MAT 114:
ELEMENTARY ALGEBRA AND TRIGONOMETRY

University of Ilorin
Centre for Open & Distance Learning

CODL

Published by the Centre for Open and Distance Learning,
University of Ilorin, Nigeria

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From the Vice Chancellor

Courseware development for instructional use by the Centre for Open and Distance Learning (CODL) has been achieved through the dedication of authors and the team involved in quality assurance based on the core values of the University of Ilorin. The availability, relevance and use of the courseware cannot be timelier than now that the whole world has to bring online education to the front burner. A necessary equipping for addressing some of the weaknesses of regular classroom teaching and learning has thus been achieved in this effort.

This basic course material is available in different electronic modes to ease access and use for the students. They are available on the University's website for download to students and others who have interest in learning from the contents. This is UNILORIN CODL's way of extending knowledge and promoting skills acquisition as open source to those who are interested. As expected, graduates of the University of Ilorin are equipped with requisite skills and competencies for excellence in life. That same expectation applies to all users of these learning materials.

Needless to say, that availability and delivery of the courseware to achieve expected CODL goals are of essence. Ultimate attention is paid to quality and excellence in these complementary processes of teaching and learning. Students are confident that they have the best available to them in every sense.

It is hoped that students will make the best use of these valuable course materials.

Professor S. A. Abdulkareem
Vice Chancellor

Foreword

Courseware remains the nerve centre of Open and Distance Learning. Whereas some institutions and tutors depend entirely on Open Educational Resources (OER), CODL at the University of Ilorin considers it necessary to develop its own materials. Rich as OERs are and widely as they are deployed for supporting online education, adding to them in content and quality by individuals and institutions guarantees progress. Doing it in-house as we have done at the University of Ilorin has brought the best out of the Course Development Team across Faculties in the University. Credit must be given to the team for prompt completion and delivery of assigned tasks in spite of their very busy schedules.

The development of the courseware is similar in many ways to the experience of a pregnant woman eagerly looking forward to the D-day when she will put to bed. It is customary that families waiting for the arrival of a new baby usually do so with high hopes. This is the apt description of the eagerness of the University of Ilorin in seeing that the centre for open and distance learning [CODL] takes off.

The Vice-Chancellor, Prof. Sulyman Age Abdulkareem, deserves every accolade for committing huge financial and material resources to the centre. This commitment, no doubt, boosted the efforts of the team. Careful attention to quality standards, ODL compliance and UNILORIN CODL House Style brought the best out from the course development team. Responses to quality assurance with respect to writing, subject matter content, language and instructional design by authors, reviewers, editors and designers, though painstaking, have yielded the course materials now made available primarily to CODL students as open resources.

Aiming at a parity of standards and esteem with regular university programmes is usually an expectation from students on open and distance education programmes. The reason being that stakeholders hold the view that graduates of face-to-face teaching and learning are superior to those exposed to online education. CODL has the dual-mode mandate. This implies a combination of face-to-face with open and distance education. It is in the light of this that our centre has developed its courseware to combine the strength of both modes to bring out the best from the students. CODL students, other categories of students of the University of Ilorin and similar institutions will find the courseware to be their most dependable companion for the acquisition of knowledge, skills and competences in their respective courses and programmes.

Activities, assessments, assignments, exercises, reports, discussions and projects amongst others at various points in the courseware are targeted at achieving the objectives of teaching and learning. The courseware is interactive and directly points the attention of students and users to key issues helpful to their particular learning. Students' understanding has been viewed as a necessary ingredient at every point. Each course has also been broken into modules and their component units in sequential order.

Courseware for the Bachelor of Science in Computer Science housed primarily in the Faculty of Communication and Information Science provide the foundational model for Open and Distance Learning in the Centre for Open and Distance Learning at the University of Ilorin.

At this juncture, I must commend past directors of this great centre for their painstaking efforts at ensuring that it sees the light of the day. Prof. M. O. Yusuf, Prof. A. A. Fajonyomi and Prof. H. O. Owolabi shall always be remembered for doing their best during their respective tenures. May God continually be pleased with them, Aameen.

Bashiru, A. Omipidan
Director, CODL

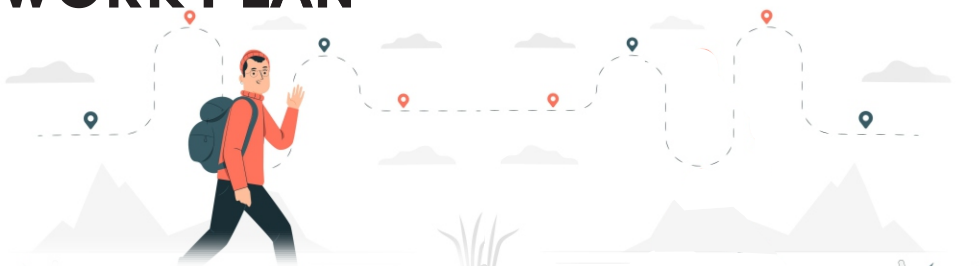
INTRODUCTION

I welcome you to Elementary Algebra and trigonometry, a second semester course. It is a 3-credit course that is available to year one undergraduate students in Faculties of Life Sciences, Physical Sciences, Engineering, Education and allied degrees. This course was designed as a foundation course for undergraduate mathematics. It consists of elementary topics from O'level mathematics and introduction to some of the rudiment topics in ad-vanced mathematics. It was prepared with the aim of introducing undergraduate students to some basic theorems and principles that will be useful in advance mathematics.

Course Goal

Your journey through this course will remind you of some basic topics like inequalities, indices and logarithm and matrices. You will also be introduced to some elementary advance topics in algebra

WORK PLAN



Learning Outcomes

At the end of this course, you should be able to:

- solve inequalities and indices problems;
- define matices and know different types of matrices;
- perform basic operations like addition and multiplication on matrices;
- find the determinant of matrices;

Course Guide

Module 1

Inequalities and Indices

Unit 1 - Inequalities

Unit 2 - Modulus of a Variable

Unit 3 - Indices

Unit 4 - Logarithms

Unit 5 - Change of Base

Module 2

Matrices

Unit 1 - Definition and Types of Matrices

Unit 2 - Addition and Subtraction of Matrices

Unit 3 - Multiplication of Matrices

Unit 4 - Determinant of a Matrix

Unit 5 - Minor, cofactor and Adjoint of a Matrix

Unit 6 - Inverse of a Matrix

Unit 7 - Application of Matrices to System of Linear Equation

Related Courses

Prerequisite: MAT 111, MAT 113

Required for: MAT 206, MAT 213

MAT 114
Elementary Algebra and Trigonometry

- obtain the inverse of matrices;
- explain binary operations and relations; and
- explain the theory of functions.

Module 3

Binary Operations and Relations

Unit 1 - Binary Operation

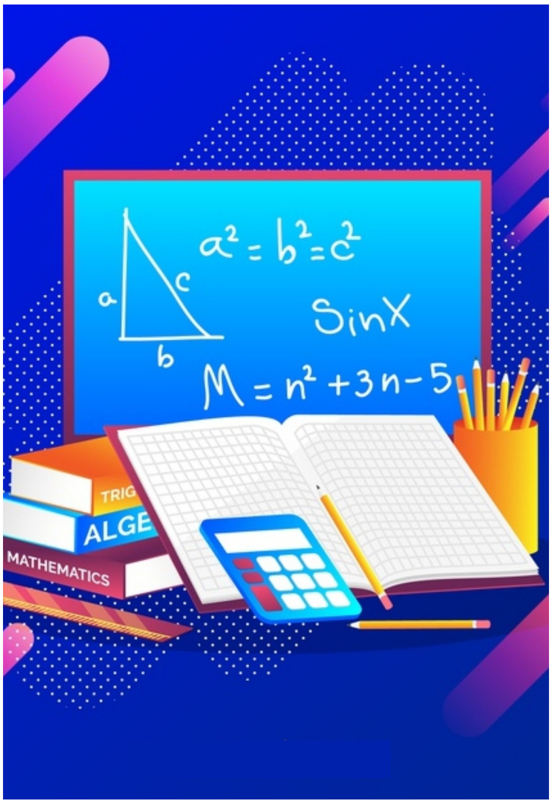
Unit 2 - Cartesian Product

Unit 3 - Relation

Module 4

Functions Theory

Unit 1 - Functions



Course Requirements

Requirements for success

The CODL Programme is designed for learners who are absent from the lecturer in time and space. Therefore, you should refer to your Student Handbook, available on the website and in hard copy form, to get information on the procedure of distance/e-learning. You can contact the CODL helpdesk which is available 24/7 for every of your enquiry.

Visit CODL virtual classroom on http://codllms.unilorin.edu.ng. Then, log in with your credentials and click on MAT 114. Download and read through the unit of instruction for each week before the scheduled time of interaction with the course tutor/facilitator. You should also download and watch the relevant video and listen to the podcast so that you will understand and follow the course facilitator.

At the scheduled time, you are expected to log in to the classroom for interaction.

Self-assessment component of the courseware is available as exercises to help you learn and master the content you have gone through.

You are to answer the Tutor Marked Assignment (TMA) for each unit and submit for assessment.

Embedded Support Devices

Support menus for guide and references

Throughout your interaction with this course material, you will notice some set of icons used for easier navigation of this course materials. We advise that you familiarize yourself with each of these icons as they will help you in no small ways in achieving success and easy completion of this course. Find in the table below, the complete icon set and their meaning.

 Introduction	 Learning Outcomes	 Main Content
 Summary	 Tutor Marked Assignment	 Self Assessment
 Web Resources	 Downloadable Resources	 Discuss with Colleagues
 References	 Futher Reading	 Self Exploration

Grading and Assessment

TMA	CA	Exam	Total
20%	20%	60%	100%

Module 1

INEQUALITIES, INDICES AND LOGARITHMS

Units

Unit 1 - Inequalities

Unit 2 - Modulus of a Variable

Unit 3 - Indices

Unit 4 - Logarithms

Unit 5 - Change of Base

UNIT 1

Inequalities

Introduction

In this unit, you will learn the concept of inequalities. It is expected that the learners should take cognizance of the fact that both equations and inequalities are mathematical sentences formed by relating two expressions to each other. In an equation, the two expressions are deemed equal which is shown by the symbol "$=$". $x = y$ implies that x is equal to y. Whereas in an inequality, the two expressions are not necessarily equal.

Thus, relational symbols like: $>$, $<$, $\leq$ or $\geq$ are employed to state the type of relationship between the expressions. $x > y$ implies that x is greater than y and $x \geq y$ means x is greater than or equal to y while $x < y$ implies that x is less than y and $x \leq y$ means x is less than or equal to y.

