This problem involves working with **discrete probability distributions**, specifically the **Poisson distribution** and the **Geometric distribution**, and their properties related to independent random variables. We'll be calculating probabilities of events, conditional probabilities, and expected values.

Problem Description:

Let A and B be two independent stochastic variables such that:

- $A \sim \text{Poisson}(3)$
- $B \sim \text{Geometric}(1/2)$

We need to solve three parts:

Part (a): Find $P(A = 2 \text{ or } B = 2)$

This asks for the probability of the union of two events, $A = 2$ and $B = 2$.
Since A and B are independent, the events $(A = 2)$ and $(B = 2)$ are also independent.
The formula for the probability of the union of two independent events is:
$P(X \text{ or } Y) = P(X) + P(Y) - P(X \text{ and } Y)$
Since X and Y are independent, $P(X \text{ and } Y) = P(X)P(Y)$.
So, $P(A = 2 \text{ or } B = 2) = P(A = 2) + P(B = 2) - P(A = 2)P(B = 2)$.

1. Probability Mass Function (PMF) for Poisson Distribution:

If $A \sim \text{Poisson}(\lambda)$, then $P(A = k) = \frac{e^{-\lambda}\lambda^k}{k!}$
For $A \sim \text{Poisson}(3)$, we need $P(A = 2)$:
$P(A = 2) = \frac{e^{-3}3^2}{2!} = \frac{9e^{-3}}{2}$

2. PMF for Geometric Distribution:

If $B \sim \text{Geometric}(p)$ (number of trials until the first success, starting from 1), then $P(B = k) = (1 - p)^{k-1}p$.
For $B \sim \text{Geometric}(1/2)$, we need $P(B = 2)$:
$P(B = 2) = \left(1 - \frac{1}{2}\right)^{2-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^1 \cdot \frac{1}{2} = \frac{1}{4}$

3. Calculate $P(A = 2 \text{ or } B = 2)$:

$P(A = 2 \text{ or } B = 2) = P(A = 2) + P(B = 2) - P(A = 2)P(B = 2)$
$P(A = 2 \text{ or } B = 2) = \frac{9e^{-3}}{2} + \frac{1}{4} - \left(\frac{9e^{-3}}{2}\right)\left(\frac{1}{4}\right)$

$P(A = 2 \text{ or } B = 2) = \frac{9e^{-3}}{2} + \frac{1}{4} - \frac{9e^{-3}}{8}$
To combine terms with e^{-3}: $\frac{4 \cdot 9e^{-3}}{8} - \frac{9e^{-3}}{8} = \frac{36e^{-3} - 9e^{-3}}{8} = \frac{27e^{-3}}{8}$
$P(A = 2 \text{ or } B = 2) = \frac{27e^{-3}}{8} + \frac{1}{4}$

Now, calculate the numerical value to four decimal places:
$e^{-3} \approx 0.049787$
$P(A = 2 \text{ or } B = 2) \approx \frac{27 \times 0.049787}{8} + \frac{1}{4}$
$P(A = 2 \text{ or } B = 2) \approx \frac{1.344249}{8} + 0.25$
$P(A = 2 \text{ or } B = 2) \approx 0.168031 + 0.25$
$P(A = 2 \text{ or } B = 2) \approx 0.418031$

Rounding to four decimal places: **0.4180**

Part (b): Find $P(B = 3 \mid A + B = 4)$

This asks for a conditional probability: $P(B = 3 \mid A + B = 4)$.
The formula for conditional probability is $P(X \mid Y) = \frac{P(X \text{ and } Y)}{P(Y)}$.
So, $P(B = 3 \mid A + B = 4) = \frac{P(B=3 \text{ and } A+B=4)}{P(A+B=4)}$.

1. Calculate the numerator $P(B = 3 \text{ and } A + B = 4)$:
If $B = 3$ and $A + B = 4$, then $A + 3 = 4$, which implies $A = 1$.
So, the event $(B = 3 \text{ and } A + B = 4)$ is equivalent to $(B = 3 \text{ and } A = 1)$.
Since A and B are independent, $P(B = 3 \text{ and } A = 1) = P(B = 3)P(A = 1)$.

