Intro to Math Physics

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This pdf is to introduce physics from mathematical perspective, including gauge theory, string theory, and mirror symmetry, but in this pdf we'll focus on physics part. and our goal is to understand the problems in, for example, GUT, quantum gravity/SUGRA in fermionic string theory, physical intuition of mirror symmetry from the few acutal examples.

Contents

1 Calculus and Linear Algebra

1.1 Differential and Functional Calculus

Definition 1.1.1 *(D'Alembert Operator)*
where D'Alembert operator $\Box$ *is defined as*

$$\Box = \partial^\mu \partial_\mu = \eta^{\mu\nu}\partial_\nu\partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

Definition 1.1.2 *(General Relativity)*
$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$

where Λ is the gravitational constant.

Definition 1.1.3 *(Covariant Derivative)*
Covariant derivative is a way of specifying derivative in tangent bundle. Covariant derivative of f at p is a scalar at p, denoted $(\nabla_v f)_p$

$\phi : [-1, 1] \to M$ *s.t.* $\phi(0) = p$ *and* $\phi'(0) = v$

$(\nabla_v f)_p = (f \circ \phi)'(0) = \lim_{t \to 0} \frac{f(\phi(t)) - f(p)}{t}$

Example 1.2 *(Covariant Derivative)*
In flat space, covariant derivative is same as

$\nabla_{\frac{\partial}{\partial \lambda}} v = \frac{\partial v}{\partial \lambda}$

but in general, covariatn derivative is subtracting normal vector

$\nabla_{\frac{\partial}{\partial \lambda}} v = \frac{\partial v}{\partial \lambda} - n$

where n is the normal vector. Or

$\nabla_{\frac{\partial}{\partial u^i}} v = \frac{\partial}{\partial u^i}(v^j e_j) = \frac{\partial v^j}{\partial u^i} e_j + v^j \frac{\partial e_j}{\partial u^i}$

where $\frac{\partial e_j}{\partial u^i} v = \Gamma^k_{ij} e_k$

If we use $g_{ij} = e_i \cdot e_j = e_j \cdot e_i = g_{ji}$, The Christoffel symbol Γ^i_{jk} is written concisely (ex: describe Γ^i_{jk} using only g_{ij}).

Definition 1.2.1 *(Parallel Transport)*
v is parallel transport if $\nabla_w v = 0$.

v is called geodesic if $\nabla_v v = 0$.

What is Levi-Civita connection? First of all, it is covariant derivative, and determining Christoffel symbol defines what covariant derivative is. What is Covariant derivative? It might be anything that satisfies product rule.

Definition 1.2.2 *(Levi-Civita Connection)*

On Riemannian manifold, there is a unqiue Levi-Civita connection. Levi-Civita connection is torsion free, and it has a metric compatibility.

Torsion freeness is that

$$\nabla_{e_i} e_j = \nabla_{e_j} e_i$$
$$\Gamma^k_{ij} = \Gamma^k_{ji}$$

Metric compatibility is that

$$\nabla_w (v \cdot u) = (\nabla_w v) \cdot u + v \cdot (\nabla_w u)$$
$$\partial_k g_{ij} = \Gamma^l_{ik} g_{jl} + \Gamma^l_{jk} g_{il}$$

$$\Gamma^m_{jk} = \frac{1}{2} g^{im} (\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk})$$

Koszul connection is a connection which defines a directional derivative for sections of a vector bundle.

Definition 1.2.3 *(Exterior Covariant Derivative)*
$d^\nabla : \Omega^k(M,E) \to \Omega^k(M,E)$

Definition 1.2.4 *(Curvature Form)*
Curvature form is $\mathfrak{g}$-valued 2-form

$$\Omega = d\omega + [\omega \wedge \omega] = D\omega$$

where D is exterior covariant derivative.

Definition 1.2.5 *(Bianchi Identity)*
Bianchi identity is an identity arising from the exterior derivative (curvature form).

- *(First Identity)*
  $D\Theta = \Omega \wedge \theta$

- *(Second Identity)*
  $D\Omega = 0$

Definition 1.2.6 *(Exterior Covariant Derivative)*
For a principal G-bundle $P \to M$, the tangent space $T_u P = H_u \oplus V_u$ has a direct sum decomposition, and $h : T_u P \to H_u$ be the projection.

If ϕ is a k-form on P, then its covariant exterior derivative $D\phi$ is a form defined by

$$D\phi(v_0, v_1, \cdots, v_k) = d\phi(hv_0, \cdots, hv_k) \text{ where } v_i \in T_u P$$

Definition 1.2.7 *(Functional Derivative)*

A set of all function is $C^0(M, \mathbb{R})$ *and an operator* J *is a morphism between them* $J : C^0(M, \mathbb{R}) \to C^0(M, \mathbb{R})$

$J[f] = \int_a^b L(x, f(x), f'(x))dx$

$\delta J = \int_a^b (\frac{\partial L}{\partial f}\delta f(x) + \frac{\partial L}{\partial f'}\frac{d}{dx}\delta f(x))dx = (\frac{\partial L}{\partial f} - \frac{\partial L}{\partial f'}\frac{d}{dx})\delta f(x)dx + \frac{\partial L}{\partial f'}(b)\delta f(b) - \frac{\partial L}{\partial f'}(a)\delta f(a)$

Definition 1.2.8 *(Partition Function)*

- *(Classical Discrete System)*
  The canonical partition function is defined as

  $Z = \Sigma_i e^{-\beta E_i}$

  where $\beta = \frac{1}{k_B T}$ *and* E_i *is the total energy of the system in the respective micro state. Also,*

  $1 = \Sigma_i \rho_i$

- *(Classical Continuous System)*

  $Z = \frac{1}{h^3}\int exp(-\beta H(q,p))d^3q d^3p$

  $Z = \frac{1}{N! h^{3N}}\int exp(-\beta \Sigma_{i=1}^N H(q_i, p_i))d^3q_1 \cdots d^3q_N d^3p_1 \cdots d^3p_N = \frac{Z_{single}^N}{N!}$

- *(Quantum Mechanical Discrete System)*

  $Z = tr(e^{-\beta \hat{H}})$

- *(Quantum Mechanical Continuous System)*

  $Z = \frac{1}{h}\int < q, p|e^{\beta \hat{H}}|q, p > dqdp$

  Note that in addition

  $1 = \int |x, p >< x, p|\frac{dxdp}{h}$

  Hence alternatively,

  $Z = \int tr(e^{-\beta \hat{H}}|x, p >< x, p|)\frac{dxdp}{h} = \int < x, p|e^{-\beta \hat{H}}|x, p > \frac{dxdp}{h}$

1.3 Hamiltonian and Lagrangian

Definition 1.3.1 *(Lagrangian)*
Lagrangian L is

$L = T - V$

where T is kinematic energy and V is potential energy.

Definition 1.3.2 *(Euler-Lagrangian Equation)*
The action functional $S : P(a, b, x_a, x_b) \to \mathbb{R}$ is defined via

$S[q] = \int_a^b L(t, q(t), \dot{q}(t))dt$

where $q(t)$ is a path. This path is a stationary point if it satisfies EL equation, or EL equation is a path that minimizes action. Formally,

$\frac{\partial L}{\partial q} - \frac{d}{dx}\frac{\partial L}{\partial q'} = 0$

See wikipedia for derivation.

Definition 1.3.3 *(Legendre Transformation)*
Hamiltonian can be related to Lagrangian by Legendre transformation. For M be a smooth n-dimensioal real manifold where physics is defined, Lagrangian needs to be spontaneously defined as well as Hamiltonian, and there are both described by $2n$-dimensional manifold T^*M and TM resp vector bundles over M, thus we have transformation map $(q, \dot{q}) \mapsto (p, q)$ where $q \in M$, $\dot{q} \in T_qM$, and $p \in T_q^*M$ are defined by

$H(q, p) = p\dot{q} - L(q, \dot{q})$

In particular, assuming pendulum, the Lagrangian is defined by $L(q, \dot{q}) = T - V = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$, plus the momentum p is simply velocy $\dot{q}$ times constant mass m, and $\dot{q} = \frac{p}{m}$, in other words $p = p(\dot{q}) = m\dot{q}$, thus by careful calculation, $H(q, p) = T + V = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2$, and notice that the cotangent bundle T^*M is a symplectic manifold, and also a phase space.

Definition 1.3.4 *(Hamiltonian)*
So far we have already introduced Lagrangian, that can describe the stanard physics model, and the obvious reason for using Hamiltonian is because Hamiltonian is mathematically convenient. Hamiltonian is mathematically derived from Lagrangian, using Legendre transformation.

Hamiltonian H is

$H = T + V$

where T is kinematic energy and V is potential energy. Hamiltonian is derived from Legendre transformation of Lagrangian.

$H(p, q, t) = \Sigma_{i=1}^{n} p_i \dot{q}^i - L(q, \dot{q}, t)$

where $p = \frac{\partial L}{\partial \dot{q}}$

Then, to summarize the following Hamilton's equation is equivalent to EL equation

- $\frac{dq}{dt} = \frac{\partial H}{\partial p}$
- $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$

Example 1.4 *(Hamiltonian)*
Hamiltonian is sum of positional energy and momentum $H = L + K$. For example, Hamiltonian or a particle is

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} k \hat{x}^2$

Definition 1.4.1 *(Hamiltonian Operator)*

If Hamiltonian operator $\hat{H}$ act on wave function $|\psi\rangle$, the particle becomes observable.

Definition 1.4.2 *(Hamiltonian Vector Field)*

For a symplectic manifold (M, ω), using a smooth function $H : M \to \mathbb{R}$, Hamiltonian vector field is defined by

$dH(-) = \omega(X_H, -)$

Definition 1.4.3 *(Poisson Bracket)*
We define Poisson bracket from a symlectic manifold (M, ω). If we don't $\omega = \Sigma_{i=1}^{n} dp_i \wedge dq^i$ For a smooth function f and g, Poisson bracket is

$\{f, g\} = \omega(X_f, X_g)$

where X_f is a vector field defined from f. Or more explicitly,

$\{f, g\} = \Sigma_{i=1}^{n} (\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i})$

Example 1.5 *(Canonical Poisson Bracket)*
In symplectic geometry, canonical coordinate is $2n$-dimensional basis $(p_1, \cdots, p_n, q_1, \cdots, q_n)$. The vector field is defined by

- $X_{p_i} = \frac{\partial}{\partial q_i}$

- $X_{q_i} = -\frac{\partial}{\partial p_i}$

the bracket will be

- $\{p_k, p_l\} = 0$
- $\{q_k, q_l\} = 0$
- $\{q_k, p_l\} = \delta_{kl}$
- $\frac{dq}{dt} = \frac{\partial H}{\partial p} = \{q, H\}$,
- $\frac{dp}{dt} = -\frac{\partial H}{\partial q} = \{p, H\}$
- $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$

1.6 Theory of Manifold

Definition 1.6.1 *(Symplectic Manifold)*

obvious.

Definition 1.6.2 *(Moment Map)*

Definition 1.6.3 *(Principal G-Bundle)*

Definition 1.6.4 *(Holonomy Group)*

Lie group valued parameter, and it calculates the gap arising from parallel transport.

Definition 1.6.5 *(Kähler Manifold)*

Kähler manifold is both symplectic, complex, and Riemannian manifold.

Definition 1.6.6 *(Calabi-Yau Manifold)*

Compact Kähler manifold, vanishing first Chern class, and Ricci flat metric.

Definition 1.6.7 *(Lagrangian Foliation)*

Definition 1.6.8 *(Orbifold)*
Orbifold is a geometry that can be obtained by quotienting the manifold by a finite group, since it's easy to find orbifold in physics. For example, Lorentizian manifold could be divided by CPT symmetry, a finite group $\mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2$.

Definition 1.6.9 *(Sasaki Manifold)*
An odd dimensional Riemannian manifold (S, g) *is Sasaki manifold if its cone* $C(S) = S \times \mathbb{R}_{\geqq 0}$ *with metric* $g_C = dr^2 + r^2 g$ *becomes kähler manifold.*

Definition 1.6.10 *(Einstein Manifold)*
A Riemannian manifold (M, g) *is Einstein manifold if its metric* g *satisfies* $Ric = \lambda g$ *where* $\lambda \in \mathbb{R}$.

1.7 Soliton and Integrable System

Definition 1.7.1 *(Lax Pair)*
Let $L(t)$ *and* $P(t)$ *be operators, acting on a Hilbert space, satisfying*

$\frac{dL}{dt} = [P, L]$

where $[P, L] = PL - LP$ *be a commutator.*

Example 1.8 *(KdV Equation)*
KdV equation is

$u_t = 6uu_x - u_{xxx}$

can be reformulated as a Lax equation

$L_t = [P, L]$

with $L = -\partial_x^2 + u$ *and* $P = -4\partial_x^3 + 6u\partial_x + 3u_x$.

In fact, L *is time-dependent Schrödinger operator with potential* u.

Example 1.9 *(Heisenberg Picture)*
$\frac{d}{dt}A(t) = \frac{i}{\hbar}[H, A(t)]$

Definition 1.9.1 *(Toda Lattice)*
If we let Hamiltonian ;

$H(p, q) = \Sigma_{m \in \mathbb{Z}}(\frac{p(n,t)^2}{2} + V(q(n+1, t) - q(n, t))$

and the equation of motion

- $\frac{d}{dt}p(n, t) = -\frac{\partial H(p,q)}{\partial q(n,t)} = e^{-(q(n,t) - q(n-1,t))} - e^{-(q(n+1,t) - q(n,t))}$

- $\frac{d}{dt}q(n,t) = \frac{\partial H(p,q)}{\partial p(n,t)} = p(n,t)$

and Toda potential $V(r) = e^r + r - 1$.

Integrability of Toda Lattice depends on existence of Lax pair.

Example 1.10 *(Integrability of Toda Lattice)*
If we let $a(n,t) = \frac{1}{2}e^{-(q(n,t)-q(n-1,t))} - e^{-(q(n+1,t)-q(n,t))/2}$ *and* $b(n,t) = -\frac{1}{2}p(n,t)$, *Toda lattice reads* $\dot{a}(n,t) = a(n,t)(b(n+1,t) - b(n,t))$ *and* $\dot{b}(n,t) = 2(a(n,t)^2 - a(n-1,t)^2)$.

Now we let

$L(t)f(n) = a(n,t)f(n+1) + a(n-1,t)f(n-1) + b(n,t)f(n)$

$P(t)f(n) = a(n,t)f(n+1) - a(n-1,t)f(n-1)$

where $f(n)$ *are shift operators.*

1.11 Linear Algebra

Generally speaking, linear algebra is an operation of vector spaces. What is vector spaces? For one thing, the only invariant of vector spaces is dimension, and we could define direct sum decomposition by choosing basis i.e. Peter-Weyl theorem for functional analysis, or for the case of finite dimension, it might be Jordan canonincal form, so this is spontaneously eigenvalue problem.

Definition 1.11.1 *(Hilbert Space)*

An infinite dimensioanl space $\mathfrak{H}$ *is called Hilbert space if it has inner product* $< \cdot, \cdot >$ *i.e.* $< f, g > = \int f\overline{g}dx$, *and* $\mathfrak{H}$ *is often interpreted as a functional space.* *Just be careful, however, the precise definition of the vector space* $\mathfrak{H}$ *is actually not unique, and it could be often interpreted as* $\mathfrak{H} = L^2(\mathbb{R}^3)$ *for wave function of 3-dim diffyq or* $\mathfrak{H} = L^2(\mathbb{R})$ *if it's one-dimensioanl, and it's often depends on context.*

Consider linear transformation in tensor product. Tensor is often used to study graph theory and braiding structure,

Definition 1.11.2 *(Tensor Network)*

Now, let's define trace, that's an invariant used for representation theory.

Example 1.12 *(Trace)*
We skip the definition of trace, it's obvious. A purpose of trace is evaluate the rank of a matrix. In particular, for an $n \times n$ *matrix* A *such that* $A^2 = A$*, the trace corresponds to its rank as* $tr(A) = rank(A)$*.*

Consider the trace commutes the matrix, $tr(AB) = tr(BA)$*, so the diagonalization* $A = P^{-1}\Lambda P$ *makes* $tr(A) = tr(\Lambda)$*. Since the eigenvalue of* A *is always* 1 *or* 0 *(solve Cayley-Hamilton from* $A^2 - A = 0$*), the diagonal matrix* Λ *will be*

$$\Lambda = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Definition 1.12.1 *(Characteristic Polynomial (Determinant))*
For an $n \times n$ *matrix* A*, its characteristic polynomial is* $p_A(t) = det(tI - A)$

There is an alternative version of characteristic polynomial using trace.

Definition 1.12.2 *(Characteristic Polynomial (Trace))*
For an $n \times n$ *matrix* A*, its characteristic polynomial is*

$$p_A(t) = \Sigma_{k=0}^{n} t^{n-k}(-1)^k tr(\wedge^k A)$$

where $tr(\wedge^k A)$ *is*

$$tr(\wedge^k A) = \frac{1}{k!} \begin{bmatrix} tr(A) & k-1 & 0 & \cdots & 0 \\ tr(A^2) & tr(A) & k-2 & \cdots & 0 \\ \cdots & \cdots & 0 & \cdots & \cdots \\ tr(A^{k-1}) & tr(A^{k-2}) & \cdots & \cdots & 1 \\ tr(A^k) & tr(A^{k-1}) & \cdots & \cdots & tr(A) \end{bmatrix}$$

Now, what is representation? Representation is a morphism, and set theoretically, a morphism consists of domain, codomain and its mapping. So the famous result is that Lie algebra representation classifies the simple Lie algebra of finite dimension using Dynkin diagram, so to classify the domain, but equally imporant is to see how the functions behave over it.

Definition 1.12.3 *(Representation)*

A group representation (ϕ, V) *is a group morphism* $\phi : G \to GL(n, V)$*, where* G *is a group and* V *is a vector space.*

The purpose of representation is to reduce the abstract algebra problem to linear algebra.

Definition 1.12.4 *(Character)*

For a representation ϕ, a character $\chi(X) = tr(\phi(X))$ is

Example 1.13 *(Representation of Finite Group)*

Mashcke's theorem.

Representation of $\mathbb{Z}_2$ makes super graded vector space.

Representation of $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 1.14 *(Modular Representation)*

If the vector space V of $GL(V)$ is positive characteristic $char(V) = p$.

Example 1.15 *(Unitary Representation)*

A unitary representation of a group G on a Hilbert space H is a map

$\pi : G \to U(H)$

where $U(H)$ is a unitary group of H such that $\pi(g)$ is a unitary operator. This means, $\pi(g) : H \to H$ is a map such that $UU^* = U^*U = Id$

From here, we will introduce some trace describing some math/physics invariant.

Example 1.16 *(Parition Function (Trace))*

Paritition function in quantum mechanical discrete system is

$Z = tr(e^{-\beta\hat{H}})$

Example 1.17 *(Curvature (Trace))*

Let F be strength field tensor. Then the trace $Tr(F^2) = Tr(F \wedge \star F)$ is the curvature, where $\star$ is Hodge star.

Example 1.18 *(Equivariant map)*

Let $S : V_\delta \to V_\delta$ be a representation.

$\int_{SO(3)} \chi_\delta(g^{-1})\delta(g)vdg$

Now, let's move on to wedge product.

Definition 1.18.1 *(Wedge Product)*

Definition 1.18.2 *(Volume Form)*

From geometric perspective, a vector is a one-dimensional arrow.
If we use wedge product,
In wedge product,

Definition 1.18.3 *(Determinant)*

For a matrix $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$
Determiant is given by wedge product

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n = det(v_1, v_2, \cdots, v_n) e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

where e_i *is the standard basis.*

Definition 1.18.4 *(Super Manifold)*

A supermanifold M *of dimension* $p|q$ *is a sheaf of topological space* M *with a sheaf of superalgebras. Usually denoted* $\mathscr{O}_M$ *locally isormophic to* $C^\infty(\mathbb{R}^p) \otimes \bigwedge^{\cdot}(\xi_1, \cdots, \xi_q)$*. So far the coordinate is* $(x_1, \cdots, x_p, \xi_1, \cdots, \xi_q)$ *and the dimension is* $n = p + q$*. (Indeed,* $\mathbb{R}^{n|m}$ *is a super vector space (or supergraded)* $\mathbb{R}^{n|m} = V_0 \oplus V_1$*).*

Definition 1.18.5 *(Odd/Even Symplectic Form)*

This is an analogy of symplectic form in a supermanifold.

Odd symplectic form is $\omega = \Sigma_i dx_i \wedge d\xi_i$

Even symplectic form is $\omega = \Sigma_i dp_i \wedge dq_i + \Sigma_j \frac{\epsilon_j}{2}(d\xi_j)^2$

where x_i*,* p_i*, and* q_i *are even coordinate and* ξ_j *is an odd coordinate, and* $\epsilon_j = \pm 1$ *is parity.*

Definition 1.18.6 *(Boson/Fermion)*

Boson = integer spin Fermion = half-integer spin Hadron =
classifcation of particle in physics

2 Elementary Physics

Before QFT, we'll start from classical mechanics including Maxwell and Einstein equations without using quantum mechanics, which is a non-quantum perspective of $U(1)$ gauge theory and SUGRA, and then we'll introduce the foundation of quantum mechanics: what it means by wave function as a quantum state of a particle.

2.1 Electron and Gravity

Definition 2.1.1 *(Maxwell Equation)*

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot B = 0$$
$$\nabla \times E = -\frac{\partial B}{\partial t}$$
$$\nabla \times B = \mu_0 (J + \epsilon_0 \frac{\partial E}{\partial t})$$

Definition 2.1.2 *(Magnetic Monopole)*
In electro-magnetic field theory, electric force and magnetic force are considered to be symmetric. However, electron and proton could independently exist, which are resp negatively/positively charged, while magnetism cannot exists as N of S separately, by the experiment up to this moment, which sounds unreasonable if symmetry is considered. We'll hypothesize the existence of magnetism only contains either N or S, which is called magnetic monopole, which is formally defined by taking dual of Maxwell equation by exchanging E and B.

We claim the existence of magnetic monopole purely theoretically, but how do we prove this is true? In gauge theory, magnetic monopole will be generated when $SU(2) \times U(1)$ electric weak theory has spontaneous symmetry breaking down to $U(1)$ electric gauge theory.

Definition 2.1.3 *(Einstein Equation)*
$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$ *where* $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

where $G_{\mu\nu}$ is Einstein tensor, Λ is the cosmological constant, $T_{\mu\nu}$ is the stress energy tensor, and $\kappa = \frac{8\pi G}{c^4}$ is the Einstein gravitational constant.

Definition 2.1.4 *(Kaluza-Klein Equation)*
KK equation is an Einstein equation, whose tensor is five dimensional space ($\mu, \nu = 1, 2, 3, 4, 5$), while assuming the solution manifold is flat. This KK equation space is miraculous since it's four-dimensional submanifold $\mu, \nu = 1, 2, 3, 4$ actually describes the general relativity of our universe, plus other diemensions $\mu = 1, 2, 3, 4, \nu = 5$ generates $U(1)$ gauge symmetry(Maxwell Equation), so we can describe both gravity and electro-magnetism at once. This idea is used to

generalize IIA string theory 4-dimensional non-compact space to 5-dimensional space in M-theory, the theory adding 1 more dimension on it. Mathematically, KK equation is given by using the replacement of metric tensor. Kaluza-Klein tensor $\tilde{g}_{ab}$ *is*

$$\tilde{g}_{ab} = \begin{bmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{bmatrix}$$

where $g_{\mu\nu}$ *is a space-time metric, and* A_μ *is gauge tensor.*

Definition 2.1.5 *(Einstein Instanton)*

Definition 2.1.6 *(de-Sitter/Anti de Sitter Space)*
DS/AdS space, mathematically noted by dS^n *or* AdS^n*, is a solution of Einstein equation*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \text{ where } T_{\mu\nu} = 0$$

and this is dS space if $\Lambda > 0$ *or AdS space if* $\Lambda < 0$

Definition 2.1.7 *(AdS/CFT Correspondence)*
AdS/CFT correspondence says the correspondence of the physics defined on $AdS^5 \times S^5$ *and that of* CFT^4*. Consider* $AdS^5 \times S^5$ *is 10-dimensional geometry, but let's still consider the boundary of* AdS^5 *defined by conformal compactification using Poincare coordinate (but idk howto do this) is* $\mathbb{R}^{1,3}$*, which is 4-dimensional, where* CFT^4 *should be defined on it. So it's geometrically the same. We also need to find the physical operation respectively on* $AdS^5 \times S^5$*(SURGRA) and* CFT^4*(QFT) sides, and consider their correspondence.*

AdS^5	CFT^4	
$\phi(x)$	$\mathscr{O}(x)$	*operators*
$g_{\mu\nu}$	$T_{\mu\nu}$	*gravity & energy momentum tensor*
p-form A_p	*operator* J_p *of degree p for current*	*1-form*

In the above assumption, respectively, AdS^5 *has* $SO(2,4)$ *symmetry,* S^5 *has* $SO(6)$ *symmetry, while* CFT^4 *has* $SO(2,4)$ *symmetry, and its R-symmetry is* $SU(4)_R \cong SO(6)$*.*

2.2 Particles and Quantum States

In physics, quantum mechanics studies interactions of particles. The particles in physics are, for instance, quark, neutrino, gauge, Higgs particles etc, which are largely classified by their spins: Ferminon(half-integer spin) and Boson(integer spin), and by using interaction of partiles, we can describe four forces, strong, weak, electro-magnetic, gravity forces in quantum way.

