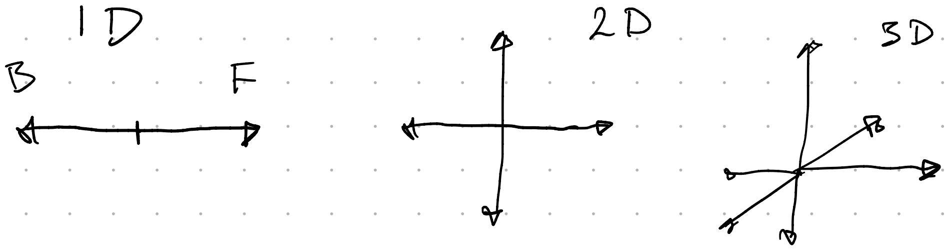
Stochastic Processes

Motivating Examples:

"A drunk man will eventually find his way home, but a drunk bird may get forever lost"

— Shizuo Kakutani



George Pólya: Do random walks eventually return home?

1D → P(Home) = 1

2D → P(Home) = 1

3D → P(Home) ≈ 0.3405

location at step n →
$$S_n = \sum_{i=1}^{n} X_i$$

$$P(X_i = e_j) = P(X_i = -e_j) = \frac{1}{2d} \quad , j = 1, 2, \dots d$$

For each dimension d, there are 2·d directions to move in, and each have equal prob. of being selected:

1D → 2·1 = 2 direction → 1/2

2D → 2·2 = 4 directions → 1/4

3D → 2·3 = 6 directions → 1/6

So we are looking for:

$$P(\exists n \in \mathbb{N} \text{ s.t. } S_n = 0) = P(S_n = 0)$$ Origin

If $P(S_n = 0) = 1 \to$ Recurrent

else Transient.

$$P(S_n = 0) = 1 \quad \text{iff.} \quad E(\#\text{ returns}) = \infty$$

Turns out that

$$P(S_{2n} = 0) \propto \frac{1}{\sqrt{n}}$$

even cases!

2D:
so you will be $\pm\sqrt{n}$ from origin.

Summary

A random proces is recurrent if we are guarenteed to return to origin and is transient if the probability of not returning > 0.

Motivating example: High, skilled, unskilled

- 80% of daughters of highly educated females are highly educated, 10% are skilled, 10% are unskilled
- 20% of daughters of skilled females are highly educated, 60% skilled, 20% unskilled
- 25% of daughters of unskilled females are highly educated, 25% skilled, 50% unskilled.

Question: What is the probability that a grand daughter of an unskilled female will be highly educated

A Stochastic/Random Process is a collection of R.V's indexed by time (or space)

Discrete: $X_0, X_1, X_2, \dots X_n$
Continuous: $\{X_t\}_{t \geq 0}$

We will be looking at two types:

1) Poisson Process
2) Discrete-Time Markov Chains

Poisson Process

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0; \infty[\}$ is called a Poisson Process with rate λ <u>if</u>:

1. $N(0) = 0$
2. $N(t)$ has independent increments
3. The number of arrivals in any interval of length $\tau > 0$ has Poisson$(\lambda \cdot \tau)$:

$$P(X=x) = \frac{e^{-\lambda\tau} \cdot (\lambda\tau)^x}{x!}$$

Example: The Number of Customers arriving at a Store can be modelled by a Poisson Process with $\lambda = 10$ customers per hour.

a) Find $P(X=2)$ between 10.00 and 10.20

$\lambda = 10$, $\tau = \frac{1}{3}$, so Poisson$\left(\frac{10}{3}\right)$

$$P(X=2) = \frac{e^{-\frac{10}{3}} \cdot \left(\frac{10}{3}\right)^2}{2!} \approx 0.2$$

b) Find $P(X=3)$ in $I_1 = [10{:}00; 10{:}20]$, $T_1 = 1/3$ and

$P(X=7)$ in $I_2 = [10{:}20; 11{:}00]$, $T_2 = 2/3$

$$P(3 \text{ in } I_1 \text{ and } 7 \text{ in } I_2) = P(3 \text{ in } I_1) \cdot P(7 \text{ in } I_2)$$

$$= \frac{e^{-\frac{10}{3}} \cdot \left(\frac{10}{3}\right)^3}{3!} \cdot \frac{e^{-\frac{20}{3}} \left(\frac{20}{3}\right)^7}{7!} \approx 0.0325$$

