Inference on Categorical Data

Goodness-of-Fit Test

Learning Objective

1. Perform a goodness-of-fit test

Goodness-of-Fit Test

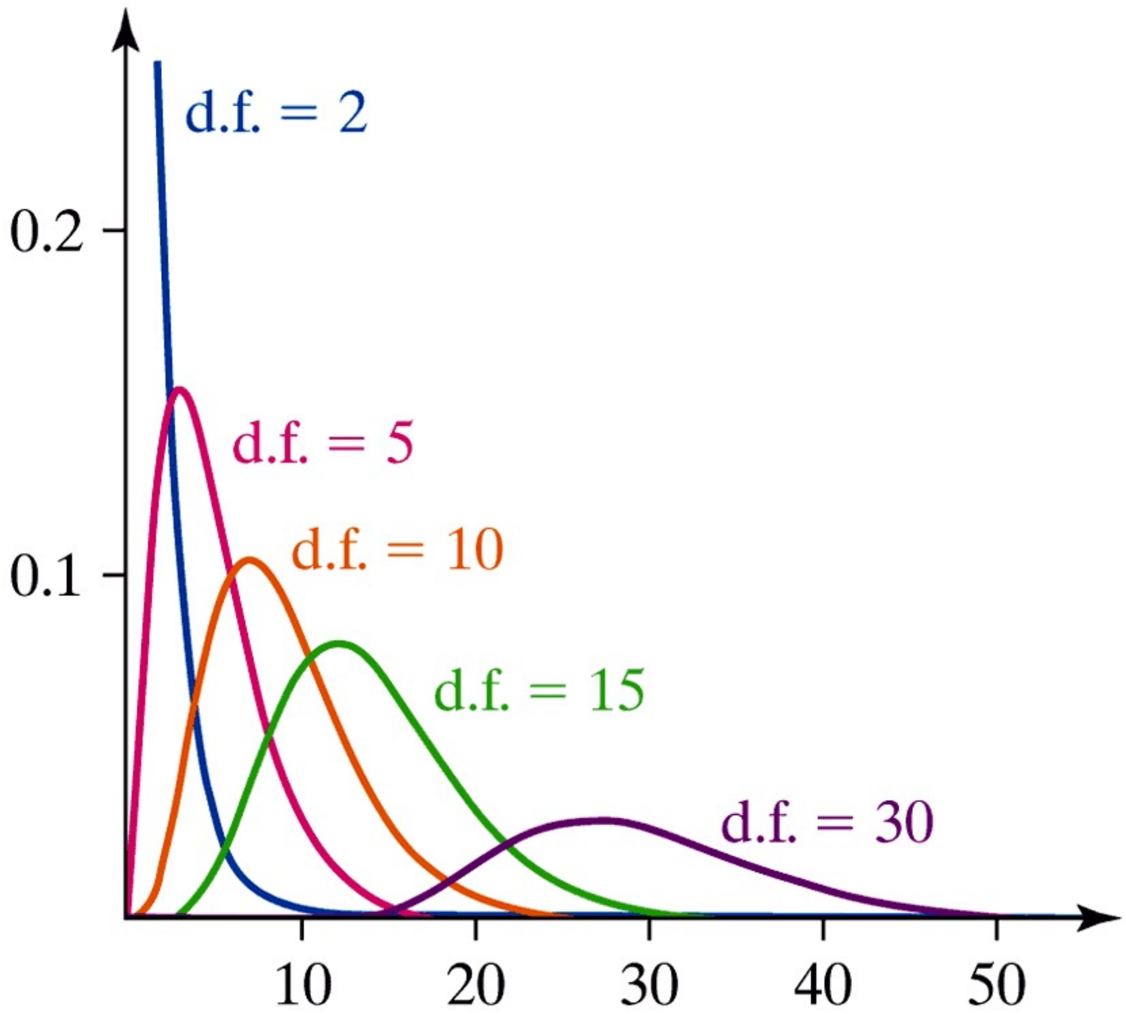
Perform a goodness-of-fit test (1 of 32)

Characteristics of the Chi-Square Distribution

1. It is not symmetric.
2. It's shape depends on the degrees of freedom, just like Student's *t*-distribution.
3. As the number of degrees of freedom increases, it becomes more nearly symmetric.
4. The values of χ^2 are nonnegative. That is, the values of χ^2 are greater than or equal to 0.

Goodness-of-Fit Test

Perform a goodness-of-fit test (2 of 32)



Goodness-of-Fit Test

Perform a goodness-of-fit test (3 of 32)

A **goodness-of-fit** test is an inferential procedure used to determine whether a frequency distribution follows a specific distribution.

Goodness-of-Fit Test

Perform a goodness-of-fit test (10 of 32)

The Goodness-of-Fit Test

To test the hypotheses regarding a distribution, we use the steps that follow.

Step 1: Determine the null and alternative hypotheses.

