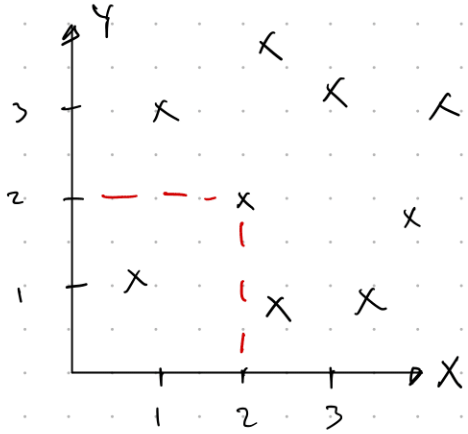
Multivariate RVs

Two Discrete RVs:

X, Y joint PMF:

$$P_{X,Y}(x_i, y_i) = f_{XY}(x_i, y_i) = P(X=x_i, Y=y_i)$$

$$R_{XY} = \{x_i, y_i \mid f_{XY}(x_i, y_i) > 0\}$$



$$f_{XY}(2,2) = 1/4$$

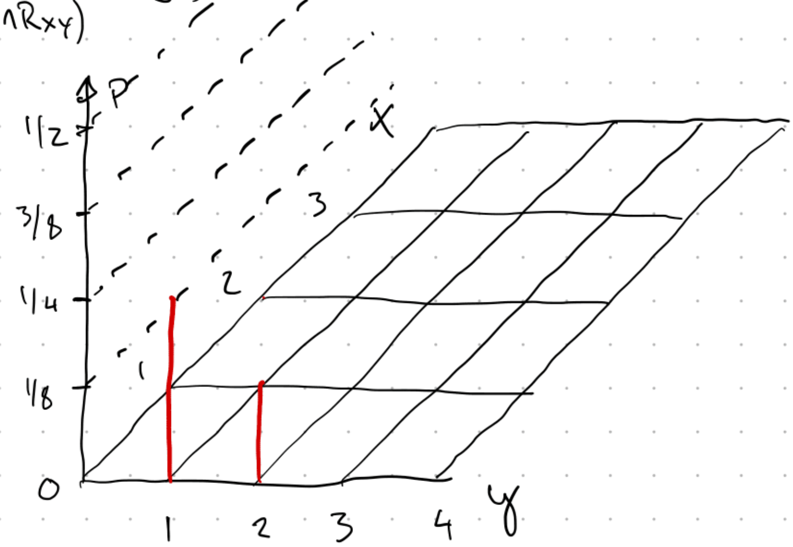
$$\sum_{(x_i, y_i) \in R_{XY}} f_{XY}(x_i, y_i) = 1$$

In general:

$$P((X,Y) \in A) = \sum_{(x_i, y_i) \in (A \cap R_{XY})} f_{XY}(x_i, y_i)$$

Example

	Y=0	Y=1	Y=2
X=0	1/6	1/4	1/8
X=1	1/8	1/6	1/6



a) Find $P(X=0, Y\leq 1) = f(0,0) + f(0,1) = 1/6 + 1/4 = \underline{\underline{5/12}}$

b) Find Marginal PMFs of X and Y:

$f_X(0) = 1/6 + 1/4 + 1/8 = \frac{13}{24}$, $f_Y(0) = 1/6 + 1/8 =$

$f_Y(1) = 1/4 + 1/6 =$

$f_X(1) = 1 - f_X(0) = \frac{11}{24}$, $f_Y(2) = 1/8 + 1/6 =$

In general Marginal PMF:

$$f_X(x) = \sum_{y_j} f_{XY}(x, y_j)$$

$$f_Y(y) = \sum_{x_i} f_{XY}(x_i, y)$$

Example cont'd:

c) Find $P(Y=1 \mid X=0) = \dfrac{P(Y=1, X=0)}{P(X=0)} = \dfrac{1/4}{13/24} = \dfrac{24}{52} = \underline{\underline{\dfrac{6}{13}}}$

d) Are X and Y independent?