Interarrival Times:

Given $N(t)$, then arrival times $X_1, X_2 \dots X_n$ are independent $X_i \sim \text{Exponential}(\lambda)$

Example: Let $\lambda = 2$

a) Find $P(X_1 > 1/2) = 1 - (1 - e^{-\lambda \cdot t}) = e^{-2 \cdot 0.5} \approx 0.37$

b) Find $P(X_1 > 3 \mid X_1 > 1) = P(X_1 > 2)$ (memoryless)

$$= e^{-2 \cdot 2} \approx 0.0183$$

c) Given third arrival occurred at $t=2$, find the probability that fourth occurs at $t=4$:

$$P(X_4 > 4 \mid X_1 + X_2 + X_3 = 2)$$

$$= P(X_4 > 2) \approx 0.0183$$

d) Let T be the first arrival after $t=10$. What is

$$T = 10 + X \quad , X \sim Exp(2)$$

$$E(T) = E(10 + X) = 10 + E(X) = 10 + \frac{1}{2} = \frac{21}{2} = \underline{10.5}$$

$$Var(T) = \frac{1}{4}$$

Merging Independent Poisson Processes

Let $N_1(t), N_2(t) \dots N_m(t)$ be m independent Poiss. Proc. with $\lambda_1, \lambda_2 \dots \lambda_m$:

$$N(t) = N_1(t) + N_2(t) + \dots N_m(t) \quad , t \in [0; \infty[$$

Then $N(t)$ is a Poiss. Proc. with rate $\lambda_1 + \lambda_2 + \dots + \lambda_m$

Example:

Let $N_1(t)$ and $N_2(t)$ be two independent Poiss. Proc. with $\lambda_1 = 1$ and $\lambda_2 = 2$.

a. Find $N(1) = 2$ and $N(2) = 5$:

$$P(N(1) = 2 \ , N(2) = 5) = \frac{e^{-3} \cdot 3^2}{2!} \cdot \frac{e^{-3} \cdot 3^5}{5!} \approx 0.023$$

b. $$P(N_1(1) = 1 \mid N(1) = 2) = \frac{P(N_1(1) = 1 \ , N(1) = 2)}{P(N(1) = 2)}$$

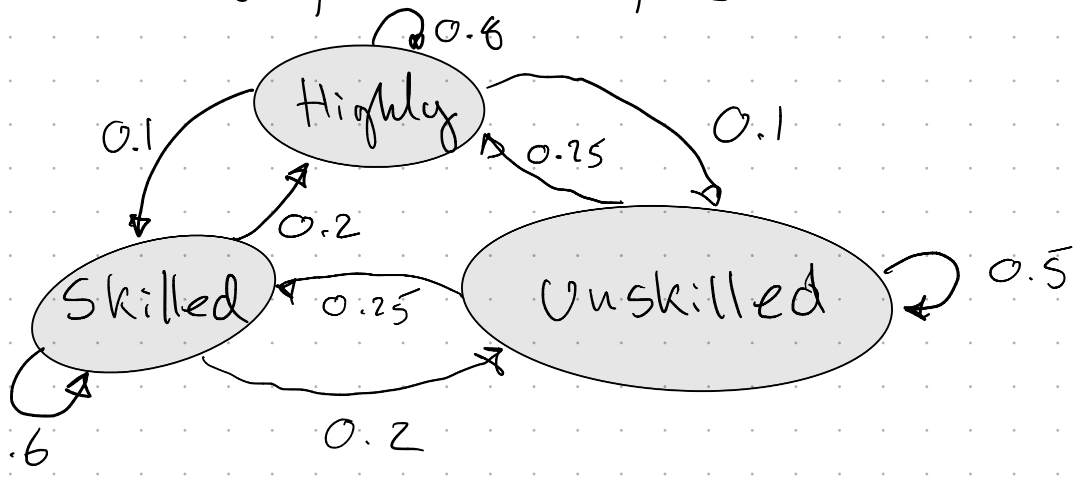
$$= \frac{P(N_1(1) = 1) \cdot P(N_2(1) = 1)}{P(N(1) = 2)} = \frac{e^{-1} \cdot 1^1 \cdot e^{-2} \cdot 2^1}{\frac{e^{-3} \cdot 3^2}{2!}}$$

$$= 2/(9/2) = 4/9$$

Markov Chains:

Recall the daughters:

- → 80% of daughters of highly educated females are highly educated, 10% are skilled, 10% are unskilled
- → 20% of daughters of skilled females are highly educated, 60% skilled, 20% unskilled
- → 25% of daughters of unskilled females are highly educated, 25% skilled, 50% unskilled.