H_0: The random variable follows a certain distribution

H_1: The random variable does not follow the distribution in the null hypothesis

Goodness-of-Fit Test

Perform a goodness-of-fit test (11 of 32)

Step 2: Decide on a level of significance, α, depending on the seriousness of making a Type I error.

Goodness-of-Fit Test

Perform a goodness-of-fit test (12 of 32)

Step 3:

a) Calculate the expected counts, E_i , for each of the k categories. The expected counts are $E_i = np_i$ for $i = 1, 2, \ldots, k$ where n is the number of trials and p_i is the probability of the ith category, assuming that the null hypothesis is true.

Goodness-of-Fit Test

Perform a goodness-of-fit test (13 of 32)

Step 3:

b) Verify that the requirements for the goodness-of-fit test are satisfied.

   1. All expected counts are greater than or equal to 1 (all $E_i \geq 1$).
   2. No more than 20% of the expected counts are less than 5.

Goodness-of-Fit Test

Perform a goodness-of-fit test (18 of 32)

***P*-value Approach**

By Hand Step 3 (continued):

c) Compute the **test statistic**:

$$\chi_0^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

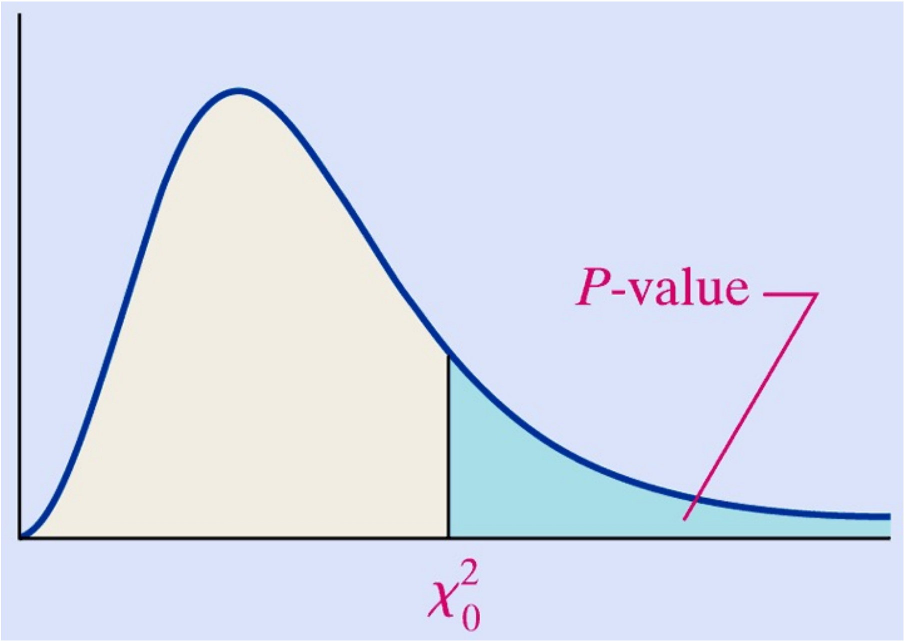
Note: O_i is the observed count for the ith category.

Goodness-of-Fit Test

Perform a goodness-of-fit test (19 of 32)

P-value Approach

d) Use Table VIII to approximate the *P*-value by determining the area under the chi-square distribution with $k - 1$ degrees of freedom to the right of the test statistic.



Goodness-of-Fit Test

Perform a goodness-of-fit test (21 of 32)

P-value Approach

Step 4: If the *P*-value < α, reject the null hypothesis.

Goodness-of-Fit Test

Perform a goodness-of-fit test (22 of 32)

P-value Approach

Step 5: State the conclusion.

Goodness-of-Fit Test

Perform a goodness-of-fit test (23 of 32)

Parallel Example 2: Conducting a Goodness-of-Fit Test

A sociologist wishes to determine whether the distribution for the number of years care-giving grandparents are responsible for their grandchildren is different today than it was in 2000.

According to the United States Census Bureau, in 2000, 22.8% of grandparents have been responsible for their grandchildren less than 1 year; 23.9% of grandparents have been responsible for their grandchildren for 1 or 2 years; 17.6% of grandparents have been responsible for their grandchildren 3 or 4 years; and 35.7% of grandparents have been responsible for their grandchildren for 5 or more years. The sociologist randomly selects 1,000 care-giving grandparents and obtains the following data.

Goodness-of-Fit Test

Perform a goodness-of-fit test (24 of 32)

Number of Years	Frequency
Less than 1 year	252
1 or 2 years	255
3 or 4 years	162
5 or more years	331

Test the claim that the distribution is different today than it was in 2000 at the $\alpha = 0.05$ level of significance.

Goodness-of-Fit Test

Perform a goodness-of-fit test (25 of 32)

Solution

Step 1: We want to know if the distribution today is different than it was in 2000. The hypotheses are then:

H_0: The distribution for the number of years care-giving grandparents are responsible for their grandchildren is the same today as it was in 2000

H_1: The distribution for the number of years care-giving grandparents are responsible for their grandchildren is different today than it was in 2000

Goodness-of-Fit Test

Perform a goodness-of-fit test (26 of 32)

Solution

Step 2: The level of significance is $\alpha = 0.05$.