For $B \sim \text{Geometric}(1/2)$, $P(B = 3) = \left(1 - \frac{1}{2}\right)^{3-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{8}$.
For $A \sim \text{Poisson}(3)$, $P(A = 1) = \frac{e^{-3}3^1}{1!} = 3e^{-3}$.

So, $P(B = 3 \text{ and } A + B = 4) = \frac{1}{8} \cdot 3e^{-3} = \frac{3e^{-3}}{8}$.

2. Calculate the denominator $P(A + B = 4)$:
This is the probability that the sum of a Poisson and a Geometric random variable equals 4.
$A + B = 4$ can occur if:

- $B = 1, A = 3 \implies P(A = 3, B = 1) = P(A = 3)P(B = 1)$
- $B = 2, A = 2 \implies P(A = 2, B = 2) = P(A = 2)P(B = 2)$
- $B = 3, A = 1 \implies P(A = 1, B = 3) = P(A = 1)P(B = 3)$

- $B = 4, A = 0 \implies P(A = 0, B = 4) = P(A = 0)P(B = 4)$
  - $P(A = 0) = \frac{e^{-3}3^0}{0!} = e^{-3}$
  - $P(A = 1) = 3e^{-3}$ (from above)
  - $P(A = 2) = \frac{9e^{-3}}{2}$ (from part a)
  - $P(A = 3) = \frac{e^{-3}3^3}{3!} = \frac{27e^{-3}}{6} = \frac{9e^{-3}}{2}$
  - $P(B = 1) = \left(\frac{1}{2}\right)^0 \cdot \frac{1}{2} = \frac{1}{2}$
  - $P(B = 2) = \frac{1}{4}$ (from part a)
  - $P(B = 3) = \frac{1}{8}$ (from above)
  - $P(B = 4) = \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{16}$

  Now, sum the products for $P(A + B = 4)$:
  $P(A + B = 4) = P(A = 3)P(B = 1) + P(A = 2)P(B = 2) + P(A = 1)P(B = 3) + P(A = 0)P(B = 4)$
  $P(A + B = 4) = \left(\frac{9e^{-3}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{9e^{-3}}{2}\right)\left(\frac{1}{4}\right) + (3e^{-3})\left(\frac{1}{8}\right) + (e^{-3})\left(\frac{1}{16}\right)$
  $P(A + B = 4) = \frac{9e^{-3}}{4} + \frac{9e^{-3}}{8} + \frac{3e^{-3}}{8} + \frac{e^{-3}}{16}$
  Find a common denominator (16):
  $P(A + B = 4) = \frac{4 \cdot 9e^{-3}}{16} + \frac{2 \cdot 9e^{-3}}{16} + \frac{2 \cdot 3e^{-3}}{16} + \frac{e^{-3}}{16}$
  $P(A + B = 4) = \frac{36e^{-3} + 18e^{-3} + 6e^{-3} + e^{-3}}{16}$
  $P(A + B = 4) = \frac{61e^{-3}}{16}$

3. Calculate $P(B = 3 \mid A + B = 4)$:
$P(B = 3 \mid A + B = 4) = \frac{\frac{3e^{-3}}{8}}{\frac{61e^{-3}}{16}}$
Cancel out e^{-3}:
$P(B = 3 \mid A + B = 4) = \frac{3/8}{61/16} = \frac{3}{8} \times \frac{16}{61}$
$P(B = 3 \mid A + B = 4) = \frac{3 \times 2}{61} = \frac{6}{61}$

Now, calculate the numerical value to four decimal places:
$\frac{6}{61} \approx 0.09836$

Rounding to four decimal places: **0.0984**

Part ©: Find $E[A + B]$ and $E[(A + B)^2]$

1. $E[A + B]$**:**
Using linearity of expectation: $E[A + B] = E[A] + E[B]$.

- For $A \sim \text{Poisson}(\lambda)$, $E[A] = \lambda$. So, $E[A] = 3$.
- For $B \sim \text{Geometric}(p)$, $E[B] = \frac{1}{p}$. So, $E[B] = \frac{1}{1/2} = 2$.

$E[A + B] = 3 + 2 = 5$
This is a positive integer.