Note 2.3 *(Is Quantization = Discretization?)*
"Quantum" in quantum mechanics originally mean something about discrete. Quantization is a mapping of a real valued function to an operator in $End(\mathfrak{H})$, which is a linear operator. A linear operator has representation matrix, thus having eigenvalues from linear algebra argument, and each eigenvalue says discreteness. The most common example should be that Schrödinger equation has discrete energy level of hydrogen atom.

Definition 2.3.1 *(Four Forces)*

- *(Strong Forces)*
  QCD (chromodynamics), $SU(3)$-symmetry. 8-dimensional gluon. Connecting proton and neutron in atomic nucleus.
- *(Weak Forces)*
  QFD (flavordynamics) $SU(2)$-symmetry. 3-dimensional Z/W-boson. Forces for nuclear reaction. β-collapse transforms proton to neutron.
- *(Electoric Magnetic Forces)*
  QED (electrodanmics) $U(1)$-symmetry. 1-dimensional photon.
- *(Gravity Forces)*
  It cannot be described in QFT. We need string theory.

Definition 2.3.2 *(Wave Function)*
In quantum mechanics, each particle has a wave function, multi-particle system can also be described by wave functions. The synthesis of the wave functions is multiplication. We cannot find the precise value of n-body particle system, but we have Hartree approximation, and synthesis of the wave function will be

$$\phi(x_1, x_2) = \phi(x_1)\phi(x_2)$$

If the particle is Fermion,

$$\phi(x_1, x_2) = \phi(x_1)\chi(x_2) - \phi(x_2)\chi(x_1)$$

or we can use determinant for easily generalizing the higher dimension.

$$\phi(x_1, x_2) = \begin{bmatrix} \phi(x_1) & \phi(x_2) \\ \chi(x_1) & \chi(x_2) \end{bmatrix}$$

Definition 2.3.3 *(Pauli Exclusion Principle)*
Pauli exclusion principle says that no two electrons on the atom have the same spin.

Definition 2.3.4 *(Quantum Number)*
Four different kind of quantum numbers can specify the quantum state of the particle.

- *Principal Quantum Number (n)*
  K,L,M orbit
  $n \in \mathbb{N}$

- *Azimuthal Quantum Number (l)*
  s, p, d, f orbits
  $0 \leqq l \leqq n-1$

- *Magnetic Quantum Number (m_l)*
  $|m_l| \leqq l$

- *Spin Quantum Number (m_s)*
  spin is $\pm\frac{1}{2}$, so two possibilities.

These quantum numbers

Definition 2.3.5 *(Klein-Gordon Equation)*
If a Klein-Gordon field $\phi(x,t)$ describes mass m free particle, then Klein-Gordon equation is

$$[\Box + \mu^2]\phi(x,t) = 0$$

where c is speed of light, $\hbar$ is reduced Plank constant, and $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ and $\mu = \frac{mc}{\hbar}$).

Definition 2.3.6 *(Radioactive Decay)*

- *(α Decay)*
  α particle (helium nucleus) separated by atomic nucleus not by strong force but by tunnel effect. α wave is a radioactive wave of α particle.

- *(β Decay)*
  β decay is an effect of neutron changing to proton by weak force. α wave is a radioactive wave of an electron.

- *(γ Decay)*
  It doesn't change the mass and charge of an atom.

Definition 2.3.7 *(Quantum State)*
In quantum mechanics, all particles are considered to be waves functions, which means quantum state $|\psi\rangle \in \mathfrak{H}$ is an element of the Hilbert space $\mathfrak{H}$. Here, the notation $|\psi\rangle$ is also called a ket notation. We'll also define bra notation as the dual notation. Bra is also a vector but is the dual, which means we define $\langle\psi| \in \mathfrak{H}^*$ so that the inner product will be $\langle\psi|\psi\rangle = 1$, and $\langle\psi|\phi\rangle = 0$ if different waves are orthogonal. The very important example of ket is quantum vacuum state $|0\rangle \in \mathfrak{H}$ is norm 1, so $||\,|0\rangle\,|| = 1$.

Definition 2.3.8 *(S-Matrix and Paritition Function)*
S-matrix is an operator s.t $|out\rangle = S\,|in\rangle$ *where* $|in\rangle$ *is in-state and* $|out\rangle$ *is out-state. Also, T-matrix is another operator* $S = 1 + iT$. *This matrix is explictly given by partition functions.*

$$T \sim \frac{\delta^n Z[J]}{\delta J^n}|_{J=0}$$

Definition 2.3.9 *(Fock Space)*
Fock space $F = \oplus_{n\in\mathbb{N}} S_\nu H^{\otimes n}$ *is the generalization of the quantum state space. Of course the Hilbert space* H *describes quantum state in* 1*-particle, and the tensor product* $H \otimes H$ *describes* 2*-particle system, so in the higher degree.* S_ν *takes values* $S_\nu = \pm 1$ *and if* $S_\nu = 1$*, it's Boson, and* $S_\nu = -1$*, it's Fermion.*

Definition 2.3.10 *(Quantum Fluctuation)*
In classical physics, vacuum space is considered to be absolute empty, but from quantum point of view, the vacuum space is actually full of noise, thus one of the consequence is Heisenberg uncertanty law denoted by $\Delta E \Delta t > \hbar$. *The particle can often appear from nothing but together with anti-particle, or they join them to compensate and disappear, and there are for example, Higgs particle, and election and positive-electron. This says, that in quantum minuscule level, there is a little distorsion of wave functions in vacuum space due to the noise, and this is called quantum correction. The quantum correction is largely classified by perturbation and non-perturbation effects.*

2.4 Superconductivity

Superconductivity is a phonomena of nullifying resistance in elements, when they are cooled down to very low temperature. This occurs since, in the very low temperature, electrons can generate cooper pairs. Cooper pair is a pair which two electrons generate. An electron is a fermion, which has $\frac{1}{2}$ spin, but once it makes a pair of two, one of them compensates the other spin, and its total spin is integer, thus cooper pair behave like one boson particle. Boson, is for example, a photon to define Coulomb force, but it doesn't require resistance, thus similarly cooper pair behaves like a gauge particle without resistance, namely there is no resistance in the current. This is superconductivity. In detail, two electrons can attract each other due to phonon, which occurs when an election can distort the lattice (crystal forms proton in equal distances, thus forming lattice structure), and the distortion attracts another election, and as a result elections generates pairs of two electrons, and in the pair one of the electrons has the opposite spin with the other. The classical perspective of superconductivity is Landau-Ginzburg theory, that is used to study B-model side of mirror symmetry according to Marc Gross, while the quantum perspective of superconductivity is Bardeen-Cooper-Schrieffer theory.

Definition 2.4.1 *(Landau-Ginzburg Model)*
A physical model of superconductivity. Landau-Ginzburg equation is

$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m}(-i\hbar\nabla - 2e\mathbb{A})^2\psi = 0$ *(LG equation)*

which is derived from variational calculus of free energy

$F = F_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}|(-i\hbar\nabla - 2e\mathbb{A})\psi|^2 + \frac{|\mathbb{H}|^2}{2\mu_0}$ *(Free energy)*

where F_n is free energy of state of normal conductivity.

2.5 Classical Dynamics

Definition 2.5.1 *(Navier-Stokes Equation)*
Generalization formula of fluid dynamics. Its calculation contains velocity of flow, mass, viscosity, and compression. The general solution for 2.5 dimension has been given, using Ladyzhenskaya's inequality, but not 3 dimension yet. 3-dimensional NS equation is millenium prize problem by CMI.

- $\frac{\partial\rho}{\partial t} + div(\rho v) = 0$
- $\frac{\partial(\rho v)}{\partial t} + div(\rho vv) = div\sigma + \rho g$

where ρ is density, σ is stress, v is a velocity, g is a gracvity.

$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{1}{\rho}grad(p) + \frac{\mu}{\rho}\delta v + \frac{\chi+\frac{1}{3}\mu}{\rho}grad(\Theta) + \frac{1}{\rho}grad(v\cdot grad(\mu)) + \frac{1}{\rho}rot(v\times grad(\mu)) - \frac{1}{\rho}v\Delta\mu + g$

where $\Theta = div(v) = \frac{1}{2}tr(e)$, χ is volume viscosity, σ is shear viscosity.

Definition 2.5.2 *(Fick's Law for Diffusion)*
Corresponding diffusion equation derived from Fick's law is

- *(First Fick's Law)*
  $J = -D(x)\nabla n(x,t)$
- *(Second Fick's Law)*
  $\frac{\partial n(x,t)}{\partial t} = \nabla\cdot(D(x)\nabla n(x,t))$

2.6 Ising Model

describing classical mechanics behavior by accumlation of particle movements, i.e. the melting the ice, evapolation.

Definition 2.6.1 *(Entropy)*

Definition 2.6.2 *(Wick Transformation)*
$\frac{1}{k_B T} \mapsto \frac{it}{\hbar}$

Definition 2.6.3 *(Ising Model)*
Assume an physical object consists of atoms forming a lattice structure Λ*, and each atom has a spin* $\sigma_k \in \{\pm 1\}$*. The Hamiltonian of the system is*

$$H(\sigma) = \Sigma_{<ij>,i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - \mu\Sigma_j h_j\sigma_j$$

picking i *and* j *are adjacent each others where* J_{ij} *is interaction,* μ *is , * h_j *is an external magnetic field interacting with it.*

$$P_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta}$$

where $\beta = \frac{1}{k_B T}$

The expectation value of f *is given by*

$$< f >_\beta = \Sigma_\sigma f(\sigma) P_\beta(\sigma)$$

Definition 2.6.4 *(Random Cluster Model)*
That's probability theory that generalizes and unifies Ising model.

3 Lie Group

We will introduce several basic real Lie group and its representation, because of studying harmonic properties of diffyq arising from physics problems. First of all, we'll briefly define Lie group representation and its properties, and let's think why this definition is important. We are interested in representation theory because we'll reduce complexity of math problem from group theory to linear algebra, because trace is an important invariant in representation, called character, in particular, also called Killing form.

Now, what kind of Lie groups we are intersted in. SO(3) is double cover of SU(2). Pauli metrices quotient by i generate $\mathfrak{su}(2)$. The Lie algebra of Lorentz group is equivalent to twice of su(2). Also, SU(2) is symplectic. So many lie group might be rooted in $\mathfrak{su}(2)$.

Definition 3.0.1 *(Lie Group Representation)*

The formal defintion of Lie group representation is as follows: For G *is a Lie group and* V *is a k-vector space, Lie group representation is a morphism*

$$\Pi : G \to GL(V)$$

Definition 3.0.2 *(Unitary Representation)*

In particular, for the vector space is a Hilbert space $V = \mathfrak{H}$, and π is a swew-adjoint operator, $\pi : G \to U(\mathfrak{H})$ is unitary representation if $\forall g \in G$, $\pi(g)$ is unitary, that is $\pi(g)^* = \pi(g)^{-1}$, or $< \pi(g)v, w > = < v, \pi(g^{-1})w >$. For example, by using Haar intgral, we could define inner product as

$< v, w > = \int_G < \pi(x)v, \pi(x)w >_1 dx$

This is indeed unitary because

$$\begin{aligned} < \pi(g)v, \pi(g)w > &= \int_G < \pi(x)\pi(g)v, \pi(x)\pi(g)w >_1 dx \\ &= \int_G < \pi(xg)v, \pi(xg)w >_1 dx \\ &= \int_G < \pi(y)v, \pi(y)w >_1 dy \\ &= < v, w > \end{aligned} \tag{1}$$

Strongly continuous/smooth/analytic unitary representation is

$g \mapsto \pi(g)\xi$ *is a norm continuous/smooth/analytic function.*

Definition 3.0.3 *(Haar Measure)*

- *(Pontryagin duality)*
  For a group G and a circle group T, a set of continuous functions is denoted by hat
  $\hat{G} = Hom(G, T)$
  and the double hat is dual as $\hat{\hat{G}} \cong G$

- *(Peter-Weyl I)*
  The set of matrix coefficients of G is dense in the space of continuous complex functions $C(G)$ on G, equipped with the uniform norm.

- *(Peter-Weyl II)*
  Let ρ be a unitary representation of a compact group G on a complex Hilbert space H. Then H splits into an orthogonal direct sum of irreducible finite-dimensional unitary representations of G.

- *(Haar measure)*
  Haar measure is a measure of a topological group G, and let G be a locally compact Hausdorff group, and we have a σ algebra generated by all open subsets of G. Let's determine the structure of Haar measure μ.

– *(Left/Right Translation)*
  $\mu(S) = \mu(gS) = \mu(Sg)$ *for* $g \in G$ *and* $S \subset G$.
– *(Compact)*
  $\mu(K) < \infty$ *for all* $K \subset G$ *compact.*
– *(Outer Regular)*
  $\mu(S) = inf\{\mu(U) : S \subset U, U \text{ is open}\}$
– *(Inner Regular)*
  $\mu(U) = sup\{\mu(K) : K \subset U, K \text{ is compact}\}$

Then, if G *is compact, there is a unique Haar measure with* $\mu(G) = 1$ *and* $\mu(U) > 0$ *for all* U.

- *(Spectrum of* C^**-algebra)*

  C^**-algebra is a Banach algebra over a field of complex numbers, together with a map* $x \mapsto x^*$ *for* $x \in A$.

Definition 3.0.4 *(Schur's Lemma)*

For a representation $\pi : G \to GL(V)$,

π *is irreducible implies* $End_G(V) = \mathbb{C}Id_V$

If π *is unitarizable, and* $End_G(V) = \mathbb{C}Id_V$, *then* π *is irreducible.*

Definition 3.0.5 *(Measure)*

$\int_G dx = 1$

$< f, g >= \int_G f(x)\overline{g(x)}dx,\ f, g \in L^2(G)$

Definition 3.0.6 *(Schur Orthogonality)*

The matrix coefficient of π *as a map* $m : G \to \mathbb{C}$ *is*

$m(g) = m_{v,w}(g) =< \pi(g)v, w >$

or alternatively this matrix is

$m_{v,w}(g) = Tr(\pi(x)L_{v,w})$ *where* $L_{v,w}(u) =< u, w > v$

$C(G)_\pi$ *is a linear span of the space of matrix coefficients* $m : G \to \mathbb{C}$.

Definition 3.0.7 *(Character)*

A character $\chi_\pi : G \to \mathbb{C}$ is a morphism, and trace of a representation $\chi_\pi(g) = Tr(\pi(g))$.

From orthogonality,

$\chi_\pi(x) = \Sigma < \pi(x)e_i, e_i >= \Sigma m_{e_i,e_i}(x)$

Also, if $\pi \sim \pi'$, the inner product is

$< \chi_\pi, \chi_{\pi'} >_{L^2}= 1$

or if $\pi \nsim \pi'$,

$< \chi_\pi, \chi_{\pi'} >_{L^2}= 0$

Definition 3.0.8 *(Peter-Weyl)*

$L^2(G) = \hat{\oplus}_{\delta \in \hat{G}} C(G)_\delta$

3.1 U(1) (Electro-Magnetic)

Definition 3.1.1 *(U(1) representation)*

On $\mathbb{C}^n$, We take Hermitian inner product

$< u, v >_{inv}= \int_0^{2\pi} < R(e^{i\theta})u, R(e^{i\theta})v > \frac{d\theta}{2\pi}$

which we prove is invariant and $< R(e^{i\theta})u, R(e^{i\theta})v >_{inv}=< u, v >_{inv}$ for all $u, v \in \mathbb{C}^n$ and $e^{i\theta} \in U(1)$. Indeed, $< u, v >_{inv}$ is an Hermitian inner product.

3.2 SO(3) (Spherical Harmonic)

Definition 3.2.1 *(SO(3))*
SO(3) is rotation.

$O(3) = \{A \in M_{3,3} | AA^T = I\}$

$SO(3) = \{A \in M_{3,3} | AA^T = I, det(A) = 1\}$

Definition 3.2.2 *(Harmonics Analysis)*

Spherical harmonics is a solution of Laplacian equation, and for example, Schrödinger equation is a diffyq of a Hamiltonian operator, which contains Laplacian.
$\hat{H}\psi = E\psi$ *where* $\hat{H} = \nabla + V$.

Definition 3.2.3 *(Spherical Harmonic Functions)*

Harmonic funciton is $f : \mathbb{R}^n \to \mathbb{C}$ *harmonic iff* $f \in C^\infty(\mathbb{R}^n)$ *and* $\nabla f = 0$*. If* $n = 3$*,*

$\mathfrak{h}_l = \{p \in P_l(\mathbb{R}^3 | \nabla p = 0\}$

where P_l *is the space of homogeneous harmonic polynomials in* $P_l(\mathbb{R}^3)$*.*

Now we define $SO(3)$*-module* $\mathfrak{h}_l$ *as the representation* $\rho_l : SO(3) \to GL(\mathfrak{h}_l)$*.*

Definition 3.2.4 *(Spherical Harmonics)*

$Y_l^m = (-1)^m (\frac{(2l+1)(l+m)!}{4\pi(l-m)!})^{\frac{1}{2}} e^{im\phi} P_l^m(cos\theta)$

where

$P_l^m(s) = \frac{(1-s^2)^{-\frac{m}{2}}}{2^l l!} \frac{d^{l-m}}{ds^{l-m}}(s^2-1)^l$*,* $|m| \leqq l$*,* $l \in \mathbb{N}$ *constitute basis for* $C(S^2)_l \subset L^2(S^2)$

3.3 Lorentz Group

As seen before, SO(3) representation for the Laplacian equation as Schrödiger equation was time-independent system, while time this the Lorentz group representation is for solving diffyq of time-dependent system e.g. Klein-Gordon equation. One of the difficulty is that Lorentz group is not compact, hence the representation might not be unitary.

Lorentz group is a Lie group structure of Minskowski space. The problem is Minkowski space is not symmetric, or it's not a metric space, since it contains imaginary number i. Conversely, we consider metrization starting from inner product (so metrization is symmetrization right?).

Definition 3.3.1 *(Indefinite Orthogonal Group)*

Indefinite orthogonal group is denoted as $O(p, q)$*. Let* $g \in M_{p+q}$ *be a diagonal matrix*

$g = diag(1, \cdots, 1, -1, \cdots -1)$ *of* p *of* 1 *and* q *of* -1*.*

We definte symmetric bilinear form

$[x, y]_{p,q} =< x, gy >= x_1y_1 + \cdots x_py_p - x_{p+1}y_{p+1} - \cdots + x_{p+q}y_{p+q}$

where $< \bullet, \bullet >$ *is an inner product on* $\mathbb{R}^{p+q}$*.*

$O(p,q) = \{A \in M_{p+q}(\mathbb{R}) | [Ax, Ay]_{p,q} = [x,y]_{p,q}, \forall p,q \in \mathbb{R}^{p+q}\}$

Or more explicitly,

$gA^T g = A^{-1}$.

Definition 3.3.2 *(Lorentz Group)*

Lorentz Group is an indefinite orthogonal group $O(1,3)$ preserving the quadratic form

$(t,x,y,z) \mapsto t^2 - x^2 - y^2 - z^2$

or let's consider $SO(1,3)$. This transformation is, in other words, an element $\Lambda \in SO(1,3)$ acts on $\mathbb{R}^{1,3}$ and

$x^\rho \mapsto x'^\rho = \Lambda^\rho_\mu x^\mu$

note Minskowski norm $\Lambda^T g \Lambda = g$

Definition 3.3.3 *(Lorentz Lie Algebra)*

J_i where $i = 1,2,3$ are generators of $SO(3)$, and we let K_i where $i = 1,2,3$ be

$$K_x = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; K_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; K_z = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

and

- $[J_i, J_j] = i\epsilon_{ijk} J_k$
- $[J_i, K_j] = i\epsilon_{ijk} K_k$
- $[K_i, K_j] = -i\epsilon_{ijk} J_k$

Definition 3.3.4 *($\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$)*

We'll claim $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$, and thus the irreducible representations of $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$ are the same.

If we use the complexification $\frac{so}{(}1,3)_{\mathbb{C}} = \frac{so}{(}1,3) \otimes \mathbb{C}$, the Lorentz algebra works nicely. So, if we use $M_i^\pm = \frac{J_i \pm iK_i}{2}$ (so that we can reconstrct original real Lie algebra $K_i = M_i^+ - M_i^-$ and $J_i = M_i^+ + M_i^-$ can be retrieved from $M_i\pm$)

Their relation is as follows

- $[M_i^{\pm}, M_j^{\pm}] = i\epsilon_{ijk}M_k^{\pm}$
- $[M_i^{+}, M_j^{-}] = 0$

Thus we hve following picture

$\mathfrak{so}(1,3) \hookrightarrow \mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$

Now, further as Lie group $SO^{+}(1,3) \cong PSL(2,\mathbb{C}) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$*, so* $SL(2,\mathbb{C})$ *is a double cover, that is,* $SL(2,\mathbb{C}) \cong Spin(1,3)$

$\mathfrak{so}(1,3) \hookrightarrow \mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \hookleftarrow \mathfrak{sl}(2,\mathbb{C})$

and finally $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$*. This means their irreducible reprensetation* $\mathfrak{so}(1,3)$ *and* $\mathfrak{sl}(2,\mathbb{C})$ *are the same.*

Definition 3.3.5 *(Representation of Lorentz Group For Klein-Gordon Equation)*

Let $\phi(x^{\mu})$ *be a scalar field, and Loretz transformation be*

$\phi'(x^{\mu}) = phi(\Lambda^{-1}x^{\mu})$

Now Lagrangian of Klein-Gordon equation $\partial_{\mu}\partial^{\mu}\phi + m^2\phi = 0$ *is*

$\frac{L}{=}\frac{1}{2}(\partial_{\mu}\phi\partial^{\mu}\phi - m^2\phi^2)$

The full group of the spacetime isometries is Poincare group.

3.4 AdS Group

Definition 3.4.1 *(de-Sitter Space)*

de-Sitter space dS_n *is maximally symmetric Lorentizian manifold with constant positive scalar curvature. It is the Lorentizian analogue of n-sphere.*

$ds^2 = -dx_0^2 + \Sigma_{i=1}^{n} dx_i^2$
dS_n *is submanifold of* $-dx_0^2 + \Sigma_{i=1}^{n} dx_i^2 = \alpha$.

Or by using Lie group, dS_n *is given by quotient*

$O(1,n)/O(1,n-1)$

Definition 3.4.2 *(Anti de-Sitter Space)*

Our universe is believed to be of positive cosmological constant, but we'll consider Anti de-Sitter space AdS_n, which is negative curvature version of dS_n, and the cosmological constant Λ is negative.

Or by using Lie group, AdS_n is given by quotient

$AdS_n = O(2, n-1)/O(1, n-1)$

Definition 3.4.3 *(Anti de-Sitter Algebra)*

We will constrct $\mathfrak{o}(2,n)$. Consider $\mathfrak{o}(1,n) \subset \mathfrak{o}(2,n)$ is a subalgebra, and explicitly, it is generated by

$$\mathfrak{H} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \leftarrow & v^t & \rightarrow \\ \cdots & \uparrow & & & \\ 0 & v & & B & \\ \cdots & \downarrow & & & \end{bmatrix}$$

where B is a skew-symemtry matrix. The complementary part of the generators are explicitly

$$\mathfrak{Q} = \begin{bmatrix} 0 & a & \leftarrow & w^t & \rightarrow \\ -a & 0 & \cdots & 0 & \cdots \\ \uparrow & \cdots & & & \\ w & 0 & & 0 & \\ \downarrow & \cdots & & & \end{bmatrix}$$

and thus generators of $\mathfrak{o}(2,n)$ are $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{Q}$. The Lie bracket structures are

- $[\mathfrak{H}, \mathfrak{Q}] \subset \mathfrak{Q}$
- $[\mathfrak{Q}, \mathfrak{Q}] \subset \mathfrak{H}$

Thus, anti de-Sitter space is reductive homogeneous space, and a non-Riemannian symmetric space.

Definition 3.4.4 *($\mathfrak{o}(3,2)$ Case)*

We consider the case of $n = 4$, then the group is $\mathfrak{G} = \mathfrak{so}(3,2)$, which is 10-dimensional. We will study the sturucture by complxification, actually identified as $\mathfrak{G}^{\mathbb{C}} = \mathfrak{so}(5,\mathbb{C})$, and we can use the same basis for both. we have standard triangular decomposition

$\mathfrak{G}^{\mathbb{C}} = \mathfrak{G}_- \oplus \mathfrak{H} \oplus \mathfrak{G}_+$

where $\mathfrak{H}$ is the diagonal two-dimensional Cartan subalgebra generated by H_1 and H_2, and $\mathfrak{G}_+$ and $\mathfrak{G}_-$ are generated by X_i^+ and X_i^- where $i = 1,2,3,4$ resp.

