Junior M Seminar – Toric Variety (Semiformal)

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Contents

This note is a summary of my prensentation for Junior Mirror symmetry seminar in fall 2022. My assignment is to introduce Toric variety which is equivalent to Mark Gross, section 3.1, p92-98, "review of toric variety", and I will introduce the foundation of Toric variety and several of their properties.

The "Intro" section is a preliminary section and it might be skipped. In this section, we review the basic algebraic variety, and rational polyhedral cones and fan, necessary component in Tropical geometry.

1 Intro

1.1 Prelim – Algebraic Variety and Scheme

Definition 1.1.1. *(Normal, Separated, Integral, Noetherian, Finite, Universally Closed)*

1. *Normal variety is a variety where each local ring is normal. For example, PID is a normal ring, and if a ring R is normal, then $R[X]$, $R[[X]]$ are normal. If a variety X is covered by an open affine cover $\{U_i\}$, then each U_i is a normal variety.*

2. *If a variety X is separated, then the diagonal morphism $\Delta : X \to X \times X$ will be an closed immersion. Infact, iff X is separated, X satisfies an separation axiom, so it is Hausdorff. For example, $Spec\mathbb{C}[X, Y]$ is Hausdorff, so the variety is a manifold.*

3. *A scheme X is integral if for every open subset $U \subset X$, the ring $\mathscr{O}_X(U)$ is an integral domain.*

4. *A scheme X is locally noetherian if it can be covered by open affine subsets $SpecA_i$ where each A_i is a noetherian ring. A scheme X is noetherian if it is locally noetherian and quasi-compact.*

5. *A morphism $f : X \to Y$ is a finite morphism if there exists a covering of Y by open affine subsets $V_i = SpecB_i$, such that for each i, $f^{-1}(V_i)$ is affine, equal to $SpecA_i$, where A_i is a B_i-algebra which is a finitely generated B_i-module.*

6. *A morphism $f : X \to Y$ is universally closed it is closed, and for any morphism $Y' \to Y$, the correspomding morphism $f' : X' = X \times_Y Y' \to Y'$ obtained by base extension is also closed.*

Definition 1.1.2. *(Proper morphism)*
f is a proper morphism if it is universally closed, separated, and finite type.

Definition 1.1.3. *(Complete Variety)*
If fan Σ is complete, then the corresponding toric variety X_Σ is complete variety, which means that for any varieties Y, the projection morphism $X \times Y \to Y$ is a closed map.

Definition 1.1.4. *(Gluing of Scheme)*
Let X_i be a family of schemes and consider X as a glued scheme. Let $\phi_i : X_i \to X$ be an embedding for each i, and U_{ij} be an open subset of X_i for all j, and if we let an isormophism $U_{ij} \cong U_{ji}$, then the scheme X is well-defined.

Definition 1.1.5. *(Abstract Algebraic Variety)*
An abstract variety is an integral separated scheme of finite type over an algebraically closed field k. If it is proper over k, we will also say it is complete.

Example 1.1.6. *(Algebraic Variety)*

1. $\mathbb{A}^n$
2. *Torus*
3. $\mathbb{P}^n$
4. $\mathbb{P}^1 \times \mathbb{P}^1$
5. *Calabi-Yau*
6. *Weighted projective space*
7. *any affine variety, projective variety, and quasi-projective variety*

Next, we will discuss the trinity of Line bundle, Divisor, and invertible sheaves, a technical heart of algebraic geometry, and these are the language to describe blow up, which is birational.

