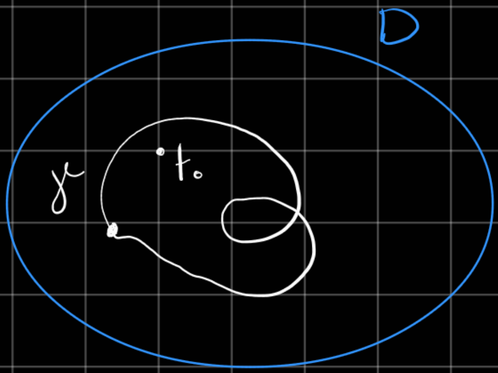
Kompleksni integral in Cauchyjeve integralske formule

$f: D \to \mathbb{C}$ hol. fje in $\gamma: [a, b] \to D$ sklenjen pot v D.
Naj bo D enostavno povezana (brez lukenj)



i) $\displaystyle\int_\gamma f(z)\,dz = 0$

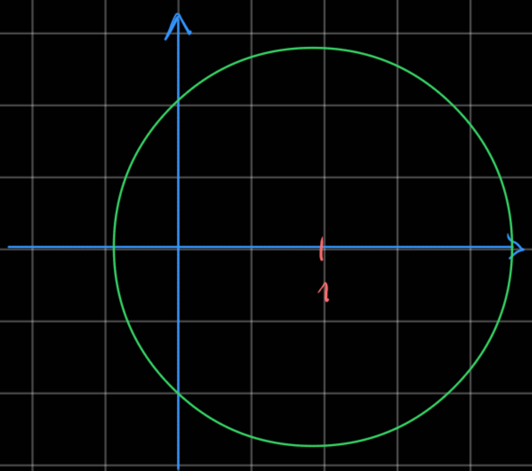
ii) $z_0 \in D$ in $z_0 \notin \gamma$

$$\frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\,dz = I_\gamma(z_0)\, f(z_0)$$

$\downarrow$ indeks poti γ okoli točke z_0. Pove nam, kolikokrat obkrožimo z_0 v + ali − meri

iii) $$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,dz = I_\gamma(z_0)\, \frac{1}{n!} f^{(n)}(z_0)$$

① Izračunaj $\displaystyle\int_K \frac{dz}{(1+z)(z-1)^3}$, kjer je K poz. orientirana krivulja dana z $K = \{z \in \mathbb{C}, |z - 1| = \frac{3}{2}\}$



$$f(z) = \frac{1}{(1+z)(z-1)^3}$$

$$D_f = \mathbb{C} \setminus \{-1, 1\}$$

K 1x v poz. meri obkroži tč. 1

$$\frac{1}{2\pi i}\oint \frac{1}{(1+z)(z-1)^3}dz = \frac{1}{2\pi i}\oint_K \frac{(1+z)^{-1}}{(z-1)^3}dz = 1\cdot\frac{1}{2!}\left((1+z)^{-1}\right)^{(2)}\Big|_{z=1}$$

$$= \frac{1}{2}\left(\frac{2}{(1+z)^3}\right) = \frac{1}{8}$$

Cauchyjeva formula za območje

Naj bo $f: D \to \mathbb{C}$, holomorfna na D in zvezna na ∂D (D ni nujno enostavno povezana). Naj bo ∂D poz. orientirana.

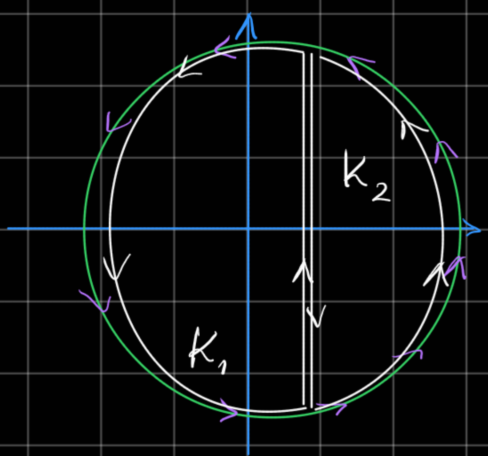
Pri koji rob polje poz. orientirano, ko je območja na levi strani.

