CSC263 Week 10

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http://goo.gl/forms/S9yie3597B

Announcement

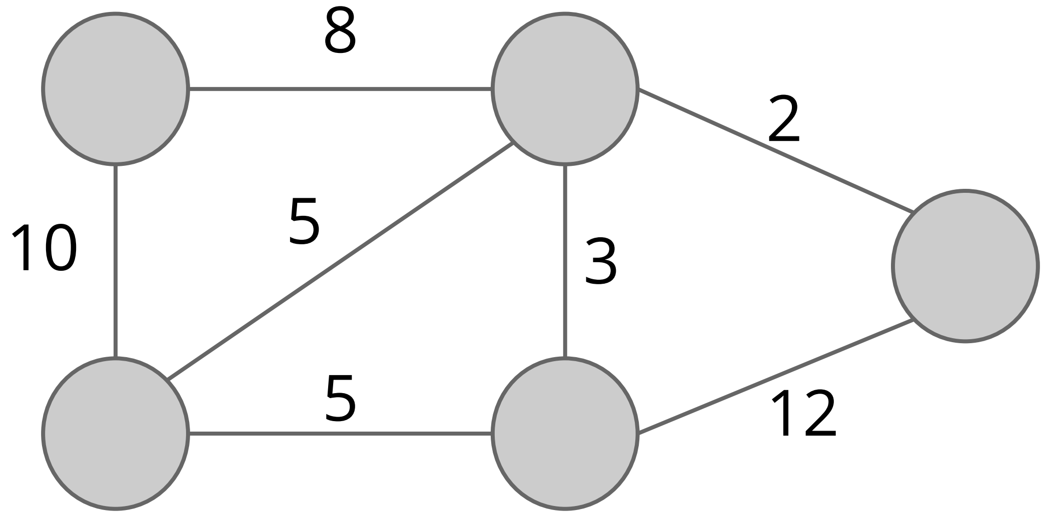
PS8 out soon, due next Tuesday

Minimum Spanning Tree

The Graph of interest today

A **connected undirected** **weighted** graph

G = (V, E) with weights **w(e)** for each $e \in E$



It has the **smallest** total weight

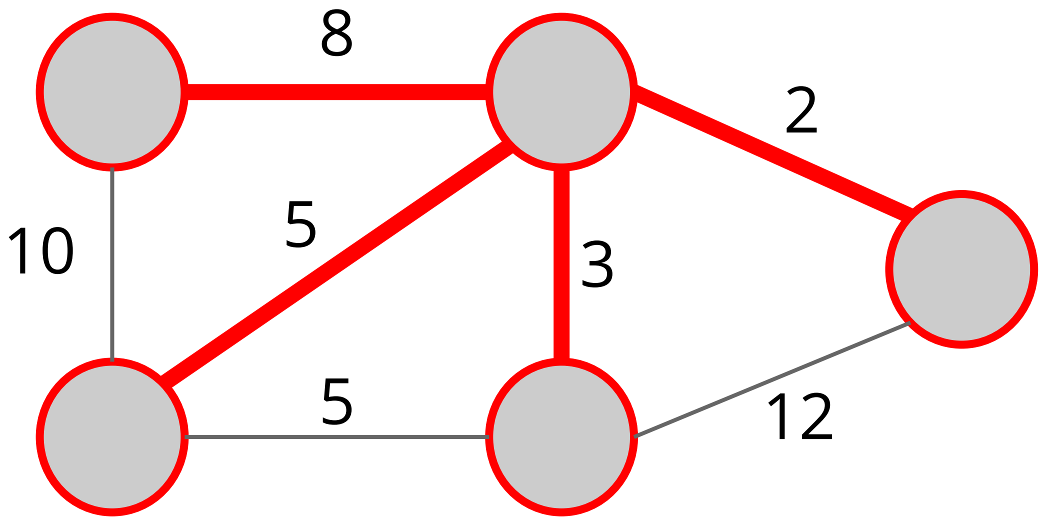
It covers **all** vertices in G

Minimum Spanning Tree

of graph G

It’s a **connected**, **acyclic** subgraph

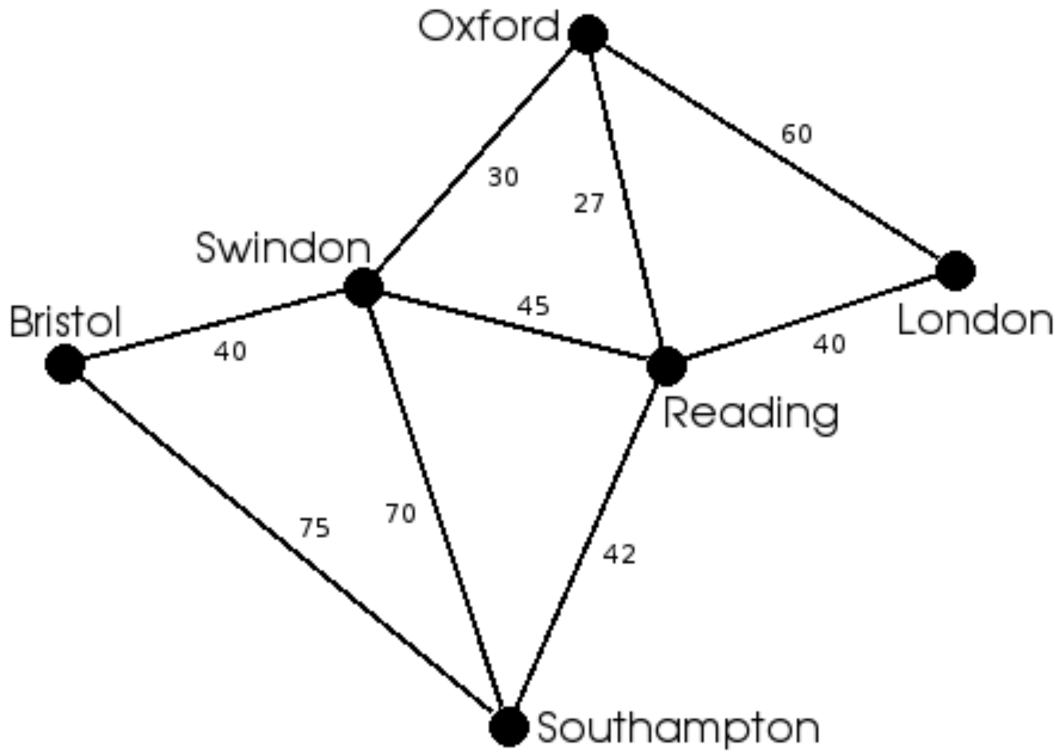
A Minimum Spanning Tree



May NOT be unique

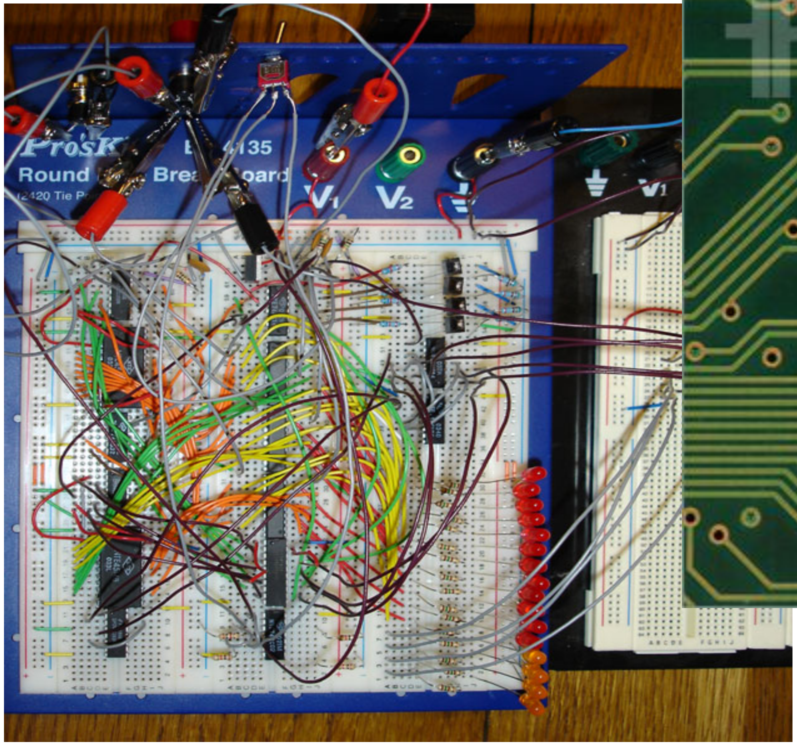
Applications of MST

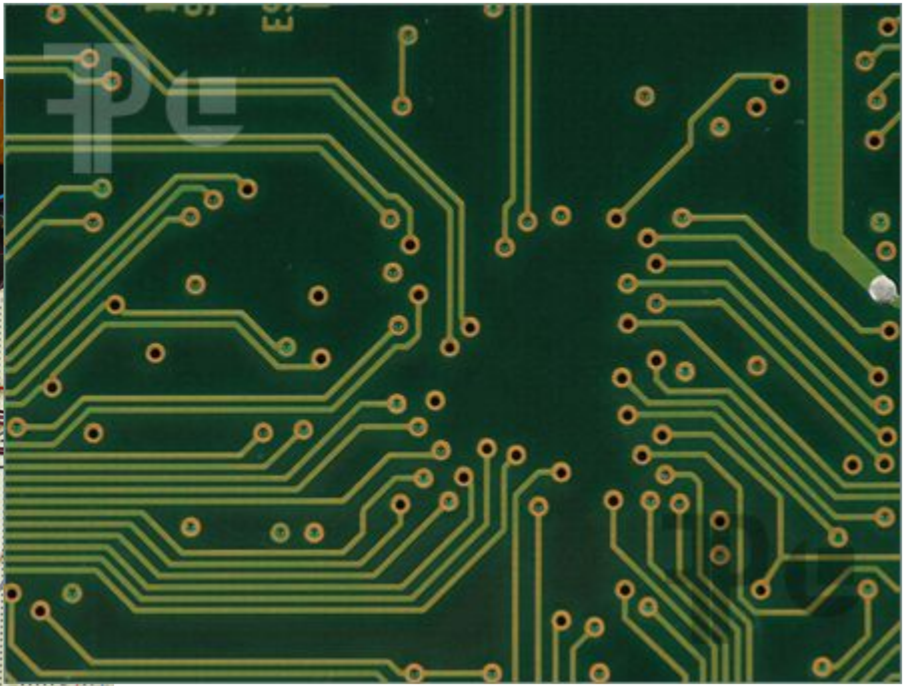
Build a road network that connects all towns and with the minimum cost.



Applications of MST

Connect all components with the least amount of wiring.



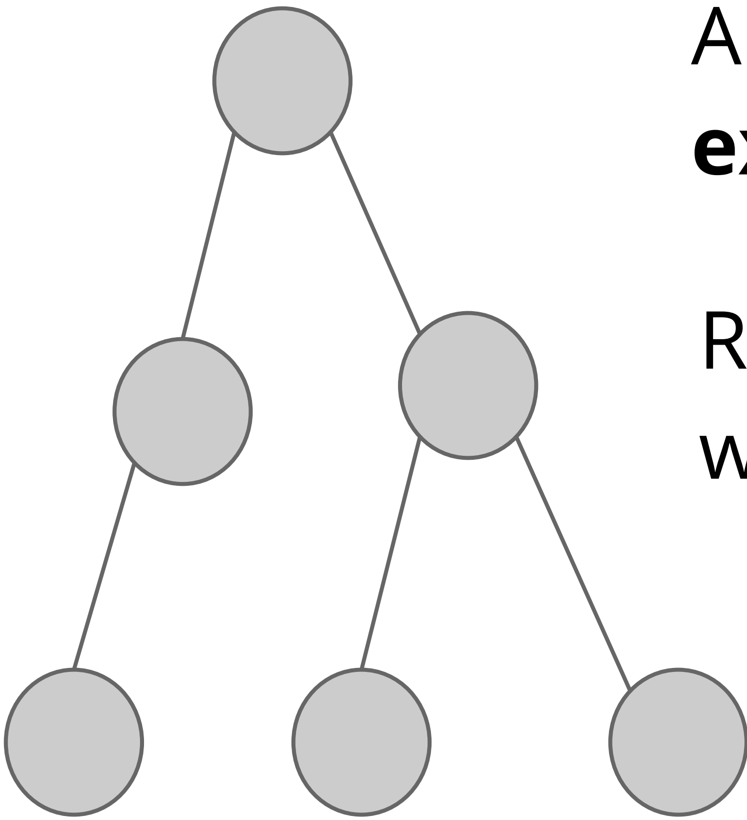


Other applications

- ➔ Cluster analysis
- ➔ Approximation algorithms for the "travelling salesman problem"
- ➔ ...

In order to understand
minimum spanning tree
we need to first understand
tree

Tree:
undirected connected acyclic graph



A tree **T** with **n** vertices has **exactly** ___**n-1**___ edges.

Removing one edge from T will ___**disconnect the tree**___.

Adding one edge to T will ___**create a cycle**___.

The MST of a connected graph G = (V, E) has **|V|**__________________ vertices.

because “spanning”

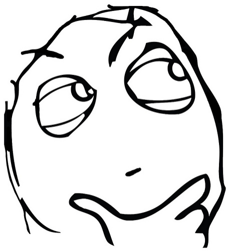
The MST of a connected graph G = (V, E) has **|V| - 1**__________________ edges.

because “tree”

Now we are ready to talk about algorithms

Idea #1

Start with **T = G.E**, then keep deleting edges until an MST remains.



Which sounds more efficient in terms of worst-case runtime?

Idea #2

Start with **empty** T, then keep adding edges until an MST is built.

Hint

A undirected simple graph G with **n** vertices can have at most ____________ edges.