Learning Outcomes

At the end of this unit, you should be able to:

1. identify inequalities
2. state the rules of inequalities
3. solve inequalities using the stated rules

Main Content

The rules governing the relationship between numbers that are not equal are being considered. For any two real numbers a and b, a is said to be greater than $b(a > b)$ if $a - b$ is positive and a is said to be less than $b(a < b)$ if $a - b$ is negative.

Hence, by definition

$a > b$, if $a - b > 0$ and $a < b$, if $a - b < 0$ (1)

Rule 1: If $a > b$, then

$a + x > b + x$ (2)

where x is any real number.

Also, if $a < b$, then

$a + x < b + x$ (3)

Therefore, as with equations, same number may be added to (or subtracted from) both sides of an inequality and the inequality sign will still be maintained.

Activity 1

$8 > 3$

Add 4 to both sides

$8 + 4 > 3 + 4$

$12 > 7$

Also, by adding (-5) to both side of

$12 > 8$

$12 + (-5) > 8 + (-5)$

$12 - 5 > 8 - 5$

$7 > 3$

Rule 2:

(i) If $a > b$, then $ax > bx$, if x is positive and $ax < bx$, if x is negative

(ii) If $a < b$, then $ax < bx$, if x is positive and $ax > bx$, if x is negative

<u>To prove for the case $a > b$</u>

If $a - b = c$, where c is positive, then

$$ax - bx = cx$$

which is positive if x is positive, but negative if x is negative. Therefore

$$ax - bx > 0, \text{ if } x > 0$$

and

$$ax - bx < 0, \text{ if } x < 0$$

which gives the required result.

Hence, when multiplying both sides of an inequality by a negative number, the inequality sign must be reversed.

Rule 3:

if $a > b$ and $c > d$, then $a + c > b + d$

Activity 2

1. $6 > 2$ and $-3 > -5$

   therefore $6 + (-3) > 2 + (-5)$

   $3 > -3$

   Note: It does not follow that $a - c > b - d$.

2. $8 > 5$ and $7 > 1$

   but $8 - 7 < 5 - 1$

Rule 4:

If a > b and b > c, then a > c

3. $6 > 4$ and $4 > 2$, therefore $6 > 2$

Rule 5:

If $a > b$ and $c > d$, where a, $b.c$ and d are all positive then $ac > bd$ and $\frac{a}{d} > \frac{b}{c}$

4. $8 > 5$ and $9 > 6$

   then $8 \times 9 > 5 \times 6 = 72 > 30$

   also $\frac{8}{6} > \frac{5}{9} = 1.667 > 0.5566$

Rule 6:

If $a > b$, where a and b are both positive then

$a^2 > b^2, a^3 > b^3, \ldots$

also $\frac{1}{a} < \frac{1}{b}, \ldots$

Indeed

$a^n > b^n$ if $n > 0$

and

$a^n < b^n$ if $n < 0$

Activity 1

1. Solve the inequalities 3x + 11 > 0 and 8 − 7x > 0

Solution

$3x + 11 > 0$ and $8 - 7x > 0$

$3x > -11$ and $-7x > -8$

Dividing through by 3 and Dividing through by (−7)

$x > -\frac{11}{3}$ $\quad$ $x < \frac{8}{7}$

Therefore,

$-\frac{11}{3} < x < \frac{8}{7}$

2. Find the values of x which satisfy $2x^2 - 7x + 9 < x2 - 2x + 3$.

Solution

Given $2x^2 - 7x + 9 < x2 - 2x + 3$

$2x^2 - 7x + 9 - x^2 + 2x - 3 < 0$

$2x^2 - x^2 - 7x + 2x + 9 - 3 < 0$

$x^2 - 5x + 6 < 0$

$(x - 2)(x - 3) < 0$

This will be true if x − 2 < 0 and x − 3 > 0 or if x − 2 > 0 and x − 3 < 0. This can be clearly seen if table showing the signs of the factors is drawn up. Hence, the original inequality is true if 2 < x < 3.

	$x < 2$	$2 < x < 3$	$x > 3$
$x - 2$	-ve	+ve	+ve
$x - 3$	-ve	-ve	+ve
$(x - 2)(x - 3)$	+ve	-ve	+ve

3. For what values of x is $\frac{1}{x-3} < -1$?

Solution

Given $\frac{1}{x-3} < -1$

In order to remove the fraction, both sides of the equality is multiplied by $(x-3)^2$ which is positive, so as to preserve the inequality sign $(x-3)^2 \frac{1}{(x-3)} < -1(x-3)^2$

$x - 3 < -(x - 3)^2$

$x - 3 + (x - 3)^2 < 0$

$x - 3 + x^2 - 6x + 9 < 0$

$x^2 - 5x + 6 < 0$

$(x - 2)(x - 3) < 0$

The table showing the signs of the factors is then given

	$x < 2$	$2 < x < 3$	$x > 3$
$x - 2$	-*ve*	+*ve*	+*ve*
$x - 3$	-*ve*	-*ve*	+*ve*
$(x - 2)(x - 3)$	+*ve*	-*ve*	+*ve*

Hence, the original inequality is satisfied by $2 < x < 3$

4. Solve the inequality $\frac{x-1}{x-2} > \frac{x-2}{x-3}$.

Solution

$\frac{x-1}{x-2} > \frac{x-2}{x-3}$

multiply both sides by $(x - 2)^2(x - 3)^2$

$(x - 2)^2(x - 3)^2. \frac{x-1}{x-2} > \frac{x-2}{x-3}. (x - 2)2(x - 3)2.$

$(x - 2)(x - 3)^2(x - 1) > (x - 2)(x - 2)^2(x - 3)$

$(x - 2)(x - 3)^2(x - 1) - (x - 2)(x - 2)^2(x - 3) > 0$

$(x - 2)(x - 3)[(x - 3)(x - 1) - (x - 2)^2] > 0$

$(x - 2)(x - 3)[x^2 - 4x + 3 - x^2 + 4x - 4] > 0$

$(x - 2)(x - 3)(-1) > 0$

$(x - 2)(x - 3) < 0$

Table of signs of the factors

	$x < 2$	$2 < x < 3$	$x > 3$
$x - 2$	-*ve*	+*ve*	+*ve*
$x - 3$	-*ve*	-*ve*	+*ve*
$(x - 2)(x - 3)$	+*ve*	-*ve*	+*ve*

Hence, the inequality is satisfied by $2 < x < 3$

5. Determine the range of values of x for which $x^2+x-2x^2+4 > \frac{1}{2}$

Solution

$x^2+x-2x^2+4 >12$

Take note that x^2+4 is the sum of two squares, which will always be positive. Hence, multiply both sides of the inequality by $x^2 + 4$ and the signs will still be preserved.

Therefore,

$x^2 + x - 2 >12(x^2 + 4)$

$2x^2 + 2x - 4 > x^2 + 4$

$x^2 + 2x - 8 > 0$

$(x + 4)(x - 2) > 0$

Table showing the sign of factors

	$x < -4$	$4< x <2$	$x > 2$
$x + 4$	$-ve$	$+ve$	$+ve$
$x - 2$	$-ve$	$-ve$	$+ve$
$(x + 4)(x - 2)$	$+ve$	$-ve$	$+ve$

Hence, the inequality is satisfied by $x < -4$ or $x > 2$

6. Solve the inequality $\dfrac{2x^2+5x+7}{3x+5} \geq 2$

Solution

Given $\dfrac{2x^2+5x+7}{3x+5} \geq 2$

multiply both sides by $(3x + 5)^2$ to obtain

$(3x + 5)(2x^2 + 5x + 7) \geq 2(3x + 5)^2$

$(3x + 5)(2x^2 + 5x + 7) - 2(3x + 5)^2 \geq 0$

$(3x + 5)[2x^2 + 5x + 7 - 6x - 10] \geq 0$

$(3x + 5)(2x^2 - x - 3) \geq 0$

$(3x + 5)(x + 1)(2x - 3) \geq 0$

$x \geq -\frac{5}{3}, x \geq -1, x \geq \frac{3}{2}$

Table showing signs of factors

	$x \geq \frac{-5}{3}$	$\frac{-5}{3} \leq x \leq -1$	$-1 \leq x \leq \frac{3}{2}$	$x \geq \frac{3}{2}$
$3x+5$	-ve	+ve	+ve	+ve
$x+1$	-ve	-ve	+ve	+ve
$2x-3$	-ve	-ve	-ve	+ve
$(3x+5)(x+1)(2x-3)$	-ve	+ve	-ve	+ve

Hence, the original inequality is true $-\frac{5}{3} < x < 1$ and $x > \frac{3}{2}$

Summary

The concept of inequalities was introduced to you in this unit with some fundamental theorem on algebra of inequalities. You should keep in mind that the sense of inequality is preserved when addition or subtraction operations are performed while the sense changes when an inequality is multiplied by a negative scalars. It is left for you to verify that the basic of laws inequality are true.

Self Assessment Questions

1. Clearly state the rules of inequalities
2. Solve the inequalities below
   (a) $3x - 11 < 2$ and $8 + 7x < 2$
   (b) $\frac{x+1}{x+2} > \frac{x+2}{x+3}$

Tutor Marked Assignment

- Find the value of x which satisfies the inequalities
  (a) $\frac{2x-1}{x+3} < \frac{2}{3}$

- Solve the inequality $\frac{2x^2+5x+7}{3x+5} \geq 2$

- Solve for x in

(a) $\frac{x-2}{x-3} > \frac{x-3}{x-5}$

References

- Arthur Lohwater (1982). ”Introduction to Inequalities”. Online e-book in PDF format.
- Hardy, G., Littlewood J. E., Polya, G. (1999). Inequalities. Cambridge Mathematical Library, Cambridge University Press. ISBN 0-521-05206-8.
- Beckenbach, E. F., Bellman, R. (1975). An Introduction to Inequalities. Random House Inc. ISBN 0-394-01559-2.
- https://en.wikipedia.org/wiki/Inequality(mathematics)
- https://www.mathplanet.com/education/pre-algebra/introducing-algebra/inequalities
- https://www.mathsisfun.com/algebra/inequality.html

Further Reading

- Hazewinkel, Michiel, ed. (2001) [1994], ”Inequality”, Encyclopedia of Mathematics, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- Graph of Inequalities by Ed Pegg, Jr., Wolfram Demonstrations Project. AoPS Wiki entry about Inequalities

UNIT 3

Modulus of a Variable

Introduction

Having been introduced to the concept of inequality in the preceding unit, you will now learn a very important application of inequality which involves the modulus. From elementary algebra, the modulus of any number returns the magnitude of the number without considering the sign in front of it.