Step 3:

a) The expected counts were computed in Example 1.

Number of Years	Observed Counts	Expected Counts
<1	252	228
1-2	255	239
3-4	162	176
≥5	331	357

Goodness-of-Fit Test

Perform a goodness-of-fit test (27 of 32)

Solution

Step 3:

Since all expected counts are greater than or equal to 5, the requirements for the goodness-of-fit test are satisfied.

The test statistic is

$$\chi_0^2 = \frac{(252-228)^2}{228} + \frac{(255-239)^2}{239} + \frac{(162-176)^2}{176} + \frac{(331-357)^2}{357}$$

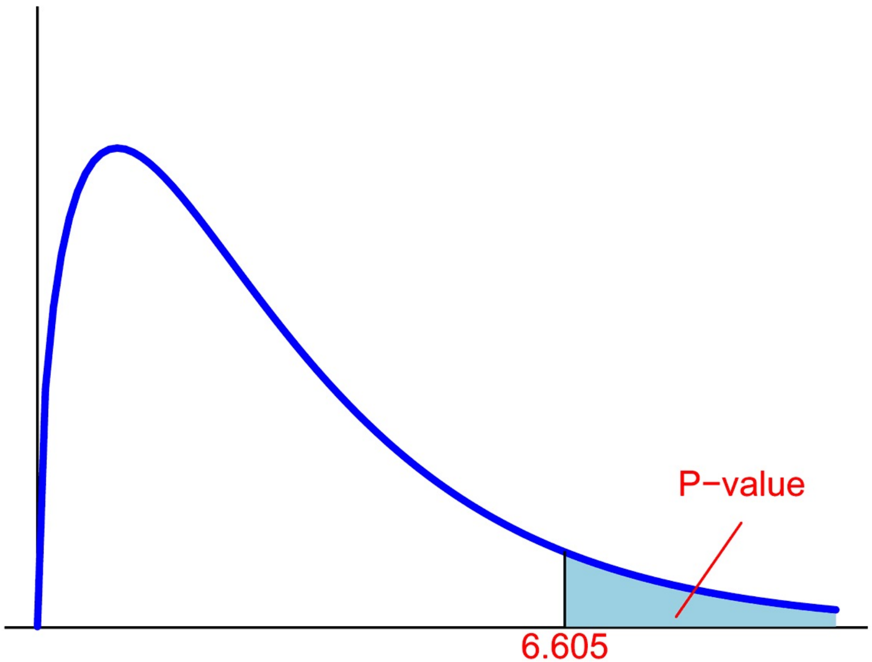
$$= 6.605$$

Goodness-of-Fit Test

Perform a goodness-of-fit test (30 of 32)

Solution: *P*-value Approach

Step 4: There are $k = 4$ categories. The *P*-value is the area under the chi-square distribution with $4 - 1 = 3$ degrees of freedom to the right of $\chi_0^2 = 6.605$. Thus, *P*-value ≈ 0.09.



Goodness-of-Fit Test

Perform a goodness-of-fit test (31 of 32)

Solution: *P*-value Approach

Since the *P*-value ≈ 0.09 is greater than the level of significance $\alpha = 0.05$, we fail to reject the null hypothesis.

Goodness-of-Fit Test

Perform a goodness-of-fit test (32 of 32)

Solution

Step 5: There is insufficient evidence to conclude that the distribution for the number of years care-giving grandparents are responsible for their grandchildren is different today than it was in 2000 at the $\alpha = 0.05$ level of significance.

Comparing Three or More Means (One-Way Analysis of Variance)
Learning Objectives

1. Verify the requirements to perform a one-way ANOVA
2. Test a hypothesis regarding three or more means using one-way ANOVA

Introduction (1 of 2)

Analysis of Variance (ANOVA) is an inferential method used to test the equality of three or more population means.

Introduction (2 of 2)

CAUTION!

Do not test H_0: $\mu_1 = \mu_2 = \mu_3$ by conducting three separate hypothesis tests, because the probability of making a Type I error will be much higher than α.

Introduction to the Practice of Statistics

Verify the requirements to perform a one-way ANOVA (1 of 12)

Requirements of a One-Way ANOVA Test

1. There must be k simple random samples; one from each of k populations or a randomized experiment with k treatments.
2. The k samples are independent of each other; that is, the subjects in one group cannot be related in any way to subjects in a second group.
3. The populations are normally distributed.
4. The populations must have the same variance; that is, each treatment group has the population variance σ^2.

