Exercise 1

Consider the Markov chain with three states, $S = \{1, 2, 3\}$, that has the following transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

a. Draw the state transition diagram for this chain.

b. If we know $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

a. The state transition diagram is shown in Figure 11.6

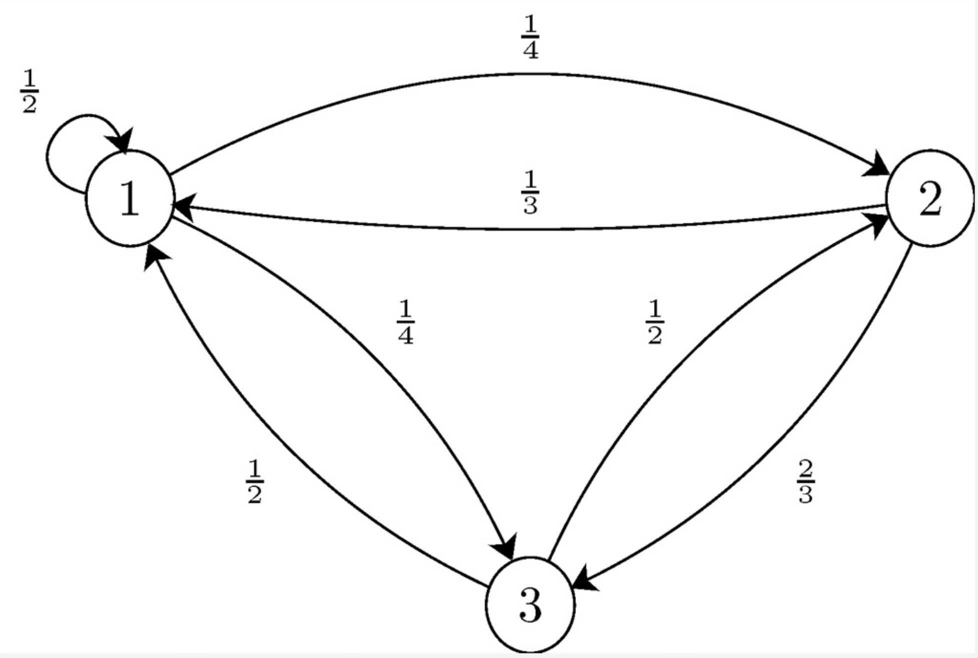


Figure 11.6 - A state transition diagram.

b. First, we obtain

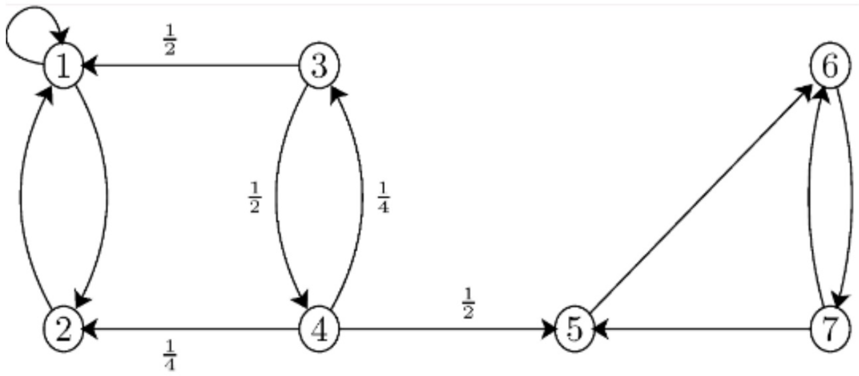
$$\begin{aligned} P(X_1 = 3) &= 1 - P(X_1 = 1) - P(X_1 = 2) \\ &= 1 - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

We can now write

$$\begin{aligned} P(X_1 = 3, X_2 = 2, X_3 = 1) &= P(X_1 = 3) \cdot p_{32} \cdot p_{21} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{12}. \end{aligned}$$

Exercise 2

Consider the Markov chain in the figure below. There are two recurrent classes, $R_1 = \{1, 2\}$, and $R_2 = \{5, 6, 7\}$.



a. Assuming $X_0 = 3$, find the probability that the chain gets absorbed in R_1.

