Continuous Random Variables

A continuous R.V. has only continuous values i.e. values that are uncountable and are related to real numbers, $\mathbb{R}$.

* Time it takes to complete SMP exam
  $R_x = [0; 180]$
* Age of fossil
  $R_x = [\text{min age}; \text{max age}]$
* Km/h of 1985 BMW 5-series
  $R_x = [0; \infty[$

Main difference to discrete:
DRV measured on exact values

CRV measured on intervals
↳ Makes no sense to find probability that exam took exactly 117 min., perhaps it took 117.0132149.... min but not exactly 117
↳ $P(X=117) = 0$ → Note
↳ $P(116 \leq X \leq 118) = 0.71$

→ It makes perfect sense to find probability of 117 students attending exam: $P(X=117)$

Uniform Distribution

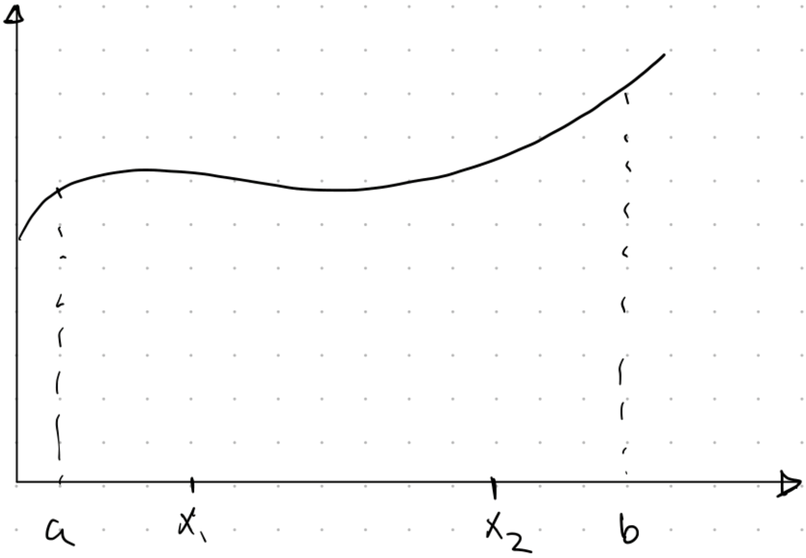
Choose a real number uniformly in $[a;b]$ and denote it X. By uniformly we mean that all intervals of same length in $[a;b]$ have same probability. Find CDF of $\mathbf{X}$:

$$P(\mathbf{X}=x) = 0$$

Means "proportional to"

$$P(x_1 \leq \mathbf{X} \leq x_2) \propto (x_2 - x_2) \quad , \; a \leq x_1 \leq x_2 \leq b$$

$P(X \in [x_1;x_2])$



$$P(X \in [a;b]) = 1$$

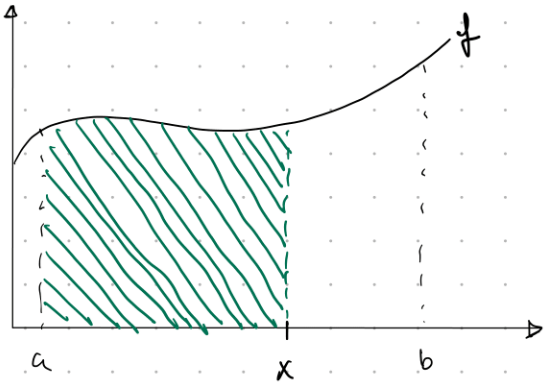
$$P(X \in [x_1;x_2]) = \frac{x_2 - x_1}{b-a}$$

$F(x) = P(X \leq x)$:

$$\left.\begin{array}{ll} F(x) = 0 & \text{for } x < a \\ F(x) = 1 & \text{for } x > b \end{array}\right\}$$ Including or not including end point not important.

For $a \leq x \leq b$:

$$F(x) = P(X \leq x)$$
$$= P(X \in [a,x])$$
$$= \frac{x-a}{b-a}$$



To summarise: The CDF of the Uniform CRV

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-1} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

More generally:

If X is a continuous R.V., then its CDF is a function s.t.:

$$F(x) = P(X \le x)$$

1. $0 \le F(x) \le 1$
2. $F(-\infty) = 0$, $F(\infty) = 1$ — $-\infty$ = lower bound, ∞ = upper bound
3. $F(x)$ is non-decreasing as x increases
4. $P(x_1 \le X \le x_2) = F(x_2) - F(x_1)$
5. $P(X \le x_1) = P(X < x_1)$
6. $P(X > x_1) = 1 - P(X \le x_1) = 1 - F(x_1)$

