Independent Discrete Random Variables Probability Calculations and Wolfram Alpha Documentation (Recalculated)

This document provides a detailed solution to calculating probabilities involving two independent discrete random variables, using the exact Probability Mass Functions (PMFs) as presented in your latest images.

Topics Covered in this Exercise:

* **Discrete Random Variables**
* **Probability Mass Function (PMF)**
* **Range of a Random Variable**
* **Independence of Random Variables**
* **Calculating Probabilities for Joint, Union, and Conditional Events**

Problem Statement: Let X and Y be two independent discrete random variables with the following PMFs:

$$P_X(k) = \begin{cases} \frac{1}{4} & \text{for } k = 1 \\ \frac{1}{8} & \text{for } k = 2 \text{ and } k = 3 \\ \frac{1}{2} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1 \text{ and } k = 2 \\ \frac{1}{3} & \text{for } k = 3 \\ \frac{1}{3} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the following. State all inputs as integers between 0 and 99 and state all sets $\{x_1, x_2, \ldots, x_n\}$ such that $x_1 < x_2 < \cdots < x_n$. Also, all resulting fractions must be irreducible.

Interpreting the PMFs:

Based on the explicit values given in the images:

PMF for X:

* $P_X(1) = \frac{1}{4}$

* $P_X(2) = \frac{1}{8}$

* $P_X(3) = \frac{1}{8}$

* $P_X(4) = \frac{1}{2}$

Let's verify the sum of probabilities for X:

$\frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{2} = \frac{2}{8} + \frac{1}{8} + \frac{1}{8} + \frac{4}{8} = \frac{2+1+1+4}{8} = \frac{8}{8} = 1$. (This PMF is valid.)

PMF for Y:

* $P_Y(1) = \frac{1}{6}$

* $P_Y(2) = \frac{1}{6}$

* $P_Y(3) = \frac{1}{3}$

* $P_Y(4) = \frac{1}{3}$

Let's verify the sum of probabilities for Y:

$\frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} = \frac{1}{6} + \frac{1}{6} + \frac{2}{6} + \frac{2}{6} = \frac{1+1+2+2}{6} = \frac{6}{6} = 1$. (This PMF is valid.)

Part a: Find the Range of X and Y, R_X and R_Y.

The range of a discrete random variable is the set of all possible values that it can take, for which its PMF is greater than 0.

For X: The non-zero probabilities are for $k = 1,2,3,4$.

So, $R_X = \{1,2,3,4\}$.

For Y: The non-zero probabilities are for $k = 1,2,3,4$.

So, $R_Y = \{1,2,3,4\}$.

Therefore, $R_X = R_Y = \{1,2,3,4\}$.

Part b: Find $P(X \leq 2 \text{ and } Y \leq 2)$

Since X and Y are independent random variables, the probability of their joint event is the product of their individual probabilities:

$$P(A \text{ and } B) = P(A) \times P(B)$$

$$P(X \leq 2 \text{ and } Y \leq 2) = P(X \leq 2) \times P(Y \leq 2)$$

Step 1: Find $P(X \leq 2)$

For discrete variable X, $P(X \leq 2)$ means $P_X(1) + P_X(2)$.

$$P(X \leq 2) = \frac{1}{4} + \frac{1}{8}$$

To sum these, find a common denominator (8):

$$P(X \leq 2) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$$

- **Wolfram Alpha Input**: `1/4 + 1/8` (Result: 3/8)

Step 2: Find $P(Y \leq 2)$

For discrete variable Y, $P(Y \leq 2)$ means $P_Y(1) + P_Y(2)$.

$$P(Y \leq 2) = \frac{1}{6} + \frac{1}{6}$$

$$P(Y \leq 2) = \frac{2}{6} = \frac{1}{3}$$

- **Wolfram Alpha Input**: `1/6 + 1/6` (Result: 1/3)

Step 3: Calculate $P(X \leq 2 \text{ and } Y \leq 2)$

Using the independence property:

$$P(X \leq 2 \text{ and } Y \leq 2) = P(X \leq 2) \times P(Y \leq 2)$$

$$P(X \leq 2 \text{ and } Y \leq 2) = \frac{3}{8} \times \frac{1}{3} = \frac{3}{24} = \frac{1}{8}$$

The fraction $\frac{1}{8}$ is irreducible. This matches the provided answer.

- **Wolfram Alpha Input**: `(1/4 + 1/8) * (1/6 + 1/6)` (Result: 1/8)

Part c: Find $P(X > 2 \text{ or } Y > 2)$

This involves the probability of a union of events.

A useful property for a union of events A and B is $P(A \cup B) = 1 - P(A^c \cap B^c)$.