$$\binom{n}{2} = \frac{n(n-1)}{2} \in \mathcal{O}(n^2)$$

Note: Here T is an edge set

Idea #1

Start with T = G.E, then keep deleting edges until an MST remains.

In worst-case, need to delete $O(|V|^2)$ edges *(n choose 2) - (n-1)*

Idea #2

In worst-case, need to add $O(|V|)$ edges

Start with **empty** T, then keep adding edges until an MST is built.

This is more efficient!

So, let's explore more of **Idea #2**, i.e.,
building an MST by **adding** edges one by one

i.e.,
we "**grow**" a tree

The generic growing algorithm

```
GENERIC-MST(G=(V, E, w)):
    T ← ∅
    while T is not a spanning tree:
        find a "safe" edge e
        T ← T ∪ {e}
    return T
```

|T| < |V|-1

What is a "safe" edge?

“Safe” means it keeps the **hope** of T growing into an MST.

“Safe” edge e for T

*Assuming **before** adding e, **T ⊆ some MST**,*
*edge **e** is safe if **after** adding **e**, still **T ⊆ some MST***

If we make sure T is always a subset of some MST while we grow it, then eventually T will become an MST!

```
GENERIC-MST(G=(V, E, w)):
    T ← ∅
    while T is not a spanning tree:
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    return T
```

Intuition

If we make sure the pieces we put together is always a subset of the real picture while we grow it, then eventually it will become the real picture!

The generic growing algorithm

```
GENERIC-MST(G=(V, E, w)):
    T ← ∅
    while T is not a spanning tree:
        find a "safe" edge e
        T ← T ∪ {e}
    return T
```

|T| < |V|-1

How to find a "safe" edge?

Two major algorithms we'll learn

➔ Kruskal's algorithm

➔ Prim's algorithm

They are both based on one theorem...

Note: Here T includes both vertices and edges

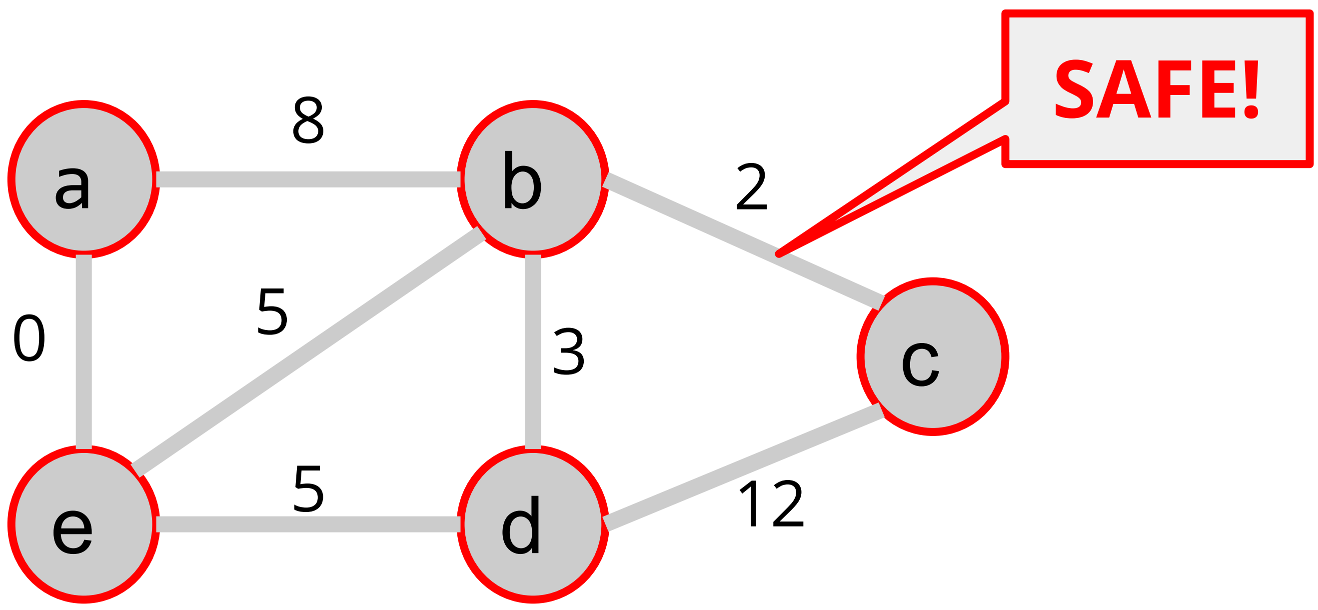
The Theorem

Let **G** be a connected undirected weighted graph, and **T** be a **subgraph** of G which is a **subset** of some MST of **G**.

