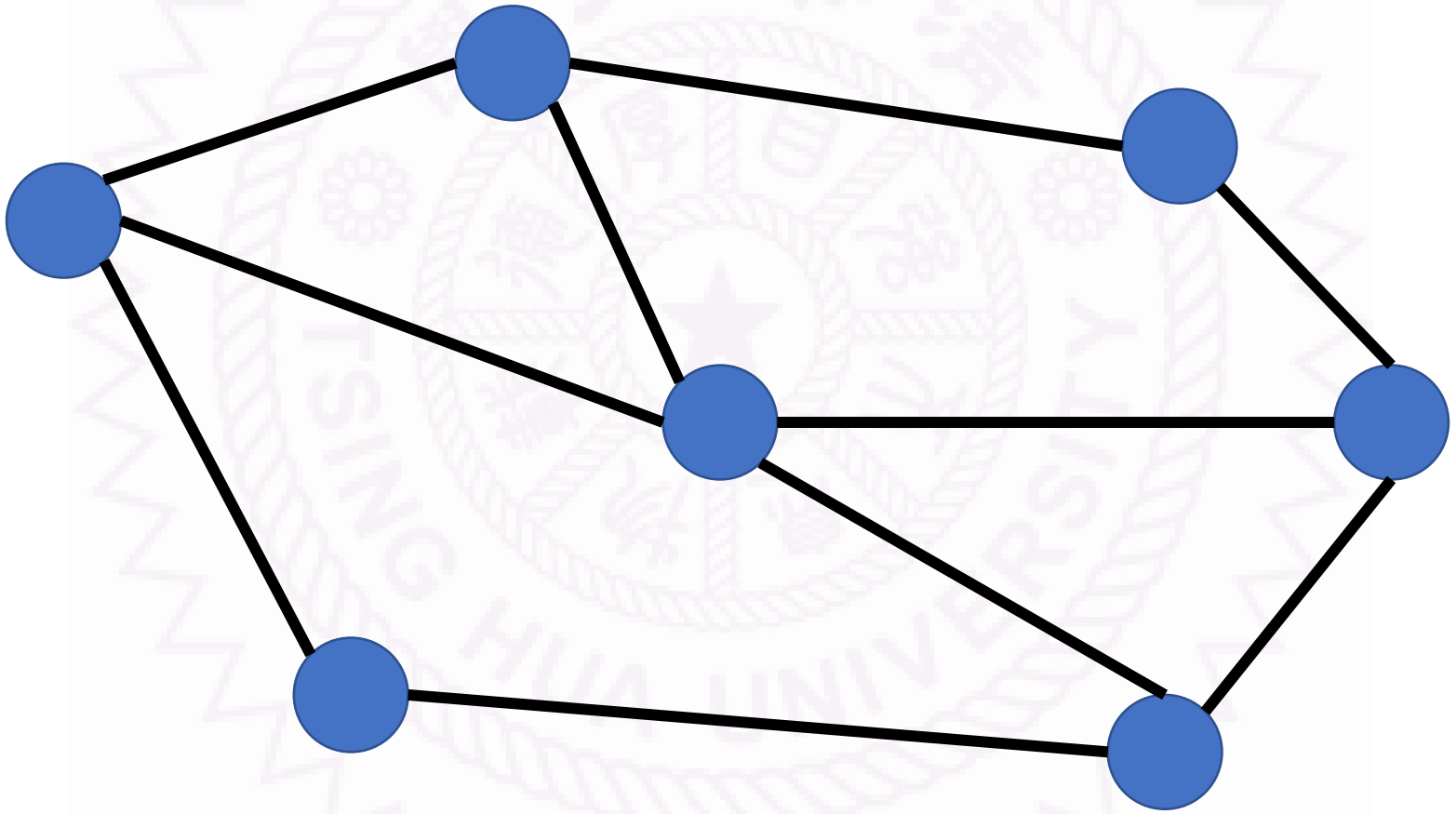
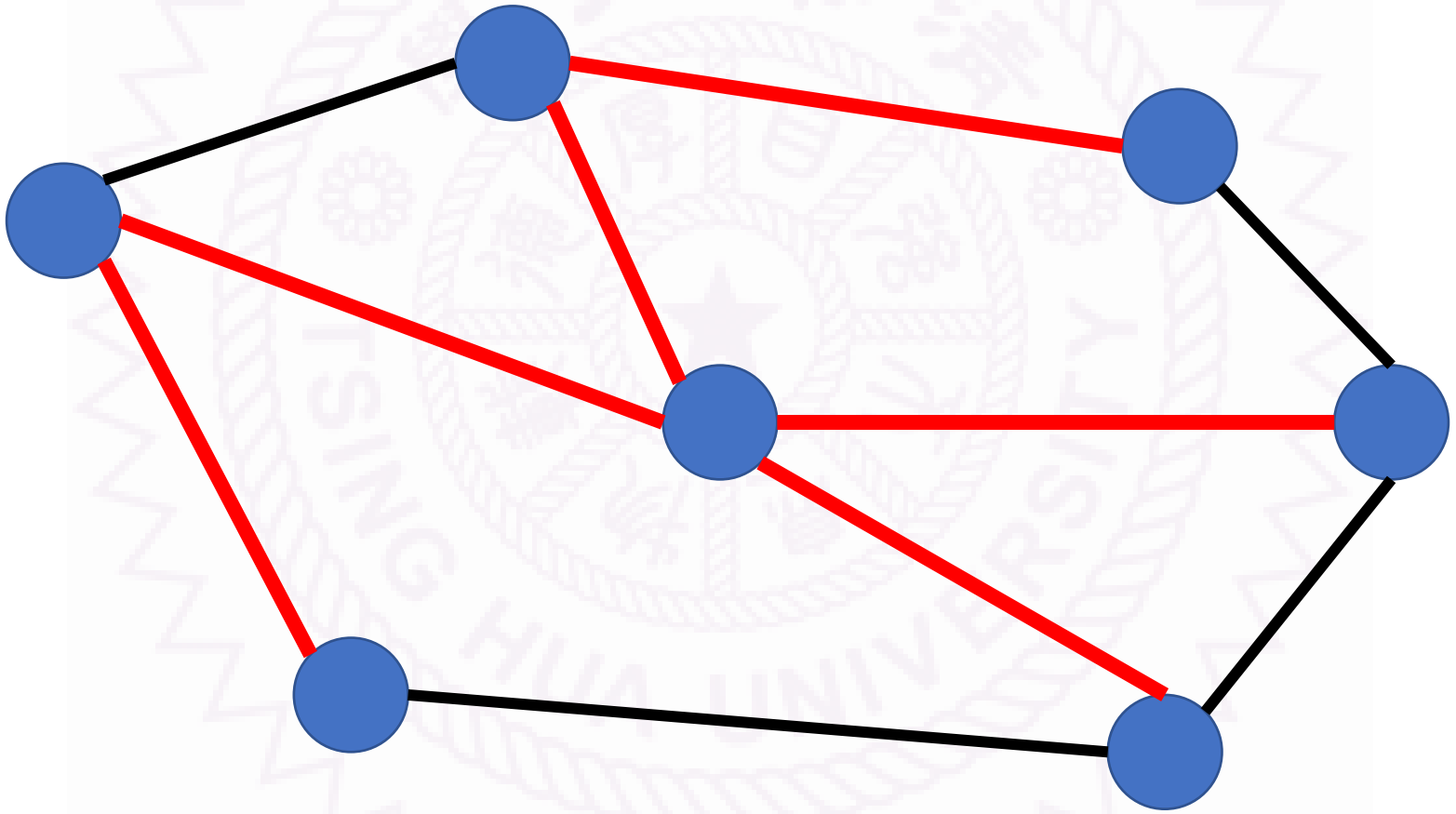
Weighted Graph

日月卦長

生成樹



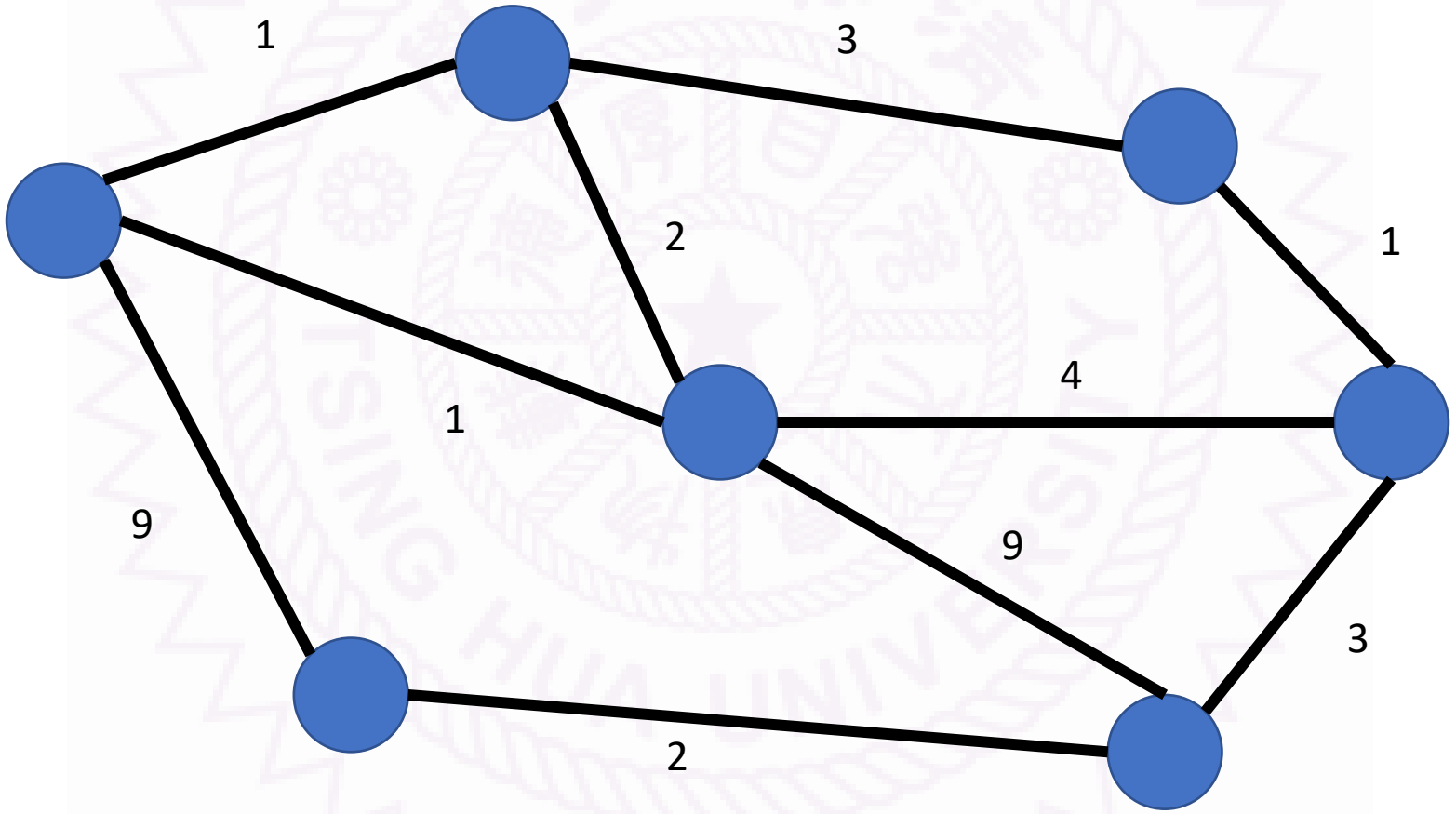
生成樹



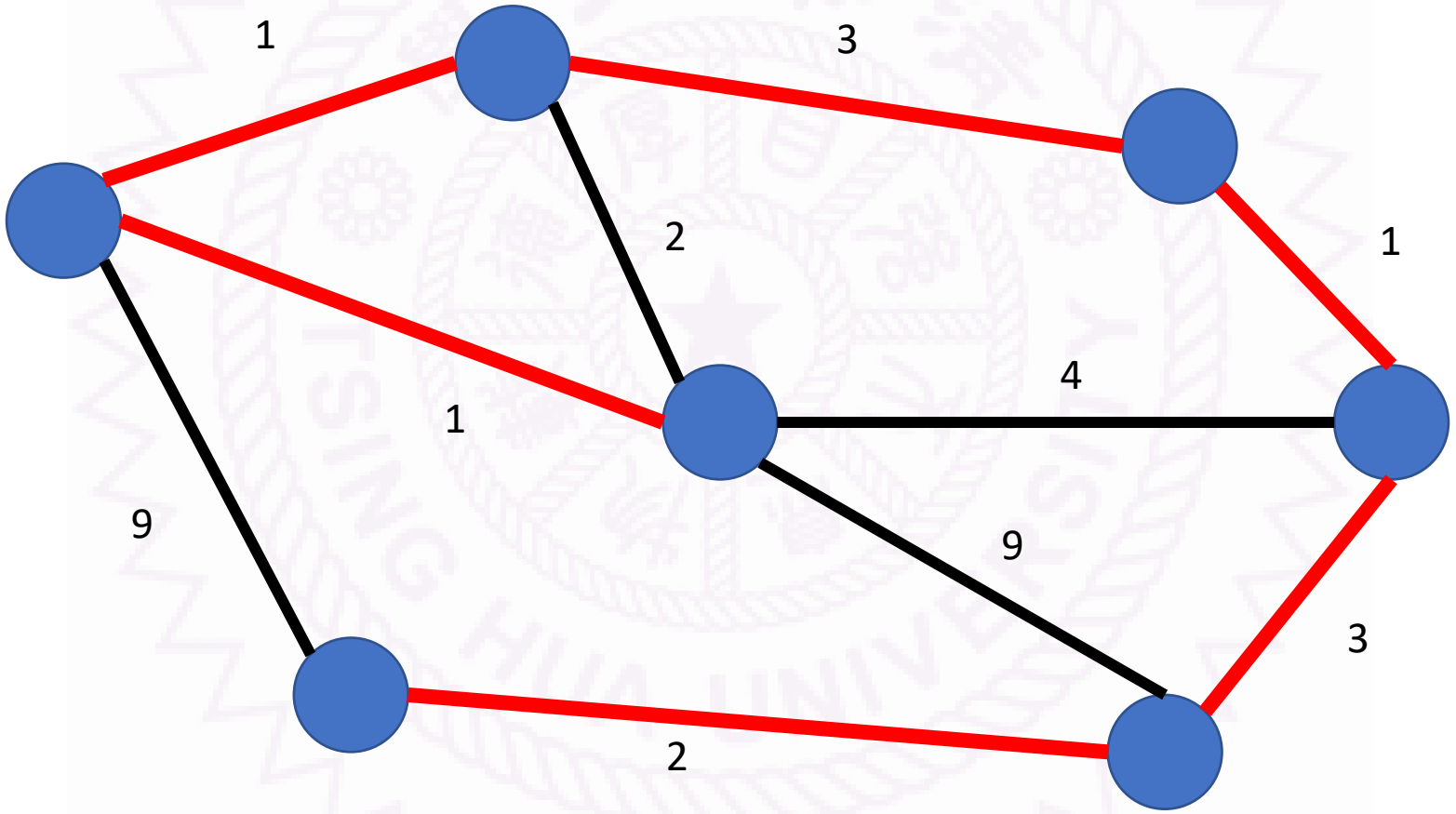
生成樹 Spanning Tree

- 無向圖 G 的生成樹 (Spanning Tree) 是具有 G 的全部頂點，但邊數最少的連通子圖。

最小生成樹



最小生成樹

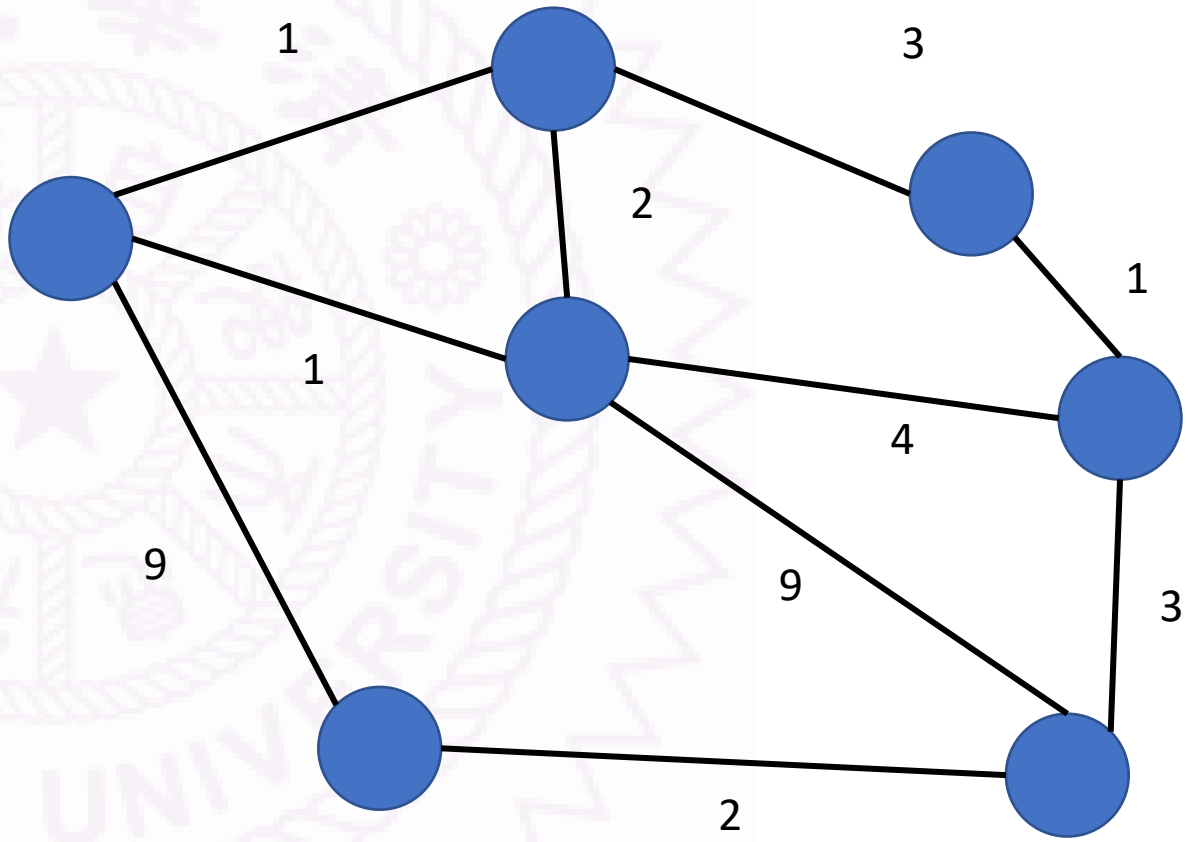


最小生成樹 Minimum Spanning Tree

- 邊權重總合最小的生成樹

Kruskal's algorithm

- Input $G = (V, E)$
- 設 $Ans = \{\ \}$
- While $E \neq \emptyset$:
  - 從 E 中拿出權重最小的邊 e
  - $E \leftarrow E - \{e\}$
  - If Ans在加入e 後沒有產生cycle:
    - $Ans \leftarrow Ans \cup \{e\}$
- 如果 $\|Ans\| = \|V\| - 1$:
  - Ans 就是最小生成樹



