

# 回溯 與剪枝

日月卦長



5	3		7					
6		1	9	5				
	9	8				6		
8			6					3
4		8		3				1
7			2					6
	6				2	8		
		4	1	9				5
		8			7	9		

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

- 每一列的數字均須包含 1~9，不能缺少，也不能重複。
- 每一宮(粗黑線圍起來的區域，通常是 3\*3 的九宮格)的數字均須包含 1~9，不能缺少，也不能重複。

# Sudoku 獨數

# 判斷數獨是否合法

## Input

```
193265478  
782314956  
456978132  
234851697  
965437281  
871692345  
319586724  
527143869  
648729513
```

## Output

```
Yes
```

# 判斷數獨是否合法

## Input

```
123456789  
234567891  
345678912  
456789123  
567891234  
678912345  
789123456  
891234567  
912345678
```

## Output

```
No
```

# 輸入資料

```
#include <iostream>
#include <string>
using namespace std;

int grid[9][9];
void input() {
    for (int r = 0; r < 9; ++r) {
        string buffer;
        cin >> buffer;
        for (int c = 0; c < 9; ++c) {
            grid[r][c] = buffer[c] - '0';
        }
    }
}
```

# 判斷 row

3	2	3	9	5	7				

bool row[9][10]

row[3]	0	1	2	3	4	5	6	7	8	9
			true	true		true		true		true

# 判斷 column

									7	
									1	
									3	
									5	
									7	
									9	

bool col[9][10]

0	1	2	3	4	5	6	7	8	9
	true								

# 判斷 subgrid


1

2

3 2 6

7 5

```
bool subgrids[3][3][10];
```

subgrids[2][1]

0	1	2	3	4	5	6	7	8	9
		true	true		true	true	true		

# 判斷 + 更新資料

```
bool row[9][10], col[9][10];
bool subgrids[3][3][10];

bool illegal(int r, int c, int num) {
    return row[r][num] || col[c][num] || subgrids[r / 3][c / 3][num];
}

void update(int r, int c, int num, bool val) {
    row[r][num] = val;
    col[c][num] = val;
    subgrids[r / 3][c / 3][num] = val;
}
```

# 判斷數獨

```
bool check() {
    for (int r = 0; r < 9; ++r) {
        for (int c = 0; c < 9; ++c) {
            if (grid[r][c] == 0) continue; // 伏筆
            if (illegal(r, c, grid[r][c]))
                return false;
            update(r, c, grid[r][c], true);
        }
    }
    return true;
}

int main() {
    input();
    cout << (check() ? "Yes\n" : "No\n");
    return 0;
}
```

# 數獨求解(輸出最小字典序)

## Input

1.....478  
7..314956  
456978132  
234851697  
965437281  
871692345  
319586724  
527143869  
648729513

## Output

193265478  
782314956  
456978132  
234851697  
965437281  
871692345  
319586724  
527143869  
648729513

# 輸入資料

```
int grid[9][9];
void input() {
    for (int r = 0; r < 9; ++r) {
        string buffer;
        cin >> buffer;
        for (int c = 0; c < 9; ++c) {
            if (isdigit(buffer[c]))
                grid[r][c] = buffer[c] - '0';
            else
                grid[r][c] = 0;
        }
    }
}
```

# 印出答案

```
void print() {
    for (int r = 0; r < 9; ++r) {
        for (int c = 0; c < 9; ++c)
            cout << grid[r][c];
        cout << '\n';
    }
}
```

# 幫每個格子編號

- $R = 6$
- $C = 3$

$$\begin{aligned} 57 &= R \times 9 + C \\ &= 6 \times 9 + 3 \end{aligned}$$

- $R = \lfloor 57/9 \rfloor = 6$
- $C = 57 \% 9 = 3$

		C									
		0	1	2	3	4	5	6	7	8	
		0	0	1	2	3	4	5	6	7	8
		1	9	10	11	12	13	14	15	16	17
		2	18	19	20	21	22	23	24	25	26
		3	27	28	29	30	31	32	33	34	35
		4	36	37	38	39	40	41	42	43	44
		5	45	46	47	48	49	50	51	52	53
		6	54	55	56	57	58	59	60	61	62
		7	63	64	65	66	67	68	69	70	71
		8	72	73	74	75	76	77	78	79	80

# 暴力枚舉所有可能性

```
bool dfs(int idx) {
    if (idx == 81) {
        memset(row, 0, sizeof(row));
        memset(col, 0, sizeof(col));
        memset(subgrids, 0, sizeof(subgrids));
        return check();
    }
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = 1; num <= 9; ++num) {
        grid[r][c] = num;
        if (dfs(idx + 1)) return true;
    }
    grid[r][c] = 0;
    return false;
}
```

# 暴力枚舉所有可能性

- 假設  $0 \sim idx - 1$  的格子都填好了
- 依序枚舉所有填滿  $idx \sim 80$  的所有可能
- 若找到合法解則回傳 true

```
bool dfs(int idx) {
    if (idx == 81) {
        memset(row, 0, sizeof(row));
        memset(col, 0, sizeof(col));
        memset(subgrids, 0, sizeof(subgrids));
        return check();
    }
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = 1; num <= 9; ++num) {
        grid[r][c] = num;
        if (dfs(idx + 1)) return true;
    }
    grid[r][c] = 0;
    return false;
}
```

# 暴力枚舉所有可能性

- 所有格子都填滿了
- 回傳是否是合法數獨

```
bool dfs(int idx) {
    if (idx == 81) {
        memset(row, 0, sizeof(row));
        memset(col, 0, sizeof(col));
        memset(subgrids, 0, sizeof(subgrids));
        return check();
    }
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = 1; num <= 9; ++num) {
        grid[r][c] = num;
        if (dfs(idx + 1)) return true;
    }
    grid[r][c] = 0;
    return false;
}
```