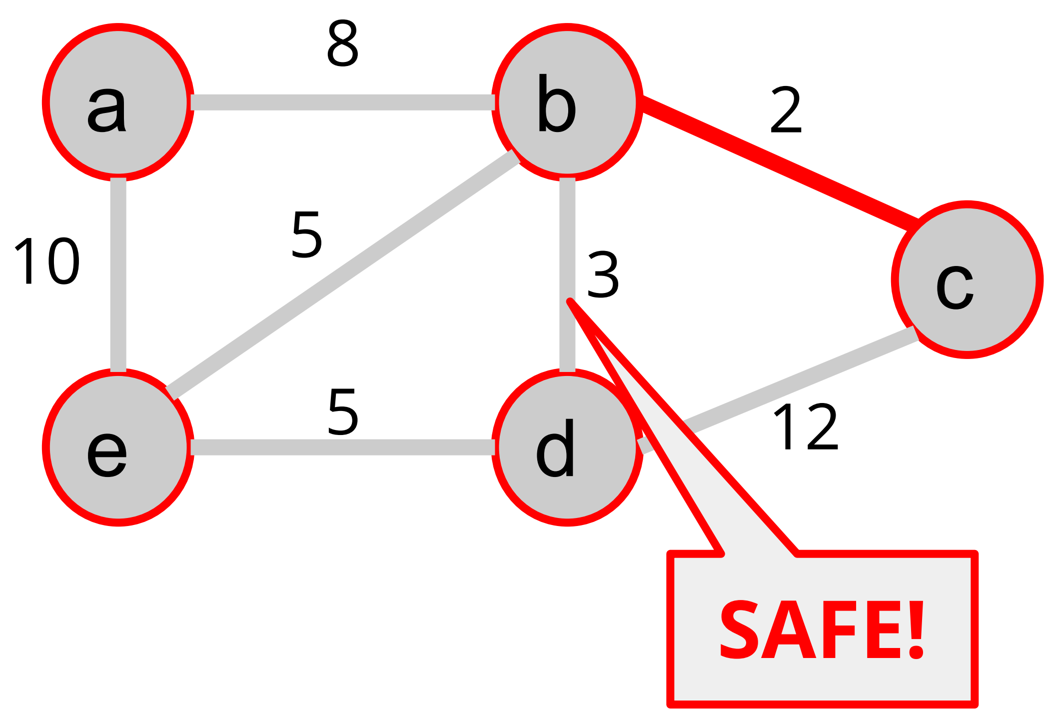
Let edge **e** be the **minimum** weighted edge among all edges that **cross** different **connected components** of **T**.

Then **e** is **safe** for **T**.

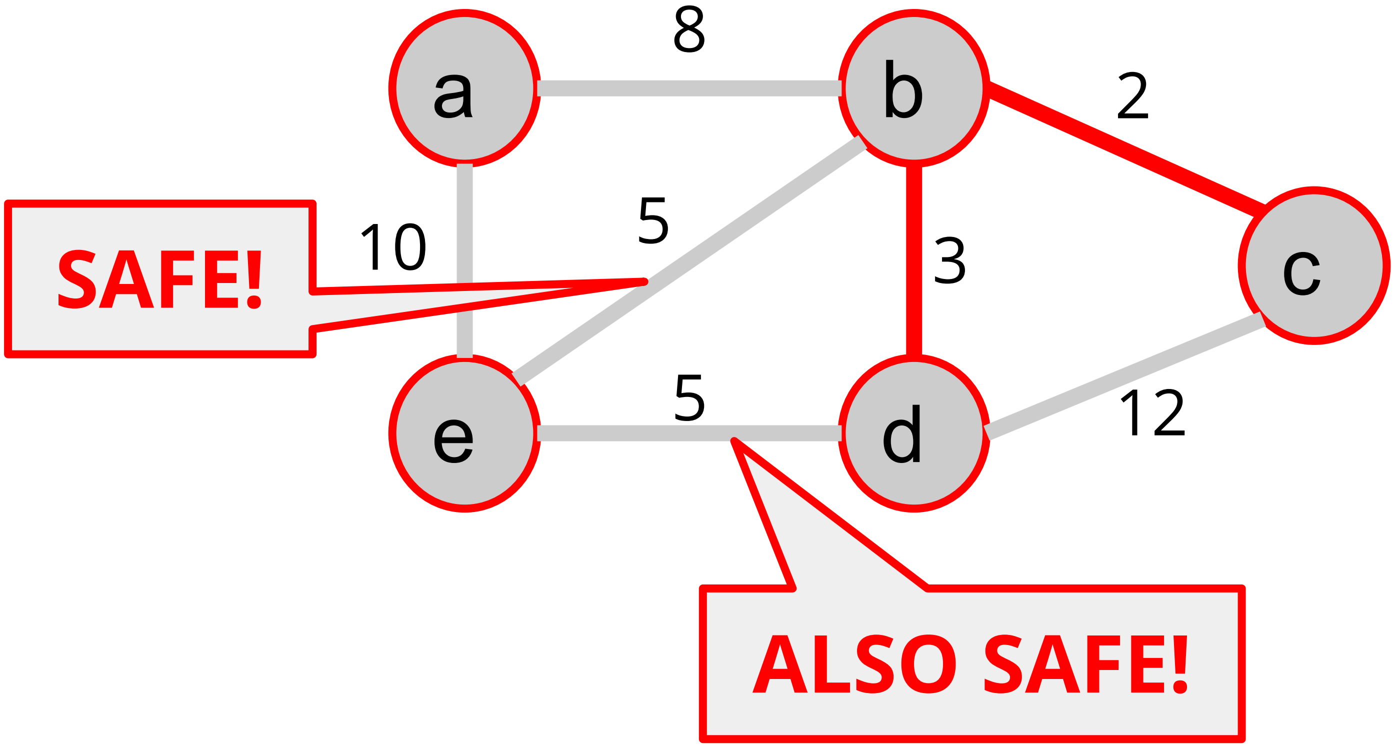
Initially, **T** (red) is a subgraph with no edge,
each vertex is a connected component,
all edges are **crossing** components,
and the minimum weighted one is ...



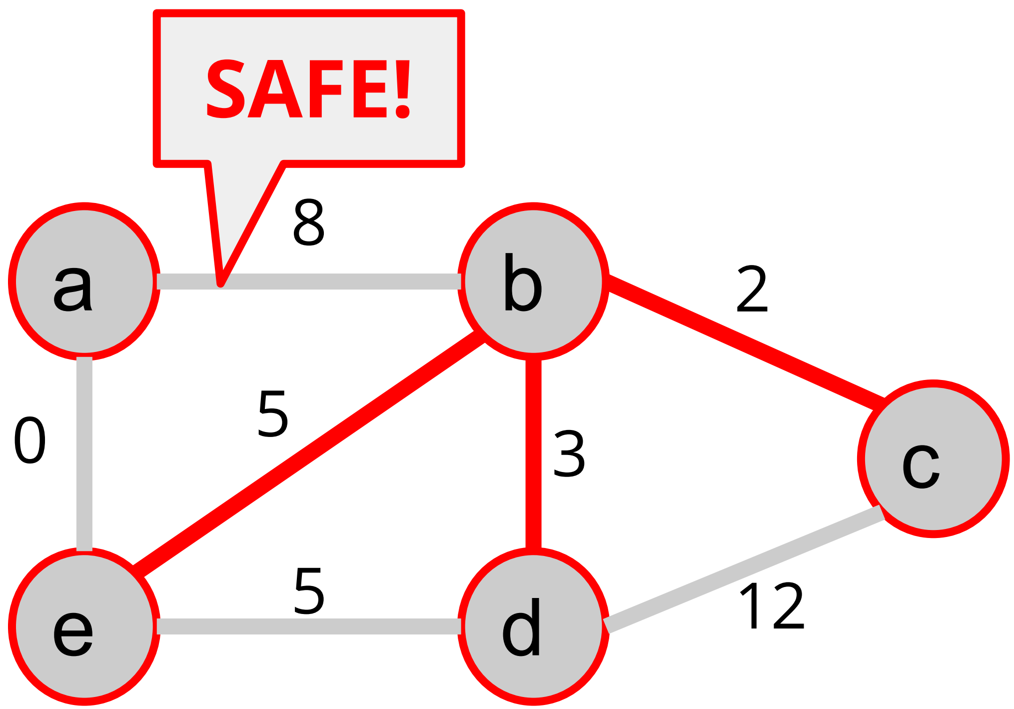
Now **b and c** in one connected component, each of the other vertices is a component, i.e., 4 components.
All gray edges are crossing components.



