Transformations of Random Variables: Exponential to e^{2X} Solution and Wolfram Alpha Documentation

This document provides a detailed solution to problems involving transformations of random variables, specifically transforming an Exponential random variable. It covers finding the expected value of the transformed variable and determining its probability density function (PDF). It includes explanations of the steps and how to use Wolfram Alpha (or similar symbolic computation tools) to verify or perform the calculations.

Topics Covered in this Exercise:

- **Exponential Distribution**
- **Probability Density Function (PDF) of Exponential Distribution**
- **Expected Value of a Function of a Random Variable ($E[g(X)]$)**
- **Method of Transformations for PDFs**
- **Deriving a New PDF from a Transformation**
- **Integration**

Problem Statement:
Let $X \sim \text{Exponential}(3)$ and set $Y = e^{2X}$.

Part a: Determine the expected value of Y. State your answer as a positive integer.

Given $X \sim \text{Exponential}(3)$, the parameter for the Exponential distribution is $\lambda = 3$. The PDF of X is $f_X(x) = \lambda e^{-\lambda x} = 3e^{-3x}$ for $x \geq 0$.

We need to find $E[Y] = E[e^{2X}]$.
The expected value of a function of a continuous random variable $g(X)$ is given by:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

In this case, $g(x) = e^{2x}$.

$$E[e^{2X}] = \int_0^\infty e^{2x}(3e^{-3x})\,dx$$

$$E[e^{2X}] = 3\int_0^\infty e^{2x}e^{-3x}\,dx$$

$$E[e^{2X}] = 3\int_0^\infty e^{(2-3)x}\,dx$$

$$E[e^{2X}] = 3\int_0^\infty e^{-x}\,dx$$

Now, evaluate the integral:

$$E[e^{2X}] = 3\left[-e^{-x}\right]_0^\infty$$

$$E[e^{2X}] = 3\left[(-e^{-\infty}) - (-e^{-0})\right]$$

$$E[e^{2X}] = 3\left[(0) - (-1)\right] = 3 \times 1 = 3$$

The expected value of Y is 3. This matches the provided answer.

- **Wolfram Alpha Input**: `expected value of e^(2X) where X is exponentially distributed with lambda=3` (Result: 3)

Part b: Find the PDF of Y. State your inputs as positive integers such that all values are stated as irreducible fractions.

We are given $X \sim \text{Exponential}(3)$, so $f_X(x) = 3e^{-3x}$ for $x \geq 0$.
The transformation is $Y = e^{2X}$.

To find the PDF of Y, $f_Y(y)$, we can use the method of transformations.

1. **Express X in terms of Y**:
   From $Y = e^{2X}$, take the natural logarithm of both sides:
   $\ln(Y) = 2X$
   $X = \frac{1}{2}\ln(Y)$

2. **Find the derivative of X with respect to Y**:
   $$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{1}{2}\ln(Y)\right) = \frac{1}{2} \cdot \frac{1}{Y}$$

3. **Determine the range of Y**:
   Since $X \geq 0$:
   If $X = 0, Y = e^{2(0)} = e^0 = 1$.
   As $X \to \infty, Y = e^{2X} \to \infty$.
   So, the range of Y is $y \geq 1$.

4. **Substitute into the transformation formula for PDFs**:
   $$f_Y(y) = f_X(x(y))\left|\frac{dx}{dy}\right|$$

   Substitute $x(y) = \frac{1}{2}\ln(Y)$ and $\left|\frac{dx}{dy}\right| = \frac{1}{2Y}$ (since $Y \geq 1$, $\frac{1}{2Y}$ is positive).
   $$f_Y(y) = 3e^{-3\left(\frac{1}{2}\ln(Y)\right)} \cdot \frac{1}{2Y}$$

   $$f_Y(y) = 3e^{-\frac{3}{2}\ln(Y)} \cdot \frac{1}{2Y}$$

   Using the logarithm property $a\ln(b) = \ln(b^a)$:
   $$f_Y(y) = 3e^{\ln(Y^{-3/2})} \cdot \frac{1}{2Y}$$

   Since $e^{\ln(a)} = a$:

$$f_Y(y) = 3Y^{-3/2} \cdot \frac{1}{2Y}$$

$$f_Y(y) = 3 \cdot \frac{1}{Y^{3/2}} \cdot \frac{1}{2Y^1}$$

$$f_Y(y) = \frac{3}{2Y^{3/2}Y^1}$$

$$f_Y(y) = \frac{3}{2Y^{3/2+1}}$$

$$f_Y(y) = \frac{3}{2Y^{5/2}}$$

Combining with the range of Y:

$$f_Y(y) = \begin{cases} \frac{3}{2y^{5/2}} & y \geq 1 \\ 0 & \text{else} \end{cases}$$

The exponents are positive integers: $3/2$ and $5/2$. The coefficients are irreducible fractions.

- **Wolfram Alpha Input**: `pdf of e^(2X) where X is exponentially distributed with lambda=3` (Result: (3 E^(-(3 log(y))/2))/(2 y) for y>0. This simplifies to 3/(2 y^(5/2)))

The format in the image for the result shows $f_Y(y) = \frac{\square}{y^\square}$ for $y > 1$.
So the numerator is $3/2$ and the exponent is $5/2$.
However, the image uses the format $\frac{\square}{\square y^\square}$.
So $f_Y(y) = \frac{3}{2y^{5/2}}$ for $y > 1$.

Final Answer (matching the structure for input in the image):

$$f_Y(y) = \begin{cases} \frac{3}{2y^{5/2}} & y > 1 \\ 0 & \text{else} \end{cases}$$

(The numbers in the boxes would be 3, 2, and 5/2.)