Poisson Process: Football Match Scoring Probabilities Solution and Wolfram Alpha Documentation

---

This document provides a detailed solution to a probability problem involving goal scoring in a football match, modeled using independent Poisson processes. It covers calculating probabilities for specific outcomes, including a 0-0 draw, at least two goals, and a specific final score. It includes explanations of the steps and how to use Wolfram Alpha (or similar symbolic computation tools) to verify or perform the calculations.

Topics Covered in this Exercise:

- **Poisson Process**
- **Poisson Distribution**
- **Parameter λ (rate per interval)**
- **Independence of Poisson Processes**
- **Sum of Independent Poisson Processes**
- **Probability Mass Function (PMF) of Poisson Distribution**
- **Complement Rule of Probability**

Problem Statement:
Two Premier League teams, A and B, are to play a match. We know that the number of goals scored by Team A is modeled by a Poisson process $N_A(t)$ with rate $\lambda_1 = 0.02$ goals per minute, and the number of goals scored by Team B is modeled by a Poisson process $N_B(t)$ with rate $\lambda_2 = 0.03$ goals per minute. The two processes are assumed to be independent. Let $N(t)$ be the total number of goals in the game up to and including time t. Assume the game lasts for 90 minutes with no overtime. State all your answers as decimal values correctly rounded off to two decimal precision for all problems below. Remember to use '.' as the decimal separator.

Introduction to Poisson Distribution:
A Poisson distribution describes the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event.
The PMF of a Poisson distribution is given by:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

where:

- X is the Poisson random variable representing the number of events.
- k is the actual number of events.
- λ is the average number of events per interval (the mean).

The duration of the game is $t = 90$ minutes. We need to calculate the mean number of goals for each team over the entire 90-minute game.

- **Mean goals for Team A (denoted λ_A)**:

$$\lambda_A = \lambda_1 \times t = 0.02 \text{ goals/min} \times 90 \text{ min} = 1.8 \text{ goals}$$

- **Mean goals for Team B (denoted λ_B)**:

$$\lambda_B = \lambda_2 \times t = 0.03 \text{ goals/min} \times 90 \text{ min} = 2.7 \text{ goals}$$

Let X be the number of goals scored by Team A ($X \sim \text{Poisson}(\lambda_A = 1.8)$).
Let Y be the number of goals scored by Team B ($Y \sim \text{Poisson}(\lambda_B = 2.7)$).
Since the processes are independent, X and Y are independent random variables.

Part a: Find the probability that no goals are scored, i.e., the game ends with a 0-0 draw.

This means Team A scores 0 goals AND Team B scores 0 goals.
Since X and Y are independent:

$$P(X = 0 \text{ and } Y = 0) = P(X = 0) \times P(Y = 0)$$

Step 1: Calculate $P(X = 0)$.

Using the Poisson PMF for X with $\lambda_A = 1.8$ and $k = 0$:

$$P(X = 0) = \frac{e^{-\lambda_A}\lambda_A^0}{0!} = \frac{e^{-1.8}(1.8)^0}{1} = e^{-1.8}$$

$$e^{-1.8} \approx 0.165298...$$

- **Wolfram Alpha Input**: `e^(-1.8)`

- **Wolfram Alpha Result**: `0.165298...`

Step 2: Calculate $P(Y = 0)$.

Using the Poisson PMF for Y with $\lambda_B = 2.7$ and $k = 0$:

$$P(Y = 0) = \frac{e^{-\lambda_B}\lambda_B^0}{0!} = \frac{e^{-2.7}(2.7)^0}{1} = e^{-2.7}$$

$$e^{-2.7} \approx 0.067205...$$

- **Wolfram Alpha Input**: `e^(-2.7)`
- **Wolfram Alpha Result**: `0.067205...`

Step 3: Calculate the joint probability $P(X = 0 \text{ and } Y = 0)$.

$$P(X = 0 \text{ and } Y = 0) = e^{-1.8} \times e^{-2.7} = e^{-(1.8+2.7)} = e^{-4.5}$$

$$e^{-4.5} \approx 0.011108...$$

Rounding to two decimal places: 0.01.

The probability that no goals are scored is 0.01.

- **Wolfram Alpha Input**: `e^(-1.8) * e^(-2.7)` (or `e^(-4.5)`)
- **Wolfram Alpha Result**: `0.011108...`
- **Wolfram Alpha Input**: `round 0.011108 to 2 decimal places` (Result: 0.01)

Part b: Find the probability that at least two goals are scored in the game.