Definition 1.1.7. *(Invertible Sheaves, Divisor, and Line Bundle)*

1. *Invertible sheaf* $\mathscr{L}$ *is a locally free sheaf of rank 1. It means that for some open neighborhood* $U \subset X$, $\mathscr{L}$ *is isomorphic to the structure sheaf* $\mathscr{L}|_U \cong \mathscr{O}_X|_U$. *In particular,* $\mathscr{L}^\vee$ *is the inverse of* $\mathscr{L}$ *which we can prove by tensor-hom adjointness, .*
2. *For a noetherian integral separated scheme which is regular in codimension one, prime divisor* Y *is a closed integral subscheme* $Y \subset X$. *Consider a family of prime divisors of the scheme* X*, generating a free abelian group denoted by* $Div(X)$. *We call an element* $D \in Div(X)$ *as a Weil divisor.*
3. *For all prime divisor* Y*, valuation* v_Y *exists. Let* $\eta \in Y$ *be the generic point, and the local ring* $\mathscr{O}_{\eta,X}$ *be a DVR. Let* $f \in K^*$ *be a non-zero rational function on* X*, then we have a principal divisor* $(f) = \Sigma v_Y(f).Y$. *Let* D *and* D' *be two Weil divisors, and if the difference* $D - D' = (f)$ *is principal, then we say* D *and* D' *are equivalent* $D \sim D'$. *The set quotient of* $Div(X)$ *by the equivalence is denoted as a class group* $Cl(X)$.
4. *Let* $\mathscr{K}$ *be a sheaf of total quotient rings of* $\mathscr{O}$. *A Cartier divisor is a global section of* $\Gamma(X, \mathscr{K}^*/\mathscr{O}^*)$.
5. *Let* D *be a Weil divisor of a normal variety* X, $O_X(D)$ *be a line bundle where for all* U $O_X(D)(U) = \{f \in \mathbb{C}(X)^* | div(f)|_U \geqq 0\} \bigcup \{0\}$. *In fact,* $\mathscr{O}_X(D)$ *is a* $\mathscr{O}_X$*-module and it is a line bundle, and it is a coherent sheaf.*
6. *For a Weil divisor* $D = \Sigma n_i[Y_i]$*, Support of* D *is* $Supp(D) = \bigcup Y_i$.

Definition 1.1.8. *(Twisting Sheaf)*
Let $\mathscr{O}_X$ be the sturcture sheaf of a projective scheme X which is given by $Proj(\bigoplus_{d \geqq 0} M_d)$, and let $\mathscr{O}_X(n)$ be a twisting sheaf $Proj(\bigoplus_{d \geqq 0} S_d)$, where $S_d = M_{d+n}$. Similarly, the analogy applies to line bundle, so $O_X(D) \times O_X(E) = O_X(D+E)$

Definition 1.1.9. *(Blow Up)*
Let X be a noetherian scheme, a coherent sheaf of ideals and let $\mathscr{P} = \bigoplus_{d \geqq 0}{}^d$ where ${}^0 = \mathscr{O}_X$. If we let $\tilde{X} = Proj(\mathscr{P})$, then $\pi : \tilde{X} \to X$ will be the blowing up of X with respect to , or blowing up of X along Y. Then:

1. *the inverse image ideal sheaf $\tilde{sheaff} = \pi^{-1}.\mathscr{O}_{\tilde{X}}$ is an invertible sheaf on $\tilde{X}$.*
2. *if Y is the closed subscheme corresponding to , and if $U = X - Y$, then $\pi : \pi^{-1}(U) \to U$ is an isomorphism.*

Definition 1.1.10. *(Ample vs Very Ample)*

An invertible sheaf $\mathscr{L}$ on a noetherian scheme X is called ample if for any coherent sheaf $\mathscr{F}$ and $n \geqq n_0$, $\mathscr{F} \otimes \mathscr{L}^n$ is generated by its global sections. Note that if the scheme X is not noetherian, the structure sheaf $\mathscr{O}_X$ and inverse sheaf $\mathscr{L}^{-1}$ do not have to be coherent.

A sheaf $\mathscr{L}$ on X is very ample relative to Y if there is an immersion $i : X \to \mathbb{P}^n_Y$ for some n such that $\mathscr{L} \cong i^*\mathscr{O}(1)$ where $\mathbb{P}^n_Y$ is a relative projective space. Particularly if $Y = Spec(A)$, then $\mathscr{L}$ admits a set of global sections s_0,...,s_n such that the corresponding morphism $X \to \mathbb{P}^n_A$ is an immersion.

Definition 1.1.11. *(Canonical Divisor/sheaf/bundle)*
Canonical sheaf is $\omega_X = \wedge^n \Omega_{X/\mathbb{C}}$. Canonical divisor is a Weil divisor K_X such that $O_X(K_X) = \omega_X$.

Example 1.1.12. *(Ampleness for Varieties)*

1. *A variety X is called a Fano variety if the anti-canonical divisor $-K_X$ is ample.*
2. *A variety X is called a Gorenstein variety if the canonical divisor K_X is ample and Cartier.*

1.2 Prelim – Combinatorics

Recall that a polynomial ring $R[X]$ is a base ring R where monoid X is applied on it. Polynomial rings represent algebraic variety, while the monoid represents the rational polyhedral cone. We study how fan, as a set of rational polyhedral cone, corresponds to some algebraic variety.

