Independent Discrete Stochastic Variables: Probability, Variance, and Covariance Solution and Wolfram Alpha Documentation

This document provides a detailed solution to problems involving two independent discrete stochastic variables, including calculating probabilities, variance, and covariance of linear combinations. It covers the properties of independent random variables and the linearity of expectation and variance. It includes explanations of the steps and how to use Wolfram Alpha (or similar symbolic computation tools) to verify or perform the calculations.

Topics Covered in this Exercise:

- **Discrete Random Variables**
- **Probability Mass Function (PMF)**
- **Probability of Events (Summing PMF values)**
- **Expected Value ($E[X]$ and $E[Y]$)**
- **Variance ($Var[X]$ and $Var[Y]$)**
- **Independence of Random Variables**
- **Properties of Variance of Linear Combinations ($Var(aX + bY)$)**
- **Covariance ($Cov[X, Y]$ and $Cov(aX + bY, cZ + dW)$)**

Problem Statement:
Let X and Y denote two independent stochastic variables. Assume the PMF of X is

$$f_X(x) = \begin{cases} 0.2 & \text{if } x = 0 \\ 0.3 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2 \\ 0 & \text{else} \end{cases}$$

and that the PMF of Y is

$$f_Y(y) = \begin{cases} 0.3 & \text{if } y = 0 \\ 0.4 & \text{if } y = 1 \\ 0.3 & \text{if } y = 2 \\ 0 & \text{else} \end{cases}$$

For all questions in this assignment, state your answers as integers between 0 and 99 such that you supply two decimal precision.

Part a: Find the following values.

Step 1: Find $P(X > 0)$.

The probability $P(X > 0)$ is the sum of probabilities for values of X greater than 0. For X, the possible values are $\{0, 1, 2\}$.

$$P(X > 0) = P(X = 1) + P(X = 2)$$

$$P(X > 0) = 0.3 + 0.5 = 0.8$$

The probability $P(X > 0)$ is 0.80.

- **Wolfram Alpha Input**: `probability X > 0 where P(X=0)=0.2, P(X=1)=0.3, P(X=2)=0.5` (Result: 0.8)

Step 2: Find $Var(X)$.

The variance of a discrete random variable is $Var(X) = E[X^2] - (E[X])^2$.

First, calculate $E[X]$:

$$E[X] = (0 \cdot P(X = 0)) + (1 \cdot P(X = 1)) + (2 \cdot P(X = 2))$$

$$E[X] = (0 \cdot 0.2) + (1 \cdot 0.3) + (2 \cdot 0.5)$$

$$E[X] = 0 + 0.3 + 1.0 = 1.3$$

Next, calculate $E[X^2]$:

$$E[X^2] = (0^2 \cdot P(X = 0)) + (1^2 \cdot P(X = 1)) + (2^2 \cdot P(X = 2))$$

$$E[X^2] = (0 \cdot 0.2) + (1 \cdot 0.3) + (4 \cdot 0.5)$$

$$E[X^2] = 0 + 0.3 + 2.0 = 2.3$$

Now, calculate $Var(X)$:

$$Var(X) = E[X^2] - (E[X])^2$$

$$Var(X) = 2.3 - (1.3)^2$$

$$Var(X) = 2.3 - 1.69 = 0.61$$

The variance of X is 0.61.

- **Wolfram Alpha Input**: `variance of discrete distribution {{0, 0.2}, {1, 0.3}, {2, 0.5}}` (Result: 0.61)

Part b: Find the following probabilities.

Step 1: Find $P(X < 2, Y > 1)$.

This is the probability of a joint event. Since X and Y are independent,
$P(A \text{ and } B) = P(A) \times P(B)$.
So, $P(X < 2, Y > 1) = P(X < 2) \times P(Y > 1)$.

First, find $P(X < 2)$:
For X, the values less than 2 are $\{0, 1\}$.

$$P(X < 2) = P(X = 0) + P(X = 1) = 0.2 + 0.3 = 0.5$$

Next, find $P(Y > 1)$:
For Y, the values greater than 1 are $\{2\}$.

$$P(Y > 1) = P(Y = 2) = 0.3$$

Now, multiply these probabilities:

$$P(X < 2, Y > 1) = 0.5 \times 0.3 = 0.15$$

The probability $P(X < 2, Y > 1)$ is 0.15.

- **Wolfram Alpha Input**: `probability X < 2 and Y > 1 where X, Y are independent and P(X=0)=0.2, P(X=1)=0.3, P(X=2)=0.5 and P(Y=0)=0.3, P(Y=1)=0.4, P(Y=2)=0.3` (Result: 0.15)

Step 2: Find $P(\{X < 2\} \cup \{Y < 2\})$.