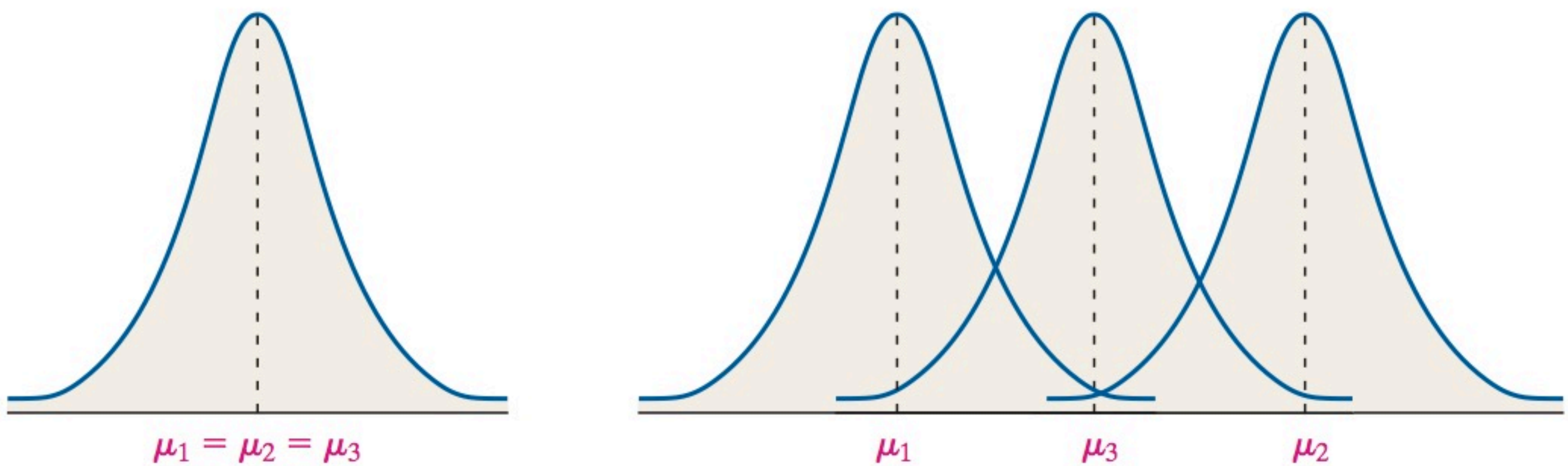
Introduction to the Practice of Statistics

Verify the requirements to perform a one-way ANOVA (2 of 12)

Testing a Hypothesis Regarding k = 3

H_0: $\mu_1 = \mu_2 = \mu_3$

H_1: At least one population mean is different from the others



Pearson

Verify the requirements to perform a one-way ANOVA (4 of 12)

Verifying the Requirement of Equal Population Variance

The one-way ANOVA procedures may be used if the largest sample standard deviation is no more than twice the smallest sample standard deviation.

Verify the requirements to perform a one-way ANOVA (5 of 12)

Parallel Example 1: Verifying the Requirements of ANOVA

The following data represent the weight (in grams) of pennies minted at the Denver mint in 1990,1995, and 2000. Verify that the requirements in order to perform a one-way ANOVA are satisfied.

Introduction to the Practice of Statistics

Verify the requirements to perform a one-way ANOVA (6 of 12)

1990	1995	2000
2.50	2.52	2.50
2.50	2.54	2.48
2.49	2.50	2.49
2.53	2.48	2.50
2.46	2.52	2.48
2.50	2.50	2.52
2.47	2.49	2.51
2.53	2.53	2.49
2.51	2.48	2.51
2.49	2.55	2.50
2.48	2.49	2.52

