# Piecewise Probability Density Function Solution and Wolfram Alpha Documentation

This document provides a detailed solution to problems involving a continuous random variable defined by a piecewise probability density function (PDF). It covers finding the constant of the PDF, calculating probabilities, determining percentiles, and computing the expected value and variance. It includes explanations of the steps and how to use Wolfram Alpha (or similar symbolic computation tools) to verify or perform the calculations.

**Topics Covered in this Exercise:**

- **Continuous Random Variables**
- **Piecewise Probability Density Function (PDF)**
- **Properties of PDFs (Total Probability = 1)**
- **Calculating Probabilities from PDF (Integration)**
- **Cumulative Distribution Function (CDF)**
- **Percentiles/Quantiles**
- **Expected Value ($E[X]$) for Continuous Random Variables**
- **Variance ($Var[X]$) for Continuous Random Variables**

**Problem Statement:**
Let a continuous random variable $X$ denote the time spent on a cell phone (in hours) per month with the following probability density function where $h \neq 0$:

$$f(x) = \begin{cases} \frac{x-10}{5h} & 10 \leq x < 15 \\ \frac{1}{h} & 15 \leq x < 20 \\ \frac{x-25}{5h} & 20 \leq x \leq 25 \\ 0 & \text{otherwise} \end{cases}$$

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## Part a: Find the value of $h$.

For $f(x)$ to be a valid PDF, the total area under its curve must be equal to 1. Also, $f(x)$ must be non-negative.
Observing the third piece of the PDF, $\frac{x-25}{5h}$ for $20 \leq x \leq 25$, the term $(x-25)$ is non-positive. For $f(x)$ to be non-negative, the numerator should be positive or zero. This implies a likely typo in the problem statement, and that the third piece was intended to be $\frac{25-x}{5h}$ to maintain symmetry and positivity. We will proceed with this common correction to ensure a valid PDF and match the provided answer.

So, we integrate $f(x)$ over its defined non-zero intervals assuming $f(x) = \frac{25-x}{5h}$ for $20 \leq x \leq 25$:

$$\int_{10}^{15} \frac{x-10}{5h}\,dx + \int_{15}^{20} \frac{1}{h}\,dx + \int_{20}^{25} \frac{25-x}{5h}\,dx = 1$$

Let's integrate each part:

1. **First integral**: $\int_{10}^{15} \frac{x-10}{5h}\,dx$

$$= \frac{1}{5h}\left[\frac{x^2}{2} - 10x\right]_{10}^{15}$$

$$= \frac{1}{5h}\left[\left(\frac{15^2}{2} - 10(15)\right) - \left(\frac{10^2}{2} - 10(10)\right)\right]$$

$$= \frac{1}{5h}\left[(112.5 - 150) - (50 - 100)\right]$$

$$= \frac{1}{5h}\left[(-37.5) - (-50)\right] = \frac{1}{5h}[12.5] = \frac{12.5}{5h} = \frac{2.5}{h}$$

   - **Wolfram Alpha Input**: `integrate (x-10)/(5h) from x=10 to 15` (Result: 2.5/h)

2. **Second integral**: $\int_{15}^{20} \frac{1}{h}\,dx$

$$= \frac{1}{h}[x]_{15}^{20} = \frac{1}{h}(20-15) = \frac{5}{h}$$

   - **Wolfram Alpha Input**: `integrate 1/h from x=15 to 20` (Result: 5/h)

3. **Third integral (corrected for positivity)**: $\int_{20}^{25} \frac{25-x}{5h}\,dx$

$$= \frac{1}{5h}\left[25x - \frac{x^2}{2}\right]_{20}^{25}$$

$$= \frac{1}{5h}\left[\left(25(25) - \frac{25^2}{2}\right) - \left(25(20) - \frac{20^2}{2}\right)\right]$$

$$= \frac{1}{5h}\left[(625 - 312.5) - (500 - 200)\right]$$

$$= \frac{1}{5h}\left[312.5 - 300\right] = \frac{1}{5h}[12.5] = \frac{2.5}{h}$$

- **Wolfram Alpha Input**: `integrate (25-x)/(5h) from x=20 to 25` (Result: 2.5/h)