Learning Outcomes

At the end of this unit, you should be able to:

1. solve problems on inequalities
2. solve problems on inequalities involving modulus
3. define the modulus of a variable

Main Content

The modulus of x is the positive number having the same magnitude as x. It is written as $|x|$.

Thus, $|2| = 2$, $|-3| = 3$, $|-2| = 2$ etc.

In general, if x is positive $|x| = x$, but if x is negative $|-x| = x$. With this notation, the range of values of x specified by the inequality $-1 < x < 1$, can be put more concisely by $|x| < 1$.

Activity 1

1. Find x if $|x+3| \leq 2$

   $x+3 \leq 2$ or $-(x+3) \leq 2$

   $x \leq 2-3$ or $x+3 \geq -2$

   $x \leq -1$ or $x \geq -5$

   $\implies -5 \leq x \leq -1$

2. Find x if $\left|\frac{1}{x+1}\right| = 1$

   $\frac{1}{x+1} = 1$ or $-\frac{1}{x+1} = 1$

   $x+1 = 1$ or $\frac{1}{x+1} = -1$

   $x = 0$ or $x = -1-1 = -2$

3. Find x if $|x+3| > 5$

   $x+3 > 5$ or $-(x+3) > 5$

   $x > 5-3$ or $x+3 < -5$

   $x > 2$ or $x < -8$

4. Find x, if $|x-1| > 3|x-2|$.

   Solution

   $|x-1| > 3|x-2|$.

   $\frac{|x-1|}{|x-2|} > 3$

Hence,

(i) $\frac{x-1}{x-2} > 3$ or (ii) $\frac{x-1}{x-2} < -3$

(i) $(x-2)^2\frac{x-1}{x-2} > 3(x-2)^2$

$(x-2)(x-1) - 3(x-2)^2 > 0$

$(x-2)[x-1-3x+6] > 0$

$(x-2)(-2x+5) > 0$

$(x-2)(2x-5) < 0$

The table of signs of the factors

	$x<2$	$2<x<\frac{5}{2}$	$x>\frac{5}{2}$
$x-2$	$-ve$	$+ve$	$+ve$
$2x-5$	$-ve$	$-ve$	$+ve$
$(x-2)(2x-5)$	$+ve$	$-ve$	$+ve$

The values of x that satisfy the first inequality is $2 < x < \frac{5}{2}$

(ii) $\frac{x-1}{x-2} < -3$

$(x-2)^2\frac{x-1}{x-2} < -3(x-2)^2$

$(x-2)(x-1) + 3(x-2)^2 < 0$

$(x-2)[x-1+3x-6] < 0$

$(x-2)(4x-7) < 0$

Table of Signs of the factors

Hence, the second inequality is satisfied by $\frac{7}{4} < x < 2$

	$x<\frac{7}{4}$	$\frac{7}{4}<x<2$	$x>2$
$x-2$	$-ve$	$-ve$	$+ve$
$4x-7$	$-ve$	$+ve$	$+ve$
$(x-2)(4x-7)$	$+ve$	$-ve$	$+ve$

Summary

In this unit, we have learnt that the modulus of both positive and negative numbers, is always the number. For example, the modulus of $-\frac{1}{5}$ is $\frac{1}{5}$. i.e. $|-\frac{1}{5}| = \frac{1}{5}$. In the same view, the modulus of 4 is 4.i.e. $|4| = 4$.

Self Assessment Questions

1. What do you understand by modulus?
2. Solve the inequality: $|x - 10| < 3$
3. When is a variable y said to have a modulus? Hence what is the modulus of 2 and −2?

Tutor Marked Assignment

- Find the value of x which satisfies the inequalities

  (a) $\frac{2x-1}{x+3} < \frac{2}{3}$

  (b) $|x + 3| > 5$
- Solve the inequality $\left|\frac{2x^2+5x+7}{3x+5}\right| \geq 2$
- Solve for x in

  (a) $|x - 1| > 3|x - 2|$

References

- Arthur Lohwater (1982). "Introduction to Inequalities". Online e-book in PDF format.
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- Graph of Inequalities by Ed Pegg, Jr., Wolfram Demonstrations Project. AoPS Wiki entry about Inequalities

UNIT 3

Indices

Introduction

A power, or an index, is used to write a product of numbers very compactly. The plural of index is indices. In this unit, you will learn how this is done. You will also learn some rules or laws that can be used to simplify expressions involving indices. You will also learn indicial equations with worked examples.

Learning Outcomes

At the end of this unit, you should be able to:

1. state clearly the rules of indices
2. solve different forms of indices using the stated rules
3. solve indicial equations

Main Content

Operations of Indices and their rules In the operation of multiplication and division with different powers of the same number, the indices are combined according to certain fundamental laws. Provided m and n are positive integers, the following rules holds.

Rule 1:

$a^m \times a^n = a^{m+n}$

Rule 2:

$a^m \div a^n = a^{m-n}$

Rule 3:

$a^0 = 1$

Rule 4:

$(a^m)^n = (a^n)^m = a^{mn}$

Rule 5:

$a^{-m} = \frac{1}{a}^m$

Rule 6:

$a^{\frac{1}{m}} = \sqrt[m]{a}$

Rule 7:

$a^{\frac{n}{m}} = (a^{\frac{1}{m}})^n = (\sqrt[m]{a})^n = (a^n)^{\frac{1}{m}} = \sqrt[m]{a^n}$

Activity 1

1. Evaluate

   (i) $27^{\frac{5}{3}}$

   (ii) $(36)^{\frac{-3}{2}}$

   (iii) $8^{\frac{7}{3}}$

   (iv) $16^{\frac{-1}{4}}$

2. Express with positive indices

   (i) $\frac{x^{-2}y^3z^{-4}}{6} \times \frac{9}{x^3y^{-3}z^4}$

   (ii) $\frac{\sqrt[3]{abc^{-4}}}{\sqrt[4]{a^3b^{-3}c}}$

3. $(4 \times 2^{n+1} - 2^{n+2})/(2^{n+1} - 2^n)$.

Solution

1. (i) $27^{\frac{5}{3}} = (3^3)^{\frac{5}{3}} = 3^5 = 243$

   (ii) $(36)^{\frac{-3}{2}} = (6^2)^{\frac{-3}{2}} = 6^{-3} = \frac{1}{216}$

(iii) $8^{\frac{7}{3}} = (2^3)^{\frac{7}{3}} = 2^7 = 128$

(iv) $16^{\frac{-1}{4}} = (2^4)^{\frac{-1}{4}} = 2^{-1} = \frac{1}{2}$

2. (i) $\frac{x^{-2}y^3z^{-4}}{6} \times \frac{9}{x^3y^{-3}z^4} = \frac{3}{2}x^{-2-3}y^{3-(-3)}z^{-4-4} = \frac{3}{2}x^{-5}y^6z^{-8} = \frac{3y^6}{2x^5z^8}$

(ii) $\frac{\sqrt[3]{abc^{-4}}}{\sqrt[4]{a^3b^{-3}c}} = \frac{a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{-4}{3}}}{a^{\frac{3}{4}}b^{\frac{-3}{4}}c^{\frac{1}{4}}} = a^{\frac{1}{3}-\frac{3}{4}} \times b^{\frac{1}{3}+\frac{3}{4}} \times c^{\frac{-4}{3}-\frac{1}{4}} = a^{-\frac{5}{12}}b^{\frac{13}{12}}c^{-\frac{19}{12}} = \sqrt[12]{\frac{a^5}{b^{13}c^{19}}}$

3. $(4 \times 2^{n+1} - 2^{n+2})/(2^{n+1} - 2^n) = (2^2 \times 2^{n+1} - 2^{n+2})/(2^{n+1} - 2^n) =$ $(2^{n+3} - 2^{n+2})/(2^{n+1} - 2^n) = 2^{n+2}(2-1)/2^n(2-1) = 2^{n+2}(1)/2^n(1) =$ $2^{n+2}/2^n = 2^{n+2} \times 2^{-n} = 2^{n+2-n} = 2^2 = 4$

Indicial Equation

This is an equation in which the unknown is an index. Some of the techniques used to solve this type of equations are demonstrated as follows:

Activity 2

1. Solve the equation $4^{2x} = 2^{6x-1}$

Solution

$4^{2x} = 2^{6x-1}$

$(2^2)^{2x} = 2^{6x-1}$

Equating the powers

$4x = 6x - 1$

$1 = 6x - 4x = 2x$

$x = \frac{1}{2}$.

Summary

In this unit, you have learnt the rules of indices. You have also learnt how to solve different forms of indices using the rules.

Self Assessment Questions

1. State all the rules of indices you know.
2. Evaluate

(a) $7^6 \times 7^4$

(b) $x^2 \times x^{-4}$

(c) $3^{\frac{1}{3}} \times 3^3$

(d) $(Z^{\frac{1}{2}})^3$

Tutor Marked Assignment

- Solve for x

(a) $3^{2x-1} = 6x$

(b) $3^{4x+2} = 9^{3x-1}$

(C) $2^{x2} = \frac{1}{4} + 8x$

References

- The Ultimate Guide to Logarithm — Theory and Applications". Math Vault. 8 May 2016. Retrieved 24 July 2019
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Further Reading

- http://www.mathcentre.ac.uk/resources/Engineering
- https://www.mathsisfun.com/definitions/index-power-.html

UNIT 4

Logarithm

Introduction

In this unit, I will be teaching you logarithms (which is the inverse function to exponentiation). The logarithm of a given number x is the exponent to which another fixed number, the base b, must be raised, to produce that number x. In the simplest case, the logarithm counts the number of occurrences of the same factor in a repeated multiplication. You should gradually follow the teaching from indices to logarithms to understand the concepts of indices and logarithms.

Learning Outcomes

At the end of this unit, you should be able to:

1. state the rules of logarithms
2. solve logarithms with its rules
3. solve indices with unknown variables

Main Content

Basic Idea of Logarithm and Rules

By theory of logarithm, the defining relation between exponentiation and logarithm is more explicitly given as

$Log_a N = x$ (1)

which implies that

$N = a^x$ (2)

by definition equivalent, we have

$a^{Log_a N} = N$

Since $a^1 = a$ and $a^0 = 1$.

