Brown Representability Theorem and Its Application

Koji Sunami

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1 Abstract

Our main purpose in this paper is to introduce the proof of the Brown representability theorem (hereafter called BRT), by following the work of Amnon Neeman. BRT is a criteria that judges the computability of compactly generated triangulated category, or in other words, the existence of homological functor. "Representation" is a branch of mathematics that transform the given algebraic structure to some linear objects or abelian groups, that computer can recognize.

I consider it is worth paying attention to this theorem, because first of all, one of our biggest concern in category theory is homomorphism, because it is the mininum unit of understanding. For example, Yoneda's lemma, Adjunction, and homological algebra are theory that explain categorical phenomena by homomorphism, and BRT is one of them. Second of all, it is because there are many examples of the compactly generated triangulated category, and we frequently encounter to this theorem. For example, Amnon Neeman claims derived category of bounded–below chain complexes of quasi-coherent sheaves is the compactly generated triangulated category, and it is used to prove Grothendieck duality theorem, which is an important result in algebraic geometry[1]. Other important examples of the compactly generated triangulated category are stable module category and stable homotopy category, that I will explain quite briefly. Another interesting feature of BRT is that it is written purely by category theory, and it improved the traditional proof of Grothendieck duality theorem, that was computational and experimental.

I put my best effort to make the whole argument self-contained. Ch2 and Ch3 are all about the foundation of category theory, although it can be skipped. Ch4 introduces the notion of triangulated categories and some of its examples. Ch5 is about the technical heart to prove the Brown representability theorem.

2 Preliminaries

2.1 Definition of category

We will define a category is a collection of a set of objects in addition to a set of morphisms, where we define the notions of an "object" and an "morphism" as á priori defined structures of the category. In other words, the objects don't have to be sets, or the morphisms don't have to be maps, either. On the other hand, it also means we are no longer required to be aware what each object or morphism signifies. Perhaps, the philosophy of category might be similar to that of objective programming in computer science, in that the category theory encapsulates the concrete information of abstract algebra, and it succeeds in restricting the domain of the algebraic argument, and as a result, we uncovered that the algebraic objects are analyzed by homological algebra, whose operation is purely categorical.

Definition (Category)**.** *A category $\boldsymbol{C}$ has*
a class whose elements are the objects in $\boldsymbol{C}$.
a class whose elements are the morphisms or arrows of $\boldsymbol{C}$.
class functions that assign to every morphism of $\boldsymbol{C}$ a domain and a codomain, which are objects of $\boldsymbol{C}$.

Functor is another fundamental principle that describes the relationship between categories.

Definition (Functor)**.** *A functor or covariant functor F from a category $\boldsymbol{C}$ to a category $\boldsymbol{D}$ assigns to each object $A \in \boldsymbol{C}$ to $F(A) \in \boldsymbol{D}$, and assigns each morphism $\alpha : A \to B$ for $A, B \in \boldsymbol{C}$ to $F(\alpha) : F(A) \to F(B)$ for $F(A), F(B) \in (D)$, so that*
(1) if $\alpha : A \to B$, then $F(\alpha) : F(A) \to F(B)$;
(2) $F(1_A) = 1_{F(A)}$ for every object A of $\boldsymbol{A}$;
(3) $F(\alpha\beta) = F(\alpha)F(\beta)$ whenever $\alpha\beta$ is defined.

Note that functors can be classied to two types. The other type of functor is called contravariant functor satisfying the following conditions instead of (3);
(3') $F(\alpha\beta) = F(\beta)F(\alpha)$ whenever $\alpha\beta$ is defined.

2.2 Basic properties of category

Perhaps, the most frequent vocabularies in category theory may be the product and the coproduct, that we can start from. In fact, broadly speaking, BRT is a claim about some relationship between the homomorphism and the coproduct. The most fundamental proposition may be the universal property.

Proposition (Universal property)**.** *Let $\boldsymbol{C}$ be a category, and $X, Y \in \boldsymbol{C}$ are the objects, where the morphism $X \to Y$ is defined. Then, there exists a unique morphism $X \prod Y \to Y$ that the diagram commutes (left).*

The dual concept is coproduct. Let the morphism be $X \to Y$. Then, there exists a unique morphism $Y \to X \coprod Y$ that the diagram commutes (right).

$$
\begin{tikzcd}
X & X \prod Y \arrow[l] \\
& Y \arrow[ul] \arrow[u, dashed, "\exists!"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[dr] & X \coprod Y \arrow[d, dashed, "\exists!"] \\
& Y
\end{tikzcd}
$$

These notions are called as universal property that is universally applied for all products and coproducts.

Proposition (Natural transformation)**.** *Let $f : X \to Y$ is a morphism inside of category $\boldsymbol{C}$, and let $F, G : \boldsymbol{C} \to \boldsymbol{D}$ are functors. We define natural transformation $\alpha : F \to G$ is a morphism from a functor to a functor where the following diagram commutes:*

$$
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"] & F(Y) \arrow[d, "\alpha_Y"] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd}
$$

where α_X is a morphism inside of category $\boldsymbol{D}$ for all $X \in \boldsymbol{C}$.