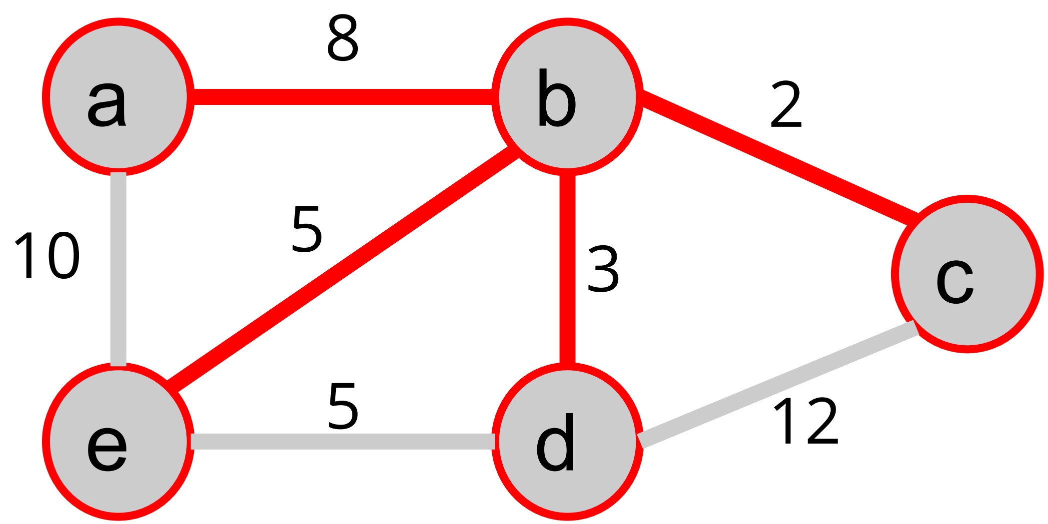
Now **b, c and d** are in one connected component, **a** and **e** each is a component. **(c, d)** is **NOT** crossing components!



Now **b, c, d and e** are in one connected component, **a** is a component.
(a, e) and **(a, b)** are crossing components.



MST grown!



Two things that need to be worried about when actually implementing the algorithm

- ➔ How to keep track of the **connected components**?
- ➔ How to efficiently find the **minimum** weighted edge?

Kruskal's and **Prim's** basically use different **data structures** to do these two things.

to be continued...

CSC263 Week 10

Thursday

Recap:
Generic MST growing algorithm

```
GENERIC-MST(G=(V, E, w)):
    T ← ∅
    while T is not a spanning tree:
        find a “safe” edge e
        T ← T ∪ {e}
    return T
```

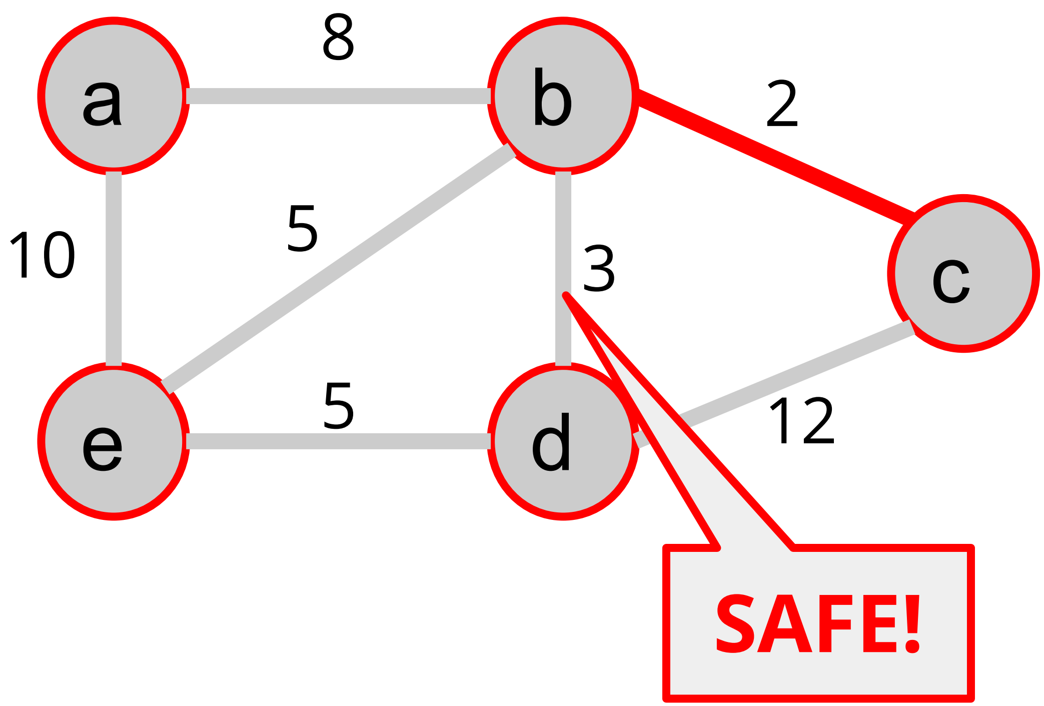
Recap: Finding safe edge

Let **G** be a connected undirected weighted graph, and **T** be a **subgraph** of G which is a **subset** of some MST of **G**.

Let edge **e** be the **minimum** weighted edge among all edges that **cross** different **connected components** of **T**.

Then **e** is **safe** for **T**.

Recap



Two things that need to be worried about when actually implementing the algorithm

- ➔ How to keep track of the **connected components**?
- ➔ How to efficiently find the **minimum** weighted edge?

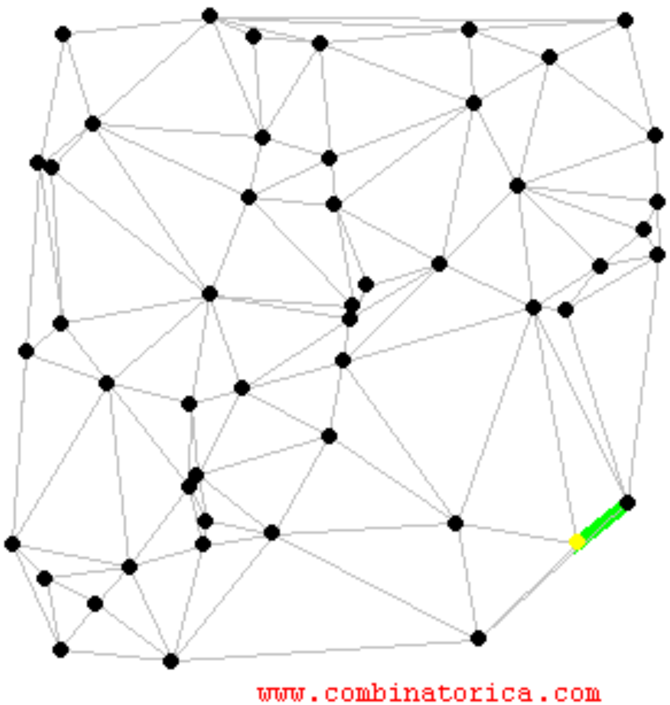
Kruskal's and **Prim's** basically use different **data structures** to do these two things.