# 暴力枚舉所有可能性

- 如果格子已經有數字了
- 就直接跳過枚舉下一個格子

```
bool dfs(int idx) {
    if (idx == 81) {
        memset(row, 0, sizeof(row));
        memset(col, 0, sizeof(col));
        memset(subgrids, 0, sizeof(subgrids));
        return check();
    }
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = 1; num <= 9; ++num) {
        grid[r][c] = num;
        if (dfs(idx + 1)) return true;
    }
    grid[r][c] = 0;
    return false;
}
```

# 暴力枚舉所有可能性

- 枚舉數字 1~9  
依序填入格子中
- 由於由小到大枚舉  
一旦找到解那就會是  
字典序最小的解

```
bool dfs(int idx) {
    if (idx == 81) {
        memset(row, 0, sizeof(row));
        memset(col, 0, sizeof(col));
        memset(subgrids, 0, sizeof(subgrids));
        return check();
    }
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = 1; num <= 9; ++num) {
        grid[r][c] = num;
        if (dfs(idx + 1)) return true;
    }
    grid[r][c] = 0;
    return false;
}
```

# 暴力枚舉所有可能性

- 為了讓當前的函數不影響其他正在遞迴的函數
- 結束遞迴時一定要把所有修改都復原

```
bool dfs(int idx) {
    if (idx == 81) {
        memset(row, 0, sizeof(row));
        memset(col, 0, sizeof(col));
        memset(subgrids, 0, sizeof(subgrids));
        return check();
    }
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = 1; num <= 9; ++num) {
        grid[r][c] = num;
        if (dfs(idx + 1)) return true;
    }
    grid[r][c] = 0;
    return false;
}
```

# 暴力枚舉所有可能性

```
int main() {
    input();
    if (check() && dfs(0))
        print();
    else
        cout << "No answer\n";
    return 0;
}
```

# 更難的測資

## Input

```
.1.....9  
...3..8..  
.....6..  
....124..  
7.3.....  
5.....  
8..6.....  
....4..2.  
...7...5.
```

## Output

```
318456279  
267391845  
459287613  
986512437  
723964581  
541873962  
872635194  
635149728  
194728356
```



回朔 = 暴力枚舉 + 剪枝  
(Backtracking)