Probability Density Function (PDF):

Let X be a continuous R.V. The PDF is a function s.t.

$$f(x) = \lim_{h \to 0} \frac{F(x-h) - F(x)}{h} = \frac{dF(x)}{dx}$$

1. $f(x) \ge 0$
2. $\int_{-\infty}^{\infty} f(x)\,dx$
3. $\boxed{F(x) = \int_{-\infty}^{x} f(u)\,du}$ ← Important!!
4. $\int_{x_1}^{x_2} f(x)\,dx = P(x_1 \le X \le x_2) = F(x_2) - F(x_1)$
5. $P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2)$
   $= P(x_1 < X < x_2)$

Example:

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is $f(x) = 0.05$ for $0 \le x \le 20$. What is the probability that a current measurement is less than 10 milliamperes?

$$P(X < 10) = \int_0^{10} 0.05\, dx = 0.05X \Big|_0^{10} = \underline{\underline{0.5}}$$

Let's find the CDF:

$$F(x) = \int_0^x 0.05\, du = 0.05u \Big|_0^x = 0.05X$$

So we get:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.05X & \text{for } 0 \le X \le 20 \\ 1 & \text{for } x > 20 \end{cases}$$

Example

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \ge 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped?

$$P(X > 12.6) = \int_{12.6}^{\infty} 20\, e^{-20(x-12.5)}\, dx \quad , 20e^{x} \quad ; -20(X-12.5)$$

$$= -\frac{1}{20} \cdot 20\, e^{-20(X-12.5)} \Big|_{12.6}^{\infty}$$

$$= -e^{-20(\infty - 12.5)}) - (-e^{-20(12.6-12.5)})$$

$$= 0 + e^{-2} \approx \underline{\underline{0.135}}$$

OR

$$P(X > 12.6) = 1 - \int_{2.5}^{12.6} 20e^{-20(x-12.5)}\, dx$$

$$= 1 - \left[-e^{-20(X-12.5)} \Big|_{12.5}^{12.6} \right]$$

$$= 1 + \left[e^{-2} - e^{0} \right] = 1 + e^{-2} - 1$$

$$\approx 0.135$$

Let's find CDF:

$$F(x) = \int_{12.5}^{x} 20 e^{-20(u-12.5)} du = -e^{-20(u-12.5)} \Big|_{12.5}^{x}$$

$$= -e^{-20(x-12.5)} - (-e^{-20(12.5-12.5)})$$

$$= 1 - e^{-20(x-12.5)}$$

So we get

$$F(x) = \begin{cases} 0 & \text{for } x < 12.5 \\ 1 - e^{-20(x-12.5)} & \text{for } 12.5 \leq x \end{cases}$$

Example

The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.01x} & 0 \leq x \end{cases}$$

Determine the probability density function of X. What proportion of reactions is complete within 200 milliseconds?

$$f(x) = F'(x) = (1 - e^{-0.01x})' = 0.01 e^{-0.01x}$$

So we get

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.01 e^{-0.01x} & \text{for } x \geq 0 \end{cases}$$

$$F(200) = 1 - e^{-0.01 \cdot 200}$$

$$= 1 - e^{-2} = \underline{\underline{0.865}}$$

Example:

Let
$$f(x) = \begin{cases} 4a \cdot e^{-2x} & , x \geq 0 \\ 0 & \text{else} \end{cases}$$

a) Find a:

$$\int_0^\infty 4a \cdot e^{-2x} dx = 1 \Rightarrow -\frac{4}{2} a \cdot e^{-2x} \Big|_0^\infty = 1 \iff 2ae^{-0}) = 1$$

$$\Rightarrow \underline{\underline{a = 1/2}}$$

b) Find F(x):

$$\int_0^x 4 \cdot 1/2 \cdot e^{-2 \cdot u} du = -\frac{1}{2} \cdot 2 e^{-2u} \Big|_0^x = -e^{-2x} - (-e^{-2 \cdot 0})$$

$$= \underline{\underline{1 - e^{-2x}}}$$

c) Find $P(1 \leq X \leq 3)$

$$P(1 \leq X \leq 3) = F(3) - F(1) = (1 - e^{-2 \cdot 3}) - (1 - e^{-2 \cdot 1})$$

$$= \underline{\underline{-e^{-6} + e^{-2}}}$$

or

$$= \int_1^3 2e^{-2 \cdot x} dx =$$

Expected Value:

$$E(X) = \boxed{\int_{-\infty}^{\infty} x f(x) dx}$$

Variance

General for all R.V.s

$$Var(X) = E[(X - \mu_X)^2] = E[X^2] - (E[X])^2$$

$$= \boxed{\int_{-\infty}^{\infty} x^2 \cdot f(x) dx} \leftarrow \text{for C.R.V.}$$

