Discrete Random Variable Problem: Car Repair Shop Solution and Wolfram Alpha Documentation

This document provides a detailed solution to finding probabilities for a discrete random variable representing the number of cars being repaired at a shop, based on given probabilistic information. It includes explanations of the steps and how to use Wolfram Alpha (or similar symbolic computation tools) to verify or perform the calculations.

Topics Covered in this Exercise:

* **Discrete Random Variables**

* **Probability Mass Function (PMF)**

* **Range of a Random Variable**

* **System of Linear Equations (for probabilities)**

* **Basic Probability Properties**

Problem Statement: Let X be the number of cars being repaired at a repair shop. We have the following information: * At any time, there are at most 3 cars being repaired. * The probability of having 2 cars at the shop is the same as the probability of having one car. * The probability of having no car at the shop is the same as the probability of having 3 cars. * The probability of having 1 or 2 cars is half of the probability of having 0 or 3 cars. * The shop can handle no more than 3 cars.

Find the following. State all inputs as integers between 0 and 99 and state all sets $\{x_1, x_2, \dots, x_n\}$ such that $x_1 < x_2 < \cdots < x_n$. Also, all resulting fractions must be irreducible.

Step 1: Define the Range of X, R_X.

The first bullet point states "At any time, there are at most 3 cars being repaired." This means the number of cars (X) can be 0, 1, 2, or 3. So, the range of X is the set:

$$R_X = \{0,1,2,3\}$$

Step 2: Formulate Equations based on the given information.

Let $P(X = x)$ denote the probability of having x cars at the shop.

1. **"The probability of having 2 cars at the shop is the same as the probability of having one car."**

$$P(X = 2) = P(X = 1) \quad (Equation\ 1)$$

2. **"The probability of having no car at the shop is the same as the probability of having 3 cars."**

$$P(X = 0) = P(X = 3) \quad (Equation\ 2)$$

3. **"The probability of having 1 or 2 cars is half of the probability of having 0 or 3 cars."** This means $P(X = 1 \text{ or } X = 2) = \frac{1}{2}P(X = 0 \text{ or } X = 3)$. Since these are mutually exclusive events, we can write:

$$P(X = 1) + P(X = 2) = \frac{1}{2}(P(X = 0) + P(X = 3)) \quad (Equation\ 3)$$

4. **The sum of probabilities for all possible outcomes must be 1.**

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1 \quad (Equation\ 4)$$

Step 3: Solve the system of equations for each probability.

Let's use substitution to find the individual probabilities.

Substitute Equation 1 into Equation 3:

$$P(X = 1) + P(X = 1) = \frac{1}{2}\big(P(X = 0) + P(X = 3)\big)$$

$2P(X = 1) = \frac{1}{2}\big(P(X = 0) + P(X = 3)\big) \quad (Equation\ 5)$

Substitute Equation 2 into Equation 5:

$2P(X = 1) = \frac{1}{2}(P(X = 0) + P(X = 0))$

$2P(X = 1) = \frac{1}{2}(2P(X = 0))\ 2P(X = 1) = P(X = 0) \quad (Equation\ 6)$

Now we have relationships between the probabilities.

Let's express all probabilities in terms of $P(X = 1)$ using Equation 1, Equation 2, and Equation 6:

* $P(X = 2) = P(X = 1)$ (from Equation 1)

* $P(X = 0) = 2P(X = 1)$ (from Equation 6)

* $P(X = 3) = P(X = 0) = 2P(X = 1)$ (from Equation 2 and Equation 6)

Substitute these expressions into Equation 4 (the sum of probabilities equals 1):

$$P(X=0)+P(X=1)+P(X=2)+P(X=3)=1$$

$\big(2P(X=1)\big)+P(X=1)+P(X=1)+\big(2P(X=1)\big)=1$

Combine the terms:

$(2+1+1+2)P(X=1)=1$

$6P(X=1)=1\ P(X=1)=\frac{1}{6}$

Now we can find all the probabilities:

* $P(X=1)=\frac{1}{6}$

* $P(X=2)=P(X=1)=\frac{1}{6}$

* $P(X=0)=2P(X=1)=2\times\frac{1}{6}=\frac{2}{6}=\frac{1}{3}$

* $P(X=3)=P(X=0)=\frac{1}{3}$

Let's verify the sum: $\frac{1}{3}+\frac{1}{6}+\frac{1}{6}+\frac{1}{3}=\frac{2}{6}+\frac{1}{6}+\frac{1}{6}+\frac{2}{6}=\frac{6}{6}=1$. The probabilities sum to 1.

Part a: Find R_X

As determined in Step 1:

$$R_X=\{0,1,2,3\}$$

Part b: Find $P_X(0)=P_X(3)$

From our calculations in Step 3:

$P_X(0)=\frac{1}{3}$

$P_X(3)=\frac{1}{3}$

So, $P_X(0)=P_X(3)=\frac{1}{3}$.

The fraction $\frac{1}{3}$ is irreducible.

(Note: The provided image answer is 2/6, which is equivalent to 1/3 but not irreducible. My answer will be the irreducible form.)

- **Wolfram Alpha Input for verification (conceptual)**: You would set up equations like `P0 = P3`, `P1 = P2`, `P1+P2 = (1/2)(P0+P3)`, `P0+P1+P2+P3=1` and solve for `P0`.

Part c: Find $P_X(1) = P_X(2)$

From our calculations in Step 3:

$P_X(1) = \frac{1}{6}$

$P_X(2) = \frac{1}{6}$

So, $P_X(1) = P_X(2) = \frac{1}{6}$.

The fraction $\frac{1}{6}$ is irreducible.

- **Wolfram Alpha Input for verification (conceptual)**: You would set up equations like `P0 = P3`, `P1 = P2`, `P1+P2 = (1/2)(P0+P3)`, `P0+P1+P2+P3=1` and solve for `P1`.