Proposition (Limit and colimit)**.** *Let a functor be $F : \boldsymbol{J} \to \boldsymbol{C}$ where $\boldsymbol{J}$ is an indexing category of $\boldsymbol{C}$ whose objects are cones of $X \in \boldsymbol{C}$, and $\boldsymbol{C}$ is an ordinary category. Then, let a tuple (L, ϕ) where $L \in \boldsymbol{C}$ and ϕ_Y is a morphism for all $Y \in \boldsymbol{C}$. This tuple is called as a cone of $\boldsymbol{J}$ if for all $X \in \boldsymbol{J}$ and for all $Y \in \boldsymbol{C}$ and $f : X \to Y$, a morphism ϕ_X and ϕ_Y are defined as $\phi_Y = F(f) \circ \phi_X$. Next, (L, ϕ) is called as a limit of diagram if for all the other cones (N, ψ), there is a unique map $u : N \to L$ that makes the following diagram commutes.*

Similarly, the colimit is the dual notion of the limit. As the right side of the diagram shows, let a functor be $F : \boldsymbol{J} \to \boldsymbol{C}$ where $\boldsymbol{J}$ is an indexing category of $\boldsymbol{C}$ whose objects are cocones of $X \in \boldsymbol{C}$, where $\boldsymbol{C}$ is an ordinary category. Then, let a tuple (L, ϕ) where $L \in \boldsymbol{C}$ and ϕ_Y is a morphism for all $Y \in \boldsymbol{C}$. This tuple is called as a cocone of $\boldsymbol{J}$ if for all $X \in \boldsymbol{J}$ and for all $Y \in \boldsymbol{C}$ and $f : X \to Y$, a morphism ϕ_X and ϕ_Y are defined as $\phi_Y = F(f) \circ \phi_X$. Next, (L, ϕ) is called as a limit of diagram if for all the other cocones (N, ψ), there is a unique map $u : L \to N$ that makes the following diagram commutes.

$$\begin{tikzcd}
& N \arrow[d, dashed, "\exists! u"] \arrow[ddl, "\psi_X"'] \arrow[ddr, "\psi_Y"] & \\
& L \arrow[dl, "\phi_X"] \arrow[dr, "\phi_Y"'] & \\
F(X) \arrow[rr, "F(f)"] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[rr, "F(f)"] \arrow[dr, "\phi_X"] \arrow[ddr, "\psi_X"'] & & Y \arrow[dl, "\phi_Y"'] \arrow[ddl, "\psi_Y"] \\
& L \arrow[d, dashed, "\exists! u"] & \\
& N &
\end{tikzcd}$$

Note that the cone is not unique. For example, some other cones are equivalent up to isomorphisms.

Proposition (Pushback and pushout)**.** *Although pushback is not a trivial statement, I will only introduce pushout, since pushback may not be appearing in the succeeding argument. In pushout, let $\boldsymbol{C}$ be a category and $X, Y, Z \in \boldsymbol{C}$ be the objects, and let morphisms $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$. Notice that some cones can be defined by colimit. We particularly call cones P as a pushout such that for all N the diagram below commutes, there are unique morphisms $P \to N$.*

$$\begin{tikzcd}
Z \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, dashed] \\
Y \arrow[r, dashed] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, "\iota_1"] \arrow[ddr] & \\
Y \arrow[r, "\iota_2"] \arrow[drr] & P \arrow[dr, dashed, "\exists!"] & \\
& & N
\end{tikzcd}$$

Pushback and pushout are similar notions with limit and colimit. Cone and cocone of $X \in \mathbf{C}$ are needed to be defined by all $Y \in \mathbf{C}$ and morphism $f : X \to Y$, while we don't need to assume the existence of Z. Conversely, in pushback and pushout, we have fixed Y.

Proposition (Monomorphism and epimorphism)**.** *A morphism μ is monomorphism if the equality of the two compositions $\mu\alpha = \mu\beta$ implies $\alpha = \beta$. Similarly, a morphism σ is epimorphism if $\alpha\sigma = \beta\sigma$ implies $\alpha = \beta$. If $\alpha\beta = 1_B$ and $\beta\alpha = 1_A$, where $\alpha : A \to B$ and $\beta : B \to A$, then α and β are both isomorphism.*

Proposition (Homological functor)**.** *Next, We call a covariant functor F is* *homological, if it induces the exact sequence from the given sequence. Then, it is cohomological if contravariant. For example, Hom functor is trivially homological, but the converse is not always true.*

Proposition (Representability)**.** *The Representability is a criteria of describing a category by Hom functor. More precisely, a functor $F : \boldsymbol{C} \to Sets$ is called representable if*

$$F \cong Hom_{\boldsymbol{C}}(-, X)$$

for some fixed element $X \in \boldsymbol{D}$.

3 Equivalence of categories

In this section, we will discuss the equivalence of categories and its applications. Its idea is purely a categorical concept, and it will be used to explain Yoneda's lemma and Adjunction, that are part of the proof in the Brown representability theorem.

3.1 Isomorphism and Equivalence

Definition (isomorphism and equivalence)**.** *We call the categories $\boldsymbol{C}$ and $\boldsymbol{D}$ are isomorphic, if for some functor $F : \boldsymbol{C} \to \boldsymbol{D}$, there exists another functor $G : \boldsymbol{D} \to \boldsymbol{C}$ such that the compositions are equal to the identity functors, such as $F \circ G = Id_{\boldsymbol{D}}$ and $G \circ F = Id_{\boldsymbol{C}}$.*

On the other hand, we call the categories $\boldsymbol{C}$ and $\boldsymbol{D}$ are equivalent if for some functor $F : \boldsymbol{C} \to \boldsymbol{D}$ there exists another functor $G : \boldsymbol{D} \to \boldsymbol{C}$ such that the compositions are isomorphic to the identity functors, such as $F \circ G \cong Id_{\boldsymbol{D}}$ and $G \circ F \cong Id_{\boldsymbol{C}}$. Or, we call the functor F as the equivalence of the categories.