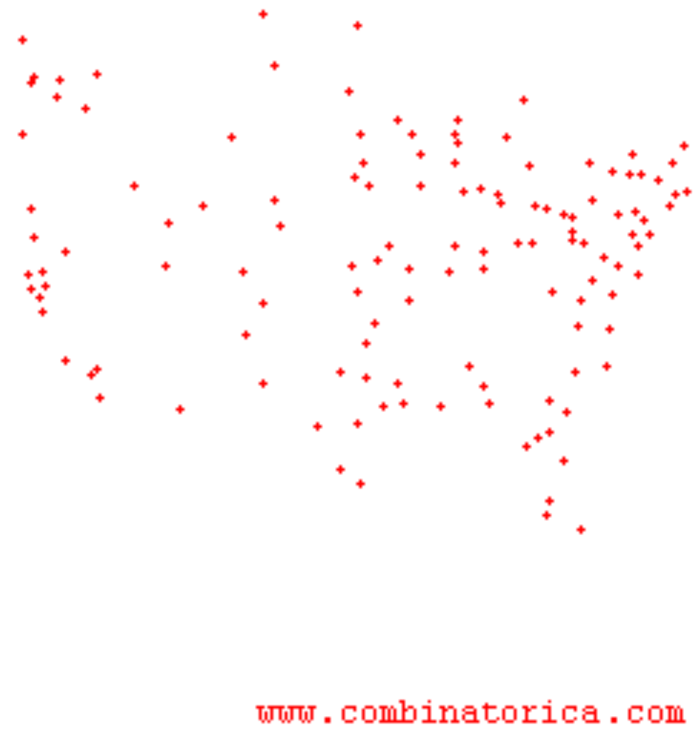
Overview: Prim’s and Kruskal’s

	Keep track of connected components	**Find minimum weight edge**
Prim’s	*Keep “one tree plus isolated vertices”*	*use priority queue ADT*
Kruskal’s	*use “disjoint set” ADT*	*Sort all edges according to weight*

Prim's

Kruskal's





https://trendsofcode.files.wordpress.com/2014/09/dijkstra.gif

https://www.projectrhea.org/rhea/images/4/4b/Kruskal_Old_Kiwi.gif

Prim’s MST algorithm

Prim’s algorithm: Idea

- Start from an arbitrary vertex as root
- Focus on growing **one** tree, add one edge at a time. The tree is one **component**, each of the other (**isolated**) vertices is a **component**.
- Add which edge? Among all edges that are **incident to the current tree** (**edges crossing components**), pick one with the **minimum** weight.
- How to get that minimum? Store all candidate vertices in a **Min-Priority Queue** whose key is the weight of the **crossing** edge (incident to tree).

```
PRIM-MST(G=(V, E, w)):
    T ← {}
    for all v in V:
        key[v] ← ∞
        pi[v] ← NIL
    Initialize priority queue Q with all v in V
    pick arbitrary vertex r as root
    key[r] ← 0
    while Q is not empty:
        u ← EXTRACT-MIN(Q)
        if pi[u] != NIL:
            T ← T ∪ {(pi[u], u)}
        for each neighbour v of u:
            if v in Q and w(u, v) < key[v]:
                DECREASE-KEY(Q, v, w(u, v))
                pi[v] ← u
```

key[v] keeps the “shortest distance” between v and the current tree

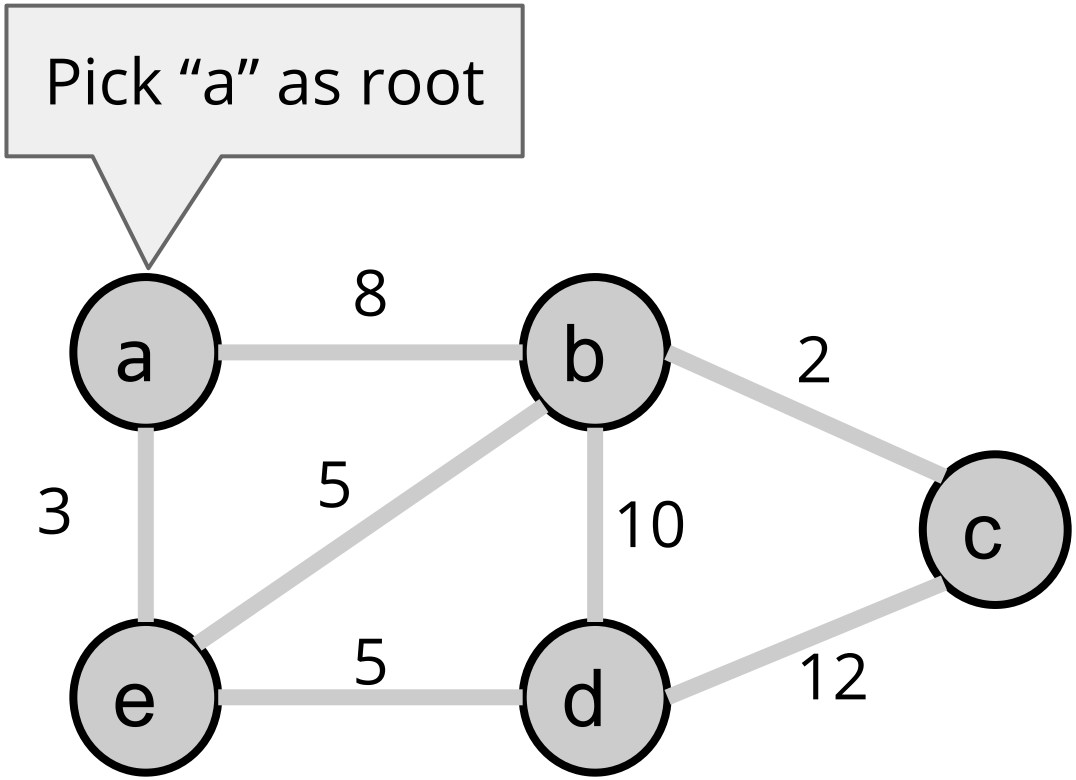
pi[v] keeps who, in the tree, is v connected to via lightest edge.

u is the next vertex to add to current tree

add edge, pi[u] is lightest vertex to connect to, “safe”

all u’s neighbours’ distances to the current tree need update

Trace an example!

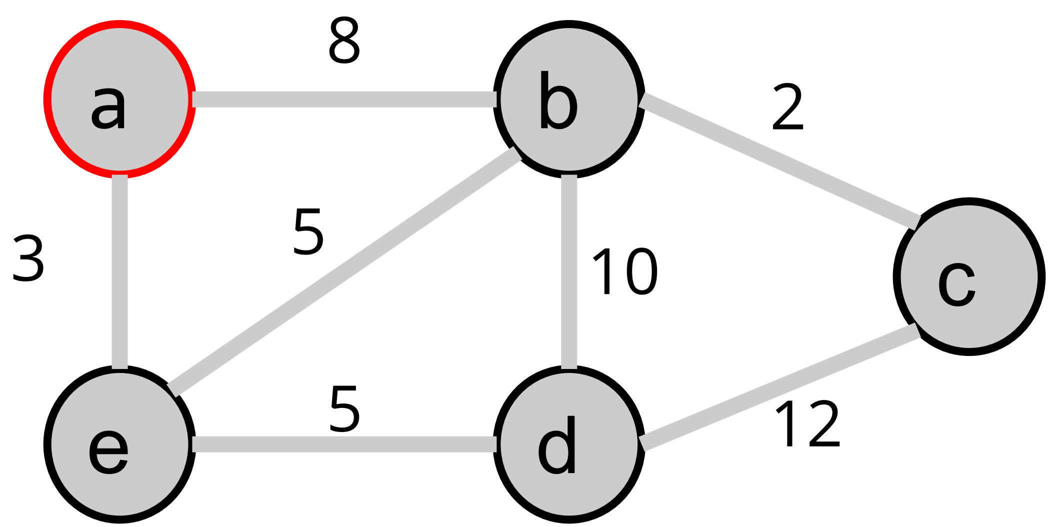


Q	key	pi
a	0	NIL
b	∞	NIL
c	∞	NIL
d	∞	NIL
e	∞	NIL

Next, ExtractMin !

