_1|\beta\gamma)w(\gamma,b_2|\alpha,b_3)$$

Or we get describe by R-matrices.

$$R = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & A^{-1} & 0 \\ 0 & A^{-1} & A^{-1} - A^{-3} & 0 \\ 0 & 0 & 0 & A \end{bmatrix}$$

where $A = e^{\eta}$*, and from here, we induce Jones polynomial wrt* A*.*

If we install R-matrix structure on Hopf algebra, it'll be quasi-triangular Hopf algebra. This is part of Hopf algebra.

Definition 5.3.2 *(Quasi-Triangular Hopf Algebra)*
There is an invertible element R of $H \otimes H$ such that

$R\Delta(x)R^{-1} = (T \circ \Delta)(x)$ *for all* $x \in H$ *where* $T(x \otimes y) = y \otimes x$.

$(\Delta \otimes 1)(R) = R_{13}R_{23}$

$(1 \otimes \Delta)(R) = R_{13}R_{12}$

where $R_{12} = \phi_{12}(R)$, $R_{23} = \phi_{23}(R)$, $R_{13} = \phi_{13}(R)$ *and* $\phi_{ij} : H \otimes H \to H \otimes H \otimes H$

- $\phi_{12}(a \otimes b) = a \otimes b \otimes 1$
- $\phi_{13}(a \otimes b) = a \otimes 1 \otimes b$
- $\phi_{23}(a \otimes b) = 1 \otimes a \otimes b$

Then R is a solution of Yang-Baxter equation.

Definition 5.3.3 *(Yang-Baxter Equation)*

Based on the preivous definition, YB equation is simply this equation.

$(1 \otimes \check{R})(\check{R} \otimes 1)(1 \otimes \check{R}) = (\check{R} \otimes 1)(1 \otimes \check{R})(\check{R} \otimes 1)$

Or

$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

where R-matrix is $R \in End(V \otimes V)$ *and each* $R_{ij} : R \otimes R \otimes R \to R \otimes R \otimes R$ *acts R on each i,j components.*

5.3.1 Quantum Group

Quantum group is a group but also Hopf algebra (quasi-triangular Hopf algebra). One example of quantum group is deformation of the universal enveloping algebra of semisimple Lie algebra.

Definition 5.3.4 *(Quantum Group)*

Let $A = (a_{ij})$ *is a Cartan matrix, and let* $q \neq 0, 1$ *be a complex number. A quantum group* $U_q(G)$ *for Lie group G whose Cartan matrix is A, λ is an element of the weight lattice, and e_i and f_i with the following relations*

- $k_0 = 1$

- $k_\lambda k_\mu = k_{\lambda+\mu}$
- $k_\lambda e_i k_\lambda^{-1} = q^{(\lambda,\alpha_i)} e_i$
- $k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda,\alpha_i)} f_i$
- $[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$ *where* $k_i = k_{\alpha_i}$, *and* $q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)}$

and the q-factorial is defined as

- $[0]_{q_i}! = 1$
- $[n]_{q_i}! = \prod_{m=1}^{n} [m]_{q_i}$ *where* $[m]_{q_i} \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$

and if we take limit $q \to 1$, *then*

$k_\lambda \to 1$ *and* $\frac{k_\lambda - k_{-\lambda}}{q - q^{-1}} \to t_\lambda$.

where $(t_\lambda, h) = \lambda(h)$ *for all* h *in CSA.*

We discuss deformation and quantization.

Example 5.4 *(Formal Deformation of Algebra)*

Deformation of k *if* $q = 0$ *is* $k[x]/(x^2)$.

In general, deformation of k *is defined over quantum plane* $k_q[x,y]/(xy = qyx)$.

Example 5.5 *(Drinfeld-Jimbo Type Quantum Group)*

$U_q(\mathfrak{g})$ *is a deformation of Lie algebra* $\mathfrak{g}$. *In other words, deformation of universal enveloping algebra* $U(\mathfrak{g})$.