The equivalence of categories is the more general concept than the isomorphism of categories, but it is more frequently studied, because isomorphism of categories and quite rare, and it is even hard to appreciate the applications. For example in equivalence of category, a natural transformation from some identity functor to another identity functor is not necessarily isomorphic. For this reasons, from here, I will primary consider the equivalence of categories.

Next, we will introduce some conditions of category theory that is logically equivalent to the equivalence of categories.

Definition (Full, faithful, and essentially surjective)**.** *Consider a morphism*

$$\begin{gathered} Hom_{\boldsymbol{C}}(X, Y) \to Hom_{\boldsymbol{D}}(FX, FY) \\ f \mapsto Ff \end{gathered}$$

Take any $f, g \in Hom_{\boldsymbol{C}}(X, Y)$, and if $Ff = Fg$ induces $f = g$, then we say the functor F is full.

Similarly, if for all $\phi \in Hom_{\boldsymbol{D}}(FX, FY)$ there exists $f \in Hom_{\boldsymbol{C}}(X, Y)$ such that $Ff = \phi$, F is faithful.

Also, if for all $Y \in \boldsymbol{D}$, there exists $X \in \boldsymbol{C}$ such that $F(X)$ is isomorphic to Y, F is called essentially surjective.

Note that if a functor is both full and faithful, we call the functor is fully faithful.

If a functor satisfies all of the previous definition, the functor is an equivalence of categories.

Proposition. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ are categories, and let $F : \boldsymbol{C} \to \boldsymbol{D}$ is a functor. The functor F is an equivalence of categories iff F is fully faithful and essentially surjective.*

Proof. Assuming F be an equivalence functor, there exist two natural isomorphisms such as $\alpha : Id_{\mathbf{C}} \cong G \circ F$ and $\beta : F \circ G \cong Id_{\mathbf{D}}$.

we will prove that F is fully faithful. First of all, let two morphisms $f, g : X \to Y$ be $Ff = Fg$, we will prove $f = g$.

$$\begin{tikzcd}
FX \arrow[r] \arrow[d, "Ff"] & GFX \arrow[r, "\alpha_X"] \arrow[d, "GFf"] & X \arrow[d, "f"] \\
FY \arrow[r] & GFY \arrow[r, "\alpha_Y"] & Y
\end{tikzcd}$$

Consider the diagram above, we have $GFf \circ \alpha_X = \alpha_Y \circ f$, because the right side of the square commutes by the definition of natural morphism. Besides, by assumption, the natural transformation α is isomorphic, morphism α_X is all invertible. For these reasons, $f = \alpha_Y \circ GFf \circ \alpha_X^{-1}$ and $g = \alpha_Y \circ GFg \circ \alpha_X^{-1}$. Now here, $Ff = Fg$ implies $GFf = GFg$, resulting $f = g$.

Next, we will prove equivalence functor is faithful. But we need to be aware to the fact that the equivalence of categories is not necessarily isomorphism of categories, so I want to discuss some possible fallacies. For example, the argument with the following diagram may be false, because, in general, we cannot assume the natural transformation $\eta : Id_{\mathbf{D}} \to Id_{\mathbf{D}}$ is isomorphic.

$$\begin{tikzcd}
FX \arrow[r, "\eta_X"] \arrow[d, "\phi"] & FX \arrow[d, "Ff"] \\
FY \arrow[r, "\eta_Y"] & FY
\end{tikzcd}$$

Recall that if α_X is invertible, $F(\alpha_X)$ is also invertible, because $F(\alpha_X) \circ F({\alpha_X}^{-1} = F(\alpha_X \circ {\alpha_X}^{-1}) = F(Id_X) = Id_{FX}$.

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[d] & GFX \arrow[d] \\
Y \arrow[r, "\alpha_Y"] & GFY
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "F(\alpha_X)"] \arrow[d, "\phi"] & FGFX \arrow[d, "FG\phi"] \\
FY \arrow[r, "F(\alpha_Y)"] & FGFY
\end{tikzcd}
\qquad
\begin{tikzcd}
FGFX \arrow[r, "\beta_{FX}"] \arrow[d, "GF\phi"] & FX \arrow[d, "Ff"] \\
FGFY \arrow[r, "\beta_{FY}"] & FY
\end{tikzcd}
$$

Without loss of generality, we can name a morphism $\phi : FX \to FY$, and the corresponding morphism is $GF\phi : FGFX \to FGFY$ because GF is a functor. Since the right diagram commutes, we describe ϕ by $\phi = F({\alpha_X}^{-1}) \circ GF\phi \circ F(\alpha_X)$. Hence, if the morphism $GF\phi$ exits, then ϕ also exists.

On the other hand, for the bottom diagrm, let morphisms $GF\phi : FGFX \to FGFY$ and $f : X \to Y$ be independently defined. Then, there exists a natural isomorphism β such that the diagram commutes. Hence, $GF\phi = {\beta_{FY}}^{-1} \circ Ff \circ \beta FX$. Therefore, by composition, existence of ϕ guarantees Ff.

The fact that the essentially surjectiveness is trivial.

We will prove the converse. We assume the function $F : \mathbf{C} \to \mathbf{D}$ is fully faithful and essentially surjective. Our goal in this proof is to define another functor $G : \mathbf{C} \to \mathbf{D}$ such that the compositions $F \circ G$ and $G \circ F$ are isomorphic to identity. Let some $Y \in \mathbf{D}$, we define the functor G so that an object $GY \in \mathbf{C}$ satisfy an isomorphism $\beta_Y : F(GY) \to Y$.

Let $\phi : X \to Y$ is a morphism.

$$
\begin{aligned}
Hom_{\mathbf{C}}(GX, GY) &\to Hom_{\mathbf{D}}(FGX, FGY) \\
f &\mapsto Ff
\end{aligned}
$$

Notice that Ff can also be written as $Ff = {\beta_Y}^{-1} \circ \phi \circ \beta_X$, since β_Y is invertible by assumption, although invertibility of β_X is not known.