1) $\oint_{\partial D} f(z)\,dz = 0$

2) $z_0 \in D$: $\frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)}{z-z_0}dz = f(z_0)$

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{1}{n!}f^{(n)}(z_0)$$

② Izračunaj $\oint_{\partial D}\frac{e^z}{z(1-z)^3}dz$, kjer je ∂D poz. orientirana rob območja $D = \{z \in \mathbb{C}, |z| \le 3\}$



$$f(z) = \frac{e^z}{z(1-z)^3}\ ;\ D_f = \mathbb{C}\setminus\{0,1\}$$

$$\oint_{\partial D}\frac{e^z}{z(1-z)^3}dz = \oint_{K_1}\frac{e^z}{z(1-z)^3}dz +$$

$$+\oint_{K_2} \frac{e^z}{z(1-z)^3}\,dz = \oint_{K_1} \frac{\frac{e^z}{(1-z)^3}}{z} + \oint_{K_2} \frac{\frac{e^z}{z}}{(1-z)^3}$$

holomorfna na K_1

$$K_1: \frac{1}{2\pi i} \oint_{K_1} \frac{\frac{e^z}{(1-z)^3}}{z}\,dz = \left(\frac{e^z}{(1-z)^3}\right)\Bigg|_{z=0} = 1$$

$$\left(\frac{e^z}{z}\right)^{(2)} = \left(\frac{e^z z - e^z}{z^2}\right)^{(1)} = \frac{z^2(e^z z + e^z - e^z) - 2z(e^z z - e^z)}{z^4}$$

$$K_2: \frac{1}{2\pi i} \oint_{K_2} \frac{\frac{e^z}{z}}{(1-z)^3}\,dz = -\frac{1}{2\pi i} \oint_{K_2} \frac{\frac{e^z}{z}}{(z-1)^3}\,dz = -\frac{1}{2!}\left(\frac{e^z}{z}\right)^{(2)}\Bigg|_{z=1}$$

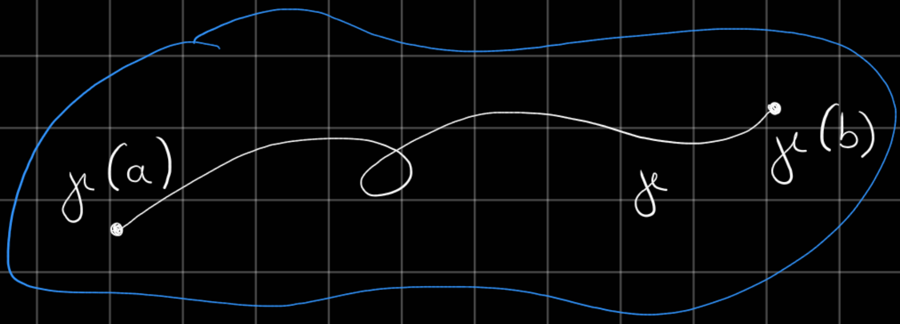
$$= -\frac{1}{2!} \frac{z^2(e^z z + e^z - e^z) - 2z(e^z z - e^z)}{z^4}\Bigg|_{z=1}$$

$$= -\frac{e}{2}$$

$$\oint_{\partial D} \frac{e^z}{z(1-z)^3}\,dz = 2\pi i - 2\pi i \frac{e}{2}$$

Nedoločeni int. hol. fje

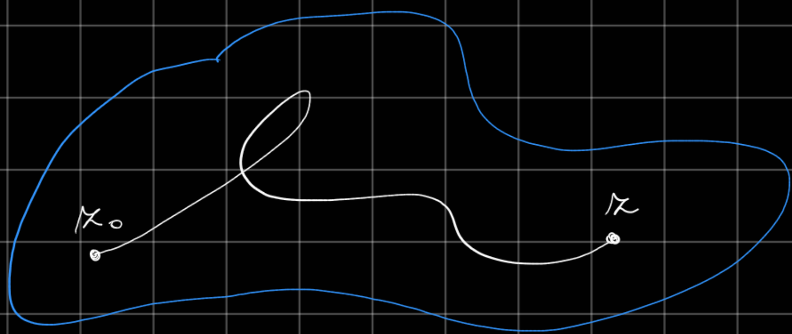
$f: D \to \mathbb{C}$ hol. in $\gamma: [a,b] \to D$ pot v D



1) $\int_\gamma f'(z)\,dz = f(\gamma(b)) - f(\gamma(a))$

2) Če sta f in D takšna, da je $\oint_\gamma f(z)\,dz = 0$ [γ polj. skl. pot]
$\Rightarrow \exists$ hol. fja $F: D \to \mathbb{C}$, da je $F'(z) = f(z)$
($\exists$ nedoločeni int. f)

Op.: Kako je nedol. int. $f(z)$ def.:



$F(z_0) = 0$

$F(z) = \int_\gamma f(z)\,dz$,

γ povezuje z_0 in z

3) Naj bo $f: D \to \mathbb{C}$ hol. in D enostavno povezana območje.
$\Rightarrow \exists$ nedoloč. int. $F: D \longrightarrow \mathbb{C}$, $F'(z) = f(z)$