Here, $A = (X > 2)$ and $B = (Y > 2)$.

So, $A^c = (X \leq 2)$ and $B^c = (Y \leq 2)$.

Therefore, $P(X > 2 \text{ or } Y > 2) = 1 - P(X \leq 2 \text{ and } Y \leq 2)$.

We use the value for $P(X \leq 2 \text{ and } Y \leq 2)$ from Part b, which was $\frac{1}{8}$.

$$P(X > 2 \text{ or } Y > 2) = 1 - \frac{1}{8} = \frac{8}{8} - \frac{1}{8} = \frac{7}{8}$$

The fraction $\frac{7}{8}$ is irreducible. This matches the provided answer.

- **Wolfram Alpha Input**: `1 - ((1/4 + 1/8) * (1/6 + 1/6))` (Result: 7/8)

Part d: Find $P(X > 2 \mid Y > 2)$

This is a conditional probability. Since X and Y are independent events, the conditional probability $P(A \mid B)$ simplifies to $P(A)$ if A and B are independent. So, $P(X > 2 \mid Y > 2) = P(X > 2)$.

Step 1: Find $P(X > 2)$

For discrete variable X, $P(X > 2)$ means $P_X(3) + P_X(4)$.

$$P(X > 2) = \frac{1}{8} + \frac{1}{2}$$

To sum these, find a common denominator (8):

$$P(X > 2) = \frac{1}{8} + \frac{4}{8} = \frac{5}{8}$$

The fraction $\frac{5}{8}$ is irreducible. This matches the provided answer.

Step 2: Calculate $P(X > 2 \mid Y > 2)$

$$P(X > 2 \mid Y > 2) = P(X > 2) = \frac{5}{8}$$

- **Wolfram Alpha Input**: `probability X > 2 where P(X=1)=1/4, P(X=2)=1/8, P(X=3)=1/8, P(X=4)=1/2` (Result: 5/8)
- **Wolfram Alpha Input for conditional**: `conditional probability X > 2 given Y > 2 where X and Y are independent and P(X=1)=1/4, P(X=2)=1/8, P(X=3)=1/8, P(X=4)=1/2 and P(Y=1)=1/6, P(Y=2)=1/6, P(Y=3)=1/3, P(Y=4)=1/3` (Result: 5/8)

Part e: Find $P(X < Y)$

To find $P(X < Y)$, we need to sum the probabilities of all pairs (x, y) from the ranges of X and Y such that $x < y$.

Since X and Y are independent, $P(X = x, Y = y) = P_X(x) \times P_Y(y)$.

The possible pairs (x, y) where $x < y$ are:

* If $X = 1$: $(1,2), (1,3), (1,4)$

* If $X = 2$: $(2,3), (2,4)$

* If $X = 3$: $(3,4)$

* If $X = 4$: (no y values are greater than 4 in R_Y)

Let's calculate the probability for each pair and sum them:

1. $P(X = 1, Y = 2) = P_X(1) \times P_Y(2) = \frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$
2. $P(X = 1, Y = 3) = P_X(1) \times P_Y(3) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$
3. $P(X = 1, Y = 4) = P_X(1) \times P_Y(4) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$
4. $P(X = 2, Y = 3) = P_X(2) \times P_Y(3) = \frac{1}{8} \times \frac{1}{3} = \frac{1}{24}$
5. $P(X = 2, Y = 4) = P_X(2) \times P_Y(4) = \frac{1}{8} \times \frac{1}{3} = \frac{1}{24}$
6. $P(X = 3, Y = 4) = P_X(3) \times P_Y(4) = \frac{1}{8} \times \frac{1}{3} = \frac{1}{24}$

Now, sum these probabilities:

$$P(X < Y) = \frac{1}{24} + \frac{1}{12} + \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24}$$

Find a common denominator, which is 24:

$$P(X < Y) = \frac{1}{24} + \frac{2}{24} + \frac{2}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24}$$

$$P(X < Y) = \frac{1 + 2 + 2 + 1 + 1 + 1}{24} = \frac{8}{24}$$

To simplify the fraction, divide both by 8:

$$P(X < Y) = \frac{1}{3}$$

The fraction $\frac{1}{3}$ is irreducible. This matches the provided answer.

- **Wolfram Alpha Input**: `probability X < Y where X is discrete with P(X=1)=1/4, P(X=2)=1/8, P(X=3)=1/8, P(X=4)=1/2 and Y is discrete with P(Y=1)=1/6, P(Y=2)=1/6, P(Y=3)=1/3, P(Y=4)=1/3` (Result: 1/3)