Then, let a morphism $\phi : X \to Y$ be defined in $\mathbf{D}$.

$$
\begin{aligned}
Hom_{\mathbf{C}}(X, Y) &\to Hom_{\mathbf{D}}(FGX, FGY) &&\to Hom_{\mathbf{C}}(GX, GY) \\
\phi &\mapsto {\beta_Y}^{-1} \circ \phi \circ \beta_X &&\mapsto f
\end{aligned}
$$

The composition implies the mapping ϕ to f exists for all f, and trivially the inverse exists. Hence, the relationship is one-to-one, and β_X has to be bijective for all $X \in \mathbf{C}$. Therefore, $FG \cong Id_{\mathbf{C}}$.

$\square$

3.2 Yoneda's lemma

Theorem (Yoneda's Lemma). *Let $F : \boldsymbol{C} \to \boldsymbol{Sets}$, and let $X \in \boldsymbol{C}$ be any object. Then the map*

$$
\begin{aligned}
Nat(h_X, F) &\to F(X) \\
\alpha &\mapsto \alpha_X(id_X)
\end{aligned}
$$

is a bijection where $h_X := Hom_{\boldsymbol{C}}(X, -)$.

Proof. In order to prove the bijectivity, we discuss both the surjectivity nad injectivity.

Let $u \in F(X)$ and we construct a natural transformation $\beta : h_x = Hom(X, Y) \to F$ so that $\beta_Y(f) = (Ff)(u)$.

Consider a diagram where $\phi : Y \to Y'$ be a morphism.

$$\begin{tikzcd}
Hom(X,Y) \arrow[r, "\beta_Y"] \arrow[d, "\phi \circ -"] & FY \arrow[d, "F(\phi)"] & & f \arrow[r, mapsto] \arrow[d, mapsto] & \beta_Y(f) = (Ff)(u) \arrow[d, mapsto] \\
Hom(X,Y') \arrow[r, "\beta_{Y'}"] & FY' & & \phi \circ f \arrow[r, mapsto] & \beta_Y(\phi \circ f) = F(\phi \circ f)(u)
\end{tikzcd}$$

The diagram implies the given morphism is surjective, because if we let f is identity morphism, the commutative diagram implies $\beta_X(id_X) = (F(id_X))(u) = (id_{f(X)})(u) = u$. Note that we have already assumed that u is an arbitrary object. Therefore, if we let any object $u \in F(X)$, then a corresponding natural transformation exists.

On the other hand, let $\alpha \in Nat(h_X, F)$ be a natural transformation, and we will particularly consider u as $u = \alpha_X(id_X)$. Since we have already proved surjectivity, there is a natural transformation β that satisfies the above commutative diagram exists. However, notice that another morphism $id_X \to \alpha_X(id_X)$ trivially exists, as shown in the next diagram.

$$\begin{tikzcd}
Hom(X,X) \arrow[r, "\alpha_X"] \arrow[d, "f \circ -"] & FX \arrow[d, "F(f)"] & & id_X \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_X(id_X) \arrow[d, mapsto] \\
Hom(X,Y) \arrow[r, "\alpha_Y"] & FY & & f \circ f \arrow[r, mapsto] & \alpha_Y(f) = (Ff)(\alpha_X(id_X))
\end{tikzcd}$$

So, from the first diagram, $\beta_Y(f) = (Ff)(\alpha_X(id_X))$, but from the second diagram, $\alpha_Y(f) = (Ff)(\alpha_X(id_X))$. it turns out $\beta = \alpha$, resulting the injectivity. Hence, a morphism $Nat(h_X, F) \to F(X)$ is bijective.

$\square$

Corollary. *Let* $\boldsymbol{C}$ *be a category. The Yoneda functor*

$$y : \boldsymbol{C} \to \boldsymbol{Sets}^{op} := Func(\boldsymbol{C}^{op}, \boldsymbol{Sets})$$
$$X \mapsto h^X = Hom_{(}C)(-, X)$$

is fully faithful.

3.3 Adjunctions

Definition. *Adjunction between $\boldsymbol{C}$ and $\boldsymbol{D}$ is a pair of functors $F : \boldsymbol{C} \to \boldsymbol{D}$ and $G : \boldsymbol{D} \to \boldsymbol{C}$ that satisfy the relationship*

$$Hom_{\boldsymbol{C}}(FY, X) \cong Hom_{\boldsymbol{D}}(Y, GX)$$

for all objects $Y \in \boldsymbol{C}$ and $X \in \boldsymbol{D}$, and the family of bijections is natural between X and Y.

Also, we call F as a left adjoint functor to G, often denoted as $F \vdash G$, and similarly, G is a right adjoint functor to F,

Grothendieck duality theorem claims the existence of the adjoint functor of derived categories of bounded–below chain complexes of quasi-coherent sheaves. We keep the explanation of adjunctions minimal to focus on the proof of BRT.

4 Triangulated Cateogry

Above arguments are the general constructions of category theory, but the simple question is if there are some interesting arguments by restricting to some specific categories. Triangulated category is, for example, a category with some additional conditions, but it is still broader notion than algebraic objects, such as ring, module, group, or chain complexes. In this section, I will dedicate to explain the struction of triangulated category.

4.1 Construction

A category $\mathbf{C}$ is a triangulated category. if $\mathbf{C}$ is an additive category with a translation functor $\Sigma : \mathbf{C} \to \mathbf{C}$ with some axioms of triangulations.

