CSC263 Week 2

If you feel rusty with probabilities, please read the Appendix C of the textbook. It is only about 20 pages, and is highly relevant to what we need for CSC263.

Appendix A and B are also worth reading.

This week topic

- ➔ ADT: **Priority Queue**
- ➔ Data structure: **Heap**

An ADT we already know

First in first serve

Queue:

- ➔ a collection of elements
- ➔ supported operations
  - ◆ Enqueue(Q, x)
  - ◆ Dequeue(Q)
  - ◆ PeekFront(Q)

The new ADT

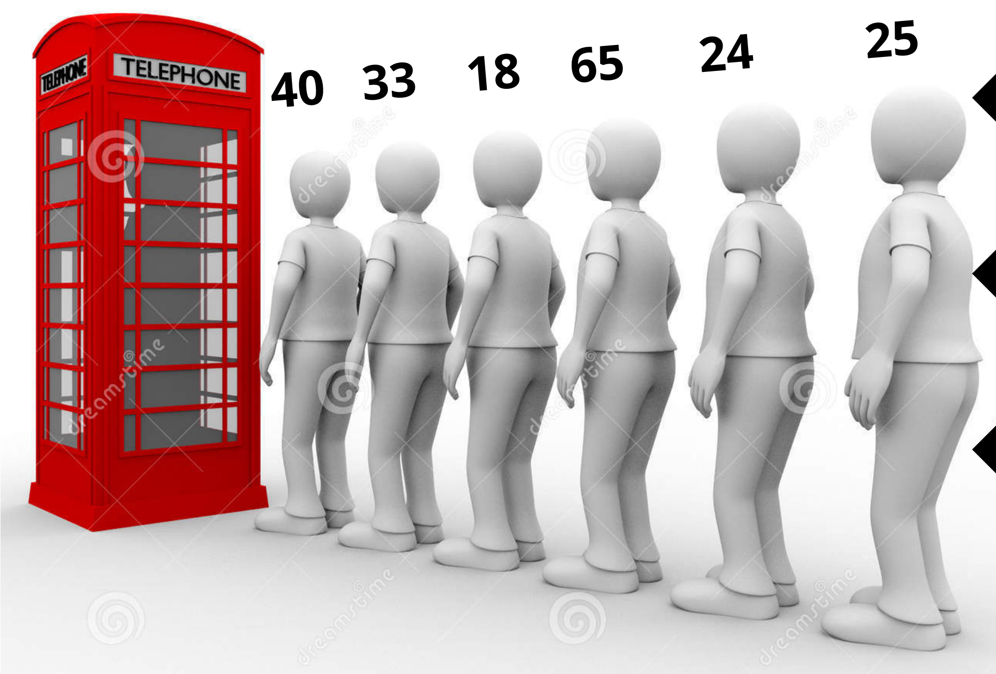
Max-Priority Queue:

→ a collection of elements **with priorities**, i.e., each element x has x.priority

→ supported operations

- **Insert**(Q, x)
  - like enqueue(Q, x)
- **ExtractMax**(Q)
  - like dequeue(Q)
- **Max**(Q)
  - like PeekFront(Q)
- **IncreasePriority**(Q, x, k)
  - increase x.priority to k

Oldest person first



Applications of Priority Queues

- ➔ Job scheduling in an operating system
  - ◆ Processes have different priorities (Normal, high...)
- ➔ Bandwidth management in a router
  - ◆ Delay sensitive traffic has higher priority
- ➔ Find minimum spanning tree of a graph
- ➔ etc.

Now, let's implement a (Max)-Priority Queue

40 –> 33 –> 18 –> 65 –> 24 –> 25

Use an **unsorted linked list**

- **INSERT(Q, x)** # x is a node
  - Just insert x at the head, which takes **Θ(1)**
- **IncreasePriority(Q, x, k)**
  - Just change x.priority to k, which takes **Θ(1)**
- **Max(Q)**
  - Have to go through the whole list, takes **Θ(n)**
- **ExtractMax(Q)**
  - Go through the whole list to find x with max priority (O(n)), then delete it (O(1) if doubly linked) and return it, so overall **Θ(n)**.

65 -> 40 -> 33 -> 25 -> 24 -> 18

Use a reversely sorted linked list

- **Max(Q)**
  - Just return the head of the list, **Θ(1)**
- **ExtractMax(Q)**
  - Just delete and return the head, **Θ(1)**
- **INSERT(Q, x)**
  - Have to linearly search the correct location of insertion which takes **Θ(n)** in worst case.
- **IncreasePriority(Q, x, k)**
  - After increase, need to move element to a new location in the list, takes **Θ(n)** in worst case.

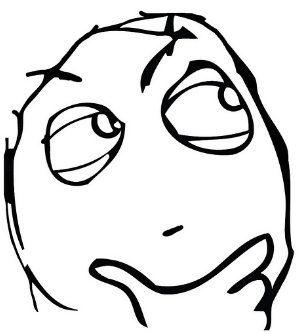
Θ(1) is fine, but Θ(n) is kind-of bad...

unsorted linked list

sorted linked list

...

Can we link these elements in a smarter way, so that we never need to do Θ(n)?

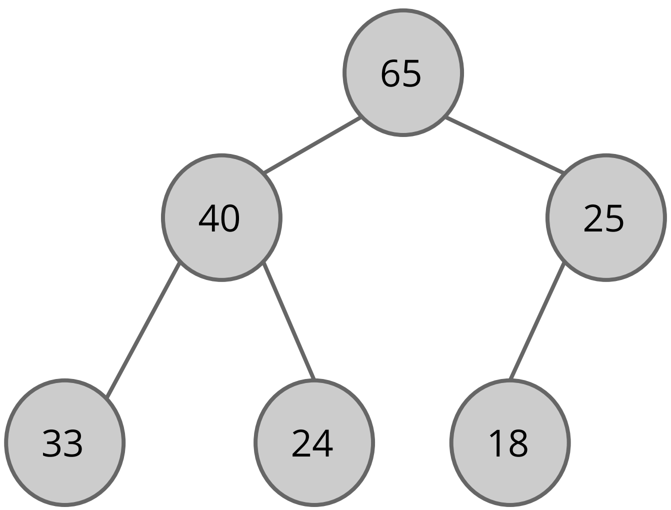


Yes, we can!

Worst case running times

	unsorted list	sorted list	Heap
Insert(Q, x)	Θ(1)	Θ(n)	Θ(log n)
Max(Q)	Θ(n)	Θ(1)	Θ(1)
ExtractMax(Q)	Θ(n)	Θ(1)	Θ(log n)
IncreasePriority (Q, x, k)	Θ(1)	Θ(n)	Θ(log n)

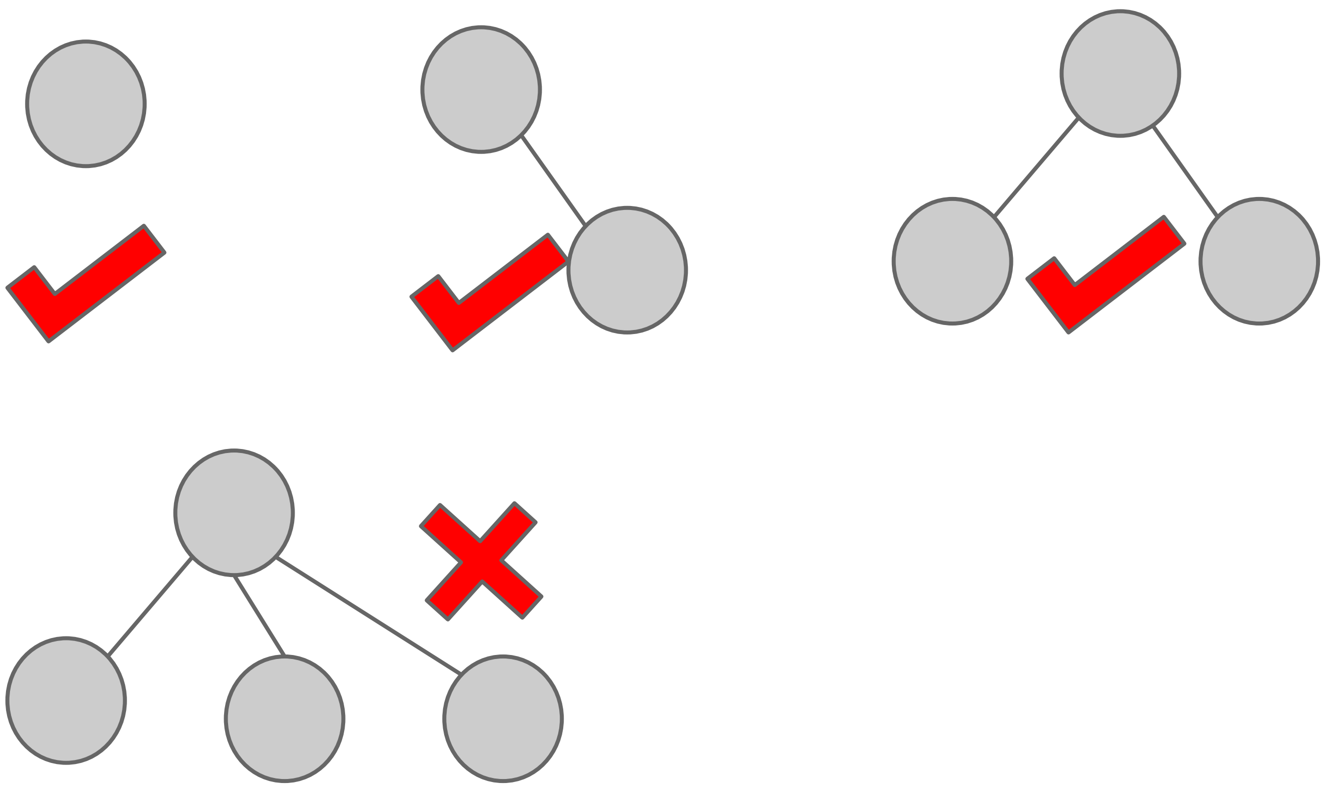
Binary Max-Heap



A **binary max-heap** is a **nearly-complete** **binary** tree that has the **max-heap property**.

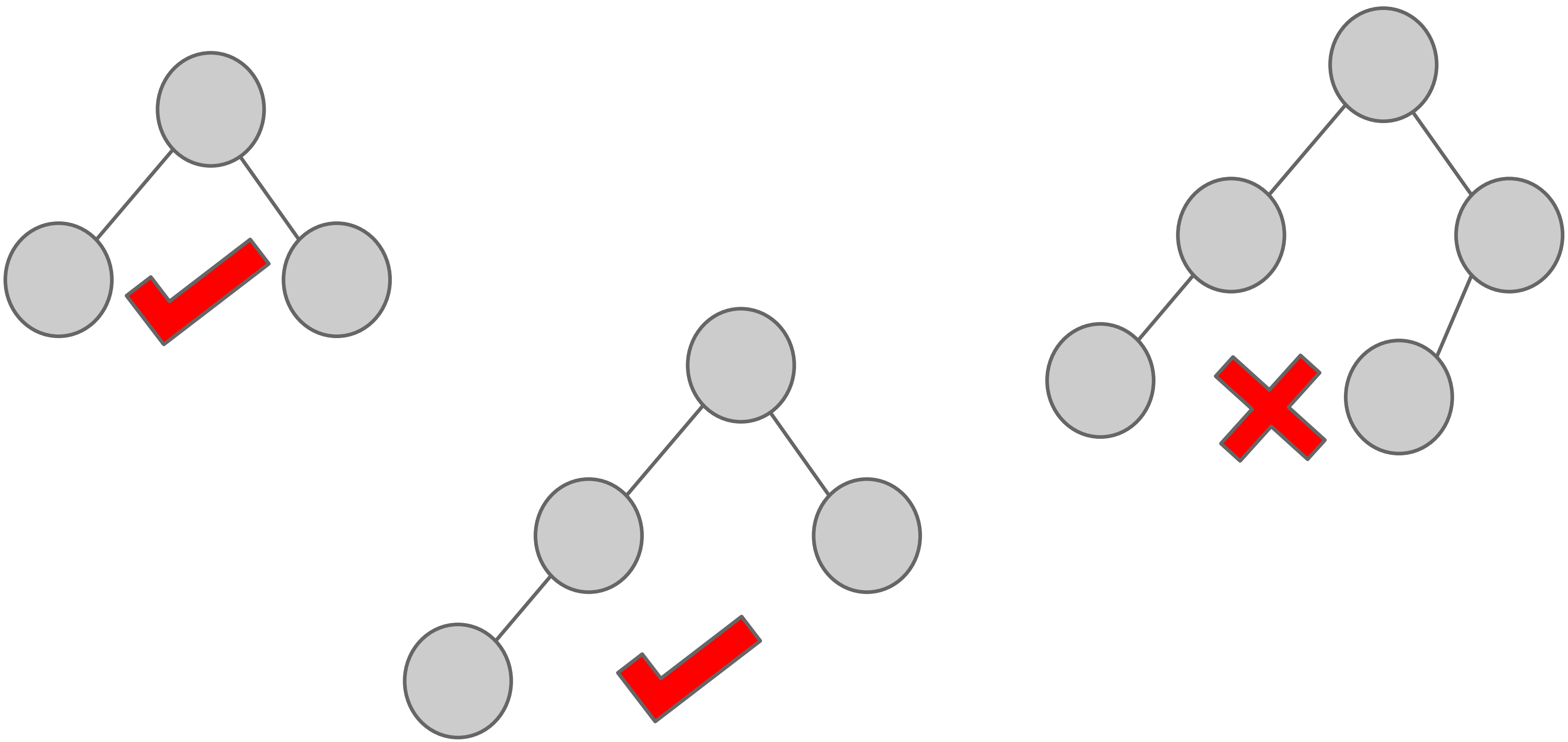
It’s a binary tree

Each node has **at most** 2 children



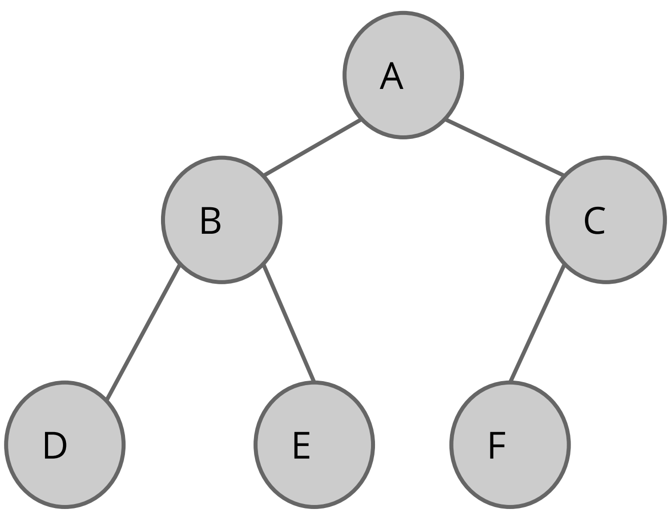
It's a nearly-complete binary tree

Each level is **completely filled**, except the bottom level where nodes are filled to as **far left** as possible



Why is it important to be a **nearly-complete binary tree?**

Because then we can **store** the tree in an **array**, and each node knows which **index** has the its parent or left/right child.