Now, sum the results and set equal to 1:

$$\frac{2.5}{h} + \frac{5}{h} + \frac{2.5}{h} = 1$$

$$\frac{10}{h} = 1$$

$$h = 10$$

The value for $h$ is $10$. This matches the provided answer.
With $h = 10$, the PDF becomes:

$$f(x) = \begin{cases} \frac{x-10}{50} & 10 \le x < 15 \\ \frac{1}{10} & 15 \le x < 20 \\ \frac{25-x}{50} & 20 \le x \le 25 \\ 0 & \text{otherwise} \end{cases}$$

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## Part b: Find $P(X < 17.5)$.

This requires integrating $f(x)$ from $10$ to $17.5$. This involves the first two pieces of the PDF.

$$P(X < 17.5) = \int_{10}^{15} \frac{x-10}{50}\,dx + \int_{15}^{17.5} \frac{1}{10}\,dx$$

1. **First integral (from Part a, with $h = 10$)**: $\int_{10}^{15} \frac{x-10}{50}\,dx = \frac{2.5}{10} = 0.25$
2. **Second integral**: $\int_{15}^{17.5} \frac{1}{10}\,dx$

$$= \frac{1}{10}[x]_{15}^{17.5} = \frac{1}{10}(17.5 - 15) = \frac{1}{10}(2.5) = 0.25$$

- **Wolfram Alpha Input**: `integrate 1/10 from x=15 to 17.5` (Result: 0.25)

Sum the results:

$$P(X < 17.5) = 0.25 + 0.25 = 0.50$$

The probability $P(X < 17.5)$ is $0.50$. This matches the provided answer.

## Part c: Find $P(X < 22.0)$.

This requires integrating $f(x)$ from $10$ to $22.0$. This involves all three pieces of the PDF.

$$P(X < 22.0) = \int_{10}^{15} \frac{x-10}{50}\,dx + \int_{15}^{20} \frac{1}{10}\,dx + \int_{20}^{22.0} \frac{25-x}{50}\,dx$$

1. **First integral (from Part a, with $h = 10$)**: $\int_{10}^{15} \frac{x-10}{50}\,dx = 0.25$
2. **Second integral (from Part a, with $h = 10$)**: $\int_{15}^{20} \frac{1}{10}\,dx = 0.5$
3. **Third integral**: $\int_{20}^{22.0} \frac{25-x}{50}\,dx$

$$= \frac{1}{50}\left[25x - \frac{x^2}{2}\right]_{20}^{22}$$

$$= \frac{1}{50}\left[\left(25(22) - \frac{22^2}{2}\right) - \left(25(20) - \frac{20^2}{2}\right)\right]$$

$$= \frac{1}{50}\left[(550 - 242) - (500 - 200)\right]$$

$$= \frac{1}{50}\left[308 - 300\right] = \frac{8}{50} = 0.16$$

   - **Wolfram Alpha Input**: `integrate (25-x)/50 from x=20 to 22` (Result: 0.16)

Sum the results:

$$P(X < 22.0) = 0.25 + 0.5 + 0.16 = 0.91$$

The probability $P(X < 22.0)$ is $0.91$. This matches the provided answer.

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## Part d: Find $x$ such that $P(X < x) = 0.95$.

We need to find $x$ such that the cumulative probability up to $x$ is $0.95$.
From Part c, we know $P(X < 22.0) = 0.91$. Since $0.95 > 0.91$, $x$ must be greater than $22.0$ and fall into the third segment of the PDF (where $20 \le x \le 25$).

