CSC263 Week 6

Larry Zhang

http://goo.gl/forms/S9yie3597B

Announcements

PS4 marks out, class average 70.3%

This week

- ➔ QuickSort and analysis
- ➔ Randomized QuickSort
- ➔ Randomized algorithms in general

QuickSort

Background

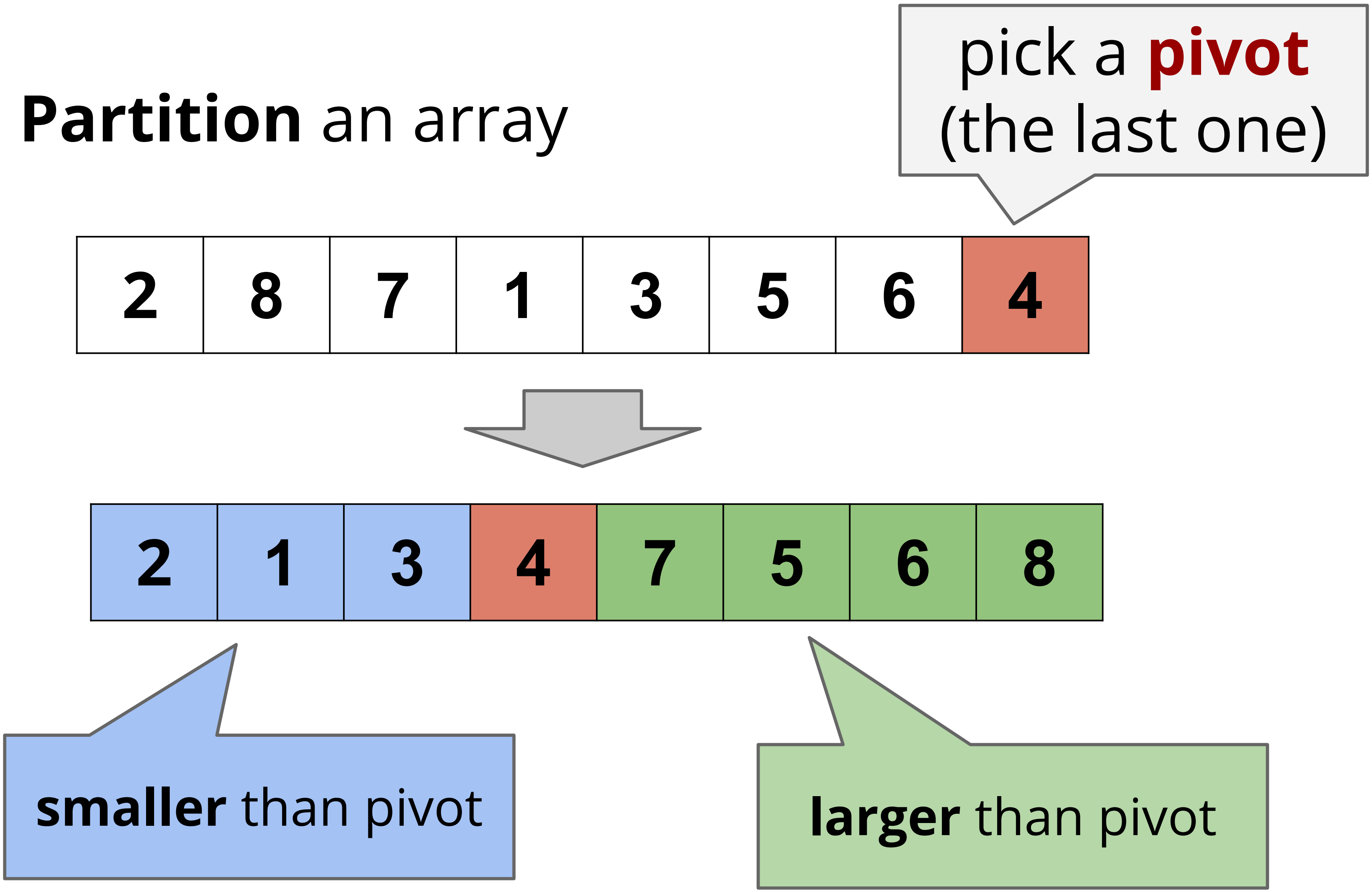
Invented by **Tony Hoare** in 1960

Very commonly used sorting algorithm. When **implemented well**, can be about 2-3 times faster than **merge sort** and **heapsort.**

Invented **NULL pointer** in 1965. Apologized for it in 2009

QuickSort: the idea

→ **Partition** an array



2	1	3	4	7	5	6	8

Recursively partition the sub-arrays **before** and **after** the pivot.

