Topics Covered:

- Probability Theory
- Discrete Stochastic Vectors
- Joint Probability Mass Functions (PMF)
- Marginal Probability Distributions (PMF)

Question (a): Find the marginal probability distributions of X and Y.

The joint probability mass function (PMF) $p_{X,Y}(x, y)$ is given by:

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{4} \cdot (1-p)p^y & \text{if } x \in 1,2,3,4 \text{ and } y \in 0,1,2,\dots \\ 0 & \text{otherwise} \end{cases}$$

Here, $p \in]0, 1[$.

Finding the marginal probability distribution of X, denoted $p_X(x)$:

To find $p_X(x)$, we sum the joint PMF $p_{X,Y}(x, y)$ over all possible values of y.

$$p_X(x) = \sum_{y=0}^{\infty} p_{X,Y}(x,y)$$

For $x \in 1, 2, 3, 4$:

$$p_X(x) = \sum_{y=0}^{\infty} \frac{1}{4}(1-p)p^y$$

This is a geometric series summation. Recall the formula for the sum of an infinite geometric series: $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$, provided $|r| < 1$. In our case, $a = \frac{1}{4}(1-p)$ and $r = p$. Since $p \in]0, 1[$, the sum converges.

$$p_X(x) = \frac{1}{4}(1-p)\sum_{y=0}^{\infty} p^y$$

$$p_X(x) = \frac{1}{4}(1-p) \cdot \frac{1}{1-p}$$

$$p_X(x) = \frac{1}{4}$$

Therefore, the marginal PMF for X is:

$$p_X(x) = \begin{cases} \boxed{\frac{1}{4}} & \text{if } x \in 1, 2, 3, 4 \\ 0 & \text{else} \end{cases}$$

Wolfram Alpha Shortcut for Geometric Series Sum:

`sum (p^y) for y from 0 to infinity` (This will give `1/(1-p)`)

Finding the marginal probability distribution of Y, denoted $p_Y(y)$:

To find $p_Y(y)$, we sum the joint PMF $p_{X,Y}(x, y)$ over all possible values of x.

$$p_Y(y) = \sum_{x=1}^{4} p_{X,Y}(x, y)$$

For $y \in 0, 1, 2, \ldots$:

$$p_Y(y) = \sum_{x=1}^{4} \frac{1}{4}(1-p)p^y$$

Since the term $\frac{1}{4}(1-p)p^y$ does not depend on x, we are simply adding this term 4 times (for $x = 1, 2, 3, 4$).

$$p_Y(y) = 4 \cdot \left(\frac{1}{4}(1-p)p^y\right)$$

$$p_Y(y) = (1-p)p^y$$

Therefore, the marginal PMF for Y is:

$$p_Y(y) = \begin{cases} \boxed{(1-p)p^y} & \text{if } y \in 0, 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

Wolfram Alpha Shortcut for Summation of a Constant:

`sum (C) for x from 1 to 4` (This will give `4C`)

Topics Covered:

- Probability Theory
- Discrete Stochastic Vectors
- Joint Probability Mass Functions (PMF)
- Marginal Probability Distributions (PMF)
- Independence of Random Variables
- Probability of Union and Intersection of Events

Question (b): Find the values below.

For the rest of the assignment, we are given the fact that $p = \frac{1}{2}$.

From the previous question, we derived the marginal PMFs:

- $p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x \in \{1, 2, 3, 4\} \\ 0 & \text{else} \end{cases}$
- $p_Y(y) = \begin{cases} (1-p)p^y & \text{if } y \in \{0, 1, 2, \dots\} \\ 0 & \text{else} \end{cases}$

Substituting $p = \frac{1}{2}$ into $p_Y(y)$:
$p_Y(y) = \left(1 - \frac{1}{2}\right)\left(\frac{1}{2}\right)^y = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^y = \left(\frac{1}{2}\right)^{y+1}$ for $y \in \{0, 1, 2, \dots\}$.

Also, as verified in the thought process, X and Y are independent random variables because $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$.

1. Calculate $P(X \leq 2)$:

To find $P(X \leq 2)$, we sum the probabilities for $X = 1$ and $X = 2$ using $p_X(x)$:

$$P(X \leq 2) = p_X(1) + p_X(2)$$

Since $p_X(x) = \frac{1}{4}$ for $x \in \{1, 2, 3, 4\}$:

$$P(X \leq 2) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$P(X \le 2) = \boxed{\frac{1}{2}}$$

Wolfram Alpha Shortcut:

`probability X <= 2 for X distributed uniformly on {1,2,3,4}`

2. Calculate $P(Y \le 1)$:

To find $P(Y \le 1)$, we sum the probabilities for $Y = 0$ and $Y = 1$ using $p_Y(y)$:

$$P(Y \le 1) = p_Y(0) + p_Y(1)$$

Using $p_Y(y) = \left(\frac{1}{2}\right)^{y+1}$:

$$p_Y(0) = \left(\frac{1}{2}\right)^{0+1} = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}$$

$$p_Y(1) = \left(\frac{1}{2}\right)^{1+1} = \left(\frac{1}{2}\right)^{2} = \frac{1}{4}$$

$$P(Y \le 1) = \frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$$

$$P(Y \le 1) = \boxed{\frac{3}{4}}$$

Wolfram Alpha Shortcut:

`probability Y <= 1 for Y distributed as geometric distribution with p=1/2`
(Note: Wolfram Alpha's geometric distribution typically defines $P(Y = k) = (1 - p)^k p$ for $k = 0, 1, \ldots$ or $P(Y = k) = p(1 - p)^{k-1}$ for $k = 1, 2, \ldots$. Our definition is $P(Y = y) = (1 - p)p^y$ which corresponds to the first type with parameter $1 - p$. So, if $p_{success} = 1 - p$, then $P(Y = y) = p_{success}(1 - p_{success})^y$. Our p is the probability of failure. With $p = 1/2, 1 - p = 1/2$, so it's $P(Y = y) = (1/2)(1/2)^y$. So you can use: `probability Y <= 1 for Y distributed as geometric distribution with P(success) = 1/2, starting from 0`)

3. Calculate $P(\{X \le 2\} \cap \{Y \le 1\})$:

Since X and Y are independent, the probability of their intersection is the product of their individual probabilities:

$$P(\{X \le 2\} \cap \{Y \le 1\}) = P(X \le 2) \cdot P(Y \le 1)$$

Using the values calculated above:

$$P(\{X \le 2\} \cap \{Y \le 1\}) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

$$P(\{X \le 2\} \cap \{Y \le 1\}) = \boxed{\frac{3}{8}}$$

This matches the given value in the prompt.

Wolfram Alpha Shortcut:

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P(X <= 2) * P(Y <= 1) where P(X <= 2) = 1/2 and P(Y <= 1) = 3/4
```

4. Calculate $P(\{X \le 2\} \cup \{Y \le 1\})$:

We use the formula for the probability of the union of two events:
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Let $A = \{X \le 2\}$ and $B = \{Y \le 1\}$.

$P(\{X \le 2\} \cup \{Y \le 1\}) = P(X \le 2) + P(Y \le 1) - P(\{X \le 2\} \cap \{Y \le 1\})$

Substitute the calculated values:

$$P(\{X \le 2\} \cup \{Y \le 1\}) = \frac{1}{2} + \frac{3}{4} - \frac{3}{8}$$

To sum these fractions, find a common denominator, which is 8:

$$P(\{X \le 2\} \cup \{Y \le 1\}) = \frac{4}{8} + \frac{6}{8} - \frac{3}{8}$$

$$P(\{X \le 2\} \cup \{Y \le 1\}) = \frac{4+6-3}{8} = \frac{10-3}{8} = \frac{7}{8}$$

$$P(\{X \le 2\} \cup \{Y \le 1\}) = \boxed{\frac{7}{8}}$$

This matches the given value in the prompt.

Wolfram Alpha Shortcut:

`1/2 + 3/4 - 3/8`

Topics Covered:

- Probability Theory
- Expected Value of a Discrete Random Variable
- Variance of a Discrete Random Variable

Question ©: Find the expected value and variance of X.

From question (a), the marginal probability mass function (PMF) of X is:

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x \in \{1,2,3,4\} \\ 0 & \text{else} \end{cases}$$

1. Find the Expected Value of X, denoted $E[X]$:

The expected value of a discrete random variable X is calculated as:

$$E[X] = \sum_x x \cdot p_X(x)$$

For our X, the possible values are $\{1,2,3,4\}$, each with probability $\frac{1}{4}$.

$$E[X] = (1 \cdot p_X(1)) + (2 \cdot p_X(2)) + (3 \cdot p_X(3)) + (4 \cdot p_X(4))$$

$$E[X] = \left(1 \cdot \frac{1}{4}\right) + \left(2 \cdot \frac{1}{4}\right) + \left(3 \cdot \frac{1}{4}\right) + \left(4 \cdot \frac{1}{4}\right)$$

$$E[X] = \frac{1}{4}(1+2+3+4)$$

$$E[X] = \frac{1}{4}(10)$$

$$E[X] = \frac{10}{4} = \frac{5}{2}$$

$$E[X] = \boxed{\frac{5}{2}}$$

Wolfram Alpha Shortcut:

`expected value of {1,2,3,4} with equal probabilities` or `mean of {1,2,3,4}`

2. Find the Variance of X, denoted $Var[X]$:

The variance of a discrete random variable X can be calculated using the formula:

$$Var[X] = E[X^2] - (E[X])^2$$

First, we need to find $E[X^2]$:

$$E[X^2] = \sum_x x^2 \cdot p_X(x)$$

$$E[X^2] = (1^2 \cdot p_X(1)) + (2^2 \cdot p_X(2)) + (3^2 \cdot p_X(3)) + (4^2 \cdot p_X(4))$$

$$E[X^2] = \left(1 \cdot \frac{1}{4}\right) + \left(4 \cdot \frac{1}{4}\right) + \left(9 \cdot \frac{1}{4}\right) + \left(16 \cdot \frac{1}{4}\right)$$

$$E[X^2] = \frac{1}{4}(1 + 4 + 9 + 16)$$

$$E[X^2] = \frac{1}{4}(30)$$

$$E[X^2] = \frac{30}{4} = \frac{15}{2}$$

Now, substitute $E[X^2]$ and $E[X]$ into the variance formula:

$$Var[X] = E[X^2] - (E[X])^2$$

$$Var[X] = \frac{15}{2} - \left(\frac{5}{2}\right)^2$$

$$Var[X] = \frac{15}{2} - \frac{25}{4}$$

To subtract, find a common denominator, which is 4:

$$Var[X] = \frac{30}{4} - \frac{25}{4}$$

$$Var[X] = \frac{30 - 25}{4}$$

$$Var[X] = \frac{5}{4}$$

$$Var[X] = \boxed{\frac{5}{4}}$$

Wolfram Alpha Shortcut:

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variance of {1,2,3,4} with equal probabilities
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Topics Covered:

- Probability Theory
- Expected Value of a Random Variable
- Variance of a Random Variable
- Covariance of Random Variables
- Properties of Expected Value, Variance, and Covariance
- Independence of Random Variables

Question (d): Find the values below.

We are given the following information:

- From part ©: $E[X] = \frac{5}{2}$ and $Var[X] = \frac{5}{4}$
- Given in this part: $E[Y] = 1$ and $Var[Y] = 2$
- From part (b), we established that X and Y are independent. This implies that $Cov(X, Y) = 0$.

1. Calculate $Var(X - Y)$:

Using the property of variance for two random variables A and B:
$Var(A - B) = Var(A) + Var(B) - 2Cov(A, B)$

Since X and Y are independent, $Cov(X, Y) = 0$.
Therefore, the formula simplifies to:
$Var(X - Y) = Var(X) + Var(Y)$

Substitute the known values:
$Var(X - Y) = \frac{5}{4} + 2$

$Var(X - Y) = \frac{5}{4} + \frac{8}{4}$
$Var(X - Y) = \frac{13}{4}$

$$Var(X - Y) = \boxed{\frac{13}{4}}$$

Wolfram Alpha Shortcut:

`variance of X - Y where variance X = 5/4, variance Y = 2, and X, Y are independent`

2. Calculate $Cov(X, X - Y)$:

Using the linearity property of covariance:
$Cov(A, B - C) = Cov(A, B) - Cov(A, C)$

Applying this to our expression:
$Cov(X, X - Y) = Cov(X, X) - Cov(X, Y)$

We know that $Cov(X, X)$ is simply the variance of X, i.e., $Cov(X, X) = Var(X)$.
Also, since X and Y are independent, $Cov(X, Y) = 0$.

Substitute the known values:
$Cov(X, X - Y) = Var(X) - 0$
$Cov(X, X - Y) = \frac{5}{4}$

$$Cov(X, X - Y) = \boxed{\frac{5}{4}}$$

Wolfram Alpha Shortcut:

`covariance of X, X - Y where variance X = 5/4 and covariance X, Y = 0`

3. Calculate $E[Y \cdot |X - 3|]$:

To find $E[Y \cdot |X - 3|]$, we use the property of expectation for independent random variables: if X and Y are independent, then $E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$.

In this case, let $g(X) = |X - 3|$ and $h(Y) = Y$.
So, $E[Y \cdot |X - 3|] = E[Y] \cdot E[|X - 3|]$.

We are given $E[Y] = 1$.

Now, we need to calculate $E[|X-3|]$.
Recall the marginal PMF for X:
$p_X(x) = \frac{1}{4}$ for $x \in \{1,2,3,4\}$ and 0 otherwise.

The expected value of $|X-3|$ is:

$$E[|X-3|] = \sum_x |x-3| \cdot p_X(x)$$

$$E[|X-3|] = |1-3| \cdot p_X(1) + |2-3| \cdot p_X(2) + |3-3| \cdot p_X(3) + |4-3| \cdot p_X(4)$$

$$E[|X-3|] = |-2| \cdot \frac{1}{4} + |-1| \cdot \frac{1}{4} + |0| \cdot \frac{1}{4} + |1| \cdot \frac{1}{4}$$

$$E[|X-3|] = 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4}$$

$$E[|X-3|] = \frac{2}{4} + \frac{1}{4} + 0 + \frac{1}{4}$$

$$E[|X-3|] = \frac{2+1+0+1}{4} = \frac{4}{4} = 1$$

Finally, substitute the values back into the expression for $E[Y \cdot |X-3|]$:

$$E[Y \cdot |X-3|] = E[Y] \cdot E[|X-3|] = 1 \cdot 1 = 1$$

$$E[Y \cdot |X-3|] = \boxed{1}$$

Wolfram Alpha Shortcut:

`expected value of |X - 3| for X distributed uniformly on {1,2,3,4}`

`expected value of Y * expected value of |X-3| where expected value Y = 1 and expected value |X-3| = 1`