Assignment 3 – Markov Chains

Exercise (a): Fill in the Missing Transition Probabilities

We are given a Markov chain with three states $S = \{1, 2, 3\}$ and a partially filled transition diagram. The task is to find the missing probabilities, ensuring that **each row sums to 1**.

The missing values are:

- P_{11}: from state 1 to 1
- P_{21}: from state 2 to 1
- P_{32}: from state 3 to 2

From state 1:

Given: $P_{12} = 0.2$, $P_{13} = 0.2$

Since total probability must be 1:

$$P_{11} = 1 - (0.2 + 0.2) = 0.6$$

From state 2:

Given: $P_{22} = 0.4$, $P_{23} = 0.3$

Then:

$$P_{21} = 1 - (0.4 + 0.3) = 0.3$$

From state 3:

Given: $P_{31} = 0.5$, $P_{33} = 0.2$

Then:

$$P_{32} = 1 - (0.5 + 0.2) = 0.3$$

Exercise (b): Transition Matrix

Using the values above, the full state transition matrix P is:

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$$

Alternatively, as a table:

From \ To	**1**	**2**	**3**
1	0.6	0.2	0.2
2	0.3	0.4	0.3
3	0.5	0.3	0.2

Exercise ©: Compute Joint Probability

We are given:

$$P(X_1 = 1) = P(X_1 = 2) = P(X_1 = 3) = \frac{1}{3}$$

We are asked to compute:

$$P(X_1 = 2,\ X_2 = 3,\ X_3 = 1)$$

Step 1: Use the Markov Property

By the **Markov property**:

$$P(X_1 = 2,\ X_2 = 3,\ X_3 = 1) = P(X_1 = 2) \cdot P(X_2 = 3 \mid X_1 = 2) \cdot P(X_3 = 1 \mid X_2 = 3)$$

Step 2: Plug in values from the matrix

- $P(X_1 = 2) = \frac{1}{3}$
- $P(2 \to 3) = P_{23} = 0.3 = \frac{3}{10}$
- $P(3 \to 1) = P_{31} = 0.5 = \frac{1}{2}$

Step 3: Multiply

$$P(X_1 = 2,\ X_2 = 3,\ X_3 = 1) = \frac{1}{3} \cdot \frac{3}{10} \cdot \frac{1}{2} = \frac{3}{60} = \frac{1}{20}$$

✅ Final Answer:

- **Numerator:** 1
- **Denominator:** 20