Base case:

1

sorted

Read textbook Chapter 7 for details of the Partition operation

Worst-case Analysis of QuickSort

T(n): the total number of **comparisons** made

For simplicity, assume all elements are distinct

A								
	2	1	3	4	7	5	6	8

Claim 1. Each element in **A** can be chosen as **pivot at most once**.

A pivot never goes into a sub-array on which a recursive call is made.

Claim 2. Elements are **only** compared to **pivots**.

That's what partition is all about -- comparing with pivot.

A	2	1	3	4	7	5	6	8

Claim 3. Any **pair** (a, b) in A, they are compared with each other **at most once**.

The only possible one happens when **a or b** is chosen as a **pivot** and the other is compared to it; after being the pivot, the pivot one will be out of the market and never compare with anyone anymore.

So, the total number of **comparisons** is **no more than** the **total number of pairs**.

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$$T(n) \leq \binom{n}{2} = \frac{n(n-1)}{2}$$

$$T(n) \in \mathcal{O}(n^2)$$

Next, show $T(n) \in \Omega(n^2)$

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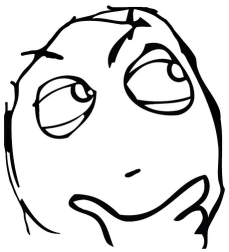
i.e., the **worst-case** running time is **lower-bounded** by some cn^2

How do you show the **tallest** person in the room is **lower-bounded** by **1 meter**?

Just find **one** person who is taller than 1m

so, just find **one input** for which the running time is some cn^2

so, just find **one input** for which the running time is some **cn^2**



i.e., find one input that results in **awful** partitions (everything on one side).

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**

Remember that we always pick the last one as pivot.

IRONY:

The worst input for QuickSort is an already sorted array.

Calculate the number of comparisons

1	2	3	4	5	6	7	8

Choose pivot **A[n]**, then **n-1** comparisons

Recurse to subarray, pivot **A[n-1]**, then **n-2** comps

Recursive to subarray, pivot **A[n-2]**, then **n-3** comps

...

Total # of comps:

$$(n-1)+(n-2)+\cdots+1=\frac{n(n-1)}{2}$$

So, the worst-case runtime

$$T(n) \geq \frac{n(n-1)}{2}$$

$$T(n) \in \Omega(n^2)$$

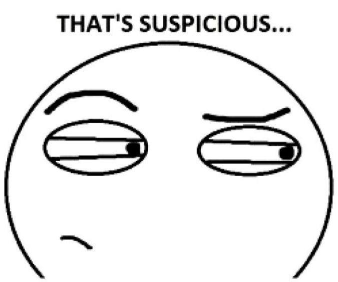
already shown $T(n) \in \mathcal{O}(n^2)$

so, $T(n) \in \Theta(n^2)$

$$T(n) \in \Theta(n^2)$$

What other sorting algorithms have **n^2** worst-case running time?

(The stupidest) Bubble Sort!

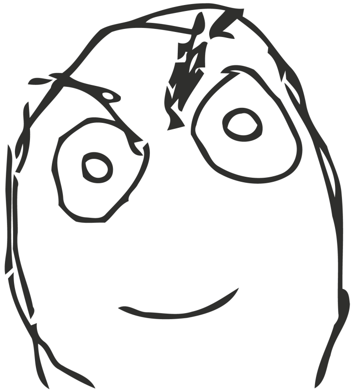


Is QuickSort really "quick" ?

Yes, in **average-case.**

Average-case Analysis of QuickSort

O(n log n)



Average over what?

Sample space and input distribution

All **permutations** of array **[1, 2, ..., n]**, and each permutation appears **equally likely**.

Not the only choice of sample space, but it is a representative one.

What to compute?

Let **X** be the random variable representing the **number of comparisons** performed on a sample array drawn from the sample space.

We want to compute **E[X]**.