So, we set up the equation:
$P(X < x) = \int_{10}^{15} \frac{x-10}{50}\,dx + \int_{15}^{20} \frac{1}{10}\,dx + \int_{20}^{x} \frac{25-t}{50}\,dt = 0.95$

1. **Sum of first two integrals**: $0.25 + 0.5 = 0.75$
2. **Third integral (from 20 to $x$)**: $\int_{20}^{x} \frac{25-t}{50}\,dt$

$$= \frac{1}{50}\left[25t - \frac{t^2}{2}\right]_{20}^{x}$$

$$= \frac{1}{50}\left[\left(25x - \frac{x^2}{2}\right) - \left(25(20) - \frac{20^2}{2}\right)\right]$$

$$= \frac{1}{50}\left[25x - \frac{x^2}{2} - (500 - 200)\right]$$

$$= \frac{1}{50}\left[25x - \frac{x^2}{2} - 300\right]$$

Now, combine everything and set equal to $0.95$:

$$0.75 + \frac{1}{50}\left(25x - \frac{x^2}{2} - 300\right) = 0.95$$

Subtract $0.75$ from both sides:

$$\frac{1}{50}\left(25x - \frac{x^2}{2} - 300\right) = 0.20$$

Multiply by $50$:

$$25x - \frac{x^2}{2} - 300 = 10$$

Rearrange into a quadratic equation:

$$-\frac{x^2}{2} + 25x - 310 = 0$$

Multiply by $-2$:

$$x^2 - 50x + 620 = 0$$

Use the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$:
Here $a = 1, b = -50, c = 620$.

$$x = \frac{50 \pm \sqrt{(-50)^2 - 4(1)(620)}}{2(1)}$$

$$x = \frac{50 \pm \sqrt{2500 - 2480}}{2}$$

$$x = \frac{50 \pm \sqrt{20}}{2}$$

$$x = \frac{50 \pm 2\sqrt{5}}{2}$$

$$x = 25 \pm \sqrt{5}$$

We know $x$ must be in the range $[20, 25]$.
$\sqrt{5} \approx 2.236$.
$25 + \sqrt{5} \approx 27.236$ (This is outside the range $[20, 25]$)
$25 - \sqrt{5} \approx 25 - 2.236 = 22.764$ (This is within the range $[20, 25]$)

So, $x = 25 - \sqrt{5} \approx 22.764$.
Rounding off correctly to the nearest integer: $23$.

The value of $x$ such that $P(X < x) = 0.95$ is $23$. This matches the provided answer.

- **Wolfram Alpha Input**: `solve integrate (x-10)/50 from 10 to 15 + integrate 1/10 from 15 to 20 + integrate (25-t)/50 from 20 to x = 0.95 for x` (Result: x = 25 - sqrt(5) approx 22.764)
- **Wolfram Alpha Input**: `round 22.764 to nearest integer` (Result: 23)

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## Part e: Find the expected value of $X$, $E[X]$.

The expected value of a continuous random variable is given by:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Using the piecewise PDF with $h = 10$ (and the corrected third piece):

$$E[X] = \int_{10}^{15} x \cdot \frac{x-10}{50}\,dx + \int_{15}^{20} x \cdot \frac{1}{10}\,dx + \int_{20}^{25} x \cdot \frac{25-x}{50}\,dx$$

1. **First integral**: $\int_{10}^{15} \frac{x^2-10x}{50}\,dx$

$$= \frac{1}{50}\left[\frac{x^3}{3} - 5x^2\right]_{10}^{15}$$

$$= \frac{1}{50}\left[\left(\frac{15^3}{3} - 5(15^2)\right) - \left(\frac{10^3}{3} - 5(10^2)\right)\right]$$

$$= \frac{1}{50}\left[(1125 - 1125) - \left(\frac{1000}{3} - 500\right)\right]$$

$$= \frac{1}{50}\left[0 - \left(\frac{1000-1500}{3}\right)\right] = \frac{1}{50}\left[-\left(-\frac{500}{3}\right)\right] = \frac{500}{150} = \frac{10}{3}$$

   - **Wolfram Alpha Input**: `integrate x*(x-10)/50 from x=10 to 15` (Result: 10/3)

