CSC263 Week 12

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Announcements

- No tutorial this week
- PS5-8 being marked
- Course evaluation:
  - available on Portal
  - http://uoft.me/course-evals

Lower Bounds

So far, we have mostly talked about **upper-bounds** on algorithm complexity,

i.e., **O(n log n)** means the algorithm takes **at most** **cn log n** time for some **c**.

However, sometime it is also useful to talk about **lower-bounds** on algorithm complexity, i.e., how much time the algorithm **at least** needs to take.

Scenario #1

You, implement a sorting algorithm with worst-case runtime **O(n log log n)** by next week.

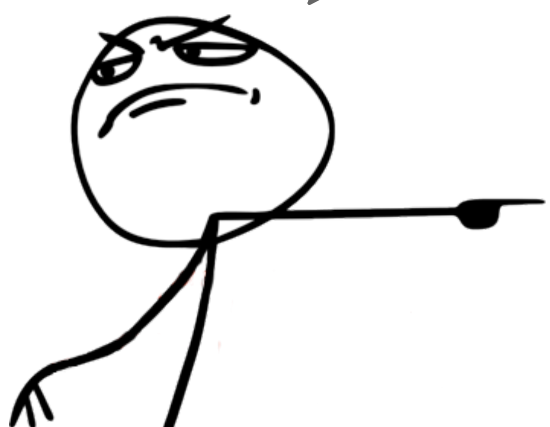
Okay Boss, I will try to do that ~

You try it for a week, cannot do it, then you are fired...

Scenario #2

You, implement a sorting algorithm with worst-case runtime **O(n log log n)** by next week.

No, Boss. **O(n log log n)** is below the **lower bound** on sorting algorithm complexity , I can't do it, **nobody** can do it!



Why learn about lower bounds

➔ Know your limit

◆ we always try to make algorithms faster, but if there is a limit that you cannot exceed, you want to know

➔ Approach the limit

◆ Once you have an understanding about of limit of the algorithm’s performance, you get insights about how to approach that limit.

Lower bounds on sorting algorithms

Upper bounds: We know a few sorting algorithms with worst-case **O(n log n)** runtime.

Is **O(n log n)** the best we can do?

Actually, yes, because the lower bound on sorting algorithms is **Ω(n log n)**, i.e., a sorting algorithm needs **at least** **cn log n** time to finish in worst-case.

actually, more precisely ...

The lower bound **n log n** applies to only all **comparison based** sorting algorithms, with **no assumptions** on the values of the elements.

It is possible to do faster than **n log n** if we make **assumptions** on the values.

Example: sorting with assumptions

Sort an array of **n** elements which are either **1** or **2**.

2	1	1	2	2	2	1

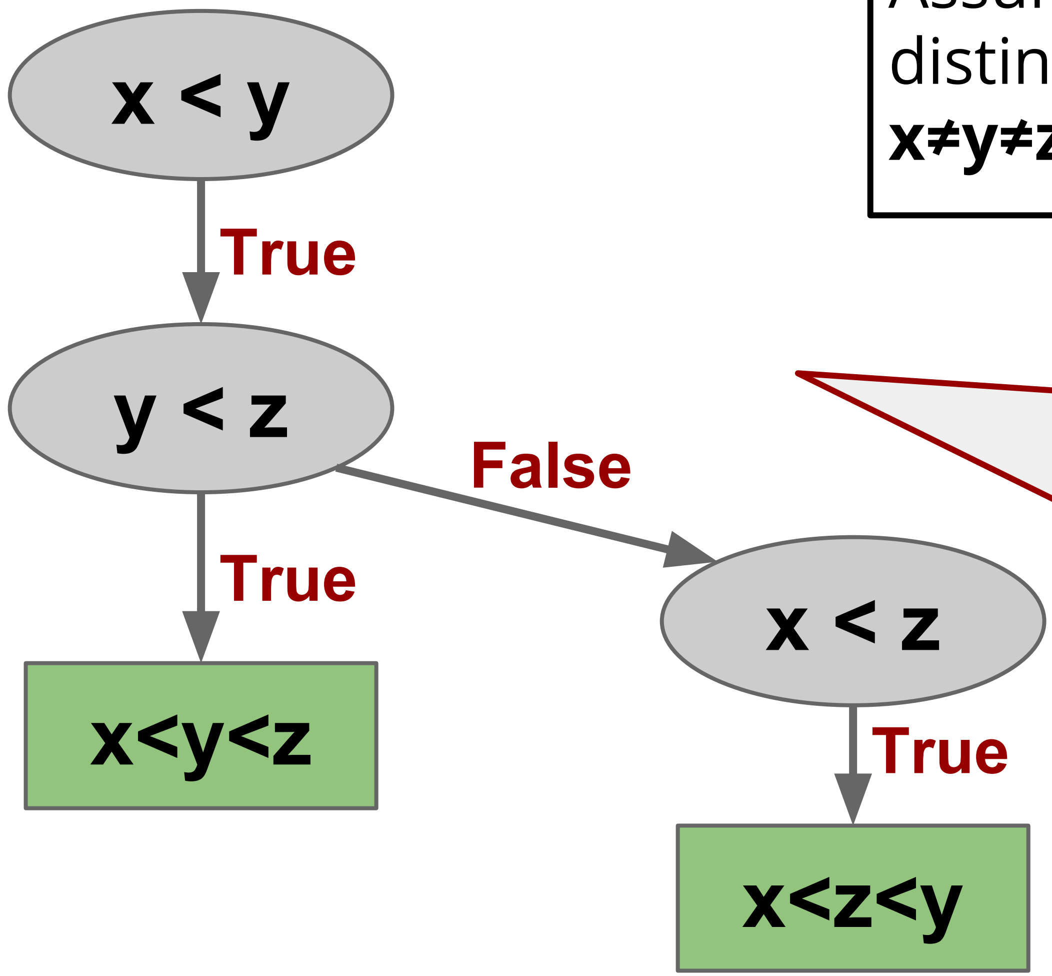
- ➔ Go through the array, count the number of 1’s, namely, **k**
- ➔ then output an array with **k** 1’s followed by **n-k** 2’s
- ➔ This takes **O(n)**.