Formal deformation of associative algebra (A, μ_0) *is*

$\mu_\hbar = \mu_0 + \hbar\mu_1 + \hbar^2\mu_2 + \cdots$

the operator is generated so that the associativity is preserved, and $(A[[\hbar]], \mu_\hbar)$.

Definition 5.5.1 *(Coassociative Coproduct)*

There are various way to decide coproducts, but here is the tiny example

- $\Delta_1(k_\lambda) = k_\lambda \otimes k_\lambda$

- $\Delta_1(e_i) = 1 \otimes e_i + e_i \otimes k_i$
- $\Delta_1(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1$

Counit ϵ is
$\epsilon(k_\lambda) = 1$ *and* $\epsilon(e_i) = \epsilon(f_i) = 0$

and the antipode S is

- $S_1(k_\lambda) = k_\lambda$
- $S_1(e_i) = -e_i k_i^{-1}$
- $S_1(f_i) = -k_i f_i$

5.5.1 KZ Equation

Quasi-Hopf algebra for KZ equation.

Definition 5.5.2 *(Quasi-Hopf Algebra)*

Quasi-Hopf Algebra is a generalization of Hopf algebra. It's antipode with Quasi-bialgebra as

$\Sigma_i S(b_i)\alpha c_i = \epsilon(a)\alpha$

$\Sigma_i b_i \beta S(c_i) = \epsilon(a)\beta$

for all $a \in A$ and where

$\Delta(a) = \Sigma_i b_i \otimes c_i$

and

$\Sigma_i X_i \beta S(Y_i)\alpha Z_i = I$

$\Sigma_j S(P_j)\alpha Q_j \beta R_j = I$

where $\Phi = \Sigma_i X_i \otimes Y_i \otimes Z_i$ *and* $\Phi^{-1} = \Sigma_j P_j \otimes Q_j \otimes R_j$.

Definition 5.5.3 *(Knizhnik-Zamolodochikov Equation)*

$fr\hat{a}kg_k$ *be affine Lie algebra with level k and dual Coxeter number h. $i, j = 1, 2, \cdots, N$, t^a be basis of $\mathfrak{g}$*
$((k+h)\partial_{z_i} + \Sigma_{j \neq i} \frac{\Sigma_{a,b}\eta_{ab} t_i^a \otimes t_j^b}{z_i - z_j}) < \Phi(v_N, z_N) \cdots \Phi(v_1, z_1) >= 0$

We'd consider alternative approach of KZ equation.

Definition 5.5.4 *(Monodromy)*
An example of monodromy is a multi-valued function $f(z) = log(z)$, since this function doesn't have the same value if we rorate around the singular point $z = 0 \in \mathbb{C}$. In fact, consider $1 = e^{2\pi i}$, $log(z) = log(z * e^{2\pi i}) = log(z) + log(e^{2\pi i}) = log(z) + 2\pi i$, thus the function can take multiple value even though it takes the same inputs. This is the idea of monodromy.

These seemingly different functions $log(z)$ and $log(z) + 2\pi i$ can be joined together by analytic continuation. Consider the domain of $log(z)$ is entire of the plane except the singular point, means that $0 \leqq \theta < 2\pi$, while the other $log(z)+2\pi i$ for $2\pi \leqq \theta < 4\pi$. If we also consider intermediate function $log(z)+\pi i$, whose domain is $\pi \leqq \theta < 3\pi$, we have non-empty intersection with both of them, hence by identity theorem, $log(z)$ and $log(z)+2\pi i$ are identified to be the same function.

Definition 5.5.5 *(Configuration Space)*
The configuraiton space is the domain of the solution of KZ equation. The configuration space is defined as

$$Conf_n(\mathbb{C}) = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$$

$Conf_n(\mathbb{C})$ creates monodromy around the singular points. The monodromy is calculated by the monodromy representation of the fundamental group $\pi_1(Conf_n(\mathbb{C})) = B_n$, which is a braid group.

Configuration space is a complex manifold.