	A	B	C	D	E	F
index:	1	2	3	4	5	6

$$\text{Left}(i) = 2i$$

$$\text{Right}(i) = 2i + 1$$

$$\text{Parent}(i) = \text{floor}(i/2)$$

Assume index starts from 1

Why is it important to be a nearly-complete binary tree?

Another reason:

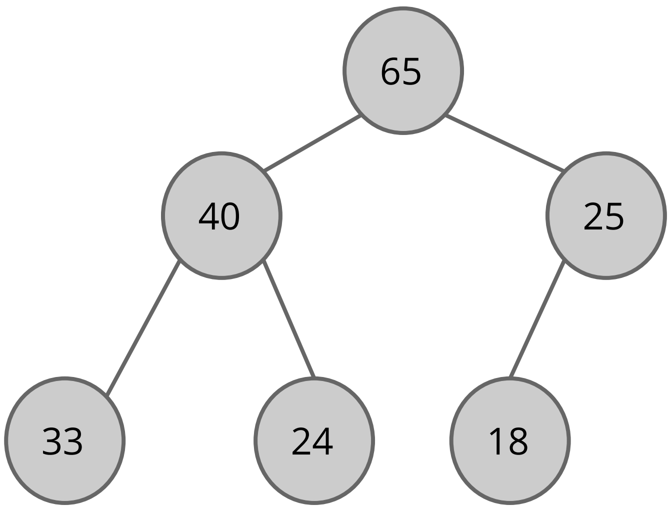
The **height** of a complete binary tree with **n** nodes is **Θ(log n)**.

This is essentially why those operations would have Θ(log n) worst-case running time.

A thing to remember…

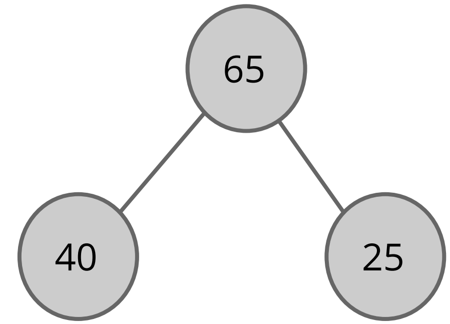
A heap is stored in an array.

Binary Max-Heap

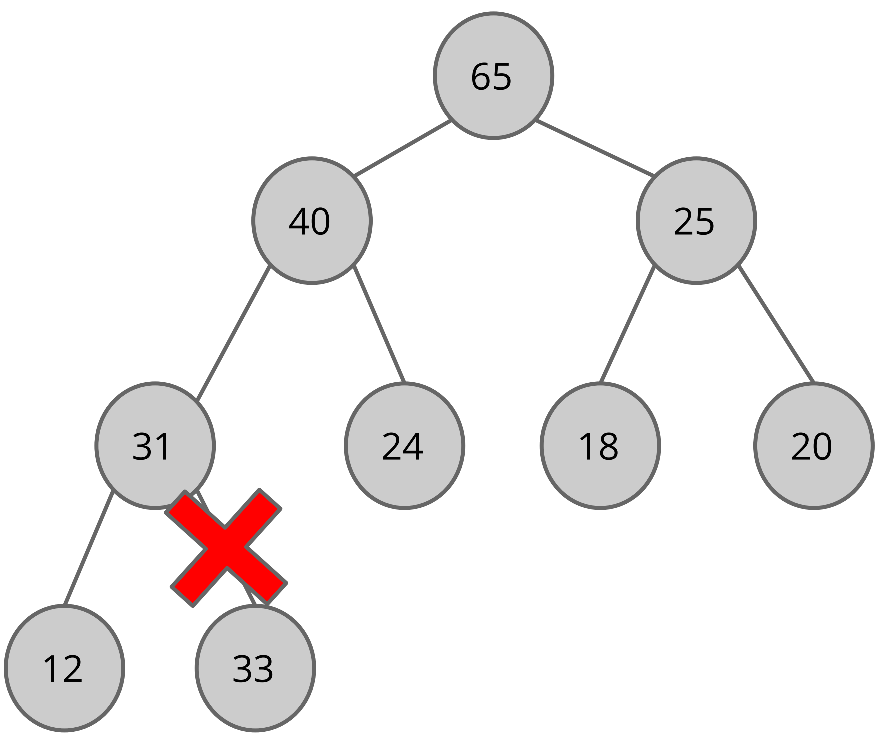


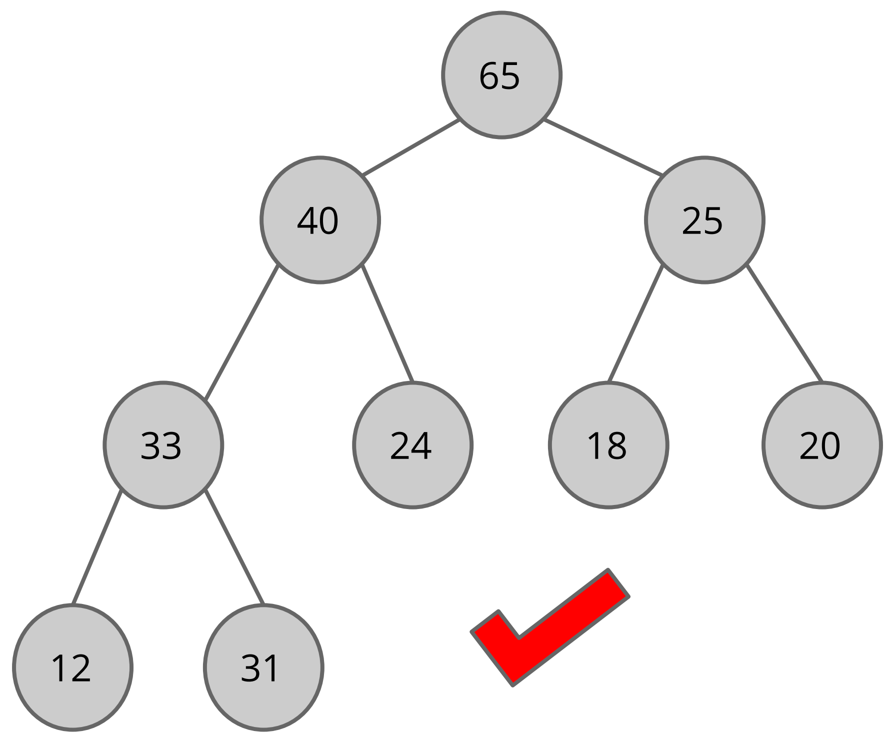
A **binary max-heap** is a **nearly-complete binary tree** that has the **max-heap property**.

The max-heap property

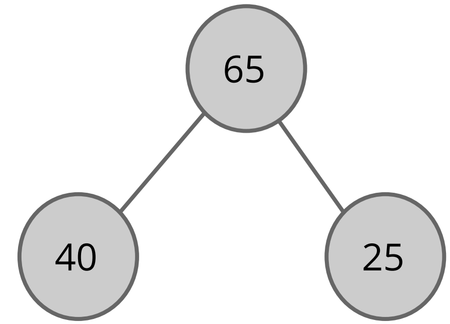


Every node has key (priority) greater than or equal to keys of its **immediate** children.



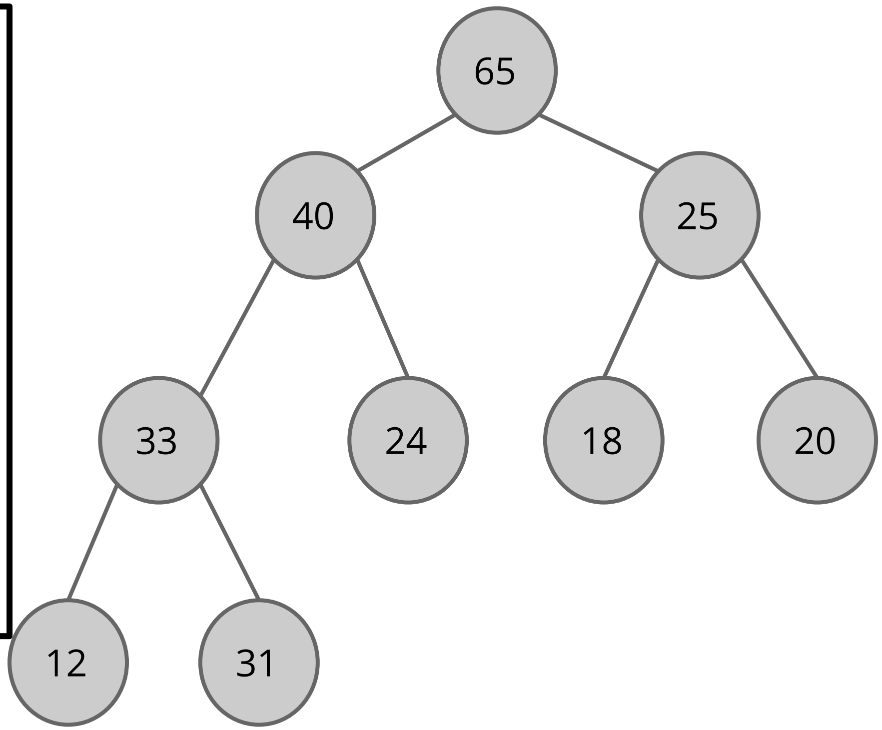


The max-heap property



Every node has key (priority) greater than or equal to keys of its **immediate** children.

Implication: every node is larger than or equal to **all its descendants**, i.e., every **subtree** of a heap is also a heap.



We have a binary max-heap defined, now let’s do operations on it.

- ➔ Max(Q)
- ➔ Insert(Q, x)
- ➔ ExtractMax(Q)
- ➔ IncreasePriority(Q, x, k)

Max(Q)

Return the largest key in Q,
in O(1) time

Max(Q): return the maximum element

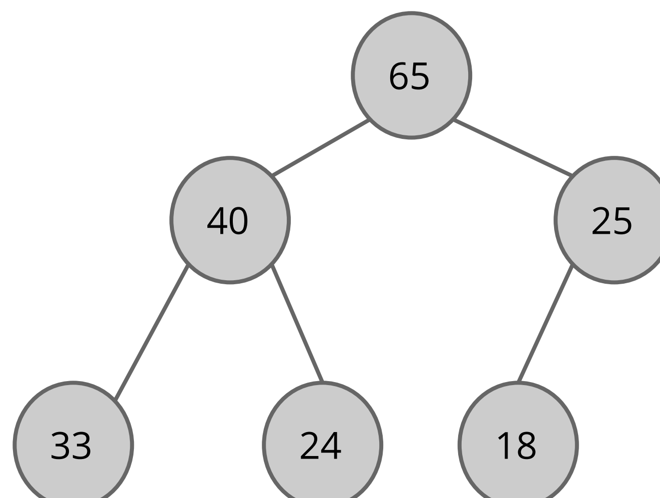
Return the **root** of the heap, i.e.,

just **return Q[1]**

(index starts from 1)

worst case **$\Theta(1)$**

Q	65	40	25	33	24	18



Insert(Q, x)

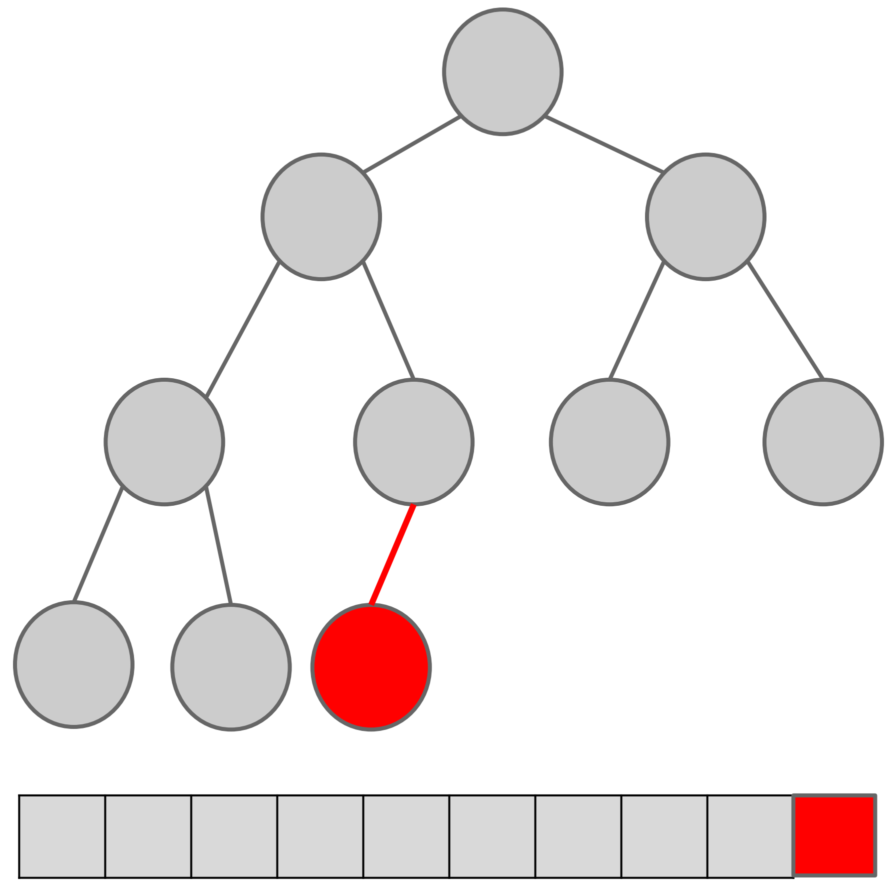
Insert node x into heap Q,
in $O(\log n)$ time

Insert(Q, x): insert a node to a heap

First thing to note:

Which spot to add the new node?

The only spot that keeps it a **complete** binary tree.