Now prove it

the worst-case runtime of comparison based sorting algorithms is in $\Omega(n \log n)$

Sort {x, y, z} via comparisons

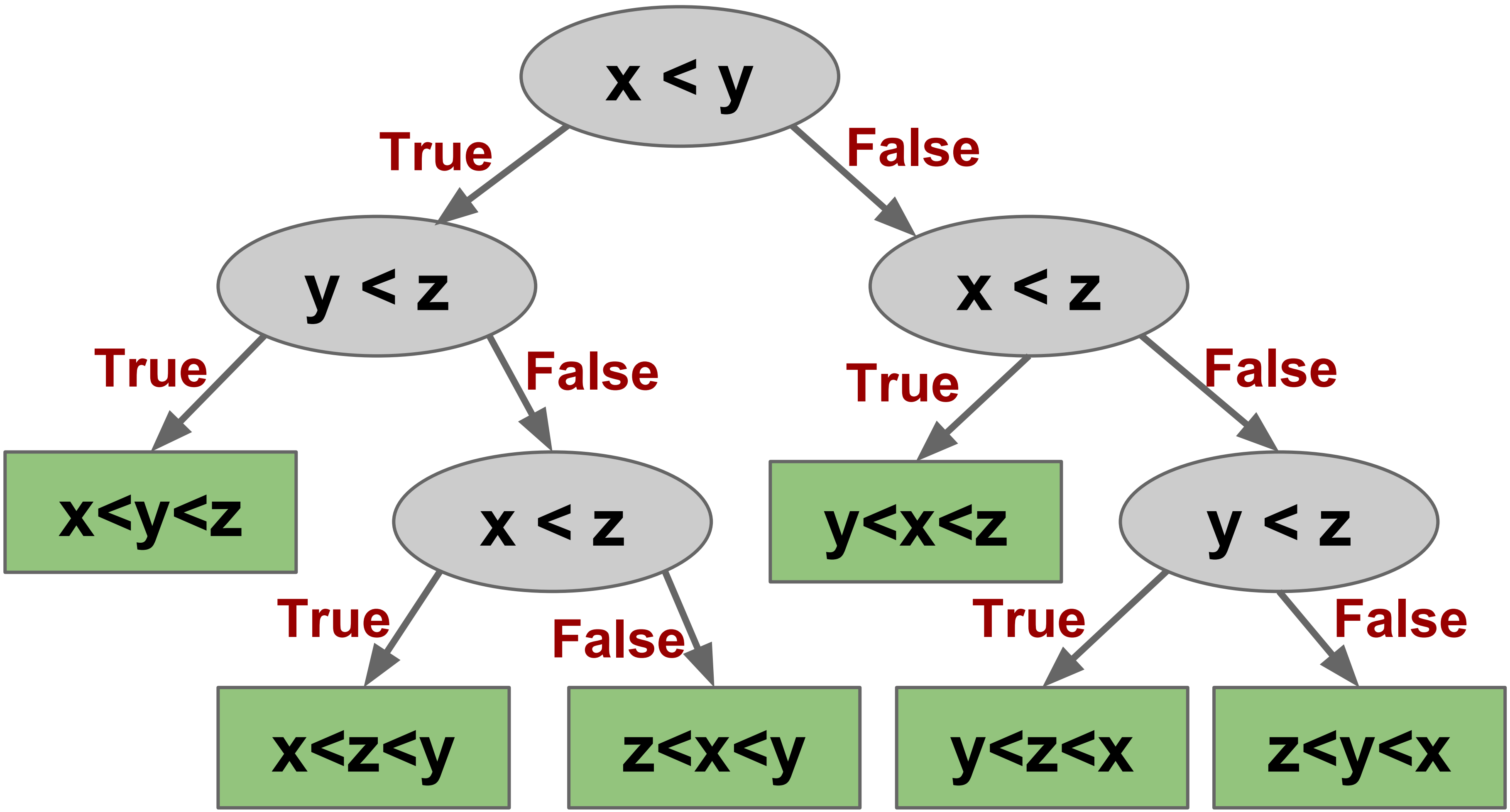
Assume **x, y, z** are distinct values, i.e., **x≠y≠z**



A tree that is used to **decide** what the sorted order of **x, y, z** should be ...

