CSC263 Week 8

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http://goo.gl/forms/S9yie3597B

Announcements (strike related)

- ➔ Lectures go as normal
- ➔ Tutorial this week
  - everyone go to BA3012 (T8, F12, F2, F3)
- ➔ Problem sets / Assignments are submitted as normal.
  - marking may be slower
- ➔ Midterm: still being marked
- ➔ **Keep a close eye on announcements**

This week’s outline

- ➔ Graph
- ➔ BFS

Graph

A really, really important ADT that is used to model **relationships** between objects.

Get that job at Google

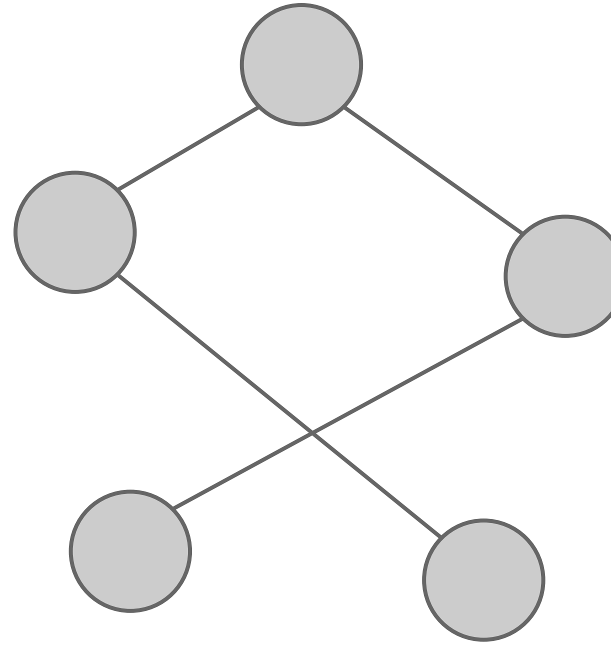
Whenever someone gives you a problem, *think* graphs. They are the most fundamental and flexible way of representing any kind of a relationship, so it's about a 50-50 shot that any interesting design problem has a graph involved in it. Make absolutely sure you can't think of a way to solve it using graphs before moving on to other solution types. This tip is important!

Reference: *http://steve-yegge.blogspot.ca/2008/03/get-that-job-at-google.html*

Things that can be modelled using graphs

- ➔ Web
- ➔ Facebook
- ➔ Task scheduling
- ➔ Maps & GPS
- ➔ Compiler (garbage collection)
- ➔ OCR (computer vision)
- ➔ Database
- ➔ Rubik’s cube
- ➔ .... (many many other things)

Definition



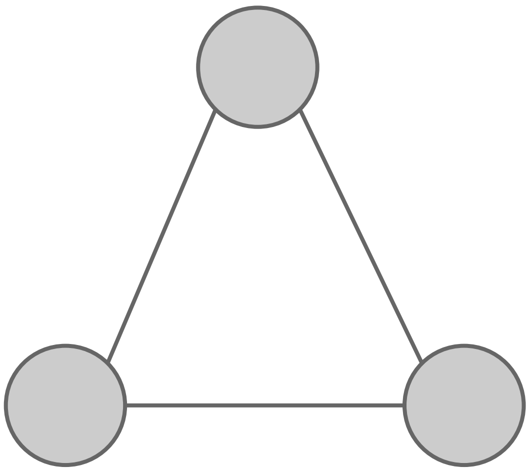
$$G = (V, E)$$

Set of **vertices**
e.g., {a, b, c}

Set of **edges**
e.g., { (a, b), (c, a) }

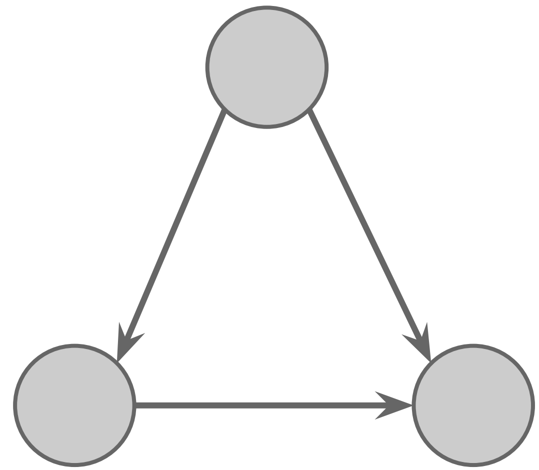
Flavours of graphs

each edge is an
unordered pair
(u, v) = (v, u)

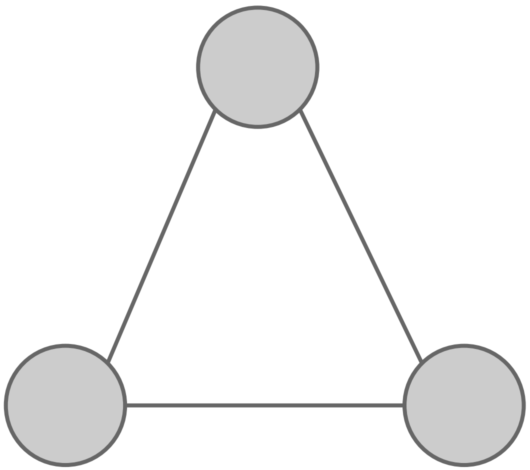


Undirected

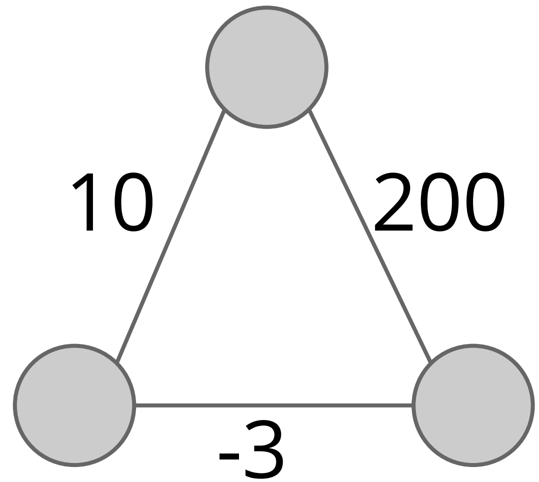
each edge is an
ordered pair
(u, v) ≠ (v, u)