Thus

$Log_a a = 1$ (3)

and

$Log_a 1 = 0$ (4)

for all $a \neq 0$

Examples: Evaluate (i) $log_3 27$ (ii)$log_{\frac{1}{3}} 27$ (iii)$log_{y^3} y$ (iv) $log_{\frac{1}{x}} x^6$ (v)$log_{\frac{1}{x}} x^n$

Solution

1. $log_3 27 = x$

   $27 = 3^x$

   $3^3 = 3^x$

   Equality of powers gives

   $x = 3$

2. $log_{\frac{1}{3}} 27 = x$

   $3^3 = (\frac{1}{3})^x$

   $3^3 = 3^{-x}$

   Equality of power

   $x = -3$

3. $log_{y^3} y = x$

   $y = y^{3x}$

   $x = \frac{1}{3}$

4. $log_{\frac{1}{x}} x^6 = k$

   $x^6 = \frac{1}{x}^k = x^{-k}$

   $x^6 = x^{-k}$

   $k = -6$

5. $log_{\frac{1}{x}} x^n = k$

$x^n = x^{-k}$

$k = -n$

the laws for the manipulation of logarithms are deduced directly from the laws of indices

$log_a xy = log_a x + log_a y$ 1

If $log_a x = M$ and $log_a y = N$

$x = a^M$ and $y = a^N$

$xy = a^M.a^N = a^{M+N}$

Therefore, $log_a xy = M + N = log_a x + log_a y$

$log_a \frac{x}{y} = log_a x - log_a y$ 2

From the initial notation

$\frac{x}{y} = \frac{a^M}{a^N} = a^{M-N}$

$log_a \frac{x}{y} = M - N = log_a x - log_a y$

$log_a x^p = p log_a x$ 3

from the initial notation

$x^p = (a^M)^p = a^{Mp}$

$log_a x^p = pM = p log_a x$

Activity 1

1. Show that $log_a(a+b)^2 = 2 + log_a(1 + \frac{2b}{a} + \frac{b^2}{a^2})$

2. Simplify (i) $\frac{log\sqrt{3}}{log9}$ (ii) $\frac{log0.2}{log25}$ (iii) $log8 - log4$ (iv) $log8 + log4$ (v) $\frac{log8-log4}{log4-log2}$

Solution

1. $log_a(a+b)^2 = log_a(a^2 + 2ab + b^2)$

   $log_a(a^2(1 + \frac{2b}{a} + \frac{b^2}{a^2}))$

   $log_a a^2 + log_a(1 + \frac{2b}{a} + \frac{b^2}{a^2})$

   $log_a(a+b)^2 = 2 + log_a(1 + \frac{2b}{a} + \frac{b^2}{a^2})$

2. (i) $\frac{log\sqrt{3}}{log9} = \frac{log3^{\frac{1}{2}}}{log3^2} = \frac{\frac{1}{2}log3}{2log3} = \frac{1}{4}$

   (ii) $\frac{log0.2}{log25} = \frac{log\frac{2}{10}}{log5^2} = \frac{log5^{-1}}{2log5} = \frac{-1log5}{2log5} = -\frac{1}{2}$

   (iii) $log8 - log4 = log\frac{8}{4} = log2$

   (iv) $log8 + log4 = log(8 \times 4) = log32 = 5log2$

   (v) $\frac{log8-log4}{log4-log2} = \frac{log(\frac{8}{4})}{log(\frac{4}{2})} = \frac{log2}{log2} = 1$

Solving an equation in which the unknown is an index with Logarithm

Some of the techniques used to solve this type of equations are demonstrated as follows:

Activity 2

1. Find x, if $3^x = 7.83$

   Take log of both sides to have

   $log3^x = log7.83$

   $xlog3 = log7.83$

   $x = \frac{log7.83}{log3}$.

2. Solve for x

(i) $3^{2x-1} = 5^x$

(ii) $7^{4x+2} = 9^{3x-1}$

Solution

(i) $3^{2x-1} = 5^x$

$log3^{2x-1} = log5^x$

$(2x-1)log3 = xlog5$

$2xlog3 - xlog5 = log3$

$x = \frac{log3}{2log3-log5}$.

(ii) $7^{4x+2} = 9^{3x-1}$

$log7^{4x+2} = log9^{3x-1}$

$(4x+2)log7 = (3x-1)log9$

$4xlog7 - 3xlog9 = -(log9 + 2log7)$

$x(4log7 - 3log9) = -(log9 + 2log7)$

$x = \frac{-(log9+2log7)}{4log7-3log9}$.

3. Find x, if $9^{x^2} = 3^{5x-2}$

$3^{2x^2} = 3^{5x-2}$

equating the powers

$2x^2 = 5x - 2$

$2x^2 - 5x + 2 = 0$

$(2x-1)(x-2) = 0$

$x = \frac{1}{2}$ or $x = 2$

4. Solve the equation

$5^{2x} - 5^{1+x} + 6 = 0$

$(5^x)^2 - 5(5^x) + 6 = 0$

Let $p = 5^x$

$p^2 - 5p + 6 = 0$

$(p-3)(p-2) = 0$

$p = 3 \quad or \quad 2$

When $p = 3$

then $5^x = 3$

$xlog5 = log3$

$x = \frac{log3}{log5}$

also when $p = 2$

then $5^x = 2$

$xlog5 = log2$

$x = \frac{log2}{log5}$.

5. Solve the simultaneous equations

$2^{x+y} = 6; \; 3^{x-y} = 4$

Solution

$2^{x+y} = 6...(1)$

$3^{x-y} = 4...(2)$

$log2^{x+y} = log6$

$(x+y)log2 = log6$

$x + y = \frac{log6}{log2} = \frac{0.7782}{0.3010} = 2.5854...(3)$

$log3^{x-y} = log4$

$(x-y)log3 = log4$

$x - y = \frac{log4}{log3} = \frac{0.6021}{0.4771} = 1.2620 \ldots \quad (4)$

$x + y = \frac{log6}{log2} \ \ldots \quad (3)$

$x - y = \frac{log4}{log3} \ \ldots \quad (4)$

Adding equation (3) and (4), we have

$2x = \frac{log6}{log2} + \frac{log4}{log3}$

$x = \frac{1}{2}\left[\frac{log6}{log2} + \frac{2log2}{log3}\right] \implies \frac{1}{2}[2.5854 + 1.2620] = \frac{1}{2}[3.8474]; x = 1.9237 \ldots \quad (5)$

Subtracting equation (4) from equation (3)

$2y = \frac{log6}{log2} - \frac{log4}{log3}$

$y = \frac{1}{2}\left[\frac{log6}{log2} + \frac{2log2}{log3}\right] \ \frac{1}{2}[2.5854 - 1.2620] = \frac{1}{2}[1.3234]; y = 0.6617 \ldots \quad (6)$

6. Solve the equation

$2^x.3^{1-x} = 6$

$2^x.3^{1-x} = 2.3$

$\frac{2^x}{2} = \frac{3}{3^{1-x}}$

$2^{x-1} = 3^{1-(1-x)}$

$2^{x-1} = 3^{1-1+x}$

$log2^{x-1} = log3^x$

$(x-1)log2 = xlog3$

$xlog2 - xlog3 = log2$

$x = \frac{log2}{log2 - log3}$

Summary

Logarithm is a convenient and important tool in mathematics with wide applications. It is used to simplify calculations. Hence, you are advised, not to only read through, but also master its application.

Self Assessment Questions

1. If $\log_a b = \log_b c = \log_c a$ show that $a = b = c$.
2. If $p^2 = qr$, show that $\log_q p + \log_r p = 2 \log_q p \log_r p$.
3. Solve $\log_3 x + \log_x 3 = \frac{10}{3}$.
4. Given that $\log_2(x - 5y + 4) = 0$ and $\log_2(x + 1) - 1 = 2 \log_2 y$, find the values of $xandy$.
5. If $2 \log_y x + 2 \log_x y = 5$. Show that $\log_y x$ is either $\frac{1}{2}or2$.
6. Solve for x;
   (a) if $2^x(3^{x-1}) = 5^{2x-1}$
   (b) if $\log_x 8 - \log_{x^2} 16 = 1$

Tutor Marked Assignment

- Find x, if $\log_x 8 - \log_{x2} 16 = 1$
- Find x, if $\log_x 3 + \log_3 x = 2.5$
- Solve the simultaneous equations $2^{x+y} = 6$; $^{3x-y} = 4$
- Express without logarithms

  (a) $\log_e A = \text{loge}\, P + K$

  (b) $\log_4 K - \log_4 P - \log_4 T + \log_{16} W$

References

- The Ultimate Guide to Logarithm — Theory and Applications". Math Vault. 8 May 2016. Retrieved 24 July 2019
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Further Reading

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- http://www.mathcentre.ac.uk/resources/Engineering
- https://www.mathsisfun.com/definitions/index-power-.html

Tutor Marked Assignment

- If $A = A_1 i + A_2 j + A_3 k$ and $B = B_1 i + B_2 j + B_3 k$, prove that $A.B = A_1 B_1 + A_2 B_2 + A_3 B_3$.

References

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- Blitzer, R. (2009) Introductory and Intermediate Algebra For College Students, Pearson Prentice Hall, Upper Saddle River, New Jersey.

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UNIT 5

Change of Base

Introduction

Another important feature of logarithm is its flexibility in usage. By definition, the logarithm of a number to a certain base is the power to which the base must be raised to obtain the number. In this unit, you will therefore learn how a logarithm can be manipulated so that both the number and the base can yield the result of the original logarithm.

Learning Outcomes

At the end of this unit, you should be able to:

1. state the rules of change of base of logarithms
2. evaluate the logarithms without using logarithm table
3. solve problems involving change of base

Main Content

Change of base

Logarithms with different bases have different values. However, the different values are related to each other as it can be seen below.

$$Let\ y = \log_b N \qquad (4)$$

$$\Rightarrow N = b^y$$

$$\log_a N = \log_a b^y = y \log_a b = \log_b N \log_a b \text{ from } (1)$$

$$\therefore \log_b N = \frac{\log_a N}{\log_a b} \qquad (5)$$

Activity 1

1. Evaluate the logarithms without using logarithm table
   (a) $\log_4 320$ in terms of x where $x = \log_2 10$
   (b) $\log_2 0.001$ where $\log_{10} 2 = 0.301$
2. Simplify $\log_a b.\log_b c.\log_c a$

Solution

1 a. $\log_4 320 = \log_4(10 \times 2^5)$

$$= \log_4 10 + \log_4 2^5$$

$$= \log_4 10 + \frac{\log_2 2^5}{\log_2 4}$$

$$= \frac{\log_2 10}{\log_2 4} + \frac{5\log_2 2}{2\log_2 2}$$

$$= \frac{x}{2} + \frac{5}{2}$$

$$= \frac{x+5}{2}$$

b. $\log_2 0.001 = \frac{\log_{10} 0.001}{\log_{10} 2} = \frac{\log_{10} 10^{-3}}{\log_{10} 2}$

$$\frac{-3}{0.3010} \times 100 = -\frac{300}{301}$$

2. Simplify $\log_a b.\log_b c.\log_c a$

Solution

$$\log_a b.\log_b c = \log_a c$$

$$\therefore \log_a b.\log_b c.\log_c a = \log_a c.\log_c a$$

$$= \log_a a = 1$$

Summary

An important result on change of base of a logarithm was highlighted with worked examples in this unit. It is expected of you to lay your hand on more problems involving change of base and logarithm at large.