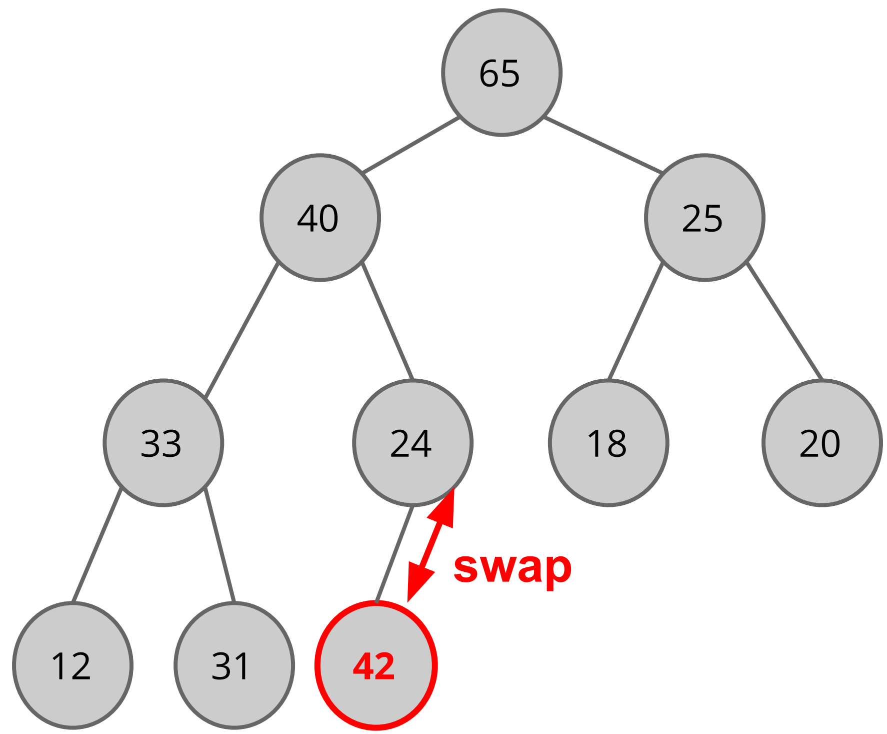


Insert(Q, x): insert a node to a heap

Second thing to note:

Heap property might be broken, how to fix it and **maintain** the heap property?

“**Bubble-up**” the new node to a proper position, by **swapping with parent.**

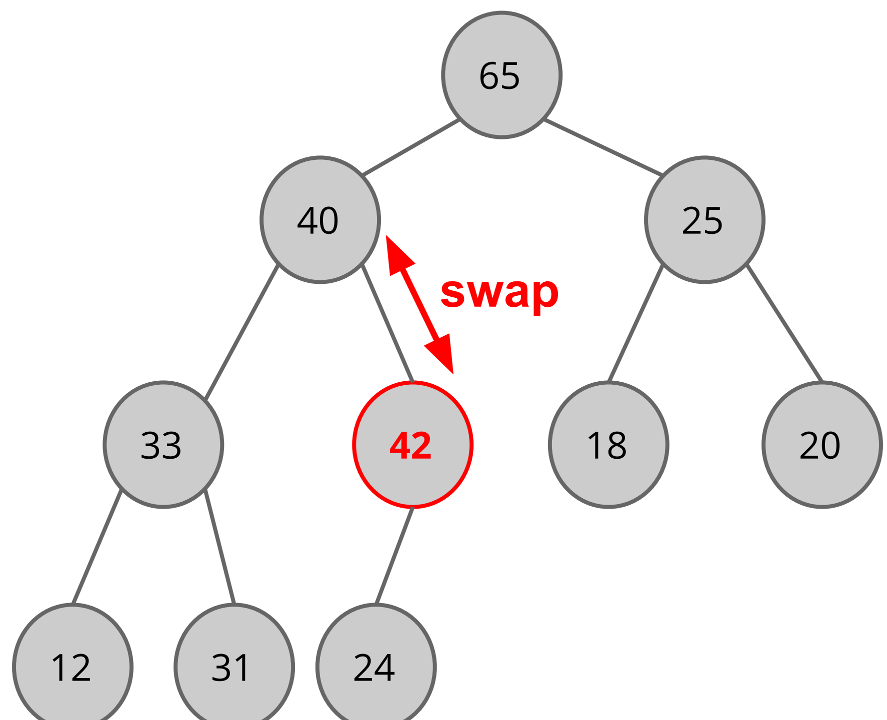


Insert(Q, x): insert a node to a heap

Second thing to note:

Heap property might be broken, how to fix it and **maintain** the heap property.

"**Bubble-up**" the new node to a proper position, by **swapping with parent.**



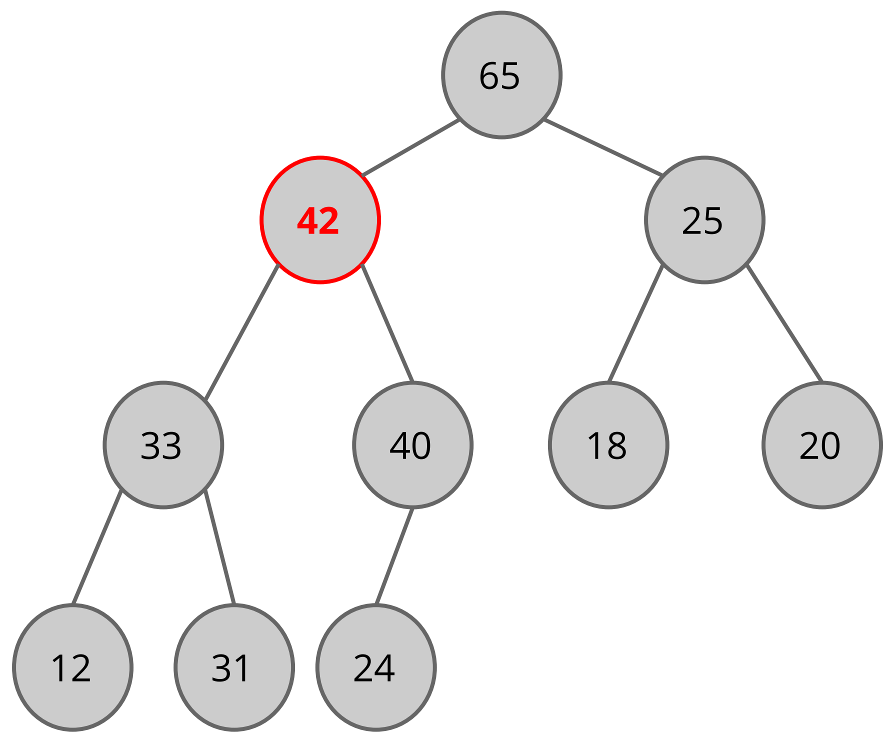
Insert(Q, x): insert a node to a heap

Second thing to note:

Heap property might be broken, how to fix it and **maintain** the heap property.

"**Bubble-up**" the new node to a proper position, by **swapping with parent.**

Worst-case:
$\Theta(\text{height}) = \Theta(\log n)$



ExtractMax(Q)

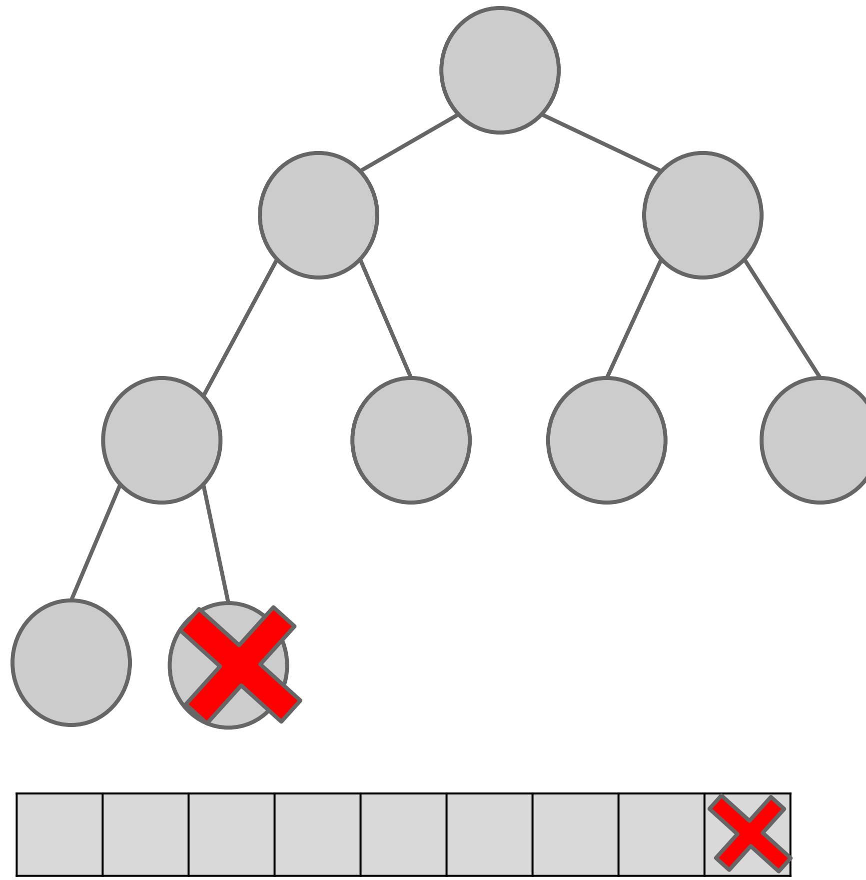
Delete and return the largest key in Q, in O(logn) time

ExtractMax(Q): delete and return the maximum element

First thing to note:

Which **spot** to remove?

The only **spot** that keeps it a **complete** binary tree.



Decrement heap size

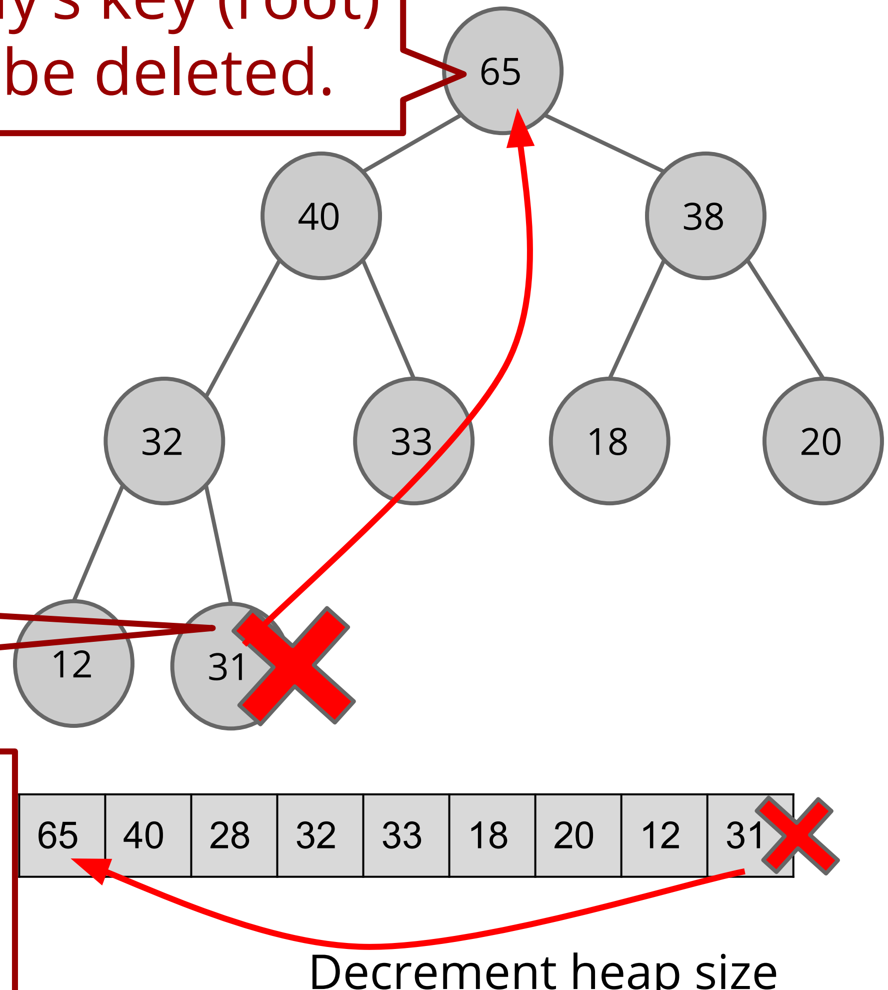
ExtractMax(Q): delete and return the maximum element

First thing to note:

Which **spot** to remove?

The only **spot** that keeps it a **complete** binary tree.

THIS guy's key (root) should be deleted.



But the last guy's key should NOT be deleted.

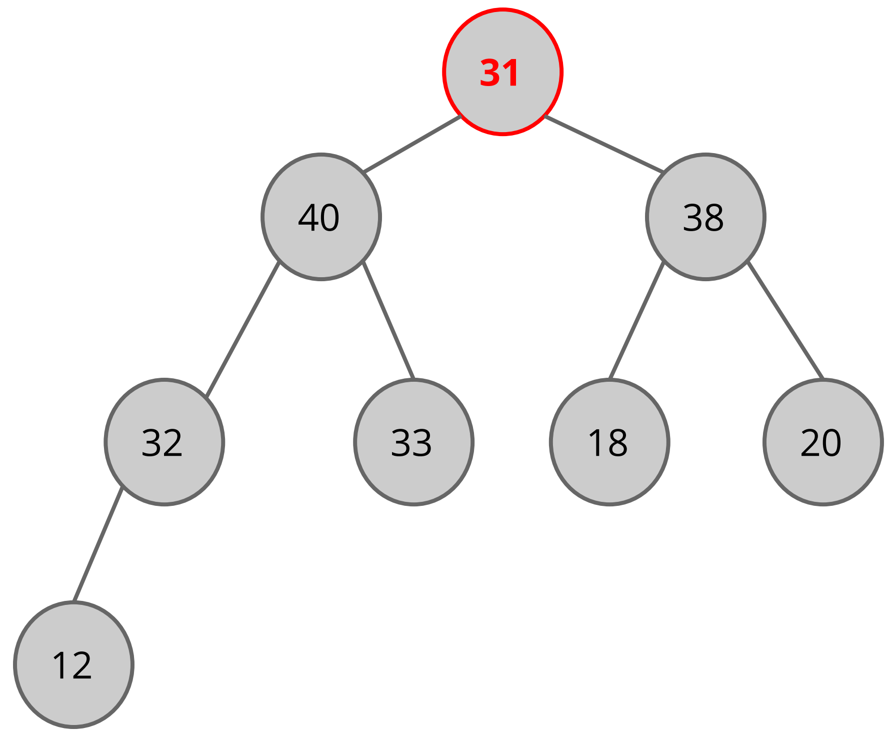
Overwrite root with the **last guy**'s key, then **delete** the last guy (decrement heap size).