These matrices are explicitly given by

$$H_1 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} \; H_2 = \begin{bmatrix} e_2 & 0 \\ 0 & -e_1 \end{bmatrix}$$
$$X_1^+ = \begin{bmatrix} \sigma_+ & 0 \\ 0 & -\sigma_+ \end{bmatrix} \; X_1^- = \begin{bmatrix} \sigma_- & 0 \\ 0 & -\sigma_- \end{bmatrix}$$
$$X_2^+ = \begin{bmatrix} 0 & \sigma_- \\ 0 & 0 \end{bmatrix} \; X_2^- = \begin{bmatrix} 0 & 0 \\ \sigma_+ & 0 \end{bmatrix}$$
$$X_3^+ = \begin{bmatrix} 0 & 1_2 \\ 0 & 0 - \sigma_+ \end{bmatrix} \; X_3^- = \begin{bmatrix} 1_2 & 0 \\ 0 & 0 \end{bmatrix}$$
$$X_4^+ = \begin{bmatrix} 0 & \sigma_+ \\ 0 & 0 \end{bmatrix} \; X_4^- = \begin{bmatrix} 0 & 0 \\ \sigma_- & 0 \end{bmatrix}$$
$$e_1 = \frac{1}{2}(1 + \sigma_3) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$e_2 = \frac{1}{2}(1 - \sigma_3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\sigma_- =^t \sigma_+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

where σ_i are 2×2 Pauli matrices. Using this basis, the AdS algebra $\mathfrak{G}$ has the following Bruhat decomposition

$$\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{M} \oplus \mathfrak{A} \oplus \mathfrak{N}_+$$

in which the four subalgebras have physical meaning related to the fact that $\mathfrak{G}$ is also the conformal algebra of the 3-dimensional Minkowski space-time M^3.

Definition 3.4.5 *(Verma Module)*

Verma module is for classification of finite dimensional representation of $\mathfrak{g}$.
Let $\mathfrak{h} \subset \mathfrak{g}$ is CSA, and R is the corresponding root system. λ is the highest weight and v is the corresponding vector.
$W_\lambda = \{Y_{\alpha_{i_1}} Y_{\alpha_{i_2}} \cdots Y_{\alpha_{i_M}} \cdot v\}$

Definition 3.4.6 *(Representation of AdS algebra)*

Using quantum numbers E_0 and s_0 called energy and spin, we characterize the explicit realization of positive energy UIR $D(E_0, s_0)$

$$(\partial_z)^{m_1} \hat{\phi} = 0$$

For example, if $E_0 = \frac{1}{2}$ and $s_0 = 0$. These are equations

$\partial_z \hat{\phi} = 0$ *where* $m_1 = 1$ $(\partial_x^2 - \partial_u \partial_v)\hat{\phi} = (\partial_0^2 - \partial_1^2 - \partial_2^2)\hat{\phi}\Box\hat{\phi}\hat{\phi}'$ *where* $m_3 = 2$

3.5 Supersymmetry Algebra(BPS State)

BPS state is a condition of a supersymmetry algebra used for supersymmetry physics, and it is an invariant which preserves mass in physics. Like Schrödinger with SO(3), consider Hilbert space $\mathfrak{H}$ and its self-adjoint operator $H : \mathfrak{H} \to \mathfrak{H}$.

Definition 3.5.1 *(BPS State)*

Let M be the mass of the state, and Z is the linear combination of the central charges, and the inequality $M \leqq |Z|$ is called BPS bound. The supersymmetry algebra is caleld BPS state if $M = |Z|$.

Definition 3.5.2 *($N = 2$ supersymmetry in dimension $d = 1$)*

We define Lie superalgebra A by

$A = A^0 \oplus A^1$ *where*

$A^0 = \mathbb{C} \cdot H$

$A^1 = \mathbb{C} \cdot Q \oplus \mathbb{C} \cdot \overline{Q}$

- $[Q, \overline{Q}] = 2H$
- $[Q, Q] = 0$
- $[\overline{Q}, \overline{Q}] = 0$
- $[Q, H] = 0$
- $[\overline{Q}, H] = 0$

$\mathbb{Z}/2\mathbb{Z}$-graded representation of A means $A^i : H^j \to H^{j+i}$, and if the representation is unitary, H is formally self adjoint operator, and Q and $\overline{Q}$ are formally self adjoint with one another.

For example, the representation is unitary if we take $Q = d$, $\overline{Q} = d^*$, and $H = \frac{1}{2}\Delta$ where

$\mathfrak{H} = \mathfrak{H}^0 \oplus \mathfrak{H}^1$ *where* $\mathfrak{H}^0 = \oplus_k \Omega^{2k}_{L^2}(M)$ *and* $\mathfrak{H}^1 = \oplus_k \Omega^{2k+1}_{L^2}(M)$.

$2 < \psi, H\psi > = < \psi, Q\overline{Q}\psi > + < \psi, \overline{Q}Q\psi > = ||Q\psi||^2 + ||\overline{Q}\psi||^2 \leqq 0$, *and the norm is nondegenerate, so $H\psi = 0$ iff $Q\psi = 0$ and $\overline{Q}\psi = 0$.*

Definition 3.5.3 *($N = (2, 2)$ supersymmetry in dimension $d = 2$)*

4 odd generators $Q^{\pm}$ *and* $\overline{Q}^{\pm}$

and 6 *even generators* P^0, P^1, B, Z, $\overline{Z}$, F

where B *is generator of* $\mathfrak{so}(1,1)$ *and* Z *and* $\overline{Z}$ *are central generators. Their algebraic relation are defined as*

- $[Q^+, \overline{Q}^+] = P^+$
- $[Q^-, \overline{Q}^-] = P^-$
- $[Q^+, Q^+] = Z$
- $[Q^-, Q^-] = \overline{Z}$

where we defined $P^{\pm} = P^0 \pm P^1 = H \pm P^1$.

The generator F *as Fermion number obeys*

- $[F, Q^{\pm}] = -Q^{\pm}$
- $[F, \overline{Q}^{\pm}] = -\overline{Q}^{\pm}$

The application of $N = 2$ and $d = 4$ supersymmetry is supersymmetric blackhole entropies.

Definition 3.5.4 *($N = 2$ and $d = 4$ supersymmetry)*

For $N = 2$ *and* $d = 4$ *superalgebra, the odd part of generators Q have relations as*

$$\{Q^A_\alpha, \overline{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B$$

$$\{Q^A_\alpha, Q^B_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\overline{Z}$$

$$\{Q_{\dot{\alpha}A}, Q_{\dot{\beta}B}\} = 2\epsilon^{\alpha\beta}\epsilon_{AB} Z$$

where α *and* $\dot{\beta}$ *are Lorentz indices, and* A *and* B *are R-symmetry indices.*

Definition 3.5.5 *(R-matrices)*

$$R^A_\alpha = \xi^{-1} Q^A_\alpha + \xi \sigma^0_{\alpha\dot{\beta}} \overline{Q}^{\dot{\beta}B}$$

$$T^A_\alpha = \xi^{-1} Q^A_\alpha - \xi \sigma^0_{\alpha\dot{\beta}} \overline{Q}^{\dot{\beta}B}$$

Consider a state ψ which has a momentum $(M,0,0,0)$, and if we apply the following operator to the state

$$(R_1^1 + (R_1^1)^\dagger)^2\psi = 4(M + Re(Z\xi^2))\psi$$

Definition 3.5.6 *($N = 1$ supersymmetry)*

Definition 3.5.7 *(Isometry Group (ISO(p,q)))*

A isometry group of a metric space is the set of all bijective isometries (e.g. distance-preserving maps).

Definition 3.5.8 *(Poincare Group)*

Poincare group is Lie group extension of Lorentz group $O(1,3)$, and it contains "translation", "rotation", and "boosts". That is,

$$\mathbb{1},\mathbb{3} \rtimes O(1,3)$$

with group multiplications

$$(\alpha, f)\cdot(\beta, g) \mapsto (\alpha + f\cdot\beta, f\cdot g)$$

Definition 3.5.9 *(Poincare Algebra)*

Continuing from the previous definition, let P be the translation, and M be the generators of the Lorentz transformations, η is Minkowski metric $(+,-,-,-)$, then the bracket will be

- $[P_\mu, P_\nu] = 0$
- $\frac{1}{i}[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu$
- $\frac{1}{i}[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}$

Definition 3.5.10 *(Dirac Matrices (Gamma Matrices))*

Dirac matrices (or gamma matrices) $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ are 4 matrices existing in 4 dimensional space as

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Let's describe its mathematicall property. If we use anti-commutator $\{\}$, the matrices describe Minkowski metric $\eta^{\mu\nu}$.

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu} I_4$$

These matrices are called Dirac matrices because it's used to describe Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

Also, the relativitstic spin matrices $\sigma^{\mu\nu}$ is given by Dirac matrices as

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$$

Definition 3.5.11 *(Supercharge)*

Supercharge Q is an operator that transform Boson to Fermion and vice versa, so this operator itself is a spin. It's described by super Poincare algebra ,and it commutes with Hamlitonian as

$$[Q, H] = 0$$

The index Q_{α} is $\alpha = 1, 2, \cdots, N$ where the number of $N \in \mathbb{N}$ depends.

Definition 3.5.12 *(Super Poincare Algebra)*

$$\{Q_{\alpha}, \overline{Q_{\dot{\beta}}}\} = 2\sigma^{\mu}_{\alpha\dot{\beta}} P_{\mu}$$

Q_{α} and $\overline{Q_{\dot{\alpha}}}$ are called super charges. σ^{μ} are Pauli matrices and P^{μ} are generators of translation. The dot $\dot{\beta}$ is to remind that this index transforms according to the inequivalent conjugate spinor representation.

Super Poincare algebra is a Poincare algebra in addition to the following structures

- $[M^{\mu\nu}, Q_{\alpha}] = \frac{1}{2}(\sigma^{\mu\nu})^{\beta}_{\alpha} Q_{\beta}$
- $[Q_{\alpha}, P^{\mu}] = 0$
- $\{Q_{\alpha}, \overline{Q_{\dot{\beta}}}\} = 2(\sigma)^{\mu}_{\alpha\dot{\beta}} P_{\mu}$

These condition will lead us to supergravity.

Definition 3.5.13 *(Representation of Poincare Group)*

3.6 Spinor (Clifford)

So, what is spin? $SU(2)$ is a double cover of $SO(3)$, and describes spin of $SO(3)$. In fact as Lie algebra, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ is isomorphic, means there is locally same property, so we need to think of it as Lie group to study global property.

Definition 3.6.1 *($SU(2)$ and $SO(3)$)*

$SO(3)$ is generated by M_θ and M_ϕ, where

$$M_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \}$$
$$M_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix} \}$$

and $0 \leqq \theta < 2\pi$ and $0 \leqq \phi < \pi$. Note that the domain of ψ is only π, but if we expand to 2π, it will be $SU(2)$. The group structure of $SU(2)$ is genearted by M_θ and M_ϕ where

$$M_\phi = \begin{bmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{bmatrix}$$
$$M_\theta = \begin{bmatrix} cos(\frac{\theta}{2}) & isin(\frac{\theta}{2}) \\ isin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{bmatrix}$$

and $0 \leqq \theta < 2\pi$ and $0 \leqq \phi < 2\pi$. So indeed, it's a double cover of $SO(3)$.

Definition 3.6.2 *(Clifford Algebra)*

Let a vector space V and scalar field K, typically $\mathbb{R}$ or $\mathbb{C}$, and the quadratic form $Q : V \to K$. For all $v \in V$,

$v^2 = Q(v) \cdot 1$, or if we take $uv + vu = d(u, v)$ where $d(x, y)$ is a bilinear form and $u, v \in V$, then all the elements $v \in V$ generates $Cl(V)$ as

$$Cl(V) = Cl(V, d) = T(V)/(v \otimes v - Q(v))$$

Proposition 3.7 *(Grading Structure)*

$Cl(V)$ is naturally graded and given as

$$Cl(V) = Cl^0(V) \oplus Cl^1(V) \oplus Cl^2(V) \oplus \cdots \oplus Cl^n(V) \text{ where } n = dim(V)$$

$$Cl^0(V) = \mathbb{R}$$

$Cl^1(V) = V$

$Cl^2(V) = \mathfrak{spin}(n)$

In particular, $Cl^2(V)$ is called a spin algebra.

Definition 3.7.1 *(Spinor Algebra)*

The spin algebra $Cl^2(V) = \mathfrak{spin}(V)$ is, in fact, a Lie algebra, whose dimension is $n(n-1)$. It naturally has a short exact sequence as Lie algebra

$0 \to \mu_2 \to \mathfrak{spin}(n) \to \mathfrak{so}(n) \to 0$

Thise sequence as Lie algebra doesn't split. in fact dim(Spin(n) = SO(n)) = $n(n-1)$ is the same dimension. Also, μ_2 is an algebraic group of roots of unity.

Definition 3.7.2 *(Spinor Group)*
Geometrically, spinor as a group Spin(V) is defined by Spin(V) = Pin(V) $\cap$ Cl^{even}.

Spin(V) is double cover of SO(V), and it's simply connected, so it's universal cover of SO(V).

Pin group Pin(V) $\subset$ Cl(V, Q) is a subset of Cl(V, Q) where Q is non-degenerate, and consisting of the elements of the form $v_1 v_2 \cdots v_k$ where v_i is $Q(v_i) \pm 1$.

Proposition 3.8 *(Pauli Matrices)*

There are 3 Pauli matrices

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
whose multiplication is

$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$

where $\epsilon_{ijk} = +1$*if (i,j,k) = (1,2,3),(2,3,1),(3,1,2)*-1*if (i,j,k) = (3,2,1),(1,3,2),(2,1,3)*0*if* $i = j$ *or* $j = k$ *or*

is 3-dimensional Levi-Civita symbol.

Also, imaginary times of Pauli matrices generate $\mathfrak{su}(2)$. Its basis is explicitly, $(i\sigma_x, i\sigma_y, i\sigma_z)$.

Also, Pauli matrices are involution as $\sigma_i^2 = \sigma_j^2 = \sigma_k^2 = I$

Also, $det(\sigma_i) = -1$, and $tr(\sigma_i) = 0$

Definition 3.8.1 *(Whitehead Tower)*

According to Postnikov tower construction, there is an exact sequnce

$Fivebrane(n) \to String(n) \to Spin(n) \to SO(n) \to O(n) \to 0$
Note that $Fivebrane(n)$ and $String(n)$ are not necessarily Lie group.

3.9 Heisenberg Group

Definition 3.9.1 *(Heisenberg Group)*

Heisenberg group H is a 3x3 matrix group whose elements are

$$\begin{bmatrix} 1 & a & c \\ 1 & 1 & b \\ 1 & 1 & 1 \end{bmatrix}$$

where $a, b, c \in R$ where R is any commutative ring. Heisenberg group is often used in one-dimensional quantum mechanics by context of Stone-von Neumann theorem.

More abstractly, Heisenberg group is created by central extension

$0 \to (\mathbb{R}, +) \to H \to (\mathbb{R}, +)$

Definition 3.9.2 *(Heisenberg Algebra)*

$$\begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$$

whose basis is X and Y and Z and

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \; Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $[X, Y] = Z$, $[X, Z] = 0$, and $[Y, Y] = 0$. Consider $X = \hat{x}$ and $Y = \hat{p}$ and $Z = i\hbar I$,

$[\hat{x}, \hat{p}] = i\hbar I$, $[\hat{x}, i\hbar I] = 0$, and $[\hat{p}, i\hbar I] = 0$.

Definition 3.9.3 *(Stone-von Neumann Theorem)*

Any unitary irresucible unitary representation can be transformed to a Schrödinger representation.

For any irreducible unitary representation π of H on a Hilbert space $\mathfrak{H}$, there is a unitary operator $\mathfrak{H} \to L^2(\mathbb{R})$ such that

$U\pi U^{-1} = \pi_S$

where π_S is a Schrödinger representation.

We consider reprensetation and how representation applies to integral.

Definition 3.9.4 *(Fock Space)*

Consider a Hilbert space with inner product as

$< f(w), g(w) >= \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(w)} g(w) e^{-|w|^2}$

Then, $< w^m, w^n >= n! \delta_{m,n}$

So $\frac{w^n}{\sqrt{n!}}$ *are orthonormal.*

Consider $a = \frac{d}{dw}$ *and* $a^\dagger = w$.

Definition 3.9.5 *(Schrödinger Representation)*
For Schrödinger representation $\pi_S : H \to L^2(\mathbb{R})$ of Heisenberg Lie algebra, explicitly given by

$\pi_S(x, y, z)\psi(q) = e^{-iz} e^{i\frac{1}{2}xy} e^{-ixq} \psi(q - y)$

$\pi_S(x, y, z)\pi_S(x', y', z') = \pi_S(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)$

- $\pi_S'(iX) = Q$
- $\pi_S'(iY) = P$
- $\pi_S'(iZ) = 1$

and so

$\pi_S'(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$

Definition 3.9.6 *(Bargmann-Fock Representation)*

Bargmann-Fock representation $\pi_{BF} : H \to \mathfrak{H}_F$ as

- $\pi_{BF}'(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{d}{dw}$

- $\pi'_{BF}(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{d}{dw}$

- $\pi'_{BF}(\frac{1}{\sqrt{2}}(iX - i(iY))) = a^{\dagger} = w$

- $\pi'_{BF}(iZ) = 1$

Now by Stone-von Neumann, we have $U : \mathfrak{H}_F \to L^2(\mathbb{R})$ called Bargmann transform, and it's given by

$(U^{-1}\psi)w = (\frac{1}{\pi})^{\frac{1}{4}} e^{-\frac{1}{2}w^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}q^2} e^{\sqrt{2}wq} \psi(q) dq$

Thus

$U \frac{d}{dw} U^{-1} = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$

/

$UwU^{-1} = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$

Definition 3.9.7 *(Heisenberg Group and Symplectic Geometry)*

Consider higher dimensional analogue of Heisenerg group with basis X_j, Y_j, and Z where $j = 1, 2, \cdots, n$, and all Lie brackets are zero except

$[X_i, Y_j] = \delta_{ij} Z$

Symmetric form S on a vector space V is non-degenerate anti-symmetric bilinear form

$V \times V \ni (v_1, v_2) \mapsto S(v_1, v_2) \in \mathbb{R}$

which is symmetric version of inner product.

Consider algebra $V \oplus \mathbb{R}$ whose bracket is

$[(v, z), (v', z')] = (0, S(v, v'))$

and one gets a corresponding Lie group

$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2} S(v, v'))$

For Symplectic basis X_j and Y_j, $S(X_j, X_k) = S(Y_j, Y_k) = 0$ and $S(X_j, Y_k) = \delta_{jk}$

3.10 CCR and CAR

Let V be a real vector space with antisymmetric bilinear form $(\cdot, \cdot)$. The unital *-algebra is

Definition 3.10.1 *(*-algebra)*

$*$-ring is a ring with a map $* : A \to A$, an antiautomorphism and an involution. Namely,

- $(x + y)^* = x^* + y^*$
- $(xy)^* = y^* x^*$
- $1^* = 1$
- $(x^*)^* = x$

**-ring is *-algebra if further $(rx)^* = r' x^* \forall r \in R x \in A$*

$(\lambda x + \mu y)^* = \lambda' x^* + \mu' y^*$ for $\lambda, \mu \in R$ and $x, y \in A$.

Definition 3.10.2 *(C*-algebra)*

Definition 3.10.3 *(CCR and CAR as *-algebra)*

- *(CCR)*
  $fg - gf = i(f, g)$
  $f^* = f$
  for any f, g in V is called canonical commutation relation (CCR) algebra.
- *(CAR)*
  $fg - gf = i(f, g)$
  $f^* = f$
  for any f, g in V is called canonical anticommutation relation (CAR) algebra.

Definition 3.10.4 *(CCR and CAR as C*-algebra)*

Let CCR algebra over H is the unital C^*-algebra generated by elements $\{W(f) : f \in H\}$ subject to

$W(f)W(g) = e^{-i(f,g)} W(f + g)$
$W(f) = W(-f)$

If H is complex Hilbert space, CCR algebra is faithfully represented on the symmetric Fock space over H by setting

$$W(f)(1, g, \tfrac{g^{\otimes 2}}{2!}, \tfrac{g^{\otimes 3}}{3!}, \cdots) = e^{\frac{-1}{2}||f||^2 - <f,g>} W(f)(1, f+g, \tfrac{(f+g)^{\otimes 2}}{2!}, \tfrac{(f+g)^{\otimes 3}}{3!})$$

for any $f, g \in H$. How's CAR algebra? Omit.

4 Quantum Field Theory

Definition 4.0.1 *(Standard Model)*
In quantum mechanics, it's considered that all physical objects consist of atoms, which consist of paricles, and in our physical system of quantum level, there are currently believed that there are totally 17 different variety of particles adding it's spin (which means totally 34 particles), and studying QFT means to calculate the particles interactions. That is, for example, Lagrangian and its action, and path integral etc. QFT is mathematicaly powerful partly because of its symmetry. The easiest example might be Maxwell equation, that is QED by the language of QFT: symmetry of electric force and magnetic force is $U(1)$, and similarly QFD describes weak force with $SU(2)$ symmetry and, QCD with $SU(3)$ for the strong force.

Now the next question is that is there any unified theory of these models? This requires to define the Lagrangian that is consistant to the whole repsective situation, and that model is called Grand Unified Theory or standard model with $U(1) \times SU(2) \times SU(3)$, but GUT has not been proved yet. Nevertheless, Weinberg-Salam theory unifies weak $SU(2)$ and electromagnetic $U(1)$ forces.

4.1 Classical Field Theory

The interest of math physics is to study symmetry, since according to Neother's theorem, Lagrangian invariant (conservation of energy) corresponds to mathematical symmetry, namely Lie group.

Definition 4.1.1 *(Lie Group)*
For a smooth manifold M, a Lie group G is a set of all infinitesimal automorphisms $G = \{\Gamma_\epsilon : M \to M\}$

Definition 4.1.2 *(Noether's Theorem)*
Noether's theorem claims the the group theoretical symmetry corresponds to .

Imagine a particle traveling on the manifold M starting at time t_1 and ending t_2, and the particle is traveling along the path $x(t)$, $t_1 \leqq t \leqq t_2$, and for each point of manifold M, Lagrangian is defined. This path has an action $S = \int L(x(t))dt$. Consider another path infinitesimally modified $x'(t) = x(t) + \epsilon(t)$, so that the path is invariant action: $\int L(x'(t))dt = S + \delta S + O(\epsilon^2)$ where $\delta S = 0$.

The matehmatical statement of Neother's theorem is

$$\frac{\partial \mathscr{L}}{\partial \dot{x_i}}\zeta_i + \mathscr{H}\tau = 0$$

where $\mathscr{L} = \mathscr{L}(q_i, \dot{q}_i, t)$, *and* $\mathscr{H} = \dot{x}_i \frac{\partial \mathscr{L}}{\partial \dot{x_i}} - \mathscr{L}$
Also, Refer to Rund-Trautman identity. Rund-Trautman identity is a gap arising from the gauge transformation $\Gamma_\epsilon(\mathscr{L})$ *from* $\mathscr{L}$. *If Rund-Trautman identity is equal to* 0, *then it means the transformation is gauge invariant.*

Definition 4.1.3 *(Gauge Theory)*

Gauge theory is a branch of field theory, where we study global/local symmetry of the geometry, which is Lagrangian invariant by the change of coordinate by physics context. More generally, gauge theory is function equivariant wrt Lie group action.

In particular, the gauge theory is Yang-Mills theory, the Lie group is $SU(n)$ *or compact Lie groups. If the Lie group is compact,* $\int_X tr(F^2)dvol_g$ *becomes positive-definite.*

Definition 4.1.4 *(Ehresmann Connection (Gauge Field))*

Under constrction.

Proposition 4.2 *(Ehresmann Connection (Gauge Field))*

Ehresmann connection ∇ *defines (gauge) covariant derivative* $\nabla_\mu = \partial_\mu - igA_\mu$ *where* g *is a coupling constant.*
Field strength (curvature) is zero everywhere.

Example 4.3 *($U(1)$ Gauge Symmetry)*

$U(1)$ *gauge symmetry describes local coordinate symmetry of Lagrangian. Consider that* $U(1) = \{e^{i\theta} | \theta \in \mathbb{R}\}$, $U(1)$ *gauge transformation is*

$$\psi(x) \mapsto e^{i\alpha(x)}\psi(x)$$

where $\psi(x)$ *is wave function, and* $\alpha(x)$ *is local symmetry, and its value changes at every point in the coordinate.*

We'll consider gauge transformation (change of coordinate) to make Lagrangian invariant. Here we use covariant derivative $D_\mu = \partial_\mu - ieA_\mu$, *and if we use covariant derivative, we can costruct and gauge invariant transformation of gauge field* A_μ.