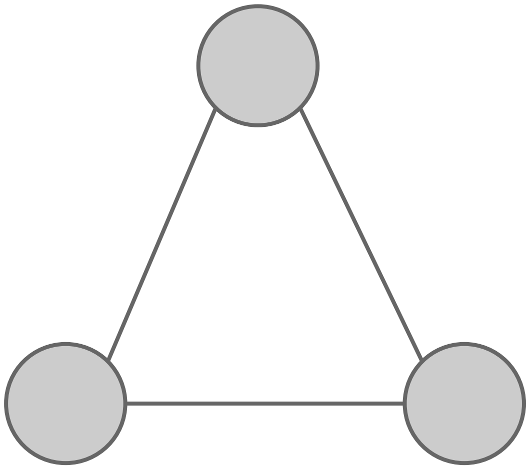
Directed



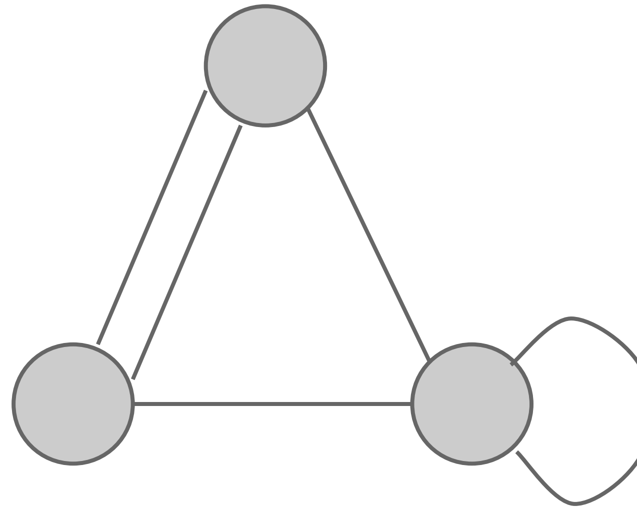
Unweighted



Weighted

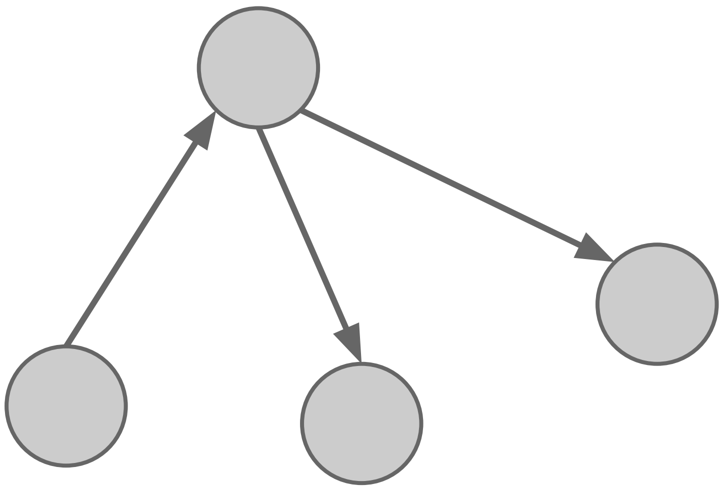


Simple

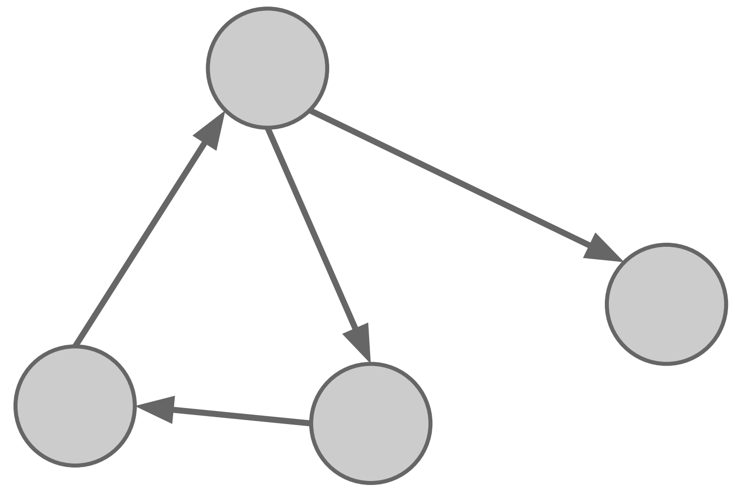


Non-simple

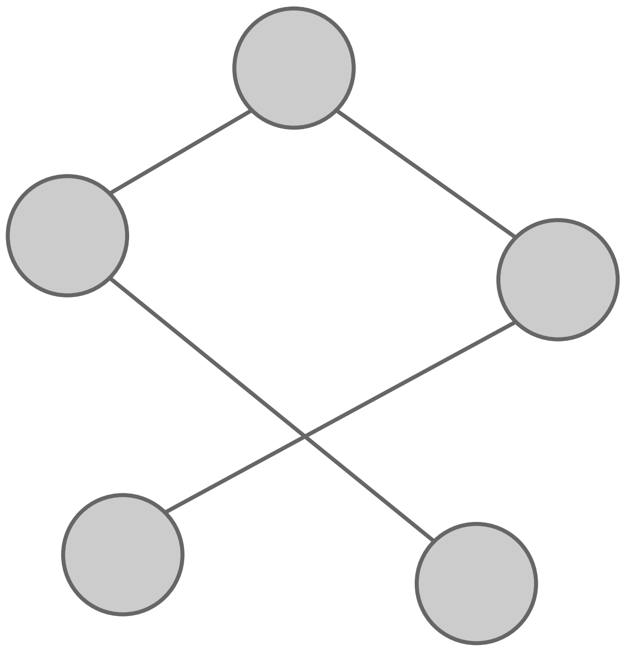
No multiple edge, no self-loop



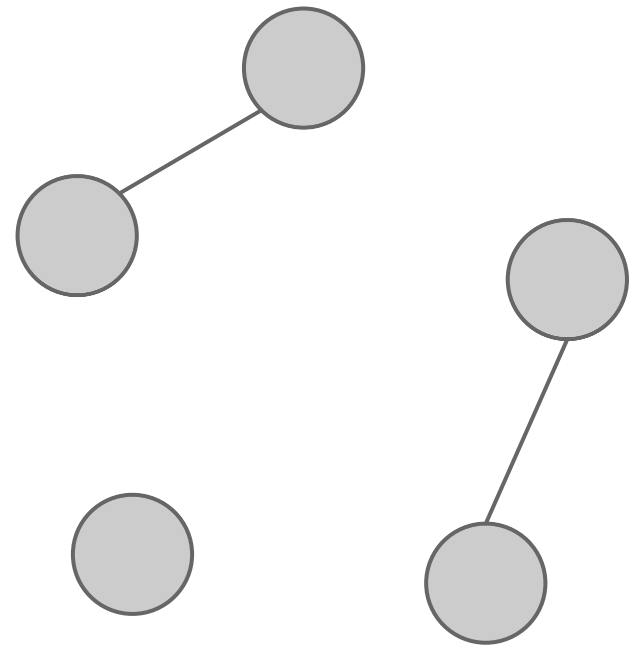
Acyclic



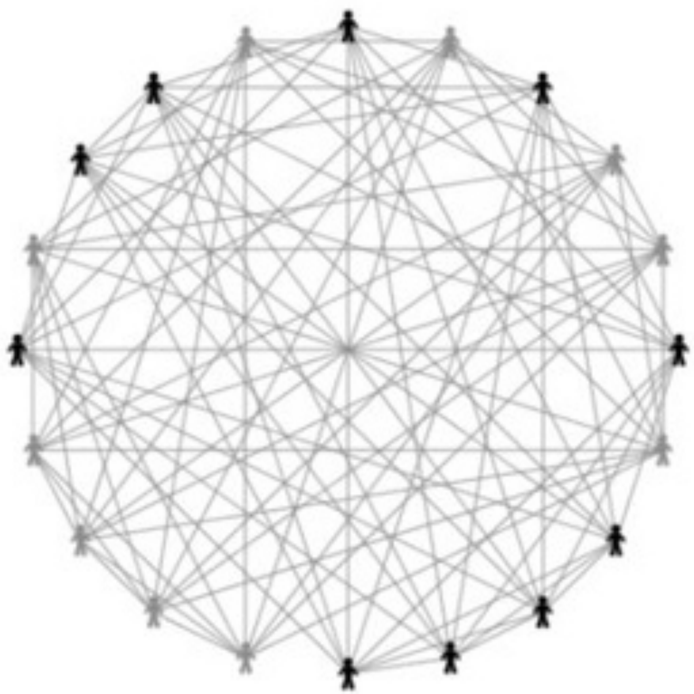
Cyclic



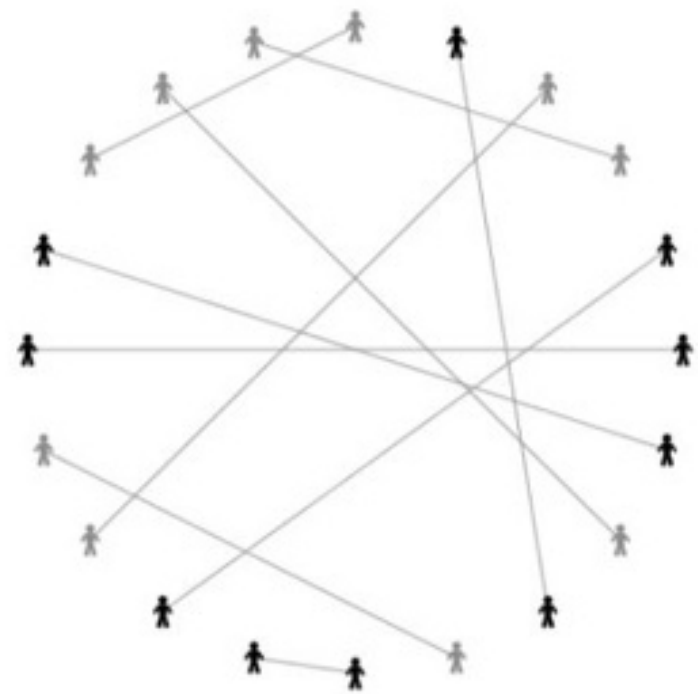
Connected



Disconnected

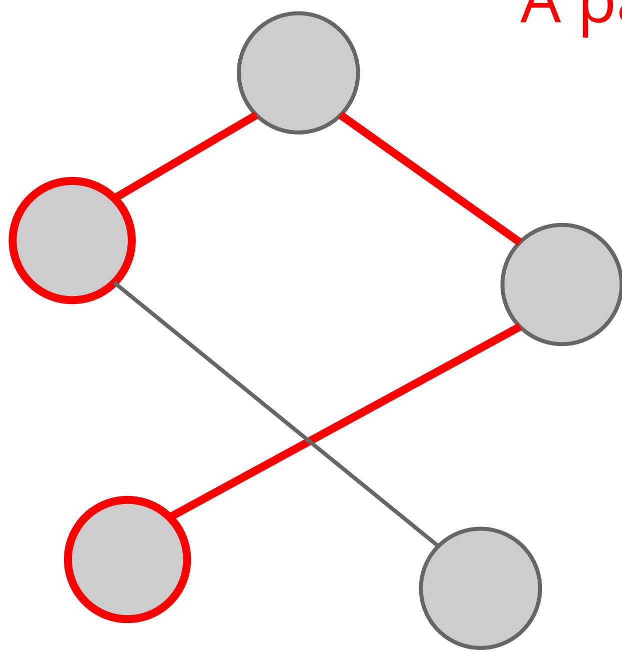


Dense



Sparse

Path



Length of path = number of edges

Read Appendix B.4 for more background on graphs.

Operations on a graph

- **Add** a vertex; **remove** a vertex
- **Add** an edge; **remove** an edge
- Get **neighbours** (undirected graph)
  - Neighbourhood(u): all $v \in V$ such that $(u, v) \in E$
- Get **in**-neighbours / **out**-neighbours (directed graph)
- **Traversal**: visit every vertex in the graph

Data structures for the graph ADT

- ➔ Adjacency **matrix**
- ➔ Adjacency **list**

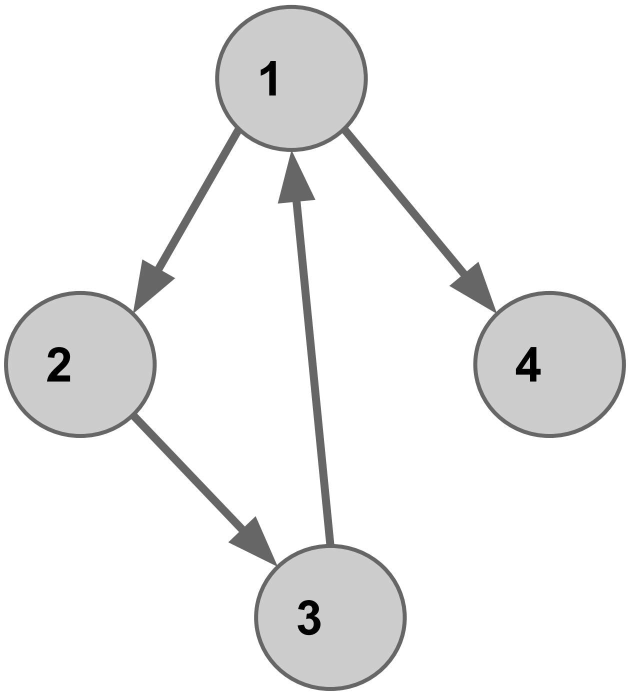
Adjacency matrix

A **|V|x|V|** matrix **A**

$$\text{Let } V = \{v_1, v_2, \ldots, v_n\}$$

$$A[i,j] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Adjacency matrix



	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	1	0	0	0
4	0	0	0	0

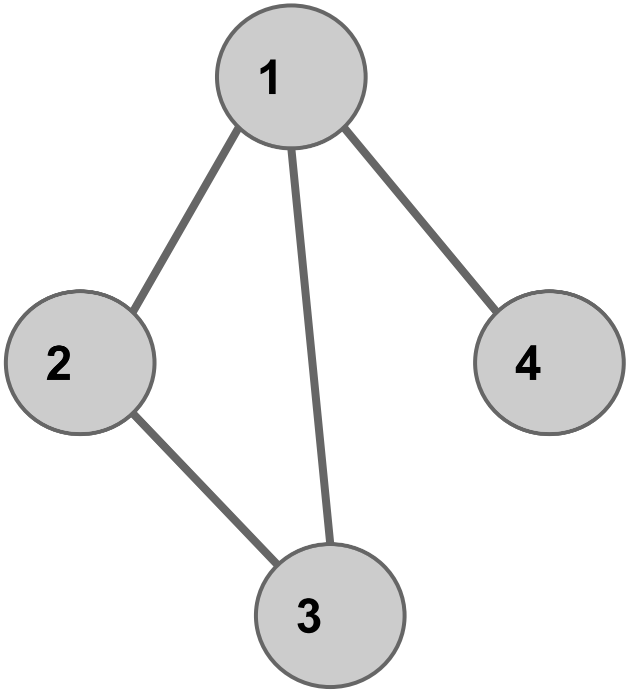
Adjacency matrix

How much space does it take?

$$|V|^2$$

	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	1	0	0	0
4	0	0	0	0

Adjacency matrix (undirected graph)



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	0
4	1	0	0	0

The adjacency matrix of an **undirected** graph is __symmetric__.

Adjacency matrix (undirected graph)

How much space does it take?