ExtractMax(Q): delete and return the maximum element

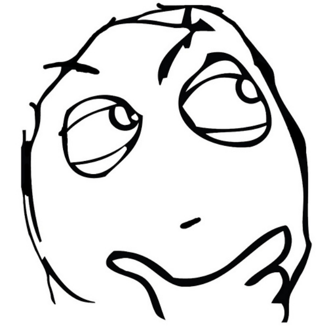
Now the **heap property** is broken again..., need to fix it.

"Bubble-down" by swapping with...

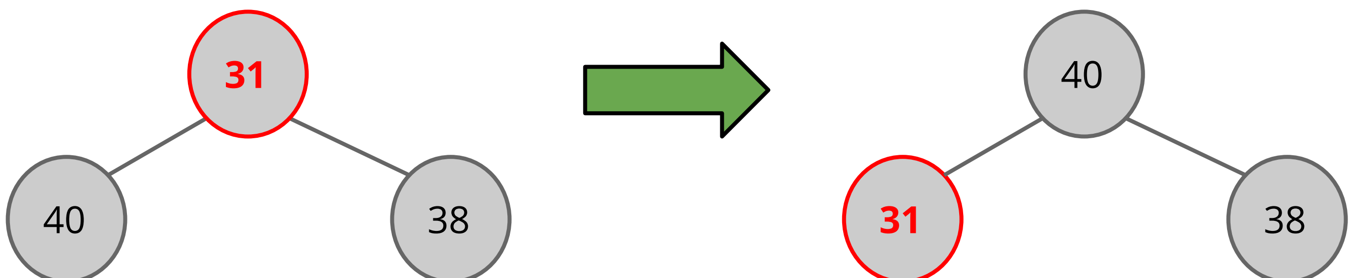
a child...



Which child to swap with?



so that, after the swap, **max-heap property** is satisfied



The “elder” child!

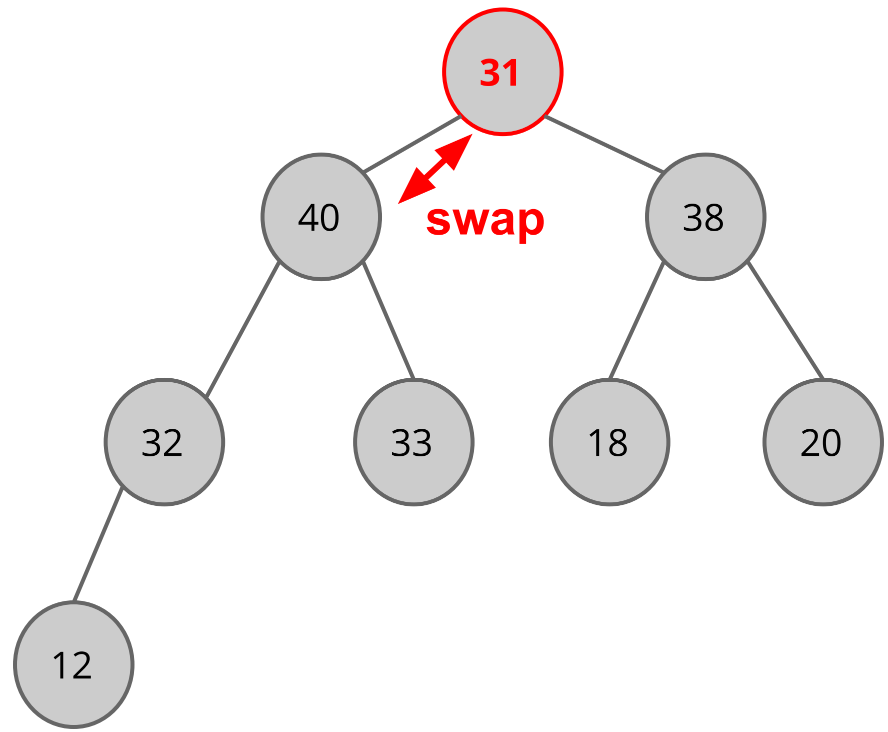
because it is the **largest** among the three

ExtractMax(Q): delete and return the maximum element

Now the **heap property** is broken again..., need to fix it.

"Bubble-down" by swapping with

the elder child

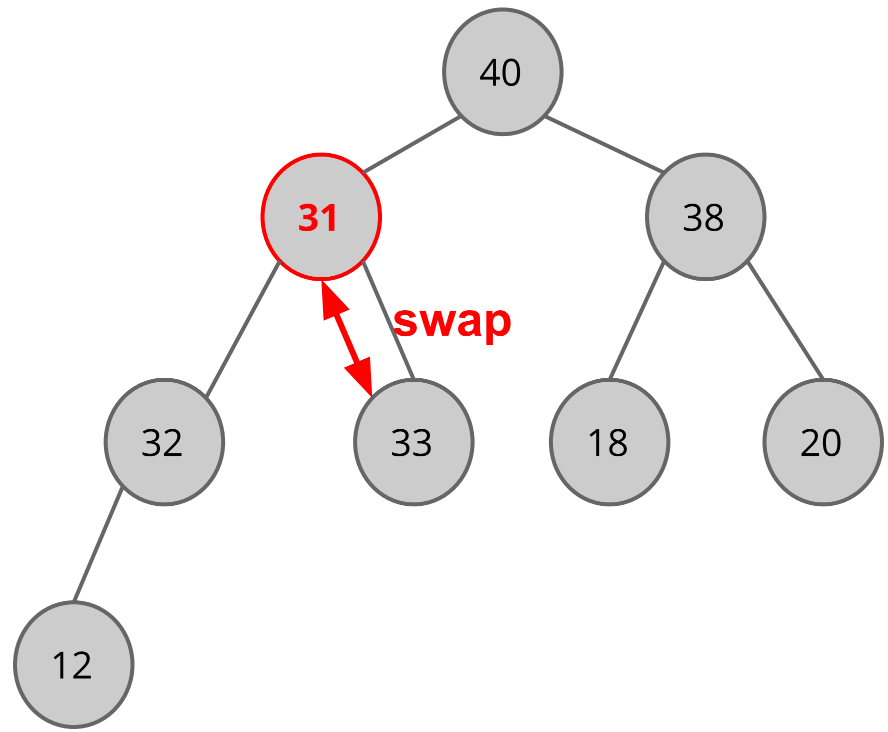


ExtractMax(Q): delete and return the maximum element

Now the **heap property** is broken again..., need to fix it.

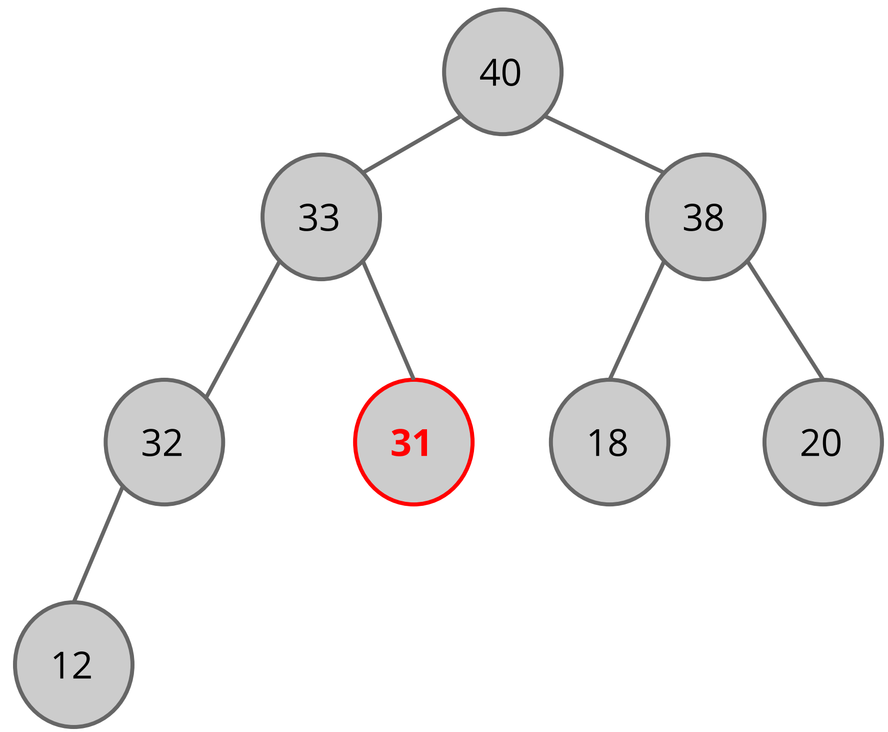
"Bubble-down" by swapping with...

the elder child



ExtractMax(Q): delete and return the maximum element

Now the **heap property** is broken again..., need to fix it.



"Bubble-down" by swapping with

the elder child

Worst case running time: Θ(height) + some constant work

Θ(log n)

Quick summary

Insert(Q, x):

➔ Bubble-up, swapping with parent

ExtractMax(Q)

➔ Bubble-down, swapping elder child

Bubble up/down is also called **percolate** up/down, or **sift** up down, or **tickle** up/down, or **heapify** up/down, or **cascade** up/down.

CSC263 Week 2

Thursday

Announcements

Problem Set 2 is out

➔ due next Tuesday 5:59pm

Additional office hours on Mondays

➔ 4 - 5:30pm (or by appointment)

A quick review of Monday

- Max-Priority Queue implementations
  - unsorted and sorted linked list -- O(1), O(n)
  - binary max-heap -- O(1), O(log n)
    - Max(Q)
    - Insert(Q, x)
      - bubble up - swapping with parent
    - ExtractMax(Q)
      - bubble down - swapping with elder child
    - IncreasePriority(Q, x, k)

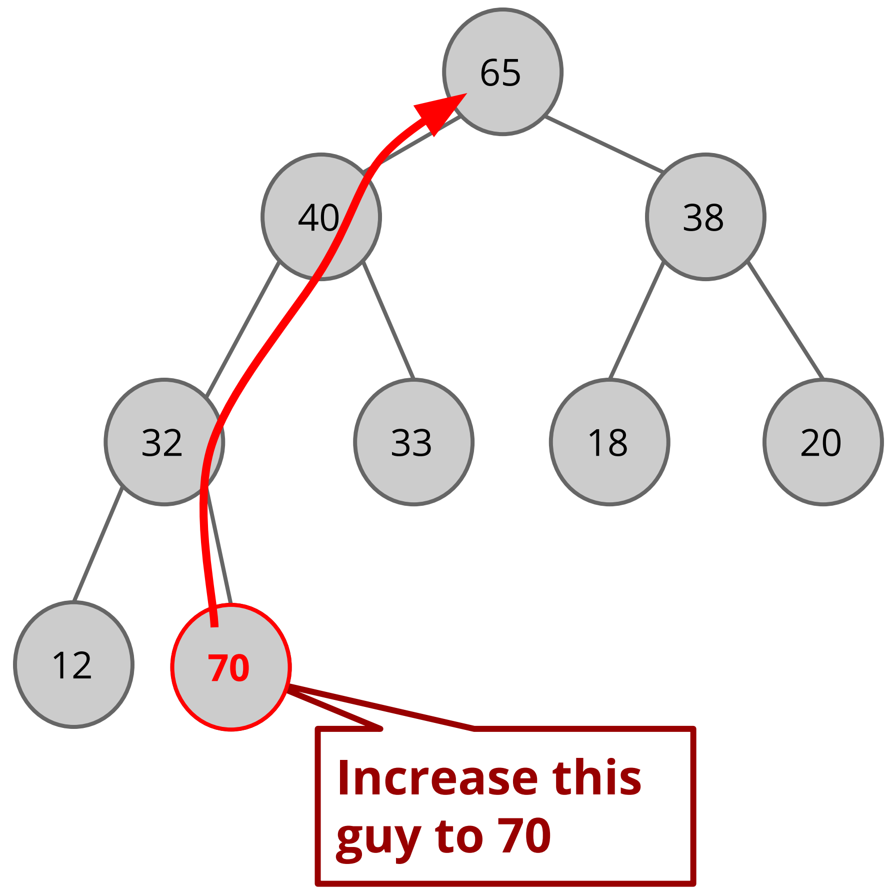
IncreasePriority(Q, x, k)

Increases the key of node x to k,
in O(logn) time

IncreasePriority(Q, x, k): increase the key of node x to k

Just increase the key, then...

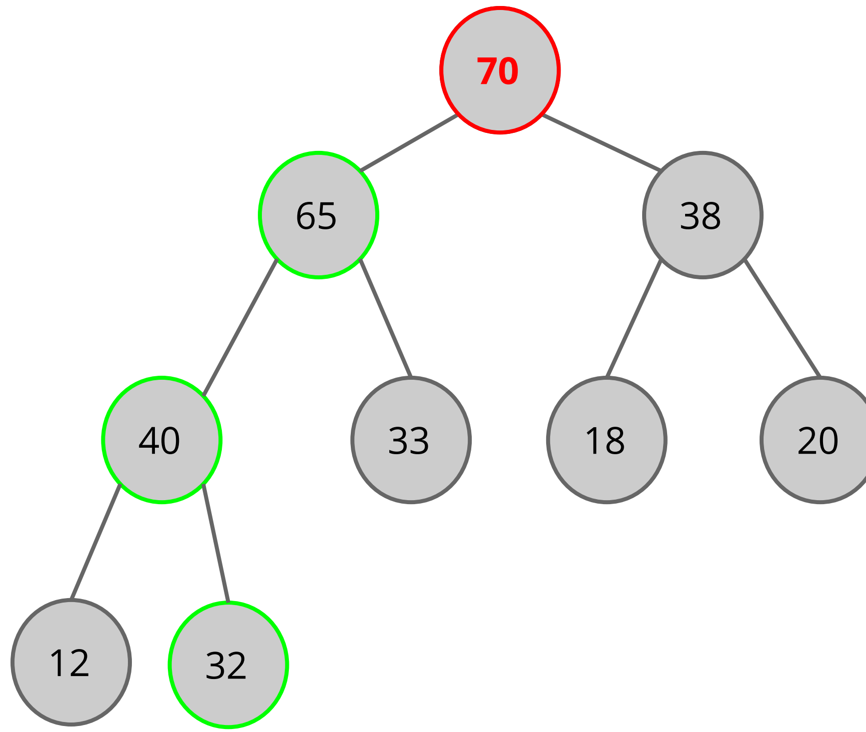
Bubble-up by swapping with parents, to proper location.



