Foundation of Algebraic Geometry

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1 Preface

This projects sounds like assembly programming for those already made by other higher-level languages, by analogy of computer programming, but I believe it not only help to summarize the overview of algebraic geometry, but also helps to see philosophically "what it is", because it's unlikely to doubt the meaning of definition of mathematical text.

When I decided to study theoretical math, I needed to find a motivation why it is more than a counting problem. Let's talk about abstract nonsense featurizes pure math. Etymonologically, abstract nonsense is the combination of "abstract" + "nonsense", where nonsense is "non"+"sense", meaning lack of sensitivity. Mathematical argument accompanies existence and uniqueness problem, (so tautologically) that can be discussed without actual existence, hence without actual sensitivity. "Abstract" in algebra considers, for instance, a polynomial ring, a set of polynomials, instead of each individual polynomial, thus "abstract" is somehow set theoretical generalization, while if number theorists prefer learning specific number, then it could be "concrete" nonsense. Is there anything like "abstract sense" instead of "abstract nonsense?" Let's consider finite set where existence and uniqueness is guaranteed. Codomain of simplicial set is Set. The forgetful functor applies to a finite-dim vector space to a finite set.

2 Math Essay

2.1 Algebraic Geometry Set Theory Flavored

2.1.1 Intersection/Union

In Euclidean topology, finite intersection and union of open sets are dimensional invariant, while intersection of closed sets just add up their codimension if transversal. This makes Chow ring. By the way, Noetherian scheme is compact. It's clear why Noetherian ring is important.

2.1.2 Cartesian Product

Consider a simplicial set. If a finite set A is a set of vertices, cartesian product AxA is a set of edges, AxAxA is a set of faces. This makes the standard resolution if abelianized. Also, what if the vertex is replaced by spec(k), or spec(k[x]) in general? We are already talking advanced algebraic geometry. Also, cartesian product has affinity to category theory.

2.1.3 Hom and Quotient

In set theory, quotient is to classify each element of the sets by equivalence classes. If a groupoid is a category, whose domain is hom, hom is a moduli space. If hom is tensor adjunct, tensor is dual of fiber product, and fiber product is fibration, which is geometric quotient. We use stack to classify fibration. This is the quotient. Tensor is kinda linear algebra, and if hom is dual to tensor, moduli space is kinda vector bundle.

2.1.4 Injectivity and Surjectivity I (Representation)

Representation is "re"+"presentation." By analogy, in abstract algebra, presentation of a group is a free group with equivalence relation. Obviously, in any groups, presentations exists, but uniqueness, so maybe "re." Naturally, surjection from the free group to that group. In category theory, Hom(X,Y) is the set of all the possible morphisms of X,Y, which means any set of morphisms of X,Y are subset of Hom(X,Y), while a representable functor Hom(X,-) is representing the whole situations (surjecitivity).

2.1.5 Injectivity and Surjectivity II (Covering Map)

The covering map is a surjective map of local homeomorphism (injectivity). Local isomorphism is etaleness. The purpose of topology is installing injectivity/surjectivity at one time.

2.2 Compactification

2.3 Etymologists' Game

What is Algebraic Geometry? This might be an etymological question. What is "algebraic", what is "geometric", and what is "algebraic geometric?" All combinations can nicely explain what we are doing for research. "Algebra" conventionally refers to non-unital associative algebra. Universal algebra, Lie algebra, linear algebra, homological algebra, contain the name of algebra, but they are not considered to be algebra, but still a branch of subjects in abstract algebra. In abstract algebra, each concept can be paraphrased by algebra. For example, group is not algebra, but there is group algebra, or exponential maps of Artinian ring generates group structure (analogy of Lie algebra/group correspondence). Lie algebra has a universal enveloping algebra. Homological algebra has dg-algebra. Hence, all abstract algebra has sort of algebrization. "Geometry" conventionally studies the shapes of the mathematical objects, namely dimension, degree, curvature etc. There are many similar but different concepts with geometry, and manifold(sheaf) is not necessarily geometric concepts, and dynamic system is independent subject from geometry. Also, topology is somewhat similar to geometry, and topology can count Euler characteristics and homotopy that can arise from homological algebra. Or fpqc topology (or fpqc morphism) is derived from flat morphism classes. "Algebraic Geometry" is anything that is both algebraic and geometric. Algebraic geometry is typically considered to be difficult like any other math subjects. By the way, what is complex geometry? Complex is named after apartment COMPLEX, nothing to do with being complicated.