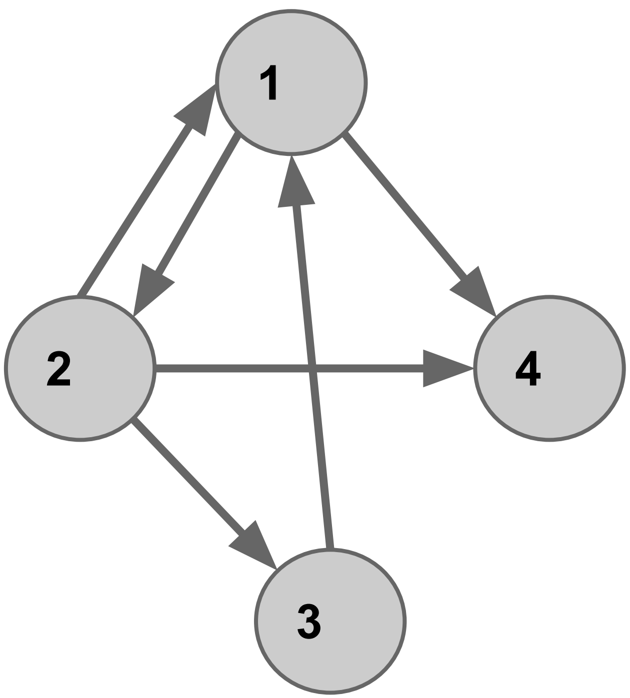
$$|V|^2$$

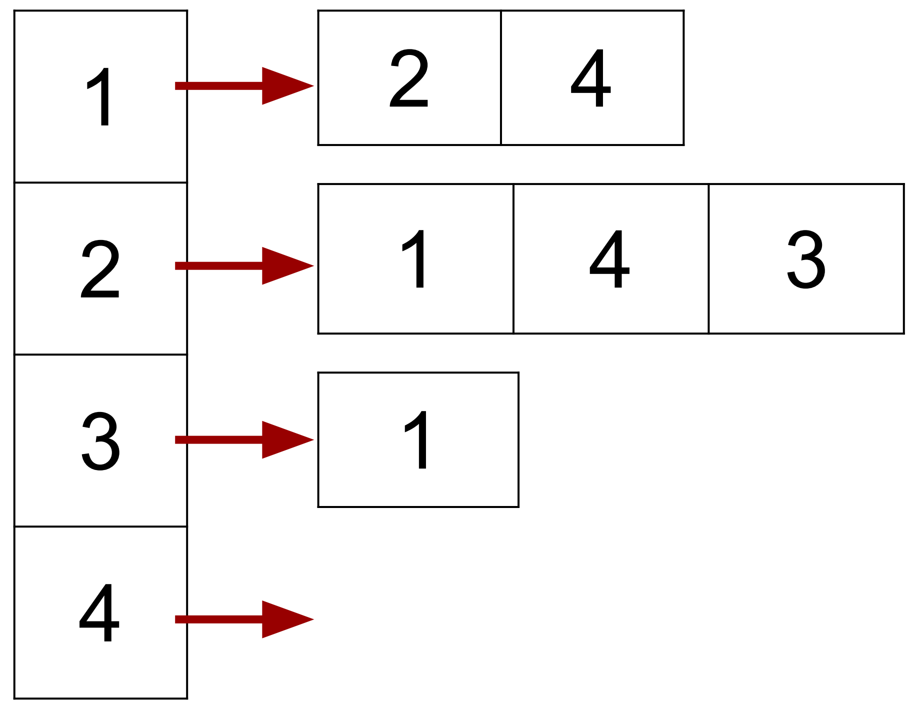
	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	0
4	1	0	0	0

Adjacency list

Adjacency list (directed graph)

Each vertex ***vi*** stores a list **A[i]** of ***vj*** that satisfies ***(vi, vj)*** $\in$ ***E***

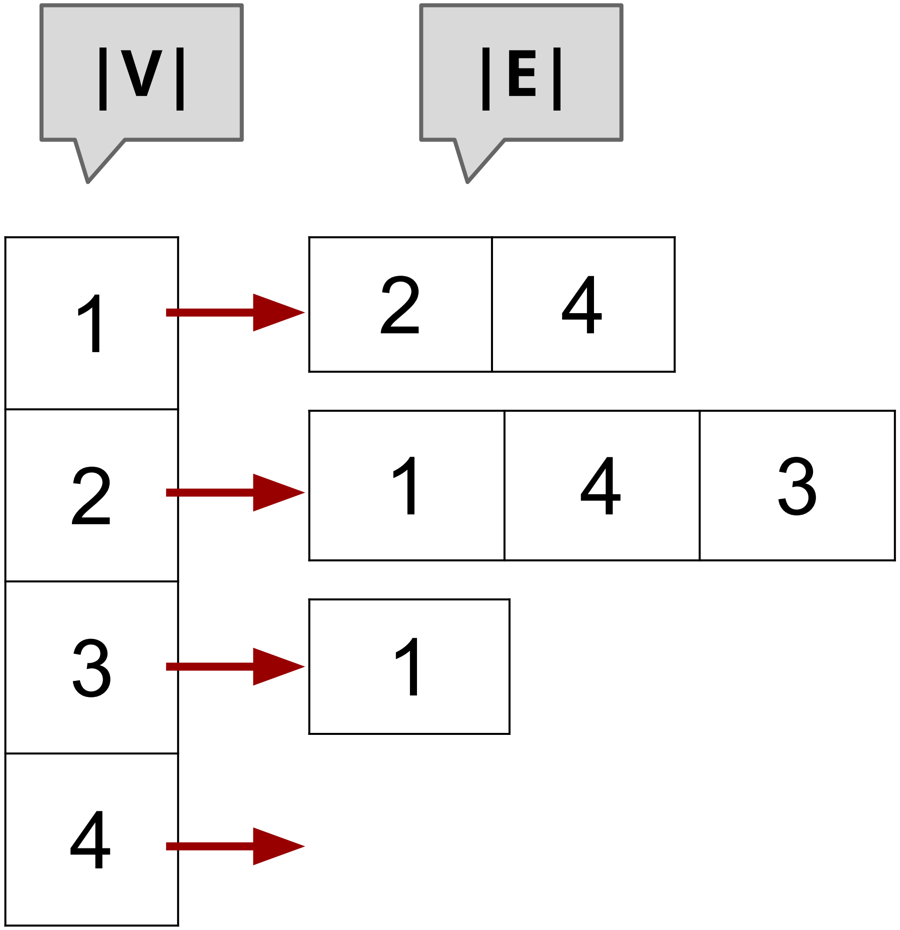




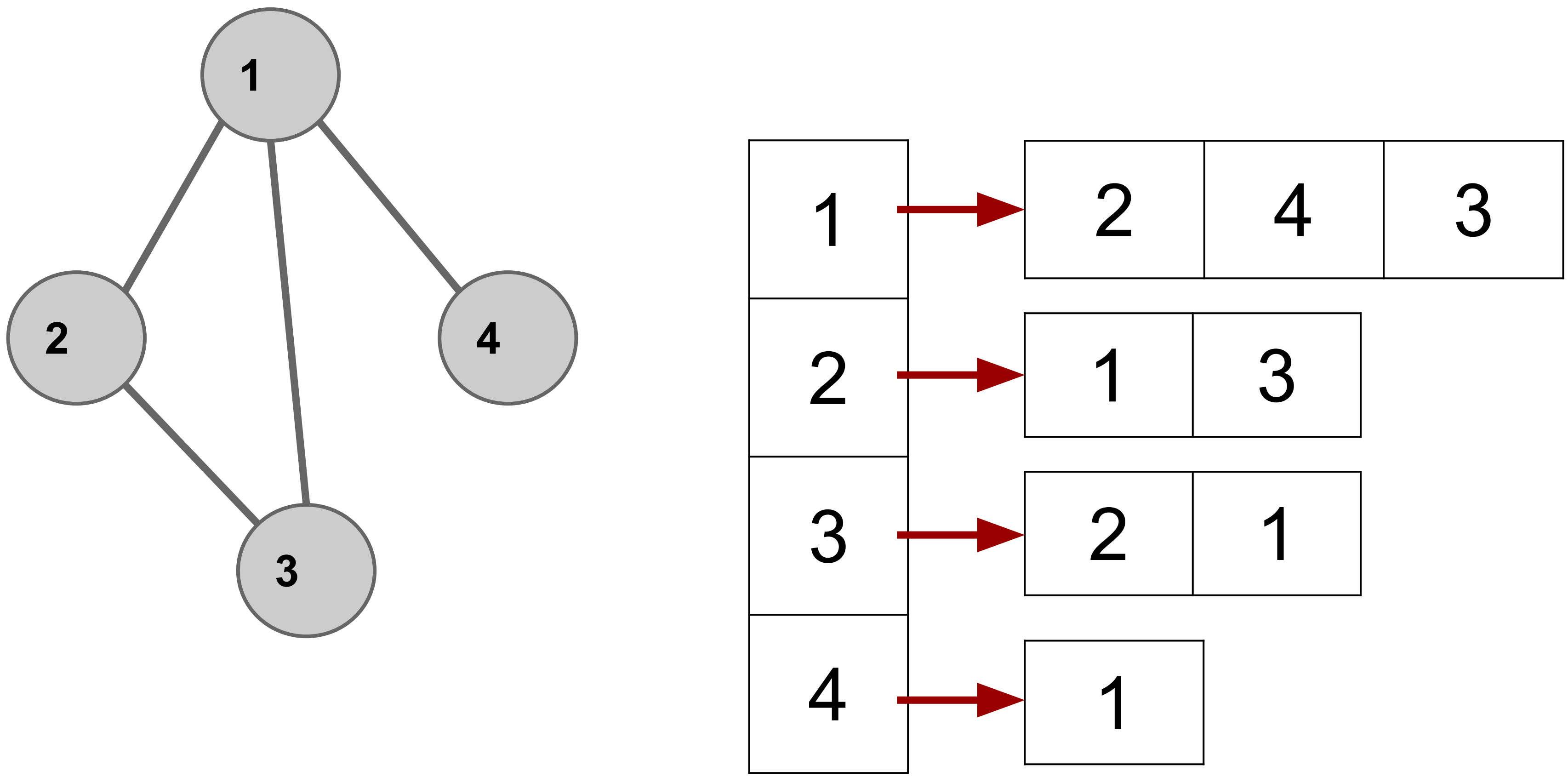
Adjacency list (directed graph)

How much space does it take?

|V|+|E|



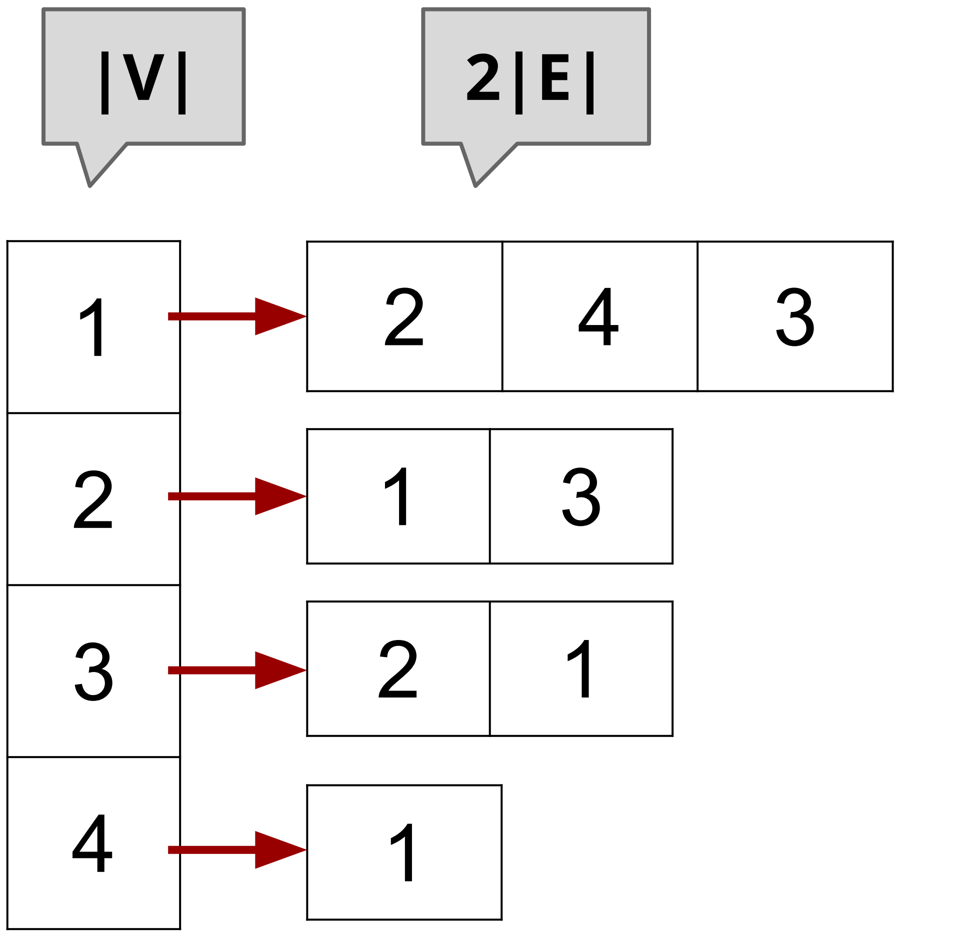
Adjacency list (undirected graph)



Adjacency list (undirected graph)

How much space does it take?