2. **Second integral**: $\int_{15}^{20} \frac{x}{10}\,dx$

$$= \frac{1}{10}\left[\frac{x^2}{2}\right]_{15}^{20} = \frac{1}{10}\left(\frac{20^2}{2} - \frac{15^2}{2}\right)$$

$$= \frac{1}{10}\left(\frac{400}{2} - \frac{225}{2}\right) = \frac{1}{10}\left(200 - 112.5\right) = \frac{1}{10}(87.5) = 8.75$$

   - **Wolfram Alpha Input**: `integrate x/10 from x=15 to 20` (Result: 8.75)

3. **Third integral**: $\int_{20}^{25} x \cdot \frac{25-x}{50}\,dx = \int_{20}^{25} \frac{25x-x^2}{50}\,dx$

$$= \frac{1}{50}\left[\frac{25x^2}{2} - \frac{x^3}{3}\right]_{20}^{25}$$

$$= \frac{1}{50}\left[\left(\frac{25(25^2)}{2} - \frac{25^3}{3}\right) - \left(\frac{25(20^2)}{2} - \frac{20^3}{3}\right)\right]$$

$$= \frac{1}{50}\left[\left(\frac{15625}{2} - \frac{15625}{3}\right) - \left(5000 - \frac{8000}{3}\right)\right]$$

$$= \frac{1}{50}\left[\left(\frac{46875 - 31250}{6}\right) - \left(\frac{15000 - 8000}{3}\right)\right]$$

$$= \frac{1}{50}\left[\frac{15625}{6} - \frac{7000}{3}\right] = \frac{1}{50}\left[\frac{15625 - 14000}{6}\right] = \frac{1}{50}\left[\frac{1625}{6}\right] = \frac{1625}{300} = \frac{65}{12}$$

- **Wolfram Alpha Input**: `integrate x*(25-x)/50 from x=20 to 25` (Result: 65/12)

Sum the expected values from each segment:

$$E[X] = \frac{10}{3} + 8.75 + \frac{65}{12}$$

$$E[X] = \frac{40}{12} + \frac{105}{12} + \frac{65}{12}$$

$$E[X] = \frac{40 + 105 + 65}{12} = \frac{210}{12} = \frac{35}{2} = 17.5$$

The expected value of $X$ is $17.5$. This matches the provided answer.

- **Wolfram Alpha Input (Direct Calculation)**: `expected value of piecewise distribution {{ (x-10)/50, 10 <= x < 15 }, { 1/10, 15 <= x < 20 }, { (25-x)/50, 20 <= x <= 25 }}` (Result: 17.5)

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## Part f: Find the variance of $X$, $Var[X]$.

The variance of a continuous random variable is given by:

$$Var[X] = E[X^2] - (E[X])^2$$

We already found $E[X] = 17.5 = \frac{35}{2}$. Now we need to find $E[X^2]$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

Using the piecewise PDF with $h = 10$ (and the corrected third piece):

$$E[X^2] = \int_{10}^{15} x^2 \cdot \frac{x-10}{50}\,dx + \int_{15}^{20} x^2 \cdot \frac{1}{10}\,dx + \int_{20}^{25} x^2 \cdot \frac{25-x}{50}\,dx$$

1. **First integral**: $\int_{10}^{15} \frac{x^3-10x^2}{50}\,dx$

$$= \frac{1}{50}\left[\frac{x^4}{4} - \frac{10x^3}{3}\right]_{10}^{15}$$

$$= \frac{1}{50}\left[\left(\frac{15^4}{4} - \frac{10(15^3)}{3}\right) - \left(\frac{10^4}{4} - \frac{10(10^3)}{3}\right)\right]$$

$$= \frac{1}{50}\left[\left(\frac{50625}{4} - \frac{33750}{3}\right) - \left(2500 - \frac{10000}{3}\right)\right]$$

$$= \frac{1}{50}\left[\left(\frac{151875-135000}{12}\right) - \left(\frac{7500-10000}{3}\right)\right]$$

$$= \frac{1}{50}\left[\frac{16875}{12} - \left(-\frac{2500}{3}\right)\right] = \frac{1}{50}\left[\frac{16875}{12} + \frac{10000}{12}\right] = \frac{1}{50}\left[\frac{26875}{12}\right] = \frac{26875}{600} = \frac{1075}{24}$$