Definition 1.2.1. *Monoid (semigroup) Monoid (or semigroup) is a set where binary operation and identity is defined, but an inverse operation is not necessarily defined (it suffice to say that group is also a monoid). For example, $\mathbb{N}$ is a monoid because for all elements $a, b \in \mathbb{N}$, the binary operation $a + b \in \mathbb{N}$ is included, and identity exists as $0 \in \mathbb{N}$*

Definition 1.2.2. *monoidal polynomial*
Let $\Bbbk$ be a field. We define nunmber $1 \in \mathbb{N}$ corresponds to undetermined object X, and any natural number n to X^n. Then, $\Bbbk[\mathbb{N}]$ contains all X^n of all natural power, thus we can say $\Bbbk[\mathbb{N}] = \Bbbk[\mathbb{N}X] = \Bbbk[X]$.

If the monoid becomes multi-dimensional, for example, $\mathbb{N}^2$, then we naturally correspond $(n, 0) \in \mathbb{N}^2$ to X^n, and similarly $(0, n) \in \mathbb{N}^2$ to Y^n for all $n \in \mathbb{N}$, thus we define $\Bbbk[\mathbb{N}^2] = \Bbbk[\mathbb{N}X, \mathbb{N}Y] = \Bbbk[X, Y]$.

Note 1.2.3. *More generally, consider that $\mathbb{Z}$ is a monoid, let's correspond $a \in \mathbb{Z}$ to X^a, so that we can define the negative degree, then thus we define $\Bbbk[\mathbb{Z}] = \Bbbk[X, X^{-1}]$. This is algebraic torus.*

Note that algebraic torus is an affine variety because $\Bbbk[X, X^{-1}] \cong \frac{\Bbbk[X,Y]}{(XY-1)}$ the coordinate ring can be naturally defined by the quotient of polynomial ideal $(XY - 1)$, so it is an affine variety.

Definition 1.2.4. *(Lattice and Vector space)*
We let $M = \mathbb{Z}^n$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N = Hom(M, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 1.2.5. *(Rational polyhedral cone)*
A rational polyhedral cone σ is a subset of $M_{\mathbb{R}}$, and the dual cone $\sigma^{\vee}$ is a subset of $N_{\mathbb{R}}$ such that $\sigma^{\vee} = \{m \in M_{\mathbb{R}} | < m, u >\geqslant 0 \forall u \in \sigma\}$.

Definition 1.2.6. *(Polytope)*
A polytope P is a convex hull of a finite subset of the Euclidean space. Particularly if each vertex of the polytope is lattice point (integer point), then we call it a lattice polytope. A polytope P is called normal if for some lattice M, $(kP) \cap M + (lP) \cap M = ((k + l)P) \cap M$ for all $k, l \in \mathbb{N}$.

Definition 1.2.7. *(Fan)*

1. *A fan is a set of rational polyhedral cones such that $\Sigma = \{\sigma\}$ and for all $\tau \subset \sigma$, τ is a face of σ, and $\sigma_1 \cap \sigma_2$ is a face of $sigma_1$ and $sigma_2$.*

2. *Support of a fan $|\Sigma|$ is $|\Sigma| = \bigcup \sigma$. We call it complete fan if the support of the fan is the entire of the space $M_{\mathbb{R}}$.*

3. *In particular, if all the rational polyhedral cones of the fan is strictly convex, we call it a normal fan.*

Proposition 1.2.8. *(Simplex Generating Fan)*
A polytope generates a fan (why?). In particular, let a full dimensional polytope $\Delta = \{(n_i)|\Sigma n_i \leqq 1, n_i \geqq 0\}$ be called n-simplex, and it generates a projective toric variety.

Definition 1.2.9. *(Star Subdivision of Fan)*
Let $\sigma = Cone(u_1, ..., u_n)$, and $u_0 = \Sigma_{i=0}^{n} u_i$. Let $\Sigma'(\sigma)$ be a set of cones that are generated by $\{u_0, ..., u_n\}$ not containing $\{u_1, ..., u_n\}$. Then star subdivision of a fan is $\Sigma^*(\sigma) = (\Sigma \setminus \{\sigma\}) \cup \Sigma'(\sigma)$

Definition 1.2.10. *(Smooth toric variety and Simplicial toric variety)*
A toric variety is smooth (resp. simplicial) if every cone is generated by a subset of an $\mathbb{Z}$-basis (resp. $\mathbb{R}$-basis) of N (resp. $N \bigotimes_Z \mathbb{R}$).

Definition 1.2.11. *Piecewise Linear function*
PL function is $\phi : |\Sigma| \to \mathbb{R}$ is a function s.t. each restriction to a cone is linear.