The decision tree for sorting {x, y, z}

a tree that contains a complete set of decision sequences

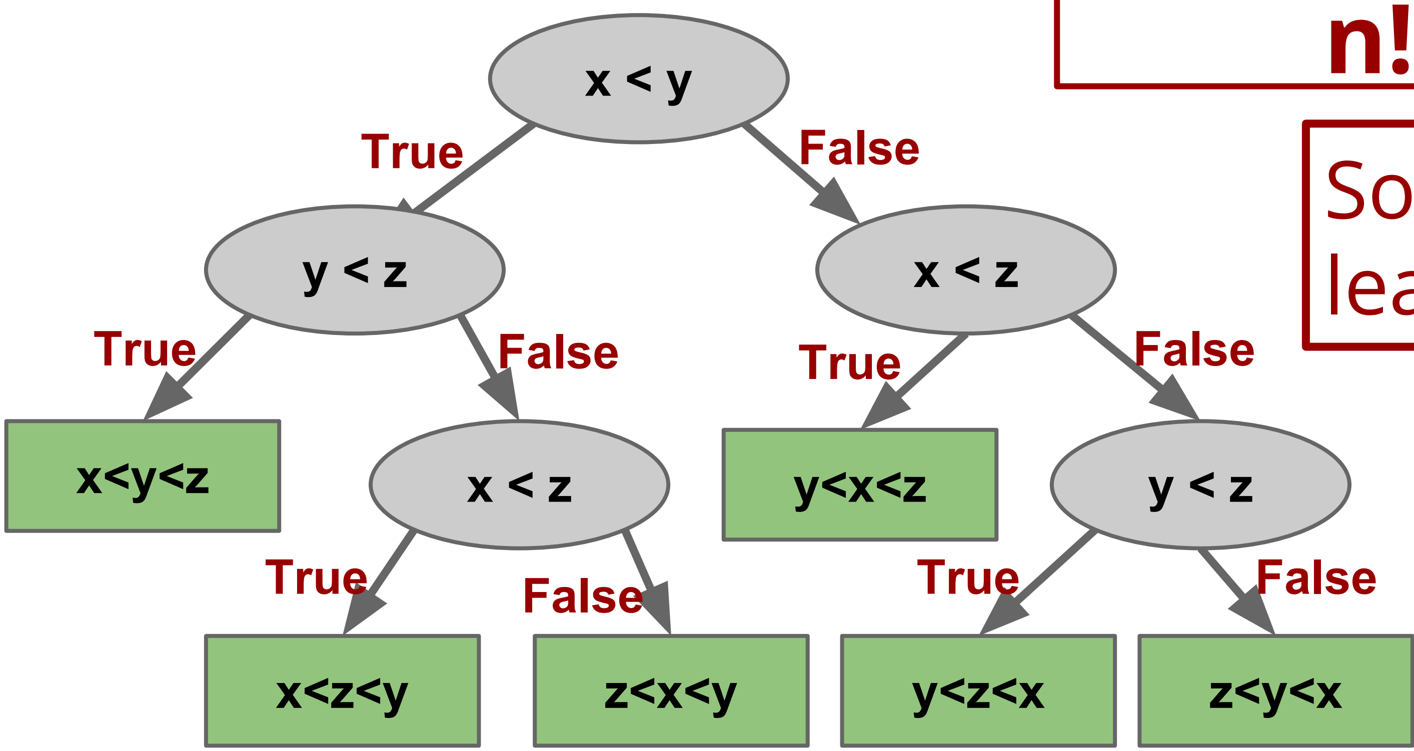


Each **leaf node** corresponds to a possible **sorted order** of {x, y, z}, a decision tree need to contain **all possible orders**.

How many possible orders for n elements?

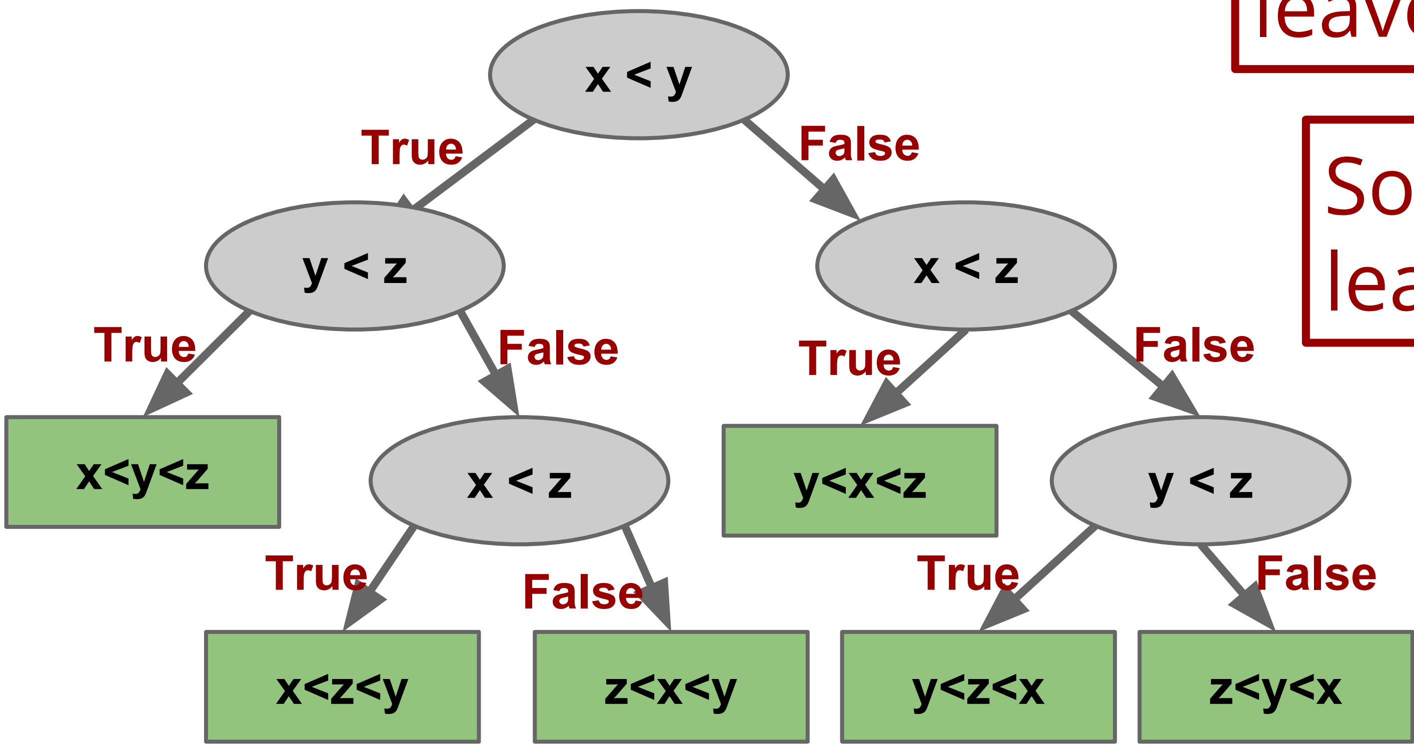
n!