Self Assessment Questions

1. State the rule of change of base of logarithm

2. Evaluate $\log_4 1024$

Tutor Marked Assignment

- Evaluate without using tables.

  (a) $\log_4 12$, given that $\log_{23} = 1.585$

  (b) $\log_3 0.01$ where $\log_{10} 3 = y$

References

- The Ultimate Guide to Logarithm — Theory and Applications". Math Vault. 8 May 2016. Retrieved 24 July 2019

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Module 2

MATRICES

Units

UNIT 1

Definition and Types of Matrices

Introduction

Array of numbers in a specific way is known as matrices. For example, the queues by JS 1 to SS 3 in an assembly make it easy for anyone to locate a student's class and colleagues. In this unit, you will be learning the term matrix and types of matrices. The singular form is matrix while the plural is matrices.

Learning Outcomes

At the end of this unit, you should be able to:

1. define a matrix
2. list the types of matrices

Main Content

A matrix is a rectangular array of numbers from a field (e.g the field of real numbers) enclosed within curved brackets or square brackets.

In general, matrix is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \vdots\vdots\vdots & a_{mn} \end{pmatrix}$$

We usually use a capital letter to denote a matrix and small letters to denote the entries. *A* matrix having *m* rows and *n* columns is said to be an $m \times n$ matrix (read as "*m* by *n*" matrix). Thus the order of such matrix is $m \times n$.

Each element (number) in a matrix is called an entry. The entry of a matrix can be specified by rows and columns of its location (i.e a_{22} means entry located at row 2 and column 2. a_{32} means entry located at row 3 and column 2).

Thus, a_{ij} is the entry located in the ith row and the jth column. The matrix A is sometimes written in a short form as

$$A = (a_{ij})_{m,n} \text{ or } A = a_{ij}$$

Types of Matrices

(a) Row Matrix is a matrix with only one row and any number of column.
e.g. (1) [2,8,4,10] is a (1×4) matrix. (2) $[a_1, a_2, a_3 . . . . , an]$ is a $(1 \times n)$ matrix.
Row matrix is sometimes called a row vector.

(b) Column Matrix is a matrix with only one column and any number of row which is sometimes called a column vector. e.g. $\begin{bmatrix} 2 \\ 8 \\ 4 \\ 10 \end{bmatrix}$ is a $(4 \text{ x } 1)$ matrix.

$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{bmatrix}$ is am $(m \text{ x } 1)$ matrix.

(c) Null Matrix or Zero matrix :- Any matrix in which all its entries are zero is called a Null matrix.

e.g $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) Square Matrix is a matrix in which the number of rows is equal to the number of columns. e.g

$$(1)\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\ (2)\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}\ (3)\begin{bmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ a_{31} & a_{32} & \dots & a_{3n}\\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nn}\end{bmatrix}$$

(e) Diagonal matrix is a square matrix in which all its entries off diagonal elements are zero e.g

$$(1)\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{bmatrix}\ (2)\begin{bmatrix}a & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 0 & 0 & d\end{bmatrix}$$

(f) Scalar matrix is a diagonal matrix in which all the diagonal elements are equal to a scalar say (k)

$$(i)\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}\ (ii)\begin{bmatrix}-5 & 0 & 0 & 0\\ 0 & -5 & 0 & 0\\ 0 & 0 & -5 & 0\\ 0 & 0 & 0 & -5\end{bmatrix}$$

i.e $A = (a_{ij})_{n*n}$ is a scalar matrix

$$\text{If } a_{ij} = \begin{cases} 0; \text{ when } i \neq j \\ k; \text{ when } i = j \end{cases}$$

(g) Identity Matrix is a square matrix in which all its diagonal elements are 1 (unity) and non-diagonal elements are zero.

$$\text{e.g } (1)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\ (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\ (3)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

(h) Symmetric Matrix is a square matrix in which $a_{ij} = a_{ji}$ e.g. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

In other words, a square matrix A is symmetric if $A = A^T$ (A is the same as its transpose)

(i) Skew Symmetric Matrix is a square matrix in which $a_{ij} = -a_{ji}$ for all values of i and j e.g.

$$\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & c \end{bmatrix}$$

(j) Triangular Matrix is a square matrix in which all the elements below its leading diagonal are zero which will be called <u>upper triangular matrix</u>. Also, a square matrix in which all elements above its leading diagonal are zero is called <u>lower triangular matrix</u>.

$$\text{e.g.} \quad \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 6 & 7 \end{bmatrix}$$

Upper triangular matrix and lower triangular matrix respectively.

(k) <u>Transpose of a Matrix</u> is a matrix obtained from the interchanging of the rows and corresponding columns and is denoted by A' or A^T e.g.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}; \; A^T = A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(l) Orthogonal Matrix is a square matrix in which the product of the matrix and its transpose gives the Identity matrix.

$$\text{e.g} \quad A.A^T = I$$

(m) Conjugate of a Matrix

$$\text{Let} A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+i2 & -i & 3-2i \end{bmatrix}$$

The conjugate of a matrix A is

$$\overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

(n) Transpose of the conjugate of a matrix A is denoted by A^Q

$$(\overline{A})^T = A^Q = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3i+2 \end{bmatrix}$$

(o) Equal Matrices.

Two matrices are said to be equal if

(i) They are of same order

(ii) The elements in the corresponding positions are equal

e.g. $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Therefore, $A = B$

(p) Singular Matrix is a square matrix whose determinant is zero. e.g.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \Rightarrow |A| = [6-6] = 0$$

Activity 1

Find the values of x, y, z and a which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Solution

Since the given matrices are equal, then the corresponding elements are equal

:-$x + 3 = 0 \qquad \Rightarrow x = -3$

$z \quad 1 = 3 \qquad \Rightarrow z = 4$

$2y + x = -7$, But, x=-3

$2y = -4$

$y = -2$

$4a - 6 = 2a$

$4a - 2a = 6$

$a = 3$

Thus $x = -3, y = -2, z = 4$ and $a = 3$.

Summary

In this unit, you have been taught basic concepts of matrices like definition of a matrix, types of matrices and equality of matrices. It is important to state that equality of matrices has very wide applications in Mathematics and engineering.

Self Assessment Questions

1. What do you understand by the word matrix?
2. When is a matrix said to be a singular matrix?
3. Differentiate between a diagonal matrix and triangular matrix

Tutor Marked Assignment

- What do you understand by skew-symmetric matrix? Give two examples of skew-symmetric matrix
- Differentiate between a diagonal matrix and an identity matrix of the same order with examples.

References

- A. Cayley, A memoir on the theory of matrices. Phil. Trans. 148 185817-37; Math. Papers II 475-496
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UNIT 2

Addition and Subtraction of Matrices

Introduction

In this unit, you will learn algebra of matrices. You will also learn addition and subtraction together with scalar multiplication of matrices.

Learning Outcomes

At the end of this unit, you should be able to:

1. add two matrices together
2. subtract two matrices from each other

Main Content

Addition and Subtraction of Matrices

In order to add or subtract matrices, they must be of the same order. Then the corresponding element of the matrices will be added or subtracted from each order. e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

Then

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \end{bmatrix}$$

Activity 1

Given that $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$

and $D = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & 4 \\ 2 & -1 & 5 \end{bmatrix}$

Find (i) $A+B$ (ii) $A-B$ (iii) $A+C$ (iv) $B+D$ (v) $C-D$

Solution

$$(i)\ A+B = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 1 & 3 \\ 1 & 1 & 5 \end{bmatrix}$$

$$(ii) A - B = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

$$(i) A+C = \begin{bmatrix} 0 & 3 & 2 \\ 2 & 2 & 6 \\ 3 & 4 & 6 \end{bmatrix}$$

Others are left for you to practice.

Scalar Multiple of a Matrix

If a matrix is multiplied by a scalar quantity *K*, then each element is multiplied by *K*.

Activity 2

Find the product of $3A$ given that $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$

Solution

$$3A = 3\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$$

$$3A = \begin{bmatrix} 6 & 9 & 12 \\ 12 & 15 & 18 \\ 18 & 21 & 27 \end{bmatrix}$$

Activity 3

Matrices A and B are such that $3A-2B = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ and

$$-4A+B = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$$

Find A and B

Solution

$$3A - 2B = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \quad (6)$$

$$-4A+B = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} \quad (7)$$

multiply (7) by 2

$$-8A+2B=2\begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} \qquad (8)$$

Add equation (6) and equation (8)

$$-5A=\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}+\begin{bmatrix} -2 & 4 \\ -8 & 6 \end{bmatrix}=\begin{bmatrix} 0 & 5 \\ -10 & 5 \end{bmatrix}$$

$$A=\frac{1}{-5}\begin{bmatrix} 0 & 5 \\ -10 & 5 \end{bmatrix}=\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$$

You are encouraged to obtain matrix B and check your result that it satisfies the two expressions (6) and equation (7) Given that matrix A is a square matrix. Then

$$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$$

where $\frac{1}{2}(A+A')$ is a symmetric matrix and $\frac{1}{2}(A-A')$ is a skew symmetric matrix.

Activity 4

Given matrix A to be square matrix, find the symmetric and skew symmetric matrix formed from A.

$$A=\begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$$

Solution

$$\text{Given } A=\begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}, \text{ then } A^T=\begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix}$$

But

$$A= \frac{1}{2}(A+A^{\,T}) + \frac{1}{2}(A - A^T)$$

$$A= \frac{1}{2}\begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}$$

$$A= \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 5 & \frac{9}{2} \\ \frac{3}{2} & \frac{9}{2} & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & \frac{5}{2} \\ -2 & 0 & -\frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

A= Symmetric Matrix + Skew Symmetric Matrix

Summary

In this unit, you have been taught basic concepts of algebra of matrices which include addition, subtraction and scalar multiplication operation matrices. You are expected to practice the self assessment questions that follow before proceeding to the next unit where you will learn more operations.

Self Assessment Questions

1. Given that $A=\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

and $D = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -2 & 3 \\ 2 & -1 & 5 \end{bmatrix}$ Find

(a) $A + B$

(b) $A + C$

(c) $B + D$

2. Given that $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

and $D = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -2 & 3 \\ 2 & -1 & 5 \end{bmatrix}$ Find

(a) $A - B$

(b) $C - D$

(c) $B - D$

Tutor Marked Assignment

- Express $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 5 & 9 & 3 \end{bmatrix}$ as a sum of symmetric and skew symmetric matrix.