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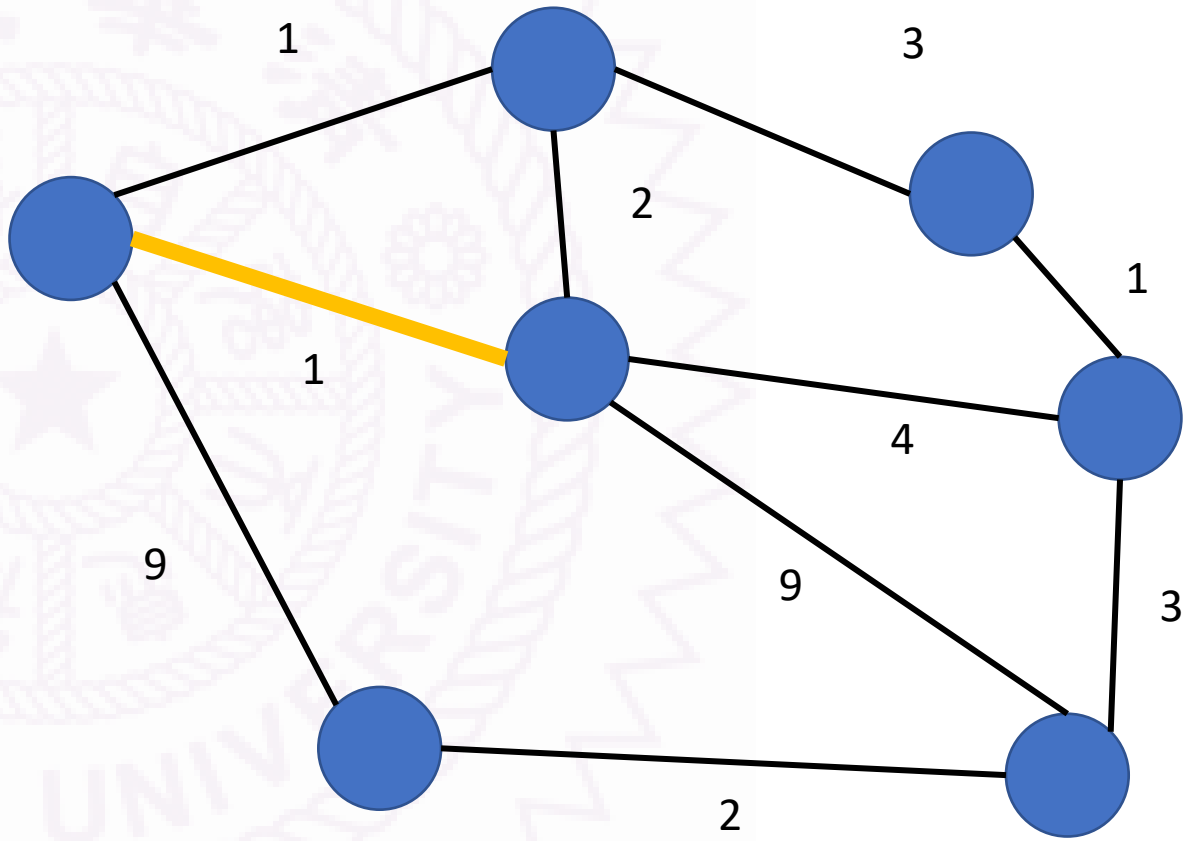

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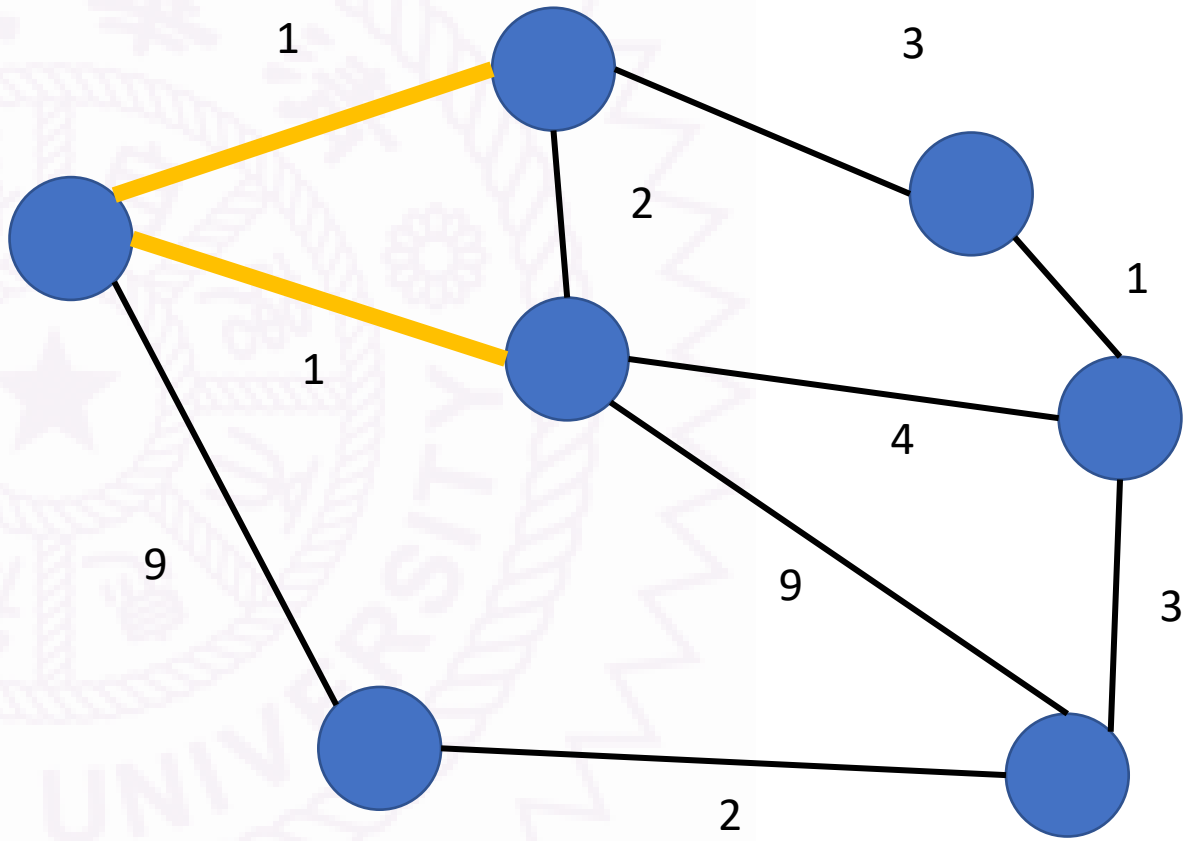

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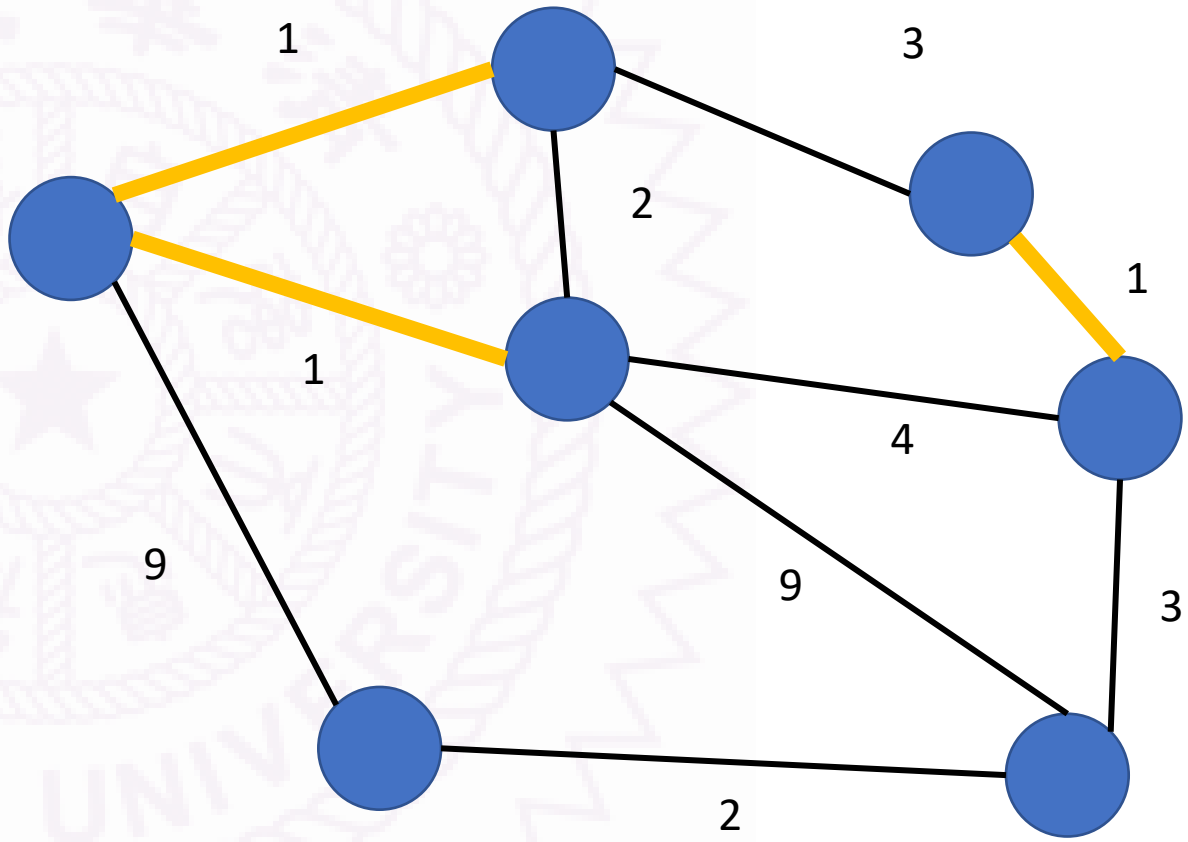

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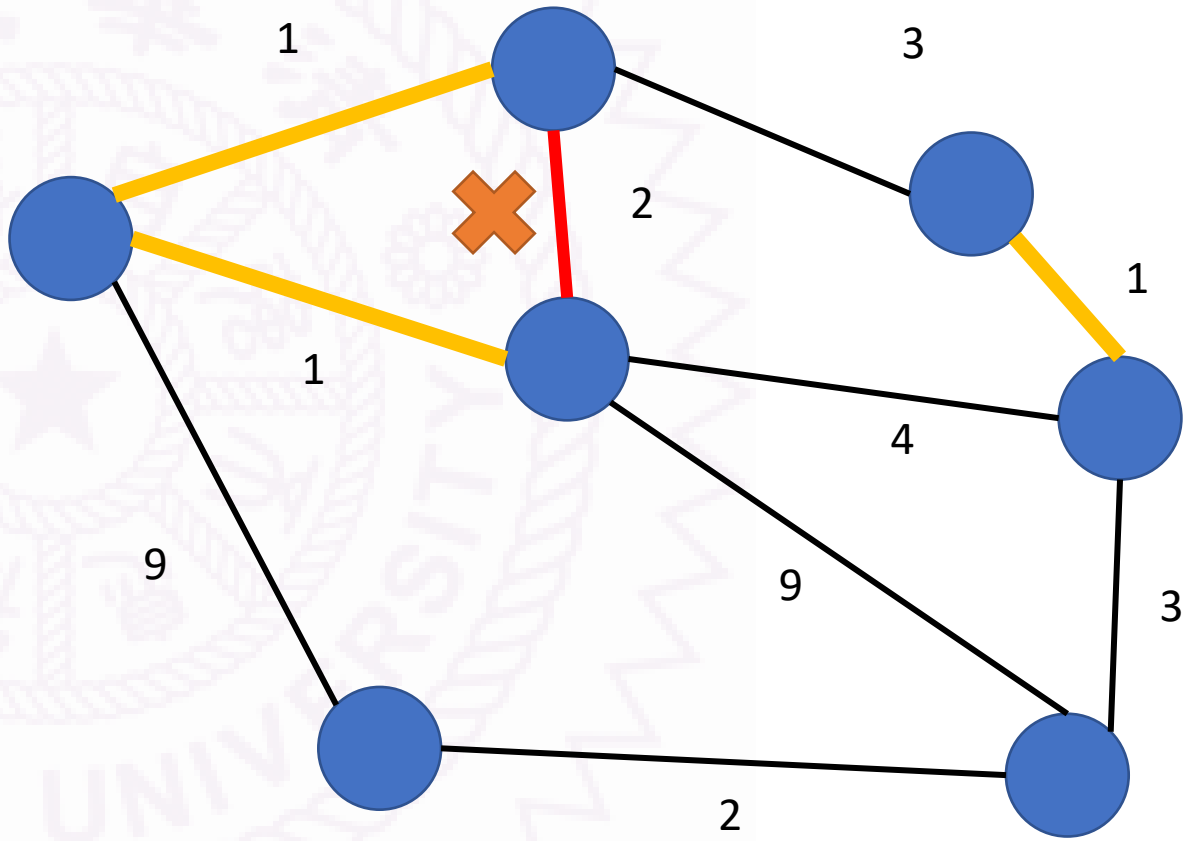

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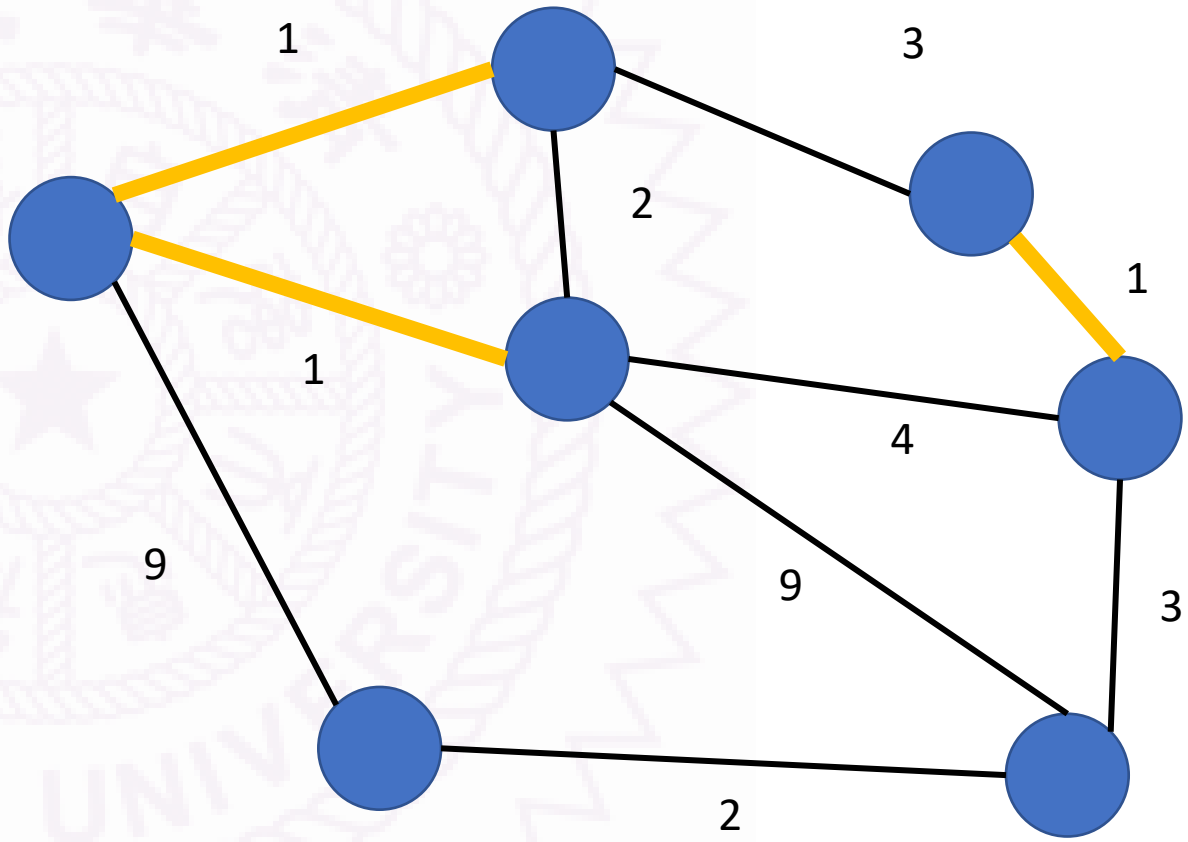

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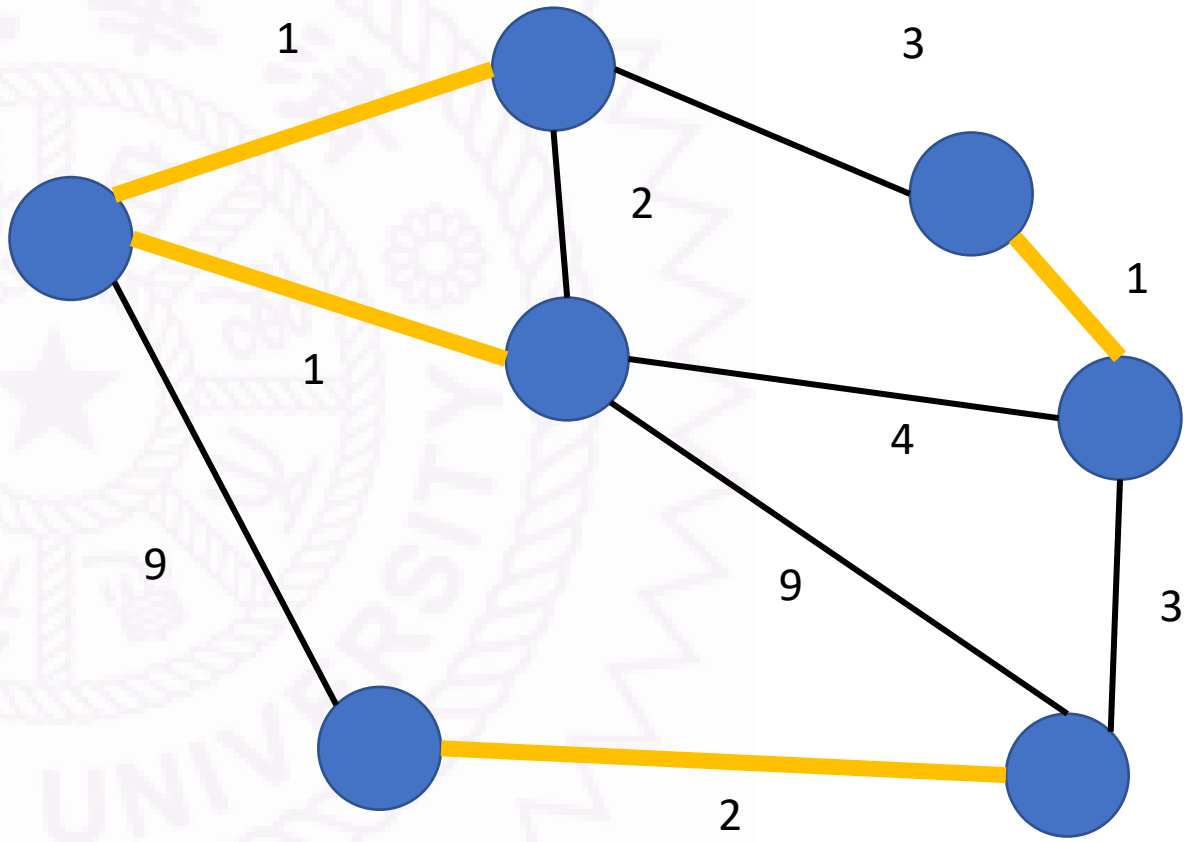

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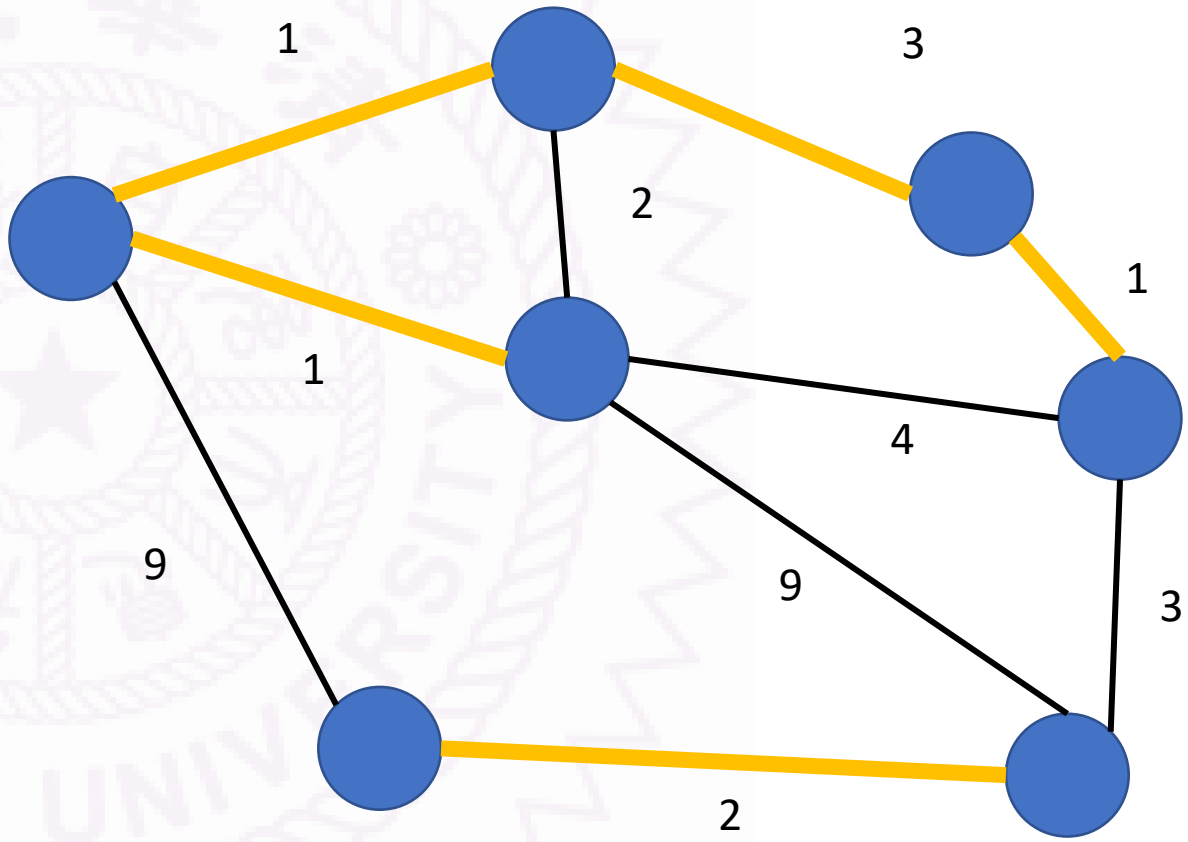

