Assignment 1

a)

$P(X>0) = 1 - P(X \leq 0) = \underline{\underline{0.8}}$

$Var(X) = \sum x_i \cdot P(x_i) - E(X)^2$

$= 0^2 \cdot 0.2 + 1^2 \cdot 0.3 + 2^2 \cdot 0.5 - (0 \cdot 0.2 + 1 \cdot 0.3 + 4 \cdot 0.5)^2$

$= \underline{\underline{0.61}}$

b)

$P(X<2, Y>1) = (P(X=0)+P(X=1)) \cdot P(Y=2)$

$= \underline{\underline{0.15}}$

$P(X<2 \cup Y<2) = P(X<2) + P(Y<2) - P(X<2) \cdot P(Y<2)$

c)

$= 0.85$

$Cov(2X-5Y, 7X+4Y+1)$

$= Cov(2X,7X) + Cov(2X,4Y) + Cov(-5Y,7X) + Cov(-5Y,4Y)$

$= 14 \cdot Var(X) + 8Cov(X,Y) - 35Cov(X,Y) - 20Var(Y)$

$= 14 \cdot 0.61 - 20 \cdot Var(Y) = 14 \cdot 0.61 - 20 \cdot (0.4 + 0.3 \cdot 2^2 - (0.4 + 0.3 \cdot 2)^2)$

$= \underline{\underline{-3.46}}$

Assignment 2:

a) $E(Y) = \int_0^1 y \cdot f_Y \, dy$, $f_Y = \int_y^1 2(x+y) \, dx = -3y^2 + 2y + 1$

$= \int_0^1 y(-3y^2+2y+1) \, dy$

$= \underline{\underline{5/12}}$

Assignment 2 (cont)

b)

$$E(XY) = \int_0^1 \int_0^x x \cdot y \cdot 2(x+y)\, dy\, dx$$

$$= \underline{\underline{\frac{1}{3}}}$$

Assignment 3:

a) $\int_0^{\infty} e^{2x} \cdot 3 \cdot e^{-3x}\, dx$, $g(x) = e^x$ and $f(x) = 3 \cdot e^{-3x}$

$= 3\int_0^{\infty} e^{-x}\, dx = \underline{\underline{3}}$

pdf of exponential $f(x) = \lambda \cdot e^{-\lambda x}$

b) Always find cdf first:

$$P(Y < y) = P(e^{2x} < y)$$

$$= P(2x < \ln y) = P(X < \tfrac{1}{2}\ln y)$$

$$= \int_0^{\frac{1}{2}\ln y} f(x)\, dx = \int_0^{\frac{1}{2}\ln y} 3 \cdot e^{-3x}\, dx$$

$$= \left(1 - e^{-3 \cdot \frac{1}{2}\ln y}\right) - \left(1 - e^{-3 \cdot 0}\right), \text{ so}$$

$$F(Y) = 1 - e^{-3 \cdot \frac{1}{2}\ln y} = 1 - y^{-3/2}$$

$$f(y) = F'(y) = \left(1 - y^{-3/2}\right)' = \underline{\underline{\frac{3}{2} y^{-5/2}}}$$

Assignment 4:

a) Because of the memoryless property:

$$P(X_5=3 \mid X_3=1, X_2=2) = P(X_5=3 \mid X_3=1)$$

And because of Markov property we just need to look at any two-step transistion from 1 to 3:

$$P^2 = \begin{bmatrix} 0.09 & 0.29 & \boxed{0.62} \\ 0 & 0.28 & 0.72 \\ 0 & 0.24 & 0.76 \end{bmatrix}$$

$$P(X_5=3 \mid X_3=1, X_2=2) = \underline{\underline{0.62}}$$

b)