An indicator random variable!

array is a permutation of [1, 2, ..., n]

$$X_{ij} = \begin{cases} 1 & \text{if the values } i \text{ and } j \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

So the total number of comparisons:

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$$

sum over all
possible pairs

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$$

$$E[X] = E\left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}\right]$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}]$$

because IRV

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \boxed{\Pr(i \text{ and } j \text{ are compared})}$$

Just need to figure this out!

$$\Pr(i \text{ and } j \text{ are compared})$$

Note: i < j

Think about the sorted sub-sequence

$$Z_{ij} : i, i+1, \ldots, j$$

A Clever Claim: ***i*** and ***j*** are compared **if and only if**, among all elements in Z_{ij}, the first element to be picked as a **pivot** is **either *i* or *j***.

$$Z_{ij} : i, i+1, \ldots, j$$

Claim: ***i*** and ***j*** are compared **if and only if**, among all elements in ***Zij***, the first element to be picked as a **pivot** is **either *i* or *j***.

Proof:

The "**only if**": suppose the first one picked as pivot as some k that is between i and j,...
then i and j will be separated into **different partitions** and will never meet each other.

The "**if**": if ***i*** is chosen as pivot (the **first one** among ***Zij***), then ***j*** will be compared to pivot ***i*** for sure, because nobody could have possibly separated them yet!

Similar argument for first choosing j

$$Z_{ij} : i, i+1, \ldots, j$$

Claim: ***i*** and ***j*** are compared **if and only if**, among all elements in ***Z_{ij}***, the first element to be picked as a **pivot** is **either *i* or *j***.

$$\Pr(i \text{ and } j \text{ are compared})$$

$$= \Pr(i \text{ or } j \text{ is the first among } Z_{ij} \text{ chosen as pivot})$$

$$= \frac{2}{j-i+1}$$

There are ***j-i+1*** numbers in $\boldsymbol{Z_{ij}}$, and each of them is **equally likely** to be chosen as the first pivot.

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$$

$$E[X] = E\left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}\right]$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}]$$

We have figured this out!

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \boxed{\Pr(i \text{ and } j \text{ are compared})}$$

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \Pr(i \text{ and } j \text{ are compared})$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j - i + 1}$$

Something close to

$$n \sum_{k=1}^{n} \frac{1}{k}$$

$$\in \mathcal{O}(n \log n)$$

Analysis Over!

Summary

The average-case runtime (**E[X]**) of QuickSort is **$O(n \log n)$**.

The worst-case runtime was **$\Theta(n^2)$**.

How do we make sure to get average-case and avoid worst-case?

We do Randomization.

CSC263 Week 6

Thursday

Announcement

- Next week: reading week
- Week after next week: Midterm
  - Feb 26 4-6pm, EX200 / EX300
  - 8.5”x11” aid-sheet, **handwritten** on **one side**
  - If have conflict, fill in this form by tomorrow http://www.cdf.toronto.edu/~csc263h/winter/tests.shtml
- Pre-test office hour
  - Feb 26, 11-1pm, 2-4pm, BA5287
  - Also go to Francois and Michelle’s office hours

Recap Tuesday

QuickSort Analysis

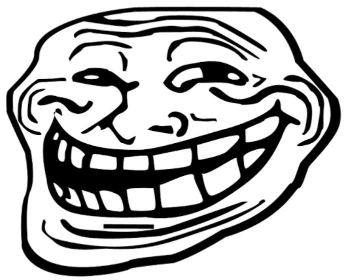
- ➔ Worst-case runtime $\Theta(n^2)$
  - ◆ worst input: already sorted array
- ➔ Average-case runtime $O(n \log n)$
  - ◆ Assume permutations of [1, 2, ..., n] chosen uniformly at random

However, in real life…

The assumption of uniform randomness is NOT really true, because it is often impossible for us to know what the input distribution really is.

QuickSort(A)

Ever worse, if the person who provides the inputs is **malicious**, they can totally only provide worst-inputs and guarantees worst-case runtime.



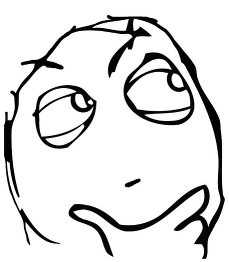
The theoretical O(nlog n) performance is no way guaranteed in real life.

How can we get some guaranteed performance in real life?

- ➔ We shuffle the input array "uniformly randomly", so that after shuffling the arrays look like drawn from a uniform distribution
- ➔ Even the **malicious** person's always-worst inputs will be shuffled to be like uniformly distributed
- ➔ This makes the **assumption** in the average-case analysis **true**
- ➔ So we are **guaranteed** the **O(n log n) expected runtime**

```
Randomize-QuickSort(A):
    permute A uniformly randomly
    QuickSort(A)
```

How exactly do we perform the permutation so that we can prove that it's going to be like uniform distribution? (Read Chapter 5.3)



Randomized Algorithms

Use randomization to guarantee expected performance

We do it everyday.