The $U(1)$ *gauge action makes* A_μ *and wave function* $D_m u\psi$ *as*

$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu \alpha(x)$

$D_\mu \psi \mapsto e^{i\alpha(x)} D_\mu \psi$

Then, the Lagrangian $L = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}$ *where* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ *is invariant under* $U(1)$ *transformation.*

Definition 4.3.1 *(Yang-Mill Functional)*
The Yang-Mills functional is

$YM(A) = \int_X ||F_A||^2 dvol_g$

where A *is a connection on a principal bundle, and* vol_g *is a volume form of* X*, typically* $dvol_g = dx^4$*.* F_A *is a strength tensor field (also called curvature) defined by* $F_A = \partial_\nu A_\mu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ *or alternatively,* $F_A = dA + A \wedge A$*. This is magnitude of field strength. For example, in electrodynamics, its unit might be* (V/m)*.*

$L_{gf} = ||F_A||^2$ *is Lagrangian, so* $YM(A)$ *is an action.* $||F_A||^2$ *is often alternatively denoted by* $L_{gf} = ||F_A||^2 = -\frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2}Tr(F^2) = -frac12Tr(F \wedge \star F = -frac12Tr(F_{\mu\nu}F^{\mu\nu})$*, that's representation theory perspective.*

What is trace? Let F_A *be a second tensor, defined over the Lie algebra whose basis be* $\{T^a\}$*, and its trace is* $Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$*, so trace is functioning as alternative to inner product. Using basis of the Lie algebra* $\{T^a\}$*,* $F_{\mu\nu}$ *is given by* $F_{\mu\nu} = F^a_{\mu\nu}T^a$*. Thus,*

$$\begin{aligned} Tr(F_{\mu\nu}F^{\mu\nu}) & \\ &= Tr(F^a_{\mu\nu}T^aF^{\mu\nu}_bT^b) \\ &= Tr(T^aT^b)F^a_{\mu\nu}F^{\mu\nu}_b \\ &= \frac{1}{2}\delta^{ab}F^a_{\mu\nu}F^{\mu\nu}_b \end{aligned} \tag{2}$$

Proposition 4.4 *(Yang-Mills Connection)*
Yang-Mills connection is a exterior covariant derivative that satisfies Yang-Mills equation $d_AF_A = 0$*. We will derive the Yang-Mills functional* $YM(A) = \int_X ||F_A||^2 dvol_g$ *makes the Yang-Mills equation* $d_A \star F_A = 0$*.*

$F_{A+ta} = F_A + td_Aa + t^2 a \wedge a$

$$
\begin{aligned}
\frac{d}{dt}(YM(A+ta))_{t=0} &= \frac{d}{dt}(\int_X < F_A + td_A a + t^2 a\wedge a, F_A + td_A a + t^2 a \wedge a > dvol_g)_{t=0} \\
&= \frac{d}{dt}(\int_X ||F_A||^2 + 2t < F_A, d_A a > +2t^2 < F_A, a\wedge a > +t^4 ||a\wedge a||^2 dvol_g)_{t=0} \\
&= 2\int_X < F_A, d_A a > dvol_g \\
&= 2\int_X < d_A^* F_A, a > dvol_g
\end{aligned} \tag{3}
$$

This connection A is a critical point of Yang-Mills functional if it vanishes for every a. Formally,

$$< d_A^* F_A, a >= 0$$

means that

$$d_A^* F_A = 0$$

Remark 4.5 *Irreducible unitary Yang-Mills connection corresponds to stable vector bundle. A vector bundle is stable vector bundle if it satisfies stability condition.*

Proposition 4.6 *(Self Dual/Anti-Self Dual Equations)*

If in particular X is 4-dimensional, the star operator becomes endomorphism

$$* : \Omega^2(X) \to \Omega^2(X)$$

Since the square of $*$-operator becomes $\pm Id$, the eigenvalue of $*$ is either 1 or -1, and we have decomposition

$$\Omega^2(X) = \Omega_+(X) \oplus \Omega_-(X)$$

Then the curvature $F_A = *F_A$ or $F_A = - * F_A$.

Proposition 4.7 *(What Yang-Mill Theory Derives Lie Structure)*

Let Lagrangian $L_{gf} = -\frac{1}{2}tr(F^2) = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}$
$tr(T^aT^b) = \frac{1}{2}\delta^{ab}$, $[T^a, T^b] = if^{abc}T^c$ where f^{abc} are structure constants.

We define covariant derivative $D_\mu = I\partial_\mu - igT^aA^a_\mu$

commutator $[D_\mu, D_\nu] = -igT^aF^a_{\mu\nu}$ can derive the relation

$F^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

By some calculation, a Bianchi identity $(D_\mu F_{\nu\kappa})^a + (D_\kappa F_{\mu\nu})^a + (D_\nu F_{\kappa\mu})^a = 0$ *holds, that's equivalent to Jacobi identity* $[D_\mu, [D_\nu, D_\kappa]] + [D_\kappa, [D_\mu, D_\nu]] + [D_\nu, [D_\kappa, D_\mu]] = 0$.

Note 4.8 *(Chern Class and Curvature)*
The integral $\int Tr(F) = \chi(M)$ *simply calculates the Euler characteristic, but our calculation is* $\int Tr(F^2) = \int Tr(F \wedge \star F)$*, it's squared. This calculates the Lagrangian density, not topological invariant. Also,* $\int Tr(F \wedge F) \neq \int Tr(F \wedge \star F)$*. If this is convenient,* $Tr(F)$ *is 1st Chern class,* $Tr(F \wedge F)$ *is the 2nd Chern class.*

Definition 4.8.1 *(Sigma Model)*

Specific Gauge theory. It depends on how we define symmetry, i.e. it could be typically $SU(N)$ *or in condensed matter physics, it could be* $O(N)$*.*

4.9 Quantum Field Theory

Similar to classical mechanics, quantum mechanics describes the state of a particle as given by position (x) mementum (p). The easiest qunatization is canonical quantization, but we also introduce several different generalizations.

Definition 4.9.1 *(Schödinger Equation)*
In quantum mechanics, let the quantum state state be $|\psi>$ *be an element in the Hilbert space, and observables are represented by a quantum state. Also, let* $\psi >= \Sigma_{n=0}^{\infty} a_n \psi_n >$

Time-independent Schödinger Equation is

$\hat{H}|\psi_n >= E_n|\psi_n >$
where E_n *is characteristic energy associated to* ψ_n *eigenstate. and*

where a_n *are constant coefficients.*

Definition 4.9.2 *(Canonical Quantization)*

For a one-particle system, canonical quantization is described by

$\{A, B\} \mapsto \frac{1}{i\hbar}[\hat{A}, \hat{B}]$

This Lie bracket encodes uncertainty principle $(\Delta x \Delta p \geqq \hbar/2)$*, and*

$[\hat{X}, \hat{P}] = \hat{X}\hat{P} - \hat{P}\hat{X} = i\hbar$

while Poisson bracket is defined in classical mechanics, and $\{x, p\} = 1$*.*

Definition 4.9.3 *(Weyl Quantization)*

Weyl transformation (Weyl quantization) is an operation in Hilbert space which is

$\Phi[f] = \frac{1}{(2\pi)^2} \int\int\int\int f(q,p)(e^{i(a(Q-q)+b(P-p))})dqdpdadb$

that is a linear transformation.

theorem:

If f is polynomial of $deg(f) \geqq 2$ and g is polynomial of any degree, then

$\Phi(\{f,g\}) = \frac{1}{i\hbar}[\Phi(f), \Phi(g)]$
that's similar to canonical quantization.

Definition 4.9.4 *(Moyal Bracket)*
We define Moyal bracket as

$\{\{f,g\}\} = \frac{1}{i\hbar}(f \star g - g \star f) = \{f,g\} + O(\hbar^2)$

where $\star$ is a star product in Moyal space.

Definition 4.9.5 *(Poisson Bivetor)*

$\Pi^0(f_1, f_2) = f_1 f_2$

$\Pi^1(f_1, f_2) = \{f_1, f_2\}$

$\Pi^n(f_1, f_2) = \Sigma_{k=0}^{n} \binom{n}{m} (\frac{\partial^k}{\partial p^k} \frac{\partial^{n-k}}{\partial q^{n-k}} f_1) \times (\frac{\partial^k}{\partial q^k} \frac{\partial^{n-k}}{\partial p^{n-k}} f_2)$

ex:

Gaussians compose hyperbolically e.g.

$\delta(p) \star \delta(q) = \frac{2}{\hbar} e^{2i\frac{qp}{\hbar}}$

Definition 4.9.6 *(Deformation Quantization)*

Given a Poisson algebra $(A, \{\cdot,\cdot\})$, its deformation quantization is an associative product $\star$ on the algebra of the formal power series $A[[\hbar]]$ with axioms

- $f \star g = fg + O(\hbar)$
- $[f,g] = f \star g - g \star f = i\hbar\{f,g\} + O(\hbar^2)$

That's called Kontsevich qunatization formula. On a Poisson manifold, we could add

$f \star g = fg + \Sigma_{k=1}^{\infty} \hbar^k B_k(f \otimes g)$

where B_k are linear bidifferential operators of degree at most k. We generalized to k-th degree because manifold is locally biholomorphic iff analytic.

In particular, for the Weyl-qunatization, the deformation quantization formula will be
$f \star g = fg + \Sigma_{k=1}^{\infty} \frac{1}{n!}(\frac{i\hbar}{2})^k \Pi^n(f \otimes g)$

Also, consider another generalization of quantization.

Definition 4.9.7 *(Prequantization)*

Let (M, ω) be a symplectic manifold, and symplectic potential θ be $d\theta = \omega$. The prequantization $Q(f)$ is defied as

$Q(f) = -i\hbar(X_f + \frac{1}{i\hbar}\theta(X_f)) + f$

where X_f is a Hamiltonian vector field associated to f. Or more concisely,

$Q(f) = -i\hbar \nabla_{X_f} + f$

with ∇ connection. For all smooth functions f and g, this prequantum operator satisfy
$[Q(f), Q(g)] = i\hbar Q(\{f, g\})$

By choice of polarization, prequantization becomes quantizaiton.

4.9.1 Path Integral

Definition 4.9.8 *(Green Function)*
Let L be a linear operator, and $\delta(x)$ is a Dirac delta function. $G(x, y)$ is a green function if

$LG(x, y) = \delta(x - y)$

Propagator measures probability that quantum state transforms from $|in\rangle$ to $\langle out|$.

Example 4.10 *(Propagator)*
A notable example of Green function is a propagator in QFT. Propagator is a function of probability of the particle moving from x to x' in time t to t'.

- *(Non-Relativistic Propagator)*
  $(H_x - i\hbar\frac{\partial}{\partial t})K(x,x',t,t') = i\hbar\delta(x-x')\delta(t-t')$
  whose solution is
  $K(x,x',t,t') =< 0|\hat{U}(t,t')|0 >$

- *(Relativistic Propagator)*
  $(\Box_x + m^2)G(x,y) = -\delta(x-y)$
  where $\Box_x = \frac{\partial^2}{\partial t^2} - \nabla^2$

- *(Feynman Propagator)*
  $G_F(x,y) = \lim_{\epsilon\to 0}\frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon}$
  $G_F(x-y) = -i < 0|T\{\Phi(x)\Phi(y)\}|0 >$

So far we have a propagator for one particle system, by using the Green function. Or it's alternatively given by partition functions, so we can generalize to n-body system.

Definition 4.10.1 *(Correlation Function)*

Let $\phi(x)$ be a scalar field state, and $|\Omega >$ be a vacuum state at every point x. The n-point correlation function (also called Green function) is the vacuum expectation value of time-ordered products

$G_n(x_1,\cdots,x_n) =< \Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega >$

There $T\{\cdots\}$ is a time-ordering operator, just reordering each state by time-ordering

$T\{A,B\} = AB$ *if* $\tau_A > \tau_B$
$T\{A,B\} = BA$ *if* $\tau_A < \tau_B$

Or we can depict the correlation function as

$G_n(x_1,\cdots,x_n) = \frac{<0|T\{\phi(x_1)\cdots\phi(x_n)\}e^{iS[\phi]}|0>}{<0|e^{iS[\phi]}|0>}$

where $|0\rangle$ is a free ground state and $S[\phi]$ is the action.

Definition 4.10.2 *(Vacuum State)*

Vacuum state (or quantum ground state) is the quantum state of the lowest energy, where quantum state is given by bra-ket notation, a vecotor in a Hilbert space thus denoted $|0\rangle$.

As noted , vacuum state is used to define a correlation function.

Definition 4.10.3 *(Partition Function)*

$Z[J] = \lim < 0|e^{-i\hat{H}T}|0 > = \int(D\phi)e^{i\int dx(L+J\phi)}$

where $J(x)$ is the fictitious source current.

Or if it's convenient we could also write

$W[J] = -iln(Z[J])$

In fact, this partition funciton describes one-particle system. Let's construct n-body correlatio function.

$$\begin{aligned}
\frac{\delta}{\delta J(x)}Z[J] &= \int(D\phi)\frac{\delta}{\delta J(x)}e^{i\int dx(L+J\phi)} \\
&= \int(D\phi)i(\int\delta(x-J)\phi(J)d^4J)e^{i\int dx(L+J\phi)} \\
&= \int(D\phi)i\phi(x)e^{i\int dx(L+J\phi)} \\
&=< 0|\hat{\phi(x)}|0 > \\
&= \frac{-i}{Z[0]}\frac{\delta Z[J]}{\delta J(x)}|_{J=0}
\end{aligned} \tag{4}$$

n-body system is

$< 0|\hat{\phi(x_1)}\hat{\phi(x_2)}\cdots\hat{\phi(x_N)}|0 >= \frac{(-i)^N}{Z[0]}\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_N)}$

Definition 4.10.4 *(Wick Rotation)*

Wick rotation is a method of finding a solution from a math problem in Minkowski space to a problem in Euclidean space, by transforming imaginary number to real number.

Definition 4.10.5 *(S-Matrix (Scattering Amplitude))*
Scattering Matrix is a matrix that describes network of n-inputs and n-outputs.

ex:

QFT

Electric circuit

Definition 4.10.6 *(Entropy Formula)*
In the closed system, the average temperature T of the closed system and heat

fluctuation δQ, entropy is $\int \frac{\delta Q}{T} = 0$. Prove $S = \Sigma p_i ln(p_i)$.

$dS = \frac{\delta Q}{T} = (\frac{\partial S}{\partial T})_P dT + (\frac{\partial S}{\partial P})_T dP = \frac{C_P}{T} dT - V\alpha_V dP$

$S = C_P ln(T) - Rln(P)$

Or the sum is

$S = \Sigma C_P ln(T) - Rln(P)$

Definition 4.10.7 *(Path Integral)*
Path integral is the explicit description of correlation function. We define path integral as

$\int_A^B e^{iS}\phi(x_1)\cdots\phi(x_n)D\phi = < A|\phi(x_1)\cdots\phi(x_n)|B >$

Path integral is similar to expectation values of A and B. Or in particular,

$\frac{\int e^{iS}\phi(x_1)\cdots\phi(x_n)D\phi}{\int e^{iS}D\phi} = < 0|\phi(x_1)\cdots\phi(x_n)|0 >$

This could be written alternatively

$G_n(x_1, \cdots, x_n) = (-i)^n \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}|_{J=0}$

or alternatively

$G_n^c(x_1, \cdots, x_n) = (-i)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}|_{J=0}$

where

$Z[J] = \int D\phi e^{iS[\phi]+i\int d^dxJ(x)\phi(x)}$

Definition 4.10.8 *(Schrödinger Picture)*

Definition 4.10.9 *(Heisenberg Picture)*

Heisenberg picture is used to calculate expectation value of an observable.

Definition 4.10.10 *(Interaction Picture)*

Definition 4.10.11 *(Feynman Diagram and Correlation Function)*

Feynman diagram describes what particles do we assume to exist in the system, and when and where they react. From the picture (coordinated by time t and position x, we can paraphrase to the correlation function)

For example, the two-point correlation function

$\langle 0| \Phi(x_1)\Phi(x_2) |0\rangle$

calculates the expectation value of a particle moving from x_1 to x_2 in time t_1 to t_2.

The four-point correlation function

$\langle 0| \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4) |0\rangle$

describes four particle system: e.g. beta collapse $n \to p + e^- + \nu_e$ makes neutron transforms to 3 particles, but totally four kind of particles in the system, so it's four-body problem.

Definition 4.10.12 *(Ward identity)*
A Ward identity is an identity between the correlation functions that follows global or gauge symmetries. More generally, Ward identity is a quantum version of classical current conservation of associated with a continuous symmetry by Noether's theorem.

In particular, in QED, $M(k) = \epsilon_\mu(k)M^\mu(k)$ be the amplitude for some QED with external photon of momentum k, where $\epsilon_\mu(k)$ is the polarization vector of the photon. Then,

$k_\mu \epsilon_\mu(k) = 0$

Definition 4.10.13 *(Polarization Vector)*

For example in QED, a photon is a wave osciallating in a direction through a polarization plane. The photon wave oscillates in 1-dimensional, but our living space is 3-dimensional, so there is the direction of oscillation called the polarization. It can be given in a vector form as follows

$|\psi\rangle = \begin{bmatrix} cos(\theta)exp(i\alpha_x) \\ sin(\theta)exp(i\alpha_y) \end{bmatrix}$

Definition 4.10.14 *(S-Matrix)*

S-matrix is a short of Scattering matrix, and S-matrix relates initial states and final states of physical system undergoing scattering process. Scattering process describes how particles change direction by collision. In particular in QFT, Feynman diagram describes how particles collide, which is exactly the scattering process.

Let S be a scattering matrix, and $\langle \Psi_{out}|$ is an outgoing wave function, $|\Psi_{in}\rangle$ in an incoming wave function. Then the relation is

$\langle \Psi_{out}| S |\Psi_{in}\rangle$

For detail, S-Matrix(video)

Definition 4.10.15 *(Faddeev-Popov Ghost)*

FP ghost field $c^a(x)$ is a notion in QFT, and works as a complementary field to make path integral consistent. For example in YM theory, its Lagrangian L_{ghost} is

$L_{ghost} = \partial_\mu \overline{c}^a \partial^\mu c^a + g f^{abc}(\partial^\mu \overline{c}^a) A^b_\mu c^c$

whose first term is kinetic term, and the second term is interaction with the gauge fields as well as the Higgs field.

4.10.1 Yang-Mill Mass Gap

Definition 4.10.16 *(Yang-Mill Mass Gap)*
In QCD, but unlike QED or QFD, particle cannot exist individually but coagulated together to gluon, and it's always observed as proton and neutron, which is weird if we consider that as comparison in QED, the electron and photon exist independently, and they are, in fact, non-divisible particles. Another problem in QCD is that why proton and neutron have bigger mass than the total sum of the particles(quarks and gluons). As a hint, Higgs mechanism defines the mass of individual particles, but this is something else.

4.11 Conformal Field Theory

The problem in CFT is that conformal symmetry of physical system is described by Virasoro algebra, and WZW model (or to put simply gauge theory) has an affine Lie algebra structure, and it has translated to Virasoro algebra by Sugawara construction. In this construction, we have the equations that calculate Lie algebra invariant, called KZ equation, and it's mathematically very intersting. The purpose of Sugawara construction is that by context Virasoro algebra is more enhanced than affine Lie algebra, because Virasoro algebra measures invariant of the correlation functions, and a correlation function is used to describe path integral of n-body system, and it's invariant with respect to the conformal transformation.

Definition 4.11.1 *(Central Charge)*

$Z(\mathscr{A}) : K(\mathscr{A}) \to \mathbb{C}$ is central charge for $\mathscr{A}$ is an abelian category.

In physics, charge usually refers to an invariant with respect to Hamiltonian. So, q is a charge such that

$\{q, H\} = 0 = \frac{dq}{dt}$

Definition 4.11.2 *(Virasoro Algebra)*

Virasoro algebra is defined by central extension of Witt algebra with central charge c. Witt algebra is

$Der(\mathbb{C}[z, z^{-1}]) = \{L_n = -z^{n+1}\frac{\partial}{\partial z} | n \in \mathbb{Z}\}$

and $[L_n, L_m] = (m - n)L_{m+n}$.

Now, Virasoro algebra is a Lie algebra with a bracket

$[L_n, L_m] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$.

Definition 4.11.3 *(Chiral CFT)*

Let $V(z)$ be a field, and

$\frac{\partial}{\partial z}V(z) = L_{-1}V(z)$

It follows that OPE

$T(y)V(z) = \Sigma_{n\in\mathbb{Z}} \frac{L_n V(z)}{(y-z)^{n+2}}$

where (L_n) is a generator of the Virasoro algebra. defines a locally holomorphic field $T(y)$ that doesn't depend on z. This $T(y)$ is called energy momentum tensor.

Let $V_\Delta(z)$ be a primary field, a field that is of lowest weight representation as

- $L_{n>0}V_\Delta(z) = 0$
- $L_0 V_\Delta(z) = \Delta V_\Delta(z)$

Theorem 4.11.1 *(Correlation Function and Conformal Identities)*

The correlation function is defined by using functional integral (path integral)

$< \phi(x_1) \cdots \phi(x_n) >= \frac{1}{Z} \int [d\phi]\phi(x_1) \cdots \phi(x_n) e^{iS[\phi]}$

where

$Z = \int[d\phi]e^{iS[\phi]}$ *and* ϕ *is a scalar field*

If we apply transformation R *to* ϕ *as* $\phi \mapsto R(\phi)$
$< R(\phi(x_1))\cdots R(\phi(x_n)) >= \frac{1}{Z}\int[d\phi]R(\phi(x_1))\cdots R(\phi(x_n))e^{iS[\phi]}$

Note that $[d\phi] = [dR(\phi)]$ *and* $S[\phi] = S[R(\phi)]$ *are invariance, and*

$< R(\phi(x_1))\cdots R(\phi(x_n)) >= \frac{1}{Z}\int[dR(\phi)]R(\phi(x_1))\cdots R(\phi(x_n))e^{iS[R(\phi)]} =< \phi(R(x_1))\cdots\phi(R(x_n)) >$

Now let using Ω*, the transformation makes*

$R(\phi(x)) = \phi'(x') = \Omega^{\frac{\Delta}{2}}\phi(x)$

in particular, $\Omega = 1$ *if* R *is conformal.*

Definition 4.11.4 *(WZW Model)*

Σ *is a Riemann Surface,* G *is a Lie group,* $k \in \mathbb{N}$*. We define* G*-WZW model on* Σ *at the level* k*. The model is a non-linear sigma model whose aciton is a functional* $\gamma : \Sigma \to G$

$S_k(\gamma) = -\frac{k}{8\pi}\int_\Sigma d^2xK(\gamma^{-1}\partial^\mu\gamma, \gamma^{-1}\partial_\mu\gamma) + 2\pi kS^{WZ}(\gamma)$

where K *is the Killing form on* G*. Its Wess-Zumino term of the action is*

$S^{WZ}(\gamma) = 1\frac{1}{48\pi^2}\int_{B^3} d^3y\epsilon^{ijk}K(\gamma^{-1}\partial_i\gamma, [\gamma^{-1}\partial_j\gamma, \gamma^{-1}\partial_k\gamma])$

where B^3 *is* $\partial B^3 = \Sigma$*.*

This WZW model has a symmetric property, generating affine Lie structure. Let $\Omega(z)$ *be any holomorhic* G*-valued function,* $\bar{\Omega}(\bar{z})$ *be any* G*-valued anti-holomorphic function.*

This $S_k(\gamma) = S_k(\Omega\gamma\bar{\Omega}^{-1})$ *is*
$J(z) = -\frac{1}{2}k(\partial_z\gamma)\gamma^{-1}$ *and* $\bar{J}(\bar{z}) = -\frac{1}{2}k\gamma^{-1}\partial_{\bar{z}}\gamma$ *are the conserved currents associated with this symmetry.*

Or we denote by $J^a(z)$ *where* a *is an adjoint indices where* $\{t^a\}$ *is the orthonomal basis of* $\mathfrak{g}$ *wrt Killing form.*

Definition 4.11.5 *(Affine Lie Algebra (or sometimes current algebra))*
$\{t^a\}$ *is orthnormal basis(wrt to the Killing form) of Lie algebra of* G *and* $J^a(z)$ *the quantization of the field* $K(t^a, \partial_z gg^{-1})$ *where* $g : (B\coprod B)/\partial B \sim S^3 \to G$*.*

Then

$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{if_c^{ab}j^c(w)}{z-w} + O(1)$

where f_c^{ab} is a coefficient of Lie bracket $[t^a, t^b] = \Sigma f_c^{ab} t^c$

$J^a(z) = \Sigma_{n\in\mathbb{Z}} J_n^a z^{-n-1}$

and the algebra generated by the $\{J_n^a\}$ is called the affine Lie algebra associated to Lie algebra of G.
$[J_n^a, J_m^b] = f_c^{ab} J_{m+n}^c + kn\delta^{ab}\delta_{n+m,0}$

We call $\hat{\mathfrak{g}} = \{J_n^a\}$ is affine Lie algebra, or alternatively $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$.

