Bernoulli Distribution: Hypothesis Testing and Confidence Interval Solution and Wolfram Alpha Documentation (Revised)

This document provides a detailed solution to problems involving a sample from a Bernoulli distribution, incorporating the specific use of the provided sample variance. It covers determining the appropriate alternative hypothesis, constructing a confidence interval for the population proportion, and performing a hypothesis test. It includes explanations of the steps and how to use Wolfram Alpha (or similar symbolic computation tools) to verify or perform the calculations.

Topics Covered in this Exercise:

- **Bernoulli Distribution**
- **Hypothesis Testing for Proportions (or Mean, with observed sample variance)**
- **Null and Alternative Hypotheses**
- **Test Statistic (Z-score)**
- **Critical Value**
- **P-value**
- **Decision Rule for Hypothesis Testing**
- **Confidence Intervals for Proportions (or Mean, with observed sample standard deviation)**
- **Standard Error**
- **Z-distribution (Standard Normal Distribution)**

Given Information:

- Sample size: $n = 1000$
- Sample mean (sample proportion for Bernoulli): $\hat{p} = \bar{x} = 0.54$
- Sample variance: $s^2 = 0.45$
  - This implies the sample standard deviation is $s = \sqrt{0.45} \approx 0.67082039$.

Part a: Determine which of the below would be an appropriate alternative hypothesis for this test.

The question asks to determine whether the outcomes are "evenly distributed" in the sample space of the Bernoulli distribution. For a Bernoulli distribution, "evenly distributed" means the probability of success (1) is equal to the probability of failure (0). This occurs when $p = 0.5$.

Therefore, the test is to see if p is significantly different from 0.5.

- **Null Hypothesis (H_0)**: The outcomes are evenly distributed. This means $p = 0.5$.
- **Alternative Hypothesis (H_1)**: The outcomes are NOT evenly distributed. This means $p \neq 0.5$.

From the given options, the most appropriate alternative hypothesis expresses this in terms of the population parameter p:

$$H_1 : p \neq \frac{1}{2}$$

Part b: Set up a 90% confidence interval for p.

To set up a confidence interval for a population proportion p (or mean of a Bernoulli variable) when the sample size is large and an observed sample variance s^2 is provided, we use the formula:

$$\text{CI} = \hat{p} \pm Z_{\alpha/2} \frac{s}{\sqrt{n}}$$

Given:

- Sample proportion $\hat{p} = 0.54$
- Sample size $n = 1000$
- Sample standard deviation $s = \sqrt{0.45} \approx 0.67082039$
- Confidence level $= 90\% \implies \alpha = 1 - 0.90 = 0.10$
- $\alpha/2 = 0.10/2 = 0.05$

Step 1: Find the critical Z-value ($Z_{\alpha/2}$).

We need the Z-score such that $P(Z \leq Z_{0.05}) = 1 - 0.05 = 0.95$.
Using a standard normal distribution table or calculator:

$$Z_{0.05} \approx 1.645$$

- **Wolfram Alpha Input**: `inverse normal distribution probability 0.95` (Result: 1.64485...)

Step 2: Calculate the standard error of the mean.

$$SE = \frac{s}{\sqrt{n}} = \frac{\sqrt{0.45}}{\sqrt{1000}} = \frac{0.67082039}{31.6227766} \approx 0.0212132$$

- **Wolfram Alpha Input**: `sqrt(0.45) / sqrt(1000)` (Result: 0.0212132...)

Step 3: Calculate the margin of error (ME).

$$ME = Z_{\alpha/2} \times SE = 1.645 \times 0.0212132 \approx 0.034880$$

Step 4: Construct the confidence interval.

$$\text{Lower Bound} = \hat{p} - ME = 0.54 - 0.034880 = 0.505120$$

$$\text{Upper Bound} = \hat{p} + ME = 0.54 + 0.034880 = 0.574880$$

Rounding to four decimal places:
Lower Bound: 0.5051
Upper Bound: 0.5749

90% Confidence Interval for p: $[0.5051, 0.5749]$
This matches the provided answer.

Part c: Assume we want to test the hypothesis mentioned above with $\alpha = 0.01$. Determine all the

values below, and determine the correct decision based on the data. Select the value closest to your result.

Hypotheses:

- $H_0 : p = 0.5$
- $H_1 : p \neq 0.5$ (Two-tailed test)

Significance Level: $\alpha = 0.01$

1. Test Statistic:

For testing a proportion when an observed sample variance s^2 is provided (and n is large), the Z-test statistic is calculated as:

$$Z = \frac{\hat{p} - p_0}{s/\sqrt{n}}$$

Given: $\hat{p} = 0.54$, $p_0 = 0.5$, $n = 1000$, $s = \sqrt{0.45} \approx 0.67082039$.

$$Z = \frac{0.54 - 0.5}{0.67082039/\sqrt{1000}} = \frac{0.04}{0.0212132}$$

$$Z \approx 1.8856$$

Test Statistic: 1.89 (rounded to two decimal places).

- **Wolfram Alpha Input**: `(0.54 - 0.5) / (sqrt(0.45) / sqrt(1000))` (Result: 1.8856...)

2. Critical Value:

For a two-tailed test with $\alpha = 0.01$, we need to find $Z_{\alpha/2} = Z_{0.01/2} = Z_{0.005}$.
This is the Z-score such that $P(Z \leq Z_{0.005}) = 1 - 0.005 = 0.995$.
Using a standard normal distribution table or calculator:

$$Z_{0.005} \approx 2.576$$

The critical values for the rejection region are ± 2.576.

Critical Value: ± 2.58 (rounded to two decimal places).

- **Wolfram Alpha Input**: `inverse normal distribution probability 0.995` (Result: 2.57582…)

3. P-value:

For a two-tailed test, the p-value is twice the probability of observing a test statistic as extreme as, or more extreme than, the calculated test statistic, in the direction of the alternative hypothesis.

$$p\text{-value} = 2 \times P(Z > |Z_{\text{test statistic}}|)$$

$$p\text{-value} = 2 \times P(Z > 1.8856)$$

Using a standard normal CDF table or calculator:

$$P(Z > 1.8856) = 1 - \Phi(1.8856) \approx 1 - 0.97034 \approx 0.02966$$

$$p\text{-value} = 2 \times 0.02966 = 0.05932$$

P-value: 0.0593 (rounded to four decimal places).

- **Wolfram Alpha Input**: `2 * (1 - normalcdf[1.8856])` (Result: 0.05931…)

4. Decision:

We compare the p-value to the significance level α:

- $p\text{-value} = 0.0593$
- $\alpha = 0.01$

Since $p\text{-value} > \alpha$ $(0.0593 > 0.01)$, we **fail to reject the null hypothesis**.

Alternatively, compare the test statistic to the critical values:

- $|Z_{\text{test statistic}}| = |1.8856| = 1.8856$
- Critical Value ($Z_{\alpha/2}$) $= 2.576$

Since $1.8856 < 2.576$, the test statistic falls within the acceptance region. Thus, we **fail to reject the null hypothesis**.

Based on this we should: Fail to reject the null hypothesis.