$$\left.\underset{\downarrow \; 6/13}{P(Y=1 \mid X=0)} \overset{?}{=} \underset{\downarrow \; 5/12}{P(Y=1)}\right\} \text{No!}$$

In general:

X and Y are independent if

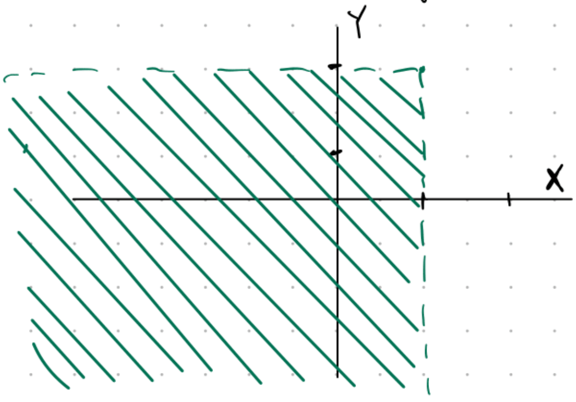
$$f_{XY}(x_i, y_j) = f_X(x_i) \cdot f_Y(y_j)$$

X, Y joint CDF:

$$F_{XY}(x, y) = P(X \le x, Y \le y)$$

$$0 \le F_{XY}(x, y) \le 1$$

$$F_{XY}(1,2) = P(X \le 1, Y \le 2)$$



Marginal CDF:

$$F_X(x) = F_{XY}(x, \infty) = \lim_{y \to \infty} F_{XY}(x, y)$$

$$F_Y(x) = F_{XY}(\infty, y) = \lim_{x \to \infty} F_{XY}(x, y)$$

Example:

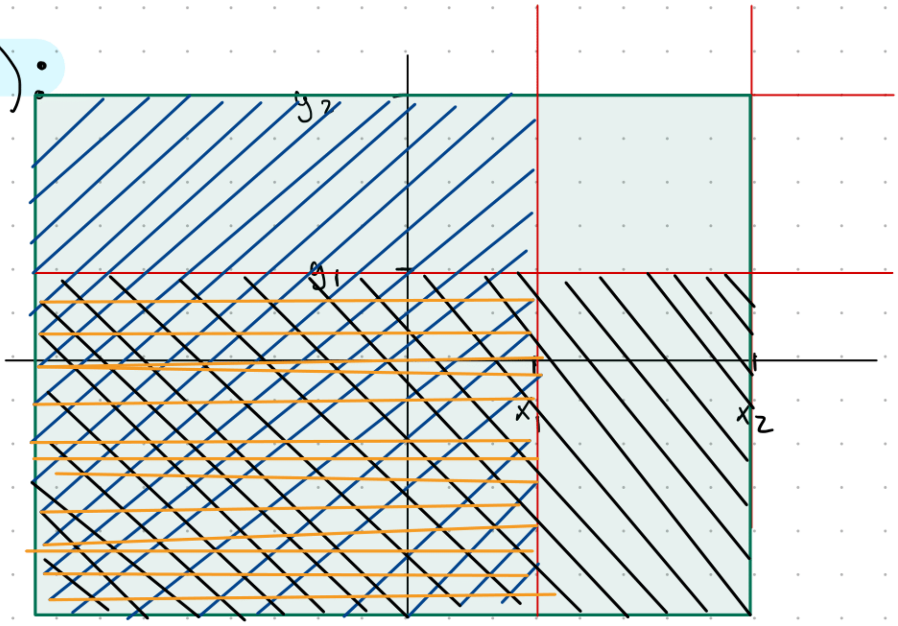
Find $F_{XY}(0.5, 1) =$

$P(X \leq 0.5, X \leq 1) =$

$f_{XY}(0,0) + f_{XY}(0,1) = \underline{\underline{5/12}}$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$1/6$	$1/4$	$1/8$
$X=1$	$1/8$	$1/6$	$1/6$