CFT is called In Chiral CFT, if the theory contains Virasoro algebra.

Definition 4.11.6 *(Energy-Momentum Tensor)*

The dependence of the field $V(z)$ on its position is assumed to be determined by

$\frac{\partial}{\partial z} V(z) = L_{-1} V(z)$

It follows that OPE

$T(y)V(z) = \Sigma_{n\in\mathbb{Z}} \frac{L_n V(z)}{(y-z)^{n+2}}$

defines a locally holomorphic field defined independent from z. From here, its OPE(operator product expansion) will be

$T(y)T(z) = \frac{\frac{c}{2}}{(y-z)^4} + \frac{T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z} + O(1)$

where c is the central charge. OPE is a Laurent series expansion associated with a two operators.

Sugawara construction is an embedding of Virasoro algebra to affine Lie algebra. If Virasoro algebra is for CFT, Sugawara construction shows that WZW model is actually CFT.

Definition 4.11.7 *(Sugawara Construction)*

Recall $\{J^a(t)\}$ is Affine Lie algebra. the energy momentum tensor $T(z)$ for the Virasoro algebra.

$T(z) = \frac{1}{2(k+h^\vee)} \Sigma_a : J^a J^a : (z)$

where ":" denotes normal ordering, $h^\vee$ is the dual Coxter number. By using OPE and Wick's theorem,

$$T(y)T(z) = \frac{\frac{c}{2}}{(y-z)^4} + \frac{T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z} + O(1)$$

which is equivalent to Virasoro algebra commutation relations. If we let central charge and its Virasoro algebra as

- $c = \frac{kdim\mathfrak{g}}{k+h^\vee}$
- $L_{n\neq 0} = \frac{1}{2(k+h^\vee)}\Sigma_a\Sigma_{m\in\mathbb{Z}}J^a_{n-m}J^a_m$
- $L_0 = \frac{1}{2(k+h^\vee)}2(\Sigma_a\Sigma^\infty_{m=1}J^a_{-m}J^a_m + J^a_0J^a_0$

then the momentum tensor will be

$$T(z) = \Sigma_{m\in\mathbb{Z}}L_nz^{-n-2}$$

Definition 4.11.8 *(Kniznik-Zamolodchikov Equations)*

Let $fr\hat{a}kg_k$ be an affine Lie algebra with level k, and dual Coxeter number h. Let v be a zero mode representation of $\hat{\mathfrak{g}}_k$ and $\Phi(v,z)$ the primary field associated with it. t^a is the basis of $\mathfrak{g}$, and t^a_i their representation on the primary field $\Phi(v_i,z)$ and η the killing form.

$$((k+h)\partial_{z_i} + \Sigma_{j\neq i}\frac{\Sigma_{a,b}\eta_{ab}t^a_i\otimes t^b_j}{z_i-z_j}) < \Phi(v_N,z_N)\cdots\Phi(v_1,z_1) >= 0$$

The KZ equation appear from Sugawara construciton of Virasoro algebra. $L_{-1} = \frac{1}{2(k+h)}\Sigma_{k\in\mathbb{Z}}\Sigma_{a,b}\eta_{ab}J^a_{-k}J^a_{k-1}$

4.12 Chern-Simon Theory

We'll construct 3-dimensional example of Chern-Simon theory, namely if $\mathfrak{g} = \mathfrak{su}(2)$, since this might be easier, because knot is defined on 3-dimensional space, neither 2 or 4-dimension or any others. This is for weak theory in physics, but what else? For the higher dimensional case, knot is unable to be defined, but we can use quantum group and its braiding structure to knottify them.

4.12.1 For 3-Dim Case

The basic idea is that for the principal bundle $P \to M$ of fiber G, we'll consider traveling of particle on M, and track the change of Lie group value. This is alternatively a traveling of Lie group, and this is a geometric problem, since Lie group is a smooth manifold, and the holonomy exactly defines Wilson loop. If especially, for the sake of convenience, we choose the differential operator as flat connection, and the holonomy becomes trivial geometrically, and Wilson loop becomes purely topological invariant. That's how it's called TQFT.

Definition 4.12.1 *(Lie Algebra Valued Differential Form)*

If $\omega \wedge \omega$ is a 1-form, then

$\omega \wedge \omega(v_1, v_2) = [\omega(v_1), \omega(v_2)] - [\omega(v_2), \omega(v_1)]$
In general, if ω is p-form, η is q-form, then

$\omega \wedge \eta(v_1, \cdots, v_n) = \frac{1}{p!q!}\Sigma_\sigma sgn(\sigma)[\omega(v_{\sigma(1)}, \cdots, v_{\sigma(p)}), \eta(v_{\sigma(p_1)}, \cdots, v_{\sigma(p+q)})]$.

Definition 4.12.2 *(Chern-Weil Homomorphism)*

For a principla G-bundle $P \to M$, Chern-Weil homomorphism is

$\mathbb{C}[\mathfrak{g}]^G \to H^*(M, \mathbb{C})$.

Let $\Omega = D\omega = d\omega + \omega \wedge \omega$ be a curvature form, and $f \in \mathbb{C}[\mathfrak{g}]^G$ is a homogeneous polynomial function of degree k $f(ax) = a^k f(x)$ for $a \in \mathbb{C}$ and $x \in \mathfrak{g}$. Then, $f(\Omega)$ is a 2k-form given by

$f(\Omega)(v_1, \cdots, v_{2k}) = \frac{1}{(2k)!}\Sigma\epsilon_\sigma f(\Omega(v_{\sigma(1)}, v_{\sigma(2)}), \cdots, \Omega(v_{\sigma(2k-1)}, v_{\sigma(2k)}))$

Thus we define a morphism

$$\begin{gathered} \mathbb{C}[\mathfrak{g}]^G \to H^*(M, \mathbb{C}) \\ f \mapsto [\overline{f}(\Omega)] \end{gathered} \tag{5}$$

An example of Chern-Weil homomorphism is Chern class.

Example 4.13 *(Chern Class)*

In differential gemetry, Chern polynomial is defined by the conneciton form. Let $E \to M$ be a vector bundle, and Ω is its conneciton form. Then

$c_t(E) = det(I - t\frac{\Omega}{2\pi i})$

and

$c_k(E) \in H^{2k}(M, \mathbb{Z})$

or

$c(E) = \Sigma c_k(E) \in H^*(M, \mathbb{Z})$

So, if $f(x) = det(I - t\frac{x}{2\pi i}) \in \mathbb{C}[\mathfrak{g}]^G$ is a character polynomial, plugging in $x = \Omega$ makes Chern-Weil homomorphism.

Definition 4.13.1 *(Chern-Simon Form)*

Let A be a connection 1-form.
$S = \frac{k}{4\pi}\int_M tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$

$F = dA + A \wedge A$

Definition 4.13.2 *(Wilson Loop)*

A Wilson loop is a gauge invariant operators. A Wilson loop is a holonomy around a loop in M, traced in given representation R of G.
$W[\gamma] = tr[Pexp(i\oint_\gamma A_\mu dx^\mu)]$

where $\gamma : [0,1] \to M$ is a loop. P is a path ordering operator.

The set of all Wilson lines is in one-to-one correspondence with the representations of the gauge group, which is:
Λ_w/W where W is Weyl group.

Definition 4.13.3 *(Knot Theory)*

A Knot is an embedding $S^1 \to M$ to a geometry M, but M needs to be compact (if we talk about $S^1 \to \mathbb{R}^3$, we spontaneously assume all the loops on $\mathbb{R}^3$ are bounded). Consider if M is compact Hausdorff, all closed subsets i.e $Im(S^1)$ are compact.

Example 4.14 *(G)*

Take $G = SU(2)$, and that's 3-dimensional real Lie group, and it's compact. Knot is definable.

4.14.1 For Higher Dimension

Now let's consider the case of higher dimensions, where we'll similarly consider knot theory, but considering by braiding structure. An algebraic way of describing braiding structure is based on noncommutativity. Swapping the neighboring path $\tau\sigma \neq \sigma\tau$ might be generally speaking different, usually makes a free group from the generators.

Definition 4.14.1 *(Braiding)*
Consider a map

$$c_{V,W} : V \otimes W \to V \otimes W$$

where R is R-matrix, and V is a representation space of $U_q(\mathfrak{su}(N))$. The braiding $c_{V,W}$ is determied by using $R \in U_q(\mathfrak{su}(N)) \otimes U_q(\mathfrak{su}(N))$ as,
$c_{V,W}(v \otimes w) = \tau \circ (\rho_V \otimes \rho_W)(R)(v \otimes w)$

where representation $\rho_1 : U_q(\mathfrak{su}(N)) \to V$*,* $\rho_2 : U_q(\mathfrak{su}(N)) \to W$ *and* $\tau(v \otimes w) = w \otimes v$

4.15 Higgs Bundle

4.15.1 Higgs Field

Higgs particle explains mass of physical system. Higgs particle doesn't have a mass itself, but it helps Boson and Ferminon particles obtain masses from spontaneous symmetry breaking. That is, Higgs field has a Mexican hat potential, it makes spontaneous symmetry breaking at the lowest energy, and this gap helps particles to have masses. Later Higgs field is more rigorously and mathematically studied by Higgs bundle.

Definition 4.15.1 *(Weinberg-Salam Theory)*

Weinberg-Salam theory unifies symmetry of weak $SU(2)$ *and electromagnetic* $U(1)$ *forces. The total symmetry* $SU(2) \times U(1)$ *may be broken due to spontaneou symmetry breaking (that I explain later).*

Definition 4.15.2 *(Higgs Field)*

Higgs Field Φ *is defined by*

$$\Phi(x) = \tfrac{1}{\sqrt{2}} \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix}$$

THe vacuum expectation value of Φ *is*

$$\Phi(x) = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v + h(x) \end{bmatrix}$$

Definition 4.15.3 *(Spontaneous Symmetry Breaking)*

Let Φ *be a Higgs field, then the Higgs field's potential*

$$V(\Phi) = \mu \Phi^\dagger \Phi + \lambda |\Phi|^4$$

whose shape is Mexican hat. At the origin, the particle can move any direction, meaning $U(1)$ *symmetry, but the particle moved to lowest potential, the symmetry no longer exist. By analogy, if pencil is standing, the pencil can move to any direction, but once pencil falls and lying down, it no longer change the direction. That's symmetry breaking.*

Proposition 4.16 *(Fermion Mass)*
The Higgs field defines interaction Lagrangian of Higgs field and Fermion

$L_{int} = -y_f \overline{\psi}_L (\frac{v+h(x)}{\sqrt{2}}) \psi_R + h.c.$

If $v \neq 0$ is non-zero vacuum expectation value, fermion has a mass.

$L_{mass} = -\frac{y_f v}{\sqrt{2}} \overline{\psi} \psi$

$m_f = \frac{y_f v}{\sqrt{2}}$

Example 4.17 *(List of Masses of Particles)*

- *(W Boson)*
  $m_W = gv$

- *(Z Boson)*
  $m_Z = \frac{v}{2} \sqrt{g^2 + g'^2}$

- *(Photon)*
  Photon doesn't have mass.

- *(Gluon)*
  Gluon doesn't have mass.

- *(Fermion)*
  $m_f = \frac{y_f v}{\sqrt{2}}$

where

- *v is vacuum expectation value of the Higgs field.*

- *g is coupling constant of $SU(2)$*

- *g' is coupling constant of $U(1)$*

4.17.1 Higgs Bundle

Higgs bundle was invented purely for mathematical convenience in studying gauge theory, that is to extract complex structure from the geometry $\mathbb{R}^{1,3}$ to discuss them algebraic geometrically. There are many applications of Higgs bundle, which is for example, correspondnece of moduli of Higgs bundles and BPS solition or Seiberg-Witen curves, and also geometri Langlands correspondence. From physical interpretation, Higgs bundle is not specifically a tool to calculate Higgs particle, but to solve Hitchin system in general.

Higgs bundle (E, ϕ) is a holomorphic vector bundle E consisting Higgs field ϕ. Physically, Higgs particle, which defines Higgs field, experimentally proved its existence in 2012 in CERN, but it was originally conjectured to explain Weinberg-Salam theory, that unifies weak force and electromagnetic force in gauge theory.

Higgs field $\phi : M \to V$ is vector valued $\phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$,

Higgs bundle connects to Geometric Langlands correspondence.

From a constructive perspective, Hitchin system is a more general form of Higgs bundle.

Background of Higgs Bundle is to describe Higgs particle.

Definition 4.17.1 *(Hitchin System)*
Tangent space to the moduli space of G-bundles at F is

$T_{[F]}\mathfrak{M} = H^1(End(F))$

which by Serre duality is dual to

$\Phi \in H^0(End(F) \otimes K)$

where K is the canonical bundle. So a pair (F, Φ) is called Hitchin pair of Higgs bundle.

Definition 4.17.2 *(Higgs Bundle)*
Assuming the geometry $\mathbb{R}^{1,3}$ and its gauge field A_μ, $\mu = 0, 1, 2, 3$, we'll consider partial compactification to change the geometry $\mathbb{R}^{1,3} \mapsto \mathbb{R}^{1,1} \times X$, so that X is a compact Riemann surface, leaving the other two dimensions, and this X and its Higgs field $\Phi = A_2 + iA_3$ generates the holomorphic vector bundle, where A_2 and A_3 are gauge field. This geometry needs to preserve gauge symmetry when it's compactified. Higgs bundle is a pair (E, Φ) where $E \to X$ is a holomorphic vector bundle of arbirary rank r, and Φ is a Higgs field, we can define as $\Phi = A_2 + iA^3$, and its potential $V(\Phi)$ is $V(\Phi) = -\mu^2 Tr(\Phi^\dagger \Phi) + \lambda Tr((\Phi^\dagger \Phi)^2)$, and in particular if the rank is $r = 1$, $V(\Phi) = -\mu^2\phi^2 + \lambda\phi^4$ is the Mexican hat.

More formally, Higgs bundle is a pair (E, Φ) where $E \to X$ is a holomorphic vector bundle of arbirary rank r, and the Higgs Field ϕ is a holomorphic 1-form $\phi \in End(E) \otimes K_X \otimes K_X$ such that $\phi \wedge \phi = 0$. That is, for example $\Phi = \Phi_1 dz + \Phi_2 dw$ makes $\Phi \wedge \Phi = [\Phi_1, \Phi_2] dz \wedge dw$, and this Lie bracket need to be zero, that's not an obvious statement.

Higgs Field is a field of energy that is assumed to exist everywhere in the universe, and Higgs field can define mass of an element.

Definition 4.17.3 *(Vacuum Expectation Value)*
The vacuum expectation value of a field $\phi(x)$ is given by using propagator $<$

$0|\phi(x)|0>$. *If the field* ϕ_H *has a spotaneous symmetry breaking e.g. Higgs field, then the VEV is* $<0|\phi_H|0>= v \neq 0$*, or the the electric field, which doesn't have a spontaneous symmetry breaking has VEV* $<0|A_\mu(x)|0>= 0$.

Proposition 4.18 *(Moduli Space of Higgs Bundle)*
In fact, the moduli space of Higgs bundle is hyperkähler manifold. Hyperhähler manifold is useful in deformation. Quantization of the moduli space of Higgs bundle makes the wave $|phi\rangle$ *of BPS state as* $(A, \phi) \mapsto |\phi\rangle$.

Definition 4.18.1 *(Hitchin Fibration)*
Hitchin fibration is a map from the moduli space of Hitchin pair(Higgs bundle) to characteristic polynomials. For the case algebraic group is $G = GL_n$, $h : (E, \phi) \mapsto (tr(\phi)), tr(\phi^2), \cdots, tr(\phi^n))$*, called Hitchin map, or in general, the morphism of the moduli space* $\mathscr{M}_H \to B = \oplus_{i=1}^n H^0(C, K_C^{\otimes i})$. *This Hitchin map is called Hithcin fibration, and* B *is called Hitchin base.*

Definition 4.18.2 *(Kobayashi-Hitchin Correspondence)*
Correspondence of stable vector bundles over complex manifolds and Einstein-Hermitian vector bundles.

Definition 4.18.3 *(Schlesinger Equation)*

Theorem 4.18.1 *(Interface)*
By nonabelian Hodge correspondence, category of flat holomorphic connections on a smooth complex projective algebraic variety, category of representations of the fundamental group of that variety, and the category of Higgs bundles over this variety are actually quivalent.

Definition 4.18.4 *(Stability Condition of Higgs Bundle)*
Let $L \subset E$ *be a rank 1* ϕ*-invariant subbundle of* E*,*
$deg(L) < \frac{1}{2} deg(\bigwedge^2 E)$

Definition 4.18.5 *(Hermitian Yang-Mill Connection)*
A hermtian metric h *on a Higgs bundle* (E, Φ) *gives rise to a Chern connection* ∇_A *and a curvature* F_A. *The condition that* Φ *is holomorphic is phrased as* $\overline{\partial_A}\Phi = 0$

$$\begin{gathered} F_A + [\Phi, \Phi^*] = \lambda Id_E \\ \overline{\partial_A}\Phi = 0 \end{gathered} \tag{6}$$

If we let a connection $D = \nabla_E + \Phi + \Phi^*$, D *is called Hermitian Yang-Mill connection.*

Definition 4.18.6 *(Non-Abelian Hodge Correspondence)*
claim 1:

A representation $\rho : \pi_1(X) \to GL(r, \mathbb{C})$ is semisimple iff the flat vector bundle $E = \hat{X} \times_\rho \mathbb{C}^r$ admits a harmonic metric. Furthermore, the representation is irreducible iff the flat vector bundle is irreducible.

claim 2:

A Higgs bundle (E, Φ) has a Hermitian metric iff it's polystable. This metric is harmonic metric, and therefore arises semisimple representation of the fundamental group, iff Chern classes $c_1(\Omega)$ and $c_2(\Omega)$ vanishes. A Higgs bundle is stable iff it admits an irreducible Hermitian Yang-Mills connection, and therefore comes from an irreducible representation of the fundamental group.

Moduli space version:

there are homeomorphism $M^{ss}_{Dol} \cong M_{dR} \cong M^{+}_{B}$ of moduli spaces which restrict to homeomorphisms $M^{s}_{Dol} \cong M^{*}_{dR} \cong M^{*}_{B}$

Definition 4.18.7 *(Geometric Langlands Correspondence)*
Might be already little mentioned in the previous exposition.

There is a conenction of Geometric Langlands correspondence and S-duality, a certain property of QFT.

Definition 4.18.8 *(S-duality)*
S-duality is short for strong-weak duality.

Connection to N=4 supersymmetry Yang-Mill theory.

Next, we will describe Yang-Mills existence and mass gap. The whole description is not fully constructed or proved yet.

Definition 4.18.9 *(Yang-Mills Existence and Mass Gap)*
It claims the construction of existence of mass of particles. For example, mass of weak bosons exist, and generated by the spontaneous symmetry breaking, and by analogy, we consider the existence of mass of gluon, that's QCD of pure Yang-Mills theory.

In other word, mass gap is the difference between second lowest energy and vacuum state. Indeed, roughly energy could be the paraphrase of mass, as $E = mc^2$.

The program claims that for any compact simple gauge group G, there exists a non-trivial quantum Yang-Mills theory on $\mathbb{R}^4$, and it has a mass gap $\Delta > 0$.

Definition 4.18.10 *(Example From Classical Theories)*
The mass gap already appears in classical mechanics.

$\Box\phi + \lambda\phi^3 = 0$

4.19 SUSY

What is supersymmetry (or called SUSY)? Moreover, what is super? Super is related to $\mathbb{Z}/2\mathbb{Z}$ grading, which is in physics considered as a spin or anti-matter, making extra symmetry in gauge symmetry in physics. Spin is geometrically spinor. In particle physics, particles can be largely classified as boson and fermion, and fermion is spin $\frac{1}{2}$. Also, the question is mathematical construction of supersymmetry in Lie bracket and vertex operators. The particular example of supersymmetry algebra is BPS state. There are many applications of BPS states: magnetic monopoles, solitons and D-branes, SUNY gauge theory, external black hole etc.

Conformal Field Theory studies Lagrangian invariance by conformal transformation in Quantum Field Theory, and this makes the QFT calculation convenient, which is for example, Operator Product exansion and n-point functions. Conformal transformation has symmetry, that can be shown by Lie algebra, which is for the case of 2-dim CFT, the symmetry is Virasoro algebra, and if it's 4-dim, the symmetry is $SO(2,4)$. To study conformal symmetry, we often consider 2-dim CFT, because it is mathematically more structureful than 4-dim CFT, even our living space is 4-dim. 2-dimension CFT studies symmetry of $\mathbb{R}^{1,1}$, ignoring the other 2-dim of our living space $\mathbb{R}^{1,3}$, but the idea is similar to that the Newton's law $F = ma$ works independently on XYZ direction.

Definition 4.19.1 *(Lagrangian Invariance)*
First clarify what exactly Conformal transformed. Under transformation $z \mapsto f(z)$ and $\overline{z} \mapsto \overline{f}(\overline{z})$,

$$\begin{aligned}
\partial\phi \mapsto (\partial f)^{-1}\partial'\phi' \ , \ \overline{\partial}\phi \mapsto (\overline{\partial f})^{-1}\overline{\partial}'\phi' \\
dz^2 = dzd\overline{z} \mapsto |\partial f|^2 dzd\overline{z} \\
\textit{thus,} \\
dz^2\partial\phi\overline{\partial}\phi \mapsto dz^2\partial'\phi'\overline{\partial'}\phi' \\
\textit{is invariant.}
\end{aligned}$$

Thus in $2D$ CFT, $dz^2\mathfrak{L}$ is ivariant under conformal transformation, where $\mathfrak{L}(\phi)$ is a free Lagrangian and $\mathfrak{L}(\phi) = \partial\phi\overline{\partial}\phi$, then naturally the action $S[\phi] = \int d^2z\partial\phi\overline{\partial}\phi$ is also invariant.

Definition 4.19.2 *(Operator Product Expansion)*
OPE is an algorithm to reduce product of quantum operators by a summation of single quantum operators. That is,

$$\mathscr{O}_i(x)\mathscr{O}_j(y) = \Sigma_k C_{12}^k(x-y)\mathscr{O}_k(y)$$

where C_{12}^k is an OPE coefficient, and it has a Laurant expansion $C_{12}^k = \Sigma_n \frac{c_n}{(x-y)^n}$.

Definition 4.19.3 *(Two Point Functions)*
Two point function is just vacuum expectation value, that is defined as

$$< \phi(x)\phi(y) >= \langle 0| \phi(x)\phi(y) |0\rangle = \frac{1}{Z} \int D\phi\phi(x)\phi(y)e^{-S[\phi]}$$

It'll be written simpler, if we use OPE.

$$< \phi(x)\phi(y) >= \Sigma_k C_{12}^k(x-y) < \phi(y) >$$

However, just be careful its conformal transformation is almost invariant, not exactly the same, since it has quantum anomally.

Definition 4.19.4 *(SCFT)*
Super CFT has virasoro symmetry but also super-virasoro symmetry. The difference between CFT and SCFT is that in SCFT there is no quantum correction appears in n-point functions.

Definition 4.19.5 *(SCFT and BPS)*
BPS state is a condition that the mass M and charge $|Z|$ are equal, and this implies that if the object is BPS state, the mass and the charge aren't affected by quantum correction. BPS operator $\mathscr{O}_{BPS}$ is an operator that makes vacuum state to the wave of BPS state as $\mathscr{O}_{BPS}|0\rangle = |BPS\rangle$. However, if this applies to any wave $|\psi\rangle$, the operator action $\mathscr{O}_{BPS}|\psi\rangle$ doesn't need to be BPS. BPS operator is, for example, $\mathscr{O}_{BPS} = tr(\phi^k)$ in $N = 4$ SYM.

Definition 4.19.6 *(R-Symmetry)*

Definition 4.19.7 *(BPS state)*

For $d = 4$ $N = 2$, extended supersymmetry algebra called BPS states have mass equal to the supersymmetry of central charge Z.

- $\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\sigma^m_{\alpha\dot\beta}P_m\delta^A_B$
- $\{Q^A_\alpha, Q^B_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\overline{Z}$
- $\{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\beta B}\} = -2\epsilon_{al\dot{p}habe\dot{t}a}\epsilon_{AB}Z$

where α β are Lorentz group indices, and A and B are R-symmetry indices.

Example 4.20 *(BPS States)*

- *Magnetic Monopoles*
- *Solitons and D-branes*

- *SUNY gauge theory*
- *External black hole*

Definition 4.20.1 *(Magnetic Monopoles)*

Definition 4.20.2 *(External black hole)*
External black hole is a black hole with a minimum possible mass and that is compatible with its charge and angular momentum. This black hole is stable and no Hawking radiation. Their black hole entropy is calculated in string theory.