進入遞迴前就發現走下去永遠找不到解  
就直接跳過這次遞迴

# 剪枝：不要做沒意義的枚舉

# 剪枝: 不要做沒意義的枚舉

9	8	7	6	5	4	3	2	1
2	4	6	1	7	3	9	8	5
3	5	1	9	2	1			

continue

# 剪枝: 不要做沒意義的枚舉

9	8	7	6	5	4	3	2	1
2	4	6	1	7	3	9	8	5
3	5	1	9	2	2			

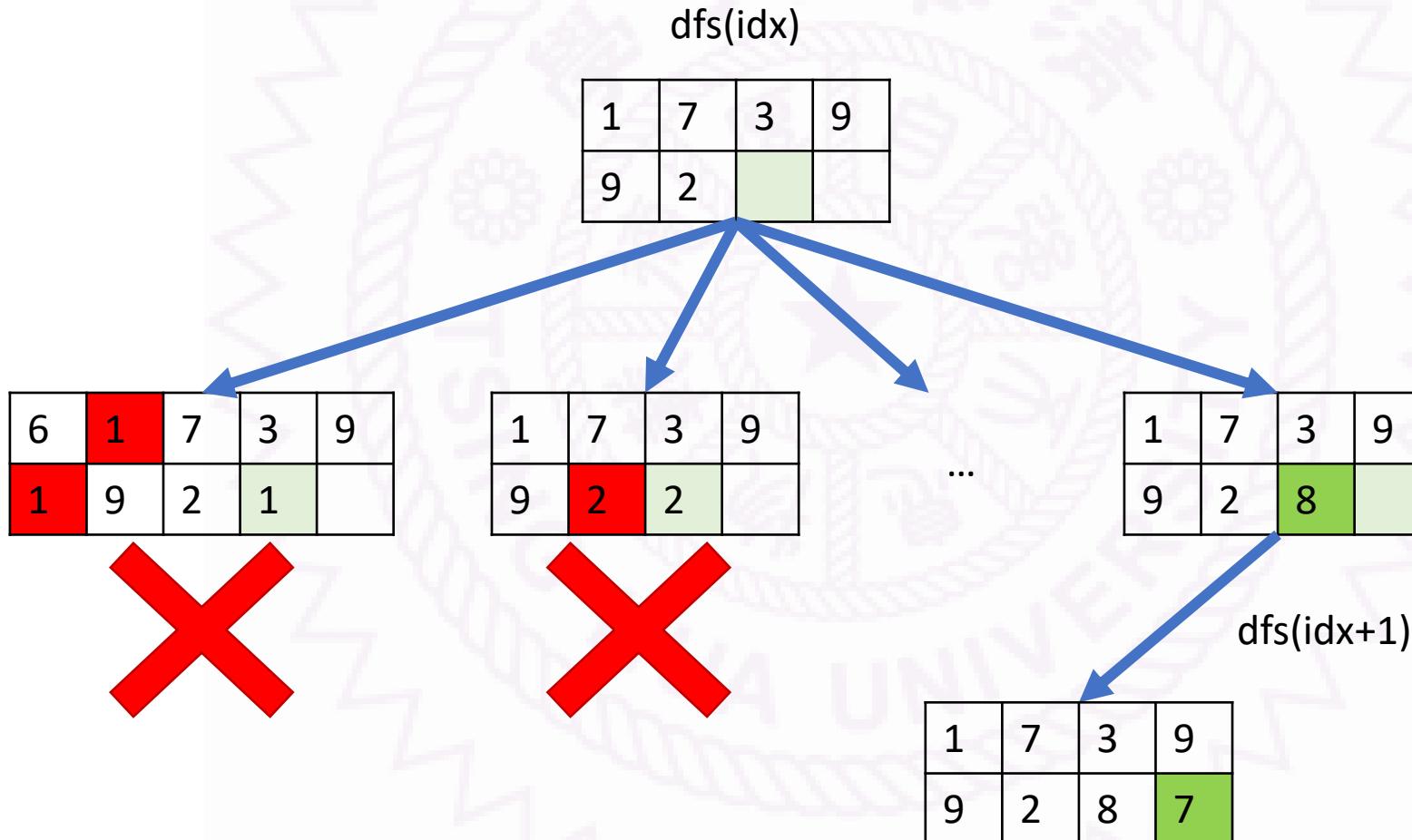
continue

# 剪枝: 不要做沒意義的枚舉

9	8	7	6	5	4	3	2	1
2	4	6	1	7	3	9	8	5
3	5	1	9	2	8			

`dfs(idx+1)`

# 剪枝: 不要做沒意義的枚舉



# 使用 Backtracking

- 如果  $\text{grid}[r][c] = \text{num}$  時  
就可以直接判斷是非法解
- 就沒必要針對  $\text{num}$  遷迴  
直接跳過 (continue)

```
bool dfs(int idx) {
    if (idx == 81) return true;
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = 1; num <= 9; ++num) {
        if (illegal(r, c, num)) continue;
        grid[r][c] = num;
        update(r, c, num, true);
        if (dfs(idx + 1)) return true;
        update(r, c, num, false);
    }
    grid[r][c] = 0;
    return false;
}
```

# 使用 Backtracking

- 進入遞迴前紀錄當前格子填入 num
- 若沒找到答案離開遞迴後把紀錄刪除

```
bool dfs(int idx) {
    if (idx == 81) return true;
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = 1; num <= 9; ++num) {
        if (illegal(r, c, num)) continue;
        grid[r][c] = num;
        update(r, c, num, true);
        if (dfs(idx + 1)) return true;
        update(r, c, num, false);
    }
    grid[r][c] = 0;
    return false;
}
```

# 使用 Backtracking

- 由於每次遞迴前都有判斷合法性
- 當 81 個格子都填完後答案一定是合法的
- 直接 return true

```
bool dfs(int idx) {  
    if (idx == 81) return true;  
    int r = idx / 9, c = idx % 9;  
    if (grid[r][c]) return dfs(idx + 1);  
    for (int num = 1; num <= 9; ++num) {  
        if (illegal(r, c, num)) continue;  
        grid[r][c] = num;  
        update(r, c, num, true);  
        if (dfs(idx + 1)) return true;  
        update(r, c, num, false);  
    }  
    grid[r][c] = 0;  
    return false;  
}
```

# 輸入資料記得 update

```
int grid[9][9];
void input() {
    for (int r = 0; r < 9; ++r) {
        string buffer;
        cin >> buffer;
        for (int c = 0; c < 9; ++c) {
            if (isdigit(buffer[c])) {
                grid[r][c] = buffer[c] - '0';
                update(r, c, grid[r][c], true);
            } else
                grid[r][c] = 0;
        }
    }
}
```

# 再難一點?

## Input

```
.....  
....3.85  
.1.2....  
..5.7...  
.4...1..  
.9.....  
5.....73  
.2.1....  

```

## Output

```
987654321  
246173985  
351928746  
128537694  
634892157  

```

就是 0 和 1

true → 1

false → 0



# 為什麼不用二進位存?

```
bool row[9][10], col[9][10];  
bool subgrids[3][3][10],
```

長度是 10 的 01 陣列

```
int row[9], col[9];  
int subgrids[3][3];
```

int: 一般電腦上是 32 個 bit 組成  
→長度是 32 的 01 陣列

# 二進位表示法

	二進位	十進位
1	0000000010	2
2	0000000100	4
3	0000001000	8
4	0000010000	16
5	0000100000	32
6	0001000000	64
7	0010000000	128
8	0100000000	256
9	1000000000	512

```
int lg(int x) {  
    switch(x){  
        case 2: return 1;  
        case 4: return 2;  
        case 8: return 3;  
        case 16: return 4;  
        case 32: return 5;  
        case 64: return 6;  
        case 128: return 7;  
        case 256: return 8;  
        case 512: return 9;  
    }  
    return -1;  
}
```

```
cout << lg(1 << 8) << endl;  
cout << __lg(1 << 8) << endl;
```

黑魔法

# 輸入資料

```
int grid[9][9];
void input() {
    for (int r = 0; r < 9; ++r) {
        string buffer;
        cin >> buffer;
        for (int c = 0; c < 9; ++c) {
            if (isdigit(buffer[c])) {
                grid[r][c] = 1 << (buffer[c] - '0');
                update(r, c, grid[r][c], true);
            } else
                grid[r][c] = 0;
        }
    }
}
```

# 輸出答案

```
void print() {
    for (int r = 0; r < 9; ++r) {
        for (int c = 0; c < 9; ++c)
            cout << __lg(grid[r][c]);
        cout << '\n';
    }
}
```

# 判斷 + 更新資料

```
int row[9], col[9];
int subgrids[3][3];

bool illegal(int r, int c, int num) {
    return (row[r] | col[c] | subgrids[r / 3][c / 3]) & num;
}

void update(int r, int c, int num) {
    row[r] ^= num;
    col[c] ^= num;
    subgrids[r / 3][c / 3] ^= num;
}
```

# 透過二進位紀錄用過的數字

- 整體上沒太大差別
- 記得枚舉數字時  
要用二進位

```
bool dfs(int idx) {
    if (idx == 81) return true;
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    for (int num = (1 << 1); num <= (1 << 9); num <<= 1) {
        if (illegal(r, c, num)) continue;
        grid[r][c] = num;
        update(r, c, num);
        if (dfs(idx + 1)) return true;
        update(r, c, num);
    }
    grid[r][c] = 0;
    return false;
}
```



使用 lowbit 減少枚舉數量

lowbit 優化

# 不能用的數字集合

- 紅色區域計算後的數字  
若第  $k$  個 bit 是 1，表示數字  $k$  不能被使用

```
bool illegal(int r, int c, int num) {
    return (row[r] | col[c] | subgrids[r / 3][c / 3]) & num;
}
```

# 可以用的數字集合

`row[r] | col[c] | subgrids[r / 3][c / 3]`

0	...	0	1	0	1	1	1	0	0	1	1	0
---	-----	---	---	---	---	---	---	---	---	---	---	---

not

1	...	1	0	1	0	0	0	1	1	0	0	1
---	-----	---	---	---	---	---	---	---	---	---	---	---