- Find x, y, z and w, given that

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ 3+w & 3 \end{bmatrix}$$

- Given that $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

  and $D = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -2 & 3 \\ 2 & -1 & 5 \end{bmatrix}$ Find

  (a) $A + D$

  (b) $B + C$

  (c) $A - C$

  (d) $C - B$

- Matrices A and B are such that $2A - B = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ and $-2A - B$

  $= \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$

  Find A and B

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UNIT 3

Multiplication of Matrices

Introduction

Multiplying two matrices is simple but requires a lot of concentration. This is because it requires the operation of addition as well as taking cognizance of which column multiplies which row to get the entry of a particular place. In this unit, you will be taught how to multiply matrices.

Learning Outcomes

At the end of this unit, you should be able to:

1. state whether two matrices can be multiplied or not
2. multiply two or more matrices.
3. state the property of matrices multiplication

Main Content

Multiplication of two Matrices

The product of two matrices A and B is only possible if the number of columns in A is equal to the number of rows in B.

Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n * p$ matrix. Then the product AB of the matrices is an $m * p$ matrix $C = (c_{ij})$ $m*p$ It is pertinent to note that matrices multiplication is not commutative because for any matrices A and B, $AB \neq BA$. Indeed, while it is possible to have matrix multiplication AB in some cases, BA may not be possible.

Activity 1

(1) If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$

obtain the product AB and BA.

Solution

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0+(-1)+4 & 0+3+8 \\ 1+(-2)+6 & 2+6+12 \\ 2+(-3)+8 & 4+9+16 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 11 \\ 5 & 20 \\ 7 & 29 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} ; BA \text{ is not possible}$$

:-$AB \neq BA$

Not possible, since the number of column in the first matrix is not equal to the number of rows in the second matrix.

(2) Given that $A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & -1 \\ -3 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

Find the products of *AB* and *BA*. What do you notice?

Solution

$$AB = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & -1 \\ -3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (1+15+9) & (2+20+6) & (3+8+3) \\ (2+20\ -3) & (4+25+(-2)) & (6+10\ -1) \\ (-3+8+6) & (-6+10+4) & (-9+4+2) \end{bmatrix}$$

$$\begin{bmatrix} 26 & 28 & 14 \\ 19 & 27 & 15 \\ 11 & 8 & -3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & -1 \\ -3 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} (1+4+(-9)) & (4+10+6) & (3-2+6) \\ (4+10-6) & (16+25+4) & (15-5+4) \\ (3+4\ -3) & (12+10+2) & (9-2+2) \end{bmatrix}$$

$$\begin{bmatrix} -4 & 20 & 7 \\ 3 & 45 & 11 \\ 4 & 24 & 9 \end{bmatrix}$$

Notice that $AB \neq BA$

Activity 2

If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$

Show that (i) $A(B+C) = AB + AC$ (ii) $(AB)C = A(BC)$.

Activity 3

Determine the values of α, β and γ when the matrix $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

Solution

Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

If A is orthogonal, then $AA^T = I$

$$A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} (0+4\beta^2+\gamma^2) & (0+2\beta^2-\gamma^2) & (0-2\beta^2+\gamma^2) \\ (0+2\beta^2-\gamma^2) & (\alpha^2+\beta^2+\gamma^2) & (\alpha^2-\beta^2-\gamma^2) \\ (0-2\beta^2+\gamma^2) & (\alpha^2-\beta^2-\gamma^2) & (\alpha^2+\beta^2+\gamma^2) \end{bmatrix}$$

But $AA^t = I$ where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence

$$\begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating corresponding elements,

$4\beta^2 + \gamma^2 = 1$ (9)

$2\beta^2 - \gamma^2 = 0$ (10)

adding equations (9) and (10)

$$6\beta^2 = 1 \qquad \Rightarrow \beta = \pm\frac{1}{\sqrt{6}}$$

Substituting for β in equation (10), we have

$$2(\pm\frac{1}{\sqrt{6}})^2 = \gamma^2 \qquad \Rightarrow \gamma = \pm\frac{1}{\sqrt{3}}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

$$\alpha^2 + (\pm\frac{1}{\sqrt{6}})^2 + (\pm\frac{1}{\sqrt{3}})^2 = 1$$

$$\frac{1}{3} + \frac{1}{6} + \alpha^2 = 1 \Rightarrow \alpha^2 = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$$

$$\alpha = \pm\frac{1}{\sqrt{2}}$$

$\therefore \qquad \alpha = \pm\frac{1}{\sqrt{2}},\ \beta = \pm\frac{1}{\sqrt{6}}$ and $\gamma = \pm\frac{1}{\sqrt{3}}$

Summary

In this unit, you have learnt the multiplication of two matrices where for instance, a given matrix *A* and another matrix *B* is said to be used in multiplying each other. You should note that matrices more than two could be used in multiplying one another. You are therefore advised to practice more examples that involve more than two matrices.

Self Assessment Questions

1. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}$

obtain the product *AB* and *BA*.

Tutor Marked Assignment

- If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 3 \\ 4 & -3 & -1 \\ 1 & 2 & -3 \end{bmatrix}$

  obtain the product AB and BA.

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UNIT 4

Determinant of a Matrix

Introduction

This is an important property of a square matrix that defines whether the matrix is singular or non-singular. In this unit, you will learn how to obtain the determinant of any $n \times n$ matrix.

Learning Outcomes

At the end of this unit, you should be able to:

1. calculate the determinant of a matrix
2. state whether a square matrix is singular or not

Main Content

Determinant of Matrix

Determinant of a matrix determines the singularity or otherwise of the square matrix. When the determinant is zero, the matrix is a singular matrix while it is non-singular otherwise. Determinant arises from the process of elimination of unknowns of simultaneous linear equation.

Consider the two linear equations in x

$$a_1x + a_2 = 0 \qquad (11)$$

$$b_1x + b_2 = 0 \qquad (12)$$

From equation (11)

$$x = -\frac{a_2}{a_1} \qquad (13)$$

Substituting for x in equation (12)

$$b_1\left(\frac{-a_2}{a_1}\right) + b_2 = 0$$

$$-a_2b_1 + a_1b_2 = 0$$

$$a_1b_2 - a_2b_1 = 0 \qquad (14)$$

Now from equation (11) and equation (12) by suppressing x, the eliminant will be written as

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = 0$$

When the rows of a_1, a_2 and b_1, b_2 are enclosed by two vertical bars, then it is called a determinant of the second order.

$$|A| = a_1 b_2 \;\; a_2 b_1$$

Also Determinant of a 3 × 3 matrix is Given

$$A = \begin{bmatrix} + & - & + \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$|A| = a_1[b_2c_3 - b_3c_2] - a_2[b_1c_3 - b_3c_1] + a_3[b_1c_2 - b_2c_1]$$

Activity 1

Find the determinant of

$$A = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & 5 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$|A| = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & 5 \\ 4 & 2 & 1 \end{bmatrix}$$

$|A| = 6[3 - 10] - 2[2 - 20] + 3[4 - 12]$

$= 42 + 36 - 24 = 30$

Summary

A singular matrix is a matrix whose determinant is zero while a nonsingular matrix has a non-zero determinant. The method of obtaining the determinant of a matrix has been highlighted in this unit. It is therefore left for you to work more exercises on how to obtain the determinant of a matrix

Self Assessment Questions

1. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 2 & 1 \\ 8 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

   (i) obtain the determinant of the matrices.

   (ii) obtain the determinant of the matrices *AB* and *BA*

Tutor Marked Assignment

- If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 3 \\ 2 & -3 & -1 \\ 8 & 2 & -3 \end{bmatrix}$

obtain the determinants of each matrices as well as that of *AB* and *BA*.

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UNIT 5

Minor, cofactor and Adjoint of a Matrix

Introduction

A knowledge of minor, cofactor and Adjoint of a matrix is particularly useful when calculating the inverse of a square matrix. Any nonsingular matrix is invertible. Hence, the need to calulate the sub determinant of each elements of the square matrix gave rise to the concept of minor. In this unit, you will learn how to calculate the minor, cofactor as well as adjoint of a square matrix.

Learning Outcomes

At the end of this unit, you should be able to:

1. calculate the minor, cofactor and adjoint of a matrix
2. apply the rule of Sarrus to higher order matrices

Main Content

Minor

The minor of an element is defined as a determinant obtained by deleting the row and column containing the element, From

$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Thus the minor a_1, a_2, a_3 are respectively $m_1 = \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}$, $m_2 = \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}$ and $m_3 = \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$ Therefore the determinant of of $A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ will be $|A|=a_1\,(minor\ of a_1)$ - $a_2(minor\ of\ a_2)+$ $a_3(minor\ of\ a_3)$

Cofactor

Cofactor=$(-1)^{r+c}$minor where r is the row number of the element and c is the column number of the element. The cofactor of any element of the ith row and jth column is $(-1)^{i+j}$minor.

Thus, the cofactor of

$a_1 = (-1)^{1+1}(b_2c_3 - b_3c_2) = +(b_2c_3 - b_3c_2)$

$a_2 = (-1)^{1+2}(b_1c_3 - b_3c_1) = -(b_1c_3 - b_3c_1)$

$a_3 = (-1)^{1+3}(b_1c_2 - b_2c_1) = +(b_1c_2 - b_2c_1)$

The determinant = a_1(cofactor of a_1) + a_2(cofactor of a_2) + a_3(cofactor of a_3)

Adjoint

The adjoint of a matrix is the transpose of the cofactor. Mathematically, for any matrix $A, Adj(A) = (cof(A))^{T}$

Activity 1

Write down the minor and cofactor of each element and also evaluate

the determinant of $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 5 & 6 \\ 3 & 5 & 2 \end{bmatrix}$

Solution

$M_{11} = Minor\ of\ element\,(1)$

$= \begin{bmatrix} -5 & 6 \\ 5 & 2 \end{bmatrix}$

$= \ 40$

Cofactor of element $(1) = A_{11} = (-1)^{1+1} = (-1)^{2} M_{11} = 40$

$M_{12} = Minor\ of\ element\ (3)$

$$= \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$$

$= \quad 10$

Cofactor of element $(3) = A_{12} = (-1)^{1+2}\ (-10) = 10$

$M_{13} = Minor\ of\ element\ (-2)$

$$= \begin{bmatrix} 4 & -5 \\ 3 & 5 \end{bmatrix}$$

$= 35$

Cofactor of element $(-2) = A_{13} = (-1)^{1+3}\ M_{13} = 35$

$M_{21} = Minor\ of\ element\ (4)$

$$= \begin{bmatrix} 3 & -2 \\ 5 & 2 \end{bmatrix}$$

$= 16$

Cofactor of element $(4) = A_{21} = (-1)^{2+1}\ M_{21} = -16$

$M_{22} = Minor\ of\ element\ (-5)$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