b. Find the expected time (number of steps) until the chain gets absorbed in R_1 or R_2. More specifically, let T be the absorption time, i.e., the first time the chain visits a state in R_1 or R_2, so find $E[T \mid X_0 = 3]$

Here, we can replace each recurrent class with one absorbing state. The resulting state diagram is shown in Figure 11.18

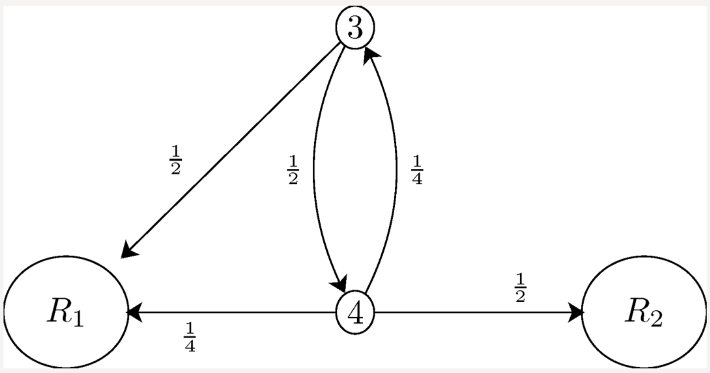


Figure 11.18 - The state transition diagram in which we have replaced each recurrent class with one absorbing state.

Now we can apply our standard methodology to find probability of absorption in state R_1. In particular, define

$$a_i = P(\text{absorption in } R_1 | X_0 = i), \quad \text{for all } i \in S.$$

By the above definition, we have $a_{R_1} = 1$, and $a_{R_2} = 0$. To find the unknown values of a_i's, we can use the following equations

$$a_i = \sum_k a_k p_{ik}, \quad \text{for } i \in S.$$

We obtain

$$\begin{aligned} a_3 &= \frac{1}{2} a_{R_1} + \frac{1}{2} a_4 \\ &= \frac{1}{2} + \frac{1}{2} a_4, \end{aligned}$$

$$\begin{aligned} a_4 &= \frac{1}{4} a_{R_1} + \frac{1}{4} a_3 + \frac{1}{2} a_{R_2} \\ &= \frac{1}{4} + \frac{1}{4} a_3. \end{aligned}$$

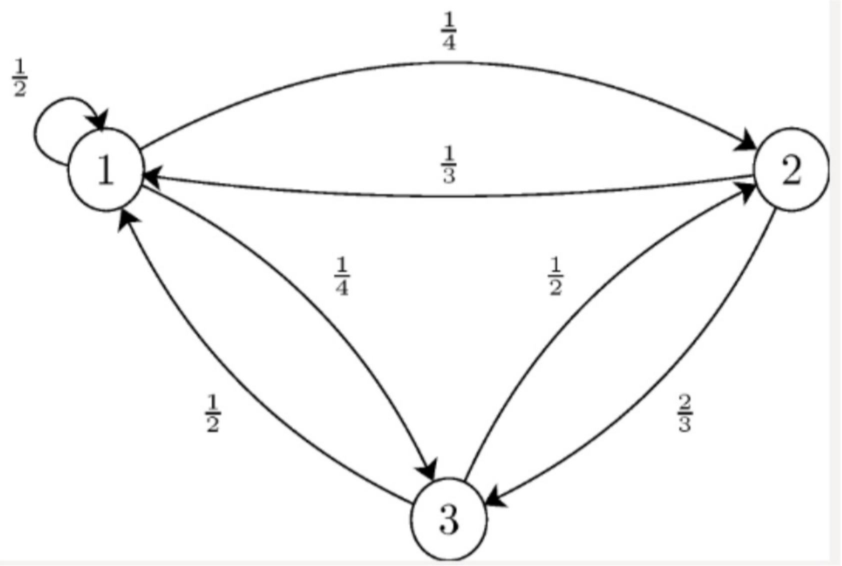
Solving the above equations, we obtain

$$a_3 = \frac{5}{7}, \quad a_4 = \frac{3}{7}.$$

Therefore, if $X_0 = 3$, the chain will end up in class R_1 with probability $a_3 = \frac{5}{7}$.