So number of leaves **L ≥ n!**



Now think about the **height** of the tree

A **binary** tree with height **h** has at most **2^h leaves**



So number of leaves **L ≤ 2^h**

So number of leaves **L ≥ n!**

So,

$2^h \geq n!$

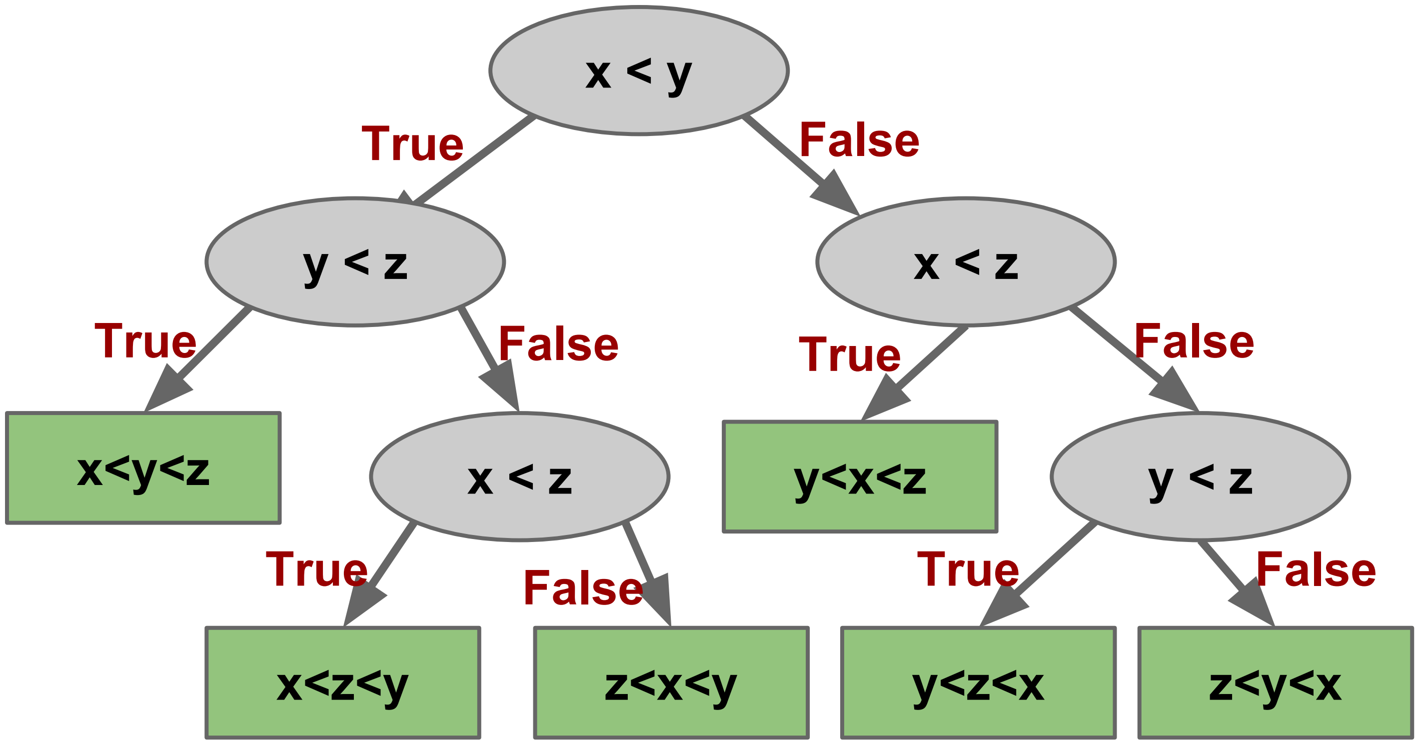
Not trivial, will show it later

So number of leaves **$L \leq 2^h$**

$h \geq \log(n!)$ $\in$ **$\Omega(n \log n)$**

So number of leaves **$L \geq n!$**

$h \in \Omega(n \log n)$



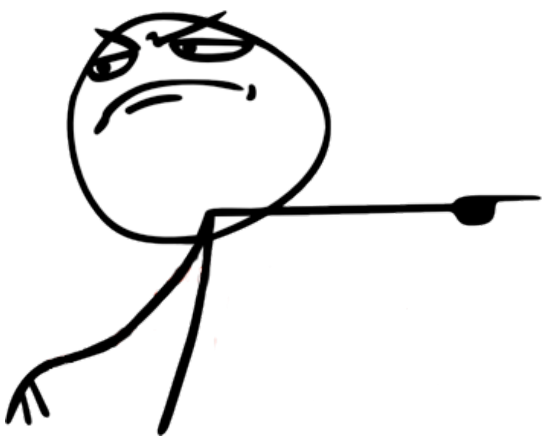
What does **h** represent, really?