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0 & 0.4 & 0.6 \\ 0.8 & 0.2 & 0 \end{bmatrix} \to P^T = \begin{matrix} 0.3 & 0 & 0.8 \\ 0.3 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{matrix}$$

$$\left.\begin{aligned} .3\pi_1 + 0.8\pi_3 &= \pi_1 \\ 0.3\pi_1 + 0.4\pi_2 + 0.2\pi_3 &= \pi_2 \\ 0.4\pi_1 + 0.6\pi_2 &= \pi_3 \end{aligned}\right\} \left\{\begin{aligned} -0.7\pi_1 + 0.8\pi_3 &= 0 \\ 0.4\pi_1 + 0.6\pi_2 - \pi_3 &= 0 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}\right.$$

$$-0.7\pi_1 + 0.8\pi_3 = 0$$

$$0.4\pi_1 + 0.6(1-\pi_1-\pi_3) - \pi_3 = 0 \quad \left.\begin{aligned} -0.7\pi_1 + 0.8\pi_3 &= 0 \quad \times 2 \\ -0.2\pi_1 - 1.6\pi_3 &= -0.6 \end{aligned}\right.$$

$$(-1.4\pi_1 + 1.6\pi) + (-0.2\pi_1 - 1.6\pi_3) = 0 + (-0.6)$$

$$-1.6\pi_1 = -0.6$$

$$\pi_1 = \frac{-0.6}{-1.6} = \underline{0.375}$$

$$-0.2(0.375) - 1.6\pi_3 = -0.6 \Rightarrow \pi_3 = \underline{\underline{0.328}} \qquad [0.375, 0.297, 0.328]$$

$$0.375 + \pi_2 + 0.328 = 1 \Rightarrow \pi_2 = \underline{\underline{0.297}}$$

Assignment 5:

a) Since we want "evenly distributed" we test $p = 1/2$ ~ p is the unknow parameter

so $H_0: p = 1/2$

$H_1: p \neq 1/2$

b) Since sample space is $\{0,1\}$ the sample mean $\bar{x}$ is an estimate of p so $\bar{x} = \hat{p}$:

$$\hat{p} \pm z_{0.95} \cdot \frac{\sqrt{0.43}}{\sqrt{1000}}$$

$$\hat{p} \pm 1.64 \cdot \frac{\sqrt{0.45}}{\sqrt{1000}} \sim [0.5051; 0.5749]$$

c)

$$z_{test} = \frac{\hat{p} - p}{\sqrt{\frac{0.45}{1000}}} = \frac{0.54 - 0.5}{\sqrt{\frac{0.45}{1000}}} = \underline{\underline{1.8856}}$$

$$z_{crit} = \Phi^{-1}(0.995) = \underline{\underline{2.5758}}$$

$$\text{p-value} = 2 \cdot (1 - \Phi(z_{crit})) = \underline{\underline{0.0593}}$$

Since $z_{crit} > \alpha$ <u>fail to reject</u>

Assignment 6:

$$X \sim \text{Binomial}(\overset{n}{20000}, \overset{p}{0.0001})$$

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \text{binom.cdf}(4, n, p)$$

$$= \text{binom.sf}(4, n, p) = 0.0526$$

Assignment 7:

a) $s(b_1) = \sqrt{\frac{\hat{\sigma}^2}{S_{xx}}}$, $s(b_0) = \sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}$

$$\hat{\sigma}^2 = \frac{SSE}{n-2} = \frac{\sum (y-\hat{y})^2}{n-2}$$

$S_{xx} = \sum x_i - \frac{1}{n}\left(\sum x_i\right)^2$ (or other formulas)

Using Pyton:

$SSE = 21.10$, $S_{xx} = 70$

$\hat{\sigma}^2 = \frac{21.10}{4} = 5.28$, $\bar{x} = 13$

$s(b_1) = \sqrt{\frac{5.28}{70}} = \underline{\underline{0.275}}$, $s(b_0) = \sqrt{5.28\left(\frac{1}{6} + \frac{13^2}{70}\right)}$

$= \underline{\underline{3.6902}}$

b) $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x$

$= 62.92 + 7.06x$

$\hat{y}_5 = 62.92 + 7.06 \cdot 5 = 98.21 \approx \underline{\underline{98}}$

Assignment 8:

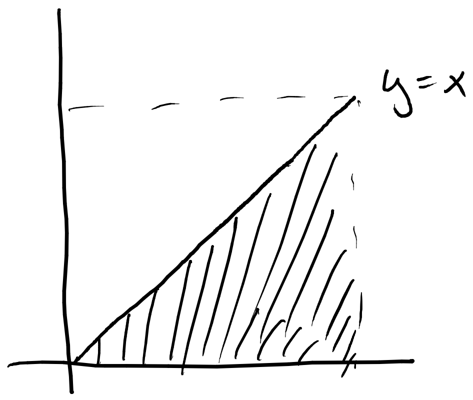
small sample, unknown variance $\rightarrow$ t-Dist.

$T_0 = \frac{313.3449 - 310}{6/\sqrt{25}} = 2.78$, so $t.cdf(2.78, 24) = 0.9949$

We have a two-sided CI so $\alpha = 2 \cdot (1 - 0.9949) = 0.01$

So C.I is 99%

Assignment 9:



$$f_{Y|X} = \frac{f_{XY}}{f_X}$$

$$f_X = \int_0^x 8xy\,dy = \left[\frac{1}{2}\cdot 8x\cdot y^2\right]_0^x$$

$$= 4x^3$$

$$f_{X>\frac{1}{2}} = \int_{1/2}^1 4x^3\,dx = \left[x^4\right]_{1/2}^1 = 1 - \frac{1}{16} = \underline{\frac{15}{16}}$$

$$f_{Y<\frac{1}{2},X>\frac{1}{2}} = \int_{1/2}^1 \int_0^{1/2} 8xy\,dy\,dx = \int_{1/2}^1 \left[\frac{1}{2}\cdot 8xy^2\right]_0^{1/2} dx$$

$$= \int_{1/2}^1 x\,dx = \left[\frac{1}{2}x^2\right]_{1/2}^1 = \frac{1}{2} - \frac{1}{8} = \underline{\frac{3}{8}}\text{ , so}$$

$$f_{X<Y|X>1/2} = \frac{3/8}{15/16} = \frac{3\cdot 16}{8\cdot 15} = \frac{3\cdot 2}{15} = \frac{1\cdot 2}{5} = \underline{\underline{\frac{2}{5}}}$$

Assignment 10:

a) $T = 90$

$\lambda_1 = 0.02$, $\lambda_2 = 0.03$, $\lambda_{1+2} = 0.05$

$$P(N_1(t)=0, N_2(t)=0) = P(N_1(t)=0)\cdot P(N_2(t)=0) \quad \text{- Independent}$$

$$= \frac{e^{-0.02\cdot 90}\cdot(0.02\cdot 90)^0}{0!}\cdot\frac{e^{-0.03\cdot 90}(0.03\cdot 90)}{0!}$$

$2^3\cdot 2^4 = 2^7$

$$= e^{-0.02\cdot 90}\cdot e^{-0.03\cdot 90} = e^{-0.05\cdot 90}$$

$$= e^{-0.05\cdot 90} = e^{-4.5} = \underline{\underline{0.01}}$$

Assignment 10 (cont)

b)

$$P(N(t) \geq 2) = 1 - P(N(t) \leq 1) = 1 - (P(N(t)=0) + P(N(t)=1))$$

$$= 1 - \left(e^{-0.05 \cdot 90} + e^{-0.05 \cdot 90} \cdot 0.05 \cdot 90\right)$$

$$= 0.9389 \approx \underline{\underline{0.94}}$$

c)

$$P(N_1(t)=1, N_2(t)=2) = P(N_1(t)=1) \cdot P(N_2(t)=2)$$

$$= \frac{e^{-0.02 \cdot 90} \cdot (0.02 \cdot 90)}{1!} \cdot \frac{e^{-0.03 \cdot 90} \cdot (0.03 \cdot 90)^2}{2!}$$

$$= \frac{e^{-0.05 \cdot 90} \cdot 0.02 \cdot 0.03^2 \cdot 90^3}{2} = 0.0729$$

$$= \underline{\underline{0.07}}$$