Definition (Additive category)**.** *Let $\boldsymbol{C}$ is a category whose morphism $\alpha, \beta \in Hom_{\boldsymbol{C}}(X, Y)$ and $\gamma, \delta \in Hom_{\boldsymbol{C}}(Y, Z)$ agrees additional operations such that $\alpha + \beta \in Hom_{\boldsymbol{C}}(X, Y)$ and $\gamma + \delta \in Hom_{\boldsymbol{C}}(Y, Z)$, and composition $(\alpha + \beta)\gamma \in Hom_{\boldsymbol{C}}(X, Z)$ and $\alpha(\gamma + \delta) \in Hom_{\boldsymbol{C}}(X, Z)$ exists. Besides, $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$ and $\alpha(\gamma + \delta) = \alpha\gamma + \alpha\delta$. If all the above satisifed, category $\boldsymbol{C}$ is called additive category.*

Definition (Translation functor)**.** *Translation functor $\Sigma : \boldsymbol{C} \to \boldsymbol{C}$ is an automorphism between objects. That means, we have inverse functor Σ^{-1}.*

Definitions (Triangle axioms)**.** *Another important feature of triangulated category is a triangle. Here, the following five axioms features triangles.*

- *TR0:*
  A sequence $X \to X \to 0 \to \Sigma X$ is distinguished.
- *TR1:*
  For all objects X and Y in D and morphism $u : X \to Y$, there exists an object $C(u)$ such that $X \to Y \to C(u)$ is a distinguished triangle. We call $C(u)$ as cone of morphism $X \to Y$.

- *TR2:*
  If $X \to Y \to Z \to \Sigma X$ is distinguished, then $Y \to Z \to \Sigma X \to \Sigma Y$ is also distinguished and vice versa.

- *TR3:*
  If a morphism from triangle to triangle where $f : X \to X'$,$g : Y \to Y'$ are defined, there exist a morphism $h : Z \to Z'$, but h does not have to be unique.

- *TR4:*
  If we let a sequence $X \to Y \to Z$ and morphism $u : X \to Y$ and $v : Y \to Z$,then we can define respective cones $C(u)$, $C(v)$, and $C(u \circ v)$, and as in the following diagram, there are three morphisms among the respective cones that consitute another triangle such that $C(v) \to C(u \circ v) \to C(u)$, and this is distinguished.

$$
\begin{tikzcd}
X \arrow[rr, "u"] & & Y \arrow[rr, "v"] \arrow[dl] & & Z \arrow[dl] \arrow[ddll, bend left] \\
& C(u) \arrow[ul] & & C(v) \arrow[ul] \arrow[ll, dashed] \arrow[dl, dashed] & \\
& & C(v \circ u) \arrow[uull, bend left] \arrow[ul, dashed] & &
\end{tikzcd}
$$

  Note that another name of TR4 is Octahedral axiom.

  Now, all above is the requirement of triangulated category. If triangulated category satifies the additional condition TR5, then we call the trinagulated category is compactly generated.

- *TR5:*

  In category $\boldsymbol{D}$, in general, an object $X \in \boldsymbol{D}$ is compact, if the diagram $Hom(-, X)$ commutes for all direct sums, namely, $Hom(\coprod_{i\in\alpha} Y_i, X) \cong \coprod_{i\in\alpha} Hom(Y_i, X)$ for any cardinal of index α and $\{Y_i\}_{i\in\alpha}$.
  Triangulated category $\boldsymbol{D}$ is compactly generated if
  (1) all infinite coproduct of object of $\boldsymbol{D}$ is included in $\boldsymbol{D}$, and

  (2) Let $X \in \boldsymbol{D}$ be any object and S be a set of objects $S \subset Obj(\boldsymbol{D})$. Then if for all $Y \in S$, $Hom(Y, X) = 0$, $X = 0$.

  For the better interpretation, I want to introduce another equivalent notation of (2).

(2)' for all X, there is a set of objects S such that for all $Y \in S$, there exist a non-zero morphism $0 \neq f \in Hom(Y[n], X)$ for some $n \in \mathbb{Z}$.

Notice that all the examples of triangulated category shown is compactly generated, so BRT is widely applicable. Indeed, our primary concern in the succeeding argument is compactly generated triangulated category.

4.2 Examples

1. Vector space

   If we define exactness of module homomorphism as distinguished, and if we let the translation functor is the identity map, then the triangle is also distinguished.

2. Homotopy cateogry

   True homotopy category $K(A)$ is induced from a category of topological space Top quotiented where its equivalent class is weak homotopy equivalence. Weak homotopy equivalence is by definition,

   and this is actually more convenient than naive homotopy catgory $hTop$.

3. derived category

   Let A be an abelian category and Derived category is noted as $D(A)$ whose objects are chain complex of A.

4. stable homotopy category

   There is a quotient morphism **Top** $\to$ **hTop***, where the quotient is defined by the homotopic equivalence. This generated category **hTop*** is called "naïve" homotopy category for some technical reasons. We instead define **SH** by the natural morphism Σ^∞ : **hTop*** $\to$ **SH**. **SH** is constructed to be a trianglated category, and it is particularly compactly generated. We name **SH** as a stable homotopy category.

5. stable module category

   Let M, N be arbitrary modules, and define stable module category $\underline{Hom}(M, N)$ as a quotient of $Hom(M, N)$ with equivalence $f \sim g$ if $f - g$ factors through a projective module. Interestingly, this category is studied by Frobenius algebra, and it is used to study modular representaion theory. $\underline{Hom}(M, N)$ is also compactly generated.