$P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2)$:



$$= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(X_1, X_2)$$

Conditioning and Independence

$$f_{X|Y}(x_i | y_i) = P(X = x_i \mid Y = y_i)$$

$$= \frac{P(X = x_i, Y = y_i)}{P(Y = y_i)} = \frac{f_{XY}(x_i, y_i)}{f_Y(y_i)}$$

Independence:

$$f_{X|Y}(x_i | y_i) = f_X(x_i)$$

$$\Updownarrow \quad \frac{f_{XY}(x_i, y_i)}{f_Y(y_i)} = f_X(x_i) \implies f_{XY}(x_i, y_i) = f_X(x_i) \cdot f_Y(y_i)$$

Conditional Expectation:

$$E(X \mid Y = y_i) = \sum_{x_i \in R_x} x_i \cdot f_{X|Y}(x_i \mid y_i)$$

Example:

Assume the following joint PMF:

$$f_{XY}(x,y) = \begin{cases} 1/3 & x=1 \wedge y=3 \\ 1/3 & x=2 \wedge y=0 \\ 1/3 & x=0 \wedge y=0 \\ 0 & \text{else} \end{cases}$$

a) Find $P_{X|Y=0}(X)$

$$P_{X|Y=0}(x_i) = \frac{P_{XY}(x_i, 0)}{P_Y(0)} \quad , \; P_Y(0) = 1/3 + 1/3 = 2/3$$

$$P_{X|Y=0}(0) = \frac{1/3}{2/3} = 1/2$$

$$P_{X|Y=0}(1) = 0$$

$$P_{X|Y=0}(2) = \frac{1/3}{2/3} = 1/2 \quad , \text{so}$$

$$P_{X|Y=0}(x) = \begin{cases} 1/2 & x=0 \\ 0 & x=1 \\ 1/2 & x=2 \\ 0 & x \notin R_X \end{cases}$$

b) Find $E[X \mid Y=0]$:

$$E[X \mid Y=0] = 0 \cdot 1/2 + 1 \cdot 0 + 2 \cdot 1/2$$

$$= \underline{\underline{1}}$$

Example:

We throw a fair coin three times. Let X denote the number of heads on first toss and let Y denote the total number of heads:

X\Y	0	1	2	3
0	1/8	1/4	1/8	0
1	0	1/8	1/4	1/8

Find the expected number of heads given that the first toss resulted in a tail.

$$P_{Y|X=0}(y) = \frac{P_{XY}(y_i, 0)}{P_X(0)}$$

$$P_X(0) = 1/2 \quad , \quad P_X(1) = 1/2$$

$$P_{Y|X=0}(0) = \frac{1/8}{1/2} = 1/4$$

$$P_{Y|X=0}(1) = \frac{1/4}{1/2} = 1/2$$

$$P_{Y|X=0}(2) = \frac{1/8}{1/2} = 1/4$$

$$P_{Y|X=0}(3) = \frac{0}{1/2} = 0 \quad , \text{ so}$$

$$E(Y|X=0) = 0 \cdot 1/4 + 1 \cdot 1/2 + 2 \cdot 1/4 + 3 \cdot 0 = 1$$

$$E(Y|X=1) = 0 \cdot 0 + 1 \cdot 1/4 + 2 \cdot 1/2 + 3 \cdot 1/4 = 2$$

Expectation

Total Expectation:

1. $E[X] = \sum_i E[X|B_i] \cdot P[B_i]$, B_1, B_2 some Partition of S
2. $E[X] = \sum_{y_i \in R_Y} E[X|Y=y_i] P_Y(y_i)$
3. $E[X] = E[E[X|Y]]$

Two Continuous RVs

Joint PDF:

$$P_{XY}((X,Y) \in A) = \iint_A f_{XY}(x,y)\,dx\,dy$$

If $A = \mathbb{R}^2$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$

Joint CDF:

$$F_{XY}(x,y) = P(X \leq x, Y \leq y)$$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial^2 x\,\partial^2 y} F_{XY}(x,y)$$

