Exponential Distribution Problem Solution and Wolfram Alpha Documentation

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This document provides a detailed solution to probability problems involving the time until the next event in a Poisson process, which follows an Exponential distribution. It covers calculating probabilities for specific time intervals, finding percentiles, and handling conditional probabilities (memoryless property). It includes explanations of the steps and how to use Wolfram Alpha (or similar symbolic computation tools) to verify or perform the calculations.

Topics Covered in this Exercise:

- **Poisson Process**
- **Exponential Distribution**
- **Probability Density Function (PDF) of Exponential Distribution**
- **Cumulative Distribution Function (CDF) of Exponential Distribution**
- **Memoryless Property of Exponential Distribution**
- **Calculating Probabilities for Continuous Variables**
- **Percentiles/Quantiles**

Please note: This document uses LaTeX syntax for mathematical expressions. To view these expressions correctly, you need a Markdown viewer or editor that supports LaTeX rendering (e.g., via MathJax or KaTeX). Standard text editors or word processors like Google Docs will show the raw LaTeX code.

Problem Statement:
Web crawlers need to estimate the frequency of changes to Web sites to maintain a current index for Web searches. Assume that the changes to a Web site follow a Poisson process with a mean of 6 days. Let a random variable X denote the time (in days) until the next change.

Introduction to Exponential Distribution:
If events occur according to a Poisson process with an average rate of λ events per unit of time, then the time between events (or time until the next event) follows an Exponential distribution with parameter λ.
The mean time between events is $1/\lambda$.

Given the mean time until the next change is 6 days, then $1/\lambda = 6$, which means $\lambda = 1/6$.

The PDF of an Exponential distribution is:

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

The CDF of an Exponential distribution is:

$$P(X \leq x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

And $P(X > x) = e^{-\lambda x}$.

For this problem, $\lambda = \frac{1}{6}$ changes per day.

Part a: What is the probability that the next change occurs in less than 4.5 days?

This asks for $P(X < 4.5)$. Using the CDF formula:

$$P(X < 4.5) = 1 - e^{-\lambda \times 4.5}$$

$$P(X < 4.5) = 1 - e^{-(1/6) \times 4.5}$$

$$P(X < 4.5) = 1 - e^{-4.5/6}$$

$$P(X < 4.5) = 1 - e^{-0.75}$$

- **Wolfram Alpha Input**: `1 - e^(-0.75)`
- **Wolfram Alpha Result**: Approximately $0.52763\ldots$

Rounding to four decimal places: 0.5276.

The probability that the next change occurs in less than 4.5 days is 0.5276. This matches the provided answer.

Part b: What is the probability that the time until the next change is greater than 9.5 days?

This asks for $P(X > 9.5)$. Using the survival function formula:

$$P(X > 9.5) = e^{-\lambda \times 9.5}$$

$$P(X > 9.5) = e^{-(1/6) \times 9.5}$$

$$P(X > 9.5) = e^{-9.5/6}$$

$$P(X > 9.5) = e^{-1.58333...}$$

- **Wolfram Alpha Input**: `e^(-9.5/6)`
- **Wolfram Alpha Result**: Approximately $0.20531...$

Rounding to four decimal places: 0.2053.

The probability that the time until the next change is greater than 9.5 days is 0.2053. This matches the provided answer.

Part c: What is the time of the next change that is exceeded with probability 90%?

This question asks for the value of x such that $P(X > x) = 0.90$.
Using the survival function formula:

$$e^{-\lambda x} = 0.90$$

$$e^{-(1/6)x} = 0.90$$

Take the natural logarithm (ln) of both sides:

$$-\frac{1}{6}x = \ln(0.90)$$

$$x = -6\ln(0.90)$$

- **Wolfram Alpha Input**: `-6 * ln(0.90)`
- **Wolfram Alpha Result**: Approximately $0.63155\ldots$

Rounding to two decimal places: 0.63.

The time of the next change that is exceeded with probability 90% is 0.63 days. This matches the provided answer.

Part d: What is the probability that the next change occurs in less than 12.5 days, given that it has not yet occurred after 3.0 days?

This is a conditional probability problem for an Exponential distribution. The Exponential distribution has the **memoryless property**, which states that the probability of an event occurring in the future is independent of how much time has already passed.
Specifically, $P(X > t + s \mid X > t) = P(X > s)$.
Or, in a general sense, if we are given that an event has not occurred by time t, the "clock resets" and the probability of it occurring within an additional Δt is the same as the probability of it occurring within Δt from time zero.

The question asks for $P(X < 12.5 \mid X > 3.0)$.
By the memoryless property, this is equivalent to $P(X < 12.5 - 3.0)$.

$$P(X < 12.5 \mid X > 3.0) = P(X < 9.5)$$

Now we calculate $P(X < 9.5)$ using the CDF formula:

$$P(X < 9.5) = 1 - e^{-\lambda \times 9.5}$$

$$P(X < 9.5) = 1 - e^{-(1/6) \times 9.5}$$

$$P(X < 9.5) = 1 - e^{-9.5/6}$$

$$P(X < 9.5) = 1 - e^{-1.58333...}$$

- **Wolfram Alpha Input**: `1 - e^(-9.5/6)`
- **Wolfram Alpha Result**: Approximately $0.79468...$

Rounding to four decimal places: 0.7947.

The probability that the next change occurs in less than 12.5 days, given that it has not yet occurred after 3.0 days, is 0.7947. This matches the provided answer.