2. $E[(A + B)^2]$**:**

We know that $\text{Var}(X) = E[X^2] - (E[X])^2$, so $E[X^2] = \text{Var}(X) + (E[X])^2$.
Thus, $E[(A + B)^2] = \text{Var}(A + B) + (E[A + B])^2$.

- We already found $E[A + B] = 5$. So $(E[A + B])^2 = 5^2 = 25$.

- For independent random variables A and B, $\text{Var}(A + B) = \text{Var}(A) + \text{Var}(B)$.

  - For $A \sim \text{Poisson}(\lambda)$, $\text{Var}(A) = \lambda$. So, $\text{Var}(A) = 3$.
  - For $B \sim \text{Geometric}(p)$, $\text{Var}(B) = \frac{1-p}{p^2}$. So, $\text{Var}(B) = \frac{1-1/2}{(1/2)^2} = \frac{1/2}{1/4} = \frac{1}{2} \times 4 = 2$.

- $\text{Var}(A + B) = 3 + 2 = 5$.

Now, calculate $E[(A + B)^2]$:
$E[(A + B)^2] = \text{Var}(A + B) + (E[A + B])^2 = 5 + 25 = 30$.
This is a positive integer.

Final Answers for Part ©:

- $E[A + B] = 5$
- $E[(A + B)^2] = 30$

Topics Covered:

- **Discrete Probability Distributions:**
  - **Poisson Distribution:** Used for modeling the number of events occurring in a fixed interval of time or space. Key properties: $E[X] = \text{Var}(X) = \lambda$. PMF: $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$.

- **Geometric Distribution:** Models the number of Bernoulli trials needed to get the first success. Key properties: $E[X] = \frac{1}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$. PMF: $P(X = k) = (1-p)^{k-1}p$.
- **Independence of Random Variables:** A crucial property that simplifies calculations for joint probabilities, sums, and variances.
- **Probability of Union of Events:** $P(X \text{ or } Y) = P(X) + P(Y) - P(X \text{ and } Y)$. For independent events, $P(X \text{ and } Y) = P(X)P(Y)$.
- **Conditional Probability:** $P(X \mid Y) = \frac{P(X \text{ and } Y)}{P(Y)}$.
- **Expected Value (Expectation):**
  - Linearity: $E[X + Y] = E[X] + E[Y]$.
  - For a single random variable: $E[X^2] = \mathrm{Var}(X) + (E[X])^2$.
- **Variance:**
  - For independent random variables: $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

WolframAlpha/Computational Check:

You can use WolframAlpha to verify individual probabilities and expected values:

- **Poisson PMF:**
  - `Poisson distribution P(X=2 | lambda=3)` will give you ≈ 0.2240.
  - `Poisson probability x=1, lambda=3` will give ≈ 0.1493.
  - `Poisson probability x=0, lambda=3` will give ≈ 0.0498.
  - `Poisson probability x=3, lambda=3` will give ≈ 0.2240.
- **Geometric PMF:**
  - `Geometric distribution P(X=2 | p=1/2)` will give 0.25.
  - `Geometric probability x=1, p=1/2` will give 0.5.
  - `Geometric probability x=3, p=1/2` will give 0.125.
  - `Geometric probability x=4, p=1/2` will give 0.0625.
- **Expected Values/Variances:**
  - `mean of Poisson(3)` will give `3`.
  - `variance of Poisson(3)` will give `3`.
  - `mean of Geometric(1/2)` will give `2`.
  - `variance of Geometric(1/2)` will give `2`.
- **Overall calculation for part (a):**

- `(9*exp(-3))/2 + 1/4 - ((9*exp(-3))/2) * (1/4)` will give `0.418031...`

- **Overall calculation for part (b):**
  - The terms in the denominator $P(A+B=4)$: `(9*exp(-3))/4 + (9*exp(-3))/8 + (3*exp(-3))/8 + exp(-3)/16` will give $\frac{61e^{-3}}{16}$.
  - The numerator $P(A=1, B=3)$: `(3*exp(-3))/8`.
  - Then, `((3*exp(-3))/8) / ((61*exp(-3))/16)` will simplify to `6/61`.

These checks confirm the numerical and fractional results.