2.4 AI and Math

This might sound like an essay, nothing technical. Primarily discussed AI and its relationship with math research.

- (Subjective Opinion)
  Computer can simply track the programming code, and it doesn't mean understanding what is actually processed. For example, if the program is hacked, the program will function in an unexpected way, but computer cannot recognize it.

- (White Revolution – Sociology for Individual Decision)
  People became extremely vigilant to insanity since COVID, and they became to prefer factory-made bread to homemade bakerly. People avoid face-to-face communication with strangers, but they voluntarily comply with the capital standard, called the Wal-Mart effect, and people started claiming sociology in every context of individual decision, and I'd also call it "White Revolution". Well, doesn't sound like category theory in math? We'd rather love category theory than abstract algebra, which is loved more than without it.

- (Bug and API)
  Software is the automation of hardware, the engineers struggle with the bug: hence software already lacks preciseness or it could be manually modified, for giving it to the client. Now, AI is the automation of software, and AI more enhanced software in that it could remove the notion of API, since language model automatically translates to it (I'm aware of tensorflow and computer architecture, but I'll talk from the application perspective), and AI is even less precise than software, so chatgpt often mistakes. On parallel, people realizes the value of traditional cram education, when they realize that cramming vocabulary and terminology (API) rather than progressive interactive communication in modern education, just as AI removed API programming. Do we pretend smart as if knowing everything since knowledge is very accesible, while giving up thinking critically and precisely?

- (Chess – Quality=Quantity?)
  Science is an activity of formulating a law from the reproductive notion that is universally considered to be truth e.g. Newton's law of motion F=ma, but the game of chess doesn't sound scientific, because every game of chess is unique, and it doesn't always go by correct strategy, while if AI defeated the chess champion, the worth of chess is not competing our strength but to express art, but AI is pattern recognition not the human intelligence equivalent. Now there are two questions: is math art or science? What is art? Or perhaps from a different perspective, the question is if math is really beautiful, and perhaps most of math algorithm is dirty, and the known algorithm is only tiny fraction of the possibly set of the solvable algorithms, being elegant, and we wonder if it's possible to automatically find new math formula: consider human correction is required to overcome overfitting/underfitting problem in chess. How's in math? Or hypothetically there is the unified theory of math is Perfectoid or Langlands program, that covers and dictates the beautiful part of math theorems, and dirty math algorithms are derived from them? Chancellor Peter Olaf Scholz realized "Perfectoid" is "Perfect" + "oid" and "oid" means that it's not.

2.5 Math Education

Math is not only a tool of computation but the way of life, and I'll define math is the harmony of spirit, power, and intelligence.

- (Spirit)
  "The man who first wrote a book index was, I suspect, a bibliophile with a grudge and a touch of the psychopath" (Oscar Wild).
- (Power)
  The idea of industrial revolution is mass production of simple products, so it's linear algebra(representation) and finger counting(0-dimensional).
- (Intelligence)
  Intelligence is Central "Intelligence" Agency, and the question is the difference between information and intelligence.

3 Set Theory

The very question is if we need to study set theory, since in fact, the problems of set theory could be naively ignorable, but the problem of set theory arises in set of sets. For example, in algebra (abstract or sigma-algebra or category theory or whatever), each algebra could be interpreted as a set of all its subalgebras, means it's the set of the sets.

3.1 Set and Universe

Definition 3.1.1 *(Russel Paradox)*

Definition 3.1.2 *()*

Definition 3.1.3 *()*

What is universe? Universe is a class, and idea to avoid Russel paradox appear in set of sets. Grothendeick universe (also Neumann universe) is a class that can relatively nicely behaves.