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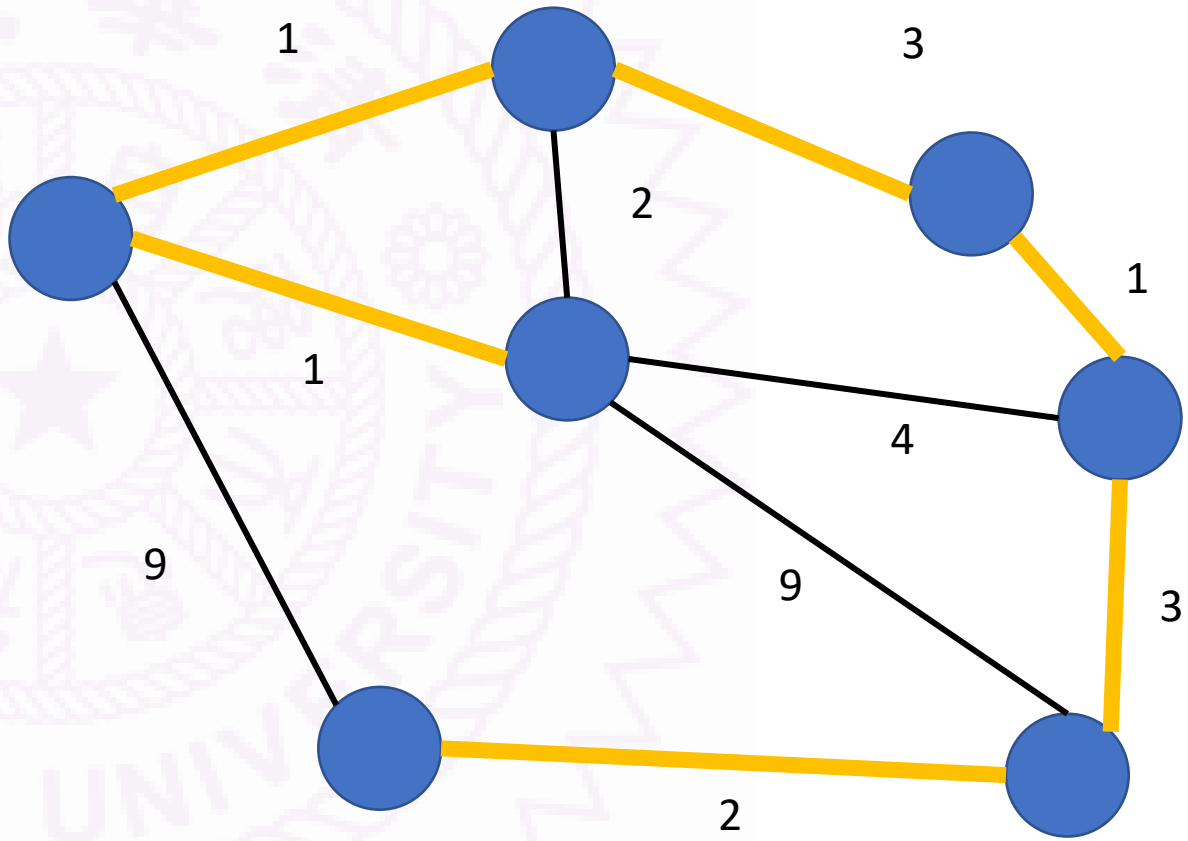
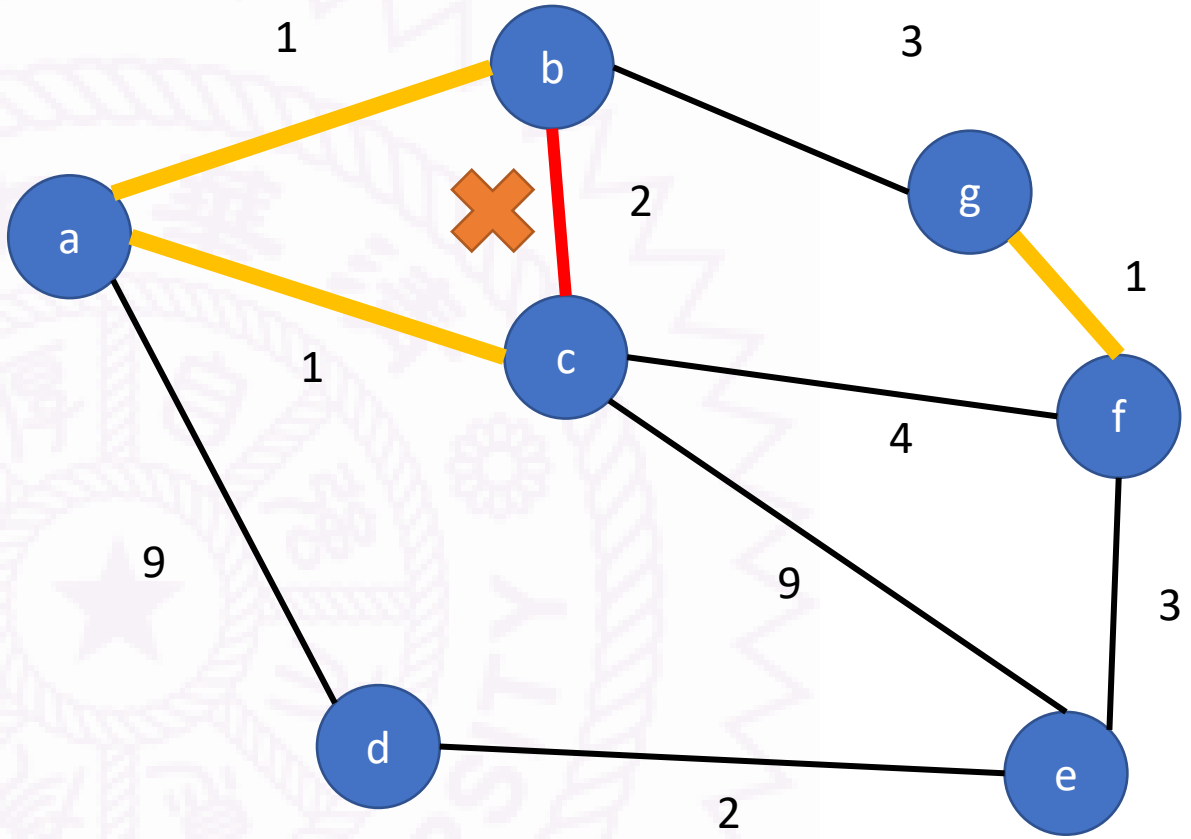
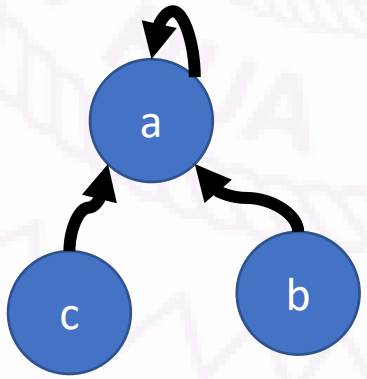


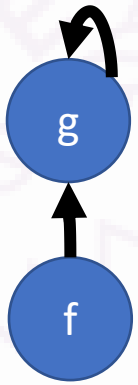
實作細節

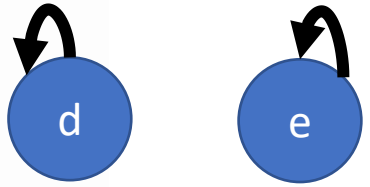
- 如何判斷環?
  ➔ Disjoint Set

- 集合處理很慢
  ➔ 直接將edge由小排到大









最小生成樹只需要 Edge 的資訊

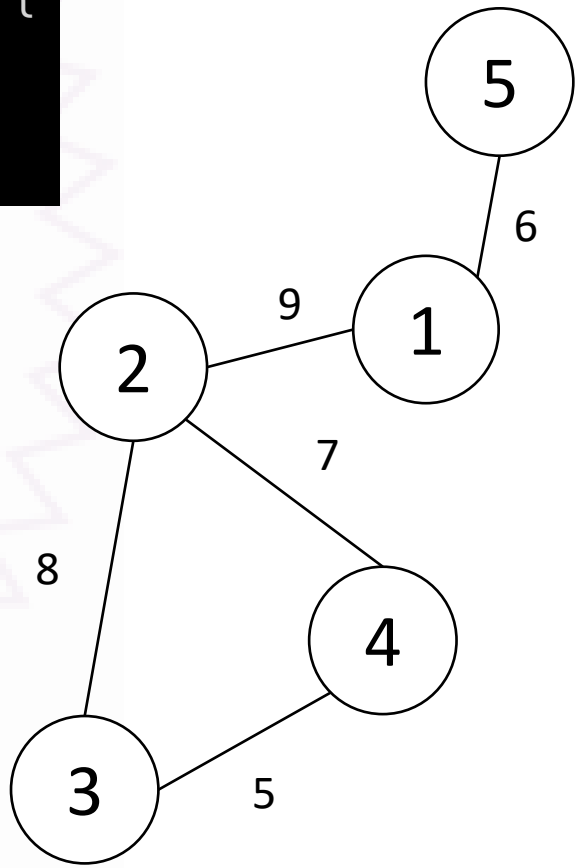
```
struct Edge {
  int u, v, cost;
  bool operator<(const Edge &other) const {
    return cost < other.cost;
  }
};
```

n 個點, m 條邊

m 條邊

```
5 5
1 2 9
2 3 8
2 4 7
1 5 6
3 4 5
```

```
vector<Edge> E;
int n, m;
cin >> n >> m;
E.resize(m);
for (int i = 0; i < m; ++i) {
  cin >> E[i].u >> E[i].v >> E[i].cost;
}
```



實作很簡單

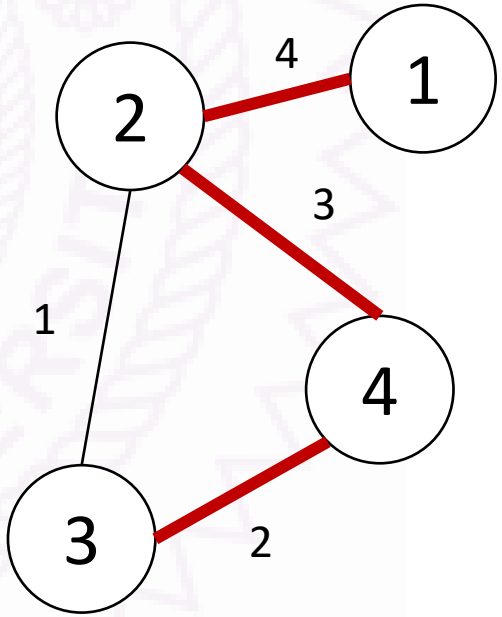
```cpp
vector<Edge> E;

long long MST(int n) {
  sort(E.begin(), E.end());
  DisjointSet DS(n);
  long long ans = 0;
  int cnt = 0;
  for (auto &e : E) {
    if (DS.Same(e.u, e.v)) continue;
    DS.Union(e.u, e.v);
    ans += e.cost;
    ++cnt;
  }
  if (cnt < n - 1) cout << "not connected\n";
  return ans;
}
```

最小瓶頸路徑

- $a \to b$ 的最小瓶頸路徑
  是邊權最大值最小的那條路徑
  可以有好幾條

- Ex: $1 \to 3$ 的最小瓶頸路徑有
  - $1 \to 2 \to 4 \to 3$
  - $1 \to 2 \to 3$



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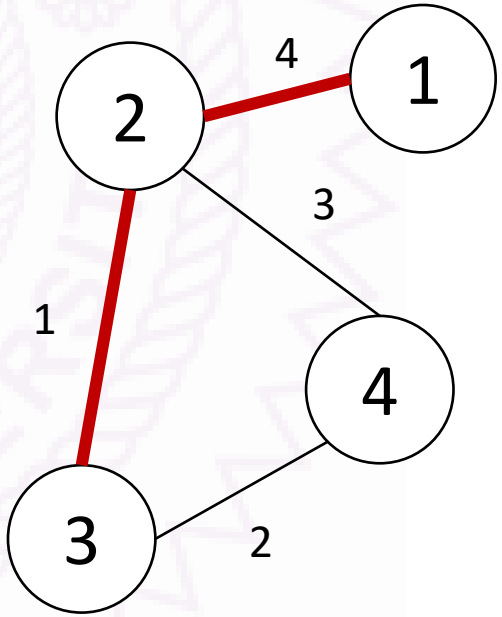
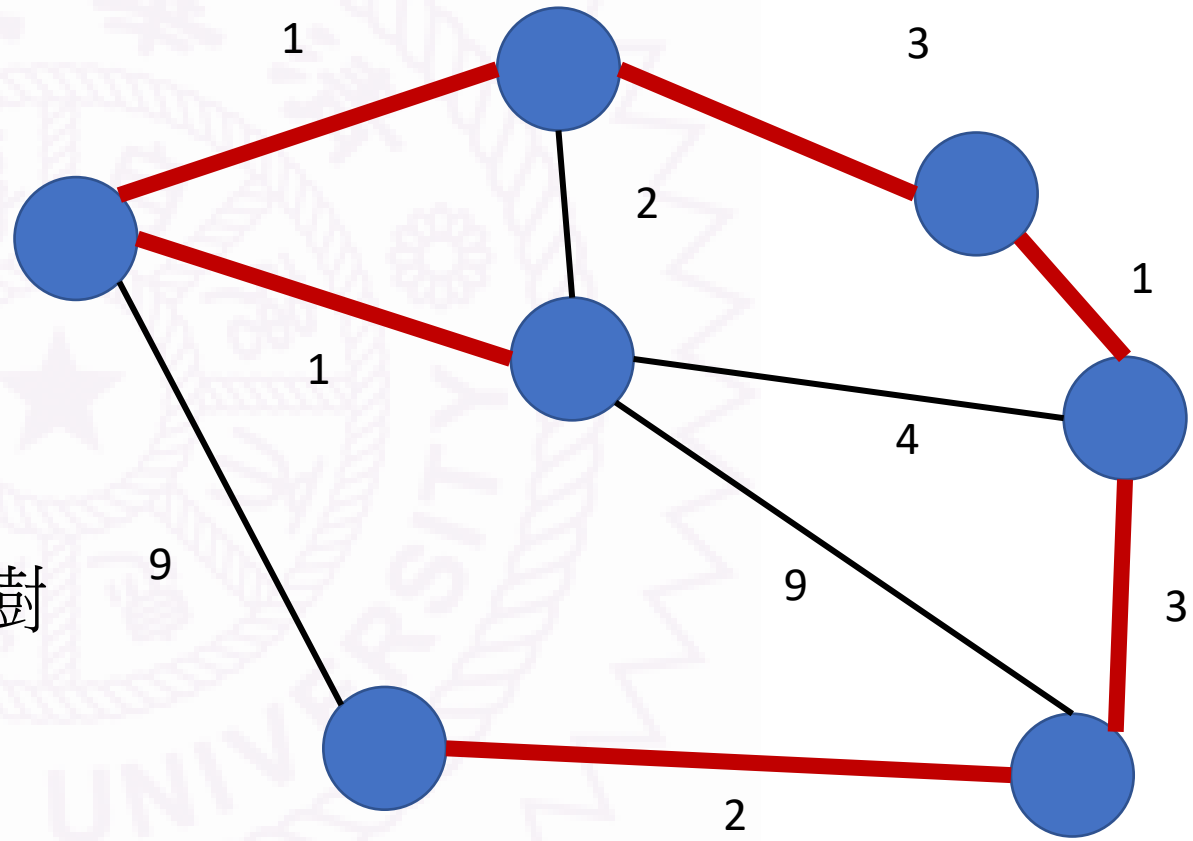


最小瓶頸生成樹

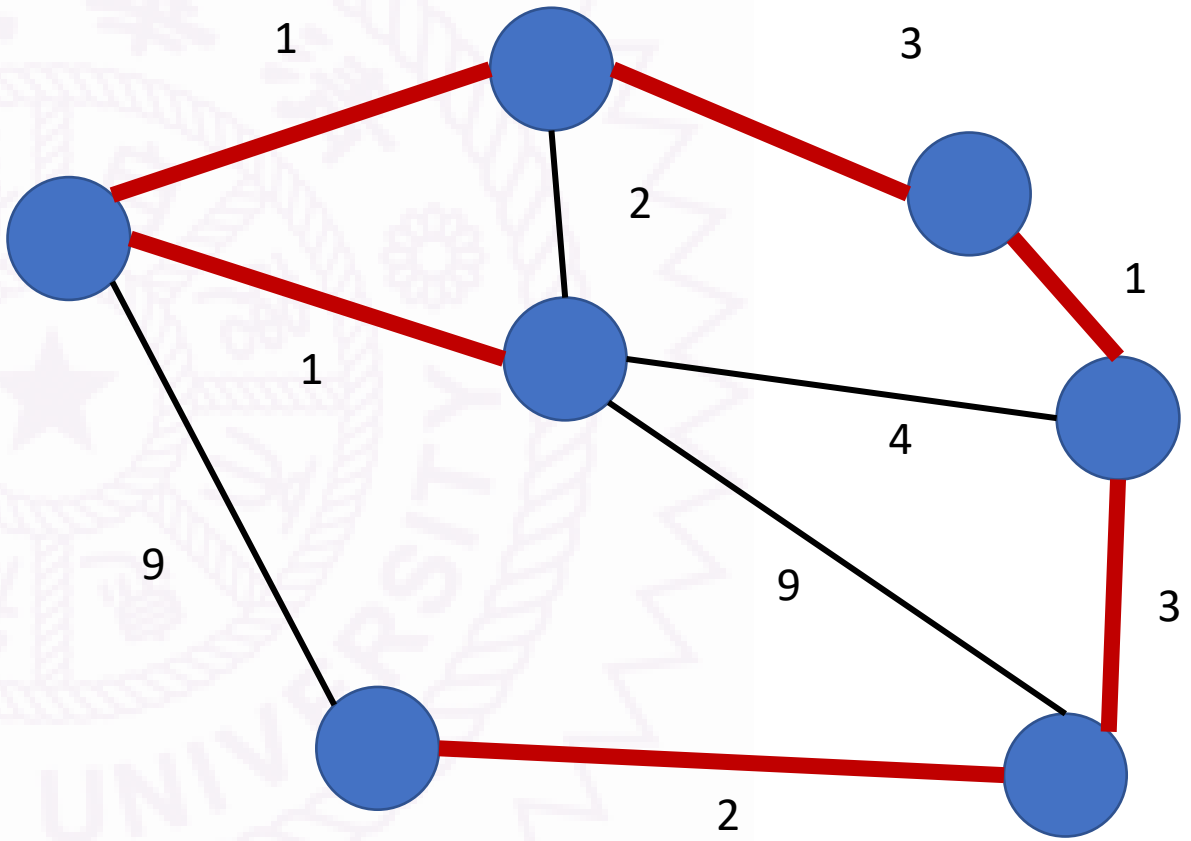
- 圖 G 的最小瓶頸生成樹 T
- 滿足：
  - T 上權重最大的邊
    是所有生成樹中最小的