$X_n = i \leftarrow$ state

$\nwarrow$ times

$P(X_n = i) = p$

Let X_0: Grand person

Let X_1: Daughter

Let X_2: Grand Daughter

What is the probability that a grand daughter of an unskilled female will be highly educated

$\hookrightarrow P(X_2 = \text{Highly educated} \mid X_0 = \text{Unskilled})$

Two important concepts:

1) States
2) Transition Probabilities

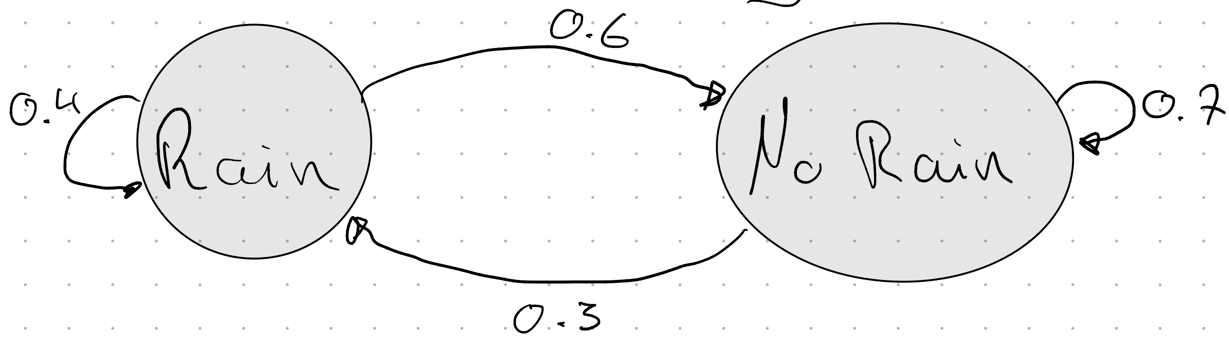
Example:

Tomorrow

Rain Today → Rain 40%, Not 60%

No Rain Today → Rain 30%, Not 70%

Transition Prob.



Formal Definition:

Discrete-Time Markov Chains

Consider the random process $\{X_n, n = 0, 1, 2, \cdots\}$, where $R_{X_i} = S \subset \{0, 1, 2, \cdots\}$. We say that this process is a **Markov chain** if

$$\left.\begin{aligned} &P(X_{m+1} = j | X_m = i, X_{m-1} = i_{m-1}, \cdots, X_0 = i_0) \\ &\qquad = P(X_{m+1} = j | X_m = i), \end{aligned}\right\} \text{Memoryless}$$

for all $m, j, i, i_0, i_1, \cdots i_{m-1}$. If the number of states is finite, e.g., $S = \{0, 1, 2, \cdots, r\}$, we call it a **finite** Markov chain.

Translated:

P(Future | Present, Past) = P(Future | Present)

We assume that the transition probabilities do not depend on time:

$$\left.\begin{aligned} p_{ij} &= P(X_{m+1} = j \mid X_m = i) \\ &= P(X_1 = j \mid X_0 = i) \\ &= P(X_2 = j \mid X_1 = i) \\ &= P(X_3 = j \mid X_2 = i) \ \ldots \end{aligned}\right\} \text{Time Homogenous}$$