Definition 4.20.3 *(Solitons and D-branes)*

Definition 4.20.4 *(Atiyah-Singer Index Theorem)*
Atiyah-Singer index theorem says that analytical index and topological index of the same elliptic differential operator on a compact manifold coincides.

There are many applications.

Definition 4.20.5 *(Fredholm Operator)*

Fredholm operator is an elliptic operator. More specifically, Let $T : X \to Y$ *be a bounded linear operator, where* X *and* Y *are Banach spaces with finite dimensional kernel* $ker(T)$ *and finite dimensional cokernel* $coker(T) = Y/ran(T)$*. We call this* T *Fredholm operator.*

The index of Fredholm operator is the integer $ind(T) = dim(ker(T)) - dim(coker(T))$

4.21 BRST Quantization

We discuss quantization using BRST cohomology, which is a tool that can calculate gauge invariance in QFT, and to extract the actual physical data, since in BRST cohomology, we can discuss elimination of gauge duplication of path integral and gauge anomally cancellation, where its duplication is measured by Fedeev-Popov ghost, and gauge anomally cancellation is a criteria whether it's actually cochain complex or not, namely $Q^2_{BRST} = 0$ or not.

Definition 4.21.1 *(Fadeev-Popov Ghost)*
For review, quantization is a process of transforming classical FT to QFT, which is in other words, the functions $\phi(p, q)$ *in classical correspond the operators* $|\phi\rangle \in End(\mathfrak{H})$*. However, the gauge invariance in classical FT doesn't necessarily correspond 1-to-1, thus the problem arises: not all data in QFT describes*

the actual physics, hypothetically mathematically described by ghost. Let's define what ghost is, and first of all, the gauge invariance in classical FT is, for simplicity, consider $U(1)$ gauge invariant,

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \lambda$$

where λ is an arbitrary differentiable function, and this transformation is $U(1)$ gauge, since this keeps invariance $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This means this gauge transformation doesn't change the actual physical phenomena. Next, I'll define the quantization of gauge transformation

$$A_\mu + \partial_\mu \lambda \mapsto |A_\mu\rangle + c(x)$$

where $c(x) \in \mathfrak{H}$ is a hypothetical function corresponds to the change of gauge transform. If the actual physics doesn't change by gauge transform, this ghost needs to be canceled out.

Definition 4.21.2 *(BRST Cochain Complex)*
I named it cochain complex, but I don't know whether $Q^2_{BRST} = 0$ or not. Let BRST operator $Q : \mathfrak{H} \to \mathfrak{H}$ be an endomorphism of the Hilbert space, which is defined by using some ghost, and the morphism Q naturally generates grading structure $\mathfrak{H} = \oplus C^i$, so that $Q : C^i \to C^{i+1}$ for all i, and each BRST class C^i are generated by i-th ghost(refer FP ghost), that's for example, 0-the ghost is the actual physical state, 1-st ghost is the ordinary ghost c^a, and the 2-nd and higher ghost is given by the product. However, this sequence is not BRST cochain complex yet without having nilpotency $Q^2 = 0$ (refer gauge anomally cancellation). The BRST operator Q is defined by

$$\begin{aligned}
QA^a_\mu &= D_\mu c^a + \partial_\mu c^a = f^{abc} A^b_\mu c^c \\
Qc^a &= -\frac{1}{2} f^{abc} c^b c^c \\
Q\bar{c}^a &= B^a \\
QB^a &= 0
\end{aligned}$$

Definition 4.21.3 *(BRST Cohomology)*
Let's assume if $Q^2_{BRST} = 0$ for the moment. The physical state can be given by 0-th cohomology as $< physicalstate > \in H^0 = \frac{ker(Q_{BRST})}{im(Q_{BRST})}$, since physical state $|\phi\rangle$ doesn't generates ghost, equivalent to say it needs to be canceled $Q|\phi\rangle$, so it should be contained in the kernel, and quotient by the image to eliminate redundancy.

Definition 4.21.4 *(Gauge Anomally)*
In BRST cohomology, $Q^2_{BRST} = 0$ iff gauge anomally cancellation works. The gauge anomally the gap arising from the 3-loop Feynman diagram of interaction

of 3 bosons, whose path integral is

$$\langle out|in\rangle = \Sigma_{n=0}^{\infty}\frac{(iS_{int})^n}{n!} = \int DA\prod_i det(i\not{D}_i[A])e^{iS_{gauge}[A]}$$

where $\not{D} = \gamma^\mu(\partial_\mu - iA_\mu)$, *and if* $\prod_i det(i\not{D}_i[A])$ *is gauge invariant, then the anomally is canceled. We'll discuss the method of computing gauge anomaly cancellation, whose step is to transform to loop integral. Anomally* M_Δ *appears in* $n = 3$ *loop, and its loop integral will be*

$$M_\Delta \sim \int \frac{d^4k}{(2\pi)^4}Tr[\frac{1}{\not{k}}\gamma^\mu + \frac{1}{\not{k}+\not{p}_1}\gamma^\nu + \frac{1}{\not{k}+\not{p}_1+\not{p}_2}\gamma^\rho]$$

Definition 4.21.5 *(Loop Integral)*
$det(i\not{D}) = exp(Tr(log(i\not{D})))$

$$Tr(log(i\not{\partial} - \not{A})) = const + \Sigma_{n=0}^{\infty}\frac{1}{n}Tr((i\not{\partial})^{-1}\not{A}))^n$$

whose $n = 3$ *term is the triple diagram.*

Example 4.22 *($U(1)$-Gauge In BRST)*

Now let's consider some mathematical enhancement of BRST cohomology.

Definition 4.22.1 *(Ward Identity)*
Ward identity is a gauge invariant in QED, and this might be interesting with BRST, since the expectation value $< Q_{BRST}(\mathscr{O}) >= 0$ *is zero for arbitrary operator* $\mathscr{O} \in \mathfrak{H}$, *where an operator* $\mathscr{O}$ *should be an element of BRST 0-th cochain* C^0. *By solving this equation* $< Q_{BRST}(\mathscr{O}) >= 0$, *the relation can be more explicitly given as.*

$$q_\mu\Gamma^\mu(p', p) = S^{-1}(p') - S^{-1}(p)$$

where p' *and* p *are momentum of electrons, and* $q = p' - p$ *is momentum of photon,* $\Gamma(p', p)$ *is a fermion-photon vertex function,* $S^{-1}(p)$ *is an inverse of propagator. Notice that the inverse of propagator is an operator same as* Q_{BRST}. *Also, the fermion-photon vertex function* $\Gamma(p', p)$ *is* $\Gamma(p', p) = \gamma^\mu + (quantumcorrection)$, *where, as usual, this quantum correction can be 1-loop correction times the coupling constant* g_s *in QCD.*

Definition 4.22.2 *(Fock Space)*
We could also generalize BRST cohomology to Fock space from Hilbert space. From the above argument, Hilbert space generates the grading $\mathfrak{H} = \oplus_{i=0}^{\infty}C^i$, *making Fock space 2-particles* $|\phi\rangle \otimes |\psi\rangle \in C^{i+j}$, *and the definition of the morphism* Q_{BRST} *extends to* $Q_{BRST}(|\phi\rangle \otimes |\psi\rangle) = (Q_{BRS}|\phi\rangle) \otimes |\psi\rangle + (-1)^i|\phi\rangle \otimes Q_{BRST}(|\psi\rangle)$, *which is indeed still* $Q_{BRST}^2 = 0$

Definition 4.22.3 *(BRST and Mirror Symmetry)*
BRST can also define mirror symmetry. Consider geometry X and its mirror dual $X^\vee$.

(A-model)
The A-model is quantum cohomology over X, that is genenerated by the quantization of the de Rham cohomology with differential operator $\mathrm{d} = Q_A$, and this de Rham cohomology is the same as BRST cohomology, means literally BRST 0-form is differential 0-form, and 1-ghost corresponds to 1-form. Q_A is left-moving supercharge from BRST, defined by $Q_A = Q_+ + \overline{Q_-}$.

(B-model)
The B-model is Dolbeault cohomology over $X^\vee$. Dolbeault cohomology can be deduced from de Rham cohomology, if it's defined over complex geometry, since the cochain complex with differential operator $\overline{\partial}$ where $\mathrm{d} = \partial + \overline{\partial}$, using Hodge decomposition, and we define $\overline{\partial}$ as $\overline{\partial} = Q_B$ from BRST cohomology with Q_B right-moving supercharge from BRST, defined by $Q_B = \overline{Q_+} + Q_-$.

5 String Theory

(Overview)
String theory is another formulation of physical model other than gauge theory, and it is a theory that generalizes gauge theory to explain the gravity, which is in other words, string theory has an ability to describe gravity (Einstein equation), and it could be better than gauge theory. In string theory, all paticles are strings, which is a 2-dimensional surface $X(\sigma, \tau)$, existing in $10D$ space. The world plane is considered to be a manifold, and its $26D$ in bosonic string theory, while it's $10D$ in ferminonic string theory. Hereafter, we'll focus on ferminonic string theory.

(Construction)
Each string $X(\sigma, \tau)$ (also called bosonic worldsheet) is a $(1,1)$-dimensional string, existing interior of that space, and a particle is described by the vibration of the strings. For example, in type IIA or IIB construction, the string is assumed to be closed (means closed on the space direction but open on the time direction), while in fermionic string theory, we add ferminon worldsheet $\phi^\mu(\sigma, \tau)$, whose chiralty differenciates IIA and IIB. Namely, the fermion worldsheet can be decomposed to left-right as $\phi^\mu(\sigma, \tau) = \phi^\mu_L(\sigma + \tau) + \phi^\mu_R(\sigma - \tau)$, and their spinor is defined by the sign of $\Gamma^{11}\phi = \pm\phi$ where Γ^{11} is $10D$ Majorana-Weyl spinor. Also, our living world can be $10D$ space given by $A^i dx^i$, and R-R potential defined by C_n form $n = 0, 1, 2, \cdots, 9$ is the subspace, and the string is living inside of C_n.

(D-brane)

Polchinski introduced a technique of transforming the closed string to an open string, that can exist on D-branes, that's called worldsheet orientation reversal projection or Ω projection, and Ω projection is given by flipping the parameter of the string $\Omega : \sigma \mapsto \pi - \sigma$, and by this operation, the closed string becomes an open string. From D-brane's point of view, the particle(string) can be seen as a point, as an intersection of open string and D-brane, thus disguising σ-model in a gauge theory, where we assume multiple of D-branes of the same shape are in the same place, and gauge invariant is generated by in which D-brane to choose one end of the open string, and for example, IIA assumes 32 of D-branes, and it has gauge theory of $SO(32)$ gauge symmetry (thus string theory is generalization of gauge theory). Finally, the D-brane D_p can be generated by C_{p+1} form. Assumging an open string is generated by the closed string existing inside of C_{p+1}, D_p is the set of points where the edge of open string exists.

(Dimension)
The space of the string theory is often considered to be $10D$, but the space can often be compactified for mathematical/physical reason. For example, if $10D$ space is compactified to fiber bundle whose fiber is $6D$ Calabi-Yau manifold, leaving $4D$ Lorenzian space $\mathbb{R}^{1,3}$, that is visually almost $4D$ if the Calabi-Yau is assumed to be very very small and invisible. Hence, compactification is often used to reduce dimensions.

(5-Different String Theories)
There are 5 different varieties of super string theories, that's $SO(32)$, $E_8 \times E_8$, $TypeI$, $TypeIIA$, and $typeIIB$, there are classified by vibration symmetry and chiralty of the strings.

String theory is a theory in which each particle is consider to be a closed/open string.
In string theory, open string is spin 1 photon, weak boson, gluon, and closed string is spin 2 graviton.

Finally, M-theory is a theory that unifies all string theories. According to M-theory, our universe is 11-dimension, consisting 2 or 5-dimensional branes.

5.1 General Construction

Definition 5.1.1 *(String and Space)*
In string theory, Boson string $X^\mu(\sigma, \tau) : \Sigma \to M$ is a 1-dimensional σ-model, and Fermion string $\psi^\mu(\sigma, \tau) : \Sigma \to M$ is another 1-dimensional σ-model, where Σ is 2-dimensional non-compact manifold, and M could be any space, but usually considered to be $\mathbb{R}^{1,9}$.

In closed string theory, the string can be treated as a periodic function, and the closed Fermion string $\psi^\mu(\sigma, \tau)$ behaves either Ramond(R) or Neveu-Schwarz(NS) sector (physically, NS behaves like boson in low-energy). In Ramond sector $\psi(\sigma+2\pi) = \psi(\sigma)$, and in Neveu-Schwarz sector $\psi(\sigma+2\pi) = -\psi(\sigma)$. The general periodic function is explicitly

The Bosonic string X^μ is given by

$X^\mu(\sigma,\tau) = x^\mu + 2\alpha' p^\mu \tau + \sqrt{\frac{\alpha'}{2}} Sigma_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau\pm\sigma)}$

The Fermionic string ψ^μ is given by

$\psi^\mu(\sigma,\tau) = Sigma_r \psi_r^\mu e^{-ir(\tau\pm\sigma)}$

where $r \in \mathbb{Z} + \frac{1}{2}$ in Neveu-Schwarz, and $r \in \mathbb{Z}$ in Ramond.

Definition 5.1.2 *(Chiralty – Wave Direction)*
In detail, the ferminonic string can have vibration of waves in either left/right waves, and in each mode of vibration, the variable is $\sigma^+ = \sigma + \tau$ and $\sigma^- = \sigma - \tau$, and for example, the vibration mode of NS-NS is the left vibration mode of NS and right vibration mode of NS, and similarly NS-R, R-NS and R-R. Mathematically, the direction of vibration of waves is defined as

$$\psi_L^\mu(\sigma^-) = \psi_L^\mu(\sigma+\tau) = \Sigma_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{ir(\sigma+\tau)}$$
$$\psi_R^\mu(\sigma^+) = \psi_R^\mu(\sigma-\tau) = \Sigma_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{ir(\sigma-\tau)}$$

and if since ferminonic wave can have direction of waves, it will be added to $\phi(\sigma,\tau) = \phi_L(\sigma+\tau) + \phi_R(\sigma-\tau)$. If the left/right vibration is the same, it's non-chiral, but if the left/right vibration is different, it's chiral. Type IIB string theory assumes chiralty, and $\psi_L^\mu(\sigma^-) \sim -\psi_R^\mu(\sigma^+)$, while type IIA has the opposte chiralty $\psi_L^\mu(\sigma^-) \sim \psi_R^\mu(\sigma^+)$.

Definition 5.1.3 *(p-Form C_p)*
p-form C_p is generated from the quantization of the string existing in the $10D$ space, explicitly $\psi \mapsto \hat{\psi} \in End(\mathfrak{H})$. This Focks space (Hilbert space) $\mathfrak{H}$ can be decomposed to $\mathfrak{H} = \mathfrak{H}_{NS-NS} \oplus \mathfrak{H}_{R-NS} \oplus \mathfrak{H}_{NS-R} \oplus \mathfrak{H}_{R-R}$, and in particular, $\mathfrak{H}_{R-R} = \oplus_p \mathfrak{H}_{C_p}$ where $p = 1, 2, \cdots, 10$. From $\hat{\psi}$ and $\hat{X}$, we extract $|C_p\rangle \in \mathfrak{H}_{R-R}$ is the R-R section of degree p (Indeed, the string ψ and X geneate the p-form C_p, and that's why the Bosonic X^μ and Ferminonic string ψ^μ are called world sheets).

In addiion, since type IIA and type IIB has different chiralty, in type IIA string theory has odd dimensional p-forms C_p, while in type IIB string theory has even dimensional p-forms C_p, since p-form is the differential form of wedge product, means that chiralty can compensate each other on even/odd degree.

Definition 5.1.4 *(Energy Momentum Tensor)*
The original definition of energy momentum tensor is

$$T(z) = -\tfrac{1}{\alpha'} : \partial X^\mu \partial X_\mu :$$

that can be the sum of the Virasoro coefficients

$T(z) = \Sigma_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$

Definition 5.1.5 *(Dimension of String Theory)*
It is known that bosonic string theory is $26D$ and ferminonic string theory is $10D$, that can be calculated by CFT. To calculate the dimension of space of string theory, first find the energy momentum tensor, so we can construct the Virasoro algebra explicitly, and we can find its central charge. In bosonic string theory, the central charge is $c - 26 = 0$, and in fermionic string theory, $c_{matter} = \frac{3D}{2}$ and $c_{ghost} = -15$, and $c_{matter} + c_{ghost} = 0$.

Definition 5.1.6 *(Wick Rotation)*
Wick rotation is to transform from Lorenzian coordinate to complex coordinate and vice versa. Wick rotation makes $\mathbb{R}^{1,1}$ to $\mathbb{C}^1$, or in general, a string can be transformed to a Riemann surface.

Notable property of Wick rotation is that it preserves compactness.

Using the Lie algebra $\mathfrak{so}(1,9)$, consider the Lie algebra representation $32_{Dirac} : \mathfrak{so}(1,9) \to GL(V)$, where $dim(V) = 32$, which can be decomposed to $32_{Dirac} = 16_{Weyl} \oplus 16_{Weyl}$.

Definition 5.1.7 *(Important Constants)*

- *(String coupling constant)*
  g_s *is defined by* $g_s = e^{<\Phi>}$ *where* Φ *is dilaton.*
- *(String length)*
  l_s *is* $T = \frac{1}{2\pi l_s^2}$ *where* T *is the tension(energy per length).*
- *(Radius)*
  $R_{11} \sim g_s l_s$
- *(Mass)*
  $M^2 \sim \frac{n}{l_s^2} + \frac{2}{l_s^2}(N_L + N_R - 2)$ *where* n *is a winding number,* N_L *and* N_R *are quantum numbers on left/right vibration mode.*
- *(Momentum)*
  $p_{11} = \frac{n}{R_{11}}$ *where* $n \in \mathbb{Z}$

Definition 5.1.8 *(Quantum Mass)*
The quantum version of gravity is similar to the classic version, but we need to add quantum correction on it. The classic version of gravity is Newton's universal gravitation, whose force depends on the mass of the objects, called classical mass, whose equivalent in string theory is that the mass can be defined by the vibration of the string, while adding quantum correction on the classical mass makes quantum mass, mathematically expressed as

$M_{quantum} = M_{classical} + < perturbative effect > + < non-perturbative effect >.$

Note this theory exists only if energy level is very low ($g_s << 1$).

Definition 5.1.9 *(Perturbative Effect)*
Assuming the very low-energy state, g_s will be very small $g_s << 1$, thus we can calculate perturbative expansion, where numerially this can be approximated using only 1-loop, since $g_s^2 >> g_s^4$ and higher term can be ignored. The quantum version of path integral can be generated by adding quantum fluctuation term on the field $\phi = \phi_{cl} + \eta$ where η is the quantum fluctuation.
$\Gamma[\phi_{cl}] = S[\phi_{cl}] + \frac{i\hbar}{2} Trlog \frac{\delta^2 s}{\delta \phi^2} + \mathscr{O}(\hbar^2)$

and the $Trlog$ term is 1-loop correction.

Definition 5.1.10 *(Partition Function)*
If we are interested in modular calculation, the 1-loop can be written by using partition functions as follows:

$Z_{1-loop} = \int_F \frac{d^2\tau}{\tau_2^2} Tr((-1)^F q^{L_0 - c/24} q^{-\overline{L}_0 - c/24})$

Definition 5.1.11 *(Non-Perturbative Effect)*
Non perturbative solution is $e^{-\frac{S[\phi_{inst}]}{\hbar}}$, by approximation of the specific reason. Around the point $\frac{\delta S}{\delta \phi}|_{\phi_{inst}}$, take path integral

$\int D\phi e^{-\frac{S[\phi]}{\hbar}} \sim e^{-\frac{S[\phi_{inst}]}{\hbar}} \cdot$(1-*loop determinant*) $\cdot$ (*higher loops*)

Definition 5.1.12 *(D-Branes)*
D-brane is a notion that appears in open string theory. Since an open string has two boundaries, and the boundaries connect to the D-brane. or more formally speaking D-branes are a class of objects of open strings can end with Dirichlet boundary conditions. This means, that from D-brane perspective the particle looks a point, thus D-brane theory is a generalization of gauge theory. In fact, in closed string theory, D-brane can also be considered, but it's required to apply Ω projection to transform closed string to open, so we can discuss as if it's open string theory.

D-branes are classified by their dimension. D_N-branes is N-dimensional cube. D1-brane is a string, D2-brane is a plane, and our living space if D_3.

Definition 5.1.13 *(Chan-Paton Factor)*
In open string theory, Chan-Paton factor creates label on open strings, since generally string theory has multiple of D-branes, and we might be interested in which D-brane do the edges of the open strings connected to. Formally, this means that an open string $|\psi\rangle$ can be labeled to $|\psi, i, j\rangle$, where $i, j = 1, 2, \cdots, N$, but this numbers i, j, cannot be absolutely randomly selected. This condition

requires the gauge symmetry

$$|\psi, i, j\rangle \mapsto U_i^k \, |\psi, k, l\rangle \, (U^\dagger)_l^j$$

Also, the length of the string l can change based on the Dbranes the open strings connect to. The length of the string can define the mass of the string.

5.2 Supergravity (SUGRA)

SUGRA is low-energy supergravity in quantum mechanics, and this is different from quantum gravity, since quantum fluctuation is ignorable. SUGRA can deduce SUSY version of Einstein equation from QFT, and let's set this to our goal. In this section, we assume that the SUGRA is $N = 2$ low-energy supergravity in $10D$ type IIB string theory, defined over the space $\mathbb{R}^{1,9}$ (of course, there are many different variety of SUGRA). Also, $N = 2$ in $10D$ means that there are 32 SUSY supercharges.

Note 5.3 *(Low-Energy/High-Energy)*
Here we'll note that they theory of gravity might be discussed differently based on energy levels. Low-energy means that the vibration of the string is very low, and the string can be considered as a point, means quantum correction can be ignoreably small, that's the topic of SUGRA. On the other hand, in high-energy, the theory of gravity can treat Big Bang where the particles are of high energy, or black hole where each individual particle has low-energy, but the density of the particles are high.

But just wonder what is the intuition of "low-energy". Low-energy is the temperature of Earth surface as $300K$ or even nuclear fusion as 10^8K. On the other hand, high-energy is mathematically $E_s \sim 10^19Gev$, where the energy is defined as $E_s \sim \frac{\hbar c}{l_s}$.

Definition 5.3.1 *(SUSY Termionology)*
SUSY called supersymmetry is an idea that relates boson and fermion with supersymmetry transformation each other, and it's used in theory with "super" name on it, In terminology, for example, SYM=SUSY+YM, SCFT=SUSY+CFT. N is number of generators of SUSY, and in $10D$ space, the number of superchages are $2^{\frac{10}{2}-1} = 16$ times N, while in $4D$ space, the number of superchages are $2^{\frac{4}{2}} = 4$ times N, the formula is little different since Majorana Weyl condition. For example, in $10D$ $N = 2$ type IIB string theory, there are totally $16N$ of supercharges.

Chiralty is a SUSY theory where only left-hand fermions ψ exist, but in general SUSY accepts both left/right-hand fermions ψ and $\overline{\psi}$.

Definition 5.3.2 *(R-R and NS-NS)*
The world sheet $X^\mu(\sigma, \tau)$ and $\psi^\mu(\sigma, \tau)$

Definition 5.3.3 *(Einstein Equation)*
In type IIB string theory, $10D$ *SUGRA is to calculate Einstein equation* $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$ $\mu, \nu = 0, 1, 2, \cdots, 9$*, that is induced from the minimization of action of graviton Lagrangian in QFT. For type IIB string theory, the action is*

$$S_{IIB} = \int d^{10}x\sqrt{-g}(R - \tfrac{1}{2}(\partial\phi)^2 - \cdots - \tfrac{1}{4}|F_5|^2) - \int C_4 \wedge H_3 \wedge F_3$$

where $\int C_4 \wedge H_3 \wedge F_3$ *is Chern-Simon term, just in case, but it doesn't appear in variational calculus, so it's not very important. Below is the list of the used variables*

- $g_{\mu\nu}$ *is* $10D$ *gravity field*
- Φ *is dilaton, determining the coupling constant* $g_s = e^{\Phi}$
- B_2 *is NS-NS* 2*-form.*
- C_0 *is R-R* 0*-form.*
- C_2 *is R-R* 2*-form. Coupling to* $D1$*.*
- C_4 *is R-R* 4*-form. Coupling to* $D3$*.* 5*-form field strength* F_5 *is self-dual, means* $F_5 = \star F_5$*.*
- C_4 *is R-R* 4*-form*
- $H_3 = dB_2$ *is NS-NS* 3*-form*
- $F_3 = C_2 - C_0 H_3$ *is R-R* 3*-form*
- *Naturally,* $C_4 \wedge H_3 \wedge F_3$ *is* 10*-form*

Definition 5.3.4 *(Graviton)*
Graviton is a particle, whose gauge transformation, if possible any, should be the transformation of $\mathbb{R}^{1,3}$ *space, and it doesn't work (It works for any inner symmetry* $SU(N)$*, so gauge theory can be defined on electro-magnetic, weak, and strong forces).*

Alternative explantion of why gravition cannot be described by gauge theory is that graviton is a particle of spin 2*. This is because the symmetry of graviton is* $SO(3,1)$ *and it's isomorphic to* $SL(2,\mathbb{C})$*, whose irreducible representation is given by* (j_L, j_R)*, and* j_L *and* j_R *corresponds to* $SU(2)$ *spin.*

Definition 5.3.5 *(Gravitino)*

5.4 M-Theory

M-theory is 11-dimensional world model, a generalization of IIA string theory and Heterotic $SO(32)$-string theory, or conversely, 5-dimensional space in M-theory as solution of KK equation for non-compact part can be reduced to 4-dimensional space in IIA. This can be done by compactifying 1-extra dimension of 5-dimesional space, making S^1-bundle, whose radius of the fiber S^1 is R, and its radius depends on the energy level, and if R is very small, that can be approximately ignoreable. This theory could be mathematically interesting, since $11D$ supergravity reduces complexity of $10D$ supergravity calculation, which is generally considered to be harder, since $10D$ supergravity includes perturbative and non-perturbative effect.