MASK

0	...	0	1	1	1	1	1	1	1	1	1	0
---	-----	---	---	---	---	---	---	---	---	---	---	---

and

0	...	0	0	1	0	0	0	1	1	0	0	0
---	-----	---	---	---	---	---	---	---	---	---	---	---

```
const int MASK = (1 << 10) - 2;
int S = MASK & ~(row[r] | col[c] | subgrids[r / 3][c / 3]);
```

# 可以用的數字集合

```
const int MASK = (1 << 10) - 2;
bool dfs(int idx) {
    if (idx == 81) return true;
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    int S = MASK & ~(row[r] | col[c] | subgrids[r / 3][c / 3]);
    for (int num = (1 << 1); num <= (1 << 9); num <<= 1) {
        if ((num & S) == 0) continue;
        grid[r][c] = num;
        update(r, c, num);
        if (dfs(idx + 1)) return true;
        update(r, c, num);
    }
    grid[r][c] = 0;
    return false;
}
```

# 重要函數 $lowbit(x)$

- $lowbit(x)$ ：  
非負整數  $x$  在二進位表示時，最靠右邊的 1 所對應的值。
- 範例：

$$20_{(10)} = 10100_{(2)}$$

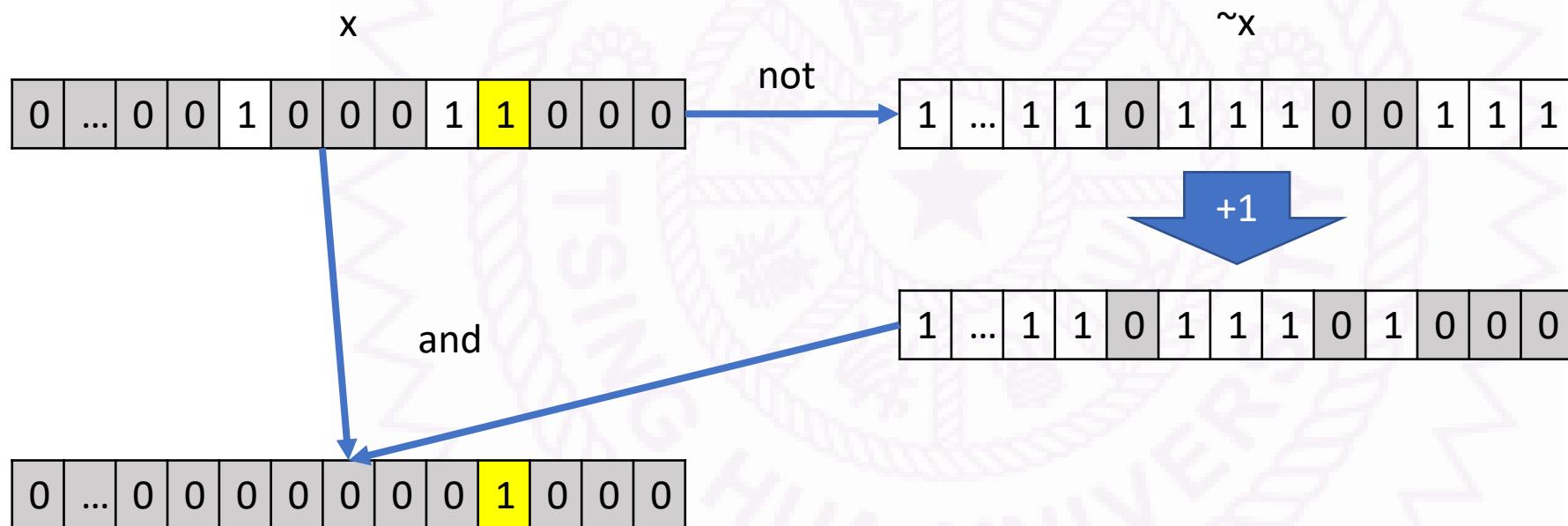
其中的兩個 bit 分別表示  $2^4$  和  $2^2$ ，因此

$$lowbit(20) = 2^2 = 4$$

# $lowbit(x)$ 計算

二補數的  $-x$

```
int lowbit(int x) { return x & (~x + 1); }
```



```
int lowbit(int x) { return x & -x; }
```

Unspecific  
Behavior  
(before C++20)

# 枚舉所有是 1 的 bit

計算 32 次

```
#include <bitset>
#include <iostream>
using namespace std;

int main() {
    int S = 0b100011000;
    cout << bitset<32>(S) << endl;
    for (int i = 0; i < 32; ++i) {
        if (S & (1 << i))
            cout << bitset<32>(1 << i) << endl;
    }
    return 0;
}
```

計算 3(是 1 的 bit 數) 次

```
#include <bitset>
#include <iostream>
using namespace std;
int lowbit(int x) { return x & -x; }

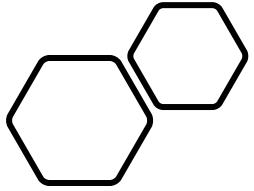
int main() {
    int S = 0b100011000;
    cout << bitset<32>(S) << endl;
    for (int num = 0; S; S ^= num) {
        num = lowbit(S);
        cout << bitset<32>(num) << endl;
    }
    return 0;
}
```

# *lowbit* 優化

```
const int MASK = (1 << 10) - 2;
int lowbit(int x) { return x & -x; }
bool dfs(int idx) {
    if (idx == 81) return true;
    int r = idx / 9, c = idx % 9;
    if (grid[r][c]) return dfs(idx + 1);
    int S = MASK & ~(row[r] | col[c] | subgrids[r / 3][c / 3]);
    for (int num = 0; S; S ^= num) {
        num = lowbit(S);
        grid[r][c] = num;
        update(r, c, num);
        if (dfs(idx + 1)) return true;
        update(r, c, num);
    }
    grid[r][c] = 0;
    return false;
}
```