Definition 1.2.12. *Cell $\mathscr{P}$*
We call cells $\mathscr{P}$ as a set of all subsets of $\sigma \in \Delta$ that satisfies the following axioms:

1. $\Delta = \bigcup_{\sigma \in \mathscr{P}} \sigma$
2. *for $\sigma \in \mathscr{P}$ and its face$\tau \subset \sigma$, $\tau \in \mathscr{P}$*
3. *$\sigma_1, sigma_2 \in \mathscr{P}$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.*

Note that face is not empty set. So $\sigma_1 \cap \sigma_2$ must not be empty (but could be still intersecting at the origin).

In addition, $\mathscr{P}_{max}$ is a set of maximal cells as a subset of $\mathscr{P}$. $\mathscr{P}^{[k]}$ is a set of k-dimensional cells of $\mathscr{P}$.

Definition 1.2.13. *very ample rational polyhedral cone and saturated*

1. *Monoid is saturated if $\forall k \in \mathbb{N}\{0\}$ and $m \in M$, $km \in S$ implies $m \in S$.*
2. *Let $P \subset N_{\mathbb{R}}$ be a lattice polytope. We call P is very ample if $\forall m \in P$, the semigroup $S_{P,m} = \mathbb{N}(P \cap M - m)$ is saturated in M.*

Theorem 1.2.14. *Gordan's lemma*
$\sigma^\vee \cap N$ is finitely generated.

According to Gordan's lemma, the toric variety $X_\Sigma = Spec \Bbbk[\sigma^\vee \cap N]$ is geometric because the coordinate ring is finitely generated (noetherian).

2 Toric variety

There are many examples of Toric varieties. For example, it could be affine Toric variety and projective toric variety. We will discuss affine toric varieties are, in particular, Torus, affine space that are directly defined by the rational polyhedral cone. Another example of Toric variety is a projective space $\mathbb{P}^2$, and its mirror symmetry is the Landau-Ginzburg model. These varities are studied by toric fan, and the complexity of the problem will be reduced to the combinatorial problems.

In this section, we will define the normal Toric variety and its correspondence with a normal fan. In general, all toric fan has a corresponding Toric variety, but not all of algebraic variety is toric variety, but if majority of the varieties we are interested are Toric variety, we have no worry about it. By the technique of Toric variety, we will study blow up, resolution of singularities, Divisor etc.

Finally, note that from this section, we assume Toric variety is always normal because if not, Torus orbit correspondence theorem might be complicated, even though some of us might have an interest of non-normal variety perspective.

2.1 Affine Toric Variety

Affine toric variety is defined over a rational polyhedral cone. $X_\sigma = Spec\Bbbk[\sigma^\vee \cup N]$ Recall that if a monoid is added as an undetermied value of k, then it will correspond to polynomial as $\Bbbk[\mathbb{N}] = \Bbbk[x]$. An affine toric variety of a fan is a collection of toric varieties of cones that are glued together.

In general, toric variety of fan is seperate, but properness depends. Toric variety of fan is also proper of over $Spec\Bbbk$ iff Σ is a complete fan.

Example 2.1.1. *X_0 is an algebraic torus where $0 \in \Sigma$ is a 0-dimensional cone, which always exists for non-empty fan Σ. Then the dual cone $0^\vee$ is entire of N, so $Spek\Bbbk[0^\vee \cap N] = Spek\Bbbk[N]$, where N is a lattice, and the negative degree is accepted.*

Example 2.1.2. *Note that not all toric varieties are affine variety, and it is vice versa. For example, projective variety $\mathbb{P}^n$ is a toric variety but it is not affine. Also, $\mathbb{V}(X^3 - Y^2 - 1)$ is an affine variety but it is not toric.*

2.2 Construction of Toric Variety

Definition 2.2.1. *(Character/Distinguished Points)*

1. *Character formula of torus T is defined by $\chi^m : {\mathbb{C}^*}^n \to \mathbb{C}^*$. $(t_1, ..., t_n) \to ({t_1}^{a_1}, ..., {t_n}^{a_n})$*
2. *One-parameter subgroup of Torus $\lambda : \mathbb{C}^* \to T$ is defined by $t \to (t^{b_1}, ..., t^{b_n})$ where $(m_1, ..., m_s) \subset M$.*

3. *Let $S_\sigma = \sigma^\vee \cap N$. For each cone σ, there exists $m, m' \in S_\sigma$ such that $m + m' \in S_\sigma \cap \sigma^\perp$, then $m, m' \in S_\sigma \cap \sigma^\perp$. We call such m or m' as a distinguished point denoted by γ_σ.*

Definition 2.2.2. *(Toric Variety)*
Toric Variety is an algebraic variety X_Σ that contains $T_N \subset X$ as open subvariety with a group action $T_N \times X_\Sigma \to X_\Sigma$. Here, T_N is a torus defined by a lattice N, X_Σ accompanies Σ as a fan. Note that in general, we do not assume that Toric variety is normal, or its torus T_N is not always Zariski dense, but hereafter we will assume so.