Exercise 3

Consider the following Markov chain



a. Is this chain irreducible?

b. Is this chain aperiodic?

c. Find the stationary distribution for this chain.

d. Is the stationary distribution a limiting distribution for the chain?

a. The chain is irreducible since we can go from any state to any other states in a finite number of steps.
b. The chain is aperiodic since there is a self-transition, i.e., $p_{11} > 0$.
c. To find the stationary distribution, we need to solve

$$\begin{aligned} \pi_1 &= \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3, \\ \pi_2 &= \frac{1}{4}\pi_1 + \frac{1}{2}\pi_3, \\ \pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2, \\ \pi_1 &+ \pi_2 + \pi_3 = 1. \end{aligned}$$

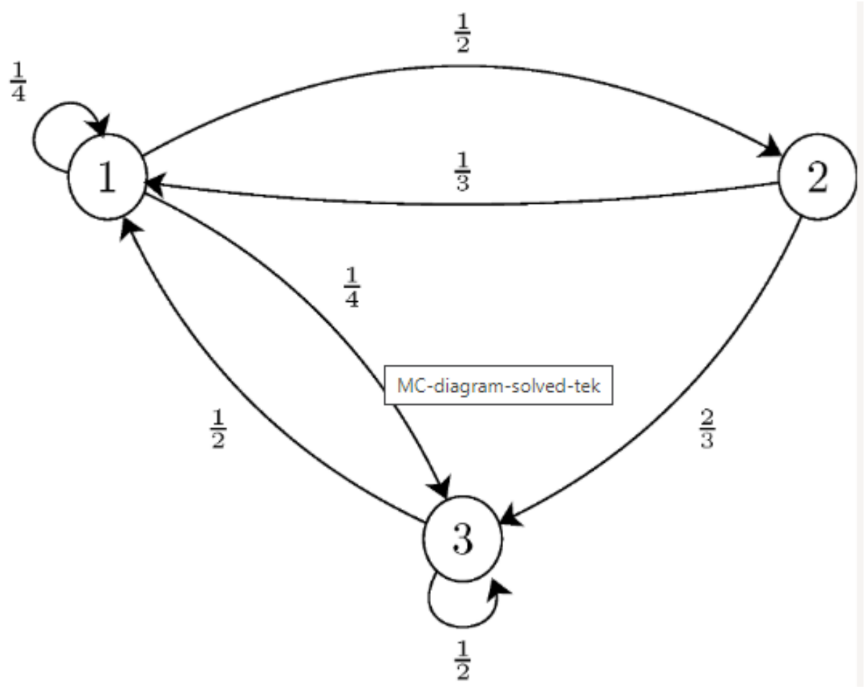
We find

$$\pi_1 \approx 0.457, \ \pi_2 \approx 0.257, \ \pi_3 \approx 0.286$$

d. The above stationary distribution is a limiting distribution for the chain because the chain is irreducible and aperiodic.

Exercise 4

Consider the following Markov chain



Assume $X_0 = 1$, and let R be the first time that the chain returns to state 1. Find $E[R \mid X_0 = 1]$.

In this question, we are asked to find the mean return time to state 1. Let r_1 be the mean return time to state 1, i.e., $r_1 = E[R|X_0 = 1]$. Then

$$r_1 = 1 + \sum_k t_k p_{1k},$$

where t_k is the expected time until the chain hits state 1 given $X_0 = k$. Specifically,

$$\begin{aligned} t_1 &= 0, \\ t_k &= 1 + \sum_j t_j p_{kj}, \qquad \text{for } k \neq 1. \end{aligned}$$

So, let's first find t_k's. We obtain

$$\begin{aligned} t_2 &= 1 + \frac{1}{3}t_1 + \frac{2}{3}t_3 \\ &= 1 + \frac{2}{3}t_3, \\ \\ t_3 &= 1 + \frac{1}{2}t_3 + \frac{1}{2}t_1 \\ &= 1 + \frac{1}{2}t_3. \end{aligned}$$

Solving the above equations, we obtain

$$t_3 = 2, \quad t_2 = \frac{7}{3}.$$

Now, we can write

$$\begin{aligned} r_1 &= 1 + \frac{1}{4}t_1 + \frac{1}{2}t_2 + \frac{1}{4}t_3 \\ &= 1 + \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot \frac{7}{3} + \frac{1}{4} \cdot 2 \\ &= \frac{8}{3}. \end{aligned}$$