   - **Wolfram Alpha Input**: `integrate x^2*(x-10)/50 from x=10 to 15` (Result: 1075/24)

2. **Second integral**: $\int_{15}^{20} \frac{x^2}{10}\,dx$

$$= \frac{1}{10}\left[\frac{x^3}{3}\right]_{15}^{20} = \frac{1}{10}\left(\frac{20^3}{3} - \frac{15^3}{3}\right)$$

$$= \frac{1}{10}\left(\frac{8000}{3} - \frac{3375}{3}\right) = \frac{1}{10}\left(\frac{4625}{3}\right) = \frac{4625}{30} = \frac{925}{6}$$

   - **Wolfram Alpha Input**: `integrate x^2/10 from x=15 to 20` (Result: 925/6)

3. **Third integral**: $\int_{20}^{25} x^2 \cdot \frac{25-x}{50}\,dx = \int_{20}^{25} \frac{25x^2-x^3}{50}\,dx$

$$= \frac{1}{50}\left[\frac{25x^3}{3} - \frac{x^4}{4}\right]_{20}^{25}$$

$$= \frac{1}{50}\left[\left(\frac{25(25^3)}{3} - \frac{25^4}{4}\right) - \left(\frac{25(20^3)}{3} - \frac{20^4}{4}\right)\right]$$

$$= \frac{1}{50}\left[\left(\frac{25^4}{3} - \frac{25^4}{4}\right) - \left(\frac{25 \cdot 8000}{3} - \frac{160000}{4}\right)\right]$$

$$= \frac{1}{50}\left[\left(\frac{25^4}{12}\right) - \left(\frac{200000}{3} - 40000\right)\right]$$

$$= \frac{1}{50}\left[\frac{390625}{12} - \left(\frac{200000 - 120000}{3}\right)\right]$$

$$= \frac{1}{50}\left[\frac{390625}{12} - \frac{80000}{3}\right] = \frac{1}{50}\left[\frac{390625 - 320000}{12}\right] = \frac{1}{50}\left[\frac{70625}{12}\right] = \frac{70625}{600} = \frac{2825}{24}$$

- **Wolfram Alpha Input**: `integrate x^2*(25-x)/50 from x=20 to 25` (Result: 2825/24)

Sum $E[X^2]$ from each segment:

$$E[X^2] = \frac{1075}{24} + \frac{925}{6} + \frac{2825}{24}$$

$$E[X^2] = \frac{1075}{24} + \frac{925 \times 4}{24} + \frac{2825}{24}$$

$$E[X^2] = \frac{1075 + 3700 + 2825}{24} = \frac{7600}{24} = \frac{950}{3}$$

$$E[X^2] = \frac{950}{3} \approx 316.6666...$$

Finally, calculate $Var[X]$:

$$Var[X] = E[X^2] - (E[X])^2$$

$$Var[X] = \frac{950}{3} - (17.5)^2$$

$$Var[X] = \frac{950}{3} - \left(\frac{35}{2}\right)^2$$

$$Var[X] = \frac{950}{3} - \frac{1225}{4}$$

To subtract these fractions, find a common denominator, which is 12:

$$Var[X] = \frac{950 \times 4}{12} - \frac{1225 \times 3}{12}$$

$$Var[X] = \frac{3800}{12} - \frac{3675}{12}$$

$$Var[X] = \frac{125}{12}$$

As a decimal, $\frac{125}{12} \approx 10.41666...$

Rounding to two decimal places: $10.42$.

The variance of $X$ is $10.42$. This matches the provided answer.