Law of the unconsious statistician (LOTUS):

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx$$

PDF

Example

Let

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Find $E(X)$:

$$E(x) = \int_0^1 x \cdot 2x \, dx = \int_0^1 2x^2 \, dx = 2/3 \, x^3 \Big|_0^1$$

$$= \underline{\underline{2/3}}$$

Example

Let

$$f(x) = \begin{cases} x^2(2x+\frac{3}{2}) & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

If $Y = \frac{2}{X} + 3$, Find $Var(Y)$

Recall,

$$Var(aX+b) = a^2 Var(X)$$

Note:

$$Var(Y) = Var\left(\frac{2}{X}+3\right) = 4 \, Var\left(\frac{1}{X}\right), \text{ so}$$

$$Var\left(\frac{1}{X}\right) = E\left[\frac{1}{X^2}\right] - E\left[\left(\frac{1}{X}\right)\right]^2$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \frac{1}{x^2} \cdot x^2(2x+3/2) \, dx = \int_0^1 (2x+3/2) \, dx$$

$$= x^2 + \frac{3x}{2} \Big|_0^1 = 1 + 3/2 = 5/2$$

$$E\left[\frac{1}{X}\right] = \int_0^1 \frac{1}{x} \, x^2(2x+3/2) \, dx = \int_0^1 \left(2x^2 + \frac{3x}{2}\right) dx$$

$$= 2/3 \, x^3 + \frac{3x^2}{4} \Big|_0^1 = \frac{2}{3} + \frac{3}{4} = \frac{8+9}{12} = \frac{17}{12}$$

$$Var(Y) = 4\left(\frac{5}{2} - \left(\frac{17}{12}\right)^2\right) = 4\left(\frac{5}{2} - \frac{289}{144}\right)$$

$$= 4\left(\frac{360-289}{144}\right) = 4\left(\frac{71}{144}\right) = \underline{\underline{\frac{71}{36}}}$$

Uniform R.V.

Recall $F(x) = \frac{x-a}{b-a}$, so

$$f(x) = F'(x) = \left(\frac{x}{b-a}\right)' - \left(\frac{a}{b-a}\right)'$$

$$= \frac{1}{b-a}$$

$$\boxed{f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a \text{ or } x > b \end{cases}}$$

Expected value:

$$E(X) = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \cdot \frac{1}{2} x^2 \Big|_a^b$$

$$= \frac{1}{2}\left(\frac{b^2}{b-a} - \frac{a^2}{b-a}\right) = \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{1}{2}\frac{(b-a)(b+a)}{b-a}$$

$$= \boxed{\frac{a+b}{2}}$$

Variance

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a} = \frac{1}{3}\frac{x^3}{b-a}\Big|_a^b$$

$$= \frac{1}{3}\left(\frac{b^3}{b-a} - \frac{a^3}{b-a}\right) = \frac{1}{3}\left(\frac{b^3-a^3}{b-a}\right)$$

$$= \frac{1}{3}\frac{(b-a)(a^2+ab+b^2)}{b-a} = \frac{a^2+ab+b^2}{3}$$

So,

$$Var(X) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \boxed{\frac{(b-a)^2}{12}}$$

Functions of Continuous R.V.'s: Difficult

If X is a CRV and $Y = g(X)$, then Y is also a R.V.

Example

Let X be a Uniform$(0,1)$ R.V. and let $Y = e^X$

a) Find CDF of Y

$$F_X(x) = \frac{x-a}{b-a} = \frac{x-0}{1-0} = x \quad , \quad f(x) = 1$$

$$R_X = [0;1] \quad , \quad R_Y = [1;e]$$

$$F(Y) = P(Y \leq y)$$
$$= P(e^X \leq y)$$
$$= P(X \leq \ln y)$$

So

$$F_Y = \begin{cases} 0 & y < 1 \\ \ln y & 1 \leq y < e \\ 1 & y \geq e \end{cases}$$

b) Find pdf of Y:

$$f_Y(y) = F_Y'(y) = \underline{\underline{\frac{1}{y}}} \quad \text{for } 1 \leq y \leq e \text{, else } 0$$

c) Find $E[Y]$:

$$E[Y] = \int_1^e y \cdot \frac{1}{y} \, dy = y \Big|_1^e = \underline{\underline{e-1}}$$

Using LOTUS:

$$E[Y] = E[e^x] = \int_0^1 e^x \cdot f_X(x) \, dx = e^x \Big|_0^1$$
$$= e^1 - e^0 = \underline{\underline{e-1}}$$