This problem continues the analysis of the Markov chain from the previous question. We will calculate the probability of specific sequences of states over multiple steps.

Problem Description Recap:

The Markov chain has three states (1, 2, 3).
The state transition matrix P (derived in the previous part) is:

$$P = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 3/4 & 1/4 & 0 \end{pmatrix}$$

Initial probabilities for X_1: $P(X_1 = 1) = 1/2$ and $P(X_1 = 2) = 1/4$. From these, we can infer $P(X_1 = 3) = 1 - 1/2 - 1/4 = 1/4$.
All answers should be irreducible fractions.

Part (b): Find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

This asks for the probability of a specific path in the Markov chain.
The probability of a sequence of states $X_1 = s_1, X_2 = s_2, \ldots, X_n = s_n$ is given by:

$$P(X_1 = s_1, X_2 = s_2, \ldots, X_n = s_n) = P(X_1 = s_1) \times P(s_1 \to s_2) \times P(s_2 \to s_3) \times \ldots \times P(s_{n-1} \to s_n)$$

We need to find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.
This expands to:
$P(X_1 = 3, X_2 = 2, X_3 = 1) = P(X_1 = 3) \times P(3 \to 2) \times P(2 \to 1)$.

- **Step 1: Identify the initial probability $P(X_1 = 3)$.**
  We are given $P(X_1 = 1) = 1/2$ and $P(X_1 = 2) = 1/4$.
  $P(X_1 = 3) = 1 - P(X_1 = 1) - P(X_1 = 2) = 1 - 1/2 - 1/4 = 1 - 2/4 - 1/4 = 1/4$.

- **Step 2: Identify the transition probability $P(3 \to 2)$.**
  From the transition matrix P, $P_{32} = 1/4$.

- **Step 3: Identify the transition probability $P(2 \to 1)$.**
  From the transition matrix P, $P_{21} = 1/2$.

- **Step 4: Calculate the path probability.**
  $P(X_1 = 3, X_2 = 2, X_3 = 1) = P(X_1 = 3) \times P(3 \to 2) \times P(2 \to 1)$
  $P(X_1 = 3, X_2 = 2, X_3 = 1) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{2}$
  $P(X_1 = 3, X_2 = 2, X_3 = 1) = \frac{1}{32}$.

The answer is an irreducible fraction: **1/32**.

Problem Description Recap:

- **The State Transition Matrix P to be used:**

$$P = \begin{pmatrix} 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

- **Initial probabilities:** $P(X_1 = 1) = 1/2$, $P(X_1 = 2) = 1/4$.
  From these, we can infer $P(X_1 = 3) = 1 - P(X_1 = 1) - P(X_1 = 2) = 1 - 1/2 - 1/4 = 1/4$.
- We need to find $P(X_1 = 3, X_3 = 1)$. All answers should be irreducible fractions.

Part ©: Find $P(X_1 = 3, X_3 = 1)$.

This asks for the joint probability of starting in State 3 at time X_1 and being in State 1 at time X_3. This involves the initial probability $P(X_1 = 3)$ and the two-step transition probability from State 3 to State 1, which is denoted as $P_{31}^{(2)}$.

The formula for this joint probability is:
$P(X_1 = 3, X_3 = 1) = P(X_1 = 3) \times P(X_3 = 1|X_1 = 3)$
And $P(X_3 = 1|X_1 = 3)$ is equivalent to the 2-step transition probability $P_{31}^{(2)}$.

So, the calculation becomes: $P(X_1 = 3, X_3 = 1) = P(X_1 = 3) \times P_{31}^{(2)}$.

- **Step 1: Identify the initial probability $P(X_1 = 3)$.**
  Given $P(X_1 = 1) = 1/2$ and $P(X_1 = 2) = 1/4$.
  Since the sum of initial probabilities for all states must be 1:
  $P(X_1 = 3) = 1 - P(X_1 = 1) - P(X_1 = 2) = 1 - 1/2 - 1/4 = 1 - 2/4 - 1/4 = 1/4$.

- **Step 2: Calculate the two-step transition probability $P_{31}^{(2)}$ using the provided matrix P.**
  This probability is the element in the 3rd row and 1st column of the matrix P^2. We obtain P^2 by multiplying P by itself:

$$P^2 = P \times P = \begin{pmatrix} 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

  To find $P_{31}^{(2)}$, we multiply the 3rd row of the first matrix (P_{31}, P_{32}, P_{33}) by the 1st column of the second matrix (P_{11}, P_{21}, P_{31}):
  $P_{31}^{(2)} = (P_{31} \times P_{11}) + (P_{32} \times P_{21}) + (P_{33} \times P_{31})$
  $P_{31}^{(2)} = \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right)$
  $P_{31}^{(2)} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$
  $P_{31}^{(2)} = \frac{3}{8}$.

- **Step 3: Calculate the final probability $P(X_1 = 3, X_3 = 1)$.**
  Multiply the initial probability by the two-step transition probability:
  $P(X_1 = 3, X_3 = 1) = P(X_1 = 3) \times P_{31}^{(2)}$
  $P(X_1 = 3, X_3 = 1) = \frac{1}{4} \times \frac{3}{8}$
  $P(X_1 = 3, X_3 = 1) = \frac{3}{32}$.

The answer is an irreducible fraction: **3/32**.

Topics Covered:

- **Markov Chains:** Understanding state transitions over multiple steps.
- **Transition Matrix (P):** The core representation of one-step transition probabilities.
- **Initial Probability Distribution:** The probability of starting in a given state.
- **Joint Probability:** Calculating the probability of two events occurring together.
- **Multi-step Transition Probabilities (P^n):** Using matrix multiplication to find probabilities of transitioning between states over multiple steps.

WolframAlpha Check:

- **To calculate the full P^2 matrix using the provided P and verify $P_{31}^{(2)}$:**
  Input: `{{1/4, 0, 3/4}, {1/2, 0, 1/2}, {1/2, 1/4, 1/4}}^2`
  WolframAlpha will return:

  $$P^2 = \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/8 & 1/2 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}$$

  You can visually confirm that $P_{31}^{(2)}$ (the element in the 3rd row, 1st column) is indeed $3/8$.

- **To verify the final probability calculation:**
  Input: `(1/4) * (3/8)`
  WolframAlpha will return `3/32`.