IncreasePriority(Q, x, k):
increase the key of node x to k

Just increase the key, then...

Bubble-up by swapping with parents, to proper location.



Worst case running time: Θ(height) + some constant work

Θ(log n)

Now we have learned how implement a priority queue using a heap

- ➔ Max(Q)
- ➔ Insert(Q, x)
- ➔ ExtractMax(Q)
- ➔ IncreasePriority(Q, x, k)

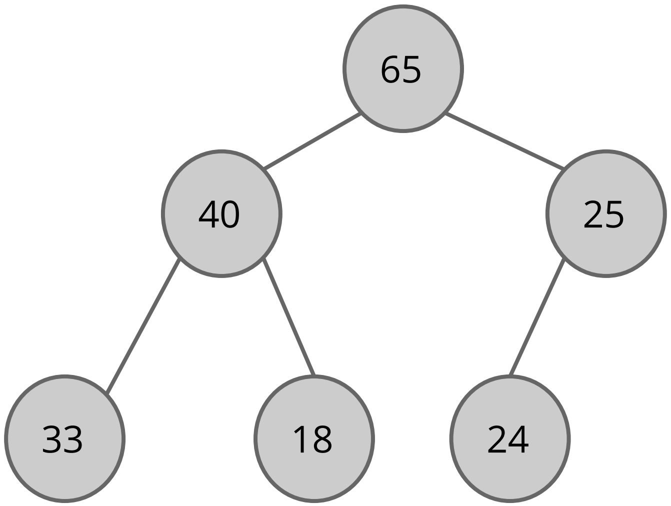
Next:

- ➔ How to use heap for **sorting**
- ➔ How to **build a heap** from an unsorted array

HeapSort

Sorts an array, in $O(n \log n)$ time

The idea



Worst-case running time: each ExtractMax is **O(log n)**, we do it **n** times, so overall it's...

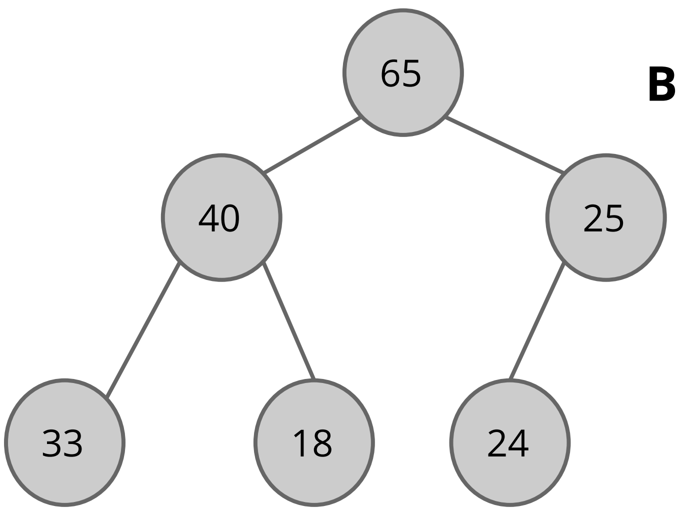
O(n logn)

How to get a sorted list out of a heap with n nodes?

Keep extracting max for n times, the keys extracted will be sorted in non-ascending order.

Now let's be more precise

What's needed: modify a max-heap-ordered **array** into a **non-descendingly** sorted **array**



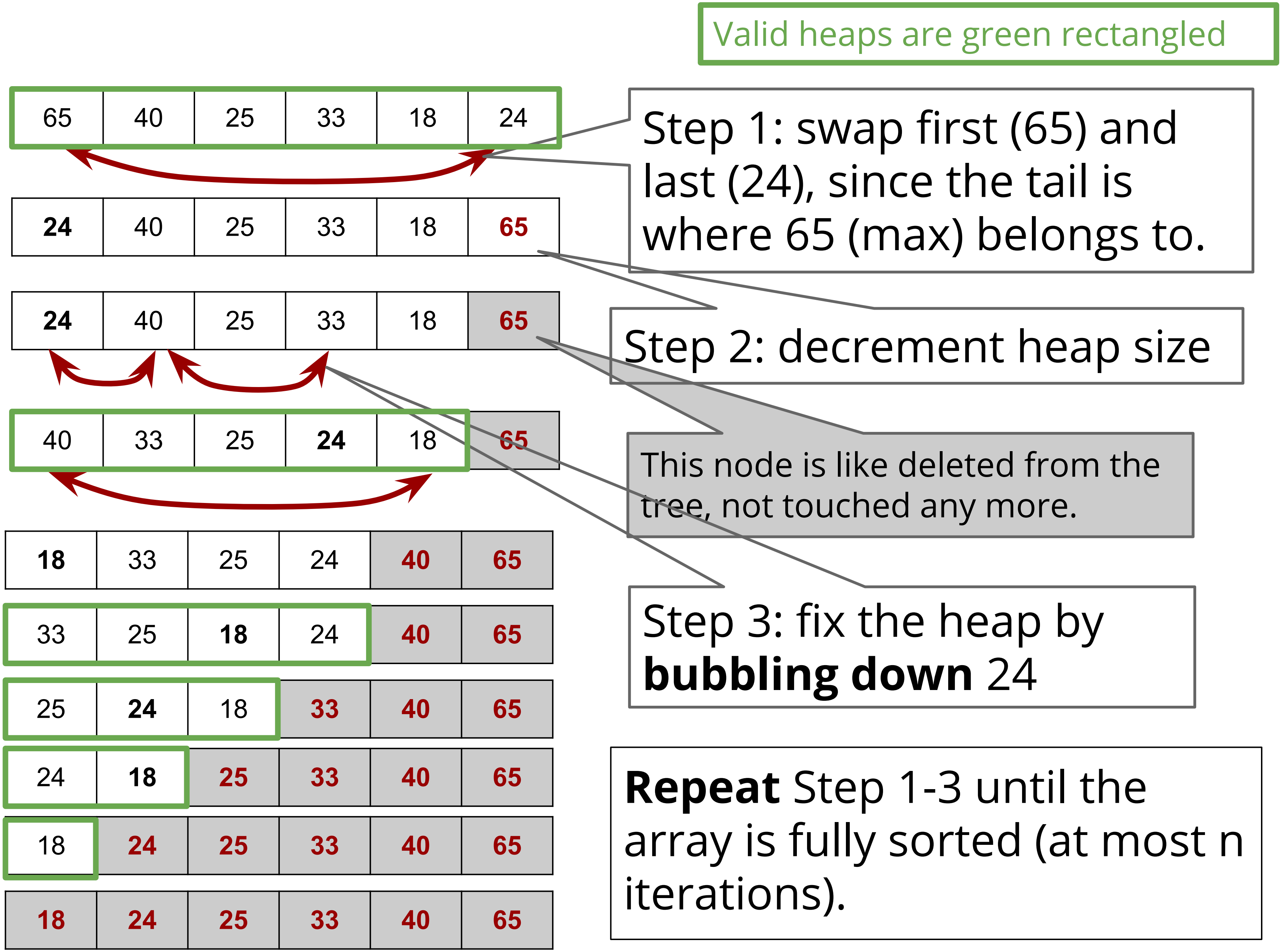
Before:

65	40	25	33	18	24

After:

18	24	24	25	40	65

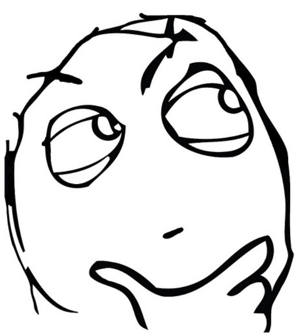
We want to do this "**in-place**" without using any extra array space, i.e., just by **swapping** things around.


Valid heaps are green rectangled
65 40 25 33 18 24
24 40 25 33 18 65
24 40 25 33 18 65
40 33 25 24 18 65
18 33 25 24 40 65
33 25 18 24 40 65
25 24 18 33 40 65
24 18 25 33 40 65
18 24 25 33 40 65
18 24 25 33 40 65
Step 1: swap first (65) and last (24), since the tail is where 65 (max) belongs to.
Step 2: decrement heap size
This node is like deleted from the tree, not touched any more.
Step 3: fix the heap by **bubbling down** 24
Repeat Step 1-3 until the array is fully sorted (at most n iterations).

HeapSort, the pseudo-code

```
HeapSort(A)
'''sort any array A into non-descending order '''
    BuildMaxHeap(A) # convert any array A into a heap-ordered one
    for i ← A.size downto 2:
        swap A[1] and A[i]  # Step 1: swap the first and the last
        A.size ← A.size - 1  # Step 2: decrement size of heap
        BubbleDown(A, 1)  # Step 3: bubble down the 1st element in A
```

Missing!



Does it work?

It works for an array A that is initially heap-ordered, it does work NOT for any array!

BuildMaxHeap(A)

Converts an array into a max-heap ordered array, in O(n) time

Convert any array into a heap ordered one

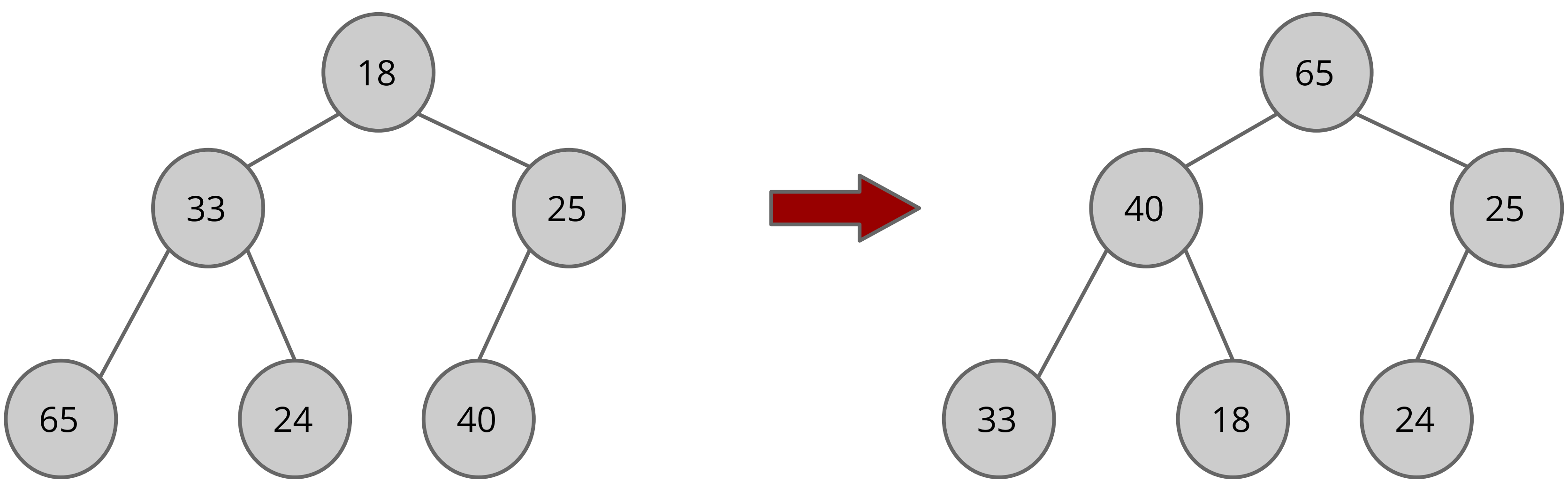
any array

18	33	25	65	24	40

heap ordered array

65	40	25	33	18	24

In other words...



Idea #1

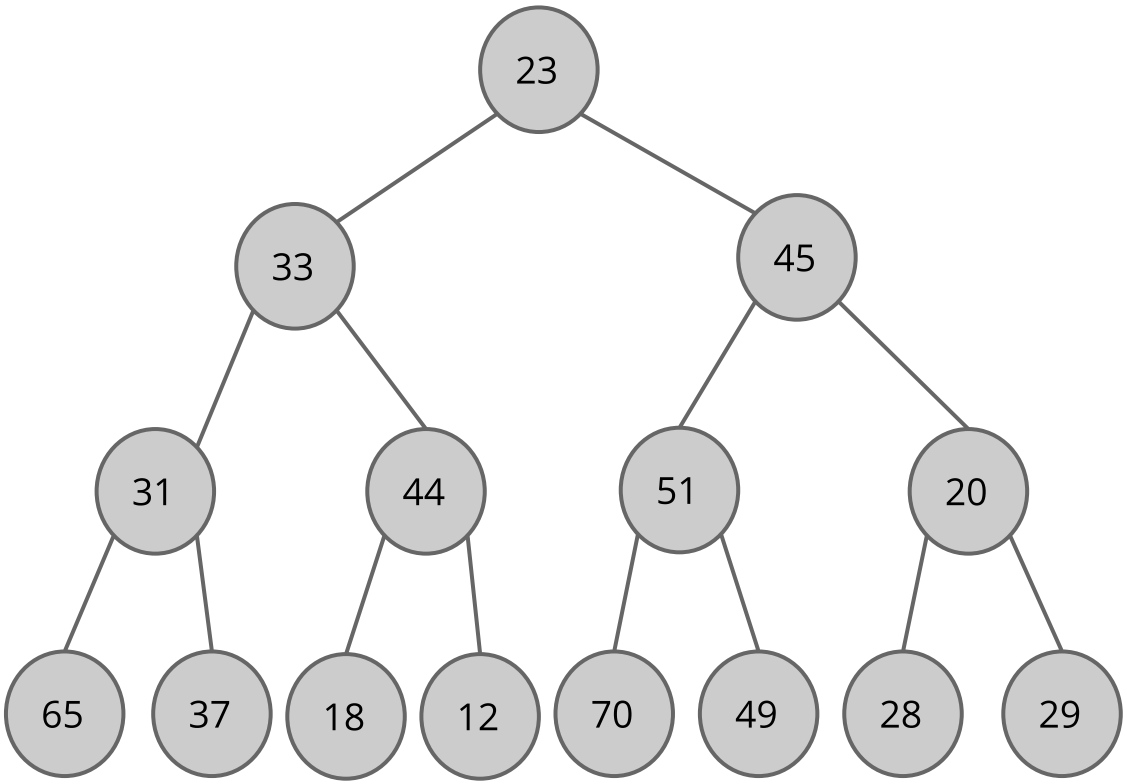
```
BuildMaxHeap(A):

    B ← empty array  # empty heap
    for x in A:
        Insert(B, x)  # heap insert
    A ← B     # overwrite A with B
```

Running time:
Each Insert takes **O(log n)**, there are **n** inserts...
so it's **O(n log n)**, not very exciting.
Not **in-place**, needs a second array.