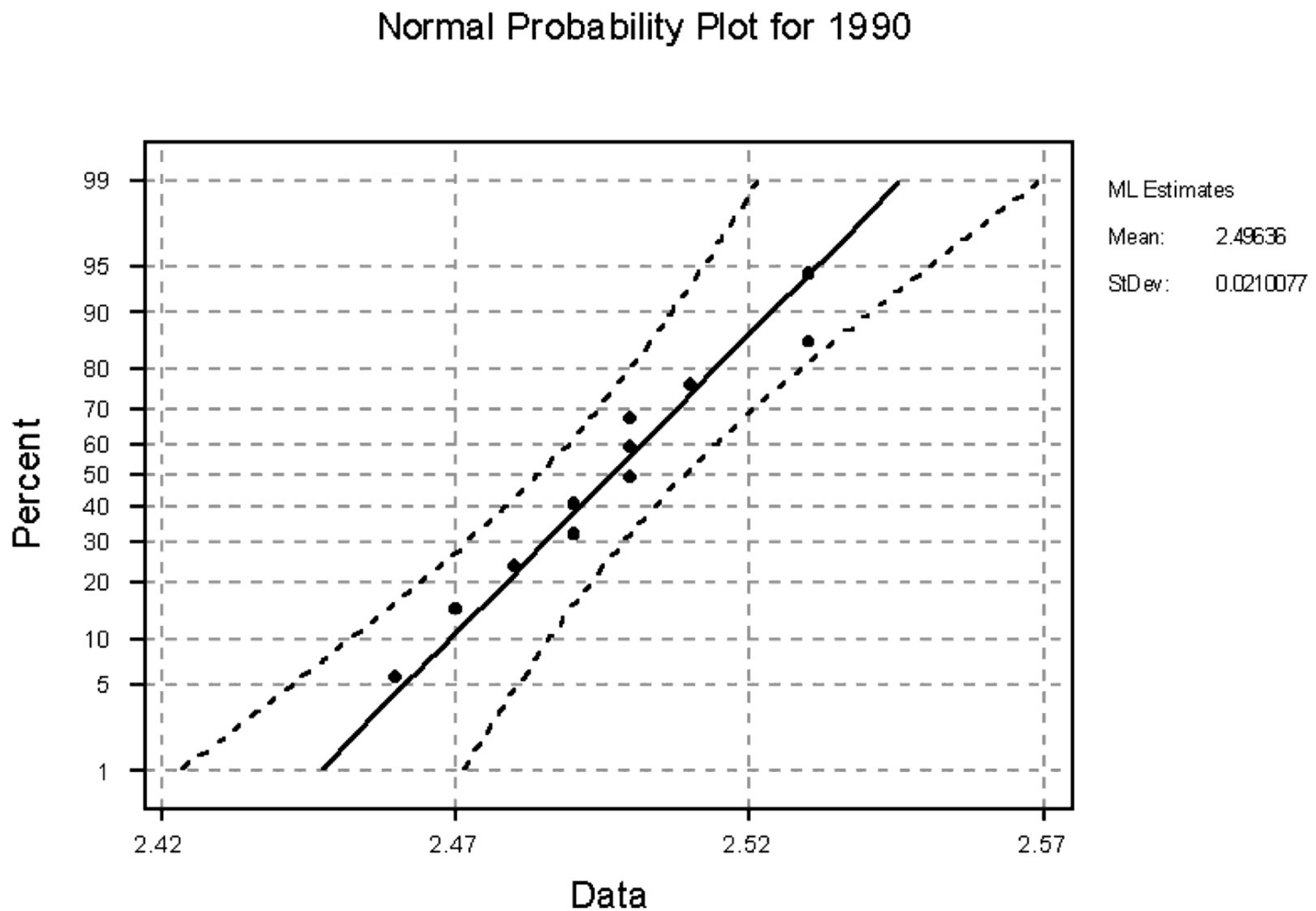
Verify the requirements to perform a one-way ANOVA (7 of 12)

Solution

1. The 3 samples are simple random samples.
2. The samples were obtained independently.
3. Normal probability plots for the 3 years follow. All of the plots are roughly linear so the normality assumption is satisfied.

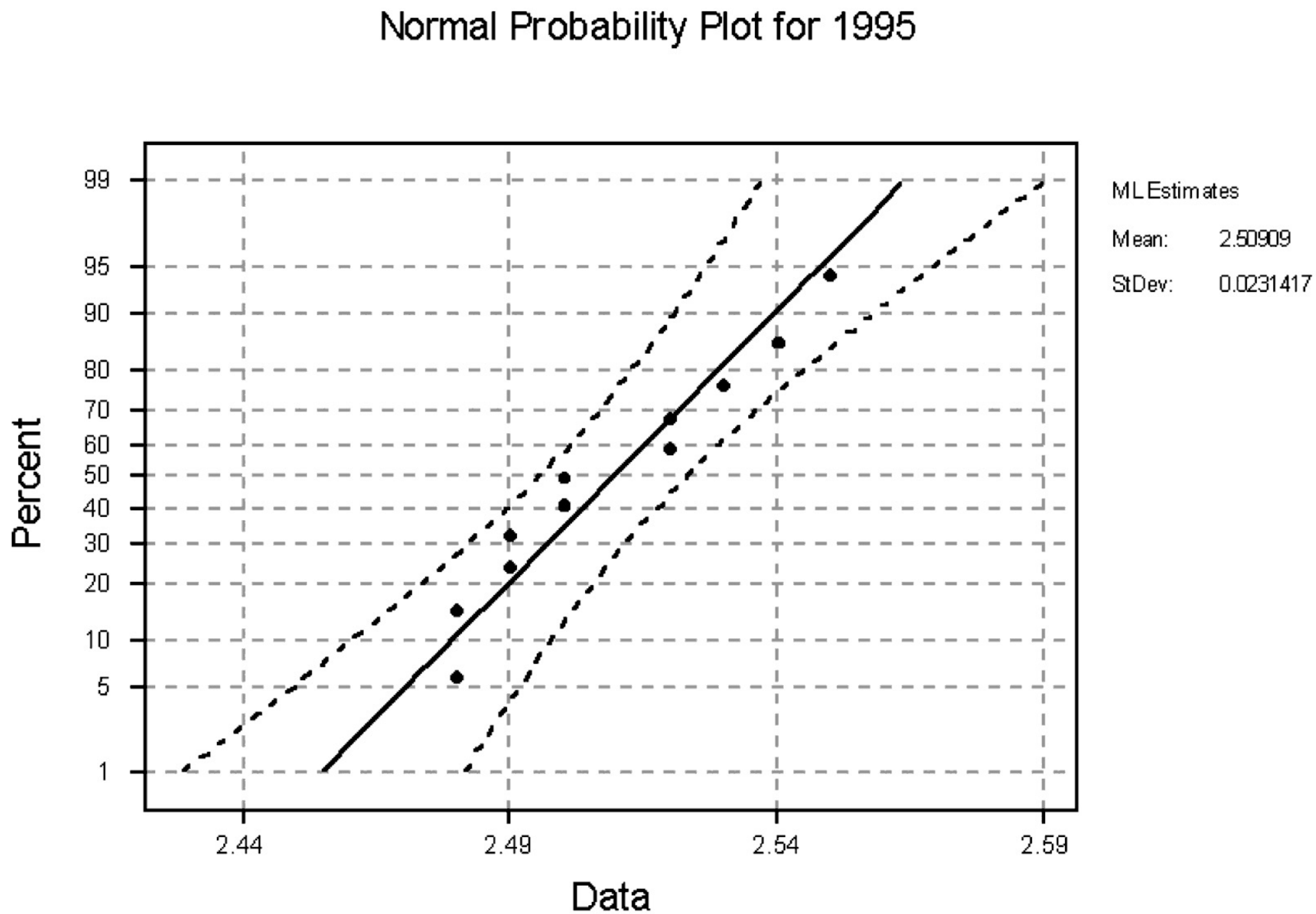
Introduction to the Practice of Statistics