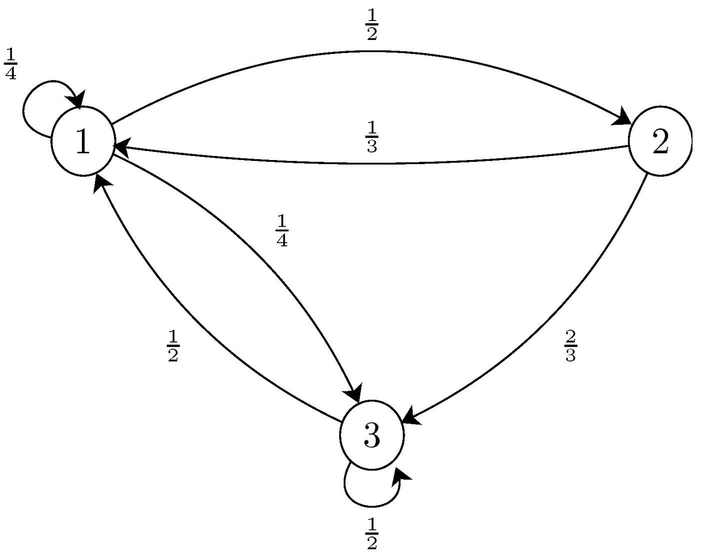
So, if the process is in state i, it will transition to state j with probability p_{ij}

Summary: Two properties

1) Memoryless
2) Time homogenous

We can describe a Markov chain in two ways

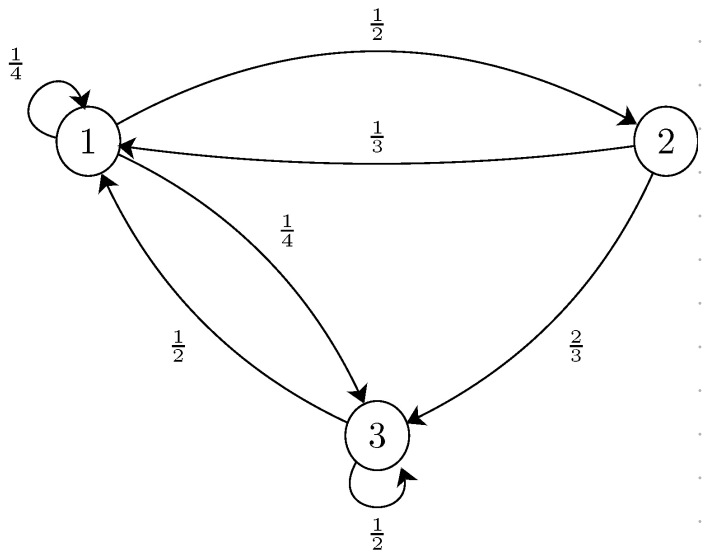
State Transition Diagram



State transition Matrix

$$P = \begin{bmatrix} p_{11} + p_{12} + \cdots + p_{1r} = 1 \\ p_{21} + p_{22} + \cdots + p_{2r} = 1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ p_{r1} + p_{r2} + \cdots + p_{rr} \end{bmatrix} 1$$

Example



$$P = \begin{array}{cc} & \text{TO} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{FROM} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} \end{array}$$

a. Find $P(X_4 = 3 | X_3 = 2)$.
b. Find $P(X_3 = 1 | X_2 = 1)$.
c. If we know $P(X_0 = 1) = \frac{1}{3}$, find $P(X_0 = 1, X_1 = 2)$.
d. If we know $P(X_0 = 1) = \frac{1}{3}$, find $P(X_0 = 1, X_1 = 2, X_2 = 3)$.

a. $P(X_4 = 3 \mid X_3 = 2) = P_{23} = 2/3$

b. $P(X_3 = 1 \mid X_2 = 1) = P_{11} = 1/4$

c. $P(X_0 = 1, X_1 = 2) = P(X_0 = 1) \cdot P(X_1 = 2 \mid X_0 = 1)$
$= 1/3 \cdot 1/2 = 1/6$

d. $P(X_0 = 1, X_1 = 2, X_2 = 3) = P(X_0 = 1) \cdot P(X_1 = 2 \mid X_0 = 1) \cdot P(X_2 = 3 \mid X_1 = 2, \cancel{X_0 = 1})$
$= P(X_0 = 1) \cdot P(X_1 = 2 \mid X_0 = 1) \cdot P(X_2 = 3 \mid X_1 = 2)$
$= 1/3 \cdot 1/2 \cdot 2/3 = 1/9$

Sometimes X_0 is used to denote the initial state of system.