The worst-case # of comparisons to sort!

$$h \in \Omega(n \log n)$$

What did we just show?

The worst-case number of comparisons needed to sort **n** elements is in **Ω (n log n)**



Lower bound proven!

Appendix: the missing piece

Show that **log (n!)** is in **Ω (n log n)**

$$\log(n!)$$

$$= \log 1 + \log 2 + \ldots + \log n/2 + \ldots + \log n$$

$$\geq \log n/2 + \ldots + \log n \quad \textbf{(n/2 + 1 of them)}$$

$$\geq \log n/2 + \log n/2 + \ldots + \log n/2 \quad \textbf{(n/2 + 1 of them)}$$

$$\geq n/2 \cdot \log n/2$$

$$\in \Omega(n \log n)$$

other lower bounds

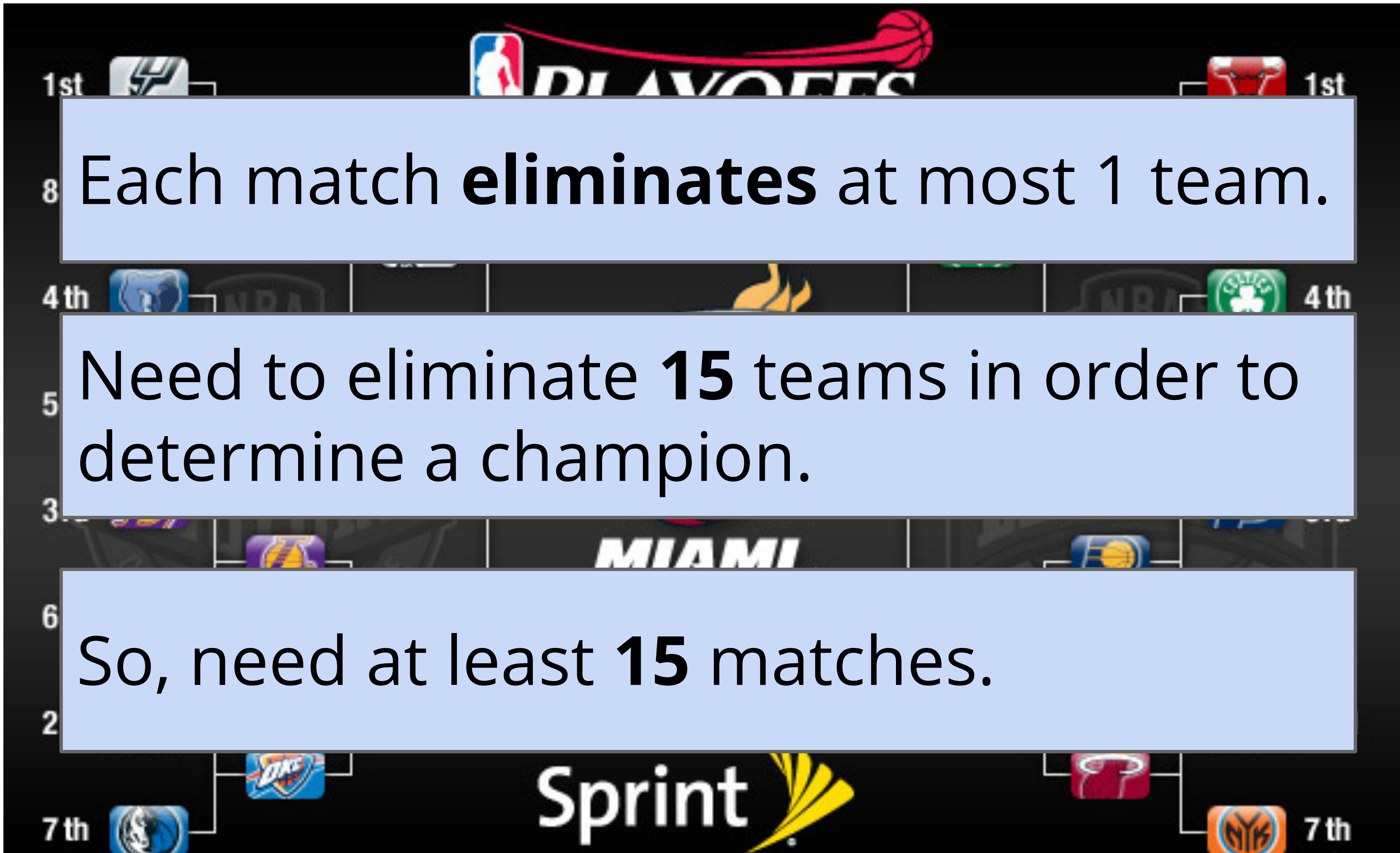
The problem

Given **n** elements, determine the **maximum** element.

How many comparisons are needed **at least**?

A similar problem

How many matches need to be played to determine a champion out of 16 teams?