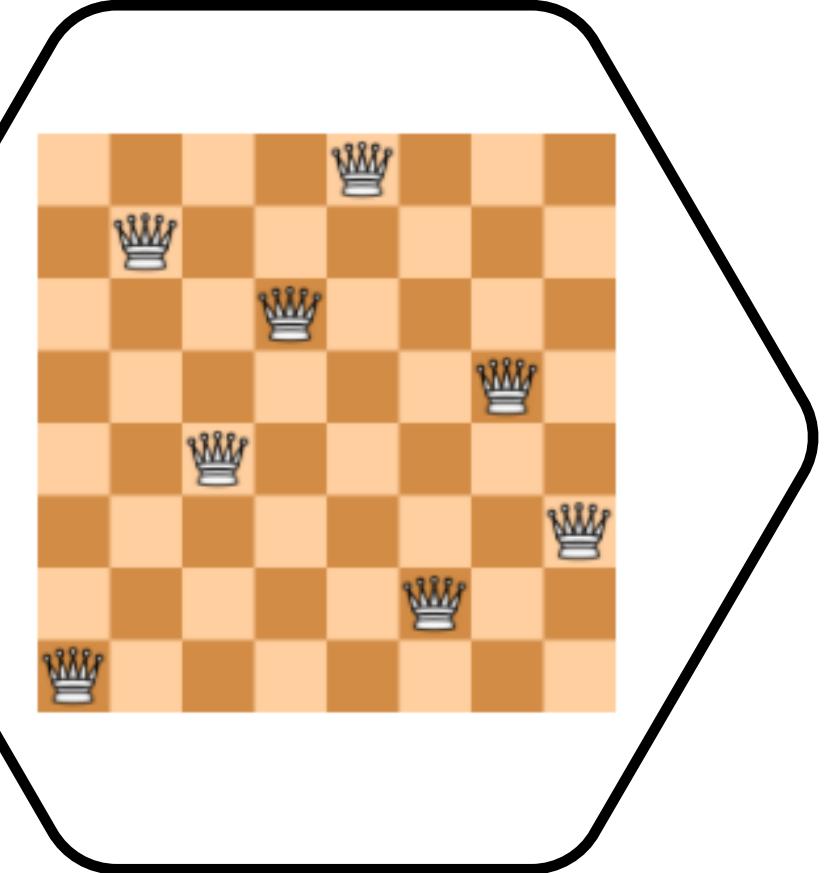
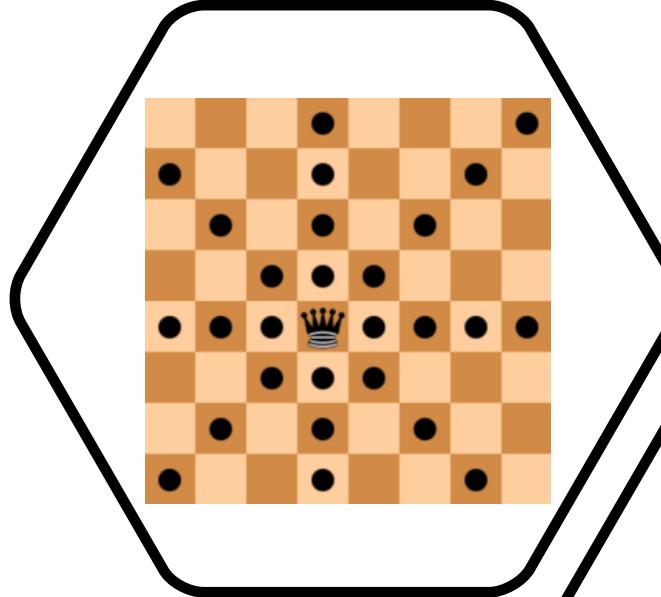
# 更難的數獨

- <https://www.spoj.com/problems/SUDOKU/>
- X 演算法
  - Dancing Links



## n皇后問題

- 在 $n \times n$ 的棋盤上，擺上 $n$ 個皇后
- 皇后能「吃掉」的範圍是米字形，如上圖
- 問你這些皇后有幾種擺法使得他們不會互相「吃掉」
- 右圖是8皇后的其中一組解