Verify the requirements to perform a one-way ANOVA (8 of 12)



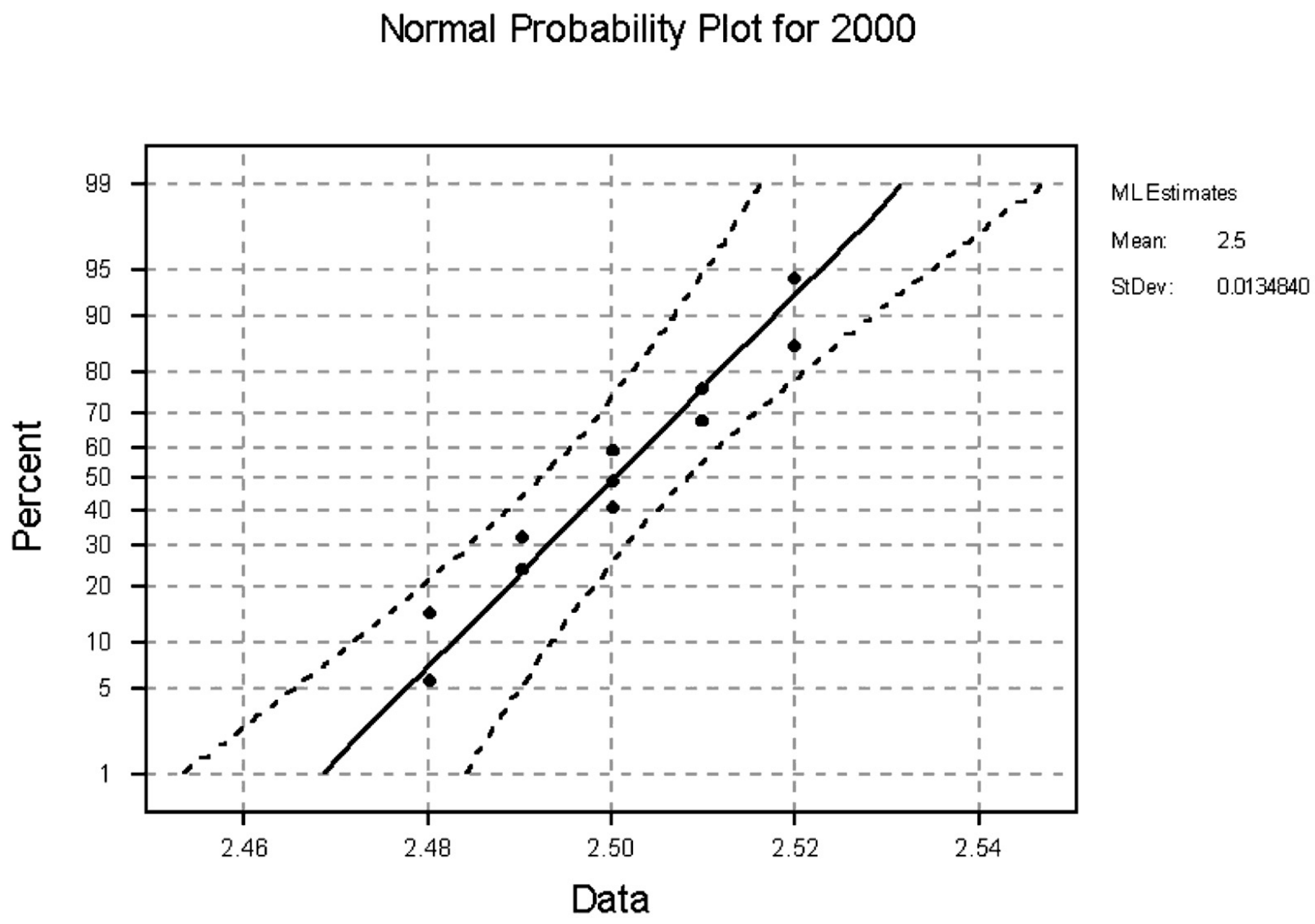
Introduction to the Practice of Statistics