Let X_0 be our initial state:

$$\pi^{(0)} = [P(X_0=1) \quad P(X_0=2) \quad \dots P(X_0=r)]$$

This row vector is the probability distribution of X_0. We can obtain the row vector:

$$\pi^{(n+1)} = \pi^{(n)} \cdot P$$
$$\pi^{(n)} = \pi^{(0)} \cdot P^n$$

Example

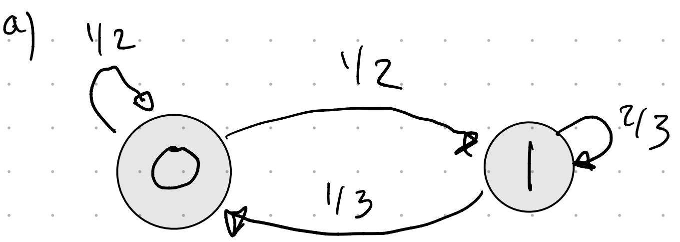
Consider a system that can be in one of two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \ \ 1 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}.$$

Suppose that the system is in state 0 at time $n = 0$, i.e., $X_0 = 0$.

a. Draw the state transition diagram.
b. Find the probability that the system is in state 1 at time $n = 3$.

a)



b)

$$\pi^{(0)} = [P(X_0=0) \quad P(X_0=1)]$$
$$= [1 \quad 0]$$
$$\pi^{(3)} = \pi^{(0)} \cdot P^3$$
$$= [1 \quad 0] \cdot \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}^3$$
$$= \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}, \text{ so}$$

$$P(X_3=1) = \frac{43}{72}$$

n-step transition matrix:

We can obtain an n-step transition matrix by raising the state transition matrix to the n'th power:

$$P^{(n)} = P^{n} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix}^{n}$$

One step:

$P(X_2 = 3 \mid X_1 = 2) = p_{23}^{(1)}$

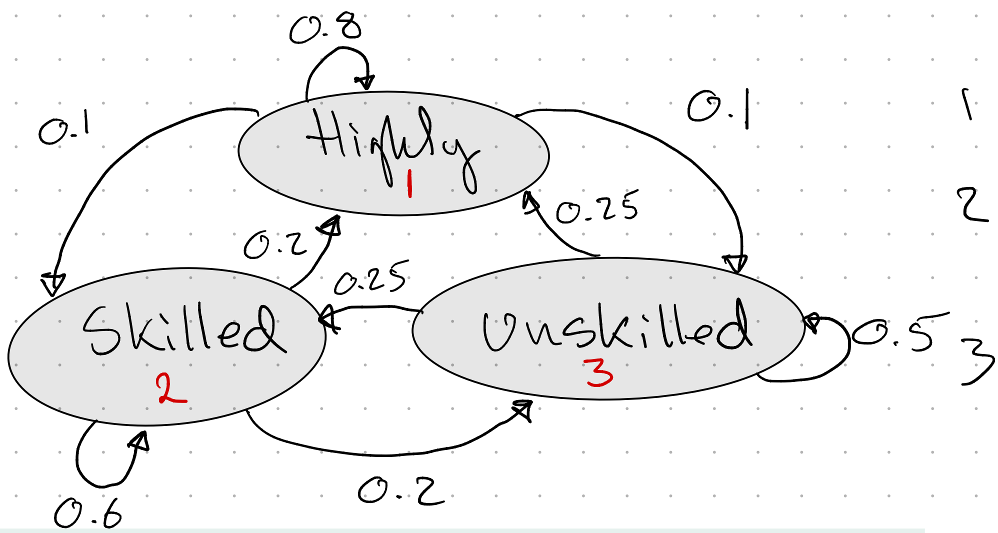
$P(X_2 = 2 \mid X_1 = 0) = p_{02}^{(1)}$

n-step

$P(X_2 = 3 \mid X_0 = 2) = p_{23}^{(2)}$

$P(X_6 = 4 \mid X_1 = 1) = p_{14}^{(5)}$.

Example: Gran daughters



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.8 & 0.1 & 0.1 \\ 2 & 0.2 & 0.6 & 0.2 \\ 3 & 0.25 & 0.25 & 0.5 \end{array}$$

What is the probability that a grand daughter of an unskilled female will be highly educated

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.685 & 0.165 & 0.15 \\ 2 & 0.33 & 0.43 & 0.24 \\ 3 & 0.375 & 0.3 & 0.325 \end{array}$$

$\rightarrow P(X_2 = \text{Highly skilled} \mid X_0 = \text{Unskilled}) = 0.375$