4) Če je $F(z)$ nedoloč. int. fje $f(z)$ in γ pot od tč. z_0 do z_1 $\Rightarrow \int_\gamma f(z)\,dz = F(z_1) - F(z_0)$

Zadnjič $\int_K \frac{dz}{z^2+9}$

Uporabili smo Cauchyjevo formula

$$F'(z) = \frac{1}{z^2+9} = \frac{1}{9\left(1+\left(\frac{z}{3}\right)^2\right)} \Rightarrow F(z) = \frac{1}{3}\operatorname{arctg}\left(\frac{z}{3}\right)$$

K poz. orient. krožnica, $R=1$ s središčem v $\emptyset$

Op.: $\oint_K f(z)\,dz \neq \emptyset \Rightarrow$ s pomočjo Cauchyjeve int. formule,

nikoli pa nismo izračunali $F'(z) = f(z)$

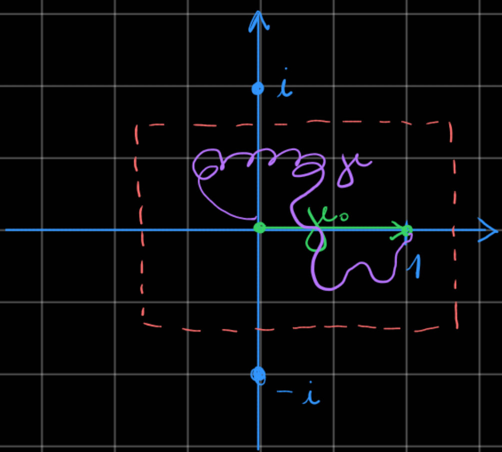
3 Dana je funkcija $f(z) = \frac{1}{1+z^2}$

a) Določi D_f

b) Izračunaj $\int_{\gamma_0} f(z)\,dz$, kjer je γ_0 daljica med 0 in 1

c) Izračunaj $\int_{\gamma} f(z)\,dz$, kjer je γ poljubna pot med 0 in 1

a)



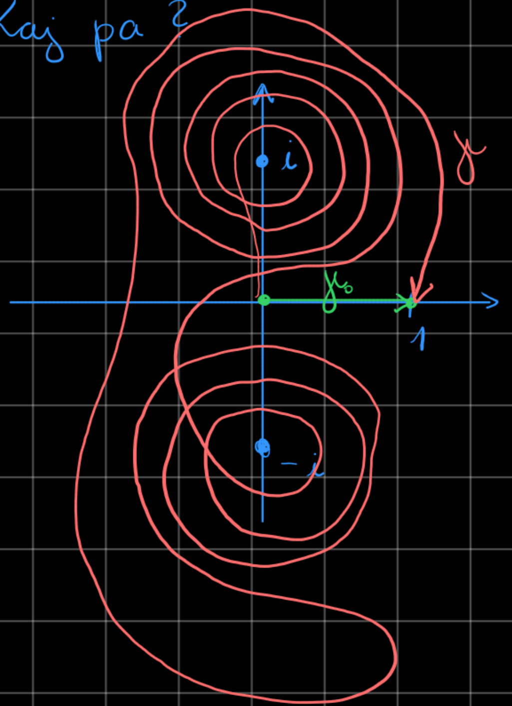
Pravokotnik brez $\pm i$ je enostavno povezano območje.

b) $\gamma_0(t) = t, \; t \in [0,1]$

$$\int_{\gamma_0} \frac{1}{1+x^2}\,dx = \int_0^1 \frac{1}{1+t^2}\,dt = \quad g(t)\Big|_0^1 = \frac{\pi}{4}$$

c) $\displaystyle\int_{\gamma'} \frac{dx}{1+x^2} = \frac{\pi}{4}$ // $\forall$, ki jih lahko definiramo na $\mathbb{C}\setminus\{-i, i\}$ dif. v γ_0

Kaj pa?



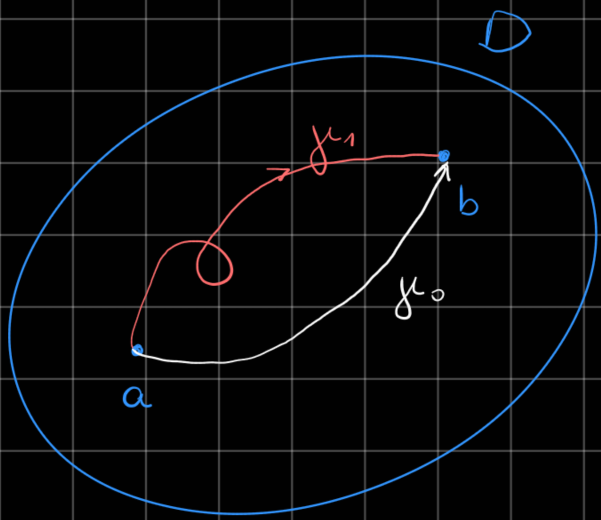
Ali je $\displaystyle\int_{\gamma} f(z)\,dz = \frac{\pi}{4}$ za $\forall \gamma$? Zdi se, da ne.