- 最小生成樹 **是** 最小瓶頸生成樹
  - 反過來不一定成立



最小瓶頸生成樹

- 圖 G 的最小生成樹 T
- 滿足：
  - T 上任兩點 a, b 的路徑同時是圖 G 中 $a \to b$ 的最小瓶頸路徑

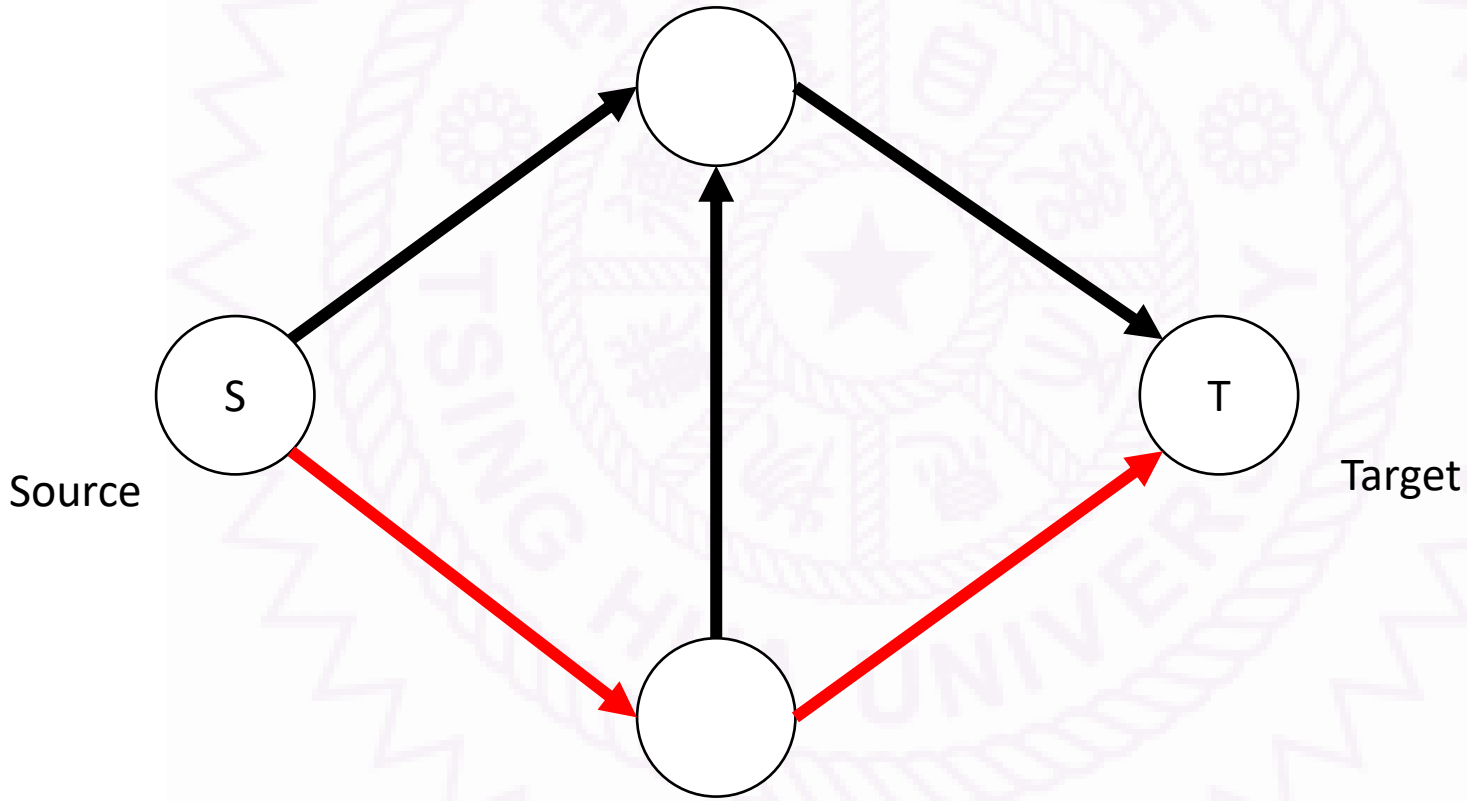


最短路徑
Shortest Path

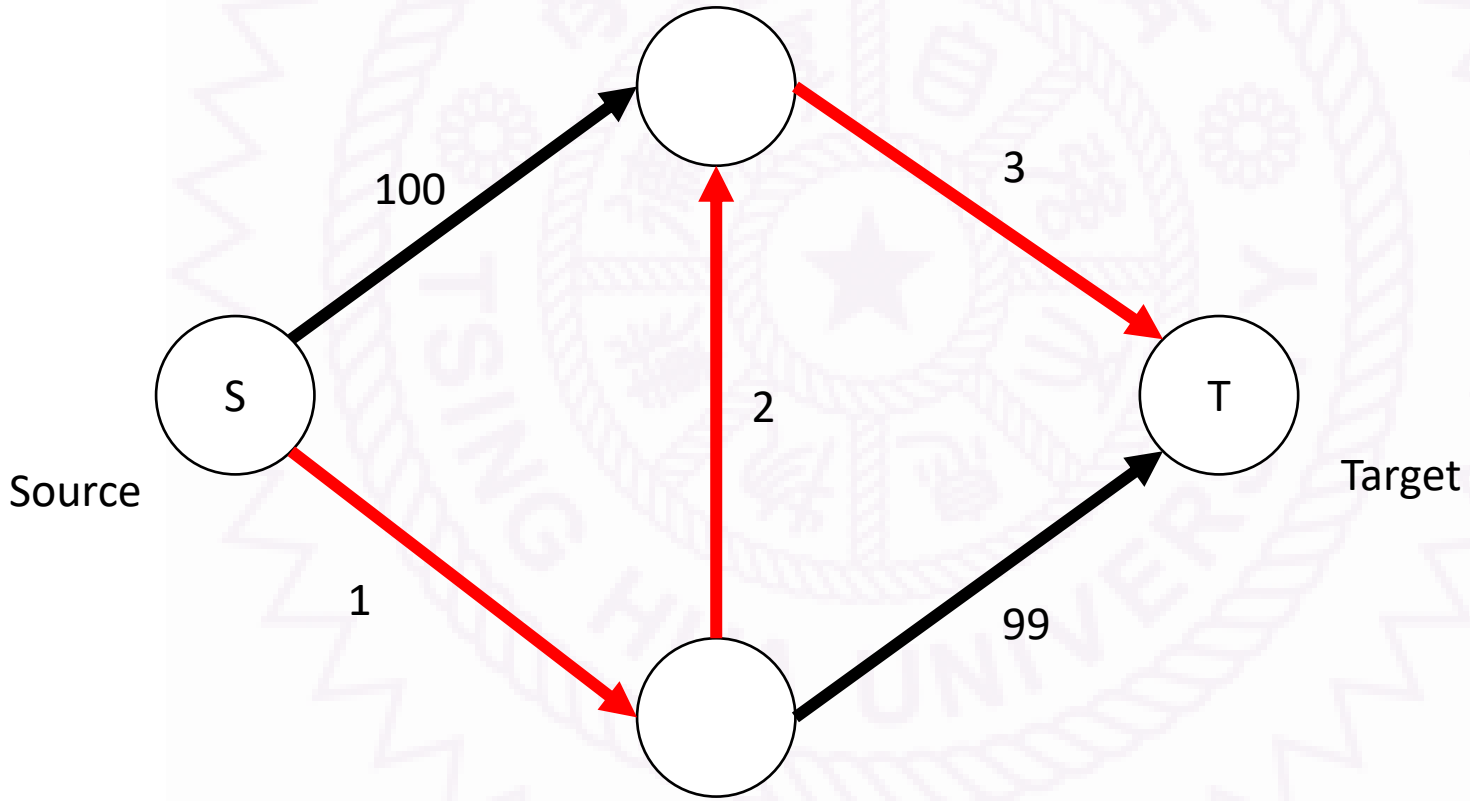
三大演算法

最常見應用：google map

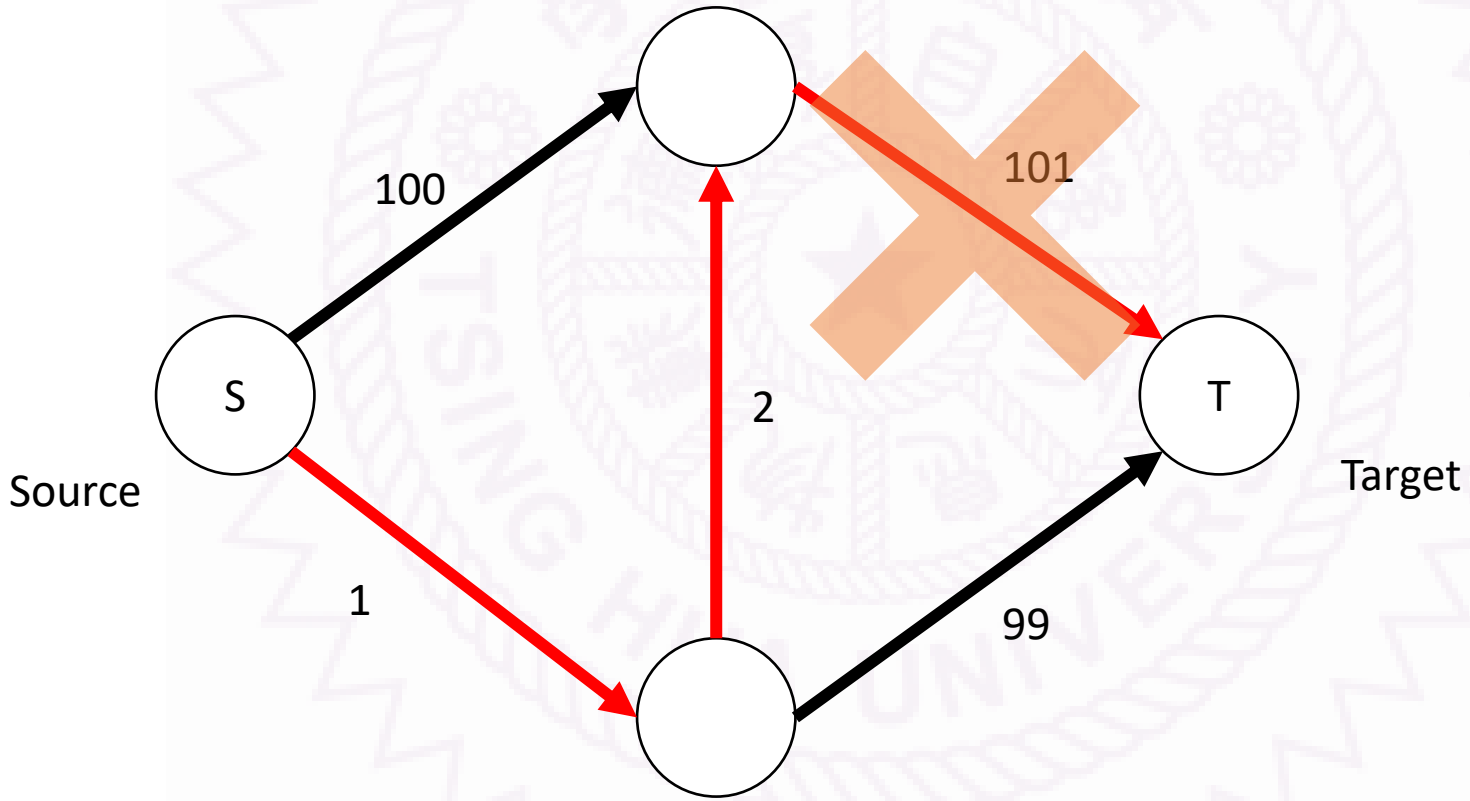
圖論上的最短路徑(BFS)



圖論上的最短路徑



貪心法顯然不行



Dijkstra

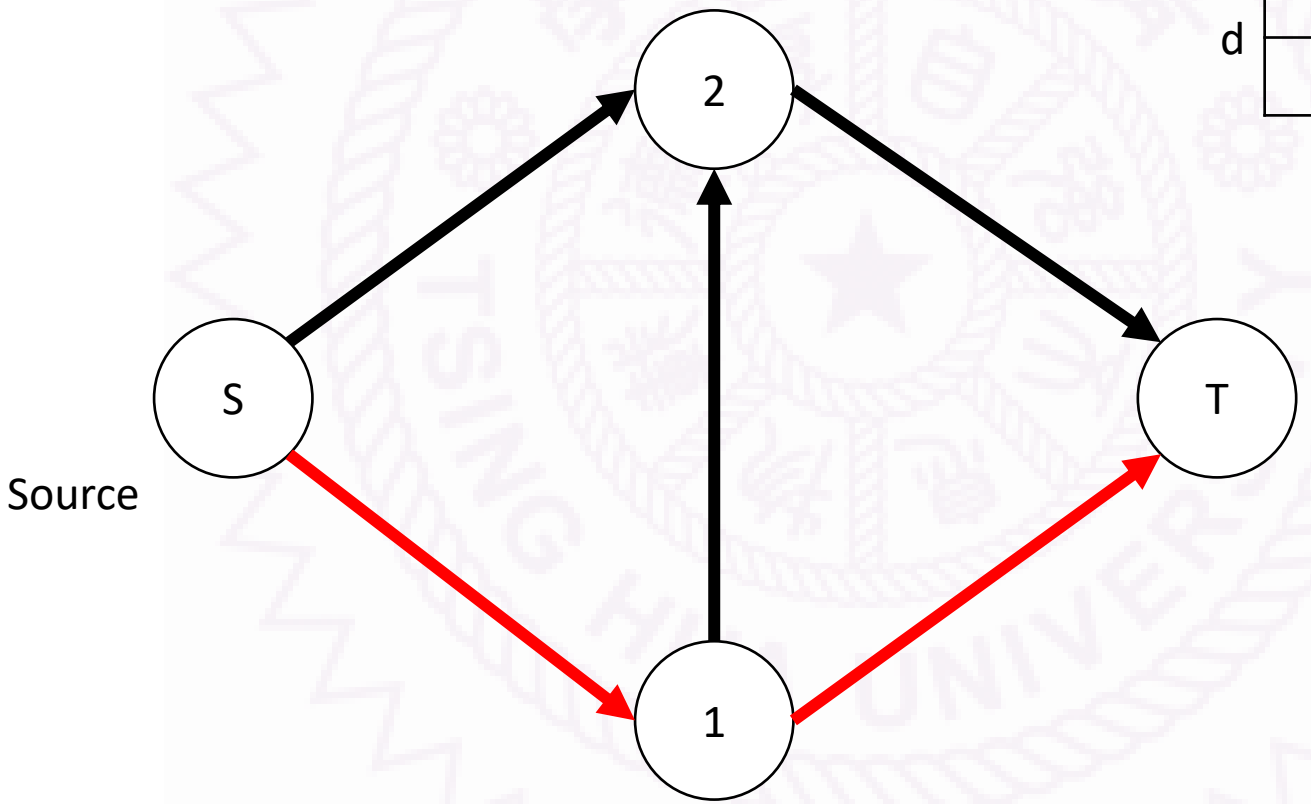
限制條件：邊的權重大於0

無權單源最短路徑

目標：
算出這表格d

d

S	1	2	T
0	1	1	2

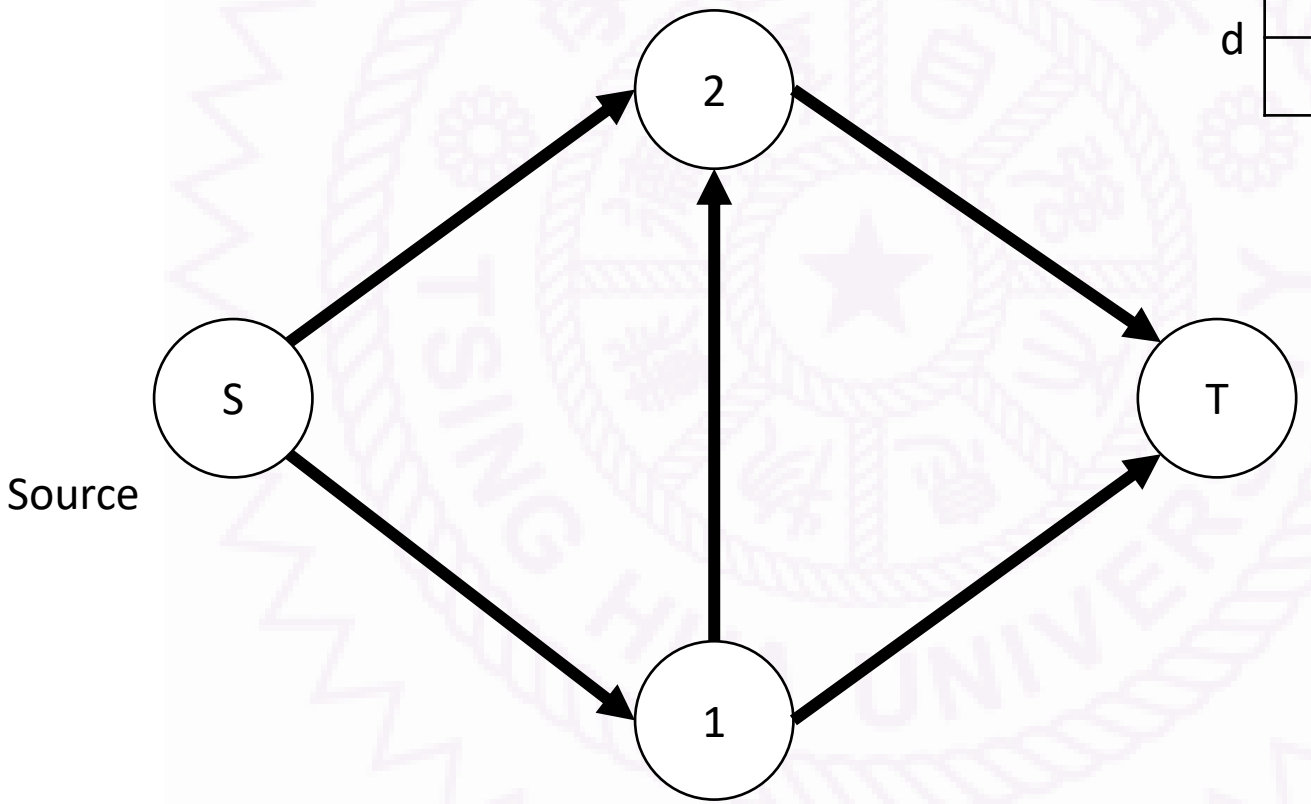


無向圖單源最短路徑

一開始除了起點
其他點都是無限大

d

S	1	2	T
0	∞	∞	∞



直接 BFS

注意此時從queue拿出來時d[u]已經是最短路徑了

```cpp
vector<long long> BFS(const vector<vector<int>> &G, int S) {
  int n = G.size(); // 假設點的編號為 0 ~ n-1
  vector<long long> d(n, INF);
  vector<bool> visited(n, false);
  queue<int> Q;
  Q.emplace(S);
  d[S] = 0; // 起點設 0
  while (Q.size()) {
    int u = Q.front();
    Q.pop();
    if (visited[u]) continue;
    visited[u] = true;
    for (auto v : G[u]) {
      if (d[v] > d[u] + 1)
        d[v] = d[u] + 1;
      Q.emplace(v);
    }
  }
  return d;
}
```