ExtractMin (#1) then update neighbours' keys

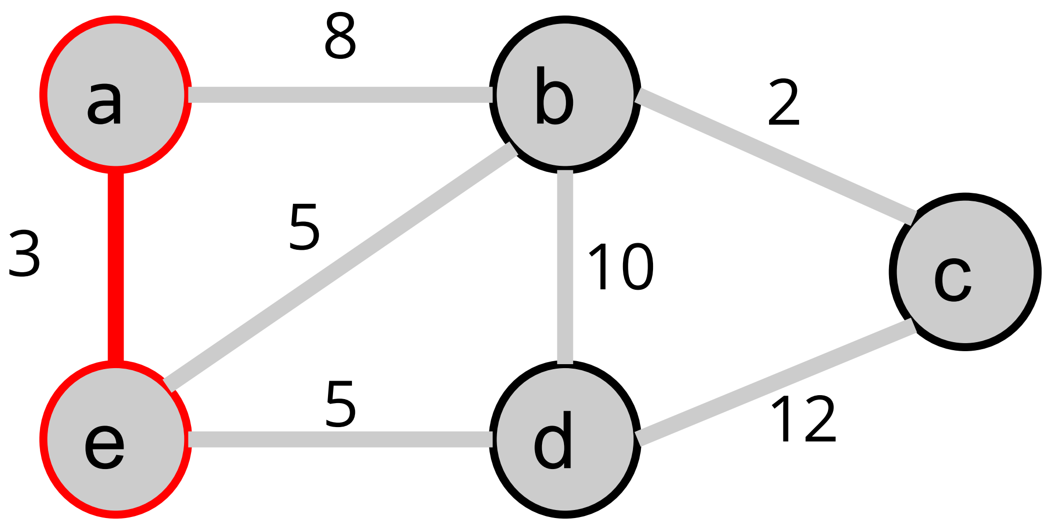
a: 0, NIL



Q	key	pi
b	∞ → **8**	NIL → **a**
c	∞	NIL
d	∞	NIL
e	∞ → **3**	NIL → **a**

ExtractMin (#2) then update neighbours' keys

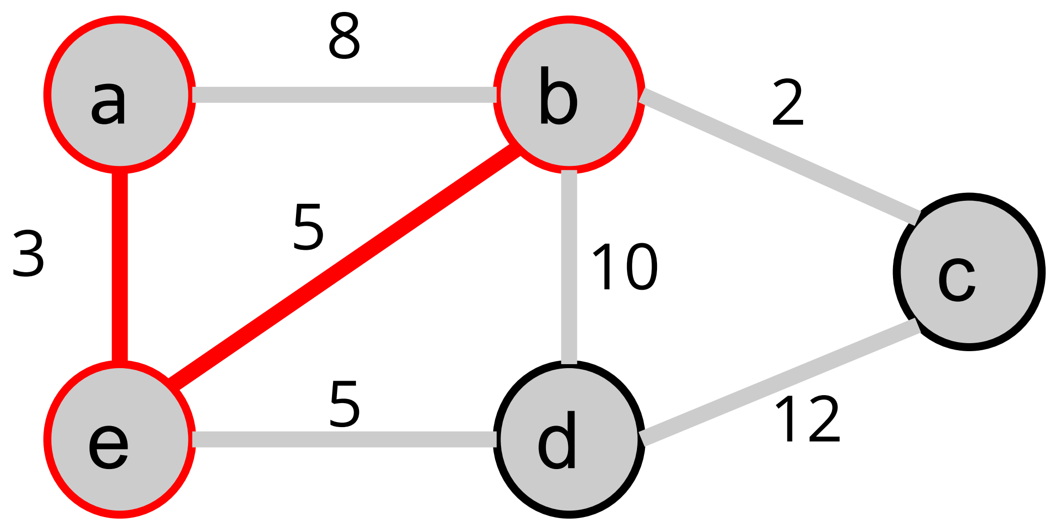
e: 3, a



Q	key	pi
b	8→**5**	a →**e**
c	∞	NIL
d	∞ →**5**	NIL →**e**

ExtractMin (#3) then update neighbours’ keys

b: 5, e

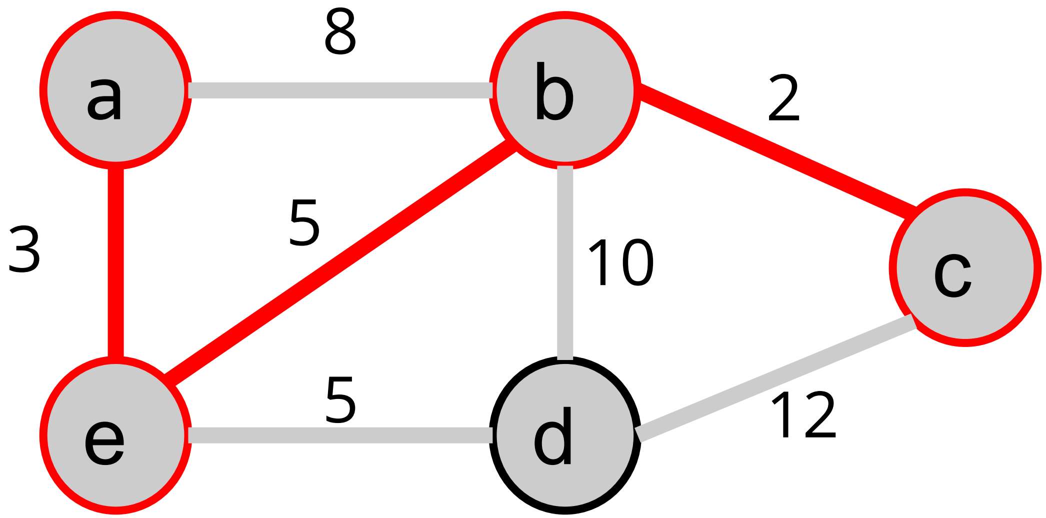


Q	key	pi
c	$\infty \rightarrow$ **2**	NIL $\rightarrow$ **b**
d	5	e

Could also have extracted d since its key is also 5 (min)

ExtractMin (#4) then update neighbours' keys

c: 2, b



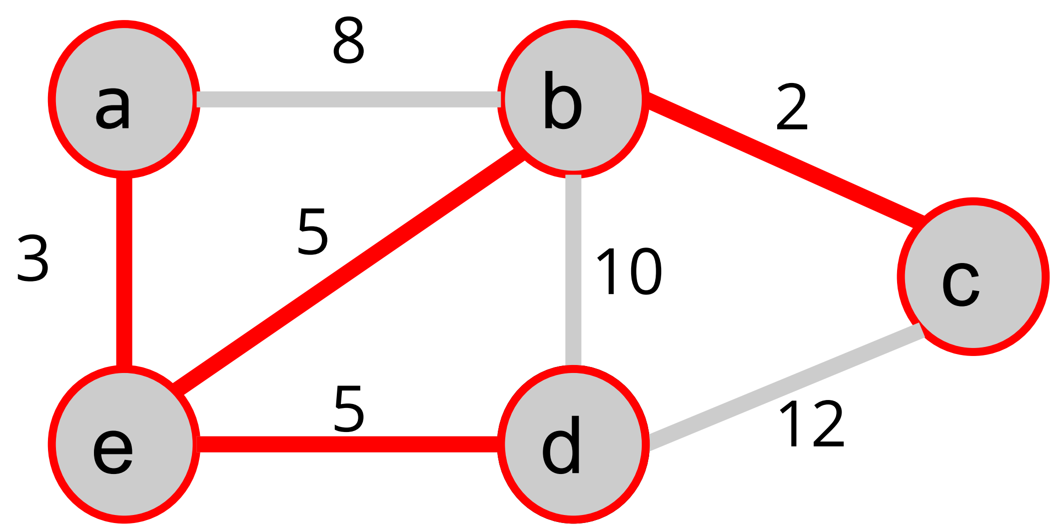
Q	key	pi
d	5	e

ExtractMin (#4) then update neighbours' keys

d: 5, e

Q	key	pi

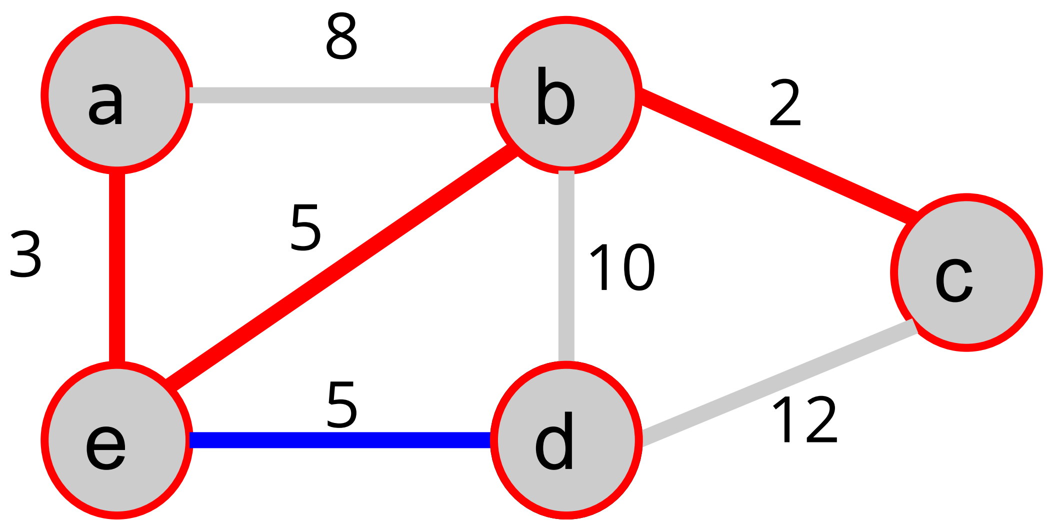
Q is empty now.