Naj bo γ poljubna pot $0 \to 1$.
Zanima nas

$$\int_{\gamma} \frac{dz}{1+x^2} = ?$$

Deformacije: $\gamma_0 : [a,b] \to D$

$\gamma_1 : [a,b] \to D$



Deformacija $\gamma_0 \to \gamma_1$ (homotopija od γ_0 do γ_1)

$H : [a,b] \times [0,1] \to D$

$H(t,0) = \gamma_0(t)$

$H(t,1) = \gamma_1(t)$

Konkatenacija $\gamma_0 * \gamma$ = zanka: začnemo v $\emptyset$, nato vzdolž γ_0 do 1, nato pa v obratni meri vzdolž γ do $\emptyset$.

$$\int_{\gamma_0 * \gamma} \frac{dz}{1+z^2} = \int_{\gamma_0 * \gamma} \frac{dz}{(z+i)(z-i)} \overset{\text{s pomočjo Cauchy. formule}}{=} \int_{K_1} \frac{dz}{(z+i)(z-i)} + \int_{K_2} \frac{dz}{(z+i)(z-i)} =$$

$$= \int_{K_1} \frac{\frac{dz}{(z+i)}}{(z-i)} + \int_{K_2} \frac{\frac{dz}{(z-i)}}{(z+i)} = 2\pi i \overbrace{\mathcal{I}_{K_1}(i)}^{m_1} \frac{1}{2i} + 2\pi i \underbrace{\mathcal{I}_{K_2}(-i)}_{m_2} \left(-\frac{1}{2i}\right)$$

$\parallel$ $\mathcal{I}_{K_1}(i)$ $\qquad$ $\parallel$ $\mathcal{I}_{K_2}(-i)$

K_1 obkroži samo i

K_2 obkroži samo $-i$

$$= m_1 \pi - m_2 \pi = \pi(m_1 - m_2)$$

$$\int_{\gamma_0} \frac{dz}{1+z^2} - \int_{\gamma} \frac{dz}{1+z^2} = \pi(m_1 - m_2) \Rightarrow \int_{\gamma} \frac{dz}{1+z^2} = \frac{\pi}{4} - \pi(m_1 - m_2)$$

Taylorjeva in Laurentova vrsta hol. fje

Taylorjeva vrsta: $f(z)$ ... hol. fja na $B(z_0, R) \Rightarrow$ na tem odprtem disku velja $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \ldots$

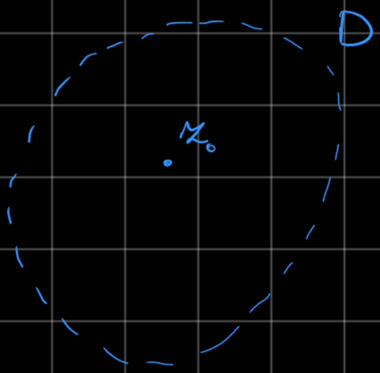
// konv. radij R

$$c_n = \frac{f^{(n)}(z_0)}{n!} \qquad \left(c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz \right)$$

$\hookrightarrow$ obkroži $1\times$ z_0

Posledica $f: D \to \mathbb{C}$ hol. fja

$z_0 \in D$

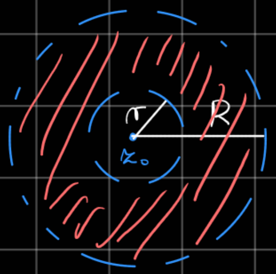


$$\Rightarrow f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Zanima nas konvergenčni radij vrste?

Laurentova vrsta

$f(z)$ je holomorfna na kolobarju $r < |z-z_0| < R$
$(r = 0,\ R = \infty)$



$\Rightarrow$ na kolobarju velja

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \dots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)}$$

$$+ c_0 + c_1 (z-z_0) + c_2 (z-z_0)^2 + \dots$$

4 Razvij $f(z) = z^2 e^{\frac{1}{z}}$ v Laurentovo vrsto okoli $z = 0$

$f(z)$ ni dif. v $z = 0$ $\Rightarrow$ f je res holomorfna na kolobarju $0 < |z| < \infty$

Uporabimo znane razvoje: $f(z) = z^2 \frac{1}{z} = z^2 \left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \dots\right)$

$$= \dots \frac{1}{24}\frac{1}{z^2} + \frac{1}{6}\frac{1}{z} + \frac{1}{2} + z + z^2$$