加速：
進到 if
才放進
queue

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  int n = G.size(); // 假設點的編號為 0 ~ n-1
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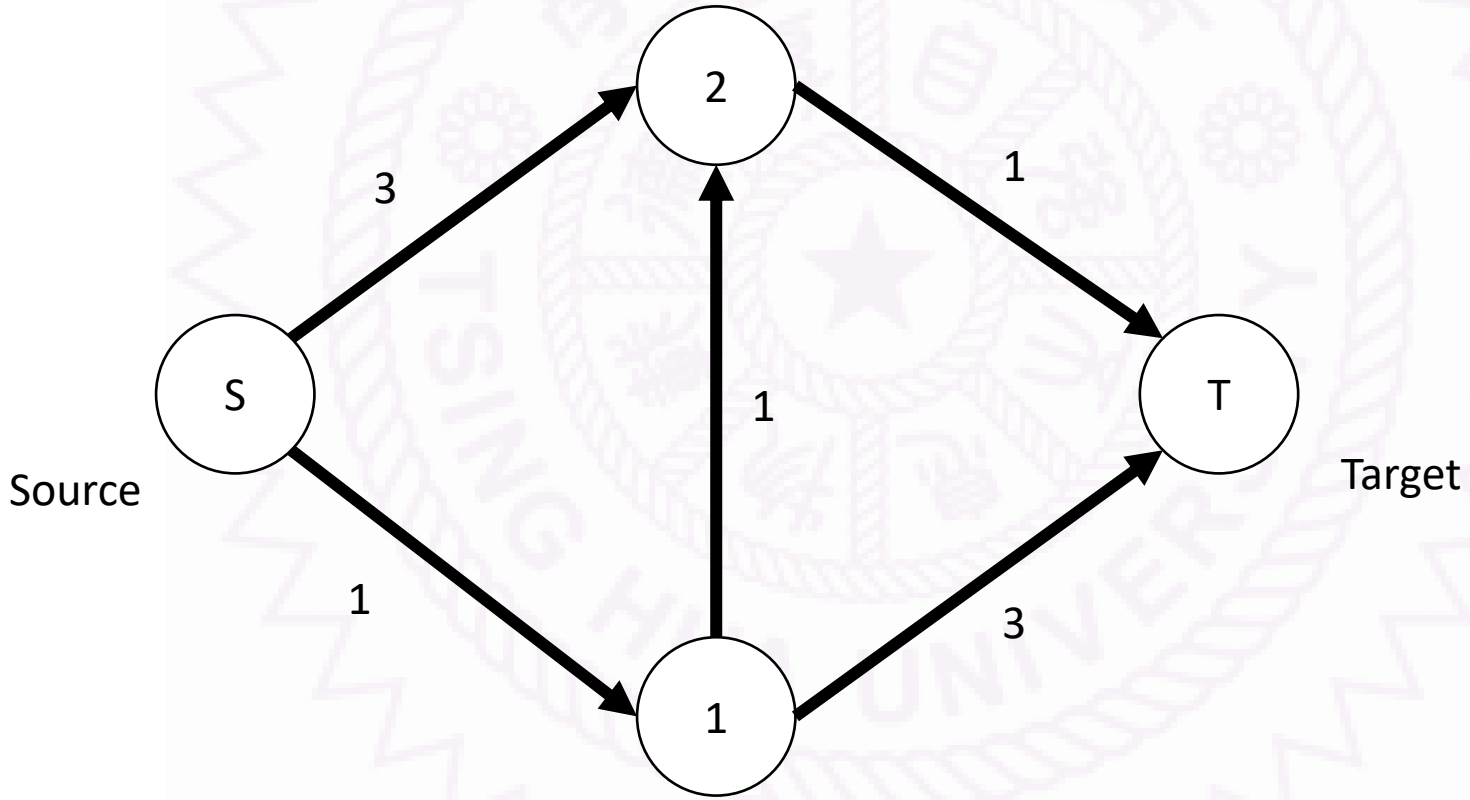
  queue<int> Q;
  Q.emplace(S);
  d[S] = 0; // 起點設 0
  while (Q.size()) {
    int u = Q.front();
    Q.pop();
    if (visited[u]) continue;
    visited[u] = true;
    for (auto v : G[u])
      if (d[v] > d[u] + 1) {
        d[v] = d[u] + 1;
        Q.emplace(v);
      }
  }
  return d;
}
```

單源最短路徑



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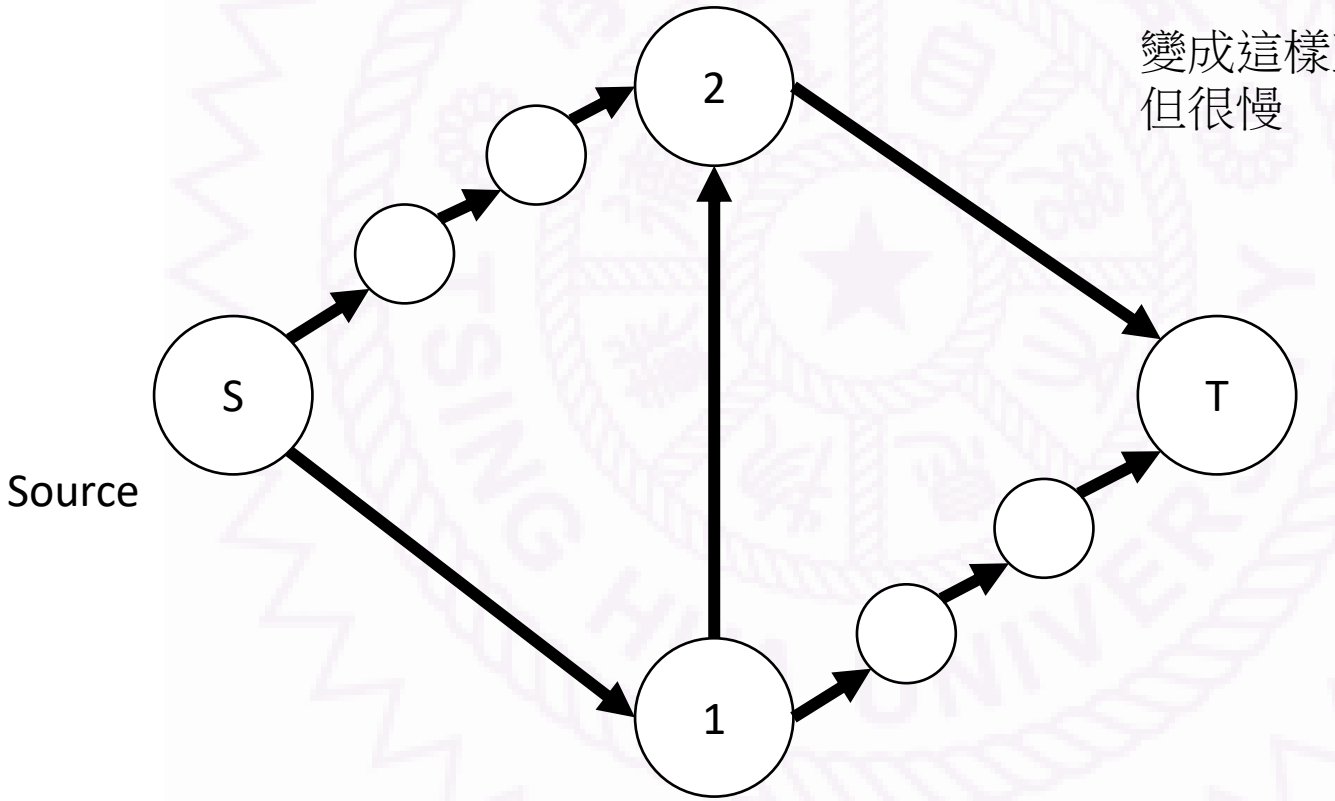
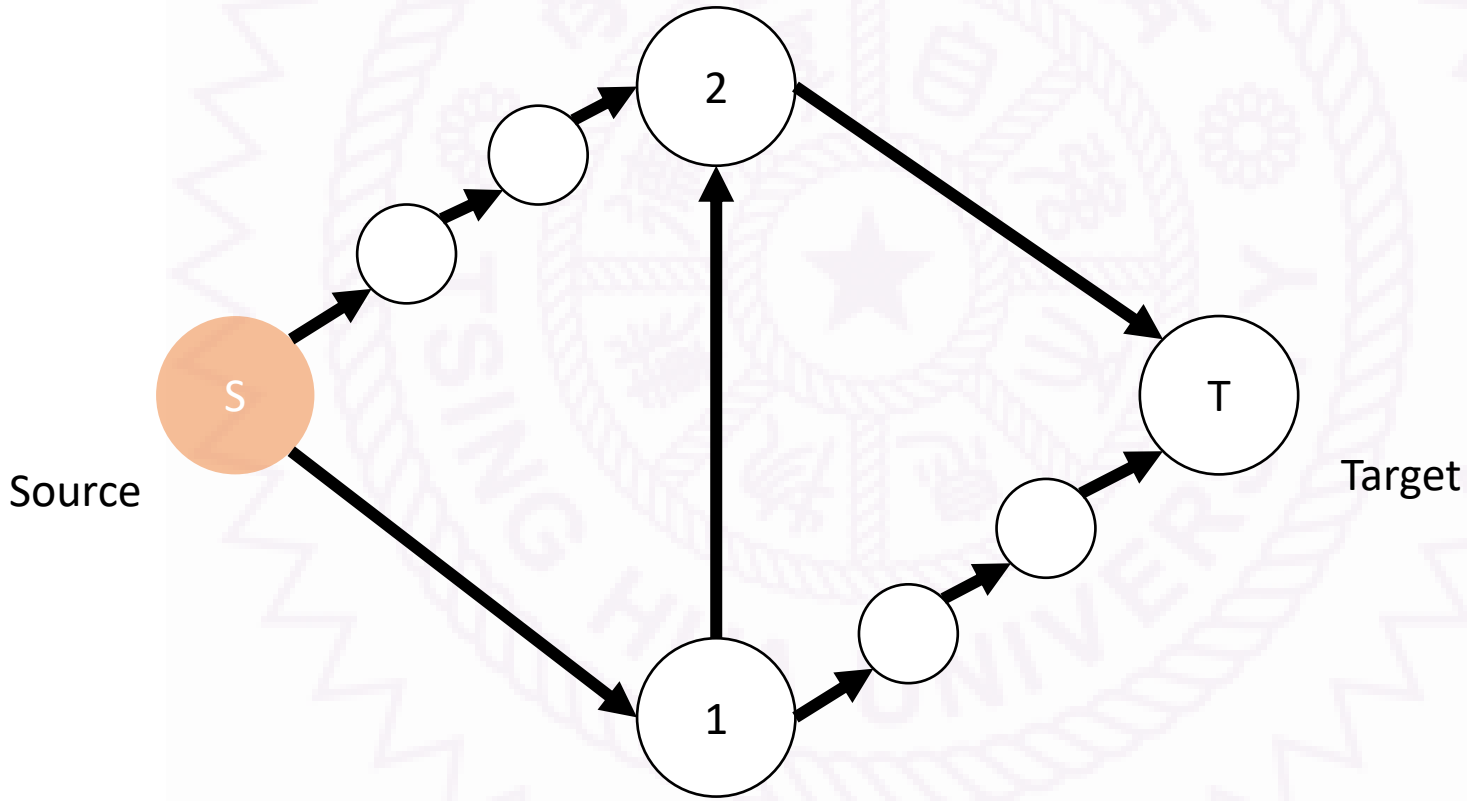


變成這樣直接用BFS就可以解了
但很慢

單源最短路徑



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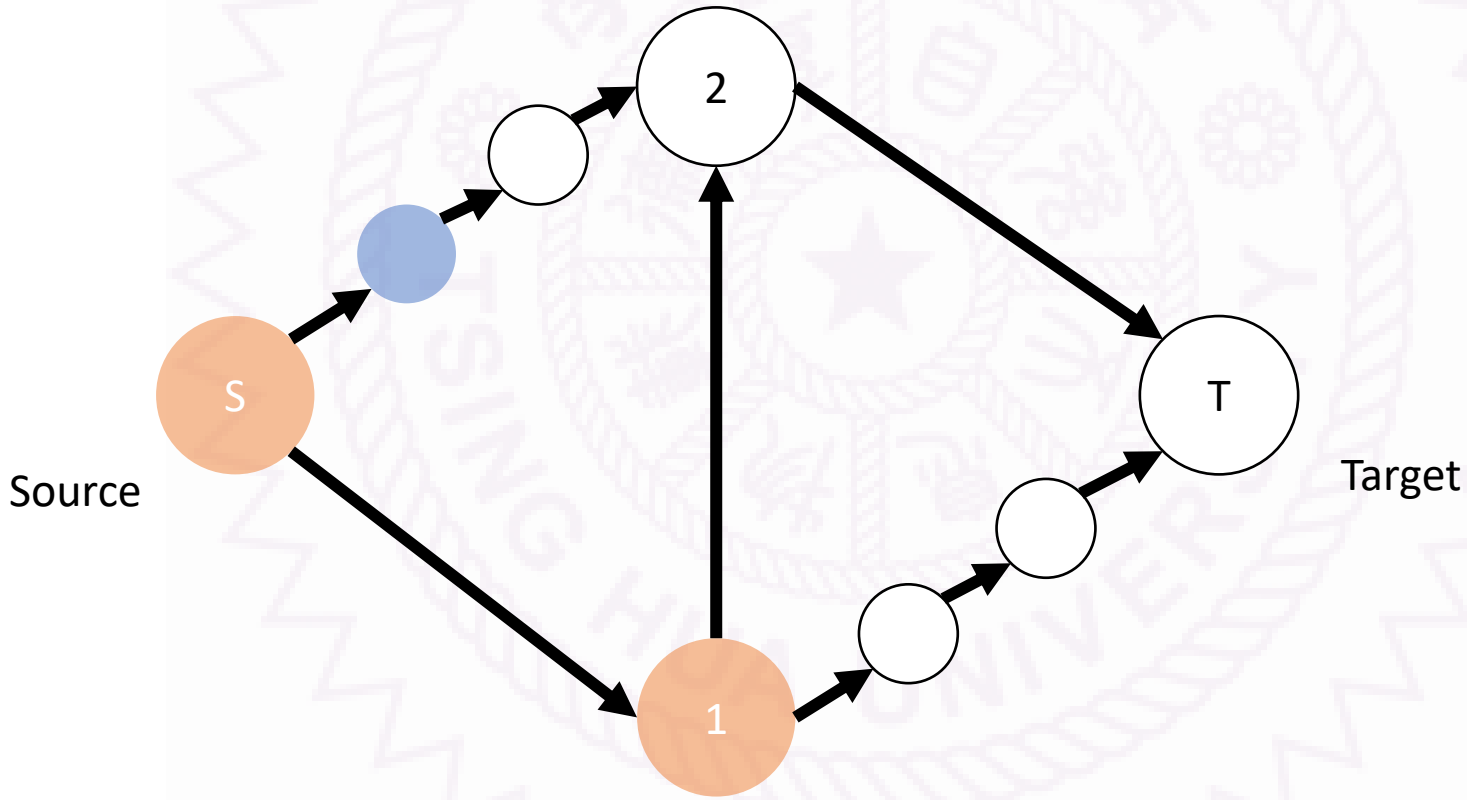

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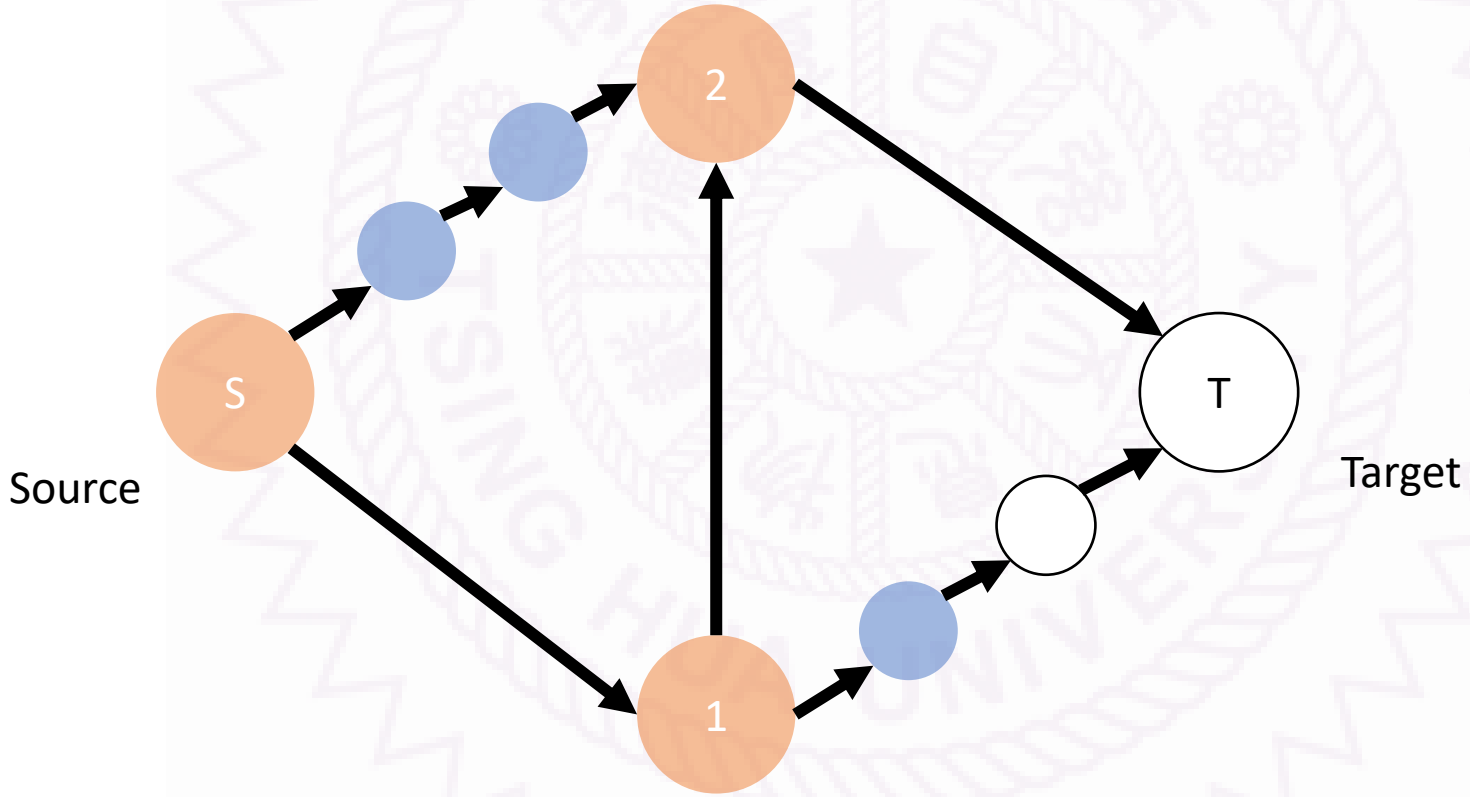

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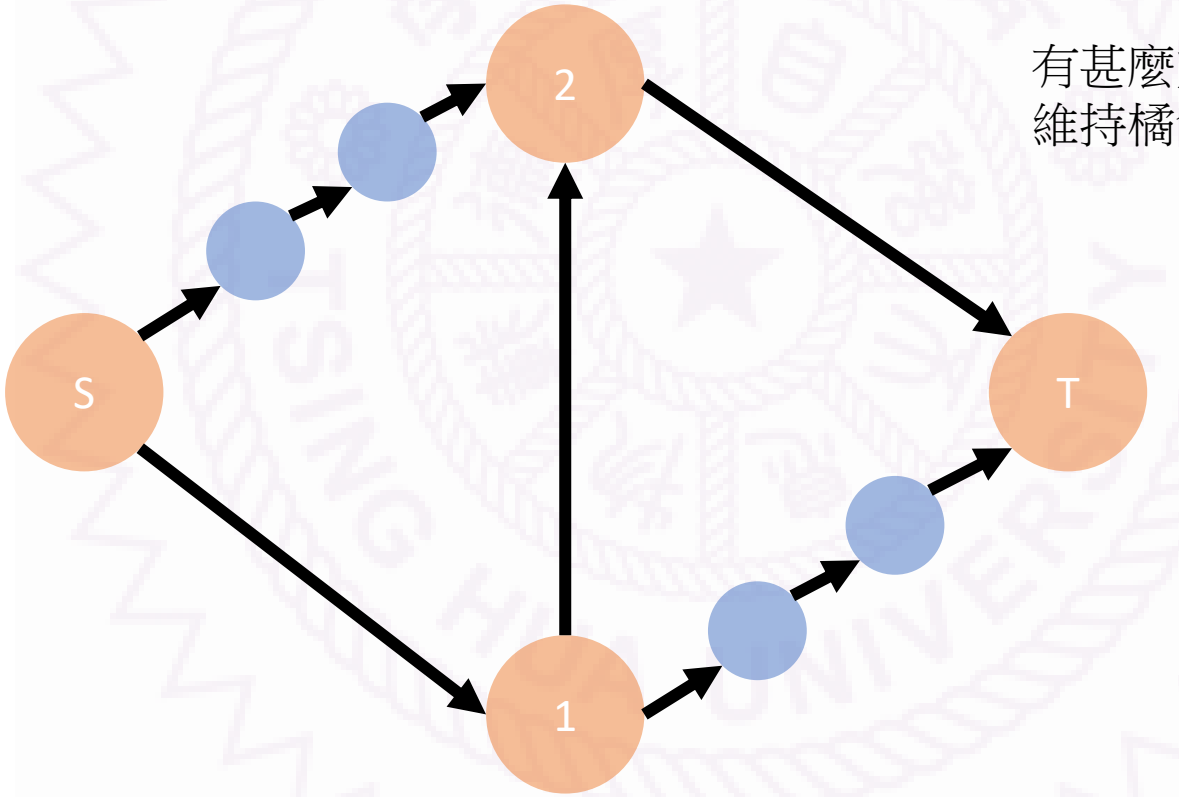
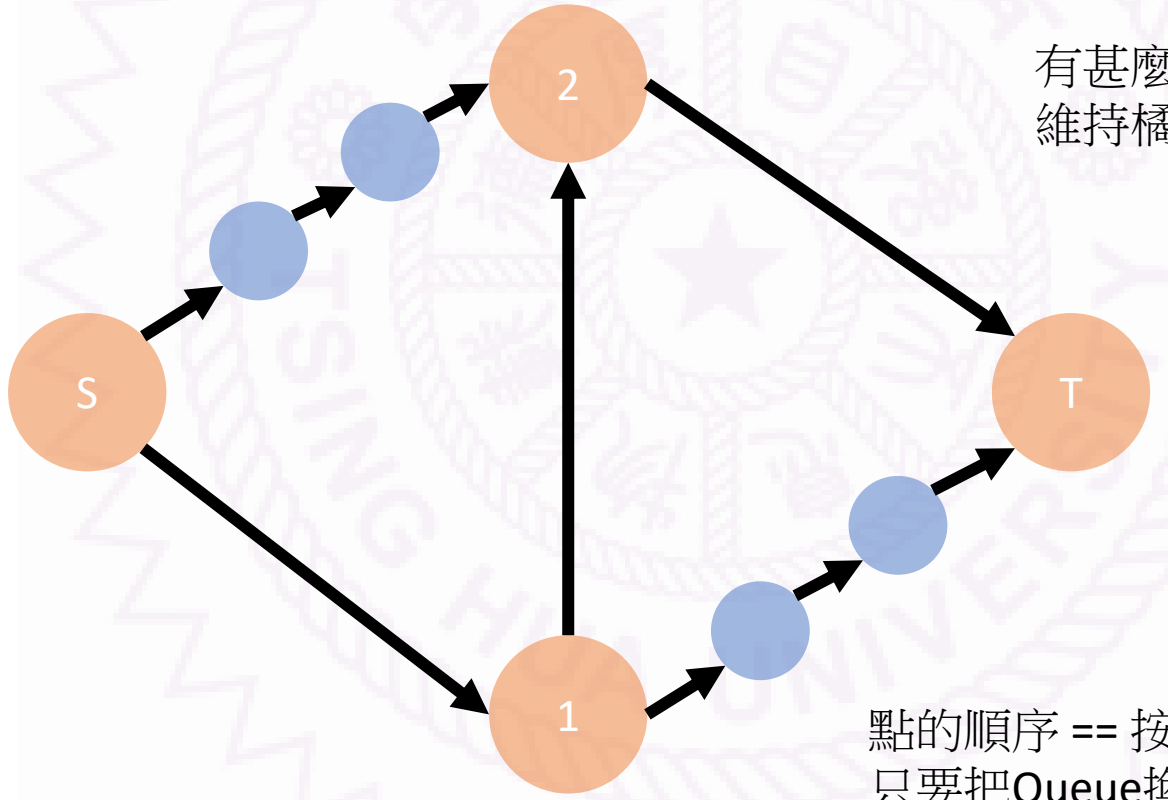


Source

Target

有甚麼方法可以跳過藍色的點
維持橘色點在BFS中的順序呢?

單源最短路徑



Source

Target

有甚麼方法可以跳過藍色的點
維持橘色點在BFS中的順序呢?

點的順序 == 按最短路徑由小排到大
只要把Queue換成min heap紀錄當前的最短路徑！

Adjacency List + 紀錄邊權重

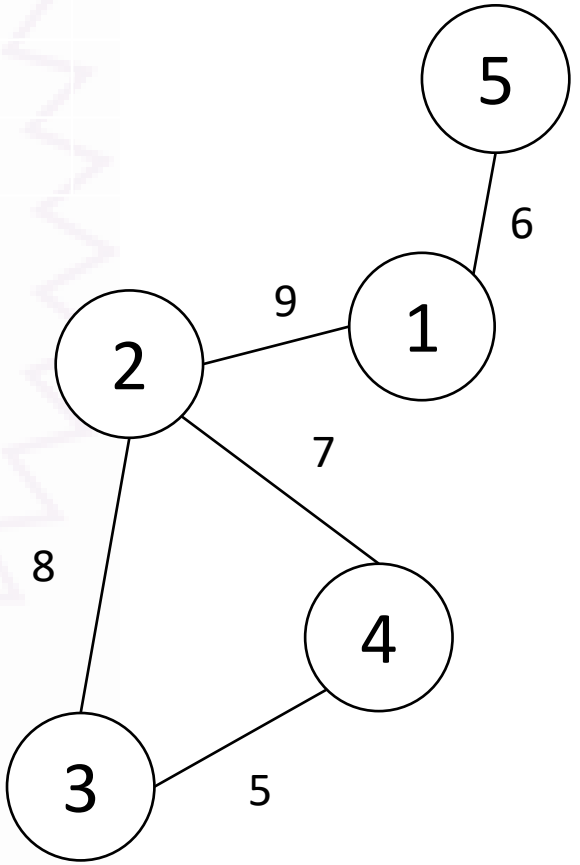
1	2,9	5,6
2	3,8	4,7
3	4,5	
4		
5		

n 個點, m 條邊

m 條邊