Each match **eliminates** at most 1 team.

Need to eliminate **15** teams in order to determine a champion.

So, need at least **15** matches.

The problem

Given **n** elements, determine the **maximum** element.

How many comparisons are needed **at least**?

Need at least **n-1** comparisons

Insight: approach the limit

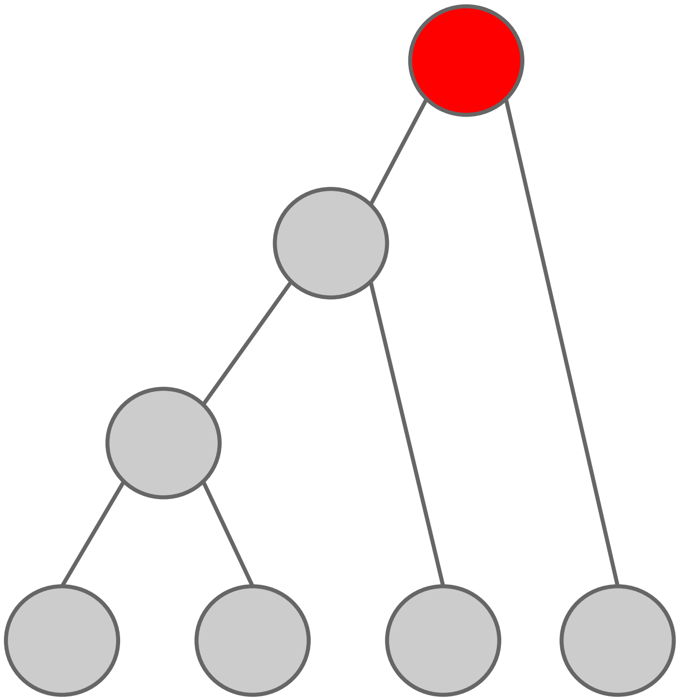
How to design a maximum-finding algorithm that reaches the lower bound **n-1** ?

➔ Make every comparison **count**, i.e., every comparison should guarantee to **eliminate a possible candidate** for maximum/champion.

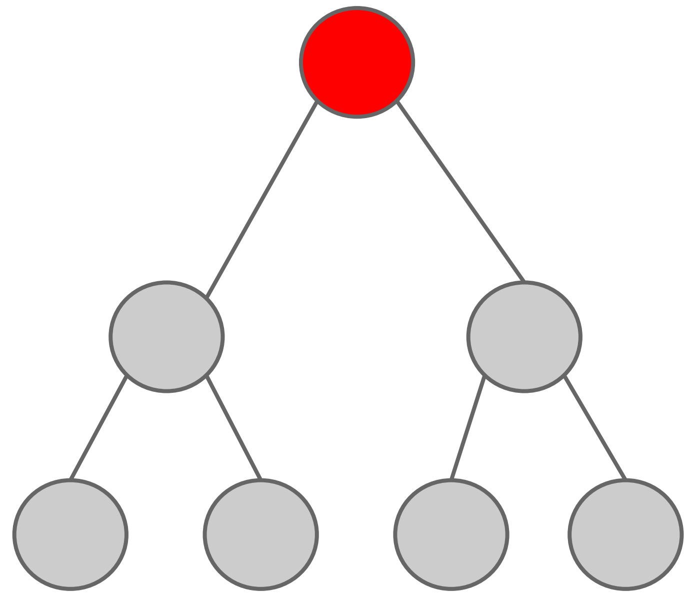
➔ No match between losers, because neither of them is a candidate for champion.

➔ No match between a candidate and a loser, because if the candidate wins, the match makes no contribution (not eliminating a candidate)

These algorithms reach the lower bound



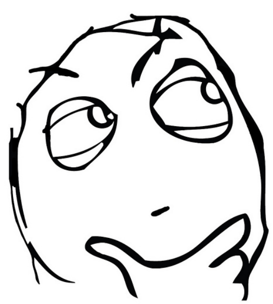
Linear scanning



Tournament

Challenge question

Given **n** elements, what is the lower bound on the number of comparisons needed to determine both the **maximum** element and the **minimum** element?



Hint: it is smaller than **2(n-1)**

The “playoffs” argument kind-of serves as a **proof** of lower bound for the **maximum-finding** problem.

But this argument may **not** work for **other** problems.

We need a more **general** methodology for **formal proofs** of **lower bounds**.

proving lower bounds using **Adversarial Arguments**

How does your opponent smartly **cheat** in this game?

➔ While you ask questions, the opponent alters their ships' positions so that they can "**miss**" whenever possible, i.e., construct the **worst possible input** (layout) **based on your questions**.

➔ They won't get caught as long as their answers are **consistent** with one possible input.

If we can prove that, no matter what sequence of questions you ask, the opponent can always craft an input such that it takes at least **42 guesses** to sink a ship.

Then we can say the **lower bound** on the complexity of the “sink-a-ship” problem is **42 guesses**, no matter what “guessing algorithm” you use.

more formally ...

To prove a lower bound **L(n)** on the complexity of problem **P**,

we show that for every algorithm **A** and arbitrary input size **n**, there exists some input of size **n** (picked by an imaginary adversary) for which **A** takes at least **L(n)** steps.

Example: search unsorted array

Problem:

Given an unsorted array of **n** elements, return the **index** at which the value is **42**.

