

Functions of Continuous R.V.s: **Difficult**

If X is a CRV and $Y = g(X)$, then Y is also a R.V.

Example

let X be a Uniform(0,1) R.V. and let $Y = e^X$

a) Find CDF of Y

$$F_X(x) = \frac{x-a}{b-a} = \frac{x-0}{1-0} = x, \quad f(x) = 1$$

$$R_X = [0; 1], \quad R_Y = [1; e]$$

$$\begin{aligned} F(Y) &= P(Y \leq y) \\ &= P(e^X \leq y) \\ &= P(X \leq \ln y) \end{aligned}$$

So

$$F_Y = \begin{cases} 0 & y < 1 \\ \ln y & 1 \leq y < e \\ 1 & y \geq e \end{cases}$$

b) Find pdf of Y :

$$f_Y(y) = F'_Y(y) = \frac{1}{y} \quad \text{for } 1 \leq y \leq e, \text{ else } 0$$

c) Find $E[Y]$:

$$E[Y] = \int_1^e y \cdot \frac{1}{y} dy = y \Big|_1^e = \underline{\underline{e-1}}$$

Using LOTUS:

$$\begin{aligned} E[Y] &= E[e^X] = \int_0^1 e^x \cdot f_X(x) dx = e^x \Big|_0^1 \\ &= e^1 - e^0 = \underline{\underline{e-1}} \end{aligned}$$

Example

$$\text{let } f_X(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $Y = \frac{1}{X}$. Find PDF.

Method 1: like above we first find CDF

Note $R_Y = [1, \infty[$

$$\begin{aligned} F(Y) &= P(Y \leq y) \\ &= P\left(\frac{1}{X} \leq y\right) \\ &= P\left(X \leq \frac{1}{y}\right) \\ &= F_X\left(\frac{1}{y}\right) = \int_0^{\frac{1}{y}} 4x^3 dx = x^4 \Big|_0^{\frac{1}{y}} \\ &= \frac{1}{y^4}, \text{ so} \end{aligned}$$

$$f_Y(y) = (y^{-4})' = -4 \cdot y^{-5} \quad \text{Must be positive}$$

$$f_Y(y) = \begin{cases} \frac{4}{y^5} & y \geq 1 \\ 0 & \text{else} \end{cases}$$

Method 2: The Method of Transformation

let X be a CRV and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotonic and differentiable function

$$(x^{-1})' = -x^{-2}$$

$$f_Y(y) = \begin{cases} \frac{f_X(x_1)}{|g'(x_1)|} & \text{where } g(x_1) = y \Rightarrow x_1 = g^{-1}(y) = \frac{1}{y} \\ 0 & \text{if } g(x) = y \text{ has no solution} \end{cases}$$

$$f_Y(y) = \frac{4x_1^3}{|-\frac{1}{x_1^2}|} = 4x_1^5 = 4\left(\frac{1}{y}\right)^5 = \frac{4}{y^5}$$

Exponential Distribution:

A CRV X is said to be exponentially distributed with $\lambda > 0$, shown as $X \sim \text{Exponential}(\lambda)$, if its PDF is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & \end{cases}$$

Is often used to model time between events. let's find CDF:

$$\begin{aligned} \int_0^x \lambda e^{-\lambda u} du &= -\frac{1}{\lambda} \cdot \lambda e^{-\lambda u} \Big|_0^x \\ &= -e^{-\lambda x} - (-e^{-\lambda \cdot 0}) \\ &= 1 - e^{-\lambda x}, \text{ so} \end{aligned}$$

$$F_Y(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

λ is called the rate parameter.

$$E(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{2}{\lambda^2}$$

Example

Jobs are sent to printer at a rate of 3 jobs per hour (an average)

a) What is expected time between jobs:

$$E(T) = \frac{1}{\lambda} = \frac{1}{3} = 0.3 = \underline{\underline{20 \text{ min}}}$$

b) What is probability that next job is sent within 5 minutes?

$$P(T < 1/12) = F(1/12) = 1 - e^{-3 \cdot 1/12} = 1 - e^{-1/4} \approx \underline{\underline{0.22}}$$