MST grown!

Correctness of Prim's

The added edge is always a "**safe**" edge, i.e., the **minimum** weight edge crossing different components (because **ExtractMin**).



Runtime analysis: Prim's

- ➔ Assume we use **binary min heap** to implement the priority queue.
- ➔ Each ExtractMin take **O(log V)**
- ➔ In total **V** ExtractMin's
- ➔ In total, check at most **O(E)** neighbours, each check neighbour could lead to a **DecreaseKey** which takes **O(log V)**

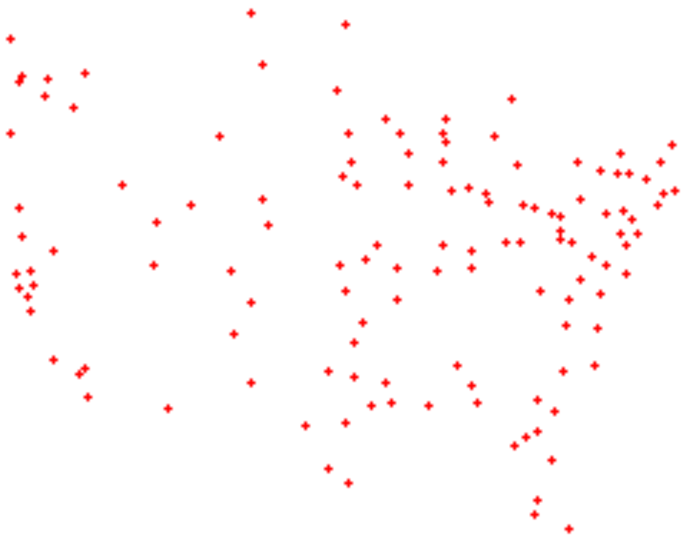
➔ **TOTAL: O((V+E)log V) = O(E log V)**

In a connected graph $G = (V, E)$

$|V|$ is in $\mathbf{O(|E|)}$ because...
$|E|$ has to be at least $\mathbf{|V|-1}$

Also, $\mathbf{log\ |E|}$ is in $\mathbf{O(log\ |V|)}$ because ...
E is at most V^2,
so $\mathbf{log\ E}$ is at most $\mathbf{log\ V^2 = 2\ log\ V}$, which is in $\mathbf{O(log\ V)}$

Kruskal’s MST algorithm



Kruskal's algorithm: idea

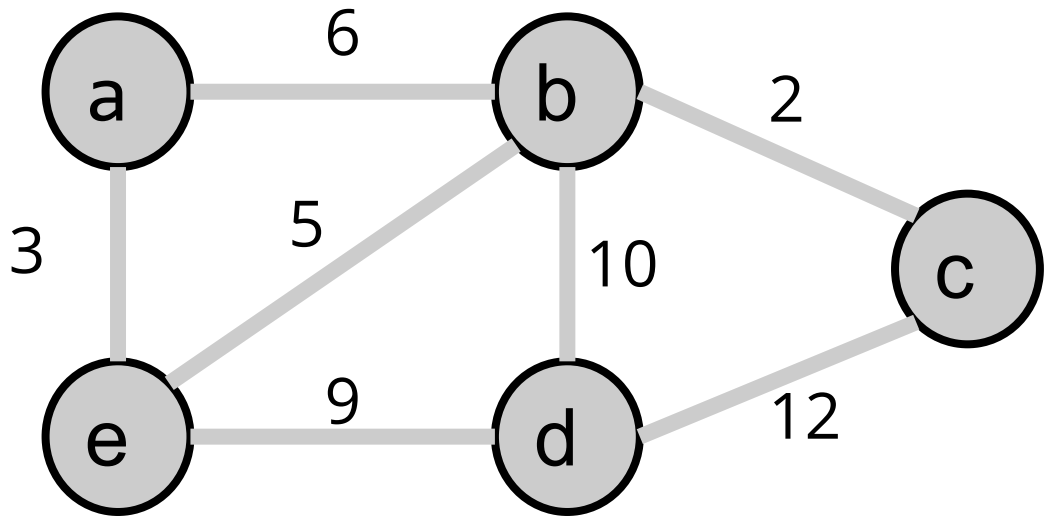
- **Sort** all edges according to **weight**, then start adding to MST from the **lightest** one.
  - **This is "greedy"!**
- Constraint: added edge must NOT cause a **cycle**
  - In other words, the two endpoints of the edge must belong to two **different** trees (components).
- The whole process is like unioning small trees into a big tree.

Pseudocode

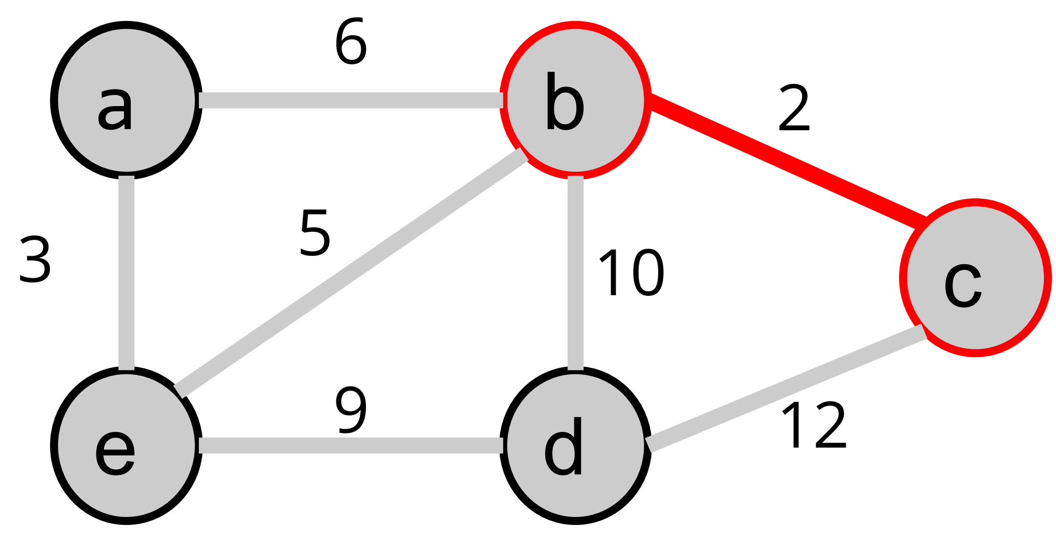
m = |E|

```
KRUSKAL-MST(G(V, E, w)):
    T ← {}
    sort edges so that w(e1)≤w(e2)≤...≤w(em)
    for i ← 1 to m:
        # let (ui, vi) = ei
        if ui and vi in different components:
            T ← T ∪ {ei}
```