WHAT IF I TOLD YOU
YOU CAN DO BETTER THAN THIS
imgflip.com

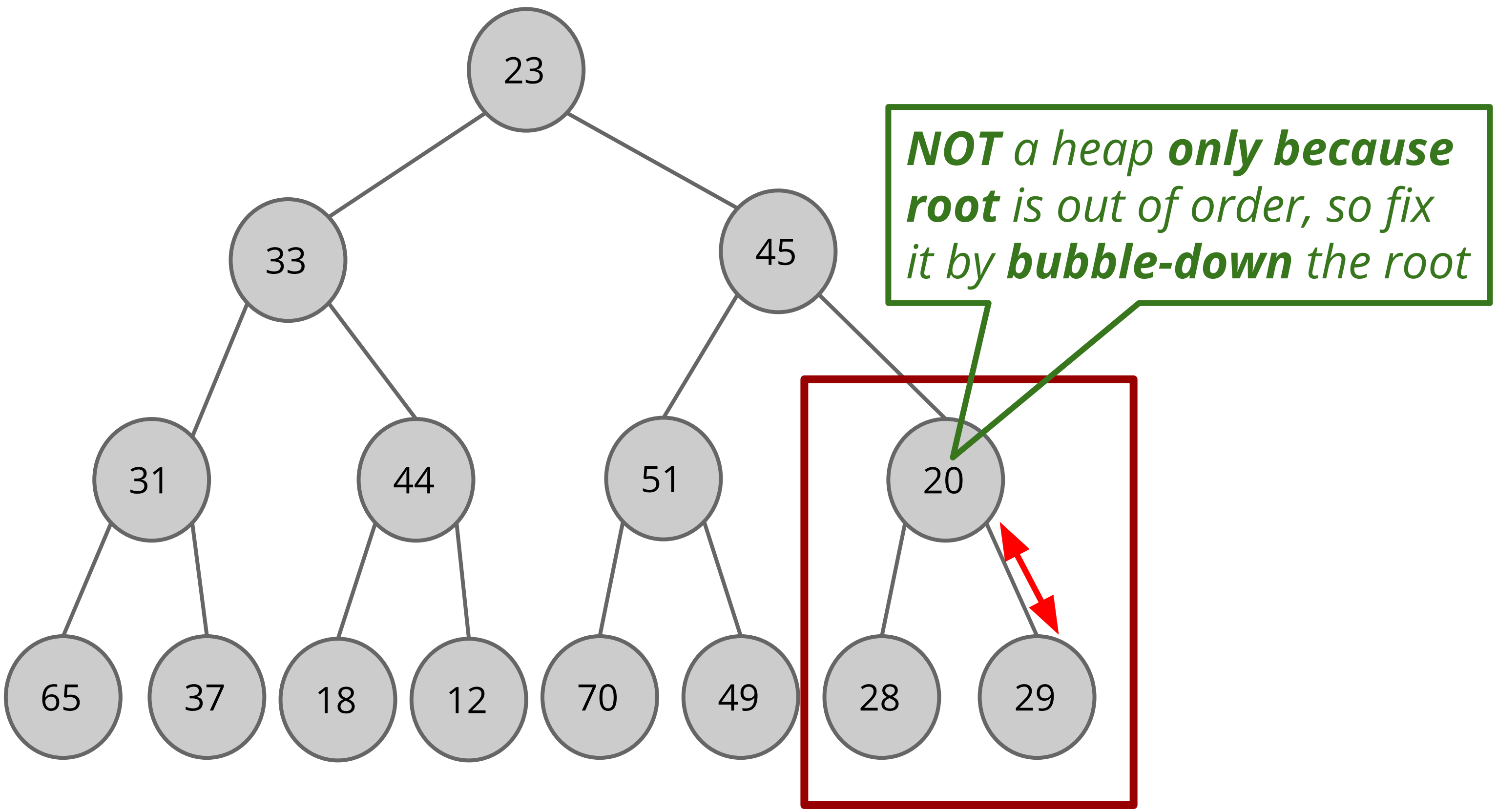
Idea #2

Fix heap order, from bottom up.



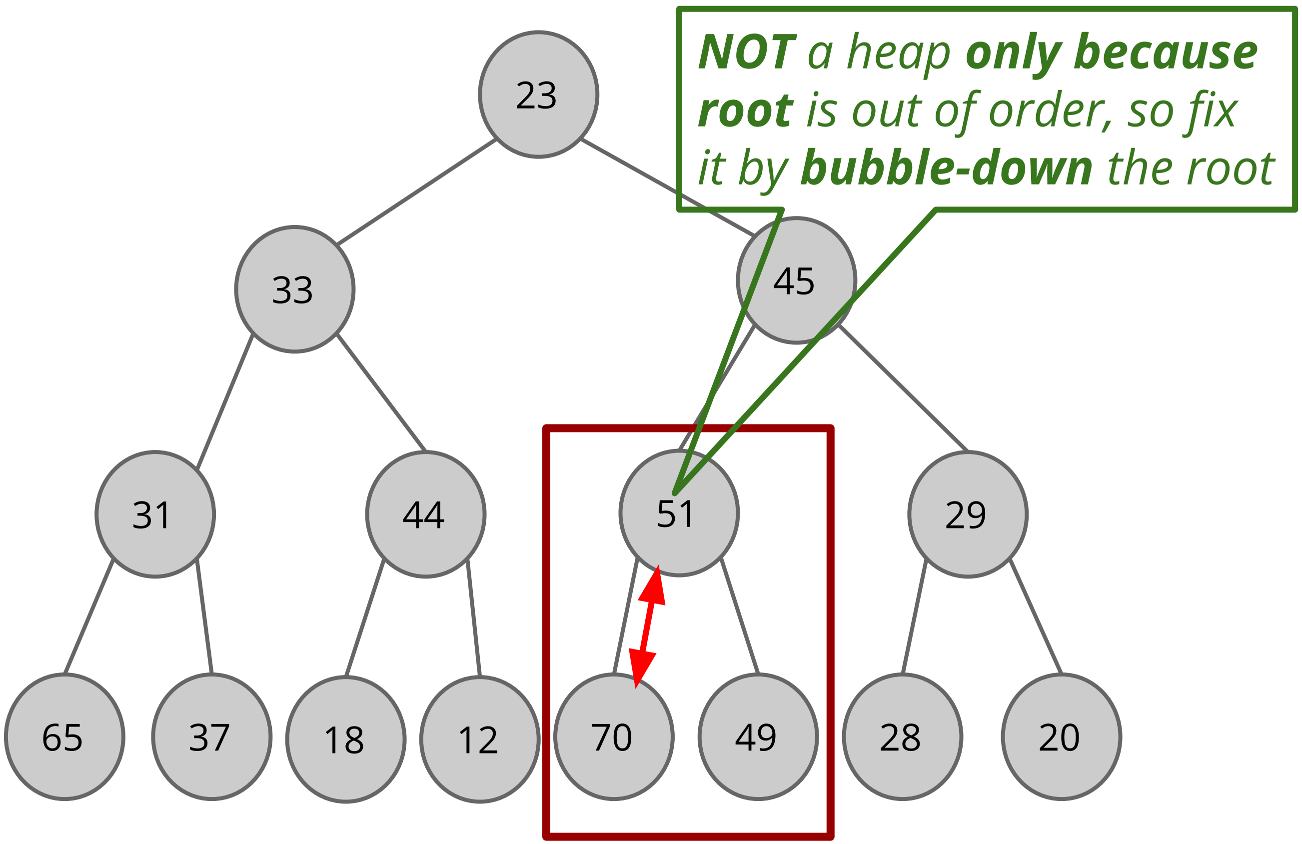
Idea #2

Adjust heap order, from bottom up.



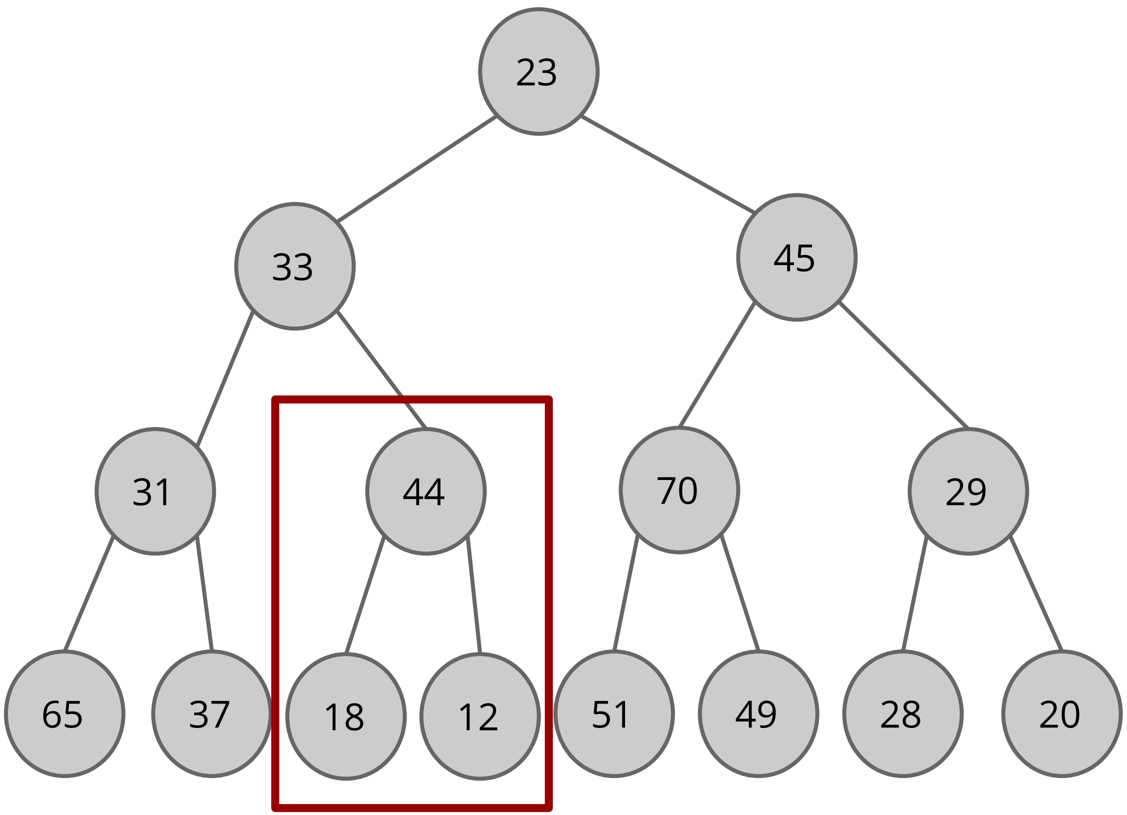
Idea #2

Adjust heap order, from bottom up.



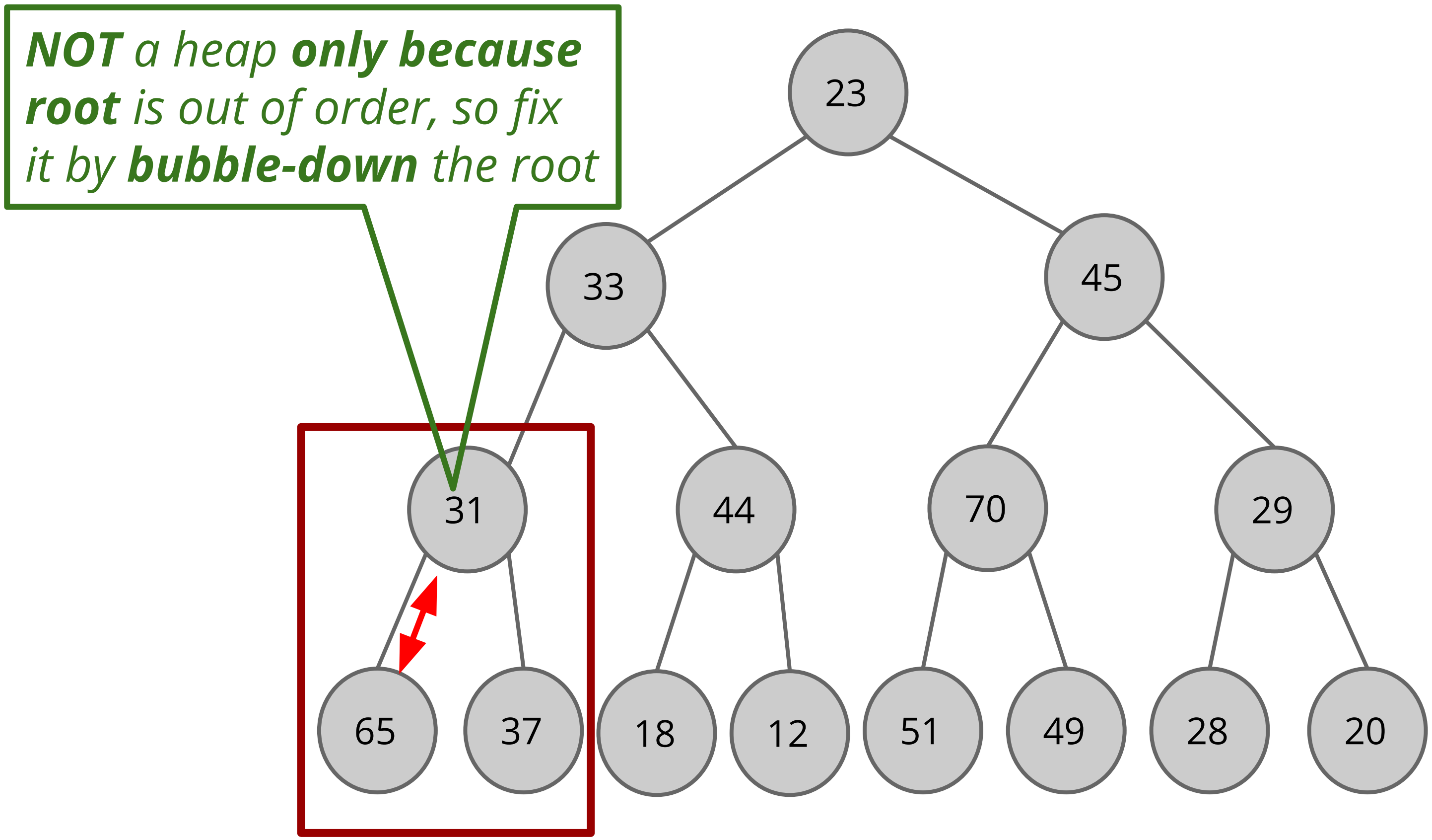
Idea #2

Adjust heap order, from bottom up.