Normal Distribution

Characteristics:

- △ Symmetric, bell shaped
- △ Continuous for all intervals s.t. any $P(X \in [a, b]) \neq 0$
- △ $-\infty \leq X \leq \infty$
- △ Two parameters, μ and σ

PDF: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

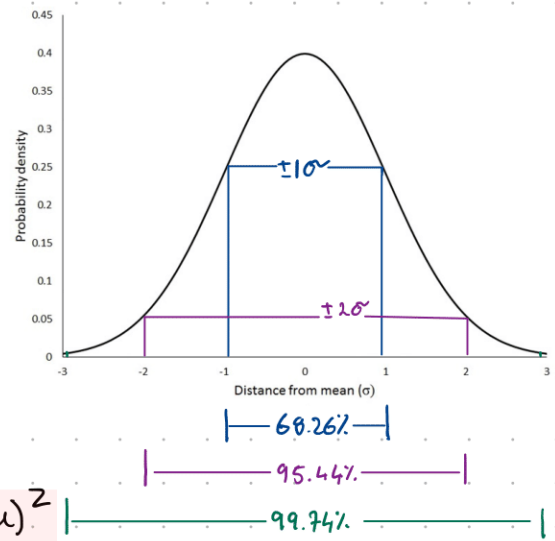
Notation: $N(\mu, \sigma^2)$ or $X \sim N(\mu, \sigma^2)$

 About $2/3$ of all cases lie in $[\mu - \sigma; \mu + \sigma]$

$$P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.6826$$

 About 95% of all cases lie in $[\mu - 2\sigma, \mu + 2\sigma]$

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.9544$$



Standard normal distribution

→ special case of normal distribution with $\mu=0$ and $\sigma^2=1$, i.e. $N(0,1)$

① Convert problem to a standardized normal variable, Z-score

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

② A table of z-scores exists

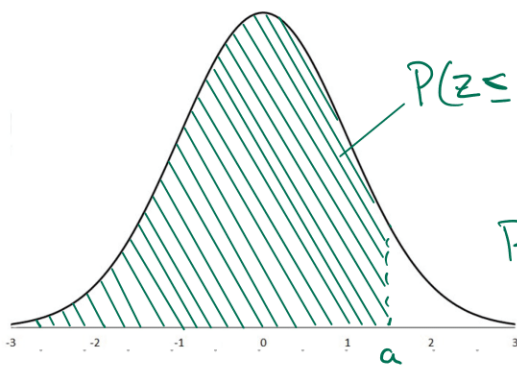
③ We can convert back:

$$X = Z \cdot \sigma + \mu$$

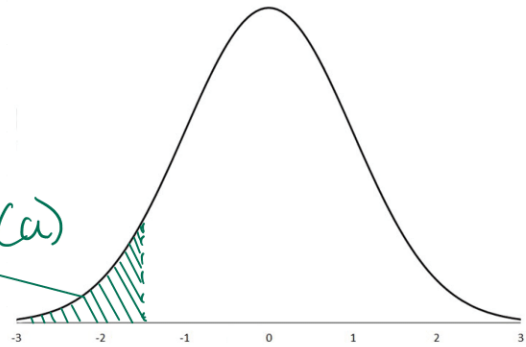
	00	01	02	03	04	05	06	07	08	09
3.4	0003	0003	0003	0003	0003	0003	0003	0003	0003	0002
3.3	0005	0005	0005	0004	0004	0004	0004	0004	0004	0007
3.2	0009	0009	0009	0009	0009	0009	0009	0009	0009	0007
3.1	0010	0009	0009	0009	0008	0008	0008	0008	0007	0007
3.0	0013	0013	0013	0012	0012	0011	0011	0011	0010	0010
2.9	0016	0016	0016	0016	0016	0016	0016	0016	0016	0016
2.8	0026	0025	0024	0023	0023	0022	0021	0021	0020	0019
2.7	0035	0034	0033	0032	0031	0030	0029	0028	0027	0026
2.6	0044	0043	0042	0041	0040	0039	0038	0037	0036	0035
2.5	0062	0060	0059	0057	0055	0054	0052	0051	0049	0048
2.4	0078	0076	0074	0073	0072	0071	0069	0068	0066	0065
2.3	0107	0104	0102	0101	0100	0099	0097	0096	0094	0093
2.2	0139	0136	0132	0129	0125	0122	0119	0116	0113	0110
2.1	0173	0169	0162	0157	0151	0145	0138	0132	0125	0118
2.0	0226	0221	0217	0212	0207	0202	0197	0192	0188	0183
1.9	0287	0281	0274	0268	0262	0256	0250	0244	0239	0233
1.8	0359	0352	0344	0337	0330	0323	0316	0310	0304	0298
1.7	0446	0437	0427	0418	0409	0401	0392	0384	0375	0367
1.6	0548	0537	0526	0515	0505	0495	0485	0475	0465	0455
1.5	0669	0657	0645	0633	0621	0609	0597	0585	0573	0561
1.4	0808	0793	0778	0764	0749	0735	0721	0707	0694	0681
1.3	0965	0951	0934	0918	0902	0886	0870	0853	0838	0822
1.2	1142	1127	1112	1095	1078	1061	1044	1027	1010	994
1.1	1357	1335	1314	1292	1271	1251	1230	1210	1190	1170
1.0	1612	1579	1547	1514	1481	1448	1415	1382	1349	1316
0.9	1941	1814	1788	1752	1716	1681	1645	1609	1573	1538
0.8	2349	2099	2061	2033	2005	1977	1949	1922	1894	1867
0.7	2874	2499	2456	2428	2399	2370	2341	2312	2283	2254
0.6	3585	3050	3015	2981	2946	2911	2876	2841	2805	2776
0.5	4524	3799	3754	3719	3684	3649	3614	3579	3544	3509
0.4	5821	4823	4778	4743	4708	4673	4638	4603	4568	4533
0.3	7421	6213	6168	6133	6098	6063	6028	5993	5958	5923
0.2	9207	7680	7635	7600	7565	7530	7495	7460	7425	7390
0.1	11250	9450	9405	9370	9335	9300	9265	9230	9195	9160
0	0000	0000	0000	0000	0000	0000	0000	0000	0000	0000
0.1	5339	5432	5447	5461	5476	5490	5505	5520	5534	5549
0.2	5549	5642	5657	5671	5686	5700	5715	5729	5744	5759
0.3	5759	5852	5867	5881	5896	5910	5925	5939	5954	5969
0.4	5969	6062	6077	6091	6106	6120	6135	6149	6164	6179
0.5	6179	6271	6285	6293	6311	6326	6340	6354	6368	6382
0.6	6382	6474	6488	6496	6514	6529	6543	6557	6571	6585
0.7	6585	6677	6691	6705	6724	6738	6752	6766	6780	6794
0.8	6794	6886	6900	6914	6934	6948	6962	6976	6990	7004
0.9	7004	7096	7110	7124	7144	7158	7172	7186	7200	7214
1.0	7214	7306	7320	7334	7354	7368	7382	7396	7410	7424
1.1	7424	7516	7530	7544	7564	7578	7592	7606	7620	7634
1.2	7634	7726	7740	7754	7774	7788	7802	7816	7830	7844
1.3	7844	7936	7950	7964	7984	7998	8012	8026	8040	8054
1.4	8054	8146	8160	8174	8194	8208	8222	8236	8250	8264
1.5	8264	8356	8370	8384	8404	8418	8432	8446	8460	8474
1.6	8474	8566	8580	8594	8614	8628	8642	8656	8670	8684
1.7	8684	8776	8790	8804	8824	8838	8852	8866	8880	8894
1.8	8894	8986	8999	9013	9033	9047	9061	9075	9089	9103
1.9	9103	9195	9209	9223	9243	9257	9271	9285	9299	9313
2.0	9313	9405	9419	9433	9453	9467	9481	9495	9509	9523
2.1	9523	9615	9629	9643	9663	9677	9691	9705	9719	9733
2.2	9733	9825	9839	9853	9873	9887	9901	9915	9929	9943
2.3	9943	9995	9999	9999	9999	9999	9999	9999	9999	9999
2.4	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
2.5	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
2.6	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
2.7	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
2.8	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
2.9	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
3.0	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
3.1	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
3.2	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
3.3	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
3.4	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999