$|V|+2|E|$



Matrix *VS* List

In term of space complexity

- ➔ adjacency matrix is $\Theta(|V|^2)$
- ➔ adjacency list is $\Theta(|V|+|E|)$

Which one is more space-efficient?

Adjacency list, if $|E| \ll |V|^2$, i.e., the graph is not very **dense**.

Matrix *VS* List

Anything that **Matrix** does better than **List**?

Check whether edge **(vi, vj)** is in **E**

➔ **Matrix**: just check if A[i, j] = 1, **O(1)**

➔ **List**: go through list A[i] see if j is in there, **O(length of list)**

Takeaway

Adjacency **matrix** or adjacency **list**?

Choose the more appropriate one depending on the problem.

CSC263 Week 8

Wednesday / Thursday

Announcements

- ➔ PS6 posted, due next Tuesday as usual
- ➔ Drop date: March 8th

Recap

- ADT: Graph
- Data structures
  - Adjacency **matrix**
  - Adjacency **list**
- Graph operations
  - Add vertex, remove vertex, ..., edge query, ...
  - **Traversal**

Graph Traversals

BFS and DFS

They are twins!

Graph traversals

Visiting **every** vertex **once**, starting from a given vertex.

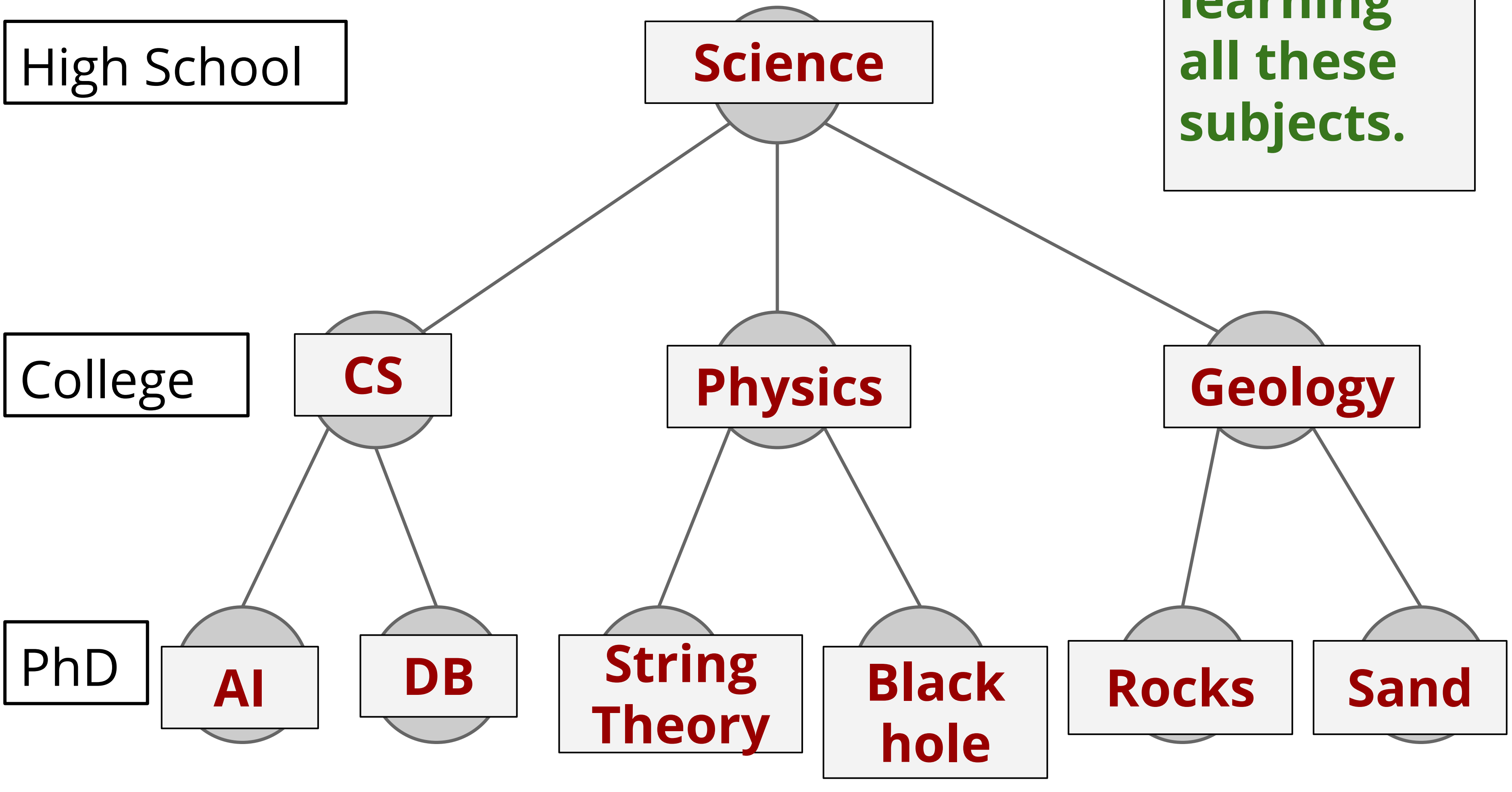
The visits can follow different **orders**, we will learn about the following two ways

- ➔ **Breadth** First Search (**BFS**)
- ➔ **Depth** First Search (**DFS**)

Intuitions of BFS and DFS

Consider a special graph -- a **tree**

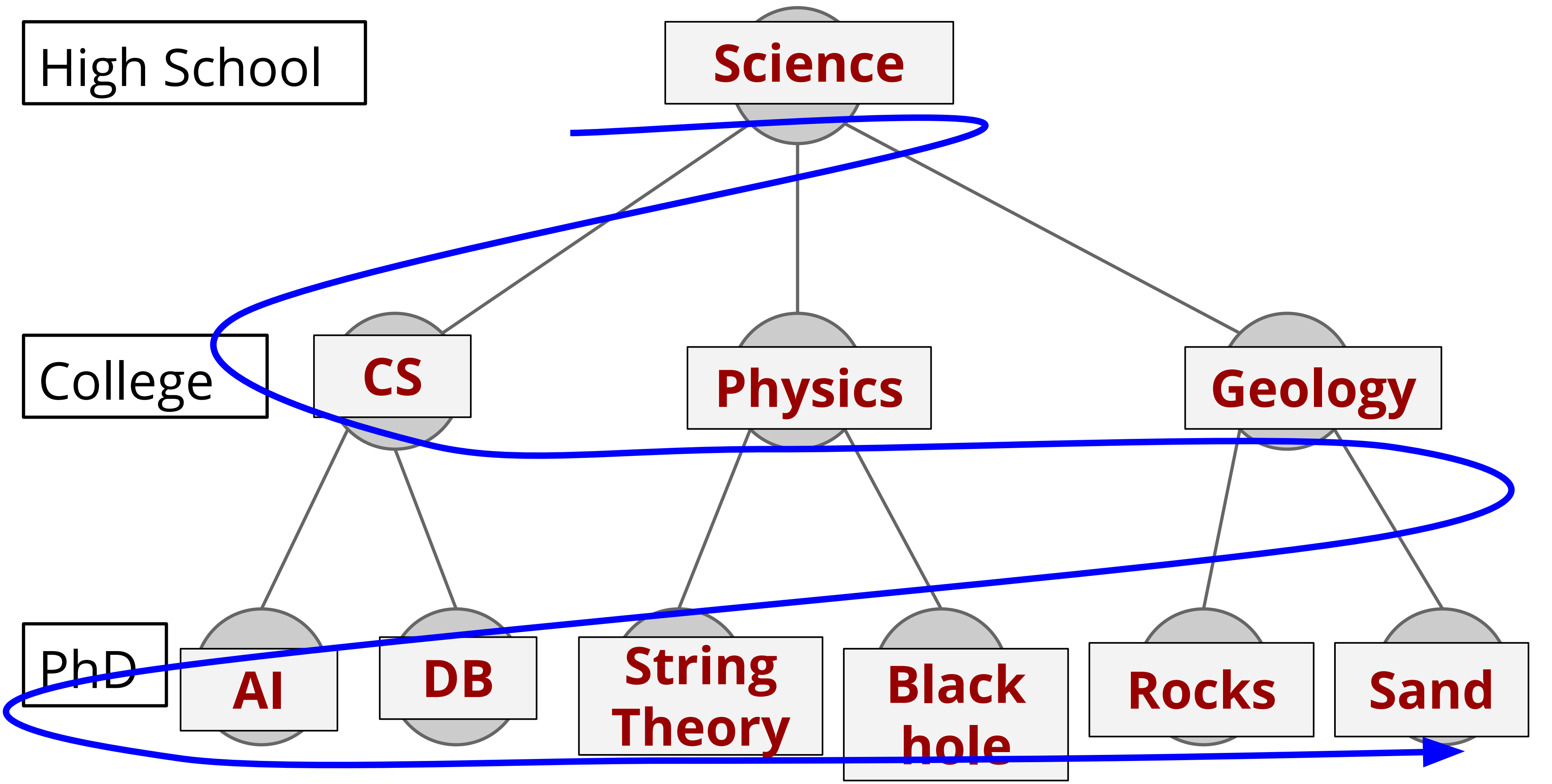
“The knowledge learning tree”



Traversing this graph means learning all these subjects.