4.3 Properties

Finally, I will introduce some important lemma about the triangulated categories that should be used to prove BRT. First of all, recall that TR1 says the existence of the cone(not necessarily unique) in each morphism, but since not all cones in the categories generate some distinguished triagules for some morphisms, it makes limit or colimit of the cones non well-defined. In order to make the whole argument consistent, we define alternative definition, such as homotopy limit and homotopy colimit, as the replacements of limit and colimit.

Definition (homotopy colimit)**.** *Let $\boldsymbol{T}$ is a compactly generated triangulated category, and we consider a sequence of objects and morphisms as $X_1 \underset{u_1}{\rightarrow} X_2 \underset{u_2}{\rightarrow} X_3 ...$,*

and notice that, by TR1, there exists a cone for each morphisms $u_i : X_i \rightarrow X_{i+1}$.

Then, we define homotopy colimit as a cone of a shift morphism $u : \coprod_i X_i \rightarrow \coprod_i X_i$ denoted by $\underline{hocolim}_i X_i$, so that the triangle

$$\coprod X_i \xrightarrow{1-shift} \coprod X_i \longrightarrow \underline{hocolim} X_i \longrightarrow \Sigma \coprod X_i$$

will be distinguished.

The relationship between limit or colimit and homotopy colimit are often studied, by considering the morphism $\underline{hocolim} X_i \rightarrow colim X_i$. On the other hand, the homotopy colimit is an invariant notion in hom functor, if we assume the category is a triangulated category as shown in the next lemma.

Lemma. *Let $\boldsymbol{T}$ be a triangulated category, and let c be a compact object in $\boldsymbol{T}$, and suppose $X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow ...$ is a sequence of objects and morphisms in $\boldsymbol{T}$. Suppose $\boldsymbol{T}$ admits small coproducts (since $\boldsymbol{T}$ is compactly generated, appearantly yes), then we claim*

$$Hom(c, \underline{hocolim} X_i) = \underline{lim} Hom(c, X_i)$$

Proof. Consider the triangle:

$$\begin{tikzcd} \coprod X_i \arrow[r, "1-shift"] \arrow[dr] \arrow[r, phantom, shift right=4ex, "(1)"] & \coprod X_i \arrow[d] \\ & \underline{hocolim} X_i \end{tikzcd}$$

Applying the homological functor $Hom(c, -)$, we get a long exact sequence (by the definition of homological functor). In particular, the induced sequence $Hom(c, \coprod X_i) \rightarrow Hom(c, \underline{hocolim} X_i) \rightarrow Hom(c, \coprod \Sigma X_i) \underset{1-shift}{\rightarrow} \prod Hom(c, \Sigma X_i)$ is exact. But since c is compact,

$$\begin{tikzcd}
Hom(c, \coprod \Sigma X_i) \arrow[r, "1-shift"] \arrow[d, "\cong"] & \coprod Hom(c, \Sigma X_i) \arrow[d, "\cong"] \\
Hom(c, \coprod \Sigma X_i) \arrow[r, "1-shift"] & \coprod Hom(c, \Sigma X_i)
\end{tikzcd}$$

But the top row is clearly injective, because this is just a shift. Consequently, the bottom row is injective. The exactness implies that γ is surjective.

Extensively, consider the following commutative diagram

$$\begin{tikzcd}
\coprod Hom(c, \Sigma X_i) \arrow[r, "1-shift"] \arrow[d, dash, "\cong"] & \coprod Hom(c, \Sigma X_i) \arrow[r, dashed] \arrow[d, dash, "\cong"] & \underline{lim} Hom(c, \Sigma X_i) \arrow[r] \arrow[d, dashed, dash, "\cong"] & 0 \\
Hom(c, \coprod \Sigma X_i) \arrow[r, dash, "\cong"] & Hom(c, \coprod \Sigma X_i) \arrow[r, "\gamma"] & Hom(c, \underline{hocolim} X_i) \arrow[r] & 0
\end{tikzcd}$$

where each rectangle commutes, and the bottom row is exact, so the top row need to be exact. Notice that the top right object is $\underline{lim} Hom(c, X_i)$, because if we let $\mathbf{J}$ be the category of diagram whose objects are $Hom(c, X_i)$ for all i, and for all shift maps, but without loss of generality, let's say, for all 1-shift maps, we have a categorical limit of the cone, that is distinguished. Then, we need to prove between the two objects in the third row $Hom(c, \underline{hocolim} X_i)$ and $underlinelim Hom(c, X_i)$ we have an isomorphism. Define two morphism from one direction to the other, and vice versa. Since the diagram commutes, the both morphisms are surjective, so injective. After all, we proved $Hom(c, \underline{hocolim} X_i) \cong \underline{lim} Hom(c, X_i)$. $\square$

5 Brown Representability Theorem

In this section, we will prove BRT. Be surprised that the proof of BRT is purely given by category theory.

Theorem (Brown Representability Theorem)**.** *Let $\boldsymbol{D}$ be a compactly generated trinangulated category and $\boldsymbol{Ab}$ be a category of abelian group, and let $H : \boldsymbol{D}^{op} \to \boldsymbol{Ab}$ be a functor. Suppose we accept the morphism*

$$H(\coprod_i X_i) \cong \prod_i H(X_i)$$

is an isomorphism for all small coproducts in $\boldsymbol{T}$. Then H is representable.

Proof. Let T be a generating set of $\mathbf{T}$. We consider a sequence of any objects and morphisms $X_0 \to X_1 \to X_2 \to X_3 ...$, except the beginning element that we let as $X_0 = \coprod_{(\alpha,t) \in U_0} t$, where $U_0 = \bigcup_{t \in T} H(t)$, where each element of U_0, and each of the elements are recognized by a pair (α, t) with $\alpha \in H(t)$.