Rules:

$$1. P(Z \leq a) = F(a)$$



$$P(Z \leq a) = F(a)$$



Example:

Find $P(Z \leq a)$ for $a = \begin{cases} -1.65 \\ -1.00 \\ 1.00 \\ 1.65 \end{cases}$

$$P(Z \leq -1.65) = 0.0495$$

$$P(Z \leq -1.00) = 0.1587$$

$$P(Z \leq 1.00) = 0.8413$$

$$P(Z \leq 1.65) = 0.9505$$

Find a for $\begin{cases} 0.6026 \\ 0.9750 \\ 0.3446 \end{cases}$

$$P(Z \leq a) = 0.6026$$

$$a = 0.26$$

$$P(Z \leq a) = 0.9750$$

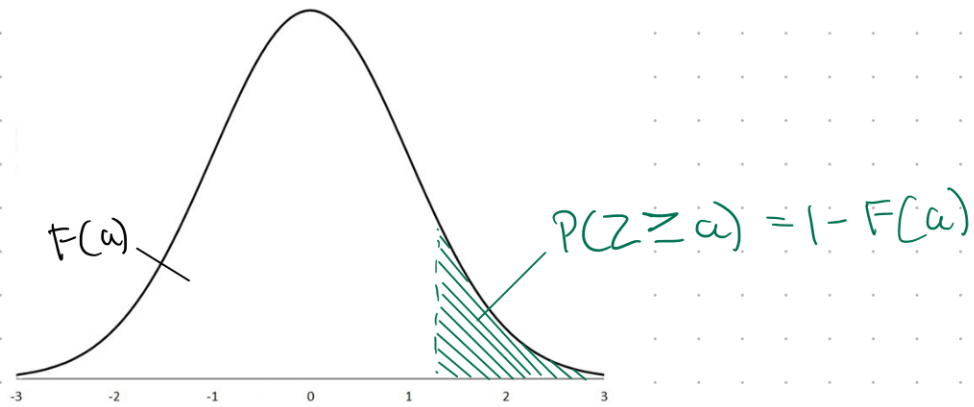
$$a = 1.96$$

$$P(Z \leq a) = 0.3446$$

$$a = -0.40$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
-0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9992	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$$2. P(Z \geq a) = 1 - F(a) = F(-a) = P(Z \leq -a)$$



Example:

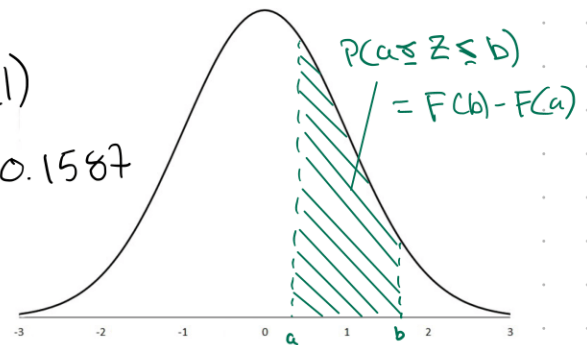
$$P(Z \geq 1.5) = P(Z \leq -1.5) = 0.0668$$

$$P(Z \geq 2) = P(Z \leq -2) = 0.0228$$

$$3. P(a \leq Z \leq b) = F(b) - F(a), \quad b \geq a$$

Example:

$$\begin{aligned} P(-1 \leq Z \leq 1.5) &= F(1.5) - F(-1) \\ &= 0.9332 - 0.1587 \\ &= 0.7745 \end{aligned}$$



Example:

Family income $\sim N(\$25000, \$10,000^2)$

If poverty level is \$10,000, what percentage of population live in poverty?

X = Family income:

Convert to Z :
$$Z = \frac{10,000 - 25,000}{10,000} = -1.5$$

So $P(Z \leq -1.5) = 0.0668 \approx 7\%$