$= 8$

Cofactor of element $(-5) = A_{22} = (-1)^{2+2}\ M_{22} = 8$

$M_{23} = Minor\ of\ element\ (6)$

$$= \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$$

$= -4$

Cofactor of element $(6) = A_{23} = (-1)^{2+3}\ M_{23} = 4$

$M_{31} = Minor\ of\ element\ (3)$

$$= \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$$

$= 28$

Cofactor of element $(3) = A_{31} = (-1)^{3+1} M_{31} = 28$

M_{32} = *Minor of element* (5)

$$= \begin{bmatrix} 1 & -2 \\ 4 & 6 \end{bmatrix}$$

$= 14$

Cofactor of element $(5) = A_{32} = (-1)^{3+2} M_{32} = 14$

M_{33} = *Minor of element* (2)

$$= \begin{bmatrix} 1 & 3 \\ 4 & -5 \end{bmatrix}$$

$= -17$

Cofactor of element $(2) = A_{33} = (-1)^{3+3} = (-1)^{3+3} M_{33} = 17$ Therefore the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{bmatrix} = 1 * [\text{Cofactor of } 1] + 3 * [\text{Cofactor of } 3] + (-2) * [\text{Cofactor of } (-2)]$$

$= 1 * (-40) + 3(10) + (-2) * (35)$

$= -40 + 30 - 70$

$= -80$

Activity 2

Find the (i) minor matrix (ii) cofactor matrix, and (iii) adjoint matrix of all the elements of the matrix A.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} = 7 - 0 = 7 \quad M_{12} = \begin{vmatrix} 4 & 0 \\ 6 & 7 \end{vmatrix} = 28 - 0 = 28$$

$$M_{13} = \begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} = 8 - 6 = 2 \quad M_{21} = \begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix} = 21 - 10 = 11$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 6 & 7 \end{vmatrix} = 14 - 30 = -16 \quad M_{23} = \begin{vmatrix} 2 & 3 \\ 6 & 2 \end{vmatrix} = 4 - 18 = -14$$

$$M_{31} = \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} = 0 - 5 = -5 \quad M_{32} = \begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} = 0 - 20 = -20$$

$$M_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 2 - 12 = -10$$

Thus, the minor matrix of the above matrix A is given as:

$$Min(A) \quad or \quad M_A = \begin{vmatrix} 7 & 28 & 2 \\ 11 & -16 & -14 \\ -5 & -20 & -10 \end{vmatrix}$$

By applying sign rule, the cofactor matrix is obtained as:

$$cof(A) \quad or \quad C_A = \begin{vmatrix} 7 & -28 & 2 \\ -11 & -16 & 14 \\ -5 & 20 & -10 \end{vmatrix}$$

The adjoint matrix is the transpose of the cofactor matrix:

$$Adj(A) \quad or Ad_A = \begin{vmatrix} 7 & -11 & -5 \\ -28 & -16 & 20 \\ 2 & 14 & -10 \end{vmatrix}$$

Activity 3

Expand the fourth order determinant of $A = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix}$

$$|A| = 0\begin{vmatrix} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$0 - 1[1(1-3) - 2(0-3) + 0(2-1)] + 2[1(0-6) - 0(0-3) + 0(4-0)] - 3[1(0-2) - 0(2-1) + 2(4-0)]$

$|A| = 0 - 1[-2 + 6 + 0] + 2[-6 - 0 + 0] - 3[-2 - 0 + 8]$

$= 0 - 4 - 12 - 18 = -34$

$|A| = -34$

Rules of Sarrus (For third order derivatives)

When given a 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ the matrix is then written in determinant form with the repetition of the first two columns as below

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{bmatrix}$$

$$= [(a_1b_2c_3) + (a_2b_3c_1) + (a_3b_1c_2) - (a_3b_2c_1) - (a_1b_3c_2) - (a_2b_1c_3)]$$

Summary

In this unit, you have been taught how to find the minor, cofactor and adjoint of a matrix, and applying such to establish results for the determinant of matrices. You also learnt Sarrus rule.

Self Assessment Questions

1. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix}$
2. Write down the minor of each element of matrices A and B in question one(1) above.
3. Write down the cofactor of each element of matrices A and B in question one(1) above.

Tutor Marked Assignment

1. Given that: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 5 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 2 & 5 \end{bmatrix}$
2. Write down the minor of each element of matrices A and B in above.
3. Write down the cofactor of each element of matrices A and B in question two(2). Using the answer of three(3), find the determinant of the matrices.
4. obtain the adjoint of the matrices

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UNIT 6

Inverse of a Matrix

Introduction

Every non-singular matrix is invertible and vice-versa. In the unit, you will learn the techniques of finding the inverse of a matrix as well its application in solving system of linear equation.

Learning Outcomes

At the end of this unit, you should be able to:

1. Obtain the inverse of a non-singular matrix.
2. solve a system of linear equations using the matrix inversion method.

Main Content

The inverse of a square matrix A denoted as $A^{-1}$is defined as: $A^{-1} = \frac{Adj(A)}{|A|}$ provided that $|A| \neq 0$.

Activity 1

Obtain the inverse of the matrix $B = \begin{bmatrix} -1 & 2 \\ -4 & 0 \end{bmatrix}$

Solution

The determinant of B is $|B| = \begin{bmatrix} -1 & 2 \\ -4 & 0 \end{bmatrix} = 8$

Minor of matrix B is obtained as $M\,in(B) = \begin{bmatrix} 0 & -4 \\ 2 & -1 \end{bmatrix}$

Cofactor of matrix B is obtained as $Cof\,(B) = \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix}$

Adjoint of matrix B is obtained as $Adj\,(B) = \begin{bmatrix} 0 & -2 \\ 4 & -1 \end{bmatrix}$

Therefore the inverse of the matrix B is given as $B^{-1} = \dfrac{Adj(B)}{|B|}$

$$B^{-1} = \frac{1}{8}\begin{bmatrix} 0 & -2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{8} \end{bmatrix}$$

Activity 2

Obtain the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{bmatrix}$

Solution

It has been obtained earlier that the determinant, minor and cofactor matrices of the above matrix is given respectively as:

$$-\,80, \begin{bmatrix} -40 & -10 & 35 \\ 16 & 8 & -4 \\ 28 & 14 & -17 \end{bmatrix}, \begin{bmatrix} -40 & 10 & 35 \\ -16 & 8 & 4 \\ 28 & -\ 14 & =17 \end{bmatrix}$$

The adjoint of the matrix is the transpose of the cofactor, hence;

$$Adj\,(A) = \begin{bmatrix} -40 & -16 & 28 \\ 10 & 8 & -14 \\ 35 & 4 & -17 \end{bmatrix}$$

Therefore the inverse of the matrix A is given as $A^{-1} = \frac{Adj\,(A)}{|A|}$

$$A^{-1} = \frac{1}{-80}\begin{bmatrix} -40 & -16 & 28 \\ 10 & 8 & -14 \\ 35 & 4 & -17 \end{bmatrix}$$

Summary

The inverse of a matrix was tackled using examples on 2 and 3 × 3 matrices. The same procedure applies for higher order matrix. Try to apply the knowledge to higher order matrix.

Self Assessment Questions

1. If $A=\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, obtain the inverse of the matrix.

Tutor Marked Assignment

- If $A=\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, obtain the inverse of the matrix.

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UNIT 7

Application of Matrices to System of Linear Equation

Introduction

System of linear equations can be expressed in matrix form. In this unit, you will learn the techniques of finding the unknown variables of a system of linear equation by using matrix inversion method as well as Crammer's rule.

Learning Outcomes

At the end of this unit, you should be able to:

1. solve problems on linear equations using matrix inversion method
2. solve a system of linear equations using Crammer's rule

Main Content

Inverse of Matrix

A system of linear equation consists of two or more equations madeup of two or more variables such that all equations in the system are considered simultaneously. The general form is:

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\
\cdot \quad \cdot \quad \cdots\cdot \qquad \cdot \\
\cdot \quad \cdot \quad \cdots\cdot \qquad \cdot \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_n
\end{array}
$$

The solution to a system of linear equation in two variables is any ordered pair that satisfies each equation independently. System of equations are classified as independent with one solution, dependent with an infinite number of solutions or inconsistent with no solution. Any system of linear equation can be represented in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

or

$$AX=C$$

where A is called the coefficient matrix and x the matrix of unknown variables.

Solution of Simultaneous Linear Equation by Matrix Inversion Method

From the general representation of the system of simultaneous linear equation: $AX = C$, if we pre-multiply both sides by the inverse of the coefficient matrix, we obtained: $A^{-1}AX = A^{-1}C$. Since the multiplication of a matrix by its inverse yield an identity matrix, thus we have, $IX = A^{-1}C$. Any matrix multiply by an identity matrix gives the same matrix, thus: $X = A^{-1}C$. Thus, the result of the simultaneous system of linear equation is obtained by pre-multiply the inverse of the coefficient matrix with the constant term on the right hand side.

Activity 1

Solve the simultaneous system of linear given as:

$$-x_1 + 2x_2 = 0$$
$$-4x_1 = 4$$

Solution

In matrix form, the system of equation becomes:

$$\begin{bmatrix} -1 & 2 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Thus:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -4 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{8} \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Therefore the solution matric is given as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$$

Solution of Simultaneous Linear Equation by determinants (CRAMER'S RULE)

Let us consider the equations

$$\begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 = d_1 \\ b_1x_1 + b_2x_2 + b_3x_3 = d_2 \\ c_1x_1 + c_2x_2 + c_3x_3 = d_3 \end{bmatrix} \qquad (15)$$

equation (15) can be form into the form of

$$AX = D$$

where A is the coefficient matrix, x is the unknown variables x_1, x_2, x_3 and D is the constant equations are equated to.

$$\text{Thus, } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

To find the values of x_1, x_2 and x_3, we first take the determinant of A. Then for x_1, we replace column 1 with the column vector D, then take the determinant of A1, So

$x_1 = \frac{|A_1|}{|A|}$, Similarly $x_2 = \frac{|A_2|}{|A|}$ and $x_3 = \frac{|A_3|}{|A|}$

Activity 2

Find the values of x_1, x_2 and x_3 from the equations below

$$2x_1 + 3x_2 - x_3 = 4$$

$$3x_1 + x_2 + 2x_3 = 13$$

$$x_1 + 2x_2 - 5x_3 = \quad 11$$

Solution

From the system of equation above

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ 1 & 2 & -5 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 \\ 13 \\ -11 \end{bmatrix}$$

$$|A| = 2(-5-4) - 3(-15-2) - 1(6-1)$$
$$= -18 + 51 - 5 = 28$$

$$A_1 = \begin{bmatrix} 4 & 3 & -1 \\ 13 & 1 & 2 \\ -11 & 2 & -5 \end{bmatrix}$$

$$|A_1| = 4(-5-4) - 3(-65+22) - 1(26+11)$$

$$-36 + 129 - 37 = 56$$

$$A_2 = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 13 & 2 \\ 1 & -11 & -5 \end{bmatrix}$$

$$|A_2| = 2(-65+22) - 4(-15-2) - 1(-33-13)$$

$$-89 + 68 + 46 = 28$$

$$A_3 = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 13 \\ 1 & 2 & -11 \end{bmatrix}$$

$$|A_3| = 2(-11-26) - 3(-33-13) + 4(6-1)$$

$$= -74 + 138 + 20 = 84$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{56}{28} = 2$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{28}{28} = 1$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{84}{28} = 3$$

Activity 3

The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we obtain 7. By adding second and third numbers to three times the first number we obtain 12. Use Crammer's Rule to find the numbers.