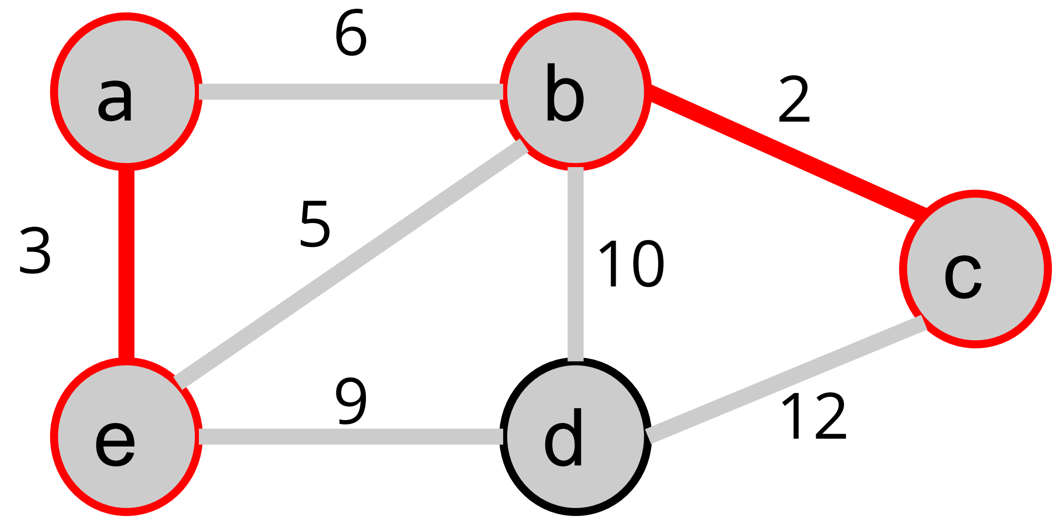
Example



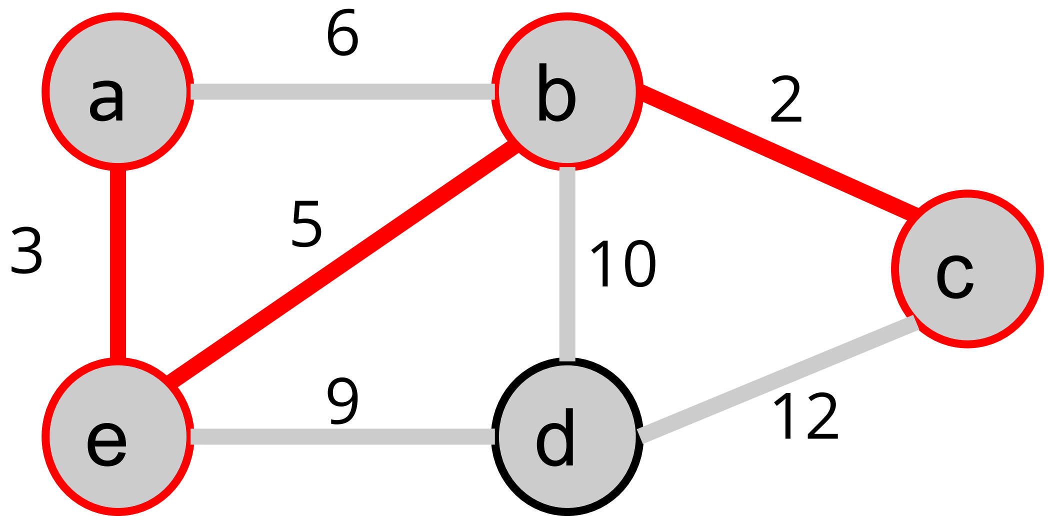
Add (b, c), the lightest edge



Add (a, e), the 2nd lightest

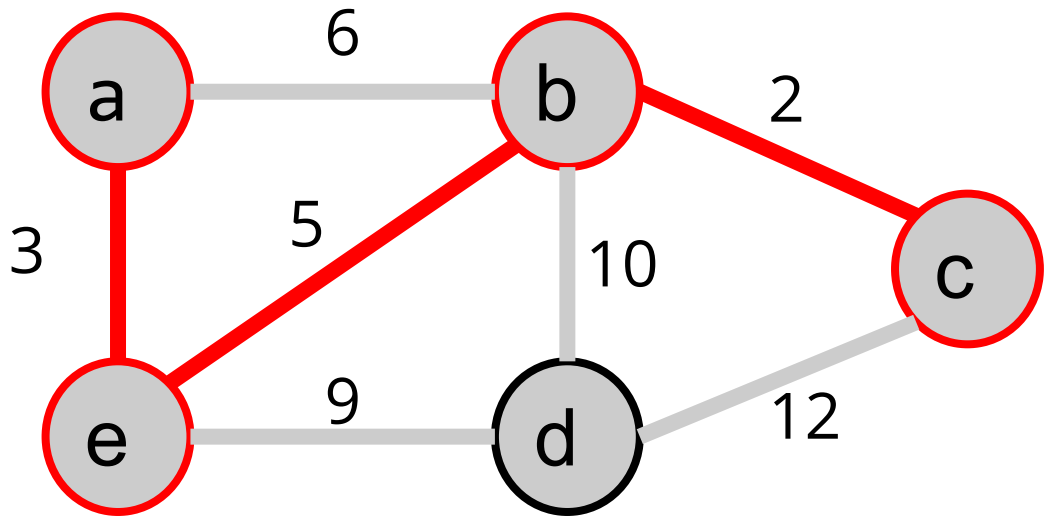


Add (b, e), the 3rd lightest



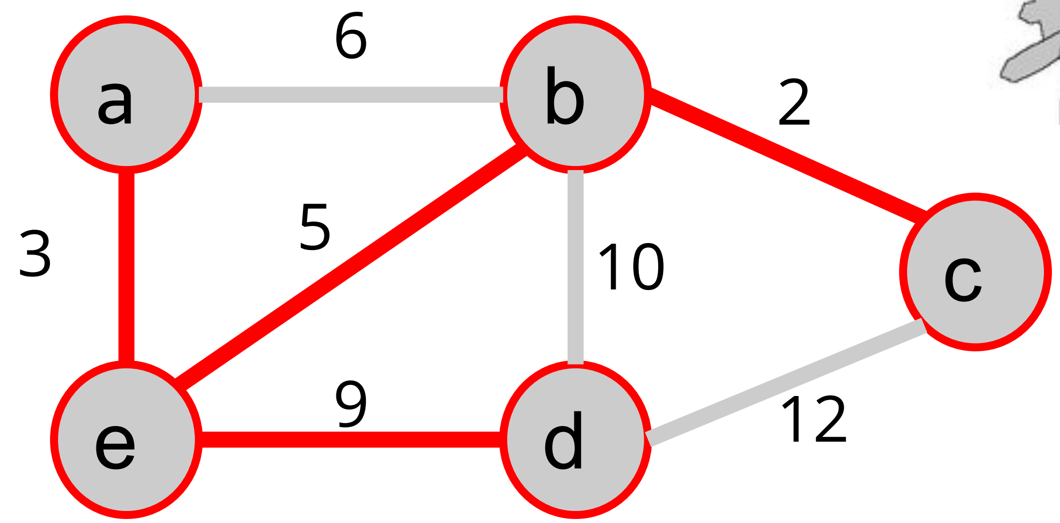
Add (a, b), the 4th lightest ...

No! a, b are in the same component
Add (d, e) instead!



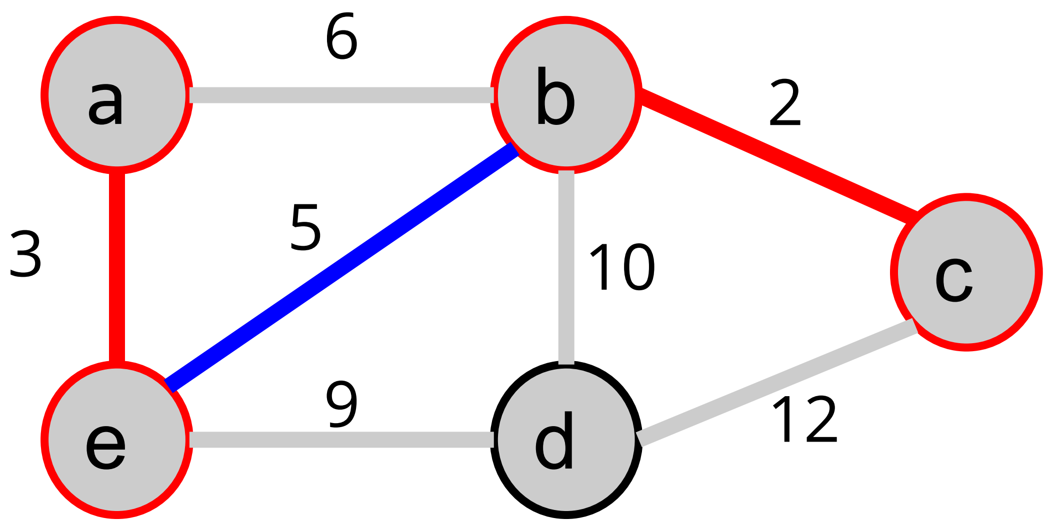
MST grown!

Add (d, e) ...



Correctness of Kruskal's

The added edge is always a "**safe**" edge, because it is the **minimum** weight edge among all edges that **cross** components



Runtime …

m = |E|

sorting takes **O(E log E)**

```
KRUSKAL-MST(G(V, E, w)):
    T ← {}
    sort edges so that w(e1)≤w(e2)≤...≤w(em)
    for i ← 1 to m:
        # let (ui, vi) = ei
        if ui and vi in different components:
            T ← T ∪ {ei}
```

How **exactly** do we do this two lines?

We need the Disjoint Set ADT

which stores a collections of nonempty disjoint sets **S1, S2, …, Sk**, each has a "representative".

and supports the following operations

- **MakeSet(x)**: create a new set {x}
- **FindSet(x)**: return the representative of the set that x belongs to
- **Union(x, y)**: union the two sets that contain x and y, if different

Real Pseudocode

m = |E|

```
KRUSKAL-MST(G(V, E, w)):
T ← {}
sort edges so that w(e1)≤w(e2)≤...≤w(em)
for each v in V:
   MakeSet(v)
for i ← 1 to m:
   # let (ui, vi) = ei
   if FindSet(ui) != FindSet(vi):
      Union(ui, vi)
      T ← T ∪ {ei}
```

Next week

- More on Disjoint Set

http://goo.gl/forms/S9yie3597B