Note that from this definition, we will naturally have the orbits and the stablizers by the torus action. An orbit of the Toric variety is $O(\sigma) = T_N.\gamma_\sigma \subset X_\Sigma$ defined for each σ. Also, since the torus action is surjective, the union of all the toric orbit $\coprod O(\sigma) = X_\Sigma$ makes X_Σ itself.

Proposition 2.2.3. *(Orbit Cone Correspondence)*

1. *There is a bijective correspondence $\{\sigma \in \Sigma\}$ and $\{T_N\text{-orbits } O(\sigma) in X_\Sigma\}$*
2. *Let $n = dim(N_\mathbb{R})$. For each cone $\sigma \in \Sigma$, $dim(O(\sigma)) = n - dim\sigma$.*
3. *The affine open subset U_σ is a union of orbits $U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$.*
4. *$\tau \preceq \sigma$ iff $O(\sigma) \subset \overline{O(\tau)}$, and $\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$*

Note 2.2.4.
This theorem says that the toric variety is purely constructible from a fan, or polytopes that generate it.

Example 2.2.5. *(Formal Construction of Affine/Proj Toric variety)*
Affine toric variety $Y_\mathscr{A}$ is defined by Zariski closure of the image of $\Phi_\mathscr{A}$, namely, $Y_\mathscr{A} = \overline{Im(\Phi_\mathscr{A})}$.

Let also $\pi : {\mathbb{C}^*}^s \to T_{\mathbb{P}^{s-1}} \subset \mathbb{P}^{s-1}$
Projective toric variety $Y_\mathscr{A}$ is defined by Zariski closure of the image of $\pi \circ \Phi_\mathscr{A}$, namely, $Y_\mathscr{A} = \overline{Im(\pi \circ \Phi_\mathscr{A})}$.

Example 2.2.6. *(Projective Space)*
'Y' shaped two dimensional complete fan defines a projective space $\mathbb{P}^2$. The fan $\Sigma = \{0, \rho_{12}, \rho_{23}, \rho_{13}, \sigma_1, \sigma_2, \sigma_3\}$ contains three of 2-full dimensional rational polyhedral cones, and this makes affine varieties $Spec(k[\sigma_i^\vee \cap N])$ glued together.

For each i, let $U_{\sigma_1} = Spec(k[X, Y])$ $U_{\sigma_2} = Spec(k[XY^{-1}, Y^{-1}])$ $U_{\sigma_3} = Spec(k[X^{-1}Y, X^{-1}])$.

Consider an open embedding $U_{\sigma_i} \to \mathbb{P}^2$ such that $(X, Y) \mapsto (X, Y, 1)$ $(X, Y) \mapsto (X, 1, Y)$ $(X, Y) \mapsto (1, X, Y)$. Then, $\mathbb{P}^2$ is covered.

Another important and relatively simple Toric surface might be a Hirzebruch surface. Let me introduce three different ways of constructing Hirzebruch surface.

Definition 2.2.7. *Hirzebruch surface*
Hirzebruch surface is a $\mathbb{P}^1$ bundle called projective bundle, associated to a sheaf $\mathscr{O} \bigoplus \mathscr{O}(-n)$, where $\mathscr{O}$ is a sheaf of projective scheme whose global section is a graded ring$S = \bigoplus S_d$, and $\mathscr{O}(-n)$ is a twisted sheaf where coordinate ring M is again a graded ring such that $M_d = S_{d+n}$.

Example 2.2.8. *Toric Easy Hirzebruch Surface Σ_0 (Simply quadrant fan)*
The Cross shaped two dimensional complete fan (four cones exist on $\mathbb{R}^2$) defines a Hirzebruch surface $\mathscr{H}_a$.
Particulary, for $a = 0$, the fan $\Sigma = \Sigma' \times \Sigma'$ is given by the product of sub fans, and $X_{\Sigma'} = X_\Sigma \times X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ is given by such product.

Example 2.2.9. *Toric definition of Hirzebruch surface Σ_r $r > 0$*
Hirzebruch surface is a ruled surface . Hirzebruch surface often appears in the compactification of two dimensional vector bundle in algebraic geometry.

Another view point of Hirzebruch surface is given by GIT quotient.