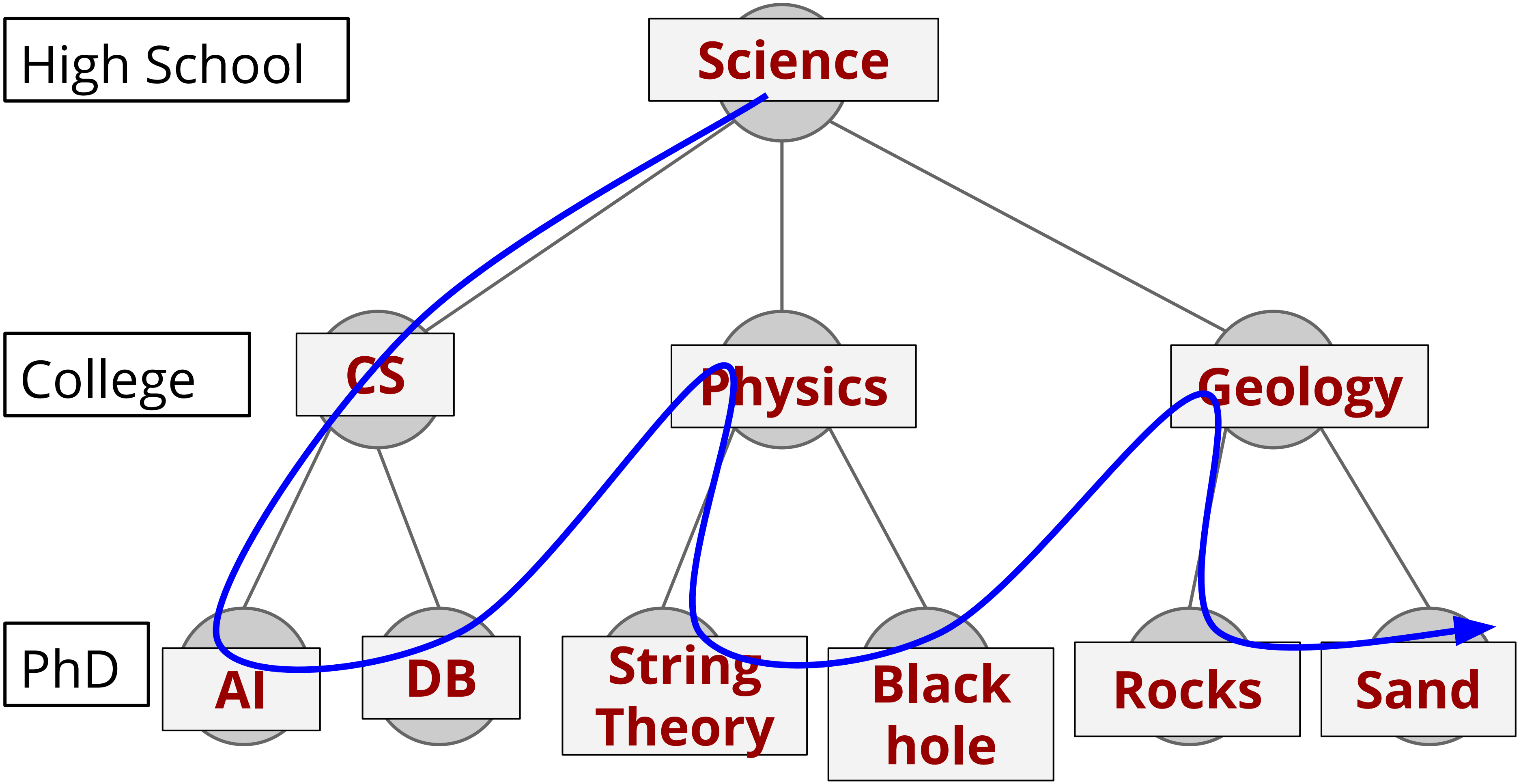
The Breadth-First ways of learning these subjects

→ Level by level, finish high school, then all subjects at College level, then finish all subjects in PhD level.

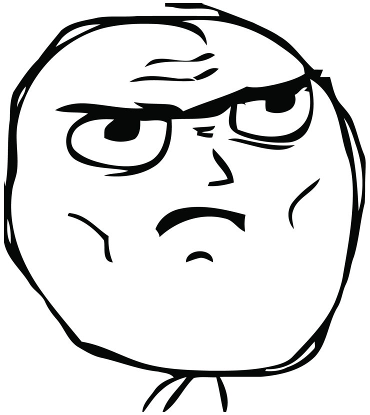


The Depth-First way of learning these subjects

➔ Go towards PhD whenever possible; only start learning physics after finishing everything in CS.



Now let's seriously start studying BFS



Special case: BFS in a tree

Review CSC148:

BFS in a tree (starting from root) is a

___**level-by-level**___________ traversal.

(NOT preorder!)

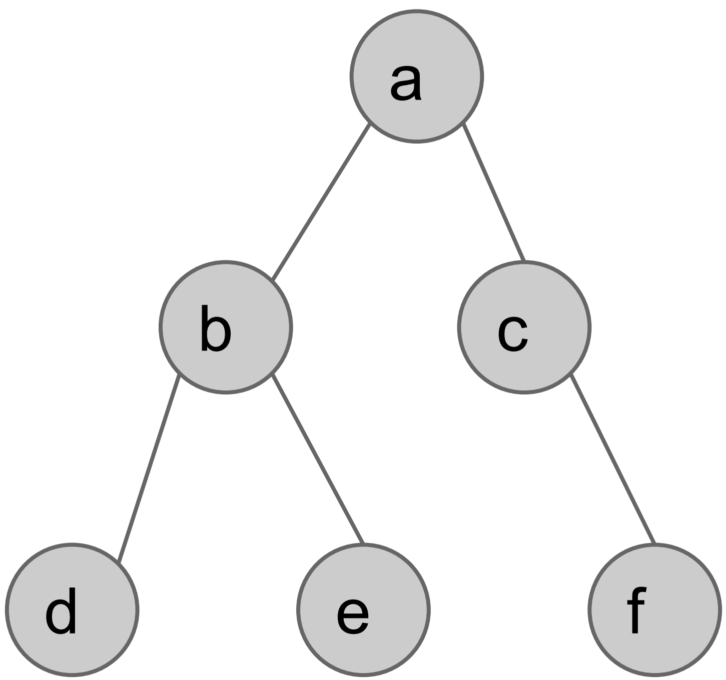
What ADT did we use for implementing the **level-by-level** traversal?

Queue!

Special case: BFS in a tree

Output:

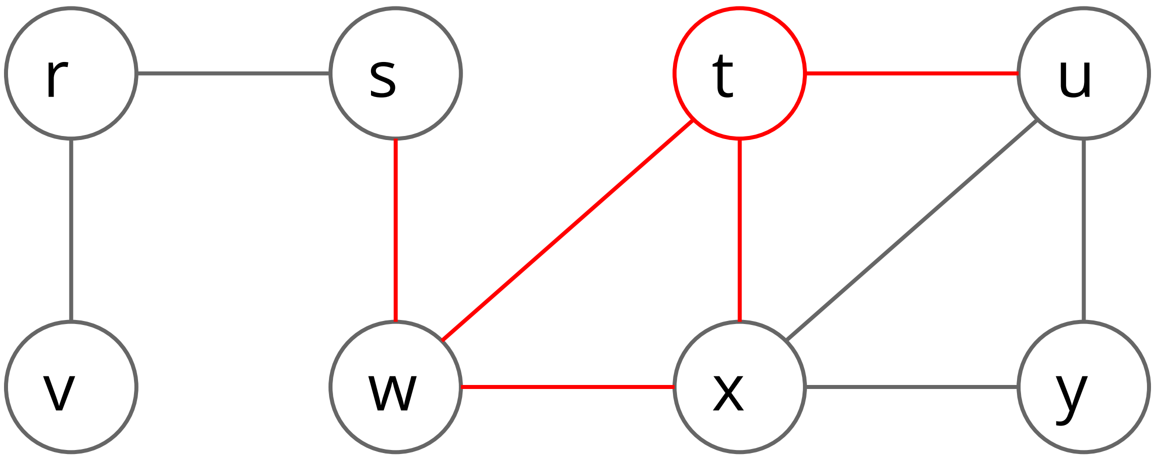
a
b
c
d
e
f

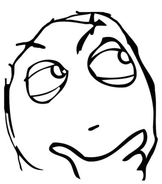



```
NOT_YET_BFS(root):
  Q ← Queue()
  Enqueue(Q, root)
  while Q not empty:
    x ← Dequeue(Q)
    print x
    for each child c of x:
      Enqueue(Q, c)
```

Queue:	a	b	c	d	e	f	**EMPTY!**
	DQ	**DQ**	**DQ**	**DQ**	**DQ**	**DQ**	

The real deal: BFS in a Graph





If we just run **NOT_YET_BFS(t)** on the above graph. What problem would we have?

It would want to visit some vertex **twice** (e.g., ***x***), which shall be **avoided**!

```
NOT_YET_BFS(root):
 Q ← Queue()
 Enqueue(Q, root)
 while Q not empty:
   x ← Dequeue(Q)
   print x
   for each neighbr c of x:
     Enqueue(Q, c)
```

How avoid visiting a vertex twice

Remember you visited it by **labelling** it using **colours**.

- ➔ **White**: “unvisited”
- ➔ **Gray**: “encountered”
- ➔ **Black**: “explored”

- ➔ Initially all vertices are **white**
- ➔ Colour a vertex **gray** the **first** time visiting it
- ➔ Colour a vertex **black** when **all** its **neighbours** have been encountered
- ➔ Avoid visiting **gray** or **black** vertices
- ➔ In the end, all vertices are **black** (sort-of)

Some other values we want to remember during the traversal...

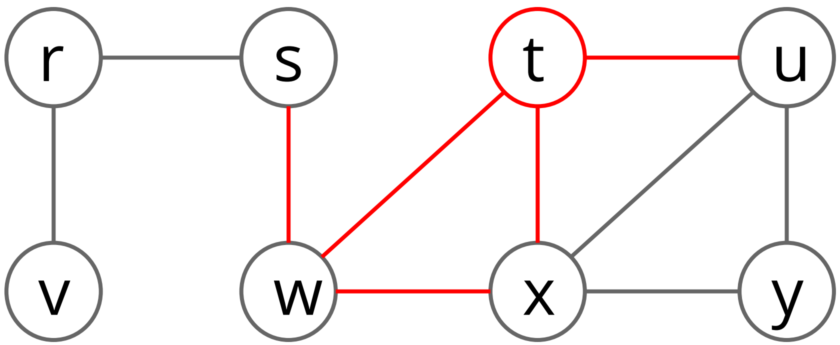
➔ **pi[v]**: the vertex from which v is encountered

- "I was introduced as **whose** neighbour?"

➔ **d[v]**: the distance value

- the distance from **v** to the source vertex of the BFS

This **d[v]** is going to be **really** useful!