Definition 3.1.4 *(Grothendieck Universe)*

- *U is a set or a proper class.*
- *$X \in U$ then $P(X) \in U$*
- $\{f | f : X \to Y\} \in U$

- $X, Y \in U$

Definition 3.1.5 *(Neumann Universe)*

- $V_0 = \emptyset$
- $V_{\alpha+1} = P(V_\alpha)$
- $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

where α, β, γ are ordinal numbers, and if we let $V = \bigcup_{\alpha \in Ord} V_\alpha$, V is called Neumann universe.

- $V_0 = \emptyset$
- $V_1 = \{\emptyset\}$
- $V_2 = \{\emptyset, \{\emptyset\}\}$
- $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Definition 3.1.6 *(Category)*

A Category $\mathscr{C}$ consists of objects $Ob(\mathscr{C})$ and $Mor(\mathscr{C})$. Category is classified by small/locally small/large categories.

- *(Small Category)*
  For the category $\mathscr{C}$, if both $Ob(\mathscr{C})$ and $Mor(\mathscr{C})$ are sets, then $\mathscr{C}$ is small category. For example, it could be the category of finite sets $FinSet$, the category of finite-dimensional vector spaces.
- *(Locally Small Category)*
  For the category $\mathscr{C}$, if all $Hom(X, Y)$ for $X, Y \in Ob(\mathscr{C})$, then $\mathscr{C}$ is locally small category. For example, it could category of Set, Grp, Mod, $Vect$, and Cat_U.
- *(Big Category)*
  $\mathscr{C}$ is large category, if it's neither. For the category $\mathscr{C}$, $Ob(\mathscr{C})$ is a set, and $Mor(\mathscr{C})$ is a class.

Example 3.2 *(Big Category)*

$C^\wedge = Fun(C^{op}, U - Set)$

is a big category.

Definition 3.2.1 *(Higher Category)*

- 0-*category is a set.*
- 1-*category is a locally small category.*
- 2-*category consists of object, morphism, and natural transformations.*
- n-*category is given inductively.*

Note: just in case, 2-*category is not category of category.*

Definition 3.2.2 *(Quasi-Category)*

- *(quasi-category)*
  Quasi-caeteory C (also called ∞-category) is a simplicial set.

- *(homotopy category)*
  Simpilcial categoyr C has an associating category hC, called homotopy category of C.
- *()*

Definition 3.2.3 *(Weak Category)*

Weak n-category are the same with strict n-categories except coherence conditions.

(Strict n-Category)

- $(f \circ g) \circ h = f \circ (g \circ h)$
- $f \circ 1 = 1 \circ f = f$

(Weak n-Category)

- *(associativity law)*
  $a_{f,g,h}(f \circ g) \circ h \Rightarrow f \circ (g \circ h)$ *is an isomorphism.*
- *(unit law)*
  $l_f : f \circ 1 \Rightarrow f$ *and* $r_f 1 \circ f \Rightarrow f$ *are isomorphism.*

Note that Associativity law has a pentagon identity.

Note weak 1-category is not homotopy category.

Definition 3.2.4 *()*

Definition 3.2.5 *(Yoneda Lemma)*

3.3 Topology and Filters

Topology, pretopology, filter, ultrafilter. These are structures defined on a set.

Definition 3.3.1 *(Topology)*

For a set X, a topology (X, τ) is a pair, $\tau = \{U \subset X\}$ is a set of open subsets of X s.t.
if $U_i \subset X$ then

- $\bigcup U_i \in \tau(X)$
- $U_1 \cap U_2 \cap \cdots \cap U_n \in \tau(X)$

Note neighbor $N(x)$ is a filter.

Pretopology is a generalization of topology, and it's defined over filters or a preclosure operators.