```
5 5
1 2 9
2 3 8
2 4 7
1 5 6
3 4 5
```

```
vector<vector<pair<int, int>>>
G;
int n, m;
cin >> n >> m;
G.assign(n + 1, {});
while (m--) {
  int u, v, cost;
  cin >> u >> v >> cost;
  G[u].emplace_back(v, cost);
}
```



Dijkstra

```cpp
vector<long long> dijkstra(const vector<vector<pair<int, int>>> &G, int S) {
  int n = G.size(); // 假設點的編號為 0 ~ n-1
  vector<long long> d(n, INF);
  vector<bool> visited(n, false);
  using QueuePair = pair<long long, int>;
  priority_queue<QueuePair, vector<QueuePair>, greater<QueuePair>> Q;
  d[S] = 0; // 起點設 0
  Q.emplace(d[S], S);
  while (Q.size()) {
    int u = Q.top().second;
    Q.pop();
    if (visited[u]) continue;
    visited[u] = true;
    for (auto [v, cost] : G[u])
      if (d[v] > d[u] + cost) {
        d[v] = d[u] + cost;
        Q.emplace(d[v], v);
      }
  }
  return d;
}
```

Dijkstra 不使用 visited

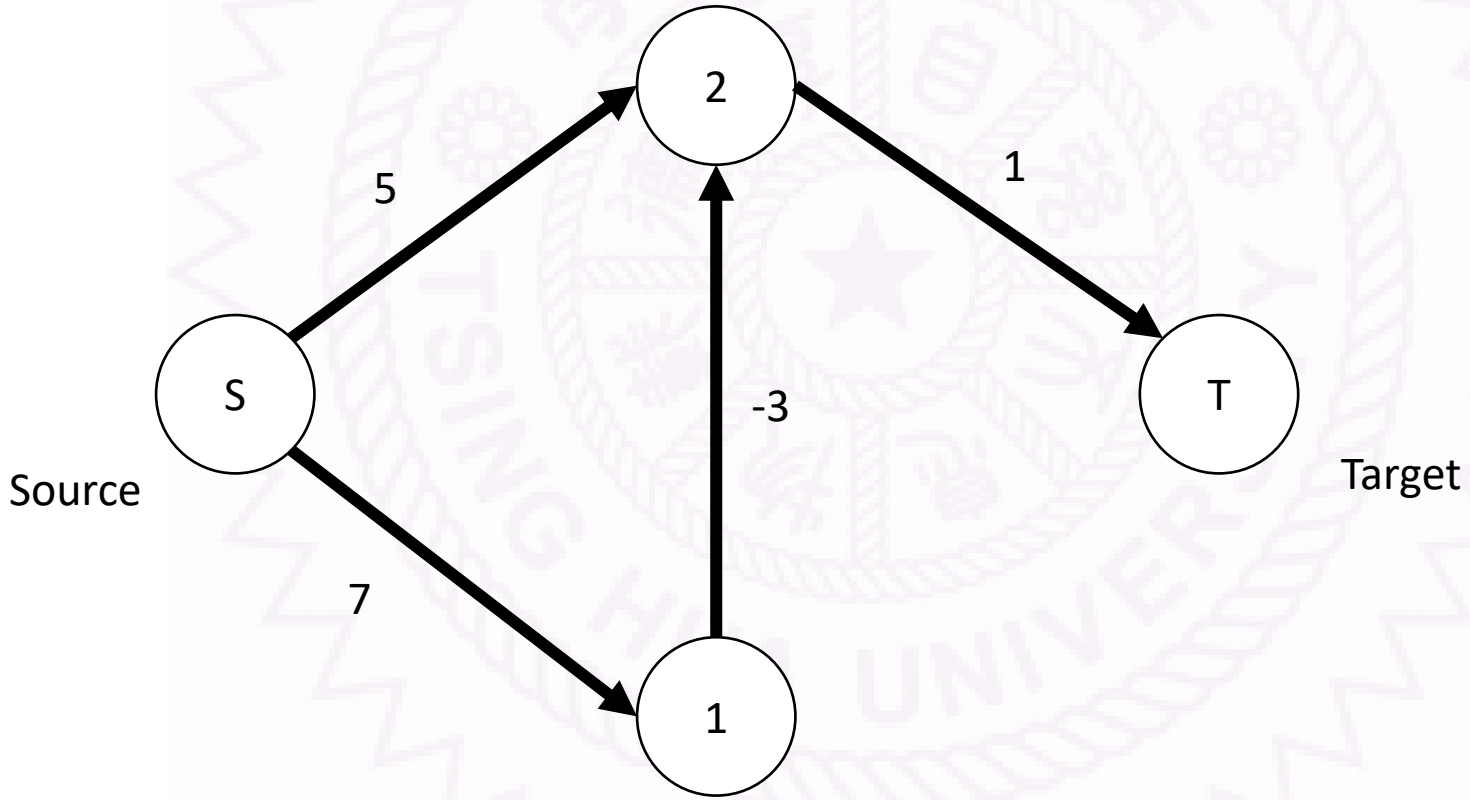
```cpp
vector<long long> dijkstra(const vector<vector<pair<int, int>>> &G, int S) {
  int n = G.size(); // 假設點的編號為 0 ~ n-1
  vector<long long> d(n, INF);
  using QueuePair = pair<long long, int>;
  priority_queue<QueuePair, vector<QueuePair>, greater<QueuePair>> Q;
  d[S] = 0; // 起點設 0
  Q.emplace(d[S], S);
  while (Q.size()) {
    auto [u_dis, u] = Q.top();
    Q.pop();
    if (d[u] < u_dis) continue;
    for (auto [v, cost] : G[u])
      if (d[v] > d[u] + cost) {
        d[v] = d[u] + cost;
        Q.emplace(d[v], v);
      }
  }
  return d;
}
```

時間複雜度

- 每條邊都有機會被丟進heap
- $O(|E|\log|V|)$

- 用費波納契堆
- $O(|E|+|V|\log|V|)$

會爛掉的例子



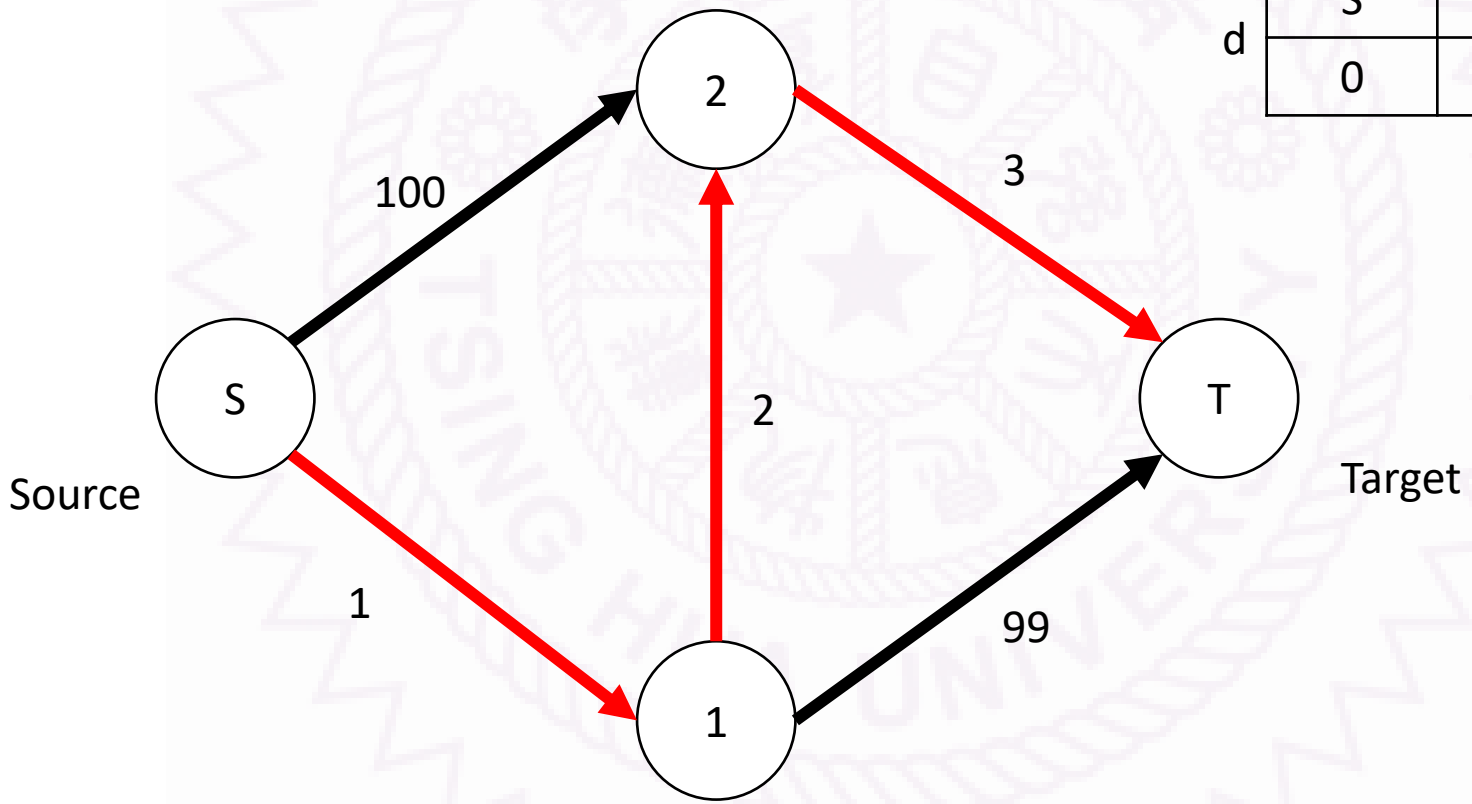
Bellman-Ford

單源最短路徑

目標：
算出這表格d

d

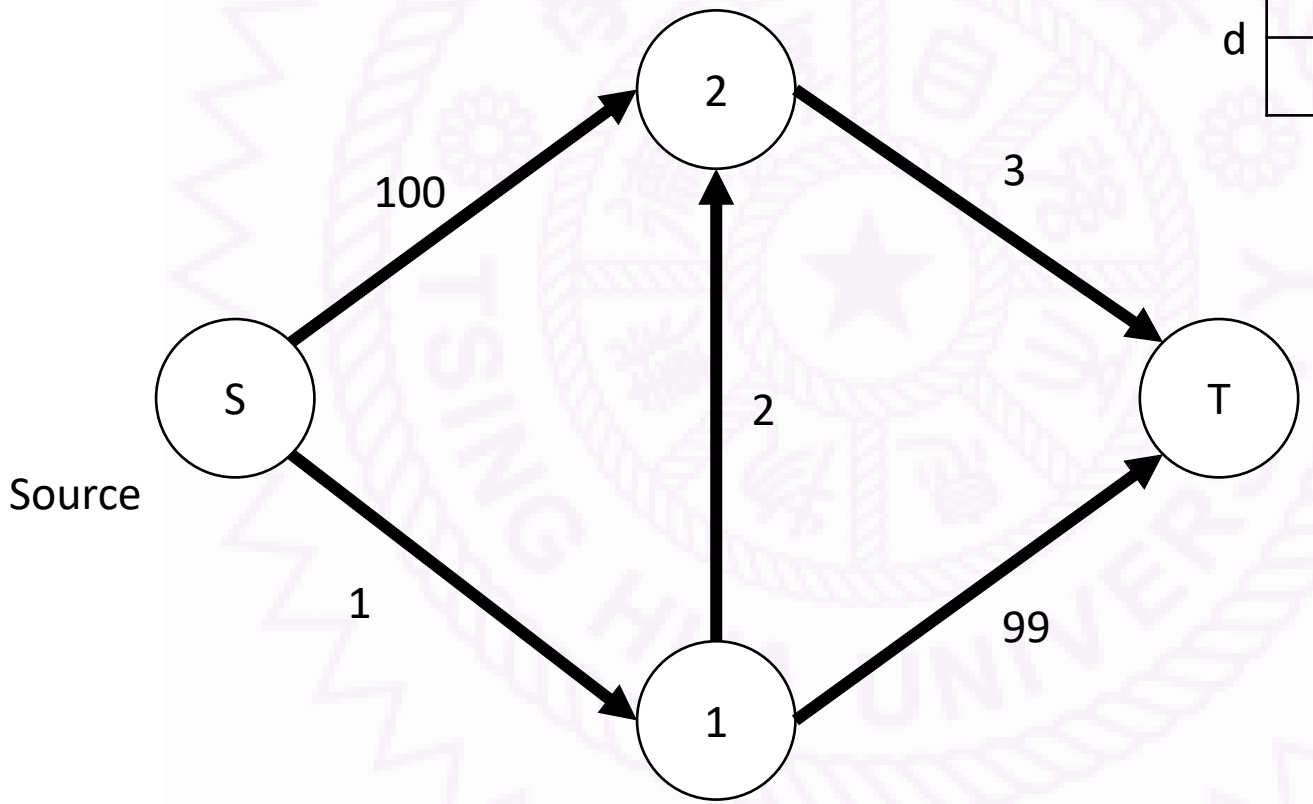
S	1	2	T
0	1	3	6



單源最短路徑

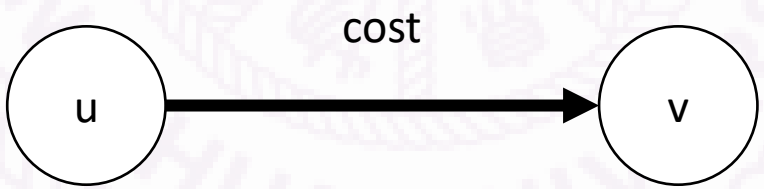
一開始除了起點
其他點都是無限大

d	S	1	2	T
	0	∞	∞	∞



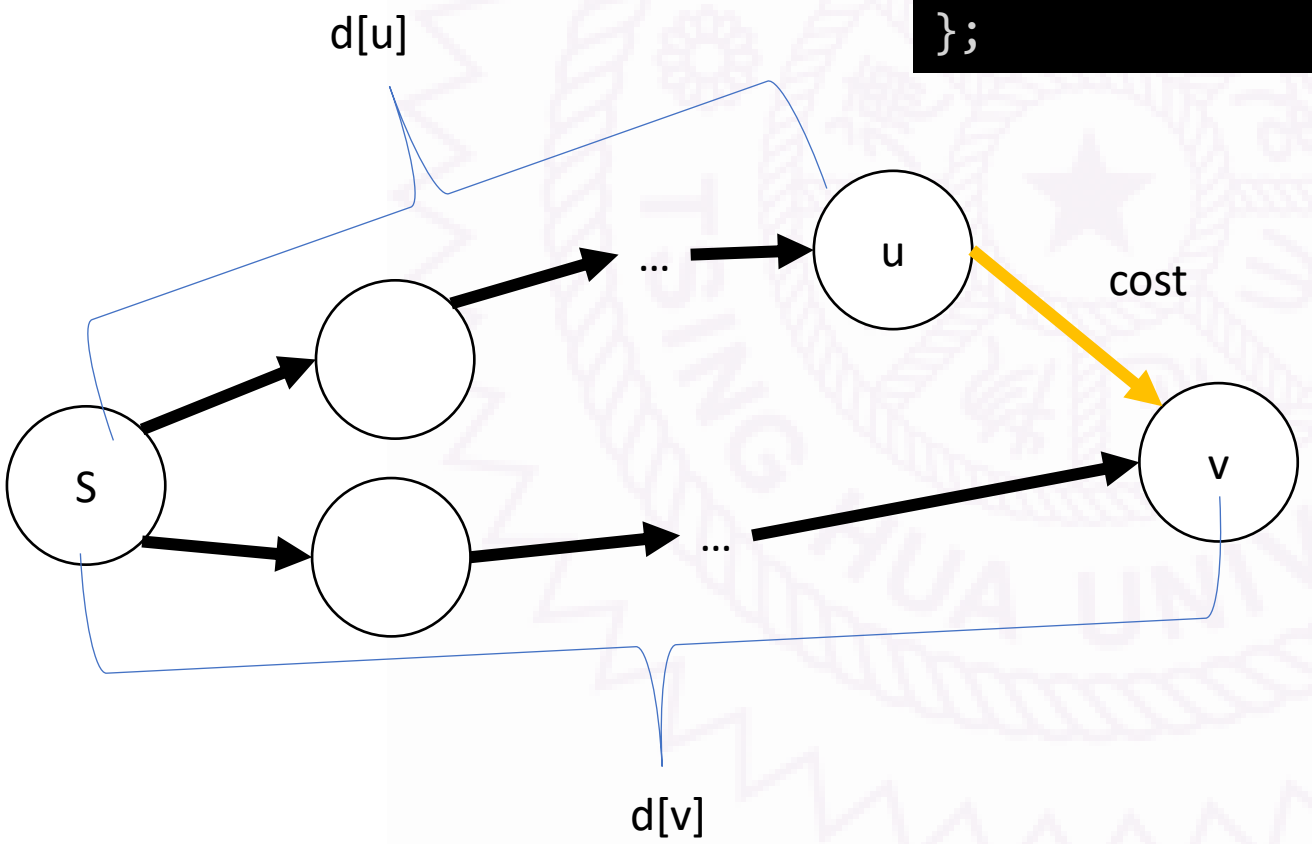
邊的資料結構