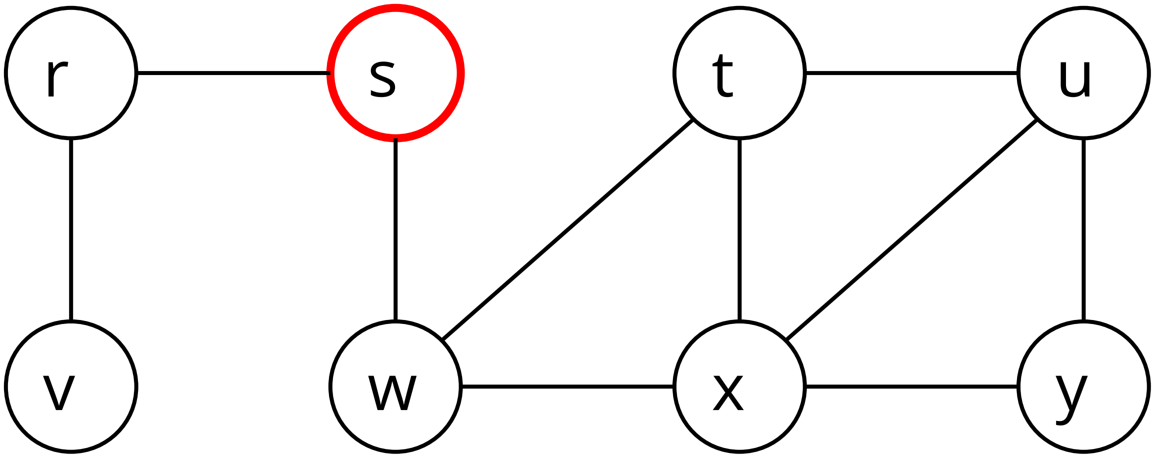


Pseudocode: the real BFS

```
BFS(G=(V, E), s):
    for all v in V:
        colour[v] ← white
        d[v] ← ∞          # Initialize vertices
        pi[v] ← NIL
    Q ← Queue()
    colour[s] ← gray   # start BFS by encountering the source vertex
    d[s] ← 0     # distance from s to s is 0
    Enqueue(Q, s)
    while Q not empty:
        u ← Dequeue(Q)
        for each neighbour v of u:
            if colour[v] = white   # only visit unvisited vertices
                colour[v] ← gray
                d[v] ← d[u] + 1   # v is "1-level" farther from s than u
                pi[v] ← u   # v is introduced as u's neighbour
                Enqueue(Q, v)
        colour[u] ← black   # all neighbours of u have been
                            # encountered, therefore u is explored
```

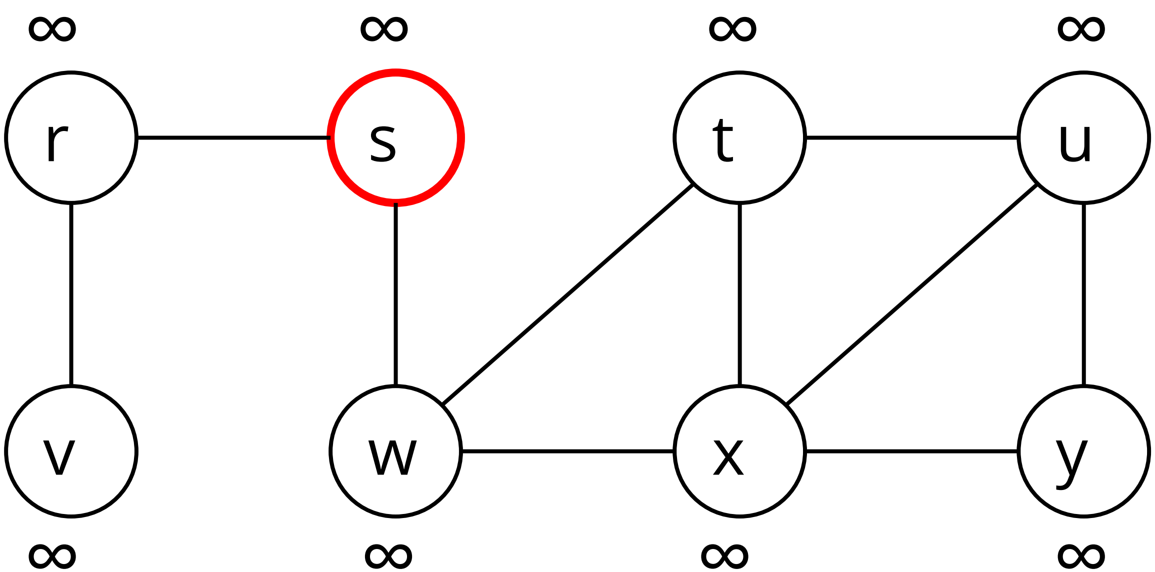
The blue lines are the same as NOT_YET_BFS

Let's run an example!



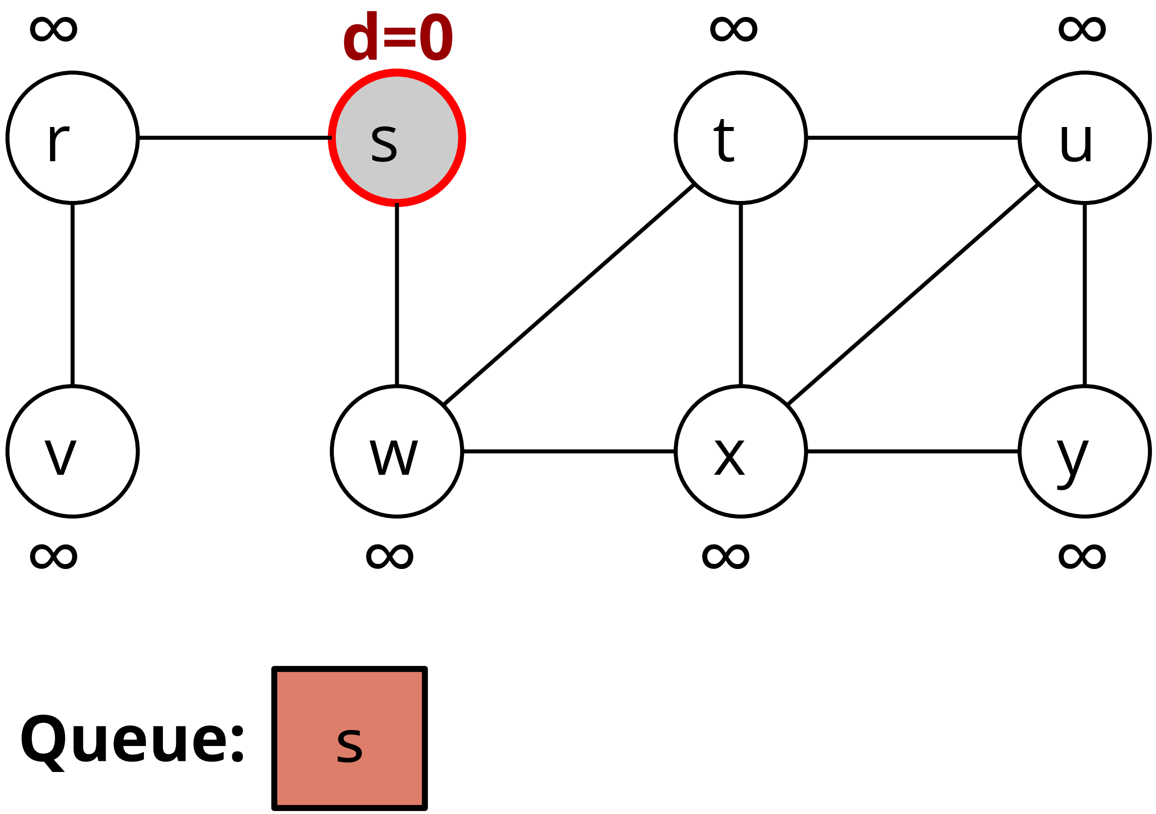
BFS(G, s)

After initialization



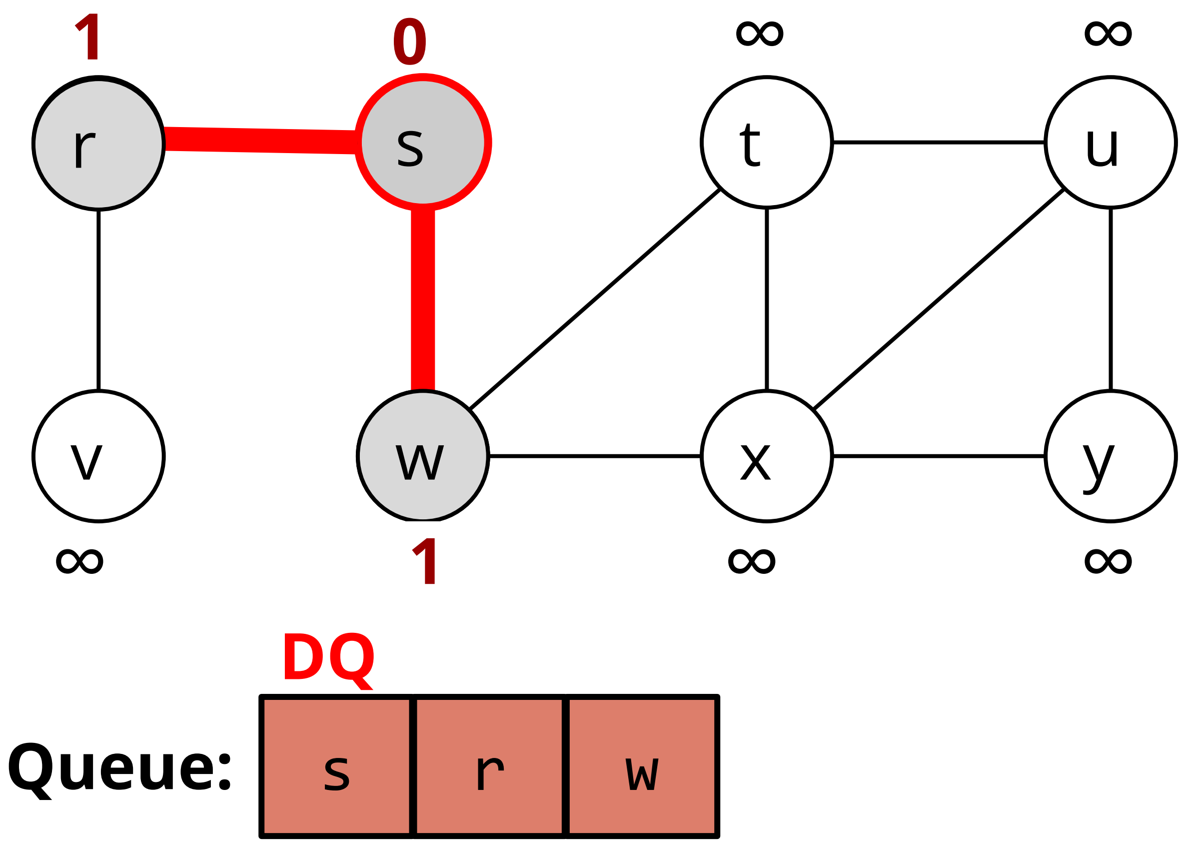
All vertices are **white** and have **d = ∞**

Start by "encountering" the source



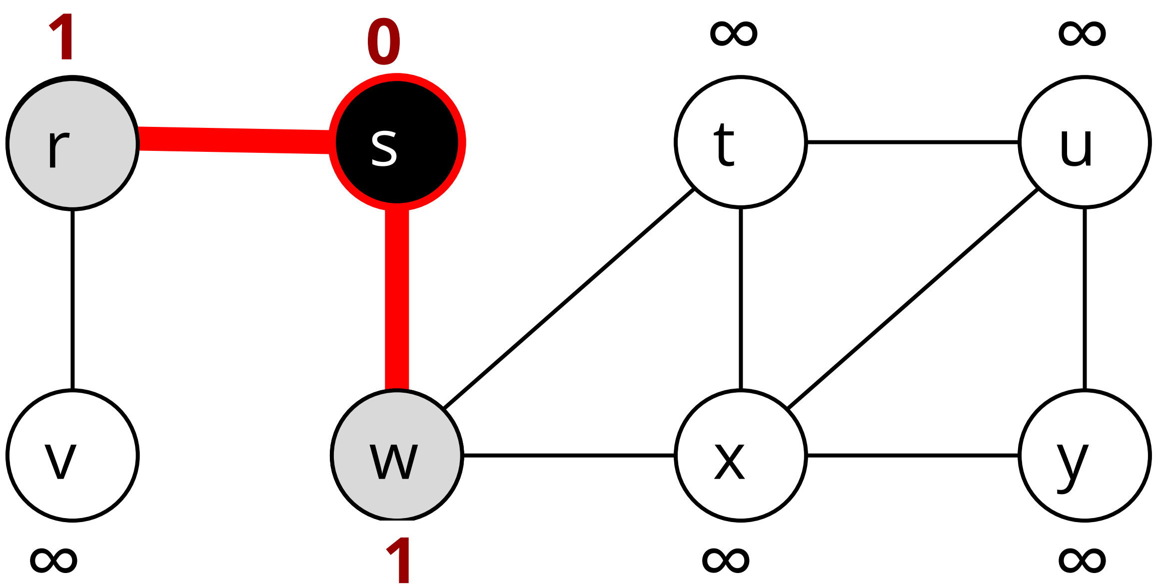
Colour the source **gray** and set its d = 0, and Enqueue it