Example

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.

b. Find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$.

a) $\int_0^1\int_0^1 (x + cy^2)\,dx\,dy = \int_0^1 \left(\frac{1}{2}x^2 + c \cdot xy^2\Big|_0^1\right) dy$

$= \int_0^1 \left(\frac{1}{2} + cy^2\right) dy = \frac{1}{2}y + \frac{1}{3} \cdot cy^3\Big|_0^1 = \frac{1}{2} + \frac{1}{3}c = 1$

$c = 3/2$

b)

$$P(0 \le X \le 1/2, 0 \le Y \le 1/2) = \int_0^{1/2}\int_0^{1/2} (x + 3/2\, y^2)\, dx\, dy$$

$$= \int_0^{1/2} \left(\frac{1}{2}x^2 + \frac{3}{2} x y^2 \Big|_0^{1/2} \right) dy = \int_0^{1/2} \left(\frac{1}{8} + \frac{3}{4} y^2 \right) dy$$

$$= \frac{1}{8} y + \frac{1}{4} y^3 \Big|_0^{1/2} = \frac{1}{16} + \frac{1}{32} = \underline{\underline{\frac{3}{32}}}$$

The joint CDF satisfies:

1) $F_X(x) = F(x, \infty)$ (marginal of x)
2) $F_Y(y) = F(\infty, y)$ (marginal of y)
3) $F_{XY}(-\infty, \infty) = 1$
4) $P(x_1 \le X \le x_2, y_1 \le Y \le y_2) =$
$F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_2)$

Marginal PDFs:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx$$

c) Find Marginal PDFs:

$$f_X(x) = \int_0^1 (x + \tfrac{3}{2} y^2)\, dy = xy + \tfrac{1}{2} y^3 \Big|_0^1 = x + \tfrac{1}{2}$$

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \int_0^1 (x + \tfrac{3}{2} y^2)\, dx = \tfrac{1}{2} x^2 + \tfrac{3}{2} x y^2 \Big|_0^1 = \tfrac{3}{2} y^2 + \tfrac{1}{2}$$

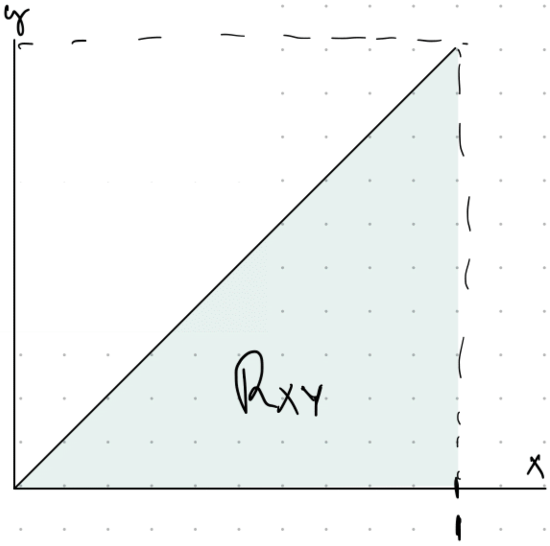
$$f_Y(y) = \begin{cases} \frac{3}{2} y^2 + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

Example

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} cx^2y & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find R_{XY} and show it in the $x-y$ plane.
b. Find the constant c.
c. Find marginal PDFs, $f_X(x)$ and $f_Y(y)$.
d. Find $P(Y \le \frac{X}{2})$.
e. Find $P(Y \le \frac{X}{4} | Y \le \frac{X}{2})$.



a) $R_{XY} = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le x \le 1\}$

b) $\int_0^1 \int_0^x c x^2 y \, dy \, dx = \int_0^1 \frac{1}{2} c x^2 y^2 \, dx \Big|_0^x$