Definition 2.2.10. *GIT definition of Hirzebruch surface*
GIT quotient $(\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\})/(\mathbb{C}^* \times \mathbb{C}^*)$ where the action $\mathbb{C}^* \times \mathbb{C}^*$ is given by $(\lambda, \mu).(l_0, l_1, t_0, t_1) = (\lambda l_0, \lambda l_1, \mu t_0, \lambda^{-n}\mu t_1)$

Definition 2.2.11. *(Toric Morphism)*
Let X_{Σ_1} and X_{Σ_2} be toric varieties, and consider a morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$. If this morphism ϕ satisfies the following commutative diagram, we call it is a toric morphism.

$$\begin{tikzcd}
T_{N_1} \times X_{\Sigma_1} \arrow[r] \arrow[d, "\phi|_{T_{N_1}} \times \phi"'] & X_{\Sigma_1} \arrow[d, "\phi"] \\
T_{N_2} \times X_{\Sigma_2} \arrow[r] & X_{\Sigma_2}
\end{tikzcd}$$

2.3 Polyhedra, Asymptote, and Degeneration

Definition 2.3.1. *Toric variety from polyhedra*
Let $\Delta \subset N_{\mathbb{R}}$ be a lattice polyhedron with at least one vertex (non-trivial). Lattice polyhedron is a polyhedoron generated by the finite intersections of some rational half-planes. We will definite a cone from the lattice polyhedron to generated a

toric variety.

Let $C(\Delta)$ be a cone generated by Δ denoted by $C(\Delta) = \overline{\{(nr, r)|n \in \Delta, r \geqq 0\}}$

Then, the vertical asymptote is defined by taking Hausdorff limit of $C(\Delta)$ as $r \to 0$. namely, divide $C(\Delta)$ by the level of height as $C(\Delta) = \overline{\bigcup(C(\Delta)_r, r)}$ where $C(\Delta)_r = \{nr|n \in \Delta\}$, and

Now we can define Asympotote of the cone using Hausdorff limit. Hausdorff limit is a set of limit of any function $Asym(C(\Delta)) = \{lim_{r\to 0} f(r)$ where f is $f(r) \in C(\Delta)_r$ for all $r \geqq 0$.

Now since $C(\Delta) \cap (N \cap \mathbb{Z})$ is naturally graded and since $\Bbbk[C(\Delta) \cap (N \cap \mathbb{Z})]$ is finitely generated, we can define the projective variety, which is $\mathbb{P}_\Delta = Proj\Bbbk[C(\Delta) \cap (N \cap \mathbb{Z})]$, which is projective over $Spec\Bbbk[Asym(\Delta) \cap N]$. $C(\Delta) \cap (N \oplus \mathbb{Z})$ is again, finitely generated by Gordan's lemma.

Notice that $\mathbb{P}_\Delta$ is also a toric variety.

Notice that, in fact, $Proj\Bbbk[C(\Delta) \cap (N \oplus \mathbb{Z})]$ is a toric variety, and moreover it is isomorphic to $X_{\check{\Sigma}_\Delta}$ for some normal fan $\check{\Sigma}_\Delta$. For example, n-dimensional tetrahedron generates a projective space $\mathbb{P}_\Delta = \mathbb{P}^n$

For any faces $\sigma \subset \Delta$, $\mathbb{P}_\sigma$ is a subset of $\mathbb{P}_\Delta$, corresponding to the toric strata $\mathbb{P}_\sigma \cong D_{\check{\sigma}}$. By induction, any $\mathbb{P}_\Delta$ is isomorphic to a toric variety.

Example 2.3.2. *Product of cone, Union of cone*
Let Δ_1, Δ_2 be simplexes, which is a polyhedron, then the corresponding toric variety is $\mathbb{P}_{\Delta_i} = \mathbb{P}^{n_i}$. Now the product $\Delta_1 \times \Delta_2$ corresponds to $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$.

More generally, $\Delta \times \mathbb{R}$ is a product of polyhedra, and $\mathbb{P}_{\tilde{\Delta}\times\mathbb{R}} = \mathbb{P}_\Delta \times Spec\Bbbk[\mathbb{Z}]$.

Also, realize $\mathbb{P}_\Delta = \bigcup \mathbb{P}_\sigma$ where $\sigma \in \Delta$ is a face.

