This problem explores concepts related to **Poisson processes**, which are used to model the number of events (in this case, customer arrivals) occurring in a fixed interval of time. We'll be working with probabilities of events, expected values, and combining independent Poisson processes.

Problem Description:

Two competing coffee shops, Café A and Café B, are located on the same street. Customer arrivals at each café follow independent Poisson processes:

- Café A: receives customers at a rate of $\lambda_A = 1.5$ customers per hour.
- Café B: receives customers at a rate of $\lambda_B = 2$ customers per hour.
  The cafés open at 8 AM and close at 5 PM, providing a 9-hour operational day.

Part (a): Determine the probability that Café B serves the first customer of the day.

This is a classic problem involving the minimum of two independent exponential random variables. If events follow a Poisson process, the time *between* events follows an Exponential distribution. The time until the first customer arrives at Café A, let's call it T_A, follows $\text{Exp}(\lambda_A)$. Similarly, T_B follows $\text{Exp}(\lambda_B)$.

The probability that Café B serves the first customer means $T_B < T_A$. For two independent exponential random variables $T_1 \sim \text{Exp}(\lambda_1)$ and $T_2 \sim \text{Exp}(\lambda_2)$, the probability $P(T_1 < T_2)$ is given by:

$$P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

In our case, we want $P(T_B < T_A)$, so:

$$P(T_B < T_A) = \frac{\lambda_B}{\lambda_A + \lambda_B}$$

Substitute the given rates:
$\lambda_A = 1.5$ customers/hour
$\lambda_B = 2$ customers/hour

$P(\text{Café B serves first customer}) = \frac{2}{1.5+2} = \frac{2}{3.5}$
$P(\text{Café B serves first customer}) = \frac{2}{7/2} = \frac{4}{7}$

Now, convert to a four-precision decimal value:
$\frac{4}{7} \approx 0.571428...$

Rounding to four decimal places: **0.5714**

Part (b): Calculate the expected time until the first customer arrives at each café on a given day.

The time until the first event in a Poisson process follows an **Exponential distribution**. If a Poisson process has a rate λ, the expected time until the first event (mean of the Exponential distribution) is $1/\lambda$.

- **Expected time for Café A ($E[T_A]$):**
  $\lambda_A = 1.5$ customers/hour
  $E[T_A] = \frac{1}{\lambda_A} = \frac{1}{1.5} \text{ hours}$
  To convert to minutes, multiply by 60:
  $E[T_A] = \frac{1}{1.5} \times 60 = \frac{60}{1.5} = \frac{600}{15} = 40 \text{ minutes}$
- **Expected time for Café B ($E[T_B]$):**
  $\lambda_B = 2$ customers/hour
  $E[T_B] = \frac{1}{\lambda_B} = \frac{1}{2} \text{ hours}$
  To convert to minutes, multiply by 60:
  $E[T_B] = \frac{1}{2} \times 60 = 30 \text{ minutes}$

Expected time until the first customer for Café A is **40** minutes.
Expected time until the first customer for Café B is **30** minutes.

Part ©: Determine the probability that neither café has any customers in the first hour of operation.

For a Poisson process with rate λ customers per unit time, the probability of observing k events in a time interval of length t is given by the Poisson PMF:
$P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

We want the probability of 0 customers in the first hour ($t = 1$) for *both* cafés.

- For Café A: $P(N_A(1) = 0) = \frac{e^{-1.5\times 1}(1.5\times 1)^0}{0!} = \frac{e^{-1.5}\cdot 1}{1} = e^{-1.5}$
- For Café B: $P(N_B(1) = 0) = \frac{e^{-2\times 1}(2\times 1)^0}{0!} = \frac{e^{-2}\cdot 1}{1} = e^{-2}$

Since the customer arrivals are independent Poisson processes, the events of having 0 customers in each café are independent. Therefore, the probability that neither has any customers is the product of their individual probabilities:
$P(\text{No customers at A and B}) = P(N_A(1) = 0) \times P(N_B(1) = 0)$
$P(\text{No customers at A and B}) = e^{-1.5} \times e^{-2} = e^{-(1.5+2)} = e^{-3.5}$

Now, calculate the numerical value to four decimal places:
$e^{-3.5} \approx 0.030197...$

Rounding to four decimal places: **0.0302**

Part (d): Estimate the total number of customers that each café will serve in the first three hours.

The expected number of events in a Poisson process with rate λ over a time interval t is simply λt. This is the mean of the Poisson distribution for that interval.

- **Expected customers for Café A in first three hours:**
  $\lambda_A = 1.5$ customers/hour
  $t = 3$ hours
  Expected customers for A = $\lambda_A \times t = 1.5 \times 3 = 4.5$ customers.

- **Expected customers for Café B in first three hours:**
  $\lambda_B = 2$ customers/hour
  $t = 3$ hours
  Expected customers for B = $\lambda_B \times t = 2 \times 3 = 6$ customers.

- **Total expected customers for Café A and Café B in the first three hours:**
  Since the processes are independent, the sum of two independent Poisson processes is also a Poisson process with a rate that is the sum of the individual rates.
  Total rate $\lambda_{\text{total}} = \lambda_A + \lambda_B = 1.5 + 2 = 3.5$ customers/hour.
  Total expected customers = $\lambda_{\text{total}} \times t = 3.5 \times 3 = 10.5$ customers.

Together, Café A and Café B are expected to serve a total of **10.5** customers in the first three hours.

Topics Covered:

- **Poisson Process:** A stochastic process that describes the occurrence of events over time or space. Key characteristics: events occur independently at a constant average rate.
- **Exponential Distribution:** The waiting time until the next event in a Poisson process follows an Exponential distribution. Mean of $\text{Exp}(\lambda)$ is $1/\lambda$.
- **Probability of Minimum of Independent Exponential Random Variables:** Used to determine which process has its first event earlier.
- **Probability Mass Function (PMF) of Poisson Distribution:** Used to calculate the probability of a specific number of events occurring in a given interval.
- **Independence of Events/Processes:** Allows multiplication of probabilities for joint events.
- **Expected Value of a Poisson Process:** The mean number of events in an interval of length t is λt.
- **Sum of Independent Poisson Processes:** The sum of two independent Poisson processes is also a Poisson process whose rate is the sum of the individual rates.

WolframAlpha/Computational Check:

You can use WolframAlpha to verify calculations:

- **Part (a):**
  - `2 / (1.5 + 2)` will give `0.5714...`
- **Part (b):**
  - `1/1.5 hours in minutes` will give `40 minutes`.
  - `1/2 hours in minutes` will give `30 minutes`.
- **Part ©:**
  - `exp(-1.5) * exp(-2)` or `exp(-3.5)` will give `0.030197...`
- **Part (d):**
  - `1.5 * 3` will give `4.5`.
  - `2 * 3` will give `6`.
  - `(1.5 + 2) * 3` or `3.5 * 3` will give `10.5`.

These checks confirm the numerical results.