Definition 3.3.2 *(Pretopology)*

Let X be a set. Neighborhood system of a pretopology on X is a collection of filters $N(x)$. Each element of $N(x)$ is called neighborhood of x. A pretopological space is a set equipped with a neighborhood system.
(preclosure operator)
$[]_p : P(X) \to P(X)$ *where $P(X)$ is a power set of X with the property*

- $[\emptyset]_p = \emptyset$
- $A \subset [A]_p$
- $[A \cup B]_p = [A]_p \cup [B]_p$

The last condition is equivalently, $A \subset B \Rightarrow [A]_p \subset [B]_p$

Proposition 3.4 *(Properties of Pretopology)*

- *(Continuity)*
  A map $f : (X, cl) \to (Y, cl')$ is called continuous, if for all subsets $A \subset X$, $f(cl(A)) \subset cl'(f(A))$
- *(Topology)*
  Pretopological space is topological space when its closure operator is idempotent.

Definition 3.4.1 *(Filter)*

Let F be a subset of Poset $(P, \leqq)$. F is called a filter if

- $F \neq \emptyset$.
- *For any* $x, y \in F$ *there exists* z *such that* $z \leqq x$ *and* $z \leqq y$*. In other words,* $z = x \wedge y \in F$
- *any* $y \in P$ *such that for any* $x \leqq y$*, then* $y \in F$.

ex:

- *(Topological Neighborhood)*
  In topological space (X, τ)*, a system neighborhood* $N(x)$ *for a point* $x \in X$ *is a filter.* τ *is a set of all the open subsets of* X*, and it's a poset.*
- *(Cauchy Filter)*
  Cauchy filter for discussing the notion of convergence of limit. The condition of convergence is equivalent to the exitence of neighborhood system N_x *of the point* $x \in X$*. The uniform space of a topological space* X *is* U_ϵ *such that* $\{(x, y) \in X \times X | d(x, y)\epsilon\}$*, and* U *uniform space if exists*$\epsilon > 0$ *s.t.* $U_\epsilon \subset U$.
  A filter F *is called Cauchy filter if for any neighborhood* $U \subset X$*, there exists* $A \in F$*, and if* $x, y \in A$*, then* $(x, y) \in U$.

Dual notion of filter is ideal.

Definition 3.4.2 *(Ideal (Order Theory))*
Let I *be a subset of Poset* $(P, \leqq)$*.* I *is called an ideal if*

- $I \neq \emptyset$.
- *For any* $x, y \in I$ *there exists* $z = x \vee y \in P$*, and* $z = x \vee y \in F$
- *any* $y \in P$ *such that for any* $x \geqq y$*, then* $y \in I$.

Definition 3.4.3 *(Ultrafilter)*

For a poset $(P, \leqq)$*, let* U *be a filter, and* U *is an ultra filter if it satisfies additional conditions, that is for any* $A \subset P$*,* $A \in U$ *or* $A^c \in U$*. (and I personally think this extra condition look like Bolzano-Weierstrauss)*

Ultra filter is a maximal filter.

Existence of ultra filter is equivalent to AOC*.*

Definition 3.4.4 *(Grothendieck Pretopology)*

Definition 3.4.5 *(Grothendieck Topology)*

3.5 Axiom Of Choice

First of all, are these theorem worth? If we are interested in applications, maybe in measure theory.

Let's construct ZFC, and then doubting it.

Definition 3.5.1 *(Axiom Of Choice)*

Axiom of choice claims for any sets A_i for $i \in I$, take an element $x_i \in A_i$, and make a set $\{x_i\}_{i \in I}$.

In other words, the equivalent definition of AOC is the existence of a choice function $f : I \to \bigcup_{i \in I} A_i$ and $f(i) \in A_i$.

Proposition 3.6 *(Exitence of Countable Subset)*

For an infinite set X, there always exists a subset $U = \{x_i\}_{i \in I}$ of finite cardinality.

pf:

X is not an empty set, so there is at least one element $x_1 \in X$. Also, $X - \{x_1\}$ is not an empty set either, since X is an infinite set, so there exists an element $x_2 \in X - \{x_1\}$, and of course $x_2 \neq x_1$. We could have similar processes, and obtain x_i for $i \in I = \mathbb{N}$ and each element is distinct i.e. $x_i \neq x_j$ if $i \neq j$. Hence a set $\{x_i\}_{i \in I} \subset X$ is a subset of X of countable cardinal.