Dequeue, explore neighbours



The red edge indicates the **pi[v]** that got remembered

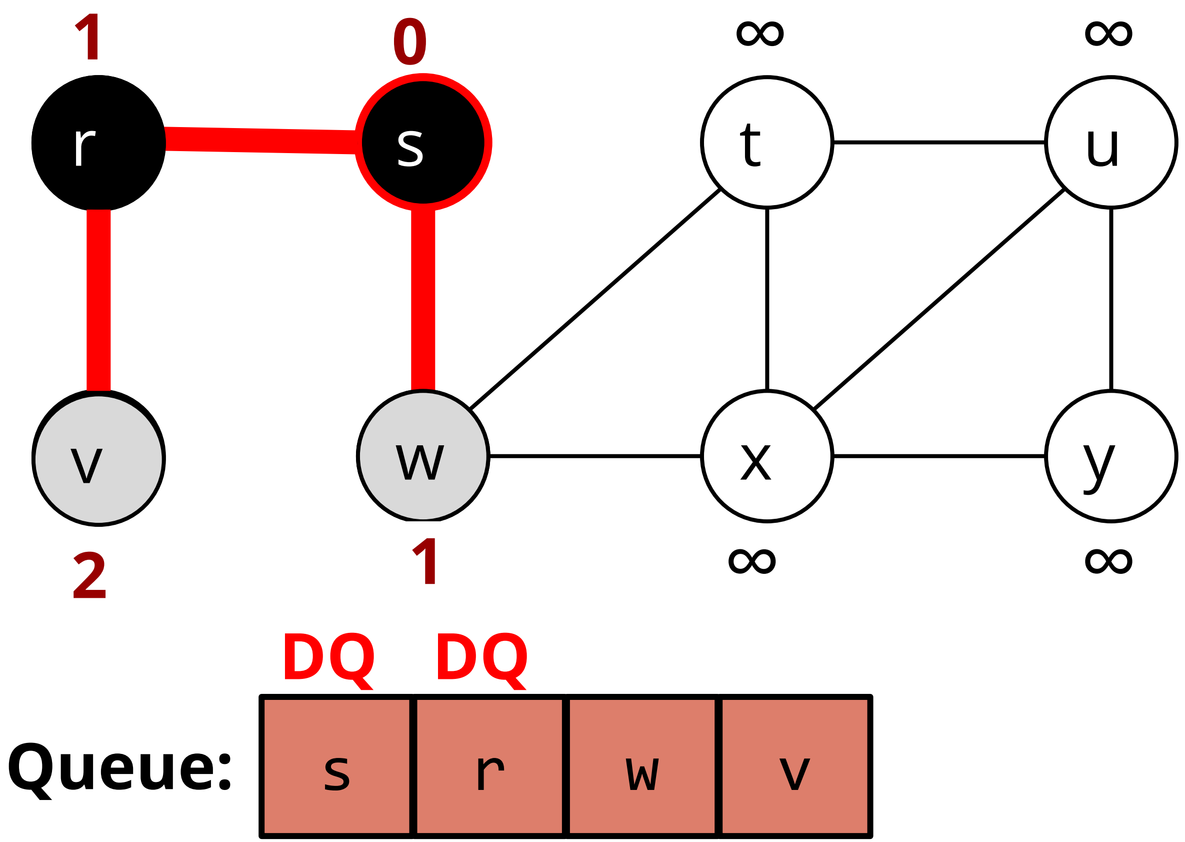
Colour black after exploring all neighbours



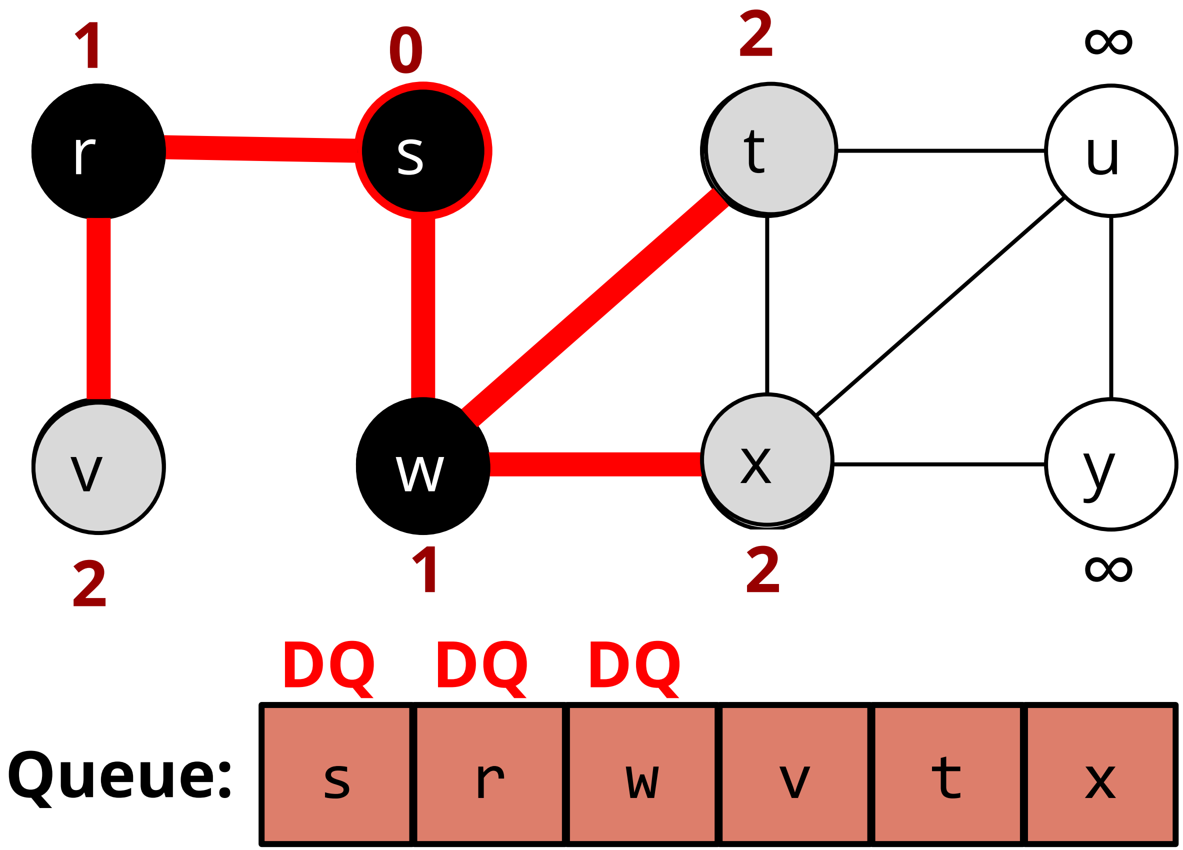
Queue:

s	r	w

Dequeue, explore neighbours (2)

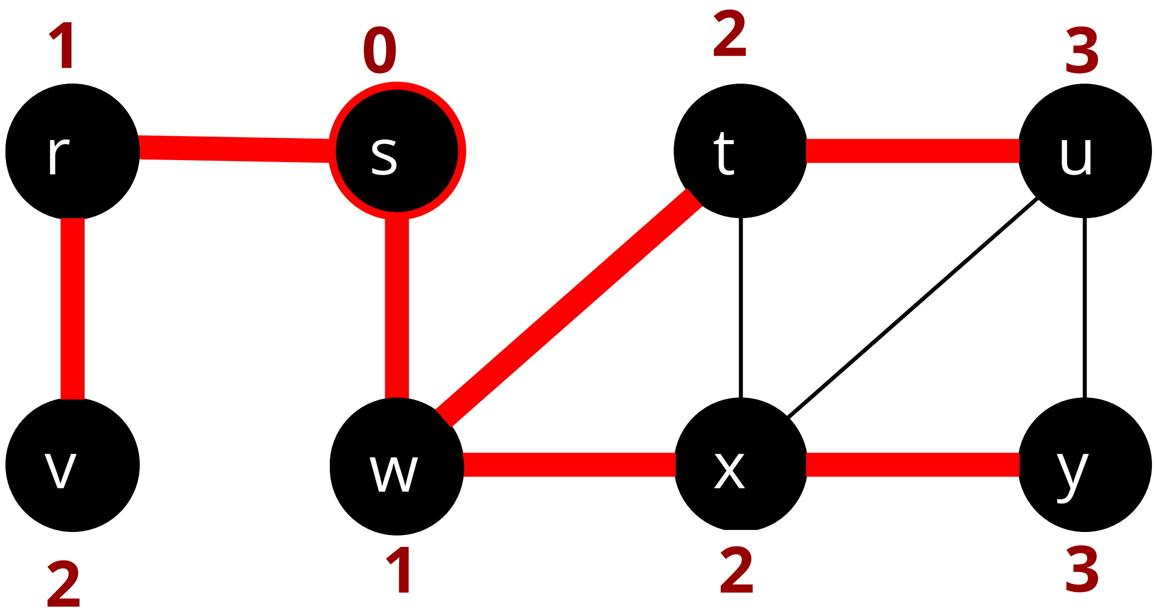


Dequeue, explore neighbours (3)



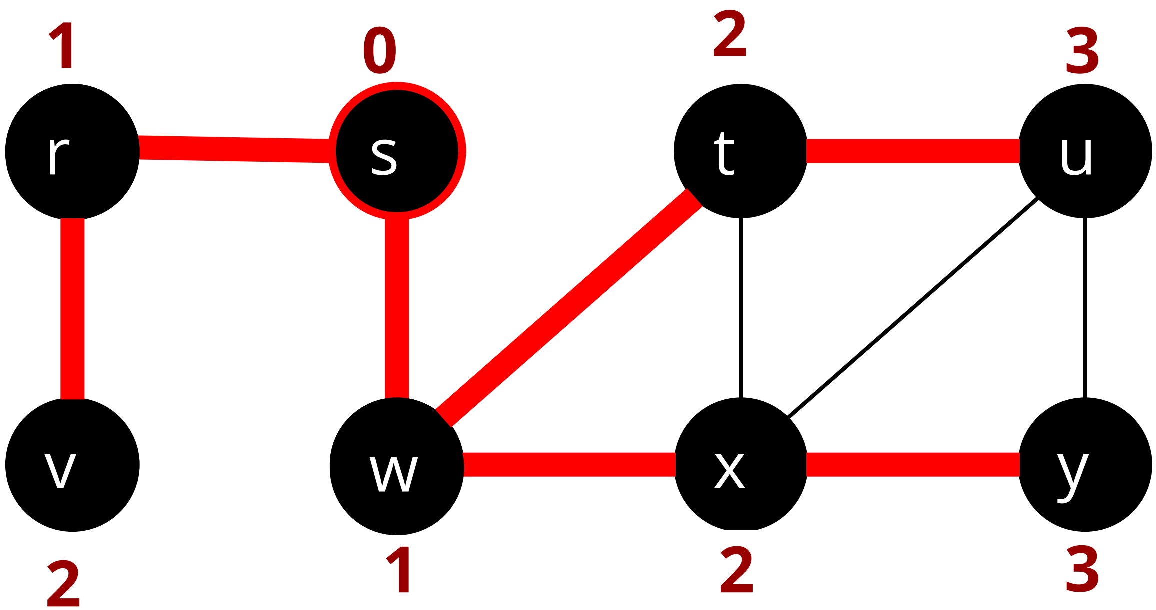
after a few more steps…

BFS done!