# 輸出 n 皇后的所有可能數

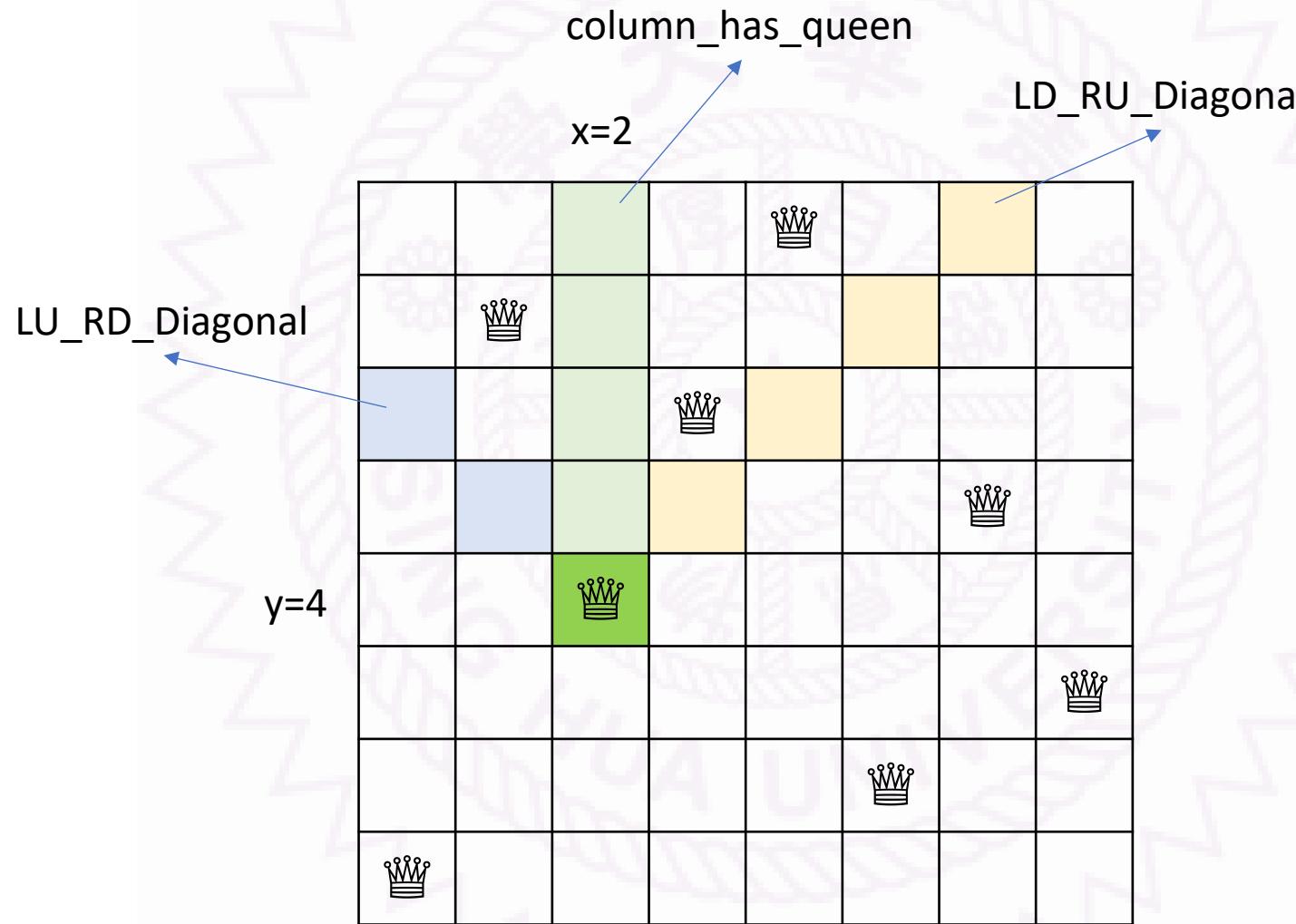
**Input**

8

**Output**

92

# 需要判斷的東西



# 需要判斷的東西

```
#include <iostream>
using namespace std;

const int MAXN = 20;

bool column_has_queen[MAXN];
bool LD_RU_Diagonal[MAXN * 2 - 1];
bool LU_RD_Diagonal[MAXN * 2 - 1];
```

# 對角線的表?

```
int n = 5;
auto LD_RU_Diagonal = [&](int y, int x)
{ return (y + x); };
auto LU_RD_Diagonal = [&](int y, int x)
{ return n - 1 + (y - x); };
void show_table(auto callback) {
    for (int y = 0; y < n; ++y) {
        for (int x = 0; x < n; ++x)
            cout << callback(y, x) << ' ';
        cout << '\n';
    }
    cout << '\n';
}
show_table(LD_RU_Diagonal);
show_table(LU_RD_Diagonal);
```

0	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8
4	3	2	1	0
5	4	3	2	1
6	5	4	3	2
7	6	5	4	3
8	7	6	5	4

# 判斷 + 更新資料

```
int n; // input

void update(int y, int x, bool val) {
    column_has_queen[x] = val;
    LD_RU_Diagonal[y + x] = val;
    LU_RD_Diagonal[n - 1 + (y - x)] = val;
}

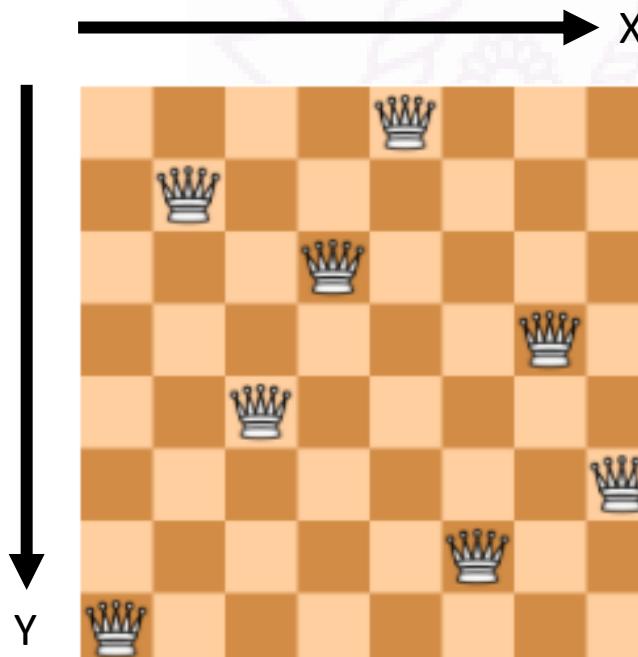
bool isValidQueenPosition(int y, int x) {
    if (column_has_queen[x])
        return false;
    if (LD_RU_Diagonal[y + x])
        return false;
    if (LU_RD_Diagonal[n - 1 + (y - x)])
        return false;
    return true;
}
```

# 遞迴找出所有答案

```
int ans;
void dfs(int y) {
    if (y == n) {
        ++ans;
        return;
    }
    for (int x = 0; x < n; ++x) {
        if (!isValidQueenPosition(y, x))
            continue;
        update(y, x, true);
        dfs(y + 1);
        update(y, x, false);
    }
}
```

```
int main() {
    cin >> n;
    dfs(0);
    cout << ans << endl;
    return 0;
}
```