(assume that **42** must be in the array)

3	**5**	**2**	**42**	**7**	**9**	**8**

Possible algorithms

- → Check through indices 1, 2, 3, ..., n
- → Check from n, n-1, n-2, ..., to 1
- → Check all odd indices 1, 3, 5, ..., then check all even indices 2, 4, 6, ...
- → Check in the order 3, 1, 4, 1, 5, 9, 2, 6, ...

Prove: the **lower bound** on this problem is **n**, no matter what algorithm we use.

3	**5**	**2**	**42**	**7**	**9**	**8**

Proof: (using adversarial argument)

- Let **A** be an **arbitrary** algorithm in which the first **n** indices checked are **i_1, i_2, ..., i_n**
- Construct (adversarially) an input array **L** such that **L[i_1]**, **L[i_2], ..., L[i_{n-1}]** are **not 42**, and **L[i_n]** is **42.**
- Because **A** is arbitrary, therefore the lower bound on the complexity of solving this problem is **n**, no matter what algorithm is used.

proving lower bounds using Reduction

The idea

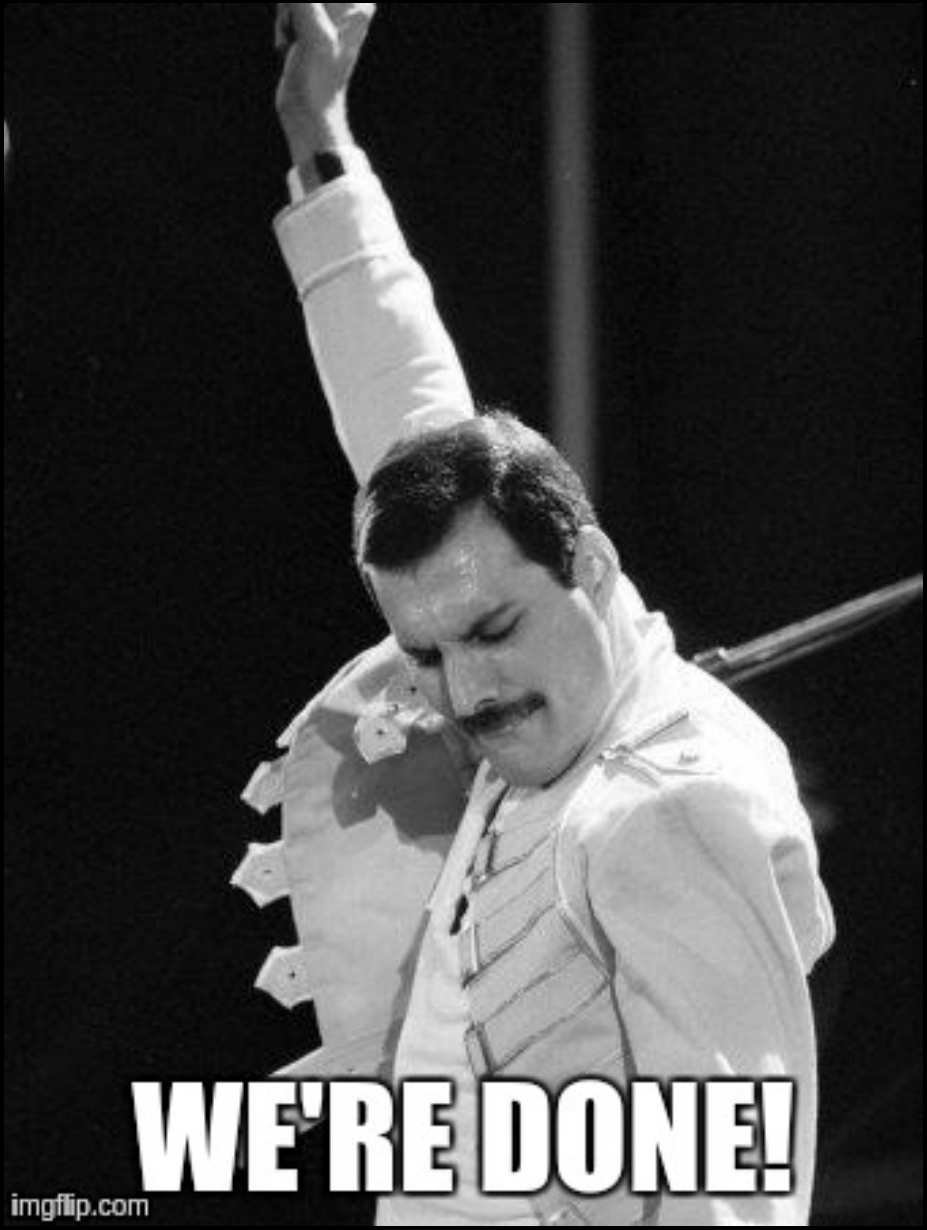
- ➔ Proving one problem's lower bound using **another** problem's **known** lower bound.
- ➔ If we know problem **B** can be solved by solving an instance of problem **A**, i.e., **A** is "harder" than **B**
- ➔ and we know that **B** has lower bound **L(n)**
- ➔ then **A** must also be lower-bounded by **L(n)**

Example:

Prove: **ExtractMax** on a binary heap is lower bounded by **Ω(log n)**.

Suppose ExtractMax can be done faster than **log n**, then HeapSort can be done faster than **n log n**, because HeapSort is basically ExtractMax **n** times

But HeapSort, as a comparison based sorting algorithm, has been proven to be lower bounded by **Ω(n log n).** Contrdiction, so ExtractMax must be lower bounded by **Ω(log n)**


WE'RE DONE!
imgflip.com

Final thoughts

what did we learn in CSC263

Data structures are the underlying skeleton of a good computer system.

