Poisson Distribution Problem Solution and Wolfram Alpha Documentation

This document provides a detailed solution to a probability problem involving patient arrivals at a clinic, which follows a Poisson distribution. It covers calculating probabilities for specific events and conditional probabilities based on the Poisson PMF. It includes explanations of the steps and how to use Wolfram Alpha (or similar symbolic computation tools) to verify or perform the calculations.

Topics Covered in this Exercise:

* **Poisson Distribution**

* **Probability Mass Function (PMF) of Poisson Distribution**

* **Mean (λ) of Poisson Distribution**

* **Calculating Probabilities for Discrete Events**

* **Conditional Probability**

Problem Statement: Let X denote the number of patients arriving at a walk-in test center between 1:00 and 2:00 PM with a mean of 10.5.

Then X follows a **Poisson** distribution.

For the following questions, state your answer as an integer between 0 and 99 such that you supply two decimal precision, correctly rounded off.

Introduction to Poisson Distribution: A Poisson distribution describes the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event. The PMF of a Poisson distribution is given by:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

where:

* X is the Poisson random variable.

* k is the actual number of events that occur.

* λ is the average number of events per interval (the mean).

* e is Euler's number (approximately 2.71828).

* $k!$ is the factorial of k.

For this problem, the time interval is 1 hour (1:00 PM to 2:00 PM), and the mean arrival rate is $\lambda = 10.5$ patients per hour.

Part a: What is the probability that fewer than 15 patients arrive at the clinic between 1:00 and 2:00 PM?

This asks for $P(X < 15)$, which means $P(X \leq 14)$.

$$P(X < 15) = P(X = 0) + P(X = 1) + \cdots + P(X = 14) = \sum_{k=0}^{14} \frac{e^{-10.5}(10.5)^k}{k!}$$

This sum is best calculated using a calculator or statistical software/tool like Wolfram Alpha that can compute cumulative Poisson probabilities.

- **Wolfram Alpha Input**: `poisson probability X < 15, lambda = 10.5` (Alternatively: `poissoncdf[10.5, 14]`)
- **Wolfram Alpha Result**: The result is approximately $0.89069...$

Rounding to two decimal places: 0.89.

The probability that fewer than 15 patients arrive is 0.89. This matches the provided answer.

Part b: What is the probability that no patients arrive between 1:00 and 1:10 PM?

The time interval is 1 hour (60 minutes), and the mean arrival rate is $\lambda = 10.5$ patients per hour.

We need to find the mean arrival rate for a 10-minute interval. A 10-minute interval is $\frac{10}{60} = \frac{1}{6}$ of an hour.

New mean for 10 minutes, $\lambda_{10min} = \lambda_{1hour} \times \frac{10}{60} = 10.5 \times \frac{1}{6} = 1.75$.

Now, we want to find the probability that $k = 0$ patients arrive in this 10-minute interval, with $\lambda = 1.75$.

$$P(X = 0) = \frac{e^{-1.75}(1.75)^0}{0!} = \frac{e^{-1.75} \times 1}{1} = e^{-1.75}$$

- **Wolfram Alpha Input**: `e^(-1.75)`

- **Wolfram Alpha Result**: Approximately $0.17377...$

Rounding to two decimal places: 0.17.

The probability that no patients arrive between 1:00 and 1:10 PM is 0.17. This matches the provided answer.

Part c: Suppose that 20 patients arrive between 1:00 and 1:30 PM. What is the probability that three more patients arrive between 1:30 and 2:00 PM?

This is a conditional probability problem involving independent intervals in a Poisson process.

Let X_1 be the number of arrivals between 1:00 and 1:30 PM. Let X_2 be the number of arrivals between 1:30 and 2:00 PM.

In a Poisson process, the number of events in non-overlapping intervals are independent. Therefore, the number of patients arriving between 1:00-1:30 PM is independent of the number of patients arriving between 1:30-2:00 PM.

The question asks for the probability that three *more* patients arrive between 1:30 and 2:00 PM, *given* that 20 patients arrived between 1:00 and 1:30 PM.

Due to independence, the information about the first interval does not affect the probability of arrivals in the second interval.

First, calculate the mean arrival rate for a 30-minute interval (1:30 PM to 2:00 PM). A 30-minute interval is $\frac{30}{60} = \frac{1}{2}$ of an hour.

New mean for 30 minutes, $\lambda_{30min} = \lambda_{1hour} \times \frac{1}{2} = 10.5 \times \frac{1}{2} = 5.25$.

Now, we want to find the probability that $k = 3$ patients arrive in this 30-minute interval, with $\lambda = 5.25$.

$$P(X_2 = 3) = \frac{e^{-5.25}(5.25)^3}{3!}$$

- **Wolfram Alpha Input**: `poisson probability X = 3, lambda = 5.25` (Alternatively: `(e^(-5.25) * (5.25)^3) / 3!`)
- **Wolfram Alpha Result**: Approximately $0.1292...$

Rounding to two decimal places: 0.13.

The probability that three more patients arrive between 1:30 and 2:00 PM is 0.13. This matches the provided answer.