Notice that U_0 is a disjoint union, since all elements has to be uniquely joined in some $H(t)$. However, I follow this notation given by Neeman. Also,

$H(X_0) = \prod_{(\alpha,t)\in U_0} H(t)$, because $H(X_0)$ is given by a product, because by the assumption, H is cohomological.

Next, we prove the morphism $Hom(X_0, t) \to H(t)$ is an isomorphism, But let's begin from the surjectivity. Consider a natural transformation $Hom(-, X_0) \to H$, we have a commutative diagram, which is:

$$\begin{tikzcd} Hom(X_0, X_0) \arrow[r] \arrow[d] & H(X_0) \arrow[d] & id_{X_0} \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_0 \arrow[d, mapsto] \\ Hom(X_0, t) \arrow[r] & H(t) & \phi_0(t) \arrow[r, mapsto] & \alpha \end{tikzcd}$$

Notice that it is suffice to say $Hom(X_0, t) \to H(t)$ is surjective, because $Hom(X_0, X_0) \to H(t)$ is a composition of a morphism $\phi 0_{X_0} : Hom(X_0, X_0) \to Hom(X_0, t)$, and both morphism are sujective. Because if $\phi 0_t : Hom(X_0, t) \to H(t)$ is not surjective, the composite will not be surjective.

First of all, consider $t \mapsto X_0$ is injective, there exists an element $\alpha_0 \in H(X_0)$ that maps to $\alpha \in H(t)$ because $t \mapsto X_0$ is injective, and H is contravariant. On the other hand, By Yoneda's lemma, the morphism $Nat(Hom(X_0, -), H) \to H(X_0)$ is bijective, So, for all the element $\alpha_0 \in H(X_0)$, there exists a morphism ϕ_0 such that $\phi 0_{X_0}(id_{X_0}) = \alpha_0$. Now for the morphism $\phi 0_{X_0} : Hom(X_0, X_0) \to H(X_0)$, an identity element maps to $id_{X_0} \mapsto \alpha_0$, for all $\alpha_0 \in H(X_0)$. So, the morphism is surjective.

For these reasons, $Hom(X_0, X_0) \to H(t)$ is surjective. Now, since the diagram commutes, the component morphism $Hom(X_0, t) \to H(t)$ has to be surjective.

Next, we expand our argument for $i \geq 0$. Now that X_i is á priori given (although not known the structure yet), we consider a natural transformation $Hom(-, X_i) \to H$. Also, we need to construct the structure of K_{i+1}. In order to make the triangle $K_{i+1} \to X_i \to X_{i+1}$ will be distinguished, we let K_{i+1} be $K_{i+1} = \coprod_{(f,t)\in U_{i+1}} t$, where $U_{i+1} = \bigcup_{t\in T} ker\{Hom(t, X_i) \to H(t)\}$, and each of the element of U_{i+1} can be recognized as a pair (f, t) where $t \in T$ and $f : t \to X_i$ is a morphism.

The morphism $Hom(-, X_i) \to H$ for all $i > 0$, we have a similar argument to conclude $Hom(t, X_i) \to H(t)$ is surjective, and $\alpha_i \in H(X_i) \mapsto \alpha \in H(t)$ exists.

$$H(X_i) \to H(K_{i+1}) = H(\coprod_{(f,t)\in U_{i+1}} t) \quad = \prod_{(f,t)\in U_{i+1}} H(t)$$

The element $\alpha_i \in H(X_i)$ maps to zero. The $f : t \to X_i$ were chosen so that the induced map $Hom(t, X_i) \to H(t)$ vanishes. But H is a homological functor, and it makes the exact sequence $H(X_{i+1}) \underset{k}{\to} H(X_i) \underset{j}{\to} H(K_i)$

where $j(\alpha_i) = 0$. It guarantees that there exists $\alpha_{i+1} \in H(X_{i+1})$ with $k(\alpha_{i+1}) = \alpha_i$ (Consider that the triangle is distinguished, the each element has to be included in the image of k). Hence, without loss of generality, we choose $\alpha_{i+1} \in H(X_{i+1})$, there is a well-defined morphism that maps $\alpha_i \mapsto \alpha_{i+1}$. For this reason, there is a corresponding natural transformation $Hom(-, X_{i+1}) \to H$ that induces commutative diagram of functors.

$$\begin{tikzcd} H(-, X_i) \arrow[d] \arrow[dr] & \\ H(-, X_{i+1}) \arrow[r] & H \end{tikzcd}$$

From the above arguments, as a result, we claim the existence of exact sequences $...Hom(-, X_{i+2}) \to Hom(-, X_{i+1}) \to Hom(-, X_i)...$ where each object has a morphism to H that commutes, where $...X_0 \to X_1 \to X_2...$ is a sequence. Recall from the lemma in the previous chapter, we have a relationship between the homotopy colimit of the sequence X_i and limit of the sequence $Hom(-, X_i)$. This will put a natural question:

Let $X = \underline{hocolim} X_i$. I assert :

(1):

There is a natural transformation $Hom(-, X) \to H$ rendering commutative the triangles

$$\begin{tikzcd} H(-, X_i) \arrow[d] \arrow[dr] & \\ H(-, X) \arrow[r] & H \end{tikzcd}$$

for every i.