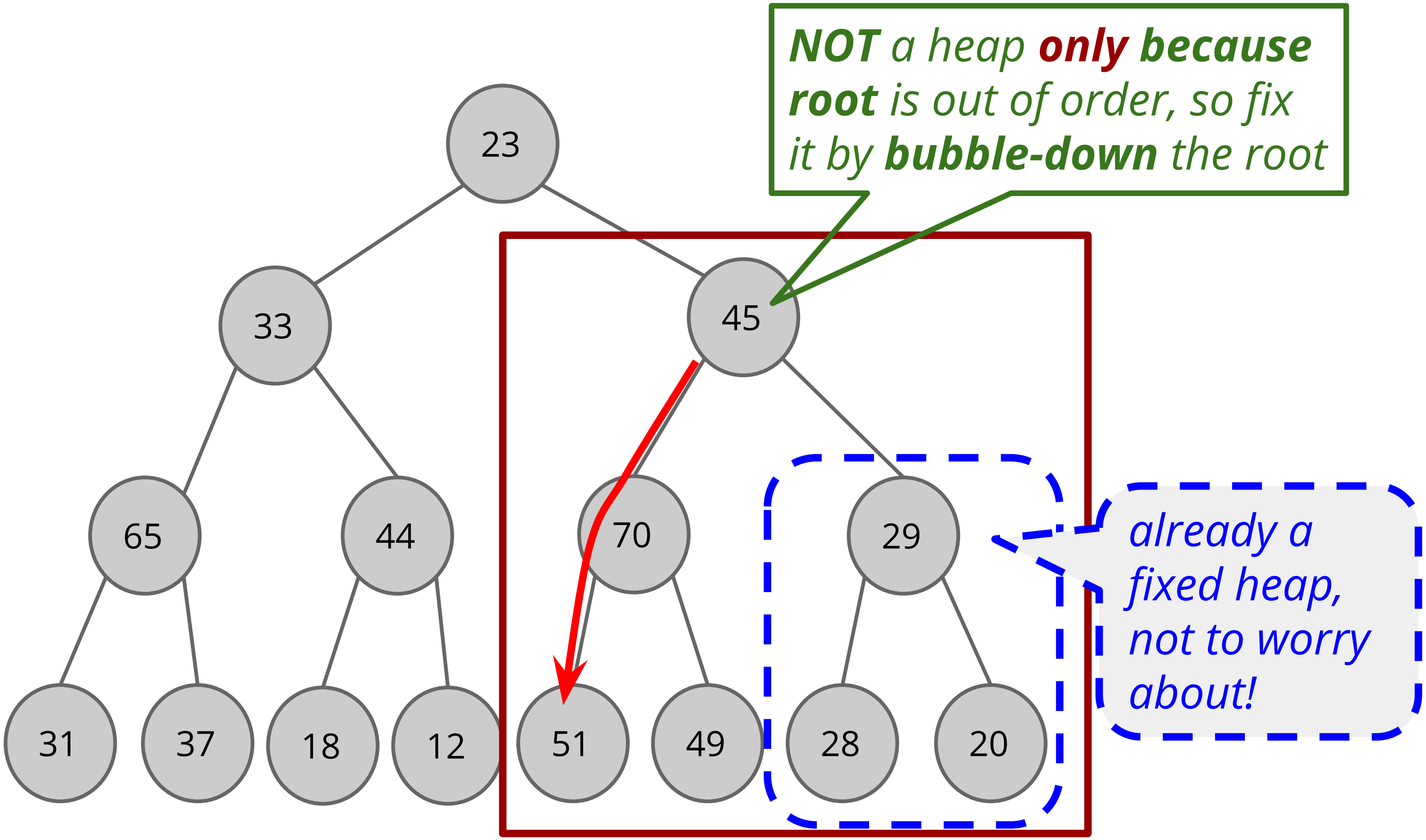
Idea #2

Adjust heap order, from bottom up.



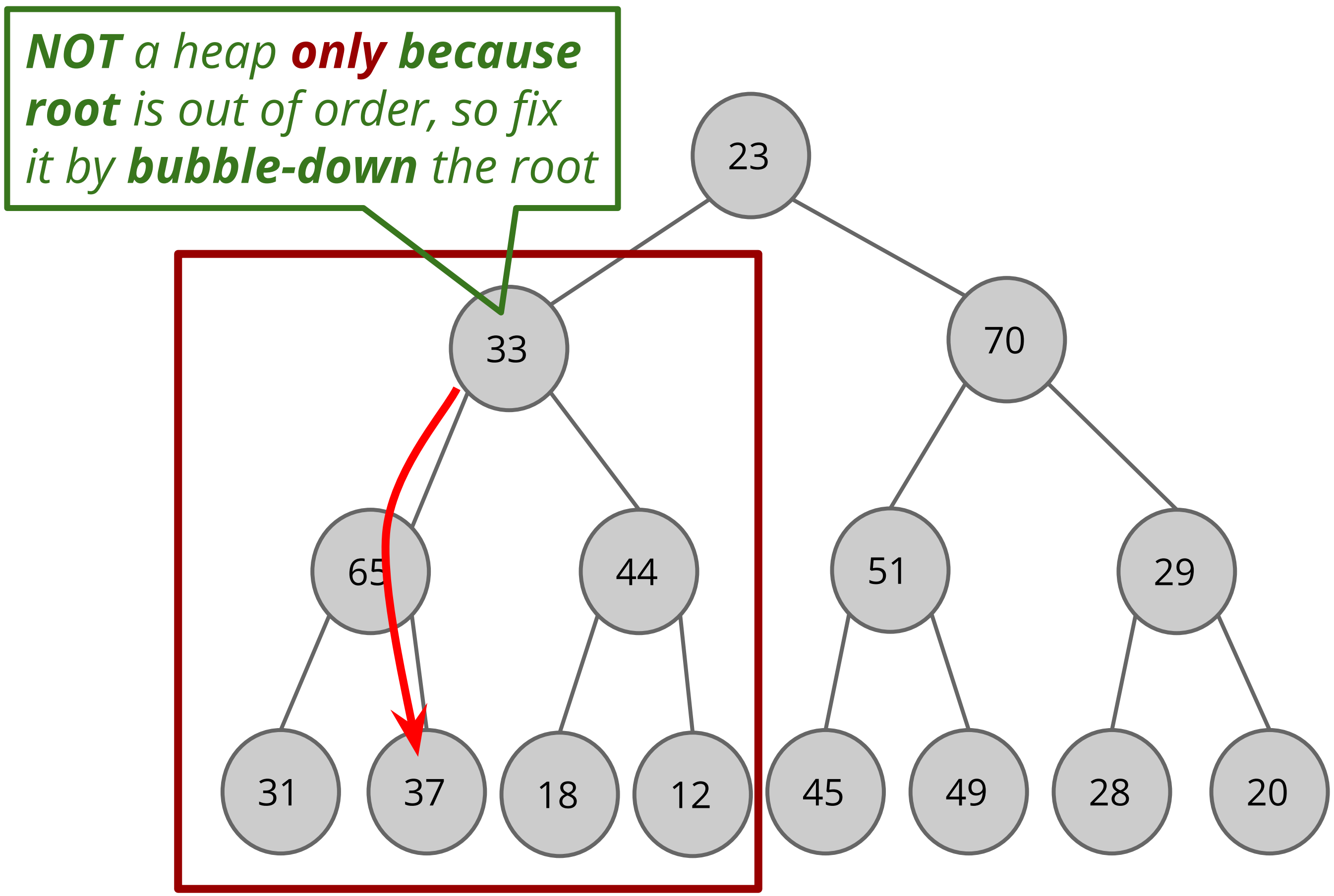
Idea #2

Adjust heap order, from bottom up.



Idea #2

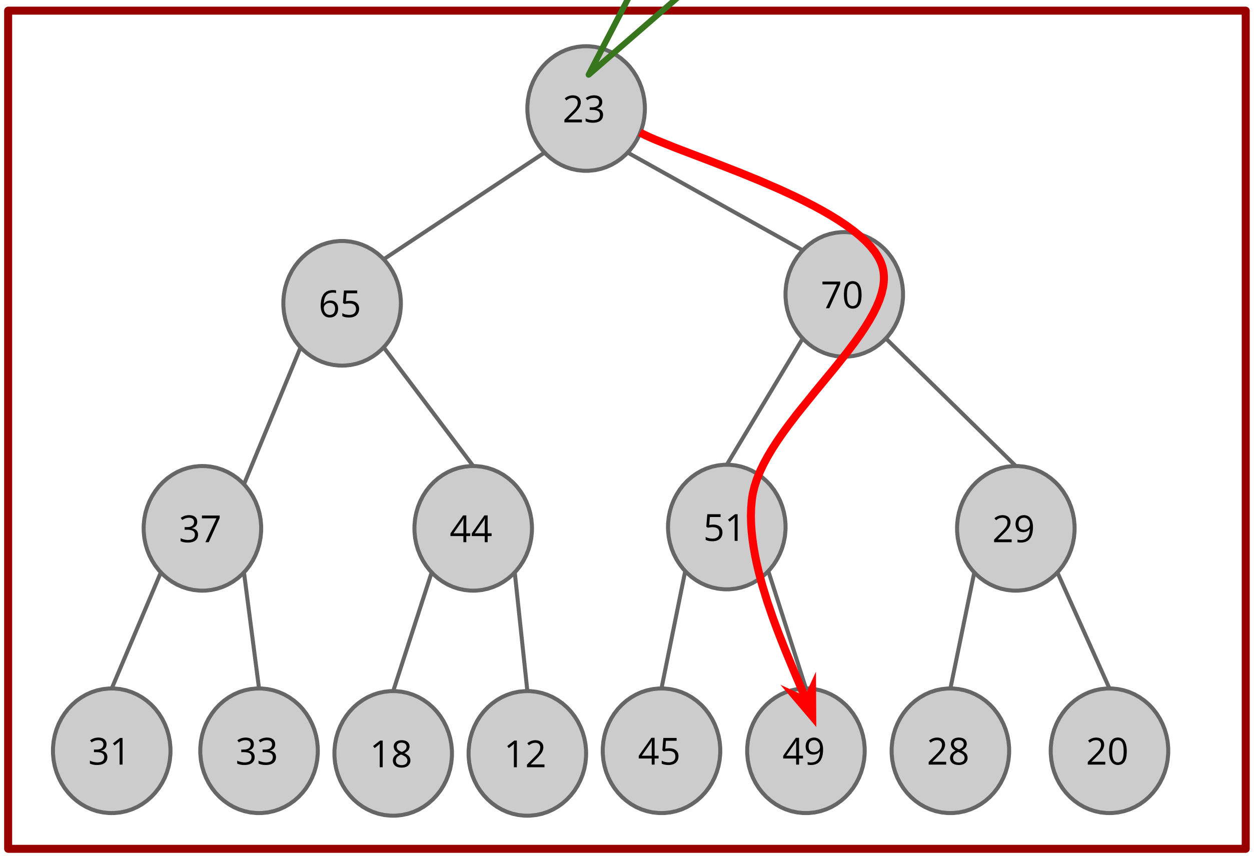
Adjust heap order, from bottom up.



Idea #2

Adjust heap order, from bottom up.

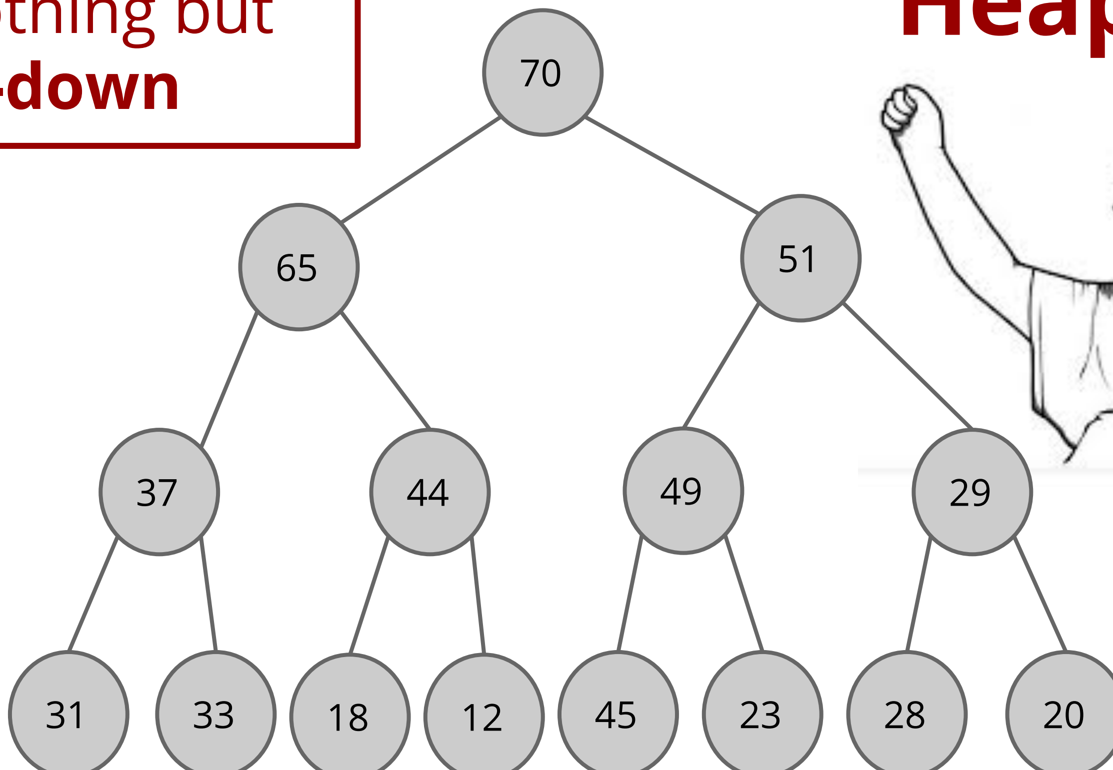
***NOT** a heap **only because root** is out of order, so fix it by **bubble-down** the root*



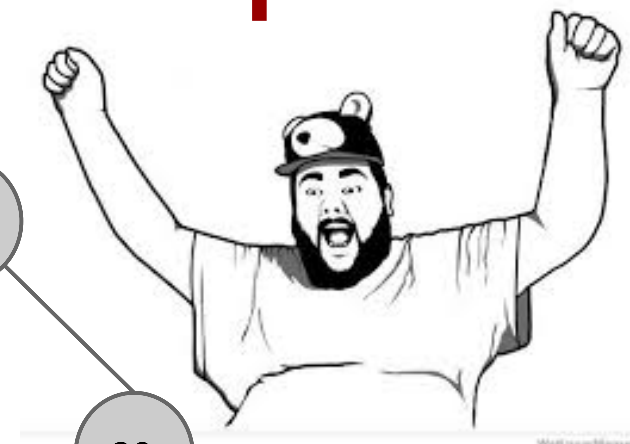
Idea #2

Adjust heap order, from bottom up.