# 解的表示法



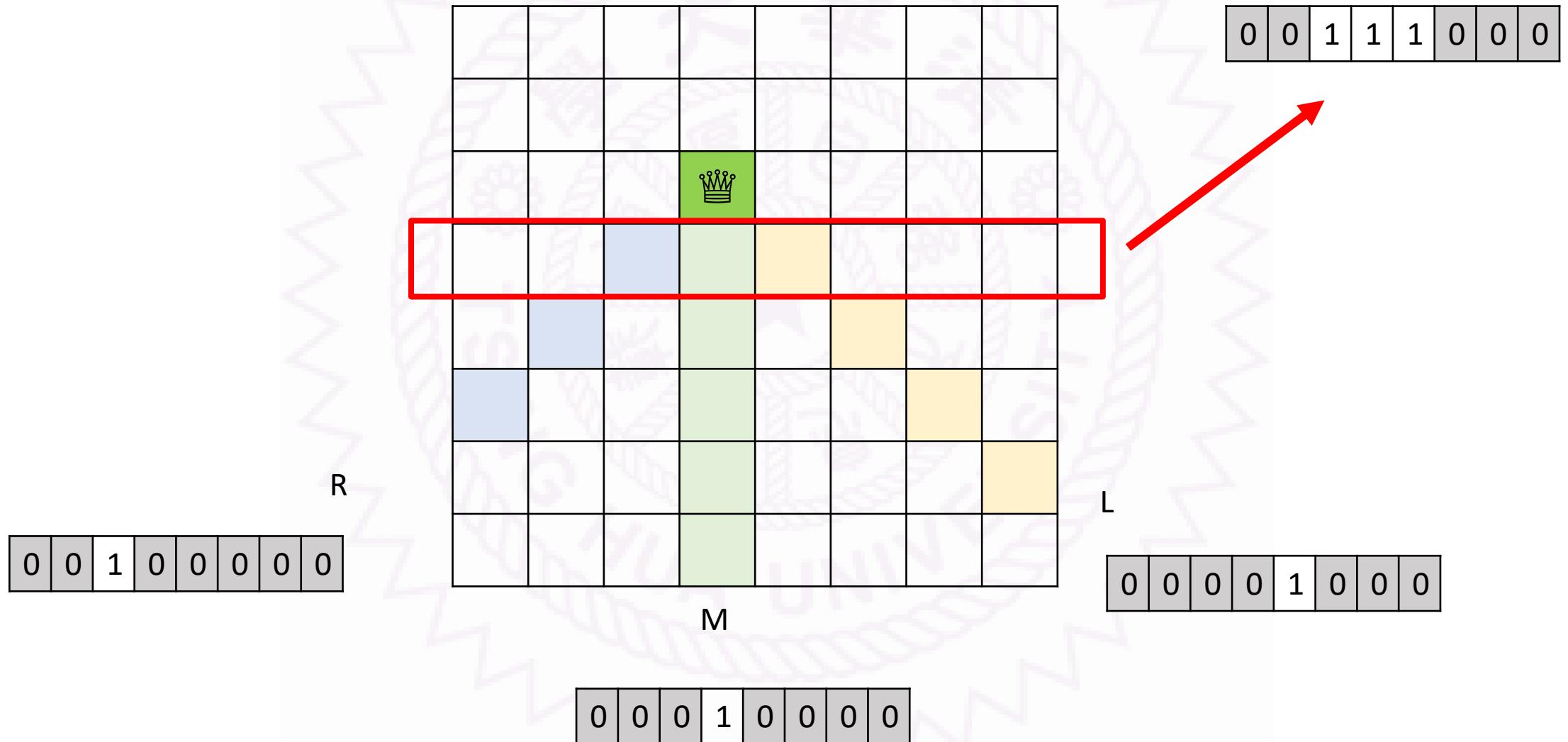
$n$  皇后的解會是某個  $0 \sim n - 1$  的全排列

$\text{row} = [4, 1, 3, 6, 2, 7, 5, 0]$

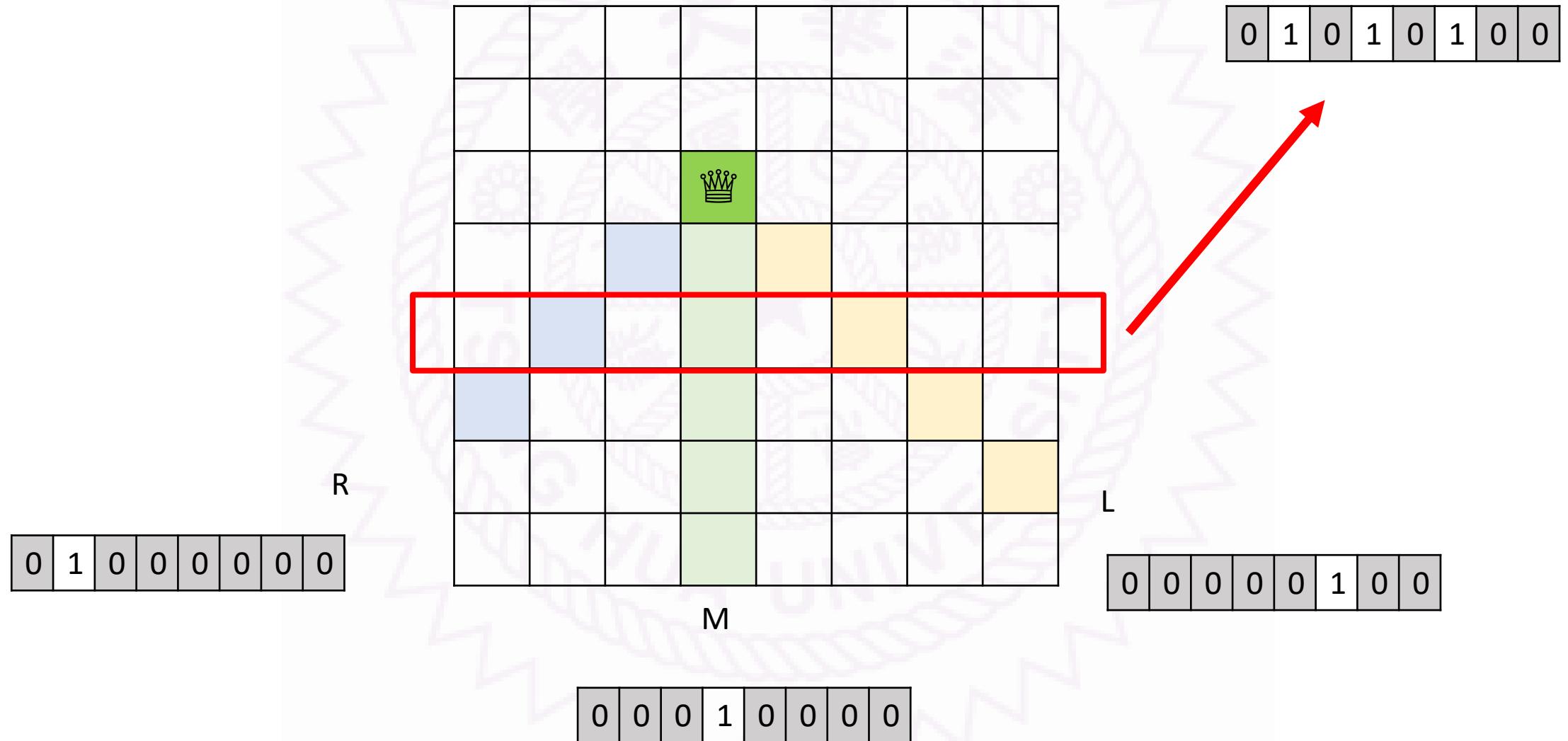
# 遞迴印出所有解

```
int row[MAXN], m;
void dfs(int y) {
    if (y == n) {
        print();
        return;
    }
    for (int x = 0; x < n; ++x) {
        if (!isValidQueenPosition(y, x))
            continue;
        update(y, x, true);
        row[m++] = x;
        dfs(y + 1);
        update(y, x, false);
        --m;
    }
}
```