```
struct Edge {
  int u, v;
  int cost;
};
```



鬆弛操作

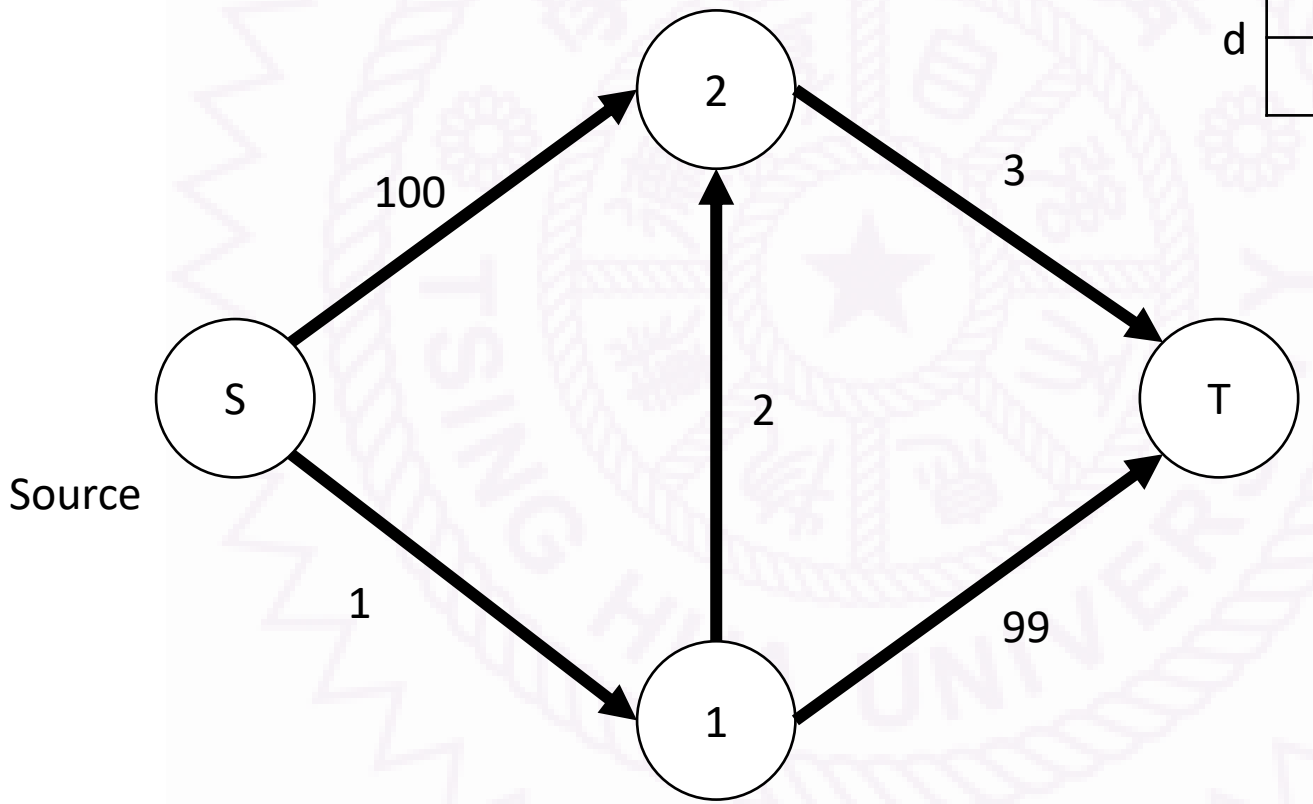
```
auto relax = [&](Edge e) {
  if (d[e.v] > d[e.u] + e.cost) {
    d[e.v] = d[e.u] + e.cost;
    return true;
  }
  return false;
};
```



暴力法：做到無法鬆弛為止

```cpp
vector<long long> bellman_ford(vector<Edge> E, int n, int S) {
  vector<long long> d(n, INF); // 假設點的編號為 0 ~ n-1
  d[S] = 0; // 起點設 0
  auto relax = [&](Edge e) { ... };
  for (;;) {
    bool update = false;
    for (auto &e : E)
      update |= relax(e);
    if (!update) break;
  }
  return d;
}
```

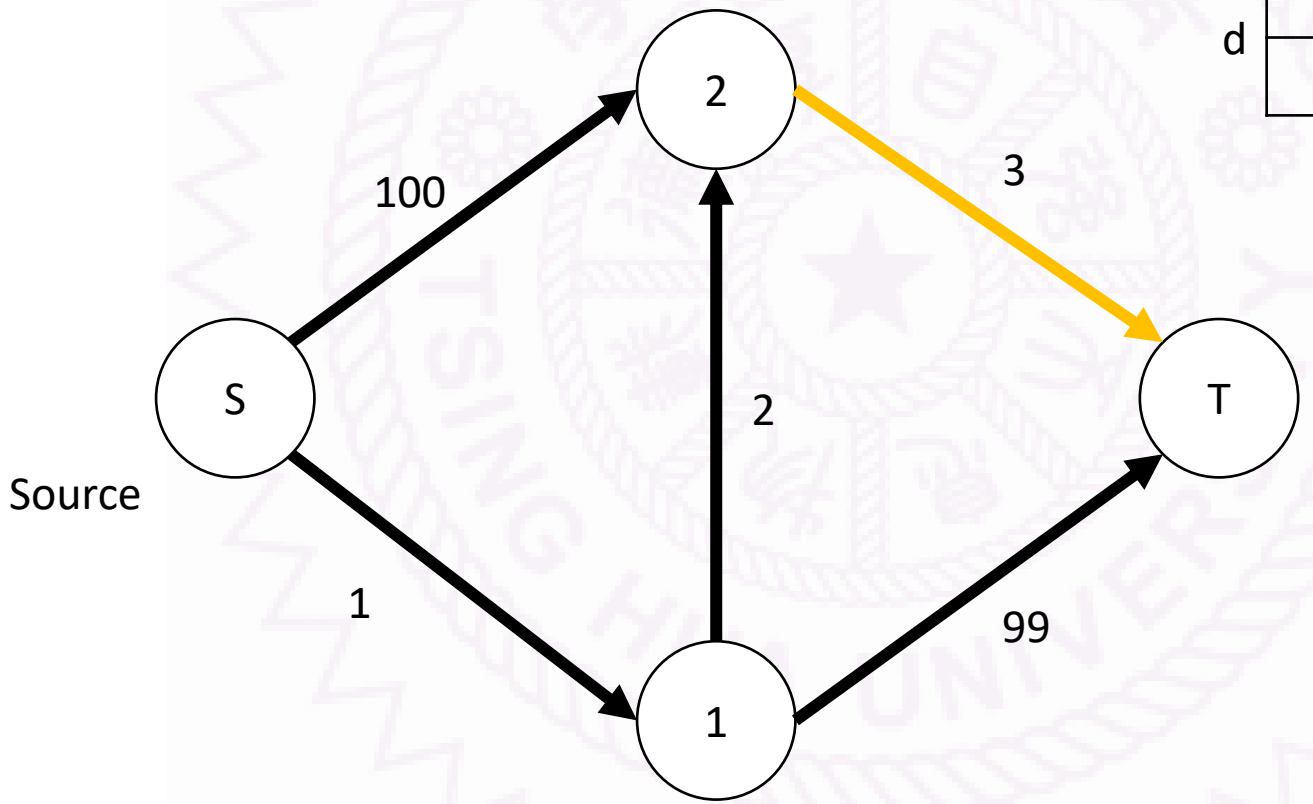
單源最短路徑



d

S	1	2	T
0	∞	∞	∞

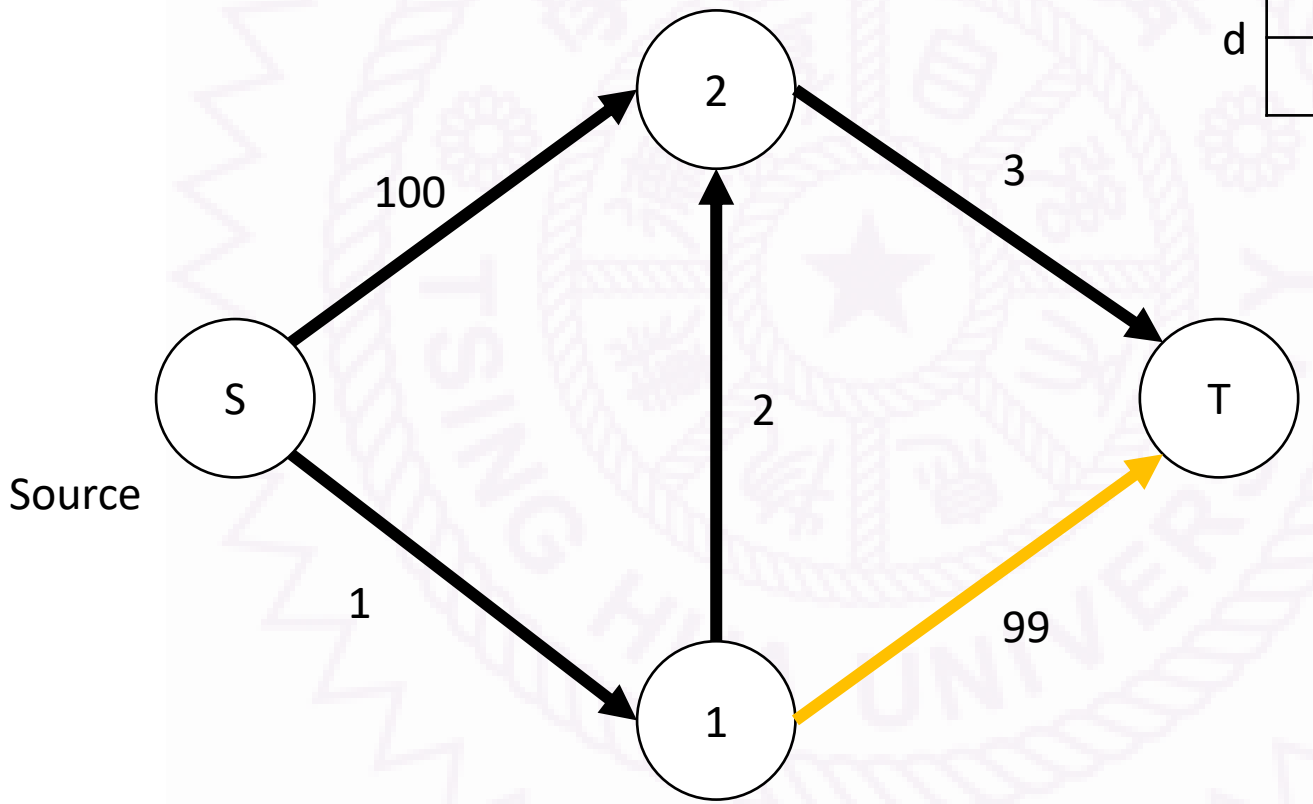
單源最短路徑



d

S	1	2	T
0	∞	∞	∞

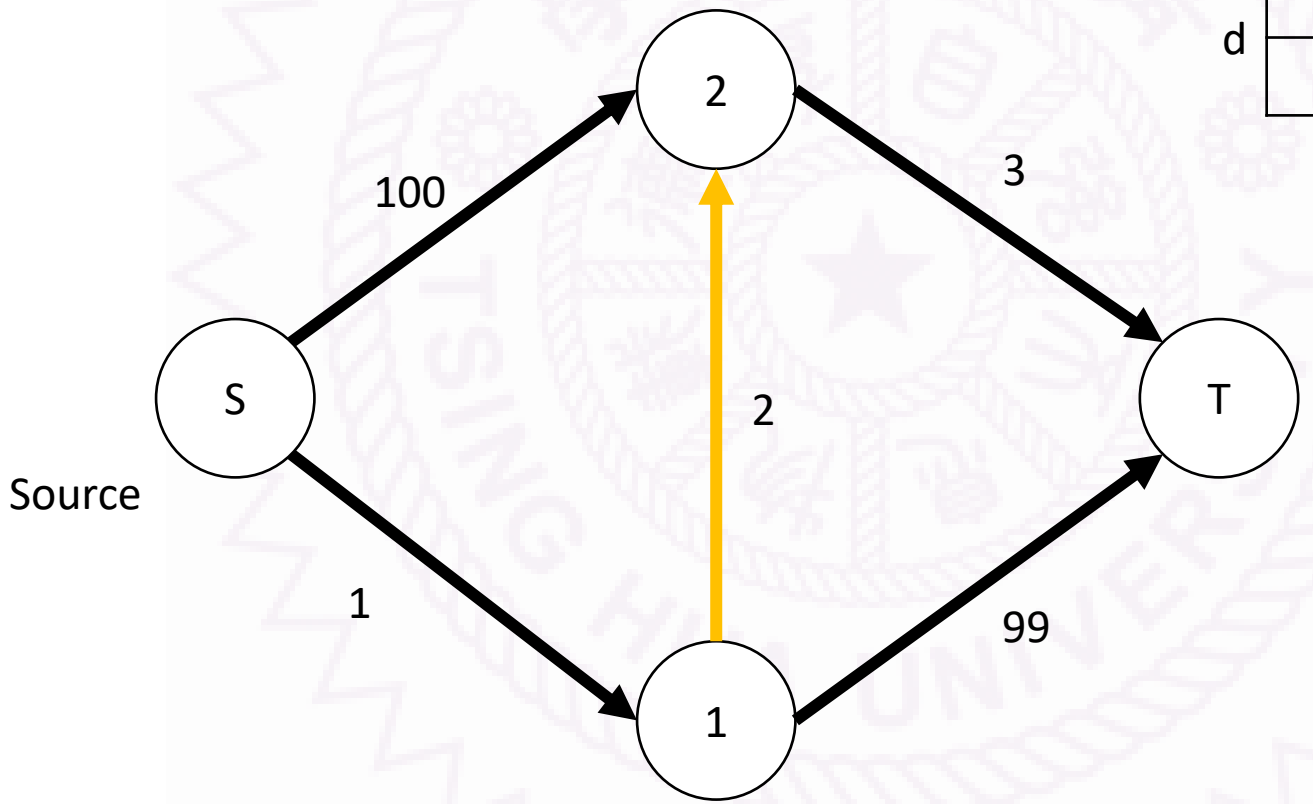
單源最短路徑



d

S	1	2	T
0	∞	∞	∞

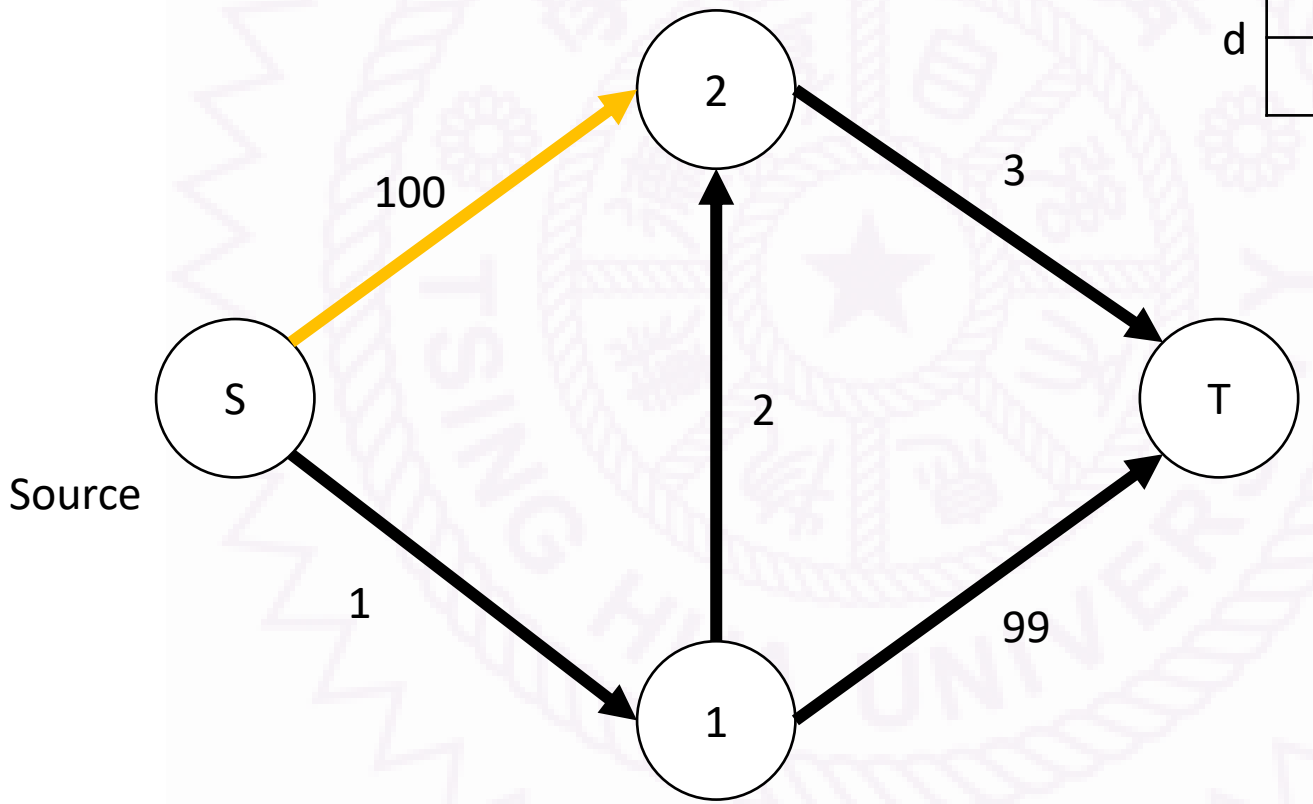
單源最短路徑



d

S	1	2	T
0	∞	∞	∞

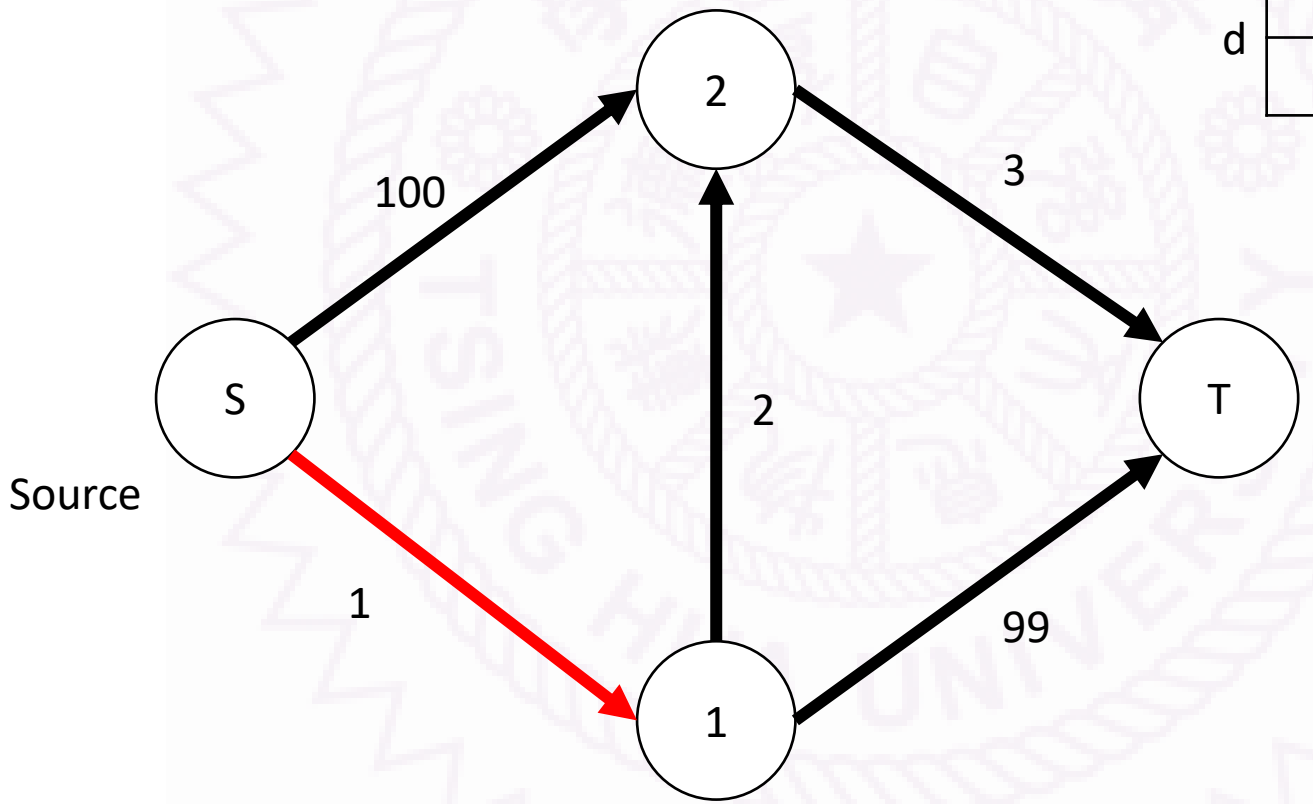
單源最短路徑



d

S	1	2	T
0	∞	100	∞

單源最短路徑

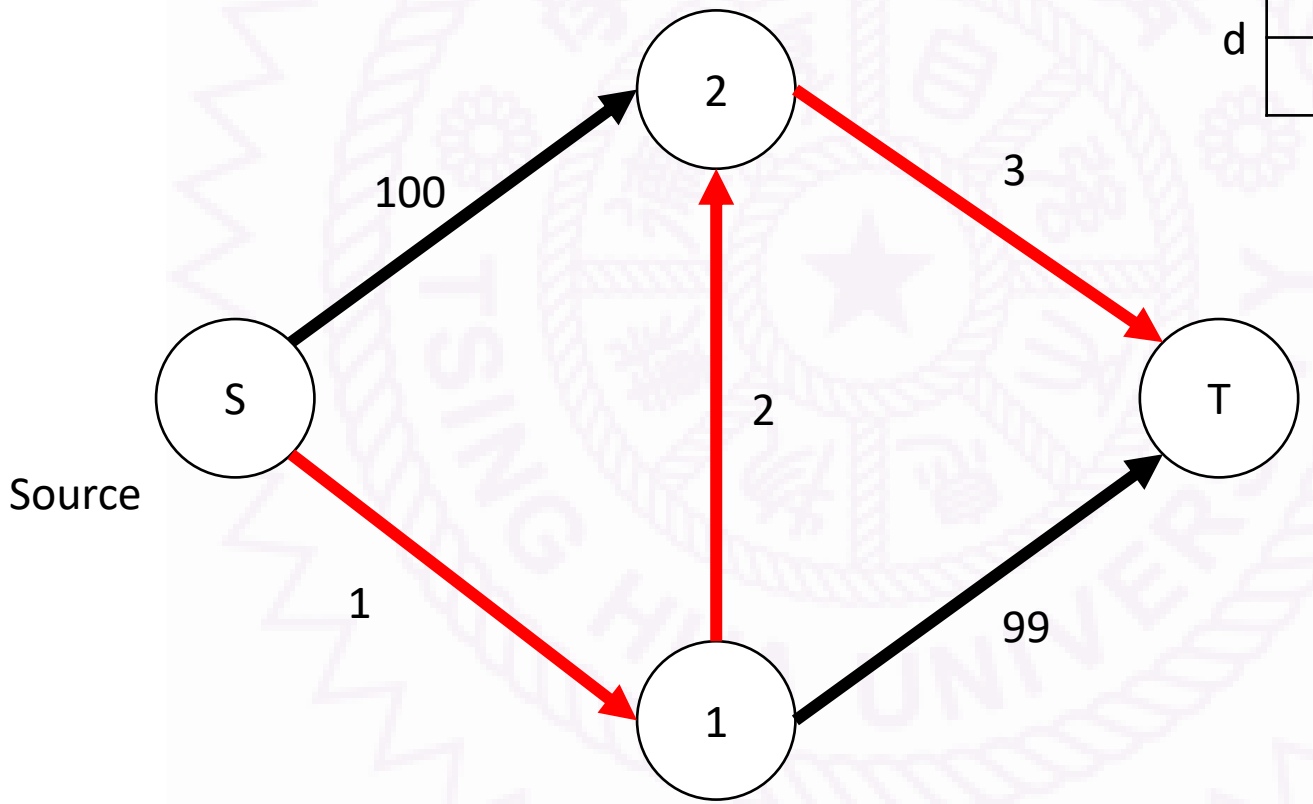


d

S	1	2	T
0	1	100	∞

最差情況下
枚舉完所有邊可以保證有一個點的最短路徑被找到

單源最短路徑



d

S	1	2	T
0	1	3	6

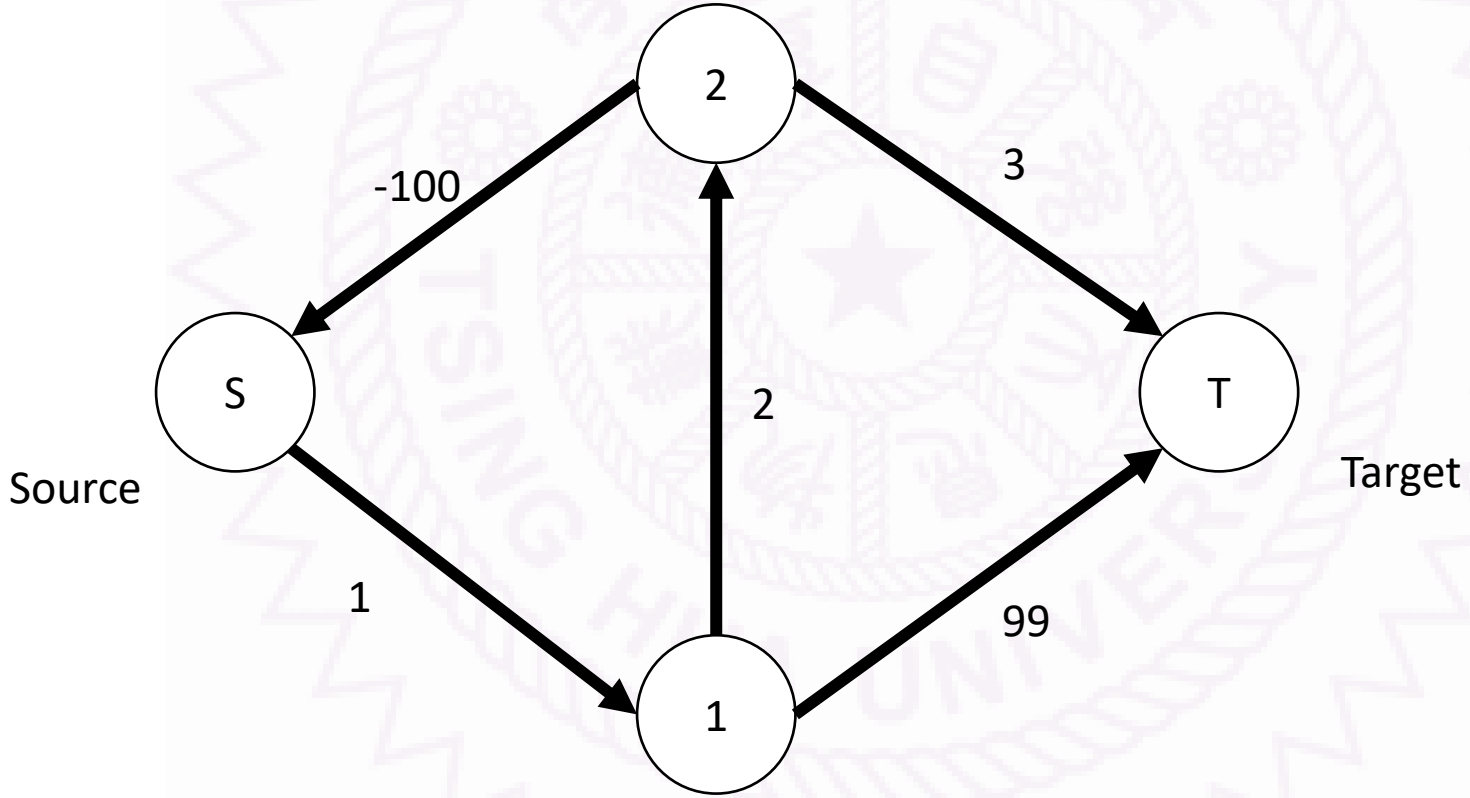
最短路徑只有n-1條邊(最短路徑樹)

Bellman Ford 最差複雜度

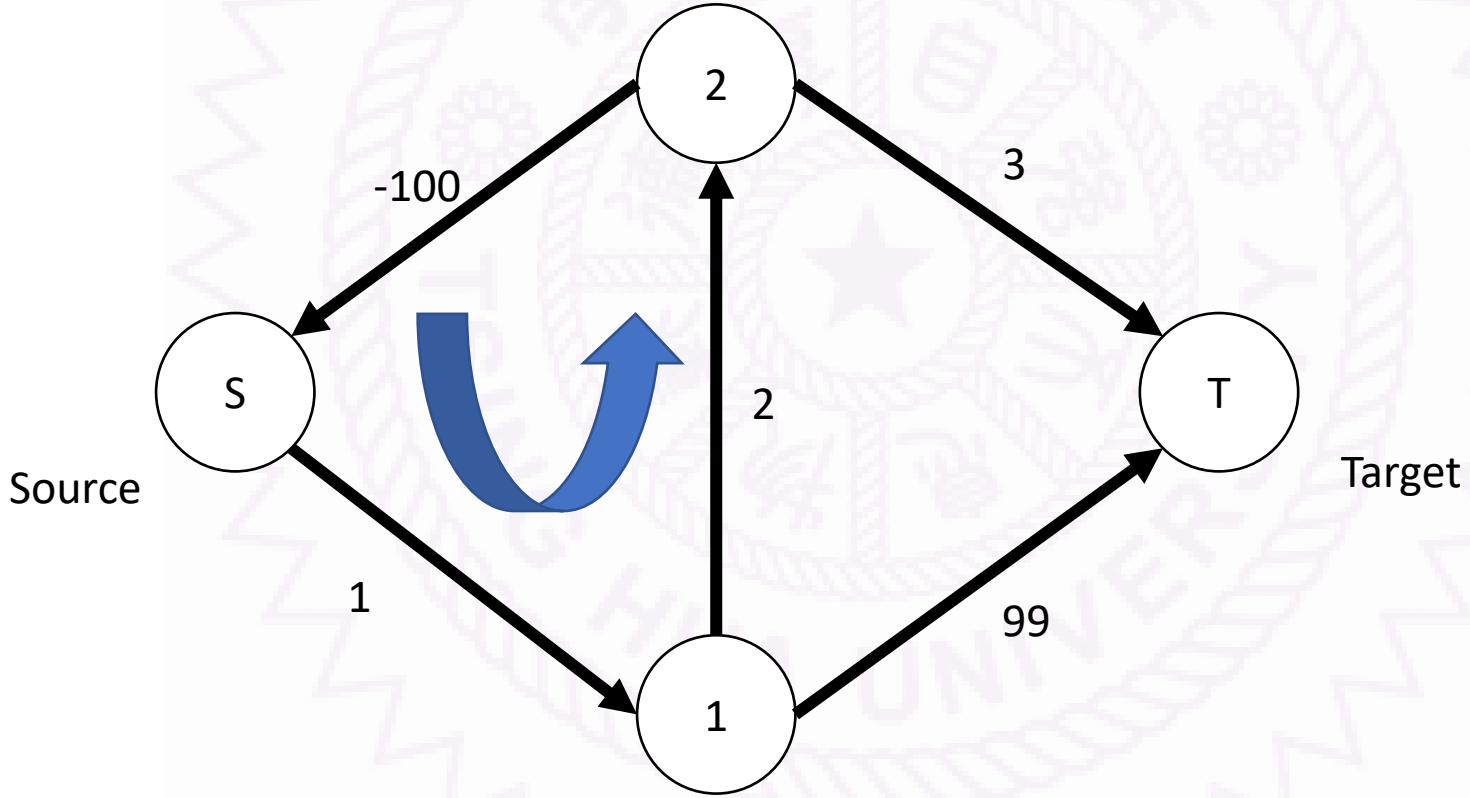
- 最短路徑的邊數 × 枚舉所有邊的時間
  - $(|V| - 1) \times |E|$
- $O(|V||E|)$

