This problem asks us to find the **mean return time** to a specific state (State 1) in a given Markov chain. This concept is a part of the study of **recurrence** in Markov chains. The mean return time to a state i, denoted as r_i, is the expected number of steps required to return to state i, given that the process started in state i.

Problem Description:

A Markov chain has states $S = \{0, 1, 2\}$ with the following transition matrix P:

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}$$

We need to find the mean return time to state 1, r_1, and round the answer to the nearest integer.

Key Concept: Mean Return Time and Stationary Distribution

For a recurrent state i in an irreducible Markov chain, the mean return time r_i is inversely related to its steady-state probability π_i:

$$r_i = \frac{1}{\pi_i}$$

where π_i is the i-th component of the steady-state probability vector $\pi = (\pi_0, \pi_1, \pi_2)$
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Steps to Solve:

1. Find the Steady-State Probability Vector $\pi = (\pi_0, \pi_1, \pi_2)$:

The steady-state probability vector satisfies two conditions:

- $\pi P = \pi$ (or $\pi(I - P) = 0$)
- $\sum_i \pi_i = 1$

Let's write out the equations from $\pi P = \pi$:

$$(\pi_0, \pi_1, \pi_2) \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} = (\pi_0, \pi_1, \pi_2)$$

This gives us the following system of linear equations:

- $\pi_0(0.8) + \pi_1(0.3) + \pi_2(0.2) = \pi_0$
- $\pi_0(0.1) + \pi_1(0.5) + \pi_2(0.4) = \pi_1$
- $\pi_0(0.1) + \pi_1(0.2) + \pi_2(0.4) = \pi_2$

And the sum constraint:

- $\pi_0 + \pi_1 + \pi_2 = 1$

Let's simplify the first two equations:

1. $0.8\pi_0 + 0.3\pi_1 + 0.2\pi_2 = \pi_0 \implies -0.2\pi_0 + 0.3\pi_1 + 0.2\pi_2 = 0$
2. $0.1\pi_0 + 0.5\pi_1 + 0.4\pi_2 = \pi_1 \implies 0.1\pi_0 - 0.5\pi_1 + 0.4\pi_2 = 0$

From equation (1), multiply by 10: $-2\pi_0 + 3\pi_1 + 2\pi_2 = 0 \quad \text{(Eq. A)}$
From equation (2), multiply by 10: $1\pi_0 - 5\pi_1 + 4\pi_2 = 0 \quad \text{(Eq. B)}$

From (Eq. B), we can express π_0 in terms of π_1 and π_2:
$\pi_0 = 5\pi_1 - 4\pi_2$

Substitute this into (Eq. A):
$-2(5\pi_1 - 4\pi_2) + 3\pi_1 + 2\pi_2 = 0$
$-10\pi_1 + 8\pi_2 + 3\pi_1 + 2\pi_2 = 0$
$-7\pi_1 + 10\pi_2 = 0$
So, $10\pi_2 = 7\pi_1 \implies \pi_2 = \frac{7}{10}\pi_1$

Now, substitute π_2 back into the expression for π_0:
$\pi_0 = 5\pi_1 - 4\left(\frac{7}{10}\pi_1\right) = 5\pi_1 - \frac{28}{10}\pi_1 = 5\pi_1 - \frac{14}{5}\pi_1$
$\pi_0 = \frac{25}{5}\pi_1 - \frac{14}{5}\pi_1 = \frac{11}{5}\pi_1$

Finally, use the normalization condition $\pi_0 + \pi_1 + \pi_2 = 1$:
$\frac{11}{5}\pi_1 + \pi_1 + \frac{7}{10}\pi_1 = 1$
To sum these fractions, use a common denominator of 10:
$\frac{22}{10}\pi_1 + \frac{10}{10}\pi_1 + \frac{7}{10}\pi_1 = 1$
$\frac{(22+10+7)}{10}\pi_1 = 1$
$\frac{39}{10}\pi_1 = 1$
$\pi_1 = \frac{10}{39}$

Now find π_0 and π_2:
$\pi_0 = \frac{11}{5}\pi_1 = \frac{11}{5} \cdot \frac{10}{39} = \frac{11 \cdot 2}{39} = \frac{22}{39}$
$\pi_2 = \frac{7}{10}\pi_1 = \frac{7}{10} \cdot \frac{10}{39} = \frac{7}{39}$

So, the steady-state probabilities are:
$\pi_0 = \frac{22}{39}, \pi_1 = \frac{10}{39}, \pi_2 = \frac{7}{39}$.
(Check: $22/39 + 10/39 + 7/39 = 39/39 = 1$. This is correct.)

2. Calculate the Mean Return Time to State 1 (r_1):
Using the formula $r_i = \frac{1}{\pi_i}$:
$r_1 = \frac{1}{\pi_1} = \frac{1}{10/39} = \frac{39}{10} = 3.9$

3. Round to the nearest integer:
Rounding 3.9 to the nearest integer gives **4**.

Topics Covered:

- **Markov Chains:** A mathematical model describing a sequence of possible events where the probability of each event depends only on the state attained in the previous event.
- **Transition Matrix:** A matrix whose elements represent the probabilities of transitioning from one state to another.
- **Steady-State Probabilities (Stationary Distribution):** The long-run probabilities of being in each state, which are independent of the initial state for ergodic chains. This is a crucial concept for understanding long-term behavior.
- **System of Linear Equations:** The method used to solve for the unknown steady-state probabilities.
- **Mean Return Time:** The expected number of steps to return to a given state. It's an important measure of how often a system visits a particular state in the long run.

WolframAlpha/Computational Check:

You can use WolframAlpha to verify the steady-state probabilities and the mean return time.

1. **Find the steady-state probabilities of the given matrix:**
   Input: `steady state probabilities of {{0.8, 0.1, 0.1}, {0.3, 0.5, 0.2}, {0.2, 0.4, 0.4}}`
   WolframAlpha will output:
   $\left\{\frac{22}{39}, \frac{10}{39}, \frac{7}{39}\right\}$
   This confirms our calculated π_0, π_1, π_2.

2. **Calculate the inverse of π_1:**
   Input: `1 / (10/39)`
   WolframAlpha will give `39/10` or `3.9`.

3. **Round to the nearest integer:**
   `round(3.9)` will give `4`.

These checks confirm the accuracy of our calculations.