This is the probability of the union of two events: $A = \{X < 2\}$ and $B = \{Y < 2\}$.
Using the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
Since X and Y are independent, $P(A \cap B) = P(A) \times P(B)$.
So, $P(\{X < 2\} \cup \{Y < 2\}) = P(X < 2) + P(Y < 2) - P(X < 2)P(Y < 2)$.

First, find $P(X < 2)$: (already calculated)

$$P(X < 2) = 0.5$$

Next, find $P(Y < 2)$:
For Y, the values less than 2 are $\{0, 1\}$.

$$P(Y < 2) = P(Y = 0) + P(Y = 1) = 0.3 + 0.4 = 0.7$$

Now, calculate $P(X < 2 \text{ and } Y < 2)$:

$$P(X < 2 \text{ and } Y < 2) = P(X < 2) \times P(Y < 2) = 0.5 \times 0.7 = 0.35$$

Finally, calculate $P(\{X < 2\} \cup \{Y < 2\})$:

$$P(\{X < 2\} \cup \{Y < 2\}) = 0.5 + 0.7 - 0.35$$

$$P(\{X < 2\} \cup \{Y < 2\}) = 1.2 - 0.35 = 0.85$$

The probability $P(\{X < 2\} \cup \{Y < 2\})$ is 0.85.

- **Wolfram Alpha Input**: `probability (X < 2) or (Y < 2) where X, Y are independent and P(X=0)=0.2, P(X=1)=0.3, P(X=2)=0.5 and P(Y=0)=0.3, P(Y=1)=0.4, P(Y=2)=0.3` (Result: 0.85)

Part c: Find the value below.

Step 1: Find $Cov(2X - 5Y, 7X + 4Y + 1)$.

Using the properties of covariance:
$Cov(aX + bY, cZ + dW) = ac \cdot Cov(X, Z) + ad \cdot Cov(X, W) + bc \cdot Cov(Y, Z) + bd \cdot Cov(Y, W)$.
Also, $Cov(U, \text{constant}) = 0$. So the $+1$ term does not affect covariance.
And $Cov(X, Y) = 0$ if X and Y are independent.
Also, $Cov(X, X) = Var(X)$.

Let $A = 2X - 5Y$ and $B = 7X + 4Y + 1$.

$$Cov(A, B) = Cov(2X - 5Y, 7X + 4Y)$$

$$= Cov(2X, 7X) + Cov(2X, 4Y) - Cov(5Y, 7X) - Cov(5Y, 4Y)$$

$$= 2 \cdot 7Cov(X, X) + 2 \cdot 4Cov(X, Y) - 5 \cdot 7Cov(Y, X) - 5 \cdot 4Cov(Y, Y)$$

$$= 14Var(X) + 8Cov(X, Y) - 35Cov(Y, X) - 20Var(Y)$$

Since X and Y are independent, $Cov(X, Y) = Cov(Y, X) = 0$.

$$Cov(2X - 5Y, 7X + 4Y + 1) = 14Var(X) - 20Var(Y)$$

Now, we need $Var(Y)$.
First, calculate $E[Y]$:

$$E[Y] = (0 \cdot P(Y = 0)) + (1 \cdot P(Y = 1)) + (2 \cdot P(Y = 2))$$

$$E[Y] = (0 \cdot 0.3) + (1 \cdot 0.4) + (2 \cdot 0.3)$$

$$E[Y] = 0 + 0.4 + 0.6 = 1.0$$

Next, calculate $E[Y^2]$:

$$E[Y^2] = (0^2 \cdot P(Y = 0)) + (1^2 \cdot P(Y = 1)) + (2^2 \cdot P(Y = 2))$$

$$E[Y^2] = (0 \cdot 0.3) + (1 \cdot 0.4) + (4 \cdot 0.3)$$

$$E[Y^2] = 0 + 0.4 + 1.2 = 1.6$$

Now, calculate $Var(Y)$:

$$Var(Y) = E[Y^2] - (E[Y])^2$$

$$Var(Y) = 1.6 - (1.0)^2$$

$$Var(Y) = 1.6 - 1.0 = 0.6$$

Finally, calculate $Cov(2X - 5Y, 7X + 4Y + 1)$:

$$Cov(2X - 5Y, 7X + 4Y + 1) = 14Var(X) - 20Var(Y)$$

$$= 14(0.61) - 20(0.6)$$

$$= 8.54 - 12.0 = -3.46$$

The covariance is -3.46.

- **Wolfram Alpha Input**: `covariance of 2X - 5Y and 7X + 4Y + 1 where X, Y are independent and P(X=0)=0.2, P(X=1)=0.3, P(X=2)=0.5 and P(Y=0)=0.3, P(Y=1)=0.4, P(Y=2)=0.3` (Result: -3.46)