Definition 5.5.6 *(Configuration Space)*
The surjective map of configuration spaces

$$Conf_{n+N}(X) \to Conf_N(X)$$

is a locally trivial fiber bundle with typical fiber $X \backslash \{x_1, \cdots, x_N\}$.

Definition 5.5.7 *(Horizontal Chord Diagrams)*
The monoid of horizontal chord diagram is the free monoid

$$D_n^{pb} = FreeMonoid((ij) | 1 \leqq i < j \leqq n)$$

and each element (ij) is a permutation of i & j.

$2T$-relation
(ij) and (kl) commutes if i,j,k,l are all distinct, namely, $(ij)(kl) = (kl)(ij)$. So $2T$-relation is

$2T = \{(ij)(kl) - (kl)(ij) | i, j, k, l$ *all distinct*$\}$

$4T$*-relation*

Consider $(ik)(ij) = (ij)(jk)$ *are the same permutation but by using different elements, and also* $(jk)(ij) = (ij)(ik)$ *is. We join them together and* $(ik)(ij) + (jk)(ij) = (ij)(jk) + (ij)(ik)$.

$4T = \{(ik)(ij) + (jk)(ij) - (ij)(jk) - (ij)(ik) | i, j, k$ *all distinct*$\}$

$A_n^{pb} = Span(D_n^{pb})/4T$

Definition 5.5.8 *(KZ Connection)*
(KZ Differential Form)
$\omega_{KZ} \in \Omega(Conf(\mathbb{C}), A_n^{pb})$ *such that*

$\omega_{KZ} = \Sigma d_{dR} log(z_i - z_j) \otimes t_{ij}$

and KZ connection is flat since

$d\omega_{KZ} + \omega_{kz} \wedge \omega_{KZ} = 0$

(KZ Connection)
$\nabla_{KZ}\phi = d\phi + \omega_{kz} \wedge \phi$

and KZ equation is to find ϕ *such that*

(KZ Equation)
$\nabla_{KZ}\phi = 0$

KZ connection is flat, which means the result of integration doesn't change by the direction of path, so analytic continuation is well-defined.

5.6 Hall Algebra

Definition 5.6.1 *(Hall Polynomial)*

A finite abelian p-group M *is a direct sum of cyclic p-power components* $C_{p^{\lambda_i}}$ *where* $\lambda = (\lambda_1, \lambda_2, \cdots)$ *is a partition of* n *called type of* M.

$g^{\lambda}_{\mu,\nu}(p)$ *is the number of subgroups* N *of* M*, such that* N *has type* ν *and the quotient* M/N *has type* ν.

This function is defiend for any p, and in fact, it is a polynomial. $g^{\lambda}_{\mu,\nu}(q) \in \mathbb{Z}[q]$.

Definition 5.6.2 *(Hall Algebra)*

An associative ring H over $\mathbb{Z}[q]$, called Hall algebra has a basis $\{u_\lambda\}$ with multiplication

$$u_\mu u_\nu = \Sigma g^{\lambda}_{\mu,\nu}(q) u_\lambda$$

Then, H is an commutative ring.

6 Algebraic Topology

Many things should be mentioned here, since algebraic topology is a huge subject, but I'm planning little with a view from sheaf theory. That is, if a sheaf if constant, means no geometric property but topology.

6.1 Perverse Sheaf

Definition 6.1.1 *(Local system)*

If X is connected, locally constant sheaf of abelian groups is constant sheaf.

6.2 TQFT

Definition 6.2.1 *(TQFT By Dimension)*

- 2-*dim TQFT*
  Frobenius algebra
- 3-*dim TQFT*
  Quantum group and tensor category
- 4-*dim TQFT*

Definition 6.2.2 *(Frobenius Algebra)*

- *(mul and add)*
  closed under multiplication and addition.
- *(Bilinearity)*
  $< a \cdot b, c >=< a, b \cdot c >$
  $Tr(a) =< e, a >$
- *(Non-degeneracy)*
  If $< a, b >= 0$, then $a = 0$ or $b = 0$

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