If you will get to design such a system yourself and make fundamental decisions, what you learned from CSC263 should give you some clues on what to do.

- ➔ Understand the nature of the system / problem, and model them into structured data
- ➔ Investigate the probability distribution of the input
- ➔ Investigate the real cost of operations
- ➔ Make reasonable assumptions and estimates where necessary
- ➔ Decide what you care about in terms of performance, and analyse it
  - ◆ “No user shall experience a delay more than 500 milliseconds” -- worst-case analysis
  - ◆ “It’s ok some rare operations take a long time” -- average-case analysis
  - ◆ “what matter is how fast we can finish the whole sequence of operations” -- amortized analysis

In CSC263, we learned to be
a computer scientist,
not just a programmer.

Original words from lecture notes of Michelle Craig

what we did NOT learn

but are now ready to learn

Awesomer kinds of heaps

- Sometimes we want to be able to **merge** two heaps into one heap, with binary heap we can do it in **O(n)** time worst-case.
- Using **binomial heap**, we can do merge in **O(log n)** time worst-case
- Using **Fibonacci heap**, we can do merge (as well as Max/Insert/IncreaseKey) in **O(1)** time amortized.

Awesomer kinds of search trees

- ➔ We learned BST and AVL tree, and there are others called red-black tree, 2-3 tree, splay tree, AA tree, scapegoat tree, etc.
- ➔ There is **B-tree**, optimized for accessing big blocks of data (like in a hard drive)
- ➔ There is **B+ tree**, which is even better than B-tree (widely used in database systems).
- ➔ You'll learn about these in CSC443.

Awesomer kinds of hashing

- **Universal hashing** which provably guarantees simple uniform hashing

- **Perfect hashing** guarantees **worst-case** O(1) time for searching, instead of **average-case** O(1) time

Shortest paths in a graph

- We learned how to get shortest paths using BFS on a graph

- We did NOT learn how to get **shortest (weighted) paths** on a weighted graph.
  - Dijkstra, Bellman-Ford, ...

- You'll learn about them in CSC358 / 373

Greedy algorithms

- ➔ We learned that Kruskal’s and Prim’s MST algorithms are greedy
- ➔ What property is satisfied by the problems that can be perfectly solved by greedy algorithms?
- ➔ Will learn in CSC373

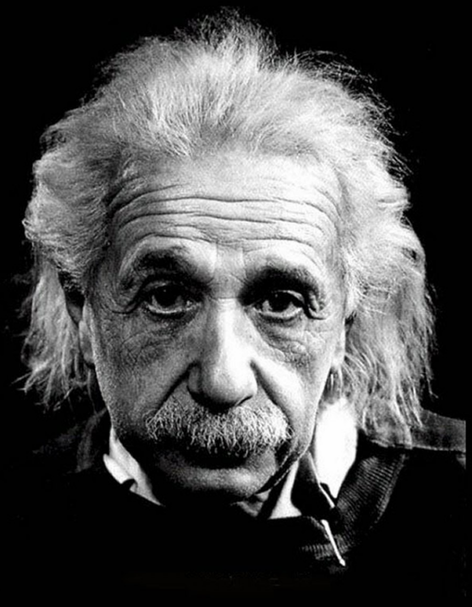
Dynamic programming

➔ Pick an interesting algorithm design problem, very likely it involves dynamic programming

➔ Will learn in CSC373

P vs NP, approximation algorithms

- ➔ We learned a bit about lower bounds.
- ➔ There are some problems, we can prove they cannot be perfectly solved in polynomial time.
- ➔ For these problems, we have to design some **approximation algorithms**.
- ➔ Will learn in CSC373 / 463



As our circle of knowledge expands, so does the circumference of darkness surrounding it.

Final Exam Prep

Topics covered: all of them

- ➔ Heaps
- ➔ BST, AVL tree, augmentation
- ➔ Hashing
- ➔ Randomized algorithms, Quicksort
- ➔ Graphs, BFS, DFS, MST
- ➔ Disjoint sets
- ➔ Lower bounds
- ➔ Analysis: worst-case, average-case, amortized.

Types of questions

- Short-answer questions testing basic understanding.
- Trace operations we learned on a data structure
- Implement an ADT using a data structure
- Analysis runtimes
  - best / worst-case
  - average-case
  - amortized cost
- Given a real-world problem, design data structures / algorithms to solve it.

Study for the exam

- ➔ Review lecture notes/slides
- ➔ Review tutorial problems
- ➔ Review all problem sets / assignments
- ➔ Practice with past exams (available at old exam repository at UofT library)
- ➔ Come to **office hours** whenever confused.

Larry’s pre-exam office hours

- ➔ All Thursdays 2-4pm
- ➔ All Fridays 2-4pm
- ➔ Monday, April 13, 4-6pm
- ➔ Tuesday, April 14, 4-6pm
- ➔ Wednesday, April 15, 4-6pm
- ➔ Monday, April 20, 4-6pm
- ➔ Tuesday, April 21, 4-6pm

Exam Time & Location

Wednesday, April 22nd, PM 2:00 - 5:00

Locations:

- ➔ A - HO: NR 25
- ➔ HU - NGO: ST VLAD
- ➔ NGU - WI: UC 266
- ➔ WL- Z: UC 273

double-sided, handwritten aid-sheet

Go to the right location.

All the best!