$= \int_0^1 \frac{1}{2} c x^4 \, dx = \frac{1}{10} c x^5 \Big|_0^1 = \frac{1}{10} c = 1 \Rightarrow c = 10$

c) $f_X(x) = \int_0^x 10x^2 y \, dy = 5x^2 y^2 \Big|_0^x = 5x^4$

$f_X(x) = \begin{cases} 5x^4 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

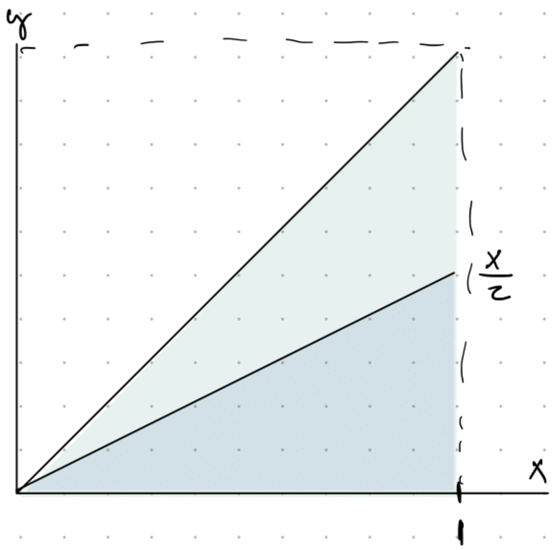
$f_Y(y) = \int_y^1 10x^2 y \, dx = \frac{10}{3} x^3 y \Big|_y^1 = \frac{10}{3} y - \frac{10}{3} y^4$

$= \frac{10}{3} y(1-y^3)$

$f_Y(y) = \begin{cases} \frac{10}{3}(1-y^3) & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$

d) $P(Y \le \frac{X}{2}) = \int_0^1 \int_0^{x/2} 10x^2 y \, dy \, dx = \int_0^1 \frac{10}{2} x^2 y^2 \, dx \Big|_0^{x/2}$

$= \int_0^1 \frac{10}{8} x^4 \, dx = \frac{1}{4} x^5 \Big|_0^1 = \frac{1}{4}$



e) $$P\left(Y \le \frac{x}{4} \mid Y \le \frac{x}{2}\right) = \frac{P\left(Y \le \frac{x}{4}, Y \le \frac{x}{2}\right)}{P\left(Y \le \frac{x}{2}\right)} = \frac{P\left(Y \le \frac{x}{4}\right)}{P\left(Y \le \frac{x}{2}\right)}$$

$$= 4 \cdot P\left(Y \le \frac{x}{4}\right)$$

$$= 4 \int_0^1 \int_0^{\frac{x}{4}} 10x^2 y \, dy \, dx = 4 \int_0^1 5x^2 y^2 \, dx \Big|_0^{\frac{x}{4}}$$

$$= 4 \int_0^1 \frac{5}{16} x^4 \, dx = 4 \left[\frac{1}{16} x^5 \Big|_0^1\right] = 1/4$$

Mean and Variance:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f_{XY}(x,y) \, dy \, dx$$

$$V[X] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx - \left(\int_{-\infty}^{\infty} x \cdot f_X(x) \, dx\right)^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f_{XY}(x,y) \, dy \, dx - \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f_{XY}(x,y) \, dy \, dx\right)^2$$

$$= E[X^2] - (E[X])^2$$

Example

Given

$$f_{X,Y}(x,y) = \begin{cases} 2-x-y & , 0 \le x,y \le 1 \\ 0 & \text{else} \end{cases}$$