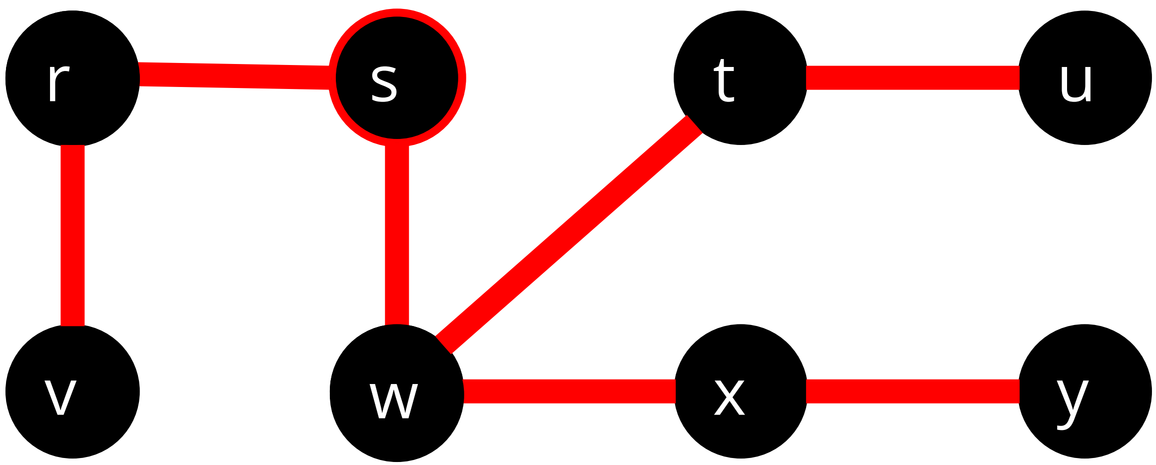
	DQ	DQ	DQ	DQ	DQ	DQ	DQ	DQ
Queue:	s	r	w	v	t	x	u	y

What do we get after doing all these?



First of all, we get to visit **every** vertex **once**.

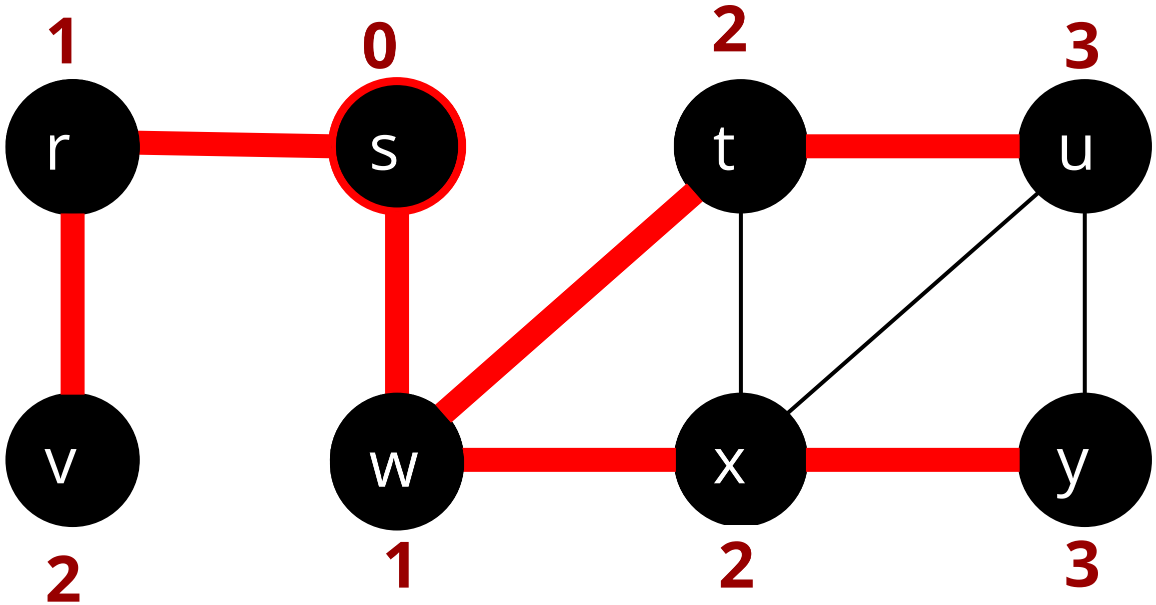
Did you know? The official name of the red edges are called **"tree edges"**.



This is called the **BFS-tree**, it's a **tree** that connects all vertices, if the graph is **connected**.

These **d[v]** values, we said they were going to be really useful.

Short path from u to s:
u → pi[u] → pi[pi[u]] → pi[pi[pi[u]]] → ... → s

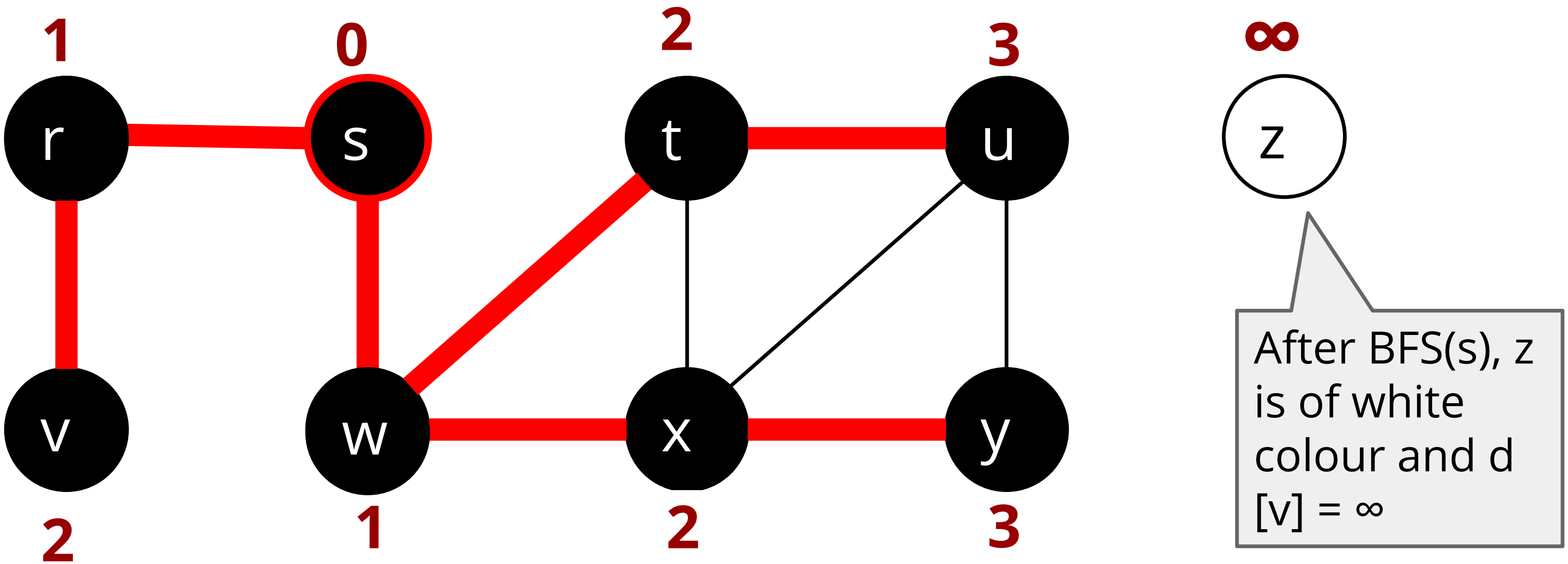


The value indicates the vertex's **distance** from the source vertex.

Actually more than that, it's the **shortest-path distance**, we can prove it.

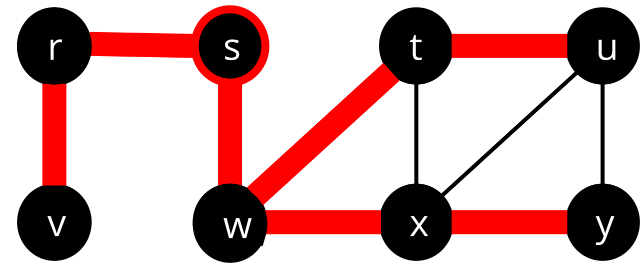
How about finding **short path** itself?
Follow the red edges, **pi[v]** comes in handy for this.

What if G is disconnected?



The infinite distance value of **z** indicates that it is **unreachable** from the source vertex.

Runtime analysis!



The total amount of work (use **adjacency list**):

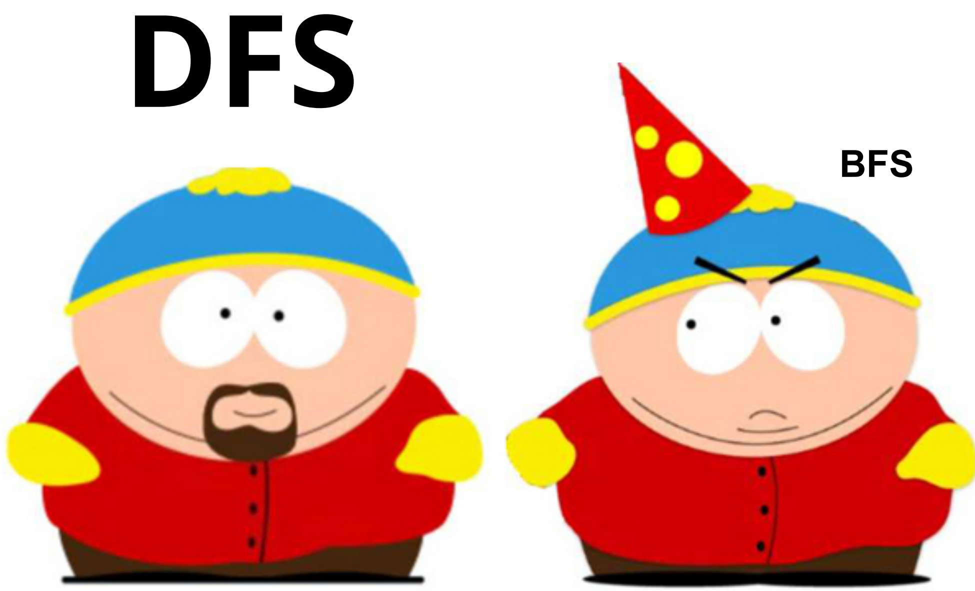
- Visit each vertex once
  - Enqueue, Dequeue, change colours, assign d[v], ..., constant work per vertex
  - in total: **O(|V|)**
- At each vertex, check all its neighbours (all its **incident edges**)
  - Each edge is checked **twice** (by the two end vertices)
  - in total: **O(|E|)**

Total runtime:
O(|V|+|E|)

Summary of BFS

- ➔ Prefer to explore **breadth** rather than **depth**
- ➔ Useful for getting **single-source shortest paths** on **unweighted** graphs
- ➔ Useful for testing reachability
- ➔ Runtime **O(|V|+|E|)** with adjacency **list** (with adjacency **matrix** it'll be different)

Next week



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