Verify the requirements to perform a one-way ANOVA (9 of 12)



Introduction to the Practice of Statistics

Verify the requirements to perform a one-way ANOVA (10 of 12)



Pearson

Test a Hypothesis Regarding Three or More Means Using One-Way ANOVA (1 of 14)

The basic idea in one-way ANOVA is to determine if the sample data could come from populations with the same mean, μ, or suggests that at least one sample comes from a population whose mean is different from the others.

To make this decision, we compare the variability among the sample means to the variability within each sample.

Test a Hypothesis Regarding Three or More Means Using One-Way ANOVA (2 of 14)

We call the variability among the sample means the **between-sample variability**, and the variability of each sample the **within-sample** variability.

If the between-sample variability is large relative to the within-sample variability, we have evidence to suggest that the samples come from populations with different means.

Introduction to the Practice of Statistics

Test a Hypothesis Regarding Three or More Means Using One-Way ANOVA (3 of 14)

ANOVA *F*-Test Statistic

$$F_0 = \frac{\text{between-sample variability}}{\text{within-sample variability}} = \frac{\text{mean square due to treatments}}{\text{mean square due to error}} = \frac{\text{MST}}{\text{MSE}}$$

Test a Hypothesis Regarding Three or More Means Using One-Way ANOVA (13 of 14)

Solution

The results of the computations for the data that led to the *F*-test statistic are presented in the following table.

ANOVA Table:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	*F*-Test Statistic
Treatment	0.0009	2	0.0005	1.25
Error	0.013	30	0.0004	
Total	0.0139	32		

Test a Hypothesis Regarding Three or More Means Using One-Way ANOVA (14 of 14)

Decision Rule in the One-Way ANOVA Test

If the *P*-value is less than the level of significance, α, reject the null hypothesis.

Test a Hypothesis Regarding Three or More Means Using One-Way ANOVA (13 of 14)

Solution

The results of the computations for the data that led to the *F*-test statistic are presented in the following table.

ANOVA Table:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	*F*-Test Statistic
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Test a Hypothesis Regarding Three or More Means Using One-Way ANOVA (14 of 14)

Decision Rule in the One-Way ANOVA Test

If the *P*-value is less than the level of significance, α, reject the null hypothesis.

Post Hoc Tests on One-Way Analysis of Variance
Learning Objective

1. Perform the Tukey Test

Post Hoc Tests on One-Way Analysis of Variance
Introduction

When the results from a one-way ANOVA lead us to conclude that at least one population mean is different from the others, we can make additional comparisons between the means to determine which means differ significantly. The procedures for making these comparisons are called **multiple comparison methods.**

Post Hoc Tests on One-Way Analysis of Variance

Perform the Tukey Test (1 of 13)

The computation of the test statistic for Tukey's test for comparing pairs of means follows the same logic as the test for comparing two means from independent sampling. However, the standard error that is used is

$$\text{SE} = \sqrt{\frac{s^2}{2} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}$$

where s^2 is the mean square error estimate (MSE) of σ^2 from the one-way ANOVA, n_1 is the sample size from population 1, and n_2 is the sample size from population 2.

Post Hoc Tests on One-Way Analysis of Variance

Perform the Tukey Test (2 of 13)

The test statistic for Tukey's test when testing H_0: $\mu_1 = \mu_2$ versus H_1: $\mu_1 \neq \mu_2$ is given by

$$q_0 = \frac{\bar{x}_2 - \bar{x}_1}{\sqrt{\dfrac{s^2}{2} \cdot \left(\dfrac{1}{n_1} + \dfrac{1}{n_2} \right)}}$$

where

$\bar{x}_2 > \bar{x}_1$

s^2 is the mean square estimate of σ^2 (MSE) from ANOVA

n_1 is the sample size from population 1

n_2 is the sample size from population 2

Post Hoc Tests on One-Way Analysis of Variance
Perform the Tukey Test (3 of 13)

The critical value for Tukey's test using a familywise error rate α is given by

$$q_{\alpha,\, v,\, k}$$

where

v is the degrees of freedom due to error $(n - k)$

k is the total number of means being compared

Post Hoc Tests on One-Way Analysis of Variance Perform the Tukey Test (4 of 13)

Parallel Example 1: Finding the Critical Value from the Studentized Range Distribution

Find the critical value from the Studentized range distribution with $v = 13$ degrees of freedom and $k = 4$ degrees of freedom with a familywise error rate $\alpha = 0.01$.