Two types of randomized algorithms

"**Las Vegas**" algorithm

→ Deterministic **answer**, random **runtime**

"**Monte Carlo**" algorithm

→ Deterministic **runtime**, random **answer**

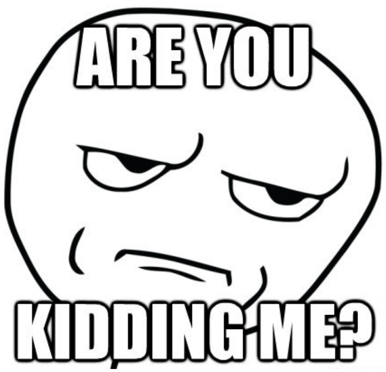
Randomized-QuickSort is a …
Las Vegas algorithm

An Example of
Monte Carlo Algorithm

"Equality Testing"

The problem

Given two binary numbers **x** and **y**, decide whether **x = y**.

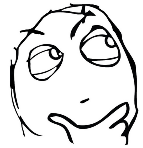



```
def equal(x, y):
    return x == y
```

No kidding, what if the **size** of **x** and **y** are **10TB** each?

The above code needs to compare ~10^{14} bits.

Can we do better?



Why assuming x and y are of the same length?

Let n = len(x) = len(y) be the length of x and y.

Randomly choose a **prime number** $p \le n^2$,
then **len(p)** $\le$ $\mathbf{\log_2(n^2) = 2\log_2(n)}$
then compare **(x mod p)** and **(y mod p)**
i.e., **return (x mod p) == (y mod p)**

Need to compare **at most 2log(n)** bits.

But, does it give the correct answer?

$\log_2(10^{14}) \approx$ **46.5**

Huge improvement on runtime!

Does it give the correct answer?

If **$(x \bmod p) \neq (y \bmod p)$**, then...

Must be **$x \neq y$**, our answer is correct **for sure**.

If **$(x \bmod p) = (y \bmod p)$**, then...

Could be **$x = y$ or $x \neq y$**, so our answer **might be** correct.

Correct with what probability?

What’s the probability of a wrong answer?

Prime number theorem

In range **[1, m]**, there are roughly **m/ln(m)** prime numbers.

So in range **$[1, n^2]$**, there are $n^2/\ln(n^2)$ = **$n^2/2\ln(n)$** prime numbers.

How many (**bad**) primes in **$[1, n^2]$** satisfy

(x mod p) = (y mod p) even if **$x \neq y$** ?

At most n

(x mod p) = (y mod p) ⇔ |x - y| is a multiple of p, i.e., p is a divisor of |x - y|.
$|x - y| < 2^n$ (n-bit binary #) so it has no more than n prime divisors (otherwise it will be larger than 2^n).

So...

Out of the **$n^2/2\ln(n)$** prime numbers we choose from, at most **n** of them are **bad**.

If we choose a **good** prime, the algorithm gives correct answer for sure.
If we choose a **bad** prime, the algorithm may give a wrong answer.
So the prob of wrong answer is less than

$$\frac{n}{n^2/(2\ln n)} = \frac{2\ln n}{n}$$

Error probability of our Monte Carlo algorithm

$$\Pr(\text{error}) \leq \frac{2\ln n}{n}$$

When $n = 10^{14}$ (10TB)

$\Pr(\text{error}) \leq 0.000000000000644$

Performance comparison (n = 10TB)

The **regular** algorithm **x == y**

➔ Perform 10^{14} comparisons

➔ Error probability: 0

The **Monte Carlo** algorithm **(x mod p) == (y mod p)**

➔ Perform < 100 comparisons

➔ Error probability: 0.000000000000644

If your boss says: “This error probability is too high!”

Run it **twice**: Perform < 200 comparisons

➔ Error prob squared: 0.000000000000000000000000215

Summary

Randomized algorithms

- ➔ Guarantees expected performance
- ➔ Make algorithm less vulnerable to malicious inputs

Monte Carlo algorithms

- ➔ Gain time efficiency by sacrificing some correctness.

Tutorial tomorrow

A mock-up midterm test!

Weekly feedback form

Let us know about your experience with A1 (what’s good / bad), so A2 can be made more likeable!

http://goo.gl/forms/S9yie3597B