Find $Var(X)$:

$$E[X] = \int_0^1 \int_0^1 x \cdot (2-x-y) \, dy \, dx = \int_0^1 \int_0^1 (2x - x^2 - yx) \, dy \, dx$$

$$= \int_0^1 \left(2x - x^2 - \frac{1}{2} y^2 x \Big|_0^1\right) dx = \int_0^1 \left(2x - x^2 - \frac{1}{2}x\right) dx$$

$$= x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^2 \Big|_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$$

$$E[X^2] = \int_0^1 \int_0^1 x^2 (2-x-y) \, dy \, dx = \int_0^1 \int_0^1 (2x^2 - x^3 - yx^2) \, dy \, dx$$

$$= \int_0^0 \left(2x^2 - x^3 - \frac{1}{2} y^2 x^2 \Big|_0^1\right) dx = \int_0^1 \left(2x^2 - x^3 - \frac{1}{2}x^2\right) dx$$

$$= \frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{6}x^3 \Big|_0^1 = \frac{2}{3} - \frac{1}{4} - \frac{1}{6} = \frac{1}{4}$$

$$Var(X) = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{36-25}{144} = \frac{1}{9}$$

Conditioning

Same principles as in discrete case. Most importantly:

1. $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

2. $P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dy$

3. $F_{X|Y}(x|y) = P(X \leq x \mid Y=y) = \int_{-\infty}^{x} f_{X|Y}(x|Y)\,dx$

4. $E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx$

5. $V[X|Y=y] = E[X^2|Y=y] - \left(E[X|Y=y]\right)^2$

Independence:

Same as discrete!

Lotus still applies:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f_{XY}(x,y)\,dy\,dx$$

Example:

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \leq x,y \leq 1 \\ 0 & \text{else} \end{cases}$$

Find $E[XY^2]$

$$E[XY^2] = \int_0^1\int_0^1 x\,y^2(x+y)\,dy = \int_0^1\int_0^1 (x^2y^2 + xy^3)\,dx\,dy$$

$$= \underline{\underline{\frac{17}{72}}}$$

Covariance and Correlation:

The covariance gives information about how X and Y are statistically related:

$$\boxed{\begin{aligned} \mathrm{Cov}(X,Y) &= E[(X-EX)(Y-EY)] \\ &= E[XY] - (EX)\cdot(EY) \end{aligned}}$$

Properties:

1. $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$
2. If X and Y are independent
$$\mathrm{Cov}(X,Y) = 0$$
3. $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$
4. $\mathrm{Cov}(aX,Y) = a\cdot \mathrm{Cov}(X,Y) \quad , \quad a \in \mathbb{R}$
5. $\mathrm{Cov}(X+c,Y) = \mathrm{Cov}(X,Y)$
6. $\mathrm{Cov}(X+Y,Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$
7. $\mathrm{Cov}(X+Y,Z+W)$
$$= \mathrm{Cov}(X,Z) + \mathrm{Cov}(X,W) + \mathrm{Cov}(Y,Z) + \mathrm{Cov}(Y,W)$$
8. More generally:
$$\mathrm{Cov}\left(\sum_{i=1}^{m} a_i X_i , \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \mathrm{Cov}(X_i, Y_j)$$

If $Z = X + Y$

9. $\mathrm{Var}(Z) = \mathrm{Cov}(Z,Z) = \mathrm{Cov}(X+Y, X+Y)$
$$= \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,Y)$$
$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\cdot \mathrm{Cov}(X,Y)$$
10. More generally:
$$\mathrm{Var}(aX+bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y)$$

greek letter 'rho' = r

The correlation coefficient, ρ_{XY} or $\rho(X,Y)$ is obtained by normalising the covariance. More specifically we use the standardised versions of X and Y:

$$U = \frac{X - EX}{\sigma_X}, \quad V = \frac{Y - EY}{\sigma_Y}$$

$$\rho_{XY} = \operatorname{cov}(U,V) = \operatorname{cov}\left(\frac{X-EX}{\sigma_X}, \frac{Y-EY}{\sigma_Y}\right)$$

$$= \operatorname{cov}\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right) \quad \text{(property 5)}$$

$$= \frac{\operatorname{cov}(X,Y)}{\sigma_X \cdot \sigma_Y} \quad \text{(property 4)}$$