Find the critical value from the Studentized range distribution with $v = 64$ degrees of freedom and $k = 6$ degrees of freedom with a familywise error rate $\alpha = 0.05$.

Post Hoc Tests on One-Way Analysis of Variance
Perform the Tukey Test (5 of 13)

Solution

a) $q_{0.01,13,4} = 5.404$

b) $q_{0.05,64,6} \approx q_{0.05,60,6} = 4.163$

Post Hoc Tests on One-Way Analysis of Variance

Perform the Tukey Test (6 of 13)

Tukey's Test

After rejecting the null hypothesis H_0: $\mu_1 = \mu_2 = \cdots = \mu_k$ the following steps can be used to compare pairs of means for significant differences, provided that

1. There are k simple random samples from k populations.
2. The k samples are independent of each other.
3. The populations are normally distributed.
4. The populations have the same variance.

Step 1: Arrange the sample means in ascending order.

Post Hoc Tests on One-Way Analysis of Variance

Perform the Tukey Test (7 of 13)

Tukey's Test

Step 2: Compute the pairwise differences $\bar{x}_i - \bar{x}_j$ where $\bar{x}_i > \bar{x}_j$.

Step 3: Compute the test statistic,

$$q_0 = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{\frac{s^2}{2} \cdot \left(\frac{1}{n_i} + \frac{1}{n_j} \right)}}$$

for each pairwise difference.

Post Hoc Tests on One-Way Analysis of Variance

Perform the Tukey Test (8 of 13)

Tukey's Test

Step 4: Determine the critical value, $q_{\alpha,v,k}$, where α is the level of significance (the familywise error rate).

Step 5: If $q_0 \geq q_{\alpha,v,k}$, reject the null hypothesis that H_0: $\mu_i = \mu_j$ and conclude that the means are significantly different.

Step 6: Compare all pairwise differences to identify which means differ.

Post Hoc Tests on One-Way Analysis of Variance

Perform the Tukey Test (9 of 13)

Parallel Example 2: Performing Tukey's Test

Suppose that there is sufficient evidence to reject H_0: $\mu_1 = \mu_2 = \mu_3$ using a one-way ANOVA. The mean square error from ANOVA is found to be 28.7. The sample means are $\bar{x}_1 = 131.8$, $\bar{x}_2 = 142.9$ and $\bar{x}_3 = 135.0$, with $n_1 = n_2 = n_3 = 9$. Use Tukey's test to determine which pairwise means are significantly different using a familywise error rate of $\alpha = 0.05$.

Post Hoc Tests on One-Way Analysis of Variance

Perform the Tukey Test (10 of 13)

Solution

Step 1: The means, in ascending order, are

$$\bar{x}_1 = 131.8, \ \bar{x}_3 = 135.0 \text{ and } \bar{x}_2 = 142.9$$

Step 2: We next compute the pairwise differences for each pair, subtracting the smaller sample mean from the larger sample mean:

$$\bar{x}_2 - \bar{x}_1 = 142.9 - 131.8 = 11.1$$

$$\bar{x}_2 - \bar{x}_3 = 142.9 - 135.0 = 7.9$$

$$\bar{x}_3 - \bar{x}_1 = 135.0 - 131.8 = 3.2$$

Post Hoc Tests on One-Way Analysis of Variance
Perform the Tukey Test (11 of 13)

Solution

Step 3: Compute the test statistic q_0 for each pairwise difference.

2-1: $$q_0 = \frac{11.1}{\sqrt{\frac{28.7}{2}\left(\frac{1}{9}+\frac{1}{9}\right)}} = 6.22$$

2-3: $$q_0 = \frac{7.9}{\sqrt{\frac{28.7}{2}\left(\frac{1}{9}+\frac{1}{9}\right)}} = 4.42$$

Post Hoc Tests on One-Way Analysis of Variance

Perform the Tukey Test (12 of 13)

Solution

3-1:

$$q_0 = \frac{3.2}{\sqrt{\frac{28.7}{2}\left(\frac{1}{9}+\frac{1}{9}\right)}} = 1.79$$

Step 4: Find the critical value using an $\alpha = 0.05$ familywise error rate with $v = n - k = 27 - 3 = 24$ and $k = 3$.
Then $q_{0.05,24,3} = 3.532$.

Post Hoc Tests on One-Way Analysis of Variance
Perform the Tukey Test (13 of 13)

Solution

Step 5: Since 6.22 and 4.42 are greater than 3.532, but 1.79 is less than 3.532, we reject H_0: $\mu_1 = \mu_2$ and H_0: $\mu_2 = \mu_3$ but not H_0: $\mu_1 = \mu_3$.

Step 6: The conclusions of Tukey's test are

$$\underline{\mu_1 \quad \mu_3} \quad \mu_2$$