Definition 3.6.1 *(Inductive Poset)*
For a poset $(X, \leqq)$, if all the order subsets have an upper limit, then it's called inductive. For example,

- $(\mathbb{N}, \geqq)$ *is inductive*
- $(\mathbb{N}, \leqq)$ *is not inductive*
- $([0, 1], \leqq)$ *is inductive*
- $((0, 1), \leqq)$ *is not inductive*
- $(P(A), \subseteq)$ *is partially inductive*

Definition 3.6.2 *(Zorn's Lemma)*

If for any poset P, if any ordered chain $C \subset P$. then there is a maximal element. If there exists an upper limit, then there is a maximal element. That is,

If for any chain $C \subset P$, there exists $u \in P$ such that $\forall c \in C$, $c \leqq u$,

then there exists $m \in P$, and $\forall x \in P$, $m \leqq x \Rightarrow x = m$.

But when does Zorn become powerful? For example, it's useless for $\mathbb{N}$, since it is an ordered set, but it doesn't apparantly have the maximal element. On the other hand, it often appears in set of set. For a set X, consider a set whose elements are subset of X as $\{S|S \subset X\}$, then its maximal element is X, and Zorn's lemma can be applied.

Example 3.7 *(Noetherian Ring)*

Consider an ascending chain condition $\cdots \subset I_i \subset I_{i+1} \subset \cdots$. *If it terminates, it's equivalent to the existence of the maximal ideal. Say,*

$I = \bigcup_{i \in I} I_i$

is an ideal (because union of inclusion relation), and by Zorn's lemma, there exists an ideal $M \in S$*, that is* $I \subset M$ *for all* I*, which means the maximal ideal. Just don't forget, we also need to claim* M *is a proper ideal: by definition, all elements of* S *is proper ideals.*

Example 3.8 *(Existence of Basis of Vector Space)*

Example 3.9 *(Existence of Ultrafilter for any Set)*

Definition 3.9.1 *(Hausdorff Maximal Principle)*

Hausdorff maximal principle claims that for a poset P*, arbitrary chain* $C \subset P$ *(also called ordered subset) can be extended to* $C_{max} \supset C$*. This is equivalent to AOC.*

Definition 3.9.2 *(Continuum Hypothesis)*

This is an hypothesis that there is no intermediate cardinal between $|\mathbb{Q}| = \aleph_0$ *and* $|\mathbb{R}| = \aleph$*.*

We cannot prove this. This means CH and ZF are independent.

Definition 3.9.3 *(Martin's Axiom)*

Martin's axiom is derived from continuum hypothesis.

Definition 3.9.4 *(Banach-Tarski Paradox)*

Banach-Tarski paradox claims that a unit sphere in $\mathbb{R}^3$ *can be splited into exactly two unit spheres.*

pf:

Consider the unit sphere can be described by $SO(3)$ and a free group of two generators F_2 also describes $SO(3)$, and the claim is that F_2 can be splitted into two. If we classify each element of F_2 by their first terms, namely, each element starts from ap or bp for some $p = a^{n_1}b^{n_2}a^{n_3}b^{n_4}\cdots$, and split them into two sets U and V, means $F_2 = U \cup V$, where

$U = \{ap | p = a^{n_1}b^{n_2}a^{n_3}b^{n_4}\cdots \text{ for some } n_i \geqq 0\}$
$V = \{bp | p = a^{n_1}b^{n_2}a^{n_3}b^{n_4}\cdots \text{ for some } n_i \geqq 0\}$

Thus,
$U = aF_2 \cong F_2$
$V = bF_2 \cong F_2$

and we make two F_2. Equivalent to say we make two $SO(3)$ out of one $SO(3)$. This is highly related to AOC.

3.10 Gödel Incompleteness Theorem

Definition 3.10.1 *(Gödel Imcompleteness Theorem)*

Proposition 3.11 *(Gödel Imcompleteness Theorem)*

ZF and AOC are independent and cannot prove one another.

References