Example 2.3.3. *Mumford Degeneration*
We continue considering the projective toric variety generated by some polyhedra. Let Δ be a polyhedron, and let $\tilde{\Delta}$ be $\tilde{\Delta} = \{(n, r) \in N_\mathbb{R} \oplus \mathbb{R}|n \in \Delta, r \geqq \phi(n)\}$ $0 \times \mathbb{R}_{\geqq 0} \subset N_\mathbb{R} \oplus \mathbb{R}$

$\Bbbk[C(\tilde{\Delta}) \cap (N \oplus \mathbb{Z} \oplus \mathbb{Z})]$ is a $\Bbbk[N]$-algebra.

Consider that $\mathbb{P}_{\tilde{\Delta}}$. Consider that $\tilde{\Delta}$ is a union of vertical faces and horizontal faces so $\tilde{\Delta} = V \cup H$. The vertical faces of the polyhedron will diminish to a point by some projection $p : N_\mathbb{R} \oplus \mathbb{R} \to N_\mathbb{R}$, while the horizontal faces will be projected homeomorphically to elements of the cell $\mathscr{P}$. Namely, the normal cones of vertical faces are contained in $M_\mathbb{R} \times \{0\}$, and in fact they correspond to the cones in Δ.

Mumford degeneration is a morphism $\pi : \mathbb{P}_{\tilde{\Delta}} \to \mathbb{A}^1_k$ which is a regular function, and it is given by a monomial z^ρ where $\rho = (0, 1) \in N_\mathbb{R} \oplus \mathbb{R}$. The primitive generators of the rays are $(m, 0)$ and $(m, 1)$ where $m \in M$, and the function π only vanishes $(m, 1)$ type. Hence $\pi^{-1}(0)$ is isomorphic to toric divisors of $\mathbb{P}_{\tilde{\Delta}}$ corresponding to the codimension one horizontal faces.

$\pi^{-1}(0) = \bigcup_{\sigma\in\mathscr{P}_{max}} \mathbb{P}_\sigma$

$\mathbb{P}_{\tilde{\Delta}} \setminus \pi^{-1}(0) \cong \mathbb{P}_{\Delta} \times_{\Bbbk} Spec\Bbbk[\mathbb{Z}] = \mathbb{P}_{\Delta} \times \mathbb{G}_M$. This means that for $t \neq 0$, $\pi^{-1}(t) \cong \mathbb{P}_{\Delta}$

2.4 Blowup and Divisor

Consider a morphism of fans $\Sigma_1 \to \Sigma_2$, then it induces a morphism of affine toric varieties $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$. If ϕ is an isomorphism and if Σ_1 is a refinement of Σ_2, then ϕ is a birational morphism. Especailly for a natural refinement $\Sigma \to \sigma$, $X_\Sigma \to X_\sigma$ is birational morphism. This is a blowup. If we apply blowup, we must increase the dimension of geometry, but since instead we can add divisors on it, the geometry might contains more structures, and it helps us compute.

Definition 2.4.1. *(Toric Strata)*
We denote toric starata as D_τ such that $D_\tau = \overline{X_\tau \setminus \bigcup_{\omega \subsetneq \tau} X_\omega}$.

Definition 2.4.2. *(Projective Space)*
Blow up of a projective variety is a projective variety.

Example 2.4.3. 1-*dimensional case*
An specific but important example is one dimensional case. Let ρ be a ray, namely a 1-dimensional cone contained in the fan $\rho \in \Sigma^{[1]}$, then D_ρ is a prime divisor because it is a prime subscheme of codimension 1, and considering that $DivX = \Sigma n_i D_{\rho_i}$ where $\rho_i \in \Sigma^{[1]}$ is any element of 1-cone in the fan Σ, Weil divisor is an element of $DivX$.

Example 2.4.4. 0-*dimensional case*
if the given fan is trivial as $\Sigma = \{0\}$, then The toric strata is $X_0 = Spec\Bbbk[0^\vee \cap N] = \Bbbk[N]$, and this is torus because N is a lattice. Indeed, toric variety must contain a torus structure.

Definition 2.4.5. *transversality*
A curve C in X_Σ is torically transverse if C is disjoint from all the toric strata of X_Σ of codimension > 1.

Example 2.4.6. *transversality*
We care toric strata because, in Gross ch4, we define transversality.

2.5 Classification

Weighted projective space is complete but not smooth.
definition of toric morphism

Theorem 2.5.1. *resolution of singularities(blow up of toric variety)*
We will construct a log morphism on toric varieties. Recall blow up of variety X at point $(0,0)$ makes an isomorphism $\tilde{X} \setminus D_\rho \cong X \setminus (0,0)$, especially a toric variety is a blow up of some affine toric variety as $\pi : \tilde{X} = X_\Sigma \to X_\sigma = X$. Let $\tilde{D} = \bigcup_{\rho \in \Sigma[1]} D_\rho$ be the union of all divisors. Consider the sheaf of monoid $M_{\tilde{X},\tilde{D}}$, $\tilde{X}$ will also be a log scheme, and π will naturally be a morphism of log scheme.