(2):

The natural transformation $Hom(-, X) \to H(t)$ is an isomorphism.

proof of (1):

Consider the triangle

$$\begin{tikzcd}
 & \coprod_i X_i \arrow[dr] & \\
\underline{hocolim} X_i = X \arrow[ur] & & \coprod_i X_i \arrow[ll]
\end{tikzcd}$$

Applying the cohomological functor H, we get an exact sequence

$$\begin{tikzcd}
H(X) \arrow[r] & H(\coprod_i X_i) \arrow[r, "1-shift"] \arrow[d, "\cong"] & H(\coprod_i X_i) \arrow[d, "\cong"] \\
 & \prod_i H(X_i) \arrow[r, "1-shift"] & \prod_i H(X_i)
\end{tikzcd}$$

The element $\prod_i \alpha_i \in \prod_i H(X_i)$ is in the kernel of 1-shift, and hence there is $\alpha \in H(X)$ that maps to it. Again by Yoneda, α corresponds to a natural transformation $Hom(-, X) \to H$.

Since α maps to $\prod_i \alpha_i \in H(\coprod X_i)$, the diagram below

$$\begin{tikzcd}
Hom(-, X_i) \arrow[d] \arrow[dr] & \\
H & Hom(-, X) \arrow[l]
\end{tikzcd}$$

commutes for all i.

proof of (2):

It remains to show that $\phi : Hom(-, X) \to H$ is an isomorphism. Since we know the $Hom(t, X_0) \to H(t)$ is surjective, it follows that $Hom(t, X) \to H(t)$ is surjective, since the diagram given in (1) commutes, and both the composite $Hom(t, X) \to H$ and the other component $Hom(t, X_i) \to Hom(t, X)$ are surjective.

So, we will only need to prove its injectivity.

Let us begin with objects $t \in T$. We will show that, for all $t \in T$, $\phi(t) : Hom(t, X) \to H(t)$ is an isomorphism. Observe that the commutative diagram

$$\begin{tikzcd}
Hom(t, X_0) \arrow[d] \arrow[dr] & \\
H(t) & Hom(t, X) \arrow[l]
\end{tikzcd}$$

Let $f \in Hom(t, X)$ with $\phi(t)(f) = 0$. Consider $Hom(t, X) = \underline{lim} Hom(t, X_i)$, because $Hom(t, X) = Hom(t, \underline{hocolim} X_i)$ by definition, and $Hom(t, \underline{hocolim} X_i) = \underline{lim} Hom(t, X_i)$, since t is compact. Since by the construction of categorical limit, there must exist a category of diagram J that derives the limit of cone, we claim for all i there exists a morphism $f_i \in Hom(t, X_i)$ so that the composite makes f with some morphism in $Hom(X_i, X)$.

But the diagram

$$\begin{tikzcd}
Hom(t, X_i) \arrow[d] \arrow[dr, "k"] & \\
Hom(t, X) \arrow[r, "j"] & H(t)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
f_i \arrow[d, mapsto] \arrow[dr, mapsto] & \\
f \arrow[r, mapsto] & 0
\end{tikzcd}$$

commutes, and $j(f_i) = f$ while $k(f) = 0$. For this reason, we found $f_i \in ker\{Hom(t, X_i) \to H(t)\} \subset Hom(t, X_i)$, that is, $(f_i, t) \in U_{i+1}$.

As the next diagram shows, there is a unique morphism h because $f_i : t \to X_i$ is factored through the map h in the triangle, because K_{i+1} is a coproduct that includes t, and the universal property applies.

$$\begin{tikzcd}
& X_i \arrow[dr, "g"] & \\
K_{i+1} \arrow[ur, "h"] \arrow[rr, "(1)"] & & X_{i+1}
\end{tikzcd}$$

Now, since the triangle is distinguished, the composition $g \circ h = 0$ must be zero, so then $g \circ f_i = 0$. Furthermore, consider the morphisms $X_i \underset{g}{\to} X_{i+1} \underset{\bar{g}}{\to} X$ prove that $f = 0$.

$$\begin{aligned}
f &= \{(\bar{g}) \circ g\} \circ f_i \\
&= (\bar{g}) \circ \{g \circ f_i\} \\
&= 0
\end{aligned}$$

Thus $\phi(t) : Hom(t, X) \to H(t)$ is an isomorphism for all $t \in T$. $\square$

We will introduce some applications of BRT very quickly from a theoretical point of view. For example, adjunction may be discussed directly.

Corollary. *Let $\boldsymbol{S}$ be a compactly generated triangulated category and $\boldsymbol{T}$ be any triangulated category. Let $F : \boldsymbol{S} \to \boldsymbol{T}$ be a triangulated functor. Suppose F respects coproducts; the natural maps*

$$F(s_\lambda) \to F(\coprod_{\lambda\in\Lambda} s_\lambda)$$

make $F(\coprod_{\lambda\in\Lambda} s_\lambda)$ a coproduct of $\boldsymbol{T}$. Then F has a right adjoint $G : \boldsymbol{T} \to \boldsymbol{S}$.

Proof. Let t be an object of $\mathbf{T}$, and consider the pseudo-functor G where $G(t)$ denotes a morphism on $\mathbf{S}$ such that $G(t) : s \mapsto Hom_{\mathbf{T}}(F(S), t)$. Since this fuctor is homological,

$$\begin{aligned} Hom_{\mathbf{T}}(F(\coprod_{\lambda\in\Lambda} s_\lambda), t) &= Hom_{\mathbf{T}}(F(\coprod_{\lambda\in\Lambda} s_\lambda), t) \\ &= Hom_{\mathbf{T}}(\prod_{\lambda\in\Lambda} F(s_\lambda), t) \end{aligned}$$

Hence, by BRT, this functor is representable. $G(t) \in \mathbf{S}$ with $Hom_{\mathbf{T}}(F(s), t) = Hom_{\mathbf{S}}(s, G(t))$.

and by standard arguments, G extends to a functor, right adjoint of F.

□

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