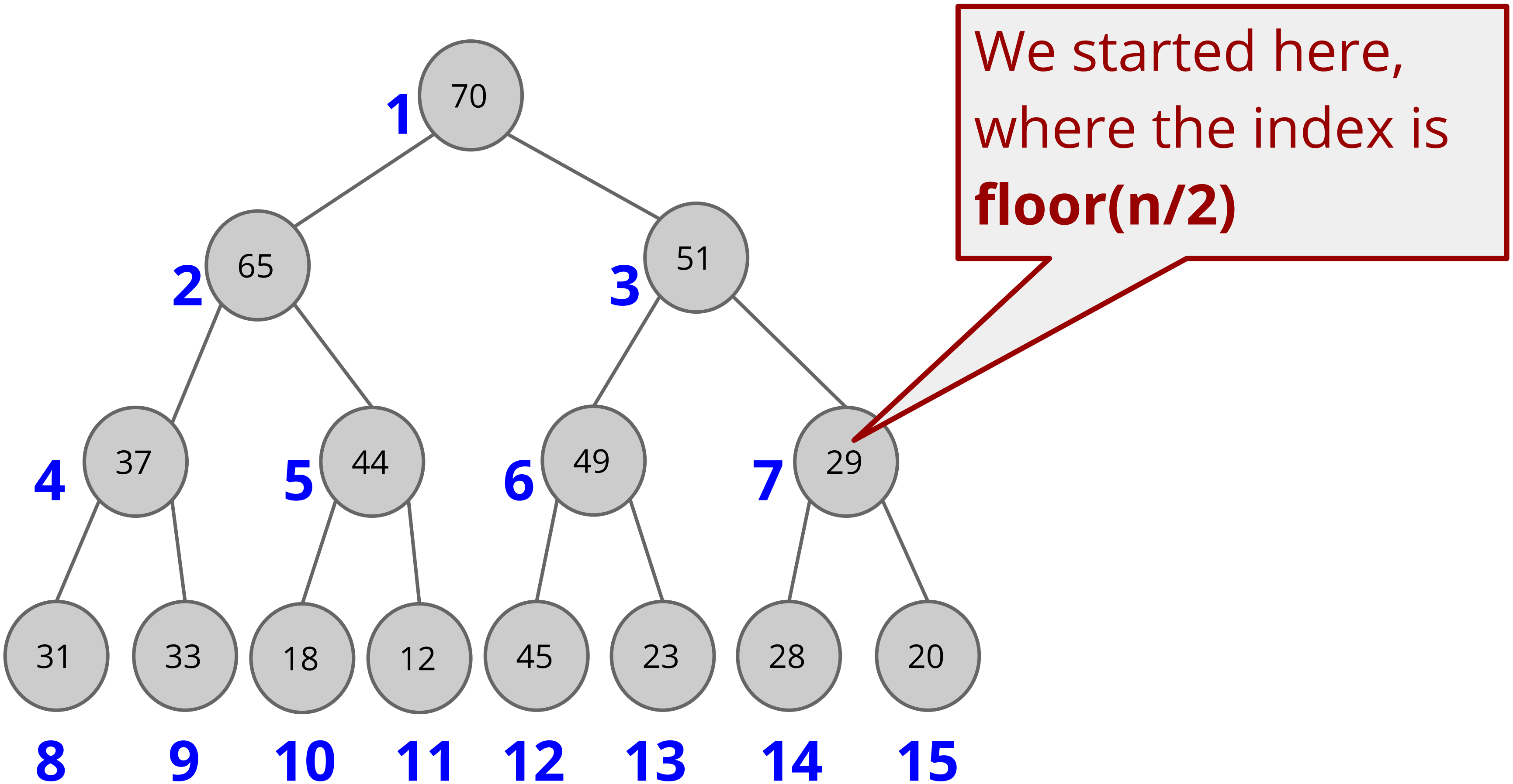
We did nothing but **bubbling-down**



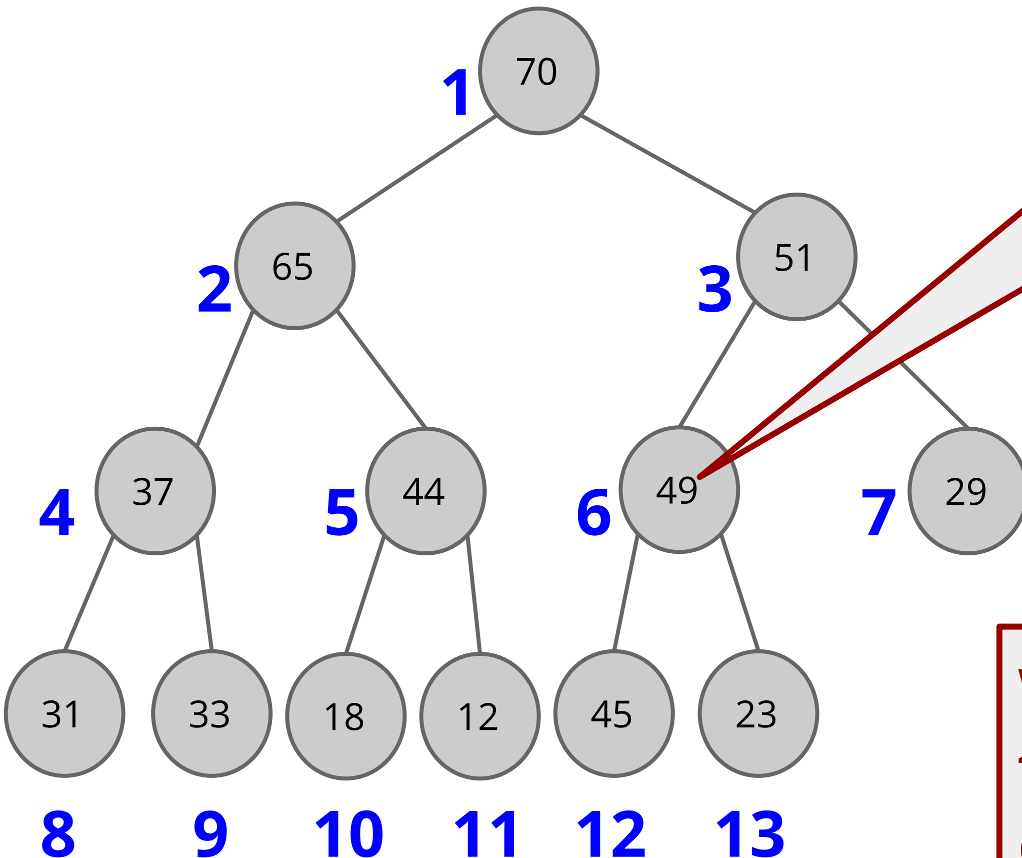
Heap Built!



Idea #2: The starting index



Idea #2: The starting index



Even the bottom level is not fully filled, we still start from **floor(n/2)**

We **always** start from **floor(n/2)**, and go down to **1**.

Idea #2: Pseudo-code!

```
BuildMaxHeap(A):

   for i ← floor(n/2) downto 1:
      BubbleDown(A, i)
```

Advantages of Idea #2:

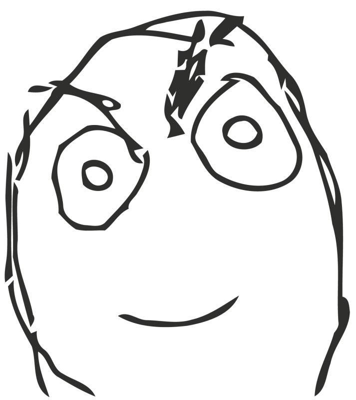
➔ It's in-place, no need for extra array (we did nothing but bubble-down, which is basically swappings).

➔ It's worst-case running time is **O(n)**, instead of O(n log n) of Idea #1.

Why?

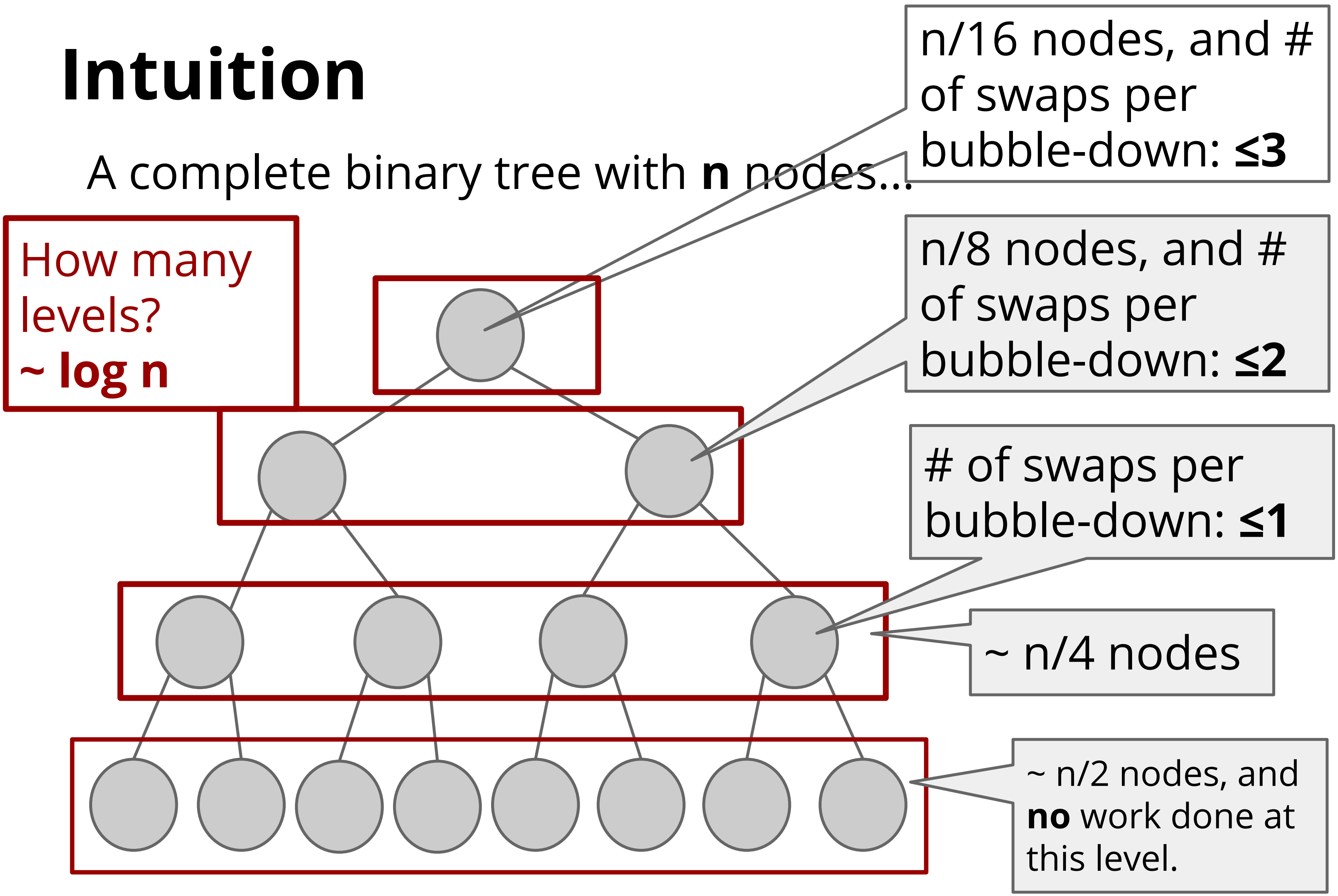
Analysis:

Worst-case running time of BuildMaxHeap(A)



Intuition

A complete binary tree with **n** nodes…



So, total number of swaps

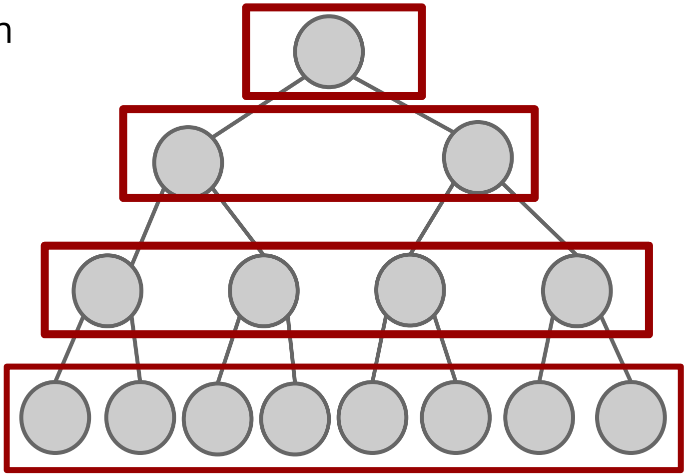
$$T(n) = 1 \cdot \frac{n}{4} + 2 \cdot \frac{n}{8} + 3 \cdot \frac{n}{16} + \dots$$

$$= \sum_{i=1}^{\log n} i \cdot \frac{n}{2^{i+1}} \leq \sum_{i=1}^{+\infty} i \cdot \frac{n}{2^{i+1}}$$

$$= n \boxed{\sum_{i=1}^{+\infty} \frac{i}{2^{i+1}}} = 1$$

same trick as
Week 1's sum

$$= n$$




BUILD MAX HEAP
YOU CAN DO IN LINEAR TIME
imgflip.com

Summary

HeapSort(A):

- ➔ Sort an unsorted array in-place
- ➔ O(n log n) worst-case running time

BuildMaxHeap(A):

- ➔ Convert an unsorted array into a heap, in-place
- ➔ Fix heap property from bottom up, do bubbling down on each sub-root
- ➔ O(n) worst-case running time

Algorithm visualizer

http://visualgo.net/heap.html

Next week

- ➔ ADT: Dictionary
- ➔ Data structure: Binary Search Tree