Definition 5.4.1 *(M2/M5-Brane and KK Compactification)*
$M2$ and $M5$-branes are geometry inside of M-theory. Under KK compactification of $M2$-brane in $11D$ becomes $F1$-string and $D2$-brane in $10D$ IIA, while $M5$-brane becomes $D4$-brane in IIA, and removing S^1 extra dimension.

Definition 5.4.2 *(Winding Number and Moduli Space of Torus)*
One of useful invariants of M-theory is the winding number of $M2$-brane on direction of a torus T^2. This torus is generated by KK-compactification of $11D$ space to S^1, but another S^1 is generated from CY compactification of the remaining $10D$ space to $4D$ (How many S^1 exist in CY manifold? Just take the fundamental group). By choosing the torus of M-theory, we can classify the possible $M2$ space, which will make the moduli problem in the moduli space of torus.

Definition 5.4.3 *(G_2-Holonomy)*
Wait, in M-theory the $11D$ geometry is Ricci-flat, but holonomy exists (common example is CY manifold is Ricci-flat but has a holonomy). Removing $4D$ fmo $11D$ M-theory, we'll be left 7-dimensional, whose gauge symmetry might be G_2 real Lie group.

5.5 F-Theory

F-theory is generalization of IIB string thery. Its model is $12D$ since it's adding $SL2$ symmetry on IIB $10D$ model.

6 List of Further Math

6.1 Mirror Symmetry

Roughly speaking, mirror symmetry is to compare two different kind of geometries. One side of geometry studies symplectic structure called A-model, while the other geometry studies complex structure called B-model. We are listing a variety of mirror symmetry.

Definition 6.1.1 *(Geometric Mirror Symmetry)*
One thing to compare geometries is their Hodge structure. Complex manifold has a Hodge decomposion of its cohomology, and it's very dual to the kähler structure, that also has Hodge decomposition. The dual of geometry X is $\check{X}$, that can be defined by purely its homological structure

$$h^{1,1}(X) = h^{n-1,1}(\check{X})$$
$$h^{n-1,1}(X) = h^{1,1}(\check{X})$$

Definition 6.1.2 *(BRST Mirror Symmetry)*
already mentioned somewhere

Definition 6.1.3 *(SCFT Mirror Symmetry)*
Let's start from some example: consider the SCFT mirror symmetry B-model side is period integral generated from LG model, and the SCFT mirror symmetry derives BRST version of mirror symmetry, which derives Maurer Cartan equation with Kodaira-Spencer deformation. BRST operator Q can be described explicitly by using Virasoro algebra L_n, and

$$Q_B = \Sigma_n c_{-n} L_n - \frac{1}{2}\Sigma_{m,n}(m-n) : c_{-m}c_{-n}b_{m+n} :$$
$$Q_B\phi = \overline{\partial}\phi + [\phi, \phi] \text{ where } \overline{\partial} = Q_B$$

Then naturally the solution of Maurer Cartan equation is equivalent to the 1*-cocycle condition $Q_B\phi = 0$.*

Definition 6.1.4 *(Path Integral and Quantum Product)*
The path integral Z of superstring theory version is defined a bit differently, that is given by

$$Z = \int_{\Sigma_g \to X} DXe^{-S[X]} = \Sigma_{g=0}^{\infty} g_s^{2g-2} Z_g$$

but this can be alternatively calculated by using Gromov-Witten invariant. That is,

$$F_g = \Sigma_\beta N_{g,\beta} q^\beta \text{ where } F_g \text{ is called free energy, and } q = e^{-t} \text{ where } t \text{ is time}$$
$$N_{g,\beta} = \int_{[\overline{\mathfrak{M}}_{X,\beta}]^{virt}} 1$$

If we use Gromov-Witten invariant, we can also define quantum product $a \star b$.

$$a \star b = a \cup b + \Sigma_{\beta \neq 0} N_{ab}^{\beta} q^\beta$$

Definition 6.1.5 *(Picard-Fuch Equation)*
Period integral is $\Pi_\gamma(z) = \int_\gamma \Omega(z)$*, while Picard Fuch equation is* $\mathfrak{L}\Pi_\gamma(z) = 0$ *where* γ *is holomorphic* 3*-form, and* Ω *is* 3*-form. Also, there is a connection of period integral to SCFT, since it defines kähler potential* $K(z, \overline{z}) = -log(i \int_X \Omega(z) \wedge \overline{\Omega(\overline{z})})$*.*

Definition 6.1.6 *(TCFT)*
TCFT is another mirror symmetry. SCFT generates BRST, which generates TCFT.

Definition 6.1.7 *(BCOV)*

Definition 6.1.8 *(Fukaya Category)*
Fukaya category is A_∞ *category. Where each object is Lagrangian submanifold of the fixed symplectic geometry* M*, and the morphisms are Floer chain groups.*

Definition 6.1.9 *(Homological Mirror Symmetry)*
This is still a conjecture, but homological mirror symmetry conjecture is $A - \infty$ *perspective on mirror symmetry. which is the equivalence of categories*

$$Fuk(X) \leftrightarrows D^b(Coh(X^\vee))$$

Both models are $A - \infty$ *categories, but if Kodaira-Spencer deformation of B-model is* $L - \infty$*,* $L - \infty$ *needs* $A - \infty$ *enhancement.*

6.2 Topological String Theory

See Marc Gross.

6.3 S-Duality

S-duality claims the equivalence of two string theories, whose coupling constant are resp g_s and $\frac{1}{g_s}$, and this is mathematically interesting, because $g_s \mapsto \frac{1}{g_s}$ is a specific example of möbius transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. The coupling constant g_s is defined as $g_s = e^{<\Phi>}$ where Φ is dilaton. Dilaton is defined by the trace of 10x10 tensor $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}\left|0\right\rangle$ where α^μ_{-1} and $\tilde{\alpha}^\nu_{-1}$ is defined by the constant coefficient of the Bosonic string, and $\mu, \nu = 1, 2, \cdots, 10$.

Defining complexified coupling constant $\tau = C_0 + \frac{i}{g_s}$, the transformation $g_s \mapsto \frac{1}{g_s}$ makes $\tau \mapsto \frac{-1}{\tau}$, that's Möbius transformation. This transformation exchanges electro charge lattice and magnetic charge lattice as $(e, m) \mapsto (-m, e)$. Here the weight lattice generates root system of Lie algebra of the gauge group, so the dual of weight lattice makes another root system. Thus its gauge group corresponds to the Langlands dual.

Definition 6.3.1 *(Montonen-Olive Duality)*

6.4 T-Duality

In string theory, under the assumption of the torus compactification $\mathbb{R}^{1,9} \mapsto \mathbb{R}^{1,9-d} \times T^d$ to consider a closed string winding around the circle (each direction of the torus), T-duality claims the invariant of energy of a closed string. In the transform of the radius R to $R \leftrightarrow \frac{\alpha}{R}$, the energy $E^2 \sim (\frac{n}{R})^2 + (\frac{wR}{\alpha'})^2 +$ (fibration mode) is invariant, under the transformation $R \mapsto \alpha R$ with the winding number $n \leftrightarrow w$ exchanged, where this radius R of the string is (roughly speaking) the average distance of the string from the center.

More broadly, the Calabi-Yau compactification can generates T-duality structure, since Calabi-Yau manifold can have torus fibration where a closed string can wind around it. However, the torus fibration of Calabi-Yau is highly controversial, and this problem is called SYZ conjecture.

Note 6.5 *(Momentum)*
Momentum is the difference from the mass square and the energy square $E^2 = p^2 + M^2$, and it's defined by $p = \frac{n}{R}$, where n is the quantum number. Here note that momentum p and energy E look to have the same unit in physics, but since we assume $c = 1$ without unit, so these are mathematically comparable.

Definition 6.5.1 *(Converting from IIA to IIB)*
The Bosonic string can be decomposed to the left/right moving waves $X^\mu(\tau, \sigma) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)$, but by T-duality, only the left moving wave will be flipped the sign, but the right moving wave doesn't change.

$$X^\mu_L(\tau + \sigma) \mapsto -X^\mu_L(\tau + \sigma)$$
$$X^\mu_R(\tau + \sigma) \mapsto X^\mu_R(\tau + \sigma)$$

In this transformation, the chiralty will be precisely flipped, thus type IIA becomes type IIB, and vice versa.

Definition 6.5.2 *(SYZ Conjecture)*
SYZ conjecture claims the fibration of Calabi Yau manifold by the 3-torus $\mathbb{T}^3 \hookrightarrow M \to B$, where B is the real 3D. SYZ conjecture is not generally prove, but is proved in K3 surface.

Definition 6.5.3 *(Fourier-Mukai Transform)*

6.6 Donaldson Thomas Theory

Definition 6.6.1 *(Quantum Correction)*
$Z = \int Z = \Sigma g_s^{2g-2} F_g$

Definition 6.6.2 *(Donaldson Thomas Invariant)*
DT invariant is a number of D6-D2-D0 BPS states, where D6 is CY 3-fold X, D2 is a curve on D6, D0 is a point over D6. We consider $v = ch(E) = ch_0(E) + ch_1(E) + ch_2(E) + ch_3(E)$ where $E \in Coh(X)$. This coherent sheaf E describes the object(set of particles not necessarily one particle), that can be alternatively given by charges $ch(E)$.

(BPS State)
For $M(E) \geqq Z(E)$, it's called BPS state if it's equal, or sheaf theoretically if $\phi(F) < \phi(E)$ for subobject $F \subset E$, then E is called stable or also called BPS state. BPS state of E can be decomposable if $Z(E) = Z(F) + Z(G)$ and $\gamma(E) = \gamma(F) + \gamma(G)$ meaning $\phi(E) = \phi(F) = \phi(G)$, when the decomposition works $E \mapsto F + G$, where $0 \to F \to E \to G \to 0$ is short exact sequence, whose decomposability works if $Ext^1(F,G) = 0$.

(Bridgeland Stability Condition)
Stability condition is a pair $(Z, \mathscr{P})$ such that $\mathscr{P}$ is a slicing of a triangulated category $\mathscr{D}$. The slicing $\mathscr{P}(\phi)$ $\phi \in \mathbb{R}$ is such that $\mathscr{P}(\phi)[1] = \mathscr{P}(\phi+1)$, and for $\phi_1 > \phi_2$ and $A \in \mathscr{P}(\phi_1)$ and $B \in \mathscr{P}(\phi_2)$, $Hom(A,B) = 0$, and existence of Harder-Narashimhan filtration, while $Z : K(\mathscr{D}) \to \mathbb{C}$ is a central charge, and it needs to satisfy for $0 \neq E \in \mathscr{P}(\phi)$, $Z(E) = m(E)exp(i\pi\phi)$ for $m(E) \in \mathbb{R}_>$. The stability condition is that $E \in \mathscr{D}$ is stable wrt $(Z, \mathscr{P})$ if for every surjection $E \to F$, $\phi(E) < \phi(F)$.

(K-theory)
$K(X) = K_0(Coh(X))$ is a Grothendieck group generated by a triangulated category $D^b(Coh(X))$, that is quotient by the every combination of derived version of short exact sequences $A \to B \to C \to A[1]$, so that $[B] = [A] + [C]$, which is by context, Chern character additive as $ch([B]) = ch([A]) + ch([C])$, and by context the quotient $K(X) = Free(D^b(Coh(X)))/ < [B] - [A] - [C] >$ makes Chern character invariant an isomorphism class. Given the stability condition $\sigma = (Z, \mathscr{P})$, DT invarant is defined by

$$DT(\gamma) = \chi(M_\sigma(\gamma), \nu)$$

where $M_\sigma(\gamma) = \{E \in \mathscr{D} | E$ is Z-stable and $[E] = \gamma\}$ is an moduli stack.

and $\gamma = ch(E)\sqrt{td(X)} \in K(X)$

Definition 6.6.3 *(Virtual Fundamental Class)*
Formally, the DT invariant can be described more simply, if we use virtual fundamental class $[M]^{vir} \in H_0(M, \mathbb{Z})$, and it's given by $DT = \int_{[M]^{vir}} 1 = \chi(M)$ Now here Euler characteristic χ calculates number of points if the space consists of 0-dimensional points, and if the original geometry M is different from 0-dimensional geometry, consider virtual fundamental class of M instead.

Definition 6.6.4 *(Wall Crossing Formula)*
If we change the stability condition σ of above, then the object can no longer be

stable, and if moreover the particle has neither the direction of decomposition or semistablity, this can be decomposed to some particles. This phenomena is given by Wall-Crossing formula.

Example 6.7 *(Application of BPS State)*

- *BPS blackhole is a black hole consists of an object of BPS state. This means, by default there is only one object of BPS state, but # can be modified by the change of stability condition.*
- *Calculation of Non-perturbative coupling constant and structure of moduli space.*
- *BPS index for calculating blackhole micro count and topological analysis*

Definition 6.7.1 *(Hall Algebra)*
For $E, F \in \mathscr{A}$ of an abelian category $\mathscr{A}$ over a finite field $\mathbb{F}_p$, the Hall algebra $H(\mathscr{A})$ is $H(\mathscr{A}) = \bigoplus_{[E]} \mathbb{Q}[E]$ where $[E]$ is an isomorphism classes of E as an object of $\mathscr{A}$. The product of representation $[E]$ and $[F]$ is given by $[E] \star [F] = \Sigma g^G_{E,F}[G]$ where $g^G_{E,F}$ is a number of possible G, which satisfies short exact sequence $0 \to E \to G \to F \to 0$.

In application, Hall algebra can define DT invariant by using quantum torus. A morphism $\int$ from Hall algebra to quantum torus $\int : [E] \mapsto q^{\chi(E,E)} x^{ch(E)}$ where $\chi(E, F) = \Sigma_i (-1)^i dim(Ext^i(E, F))$ is an Euler form, but $q^{\chi(E,E)} = \Omega(E) \in \mathbb{Z}$ also denoted by $\chi(\mathfrak{M}_E)$, which is DT invariant. The quantum torus appears in Hall algebra, since $g^E_{M,N}$ appears in formal deformation $g^E_{M,N} = \begin{bmatrix} m+n \\ n \end{bmatrix}_q$.

Definition 6.7.2 *(Cohomological Hall Algebra)*
(Quiver Representation)
For a quiver $Q = (Q_0, Q_1, s, t)$, quiver representation $Rep(Q)$ is a mapping of quiver morphism to linear maps, that is $a : i \to j$ corresponds to a linear map $\phi_a : V_i \to V_j$ for arbitrary vector spaces V_i and V_j and arbitrary morphism ϕ_a. For determining the dimension of vector spaces of Q_0, we define the index $d = (d_i)_{i \in Q_0}$, and we let quiver representation of d $Rep(Q, d) = \prod_{a \in Q_1} Hom(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}})$. To eliminate the double counting due to the base change, we'll also consider quotient stack $[Rep(Q, d)/G_d]$ by $G_d = \prod_i GL_{d_i}(\mathbb{C})$.

(Cohomological Hall algebra)
Cohomological Hall algebra is defined by $H_Q = \oplus_d H^*_{G_d}(Rep(Q, d))$, where the equivariant cohomology $H^*_{G_d}(Rep(Q, d))$ is the same as the cohomology of $H^*(\mathfrak{M}_d)$. where $\mathfrak{M}_d = [Rep(Q, d)/G_d]$ is the moduli space (Why is this a moduli space? $Rep(Q, d)$ is in fact just an affine scheme, whose GIT is moduli space). As an algebra, while the addition is canonical, the product is given by

$$m_{d_1,d_2} : H^*_{G_{d_1}}(Rep(Q,d_1)) \otimes H^*_{G_{d_2}}(Rep(Q,d_2)) \to H^*_{G_{d_1+d_2}}(Rep(Q,d_1+d_2))$$
$$(\alpha,\beta) \mapsto a \star b = q_*(p^*(\alpha \boxtimes \beta))$$
$$\textit{where these morphisms } p^* \textit{ and } q_* \textit{ are defined in}$$
$$Rep(Q,d_1) \times Rep(Q,d_2) \leftarrow_p Ext(d_1,d_2) \to_q Rep(Q,d_1+d_2)$$
$$\textit{where } Ext(d_1,d_2) = \{(V,V_1) | V \in Rep(Q,d_1+d_2), V_1 \subset V \textit{ subrepresentation of dimension } d_1\}$$
$$\textit{where } Ext(d_1,d_2) \textit{ is not exactly extension but abuse of notation.}$$

Definition 6.7.3 *(Motivic Hall Algebra)*

6.8 ADHM Construction

Definition 6.8.1 *($D(-1)$ Instanton)*
$D(-1)$ is a $0D$ point existing in the $10D$ space in string theory, called instanton. This point is showing point of the non-perturbative effect. This instanton is typicalled considered to exist finitely many.

Instanton is often considered in an open string whose boundaries exist on some $D(-1)$, which means both edges connects to $D(-1)$ and $D(-1)$, or connects to $D(-1)$ and $D3$. If at least one side of an open string is $D(-1)$, then the open string can only exist in the universe instant amount of time. The vibration of this open string can only be measured by its zero mode, in other words, the string is actually no vibration, not only positional data.

Definition 6.8.2 *(Moment Map)*
(M,ω) is a symplectic manifold, and a Lie group G acts on M via symplectomorphism (each action of $g \in G$ preserves ω).

$< -, - >: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$

Any $\xi \in \mathfrak{g}$ induces a vector field $\rho(\xi)$ on M, defined locally as

$\rho(\xi))_x = \frac{d}{dt}|_{t=0} exp^{t\xi} \cdot x$

Since $\iota_{\rho(\xi)}\omega$ is closed and if moreover exact, then there exists $H_\xi : M \to \mathbb{R}$. Now momentum map $\mu : M \to \mathfrak{g}^*$ is a map

$d(< \mu, \xi >) = \iota_{\rho(\xi)}\omega$

for all $x \in \mathfrak{g}$ and $< \mu, \xi > (x) =< \mu(x), \xi >$.

Definition 6.8.3 *(ADHM Space)*
Assuming there exists k amount of instanton $D(-1)$ and N amount of $D3$ branes, ADHM construction as a collection (B_1, B_2, I, J) where each morphisms

are $I, J : \cdot \rightleftarrows \cdot$ is $k \times N$ and $N \times k$ matrices, while $B_1, B_2 : \cdot \to \cdot$ is an $k \times k$ endomorphism. These morphisms need to satisfy the equations below

$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0$$
$$\mu_{\mathbb{C}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger + JJ^\dagger = 0$$

since AHDM space can be defined using the inverse of momentum maps $\mathfrak{M}_{k,n} = \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(0)/U(k)$ where this $U(k)$ action (also called symplectic action) is defined as $(B_1, B_2, I, J) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1})$.

Definition 6.8.4 *(AHDM Space)*
Furthermore, the ADHM moduli space is given by $\mathfrak{M}_{ADHM} = \{(B_1, B_2, I, J) | \mu_{\mathbb{C}} = 0, \mu_{\mathbb{R}} = 0\}/U(k)$ where $(B_1, B_2, I, J) \in Mat(k) \oplus Mat(N) \oplus Mat(k, N) \oplus Mat(N, k)$. This ADHM moduli space is, in fact, an algebraic variety with flat hyperkähler structure.

Definition 6.8.5 *(Quiver variety)*
Quiver variety is a generalization of ADHM moduli space.

6.9 TQFT

Definition 6.9.1 *(Cohomological Field Theory)*
Calabi Yau category??

Definition 6.9.2 *(4D CFT)*
$4D$ CFT looks boring unlike $2D$ CFT, since its symmetry is $SO(2, 4)$, but combining with Wilson loop in TQFT, it makes some other structures e.g. Kac-Moody algebra, Virasoro algebra, Yangian symmetry, and line defects etc.

6.10 Seiberg-Witten Theory

Seiberg-Witten theory is a simplified version of Donaldson-Thomas theory, since DT invariant has $SU(2)$ symmetry, but SW invariant only has $U(1)$ symmetry together with Spinor, thus computationally easier.

Definition 6.10.1 *($Spin^c$)*
Ordinary spinor $Spin(n)$ is a double cover of $SO(n)$. Spin c $Spin^c(n)$ is $Spin^c(n) = \frac{Spin(n) \times U(1)}{\mathbb{Z}_2}$. Here I'll list popular example of spinors.

- $spin(3) \cong SU(2)$
- $spin(4) \cong SU(2) \times SU(2)$
- $SO(4) \cong \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}$

Definition 6.10.2 *(Seiberg-Witten Moduli Space) For defining the Seiberg-Witten moduli space, we'll define Seiberg-Witten equation as below*

$$D_A\psi = 0F_A^+ = \sigma\psi$$

defined over the 4*-dimensional geometry, where* A *is a* $U(1)$ *gauge-connection,* ψ *is positive-spinor field,* D_A *is a Dirac operator,* F_A^+ *is a curvature of self-dual part,* $\sigma(\psi)$ *is quadratic form. All the possible pair of solution space* (A,ψ) *generates the moduli space* $\mathscr{M} = \{(A,\phi)\}$*, but let's take the quotient by the gauge redundancy* $U(1)$*, so* $\mathscr{M}_{\mathfrak{s}} = \{(A,\phi)\}/\mathfrak{G}$ *where* $\mathfrak{G} = U(1)$*. Now from SW moduli space, the counting SW invariant, in other words, BPS invariant is possible. Below is the formula for SW invariants.*

$$SW_X(\mathfrak{s}) = \Sigma_{p\in\mathscr{M}_{\mathfrak{s}}} \pm 1 \qquad \text{for } dim(\mathscr{M}_{\mathfrak{s}}) = 0$$

$$SW_X(\mathfrak{s})(\alpha_1,\alpha_2,\cdots,\alpha_k) = \int_{\mathscr{M}_{\mathfrak{s}}} \alpha_1 \wedge \alpha_2 \wedge \cdots, \wedge\alpha_k \qquad \text{for } dim(\mathscr{M}_{\mathfrak{s}}) > 0$$

where $\alpha_i \in H^*(\mathscr{M}_{\mathfrak{s}})$ *and* $dim(M_{\mathfrak{s}}) = \frac{1}{4}(c_1(\mathfrak{s})^2 - (2\chi(X) + 3\sigma(X)))$.

Definition 6.10.3 *(Magnetic Monopole)*
Highly relevant to Seiberg Witten theory.

6.11 Higgs/Couloumb Branch

Higgs/Couloumb Branch correspondence appears in $3D$ mirror symmetry of $3D$ and $N = 4$ supersymmetry (while ordinary Calabi-Yau mirror symmetry is $2D$ and $N = (2,2)$).

Definition 6.11.1 *()*
Single moduli space $\mathfrak{M}$ *of vacua can be decomposed to Higgs branch* $\mathfrak{M}_H$ *and Couloumb branch* $\mathfrak{M}_C$.
$dim_{\mathbb{H}}(\mathfrak{M}_H) + dim_{\mathbb{H}}(\mathfrak{M}_C) =$ *rank of gauge group*

Definition 6.11.2 *(Quiver Gauge Theory)*
In quiver gauge theory, we use quiver and its representation to describe the gauge theory of non-simple Lie groups. The point in the quiver shows the simple Lie group as gauge subgroup, and the arrow in the quiver shows the representation.

$$\circ_{U(1)} \longrightarrow \circ_{SU(2)}$$

For example, if the gauge group is $G = U(1) \times SU(2)$*, the corresponding quiver is the above diagram, and we have 3 dots from the number of simple Lie groups. A bifundamental field is denoted* $(1,\overline{2}) : U(1) \times SU(2) \to Hom(V_1, V_2)$

representation defined by

$$(1,\overline{2}) : U(1) \times SU(2) \to End(V)$$
$$\phi \mapsto g_2 \phi g_1^{-1}$$

where an SU(2) fundamental representation carrying a U(1) charge, and thus $dim(V_1) = 1$ *and* $dim(V_2) = 2$, *and* $\phi \in Hom(V_1, V_2)$. *Or if the gauge group is* $G = U(1) \times SU(2) \times SU(3)$, *the quiver diagram will be as follows, and guess what the group action like*

$$\circ_{U(1)} \longrightarrow \circ_{SU(2)} \longrightarrow \circ_{SU(3)}$$

The next step is how to extract Higgs/Couloumb branch from the quiver gauge theory.