```cpp
vector<long long> bellman_ford(vector<Edge> E, int n, int S) {
  vector<long long> d(n, INF); // 假設點的編號為 0 ~ n-1
  d[S] = 0; // 起點設 0
  auto relax = [&](Edge e) { ... };
  for (int t = 1; t <= n - 1; ++t) {
    for (auto &e : E) relax(e);
  }
  return d;
}
```

等等，這樣會發生甚麼事



只要一直繞路徑長就會變成負無限大



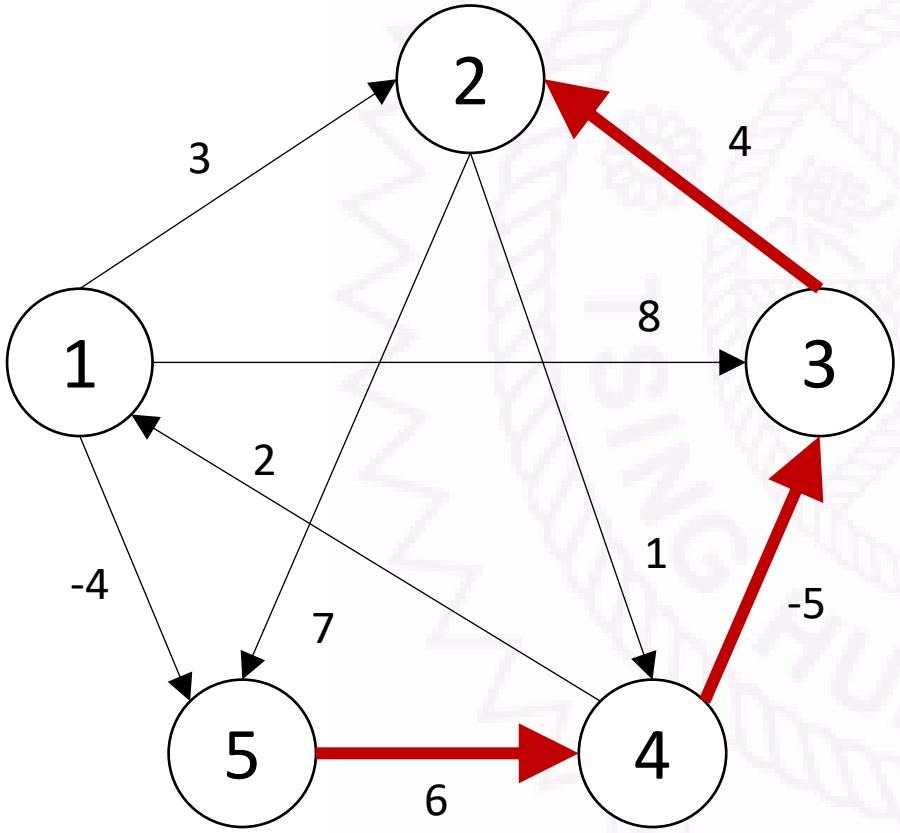
改良算法

```cpp
vector<long long> bellman_ford(vector<Edge> E, int n, int S) {
  vector<long long> d(n, INF); // 假設點的編號為 0 ~ n-1
  d[S] = 0; // 起點設 0
  auto relax = [&](Edge e) { ... };
  for (int t = 1; t <= n; ++t) {
    bool update = false;
    for (auto &e : E)
      update |= relax(e);
    if (t == n && update) return {};
  }
  return d;
}
```

Floyd Warshall

全點對最短路徑

目標輸出



表格 D

	1	2	3	4	5
1	0	1	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

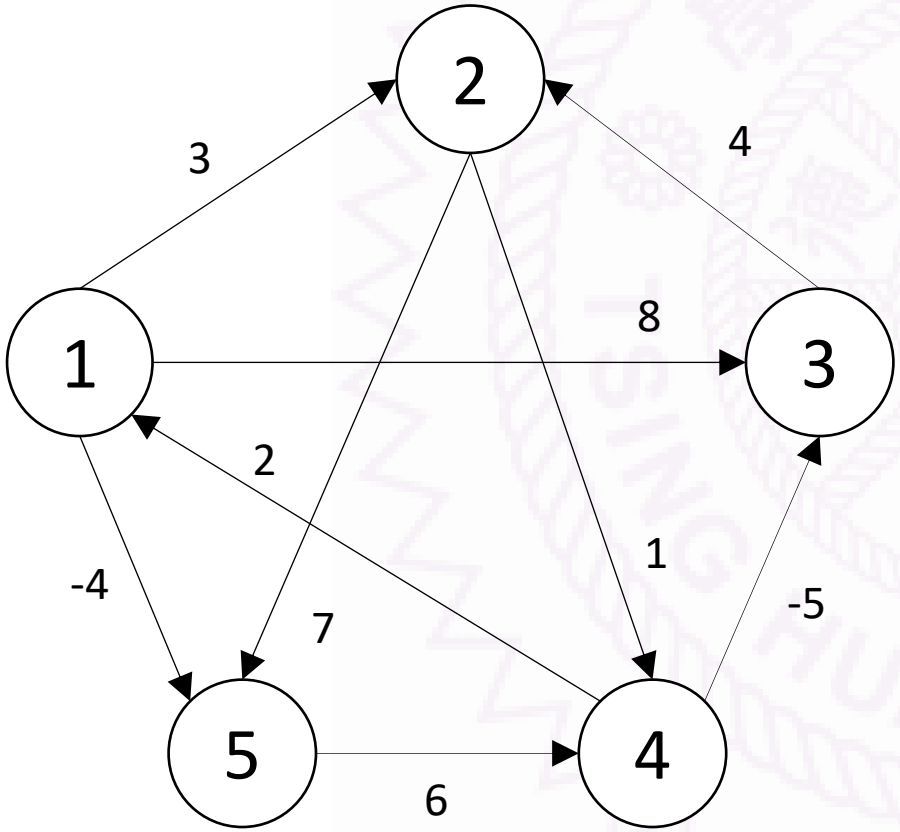
定義

- 假設點的編號為$0 \sim n-1$

- $D(k,u,v)$ ：
  從 u 點走到 v 點，途中能經過的點編號都小於 k
  這種情況下的最短路徑

- $D(0,u,v)$ ：
  - 如果 u v 有邊聯通 ➔ (u,v) 的權重
  - Else ∞

輸入格式 – 直接用 Adjacency Matrix

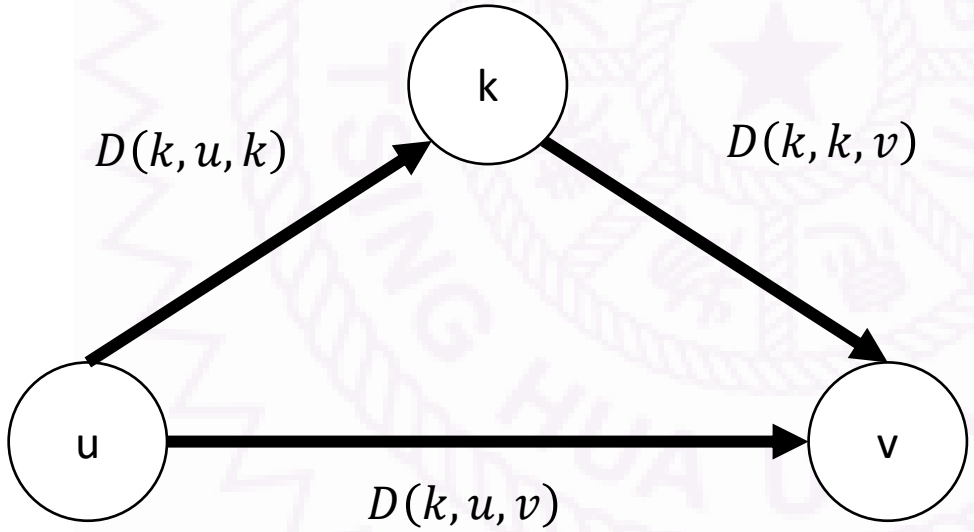


表格 D_0

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

計算 $D(k+1,u,v)$

- $D(k+1,u,v) = \min\big(D(k,u,v),\ D(k,u,k)+D(k,k,v)\big)$



造著公式寫

```
for (int k = 0; k < n; ++k)
  for (int u = 0; u < n; ++u)
    for (int v = 0; v < n; ++v)
      D[k + 1][u][v] = min(D[k][u][v], D[k][u][k] + D[k][k][v]);
```

空間優化

```cpp
void floyd_warshall(vector<vector<long long>> &D) {
  int n = D.size(); // 假設點的編號為 0 ~ n-1
  for (int k = 0; k < n; ++k)
    for (int u = 0; u < n; ++u)
      for (int v = 0; v < n; ++v)
        D[u][v] = min(D[u][v], D[u][k] + D[k][v]);
}
```

https://www.zhihu.com/question/292283275/answer/484871888

被封印的第四演算法 – SPFA

- Shortest Path Faster Algorithm
  - 中國人在 1994 年取的名子
- Algorithm D
  - Edward F. Moore 在 1959 年發表

- 本質上是 Bellman Ford ，所以支援負邊
- 「平均」時間 $O(|E|)$
- 但能構造出一個稀疏圖但需要 $O(|V|^2)$ 的例子
- 各位請自行學習，斟酌使用，進階課程有機會用到

SPFA

```cpp
vector<long long> spfa(vector<vector<pair<int, int>>> G, int S) {
  int n = G.size(); // 假設點的編號為 0 ~ n-1
  vector<long long> d(n, INF);
  vector<bool> in_queue(n, false);
  vector<int> cnt(n, 0);
  queue<int> Q;
  d[S] = 0;
  auto enqueue = [&](int u) {
    in_queue[u] = true; Q.emplace(u);
  };
  enqueue(S);
  while (Q.size()) {
    int u = Q.front(); Q.pop();
    in_queue[u] = false;
    for (auto [v, cost] : G[u])
      if (d[v] > d[u] + cost) {
        if (++cnt[u] >= n) return {}; // 存在負環
        d[v] = d[u] + cost;
        if (!in_queue[v]) enqueue(v);
      }
  }
  return d;
}
```