Let N_{total} be the total number of goals scored in the game. Since X and Y are independent Poisson random variables, their sum $N_{total} = X + Y$ also follows a Poisson distribution with a mean equal to the sum of their individual means.

$$\lambda_{total} = \lambda_A + \lambda_B = 1.8 + 2.7 = 4.5 \text{ goals}$$

So, $N_{total} \sim \text{Poisson}(\lambda_{total} = 4.5)$.

We want to find $P(N_{total} \geq 2)$.
It's easier to calculate using the complement rule:

$$P(N_{total} \geq 2) = 1 - P(N_{total} < 2)$$

$$P(N_{total} < 2) = P(N_{total} = 0) + P(N_{total} = 1)$$

Step 1: Calculate $P(N_{total} = 0)$.

Using the Poisson PMF for N_{total} with $\lambda_{total} = 4.5$ and $k = 0$:

$$P(N_{total} = 0) = \frac{e^{-4.5}(4.5)^0}{0!} = e^{-4.5}$$

$$e^{-4.5} \approx 0.011108...$$

- **Wolfram Alpha Input**: `e^(-4.5)`
- **Wolfram Alpha Result**: `0.011108...`

Step 2: Calculate $P(N_{total} = 1)$.

Using the Poisson PMF for N_{total} with $\lambda_{total} = 4.5$ and $k = 1$:

$$P(N_{total} = 1) = \frac{e^{-4.5}(4.5)^1}{1!} = 4.5e^{-4.5}$$

$$4.5e^{-4.5} \approx 4.5 \times 0.011108 = 0.049986...$$

- **Wolfram Alpha Input**: `4.5 * e^(-4.5)`
- **Wolfram Alpha Result**: `0.049986...`

Step 3: Calculate $P(N_{total} < 2)$.

$$P(N_{total} < 2) = P(N_{total} = 0) + P(N_{total} = 1) \approx 0.011108 + 0.049986 = 0.061094$$

Step 4: Calculate $P(N_{total} \geq 2)$.

$$P(N_{total} \geq 2) = 1 - P(N_{total} < 2) \approx 1 - 0.061094 = 0.938906$$

Rounding to two decimal places: 0.94.

The probability that at least two goals are scored in the game is 0.94.

- **Wolfram Alpha Input (Direct Poisson CDF)**: `poisson probability X >= 2, lambda = 4.5`
- **Wolfram Alpha Result**: `0.938906...`
- **Wolfram Alpha Input**: `round 0.938906 to 2 decimal places` (Result: 0.94)

Part c: Find the probability of the final score being: Team A: 1, Team B: 2.

This asks for $P(X = 1 \text{ and } Y = 2)$.
Since X and Y are independent:

$$P(X = 1 \text{ and } Y = 2) = P(X = 1) \times P(Y = 2)$$

Step 1: Calculate $P(X = 1)$.

Using the Poisson PMF for X with $\lambda_A = 1.8$ and $k = 1$:

$$P(X = 1) = \frac{e^{-1.8}(1.8)^1}{1!} = 1.8e^{-1.8}$$

$$1.8e^{-1.8} \approx 1.8 \times 0.165298 = 0.2975364...$$

- **Wolfram Alpha Input**: `1.8 * e^(-1.8)`
- **Wolfram Alpha Result**: `0.297536...`

Step 2: Calculate $P(Y = 2)$.

Using the Poisson PMF for Y with $\lambda_B = 2.7$ and $k = 2$:

$$P(Y=2) = \frac{e^{-2.7}(2.7)^2}{2!} = \frac{e^{-2.7} \times 7.29}{2} = e^{-2.7} \times 3.645$$

$$3.645e^{-2.7} \approx 3.645 \times 0.067205 = 0.245058...$$

- **Wolfram Alpha Input**: `(e^(-2.7) * (2.7)^2) / 2`
- **Wolfram Alpha Result**: `0.245058...`

Step 3: Calculate the joint probability $P(X = 1 \text{ and } Y = 2)$.

$$P(X = 1 \text{ and } Y = 2) = P(X = 1) \times P(Y = 2) \approx 0.297536 \times 0.245058 = 0.072917...$$

Rounding to two decimal places: 0.07.

The probability of the final score being Team A: 1, Team B: 2 is 0.07.

- **Wolfram Alpha Input**: `(1.8 * e^(-1.8)) * ((e^(-2.7) * (2.7)^2) / 2)`
- **Wolfram Alpha Result**: `0.072917...`
- **Wolfram Alpha Input**: `round 0.072917 to 2 decimal places` (Result: 0.07)