Definition 2.5.2. *(Toric Variety)*
X_Σ is a toric variety if it is an algebraic variety and it contains an algebraic torus as open dense subvariety. In fact, X_Σ is given by a fan.

Definition 2.5.3. *(Refinement and Blowup)*
Let Σ be a toric fan and Σ' be a refinement of Σ. Then, $X'_\Sigma \to X_\Sigma$ is a blow up.

The fan of Hirzebruch H_0 is $\Sigma \times \Sigma$, so it is $H_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.5.4. *(Projective Space from a fan)*
The complete fan of "ts" shape corresponds to a projective variety. Consier the open embedding of the three two-dimensional cones into $\mathbb{P}^2$, then it has a homogeneous coordinate. So indeed it is projective space. is a projective space.

Proposition 2.5.5. *(Complete)*

1. *X_Σ is proper over Spec$\mathbb{k}$ if and only if Σ is a complete fan. Moreover, refinement of the fan makes proper toric morphism, hence blow up is proper morphism.*

2. *X is a complete variety iff in compact in classical topology iff its fan is complete.*

Proposition 2.5.6. *(Toric Chow lemma)*
For a complete toric fan Σ, there is a refinement Σ' that the toric variety X'_Σ is a projective variety.

Remark 2.5.7.
All the 2-dimensional complete toric surfaces are projective variety. However, not all complete toric varieties are not projective. For example, a fan that is generated by $(1,0,1),(0,1,1),(-1,-1,1),(1,0,-1),(0,1,-1),(-1,-1,-1)$ has a six minimal cones $Cone(u_1,u_2,u_3)$, $Cone(u_1,u_2,u_4)$, $Cone(u_2,u_4,u_5)$, $Cone(u_1,u_3,u_4,u_6)$, $Cone(u_2,u_3,u_5,u_6)$, $Cone(u_4,u_5,u_6)$. This fan is a complete fan, but it does not make a projective variety.

Example 2.5.8.

1. *$\mathbb{P}(1,1,2)$ is compact but not smooth*

2. *Hirzebruch surface is compact smooth*

Definition 2.5.9. *(Basepoint Free)*
Let X be a sheme, and $\mathscr{L}$ be a line bundle over X. A subspace $W \subset \Gamma(X,\mathscr{L})$ is basepoint free if for all $p \in X$, there is $s \in W$ such that $s(p) \neq 0$.

Proposition 2.5.10. *(Basepoint Free)*
For a Cartier divisor D, the followings are TEAE:

1. *$\mathscr{O}_X(D)$ is generated by global sections.*

2. *D is basepoint free, meaning that $\Gamma(X, \mathscr{L})$ is basepoint free.*

3. *for every $p \in X$, there is $s \in \Gamma(X, \mathscr{O}_X(D))$ with $p \notin Supp(div_0(s))$ where $div_0(s)$ is a divisor of zero.*

Definition 2.5.11. *(Gorenstein Fano Variety)*
For a complete normal variety X, if the canonical divisor K_X is ample and the anticanonical divisor $-K_X$ is ample and Cartier, then X is Gorenstein Fano variety.

In fact, Gorenstein Fano variety is projective variety.

Definition 2.5.12. *(Resolution of Singularties)*
We call a morphism $\phi : Y \to X$ be a resolution of singularities of X if Y is a smooth surface and ϕ induces an isomorphism of surfaces $Y \setminus \phi^{-1}(X_{sing}) \cong X \setminus X_{sing}$

Definition 2.5.13. *(Calabi-Yau variety)*
Smooth projective variety Y is Calabi-Yau if the canonical sheaf ω_Y is trivial, and the homology of degree $1 \leqq i \leqq n-1$ is trivial.

3 future

1. convexity and ampleness of divisor.
2. ghost sheaf (log scheme) construction and application
3. toric mori program.
4. why we care projective toric vareity
5. is blowing up to projective space is itself?
6. ex: non-smooth toric variety
7. relating Landau-Ginzburg model and Calabi Yau manifold
8. kähler manifold.
9. Hodge decomposition
10. intro to intersection theory
11. blow up and projective toric variety.
12. resolution of singularities and Goreinstein Fano
13. torus orbit correspondence
14. log 1-form from toric perspective.
15. kahler differential from toric perpsective

References

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[3] Robin Hartshorne (1977) *Algebraic Geometry.*