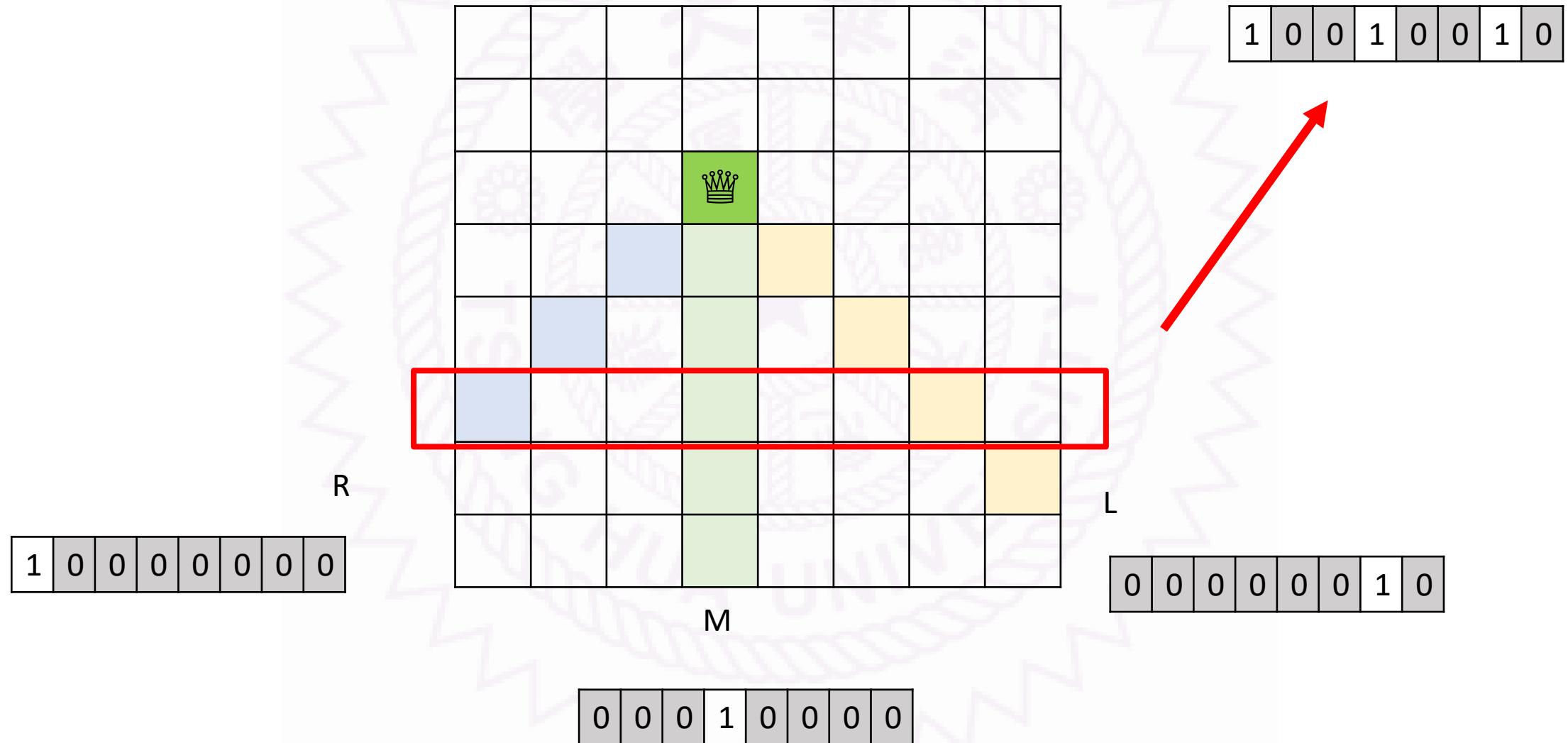
# 不能放的區域能不能用二進位表示?



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# 利用 shift 操作

```
#include <bitset>
#include <iostream>
using namespace std;

int main() {
    int L = 0, M = 0, R = 0;
    L = M = R = 1 << 12;
    for (int i = 0; i < 20; ++i) {
        cout << bitset<32>(L | M | R) << endl;
        L <<= 1;
        R >>= 1;
    }
    return 0;
}
```

# 位元運算加速

```
#include <iostream>
using namespace std;

int MASK;
int ans;
void dfs(int M, int L, int R);

int main() {
    int n = 0;
    cin >> n;
    MASK = (1 << n) - 1;
    dfs(0, 0, 0);
    cout << ans << endl;
    return 0;
}
```

# 位元運算加速

```
int lowbit(int x) { return x & -x; }
void dfs(int M, int L, int R) {
    if (M == MASK) {
        ++ans;
        return;
    }
    int Legal = MASK & ~(M | L | R);
    for (int num = 0; Legal; Legal ^= num) {
        num = lowbit(Legal);
        dfs(M | num, (L | num) << 1, (R | num) >> 1);
    }
}
```