Definition 6.11.3 *(Higgs/Coulomb Branch)*
Let $E \to X$ *where* $\phi \in H^0(X, End(E) \otimes K_X)$ *be a Higgs bundle. For the quiver gauge theory,* E_1 *is a rank* 1 *line bundle, and* E_2 *is a rank* 2 *vector bundle.*

$$\circ_{U(1),E_1} \longrightarrow \circ_{SU(2),E_2}$$

where (E_1, E_2, ϕ) *is a moduli space where* $\phi \in H^0(X, Hom(E_1, E_2) \otimes K_X)$ *is a bifundamental field. From the moduli space* $\mathfrak{M}_{Higgs}\{(E_1, E_2, \phi)\}/(gaugegroup)$, *take* $\phi \neq 0$ *case, whose stability condition is Higgs bundles, also called mathematically Nakajima quiver variety. Also, the moduli space* $\mathfrak{M}_{Higgs}\{(E_1, E_2, \phi)\}/(gaugegroup)$ *is called Couloumb branch, also called mathematically BFN construction, both given algebraic geometrically.*

In other words, Higgs branch is a moduli space of non-zero value of VEV $<\tilde{q}> = \langle 0| \hat{\tilde{q}} |0\rangle$ *and* $<q> = \langle 0| \hat{q} |0\rangle$, *where q satisfies D-term condition* $q^\dagger \sigma q = 0$, *while Coulomb branch is a moduli space of VEV* $<\hat{\phi}> = \langle 0| \hat{\phi} |0\rangle$.

Definition 6.11.4 *(Nakajima Quiver Variety)*

Definition 6.11.5 *(Braverman-Finkelberg-Nakajima Construction)*
By BFN construction, Coulomb branch is defined by the scheme spectrum $\mathfrak{M}_C(G, N) = Spec(H^{G_*^{\mathscr{O}}}(\mathfrak{R}_{G,N}))$ *where* $\mathfrak{R}_{G,N} = \{(g, s) | g \in G_{\mathscr{K}}, s \in N_{\mathscr{K}}, g \cdot s \in N_{\mathscr{O}}\}/G_{\mathscr{O}}$ *is a moduli space of monopole configuration. G is the gauge group and N is the representation, and* $Gr_G = G_{\mathscr{K}}/G_{\mathscr{O}}$ *where* $G_{\mathscr{K}}$ *is loop group and* $G_{\mathscr{O}}$ *is formal loop group.* $H^{G_*^{\mathscr{O}}}(\mathfrak{R}_{G,N})$ *is an equivariant Borel-Moore cohomology.*

Definition 6.11.6 *(Some Properties)*

- $dim_{\mathbb{H}}(\mathfrak{M}_H) + dim_{\mathbb{H}}(\mathfrak{M}_C) =$ *rank of gauge group*

- *Higgs branch of mirror theory A $\cong$ Coulomb Branch of Theory B*
- *Not all Higgs branches are Higgs bundles and vice versa.*

6.11.1 McKay and DuVal

McKay correspondence and DuVal singularity are also related to mirror symmetry. McKay correspondence claims the correspondence of finite group $G \subset SL(2, \mathbb{C})$ and simple Lie groups of type ADE, and DuVal singularity is a classification of complex 2-dim Calabi-Yau manifold classified by type ADE.

The Lie algebra of type ADE is the synthesis of $\mathfrak{sl}_2$ of type A1, which is well-knowned, and the quiver structure of type ADE, also called simply laced dynkin diagram, can determine the bigger simple Lie algebra of type ADE, and each vertex in the dynkin diagrams corresponds to $\mathfrak{sl}_2$. The famous application is to construct the quantum group $U_q(\mathfrak{g})$ of $\mathfrak{g}$ is type ADE.

First of all, $\mathfrak{sl}_2(\mathbb{C})$ is very clear mathematically, but physics is often real geometric. In fact, the gauge theory uses $\mathfrak{su}_2(\mathbb{R})$ symmetry, which is real geometric, but algebraically it might be replaced to $\mathfrak{sl}_2(\mathbb{C})$ by complexification.

$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2(\mathbb{R}) \otimes \mathbb{C}$

First, Dynkin diagram generated by Cartan matrix. Discrete Laplace operator generates Cartan matrices.

Definition 6.11.7 *(Discrete Laplace Operator)*

Discrete Laplace operator is a discrete version of continuous Laplace operator. Discrete Laplace operator is used in graph theory, and simply take the difference of functional value of neighboring points. For a given graph $\Gamma = (V, E)$ and a function $\phi : \Gamma \to \mathbb{R}$,

$\Delta\phi(v) = \Sigma_{w\in\Gamma, d(v,w)=1}(\phi(v) - \phi(w))$

where $v \in \Gamma$ is a vertex, and $d(v, w)$ is a distance function of two points.

Here, it's possible to recognize ϕ as a vector and Δ as a matrix to think this problem linear algebraically.

Definition 6.11.8 *(Laplacian Matrix)*
We let the Laplacian matrix $L = \{L_{i,j}\}$ as

$$L_{i,j} = \begin{cases} deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_j \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

deg is the degree function, in which each vertex is counted by how many edges connect with it, and we could write simply $L = D - A$ where D is degree matrix and A is adjacency matrix.

Definition 6.11.9 *(Gabriel Theorem)*

A simple finite dimensional Lie algebra has a connected Dynkin diagram (I'll omit how to derive it), and Dynkin diagram is a graph. and by Gabriel theorem, dynkin diagram is classified by A-G type. Type A,D,E is called simply-laced graph.

Definition 6.11.10 *(McKay Correspondence)*
The GIT quotient $\mathbb{C}^2/G$ for some the finite group G is a singular algebraic variety, containing important information in its resolution of singularity. and for the case of $G = S^5$, the length of resolution of singularity is 8, which is in other words, the singular algebraic variety consists of 8 exceptional curves(divisors), and this corresponds to the dimension $8 = dim_{\mathbb{C}} E8$. The intersection of these 8 exceptional curves combinationally defines McKay quiver, and this claims the one-to-one correspndence of the Mckay graph to extended Dynkin diagram of ADE type Lie algebras. For the group represntation of a finite subgroup $G \subset SL(2, \mathbb{C})$, the Mckay graph Γ_G is a graph whose verticies correspond each character of irreducible representations $\{\chi_1, \cdots, \chi_d\}$ (indeed, we only have finite possible variety of irreducible representations), and the arrows n_{ij} are calculated by

$$n_{ij} = < V \otimes \chi_i, \chi_j > = \frac{1}{|G|} \Sigma_{g \in G} V(g) \chi_i(g) \overline{chi_j(g)}.$$

and Γ_G is undirected because $n_{ij} = n_{ji}$. McKay graph of some finite subgroup of $SO(3)$ corresponds Dynkin diagram of ADE type. as follows.

- $A_n = \mathbb{Z}_{n+1}$
- $D_n = D_{2(n-2)}$ *(Dihedral Group if $n \geqq 4$)*
- $E_6 = T$ *(Tetrahedral)*
- $E_7 = O$ *(Octahedral)*
- $E_8 = I$ *(Icosahedral)*

Definition 6.11.11 *(Du Val Singularity)*
Du Val singularity is a calassification of 2-dim Calabi-Yau manifold, if it has singular points, that can be classified using type ADE. The application of Du Val singularity is, in particular, type IIB string theory, generated from the compactification of D7 into D3 makes 2-dim Calabi-Yau, and the Lie group of type ADE that corresponds the singular Calabi-Yau is the actual gauge symmetry of the string theory for this, but if the Calabi-Yau is smooth the gauge group is

trivial.

- $A_n = w^2 + x^2 + y^{n+1} = 0$
- $D_n = w^2 + y(x^2 + y^{n-2}) = 0$
- $E_6 = w^2 + x^3 + y^4 = 0$
- $E_7 = w^2 + x(x^2 + y^3) = 0$
- $E_8 = w^2 + x^3 + y^5 = 0$

Definition 6.11.12 *(Symplectic Singularity)*

Definition 6.11.13 *(Blow up)*
For a scheme X, the blow up $\pi : \tilde{X} \to X$ with respect to a $\mathscr{I}$, as a coherent sheaf of ideals on X.

$\tilde{X} = Proj \oplus_{n=0}^{\infty} \mathscr{I}^n$

Example 6.12 *(Blow Up)*

- *($\mathbb{P}^2$)*
  The blowing of $\mathbb{P}^2$ at a point $P \in \mathbb{P}^2$ is $X = \{(Q,l) | Q \in \mathbb{P}l \in Gr(1,2)\} \subset \mathbb{P}^2 \times Gr(1,2)$.
  If $Q \neq P$, choosing Q can uniquely define a line that goes through both P and Q, while if $Q = P$, there could be any lines. namely, we only modify the origin.

  More concretely, for $P = [P_0 : P_1 : P_2]$, take $l = [L_0 : L_1 : L_2]$ is the set of all $[X_0 : X_1 : X_2]$ such that $X_0L_0 + X_1L_1 + X_2L_2 = 0$. Therefore, the blow up can be described as

  $$X = \{([X_0 : X_1 : X_2], [L_0 : L_1 : L_2]) | P \cdot L = 0, X \cdot L = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2.$$

- *($\mathbb{A}^2$)*

  $$X = Proj \oplus Sym^r_{k[x,y]} \mathfrak{m}/\mathfrak{m}^2 = Projk[x,y][z,w]/(xz - yw)$$

  where x and y are degree 0 and z and w are degree 1.

  Is this just replacing projective tangent bundle the origin? and otherwise all the same?

- *($\mathbb{C}^2$)*

  In general, blow up of $\mathbb{C}^n$ is $\tilde{\mathbb{C}^n} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$ with equation $x_i y_j = x_j y_i$. Blow up is a morphism $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ induced by the projection $\pi : \mathbb{C}^n \times \mathbb{P}^{n-1} \to \mathbb{C}^n$ in

- *($\mathbb{R}^2$)*
  Real blow up of $\mathbb{R}^2$ at its origin is a Möbius strip. Similarly, real blow up of a 2-sphere S^2 is a real projective plane $\mathbb{RP}^2$. Indeed, Möbius strip is $\mathbb{RP}^2$ removed the disk D^2.

- *(Δ)*
  $\Delta \subset X \times X$ is the diagonal, whose blow up is you know what.

  $Bl_\Delta X$

6.13 Yang Baxter Equation

One major purpose of Yang-Baxter equation is to describe statistical mechanics of the lattice structure of ice melting or magnetism, and YB equation calculates partition functions, that measures entropy, and another is that it's also used to describe quantum mechanics, so it's to connect between them. YB equation is mathematically, the identity of R-matrices, that's used to define an operator of the transpose $V \otimes W \to W \otimes V$, where V and W are representation spaces, hence YB generates braiding structure.

The physics interpretation of YB equation is six-vertex model. Suppose the ice structure is two dimensional, where each position is a water molecule (H2O), that's electrically unevenly charged, namely the position of hydrogen wrt oxygen, by the four directions (up/down/right/left), but as a total, it's neutrally charged. Our problem is to determine the direction of charge (up/down/right/left) of each molecule, that's YB equation, and we have six different combination of charges. Now, in physics, ice melting is entropy problem, thus here we define a partition function.

On the other hand, YB equation describes quantum mechanics if R is scattering matrix.

Definition 6.13.1 *(Partition Function)*

Of partition function, Boltzmann weight is $e^{\frac{\epsilon(a,b,c,d)}{k_B T}}$ where $\epsilon(a,b,c,d)$ is the energy of the state. A particle has four legs, and each of them has Boltzmann weights, and If few particles are connected, which is called tensor network, the weight is given by adding each weights. The partition function is simply the sum

$\Sigma e^{\frac{\epsilon(a,b,c,d)}{k_B T}}$

YB equation describes the 2d lattice structure, so supposely define a ice molecule on aligned in a row, and then determining the column. The the position of H2O molecule in the second row is somewhat determined by the first row, but not absolutely, and we have some freedom of positions. We define transfer matrix for systematically defines the entropy structure.

Definition 6.13.2 *(Transfer Matrix)*
Consider $M \times N$ 2d lattice, and if we determine the order of 1-st row, we will automatically determine the 2-nd row, that's precisely determined by transfer matrix. Transfer matrix is given by

$$T_{\sigma,\sigma'} = \Sigma \prod w(a,b,c,d).$$

and the partition function is $Z = Tr(T^N)$ where N is piling up N times.

Proposition 6.14 *(R-Matrix and T-Matrix)*
$T = \prod_i R(u_i)$

R-matrix describes how one particle relates to the neighboring molecules. Pixel by pixel, not row by row.

Definition 6.14.1 *(R-Matrix)*

$$\Sigma w(a_1,\alpha|a_2,\gamma)w(\gamma,\beta|a_3,b_3)w(\alpha,b_1|\beta,b_2)\Sigma w(a_2,\beta|a_3\alpha)w(a_1,b_1|\beta\gamma)w(\gamma,b_2|\alpha,b_3)$$

Or we get describe by R-matrices.

$$R = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & A^{-1} & 0 \\ 0 & A^{-1} & A^{-1} - A^{-3} & 0 \\ 0 & 0 & 0 & A \end{bmatrix}$$

where $A = e^{\eta}$, and from here, we induce Jones polynomial wrt A.

If we install R-matrix structure on Hopf algebra, it'll be quasi-triangular Hopf algebra. This is part of Hopf algebra.

Definition 6.14.2 *(Quasi-Triangular Hopf Algebra)*
There is an invertible element R of $H \otimes H$ such that

$R\Delta(x)R^{-1} = (T \circ \Delta)(x)$ *for all $x \in H$ where $T(x \otimes y) = y \otimes x$.*

$(\Delta \otimes 1)(R) = R_{13}R_{23}$

$(1 \otimes \Delta)(R) = R_{13}R_{12}$

where $R_{12} = \phi_{12}(R)$, $R_{23} = \phi_{23}(R)$, $R_{13} = \phi_{13}(R)$ and $\phi_{ij} : H \otimes H \to H \otimes H \otimes H$

- $\phi_{12}(a \otimes b) = a \otimes b \otimes 1$
- $\phi_{13}(a \otimes b) = a \otimes 1 \otimes b$
- $\phi_{23}(a \otimes b) = 1 \otimes a \otimes b$

Then R is a solution of Yang-Baxter equation.

Definition 6.14.3 *(Yang-Baxter Equation)*

Based on the preivous definition, YB equation is simply this equation.

$(1 \otimes \check{R})(\check{R} \otimes 1)(1 \otimes \check{R}) = (\check{R} \otimes 1)(1 \otimes \check{R})(\check{R} \otimes 1)$

Or

$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

where R-matrix is $R \in End(V \otimes V)$ and each $R_{ij} : R \otimes R \otimes R \to R \otimes R \otimes R$ acts R on each i,j components.

6.14.1 Quantum Group

Quantum group is a group but also Hopf algebra (quasi-triangular Hopf algebra). One example of quantum group is deformation of the universal enveloping algebra of semisimple Lie algebra.

Definition 6.14.4 *(Quantum Group)*

Let $A = (a_{ij})$ is a Cartan matrix, and let $q \neq 0, 1$ be a complex number. A quantum group $U_q(G)$ for Lie group G whose Cartan matrix is A, λ is an element of the weight lattice, and e_i and f_i with the following relations

- $k_0 = 1$
- $k_\lambda k_\mu = k_{\lambda+\mu}$
- $k_\lambda e_i k_\lambda^{-1} = q^{(\lambda,\alpha_i)} e_i$
- $k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda,\alpha_i)} f_i$
- $[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$ *where* $k_i = k_{\alpha_i}$, *and* $q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)}$

and the q-factorial is defined as

- $[0]_{q_i}! = 1$

- $[n]_{q_i}! = \prod_{m=1}^{n}[m]_{q_i}$ *where* $[m]_{q_i} \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$

and if we take limit $q \to 1$, *then*

$k_\lambda \to 1$ *and* $\frac{k_\lambda - k_{-\lambda}}{q - q^{-1}} \to t_\lambda$.

where $(t_\lambda, h) = \lambda(h)$ *for all* h *in CSA.*

We discuss deformation and quantization.

Example 6.15 *(Formal Deformation of Algebra)*

Deformation of k *if* $q = 0$ *is* $k[x]/(x^2)$.

In general, deformation of k *is defined over quantum plane* $k_q[x,y]/(xy = qyx)$.

Example 6.16 *(Drinfeld-Jimbo Type Quantum Group)*

$U_q(\mathfrak{g})$ *is a deformation of Lie algebra* $\mathfrak{g}$. *In other words, deformation of universal enveloping algebra* $U(\mathfrak{g})$.

Formal deformation of associative algebra (A, μ_0) *is*

$\mu_\hbar = \mu_0 + \hbar\mu_1 + \hbar^2\mu_2 + \cdots$

the operator is generated so that the associativity is preserved, and $(A[[\hbar]], \mu_\hbar)$.

Definition 6.16.1 *(Coassociative Coproduct)*

There are various way to decide coproducts, but here is the tiny example

- $\Delta_1(k_\lambda) = k_\lambda \otimes k_\lambda$
- $\Delta_1(e_i) = 1 \otimes e_i + e_i \otimes k_i$
- $\Delta_1(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1$

Counit ϵ *is*
$\epsilon(k_\lambda) = 1$ *and* $\epsilon(e_i) = \epsilon(f_i) = 0$

and the antipode S *is*

- $S_1(k_\lambda) = k_\lambda$
- $S_1(e_i) = -e_i k_i^{-1}$
- $S_1(f_i) = -k_i f_i$

6.17 Monstrous Moonshine

Monstrous Moonshine is a mathematical formulation derived from math physics using vertex algebra. Vertex algebra is a formal Laurent series where the coefficients are vectors.

Definition 6.17.1 *(Λ-bracket)*
For an affine vertex algebra $V^k(\mathfrak{g})$ of level k associated with $\mathfrak{g}$, $V^k(\mathfrak{g})$ is generated by the parity reversed vector superspace $\overline{\mathfrak{g}}$ of $\mathfrak{g}$ and the Λ-bracket is given by

$$[\overline{a}\Lambda\overline{b}] = (-1)^{p(a)}(\overline{[a,b]} + k\chi(a|b)) \text{ for } a, b \in \mathfrak{g}$$

Definition 6.17.2 *($\mathbb{C}[\nabla]$ Algebra)*
Let R be $\mathbb{C}[\nabla]$-module, where $\nabla = (\partial, D^1, D^2, \cdots, D^n)$ consists of an even operator ∂ and odd operators D^i such that

$$[D^i, D^j] = \delta_{i,j}2\partial \text{ for } i, j = 1, 2, \cdots, n$$

Note that the bracket is a supercommutator. In addition, consider a tuple $\Lambda = (\lambda, \chi^1, \chi^2, \cdots, \chi^n)$ of an even formal variable λ and odd formal variables χ^i subject to the relations

$$[\chi^i, \chi^j] = -\delta_{i,j}2\lambda$$

bracket is again supercommutator.

Now a Λ-bracket $R \otimes R \to \mathbb{C}[\Lambda] \otimes R$ is degree $\overline{n}$ map with sesquilinearity

$$[D^i a\Lambda b] = (-1)^{n+1}\chi^i[a\Lambda b]$$
$$[a\Lambda D^i b] = (-1)^{p(a)+n}(D^i + \chi^i)[a\Lambda b]$$
$$[D^i, \chi^j] = \delta_{i,j}2\lambda \ [D^i, \lambda] = 0$$

Definition 6.17.3 *(Supersymmetric(SUSY) Lie Superalgebra)*

An $N = n$ SUNY Lie conformal algebra is $[Z]/2[Z]]$-graded $\mathbb{C}[\nabla]$-module R with a Λ-bracket which is a $\mathbb{C}$-linear map of degree $\overline{n}$:

$$[\cdot\Lambda\cdot] : R \otimes R \to \mathbb{C}[\Lambda] \otimes R, \ a \otimes b \mapsto [a\Lambda b]$$

- *(Skew-Symmetry)*
  $[b\Lambda a] = (-1)^{p(a)p(b)+n+1}[a - \nabla - \Lambda b]$
- *(Jacobi Identity)*
  $[a\Lambda[b\Gamma c]] = (-1)^{(p(a)+1)n}[[a\Lambda b]\Lambda + \Gamma c] + (-1)^{(p(a)+n)(p(b)+n)}[b\Gamma[a\Lambda c]]$

for $a, b, c \in R$ and $\Gamma = (\gamma, \eta^1, \eta^2, \cdots, \eta^n)$.

Definition 6.17.4 *(Supersymmetric(SUSY) Vertex Algebra)*

An $N = n$ SUSY vertex algebra is a pair $(V, \nabla, [\cdot, \Lambda, \cdot], |0\rangle, ::)$ such that

- *($N = n$ SUSY Lie conformal algebra)*
  $(V, \nabla, [\cdot\Lambda\cdot])$ is an $N = n$ SUSY Lie conformal algebra.
- *(Unital differential superalgebra)*
  $(V, \nabla, |0\rangle, ::)$ is a unital differential superalgebra satisfying the following properties
  - *(Quasi-commutativity)*
    $: ab : -(-1)^{p(a)p(b)} : ba := \int_{-\nabla}^{0}[a\Lambda b]d\Lambda$
  - *(Quasi-associativity)*
    $:: ab : c- : a : bc ::=: (\int_0^{\nabla} d\Lambda a)[b\Lambda c] : +(-1)^{p(a)p(b)} : (\int_0^{\nabla} d\Lambda b)[a\Lambda c] :$
- *(Λ and $::$)*
  Λ and $::$ are related by
  - *(Non-commutative Wick Formula)*
    $[a\Lambda : bc :] =: [a\Lambda b]c : +(-1)^{(p(a)+n)p(b)} : b[a\Lambda c] + \int_0^{\Lambda}[[a\Lambda b]\Gamma c]d\Gamma$
    where the integral $\int_0^{\Lambda} d\Gamma = \partial_{\eta^1}\partial_{\eta^2}\cdots\partial_{\eta^n}\int_0^{\lambda} d\gamma$

Definition 6.17.5 *(Vertex Algebra/Vertex Operator Algebra)*
A vector space V, called the space of states.

- *(Identity)*
  $1 \in V$ or denoted $|0\rangle$ or $|\Omega\rangle$
- *(Translation)*
  $T : V \to V$
- *(Linear Multiplication)*
  $Y : V \otimes V \to V((z))$
  where $V((z))$ is the space of all formal Laurent series with coefficients in V.
  $\cdot_n : u \otimes v \mapsto u_n v$ where $u_n \in End(V)$
  such that $u_n v = 0$ for all $n < N$. Then,
  $u \otimes v \mapsto Y(u, z)v = \Sigma_{n\in\mathbb{Z}} u_n v z^{-n-1}$

Note that formal Laurent series is $f(z) = \Sigma_{n=N}^{\infty} a_n (z - c)^n$ for some integer $N \in \mathbb{Z}$. In general, if $N = -\infty$, $f(z)$ is Laurent series.

Axioms:

- *(Identity)*
  For any $u \in V$, $Y(1, z) = u$ and $Y(u, z) = 1 \in u + zV[[z]]$
- *(Translation)*
  $T(1) = 0$ and for any $u, v \in V$,
  $[T, Y(u, z)]v = TY(u, z)v - Y(u, z)Tv = \frac{d}{dz}Y(u, z)v$
- *(Locality (or Jacobi identity))*
  there exists a positive integer $N \in ⫙_{\geqq 0}$ $(z - x)^N Y(u, z)Y(v, x) = (z - x)^N Y(v, x)Y(u, z)$

(Vertex Operator Algebra)

Vertex operator algebra is a vertex algebra equipped with a conformal element $\omega \in V$, such that the vertex operator $Y(\omega, z)$ is the weight two Virasoro field $L(z)$

$$Y(\omega, z) = \Sigma_{n\in\mathbb{Z}}\omega_n z^{-n-1} = L(z) = \Sigma_{n\in\mathbb{Z}}L_n z^{-n-2}$$

and satisfies the following properties:

$$[L_m, L_n] = (m - n)L_{m+n} + \tfrac{1}{12}\delta_{m_n,0}(m^3 - m)cId_V$$

L_0 acts semisimply with integer eigenvalues that are bounded below.

$$L_{-1} = T$$

Definition 6.17.6 *(Operator Product Expansion)*

$$Y(A, z)Y(B, w)C = \Sigma_{n\in\mathbb{Z}}\tfrac{Y(A_{(n)}\cdot B, w)}{(z-w)^{n+1}} \cdot C$$

Example 6.18 *(Monster Vertex Algebra)*

Monster vertex algebra (also called Moonshine Module) is a vertex algebra acted on Monster group.

Definition 6.18.1 *(K3 Surface and CFT)*
symmetry is Mathieu group M24.

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