Solution

Let the numbers be *x*, *y*, and *z*

$$x + y + z = 6 \qquad (16)$$

$$x + 2z = 7 \qquad (17)$$

$$3x + y + z = 12 \qquad (18)$$

Summary

Application of algebra of matrices were used to solve system of linear equations using two methods. You are encouraged to get more practice questions from the references listed for better understanding.

Self Assessment Questions

1. Find the values of x_1, x_2 and x_3 from the equations below using both the matrix inversion method and Cramer's rule

$$3x_1 + 4x_2 - 2x_3 = 5$$
$$4x_1 + 2x_2 - 2x_3 = 14$$
$$2x_1 + 3x_2 - 6x_3 = -12$$

Tutor Marked Assignment

- Find the values of x_1, x_2 and x_3 from the equations below using both the matrix inversion method and Cramer's rule

$$2x_1 + 4x_2 - 2x_3 = 6$$
$$x_1 + 2x_2 - 3x_3 = 8$$
$$x_1 + 3x_2 - 6x_3 = -12$$

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Module 3

BINARY OPERATIONS AND RELATIONS

Units

UNIT 1

Binary Operations

Introduction

You are already familiar with arithmetic operations like addition, subtraction, division, and multiplication. You are also familiar with exponential function, log function etc. In this unit, you will learn about the binary operations. As the name suggests, binary stands for two, while "operation" simply means procedure. Does that mean that we can use two functions simultaneously using binary operation? Let us find out.

Learning Outcomes

At the end of this unit, you should be able to:

1. state the properties of binary operations
2. solve set problems using the properties of binary operations

Main Content

Just as we get a number when two numbers are either added or subtracted or multiplied or are divided. The binary operations associate any two elements of a set. The resultant of the two are in the same set. Binary operations on a set are calculations that combine two elements of the set (called operands) to produce another element of the same set. In other words, binary operation is any operation or sign that combines any two elements of a given set according to some clearly defined rules. The binary operations $*$ on a non-empty set A are functions from $A \times A$ to A. The binary operation, $* : A \times A \to A$. It is an operation of two elements of the set whose domains and co-domain are in the same set.

Properties of Binary Operations

1. **Closure property**

(a) Suppose the binary operation of addition " + " is defined on a set of real numbers. We notice that whenever two real numbers are added, we still obtain a real number under those circumstances, we say that the operation of addition " + " is closed on the set of real numbers.

(b) Suppose we define the binary operation of addition " + " on the set $A = \{2, 3, 5, 7\}$, we notice that $3 + 7 = 10$, but $10 \notin A$, so " + " is not closed on the set A. In general, the binary operation Δ defined on a set S is closed on S if and only if $a \Delta b \in S \; \forall \; a, b \in S$.

Activity 1

(a) The binary operation $*$ is defined on the set $S = \{1, 2, 3, 4, 5\}$ by $a * b = a + 3b + ab$ $\forall \; a, b \in S$. Is the operation $*$ closed on S?

Answer From the equation $a * b = a + 3b + ab$

$3 * 4 = 3 + 3(4) + (3)(4)$

$3 * 4 = 3 + 12 + 12 = 27$

But $27 \notin S$, thus, the operation $*$ is not closed.

(b) The binary operation Θ is defined on the set $B = \{a, b, c, d\}$ by the table below.

Θ	a	b	c	d
a	a	b	c	d
b	c	d	b	c
d	m	c	a	b

Is the operation Θ closed on the set B?

Answer $m \notin B$, so the operation Θ is not closed on the set B.

(c) The operation $*$ is defined on the set $S = \{1, 3, 5\}$ by the table below.

$*$	1	3	5
1	3	1	5
3	1	3	5
5	5	5	5

Is the operation $*$ closed in S?

Answer Since all the answers in the table are members of a set S, the operation $*$ is closed on the set S.

2. **Commutative Property**: The binary operation $*$ defined on the set S is said to be commutative if and only if $a * b = b * a \ \forall \ a, b \in$.

Activity 2

The binary operation Θ is defined on the set R of real numbers by $a\Theta b = a^2 + b + ab$, where $a, b \in R$.

Evaluate

(a) $3\Theta 2$

(b) $2\Theta 3$

(c) what conclusion can you draw from the result in (a) and (b)?

Answer

(a) $3\Theta 2 = 3^2 + 2 + 3(2)$

$= 9 + 2 + 6 = 17$

(b) $2\Theta 3 = 2^2 + 3 + 2(3)$

$4 + 3 + 6 = 13$

(c) From the result of (a) and (b), $3\Theta 2 \neq 2\Theta 3$ Hence the operation Θ is not commutative.

Activity 3

A binary operation o is defined on the set R of real numbers by $a \, o \, b = a + b - 2ab$; $a, b \in R$. Is the operation o commutative?

Answer

$a \, o \, b = a + b - 2ab \cdots$(i)

$b \, o \, a = b + a - 2ba \cdots$(ii)

Since $a + b - 2ab = b + a - 2ba$,

$a \, o \, b = b \, o \, a$

Hence the operation o is commutative.

3. **Associative Property**: If the binary operation Δ is closed on the set S such that: $(a \Delta b) \Delta c = a \Delta (b \Delta c) \; \forall \, a, b, c \in S$, then the operation Δ is said to be associative.

Activity 4

The binary operation Θ is defined on the set R of real numbers by $a \Theta b = a+b \; ab \; \forall \, a, b \in R, a \neq 0, b \neq 0$.

Find

(i) $(2 \Theta 5) \Theta 7$ (ii) $2 \Theta (5 \Theta 7)$

What is your conclusion?

(i) $(2 \Theta 5) \Theta 7$

$$= \left(\frac{2+5}{(2)(5)}\right) \Theta 7$$

$$= \frac{7}{10} \Theta 7$$

$$= \frac{\frac{7}{10} + 7}{\frac{7}{10} . 7}$$

$$= \frac{\frac{77}{10}}{\frac{49}{10}}$$

$$= \frac{77}{49}$$

$$= \frac{11}{7}$$

(ii) $2 \ominus (5 \ominus 7)$

$$= 2 \ominus (\frac{5+7}{5 \times 7})$$

$$= \frac{2 + \frac{12}{35}}{2.\frac{12}{35}}$$

$$= \frac{\frac{82}{35}}{\frac{24}{35}}$$

$$= \frac{82}{24}$$

$$= \frac{41}{12}$$

From the results, the binary operation Θ is not associative.

Activity 5

The binary operation Θ is defined on the set R of real numbers by $x \,\Theta\, y = x + y + xy$, where $x, y \in R$. Is the operation Θ associative?

Answer

Let $z \in R$, (since associative property deals with three variables)

$(x \,\Theta\, y) \,\Theta\, z = x \,\Theta\, (y \,\Theta\, z)$

from RHS,

$(x \,\Theta\, y) \,\Theta\, z = (x + y + xy) \,\Theta\, z$

$= (x + y + xy) + z + z(x + y + xy)$

$= x + y + xy + z + zx + zy + xyz$

$= x + y + z + xz + xy + yz + xyz$

from LHS,

$x \,\Theta\, (y \,\Theta\, z) = x \,\Theta\, (y + z + zy)$

$= x + (y + z + zy) + x(y + z + zy)$

$= x + y + z + zy + xy + xz + xyz$

$= \mathrm{x} + \mathrm{y} + \mathrm{z} + xy + \mathrm{xz} + zy + xyz$

Since LHS = RHS, the operation is associative.

4. **Distributive Property**: The distributive property uses two binary operations on the same set. If the binary operation $*$ and Θ are defined on the set S such that:

$a * (b \Theta c) = (a * b) \Theta (a * c) \forall a, b, c \in S$, then the operation $*$ is distributive over the operation Θ.

Activity 6

The binary operations of addition " + " and " × " are defined on the set R of real numbers. Determine whether or not

(a) " × " is distributive over " + "

(b) " + " is distributive over " × "

Answer

(a) Let a, b, c be any real number, then the equation $a \times (b + c) = (a \times b) + (a \times c)$ is always positive, hence " × " is distributive over " + " on the set R.

(b) The equation $a + (b \times c) = (a + b) \times (a + c)$ is not poistive on the set R.

Hence " + " is not distributive over " × "

Activity 7

The binary operation Δ and $*$ defined on the set R of real number by $a \Delta b = a\,2 - b\,2$ and $a * b = a + 2b, \forall a, b \in R$.

Evaluate

(i) $2 \Delta (3 * 4)$

(ii) $(2 \Delta 3) * (2 \Delta 4)$.

What is your conclusion?

(I) $(2 \Delta (3 * 4)) = 2 \Delta [3 + 2(4)]$

$= 2 \Delta 11$

$= 2^2 - 11^2$

$4 - 121$

$= -117$

(ii) $(2 \Delta 3) * (2 \Delta 4) = [2^2 - 3^2] * [2^2 - 4^2]$

$= [4 - 9] * [4 - 16]$

$= -5 * -12$

$= -5 + 2(-12)$

$= -5 - 24$

$= -29$

Since $2 \Delta (3 * 4) \neq (2 \Delta 3) * (2 \Delta 4)$, the operation Δ is not distributive over the operation $*$.

5. **Identity Property**: The set P, is said to have an identity element e under a given operation $*$ if and only if

$$e * x = x * e = x, \forall e, x \in P$$

The identity element, if it exists, is unique. That is to say, the set P has only one identity element.

Note: Naturally, identity of addition is zero (0), while that of multiplication is 1. It is simply the neutral number that, when combined with any other element of a set under a given operation, leaves the value of the element unaltered.

Activity 8

Find the identity element of the set Z of positive integers under the binary operation of addition " + ".

Answer

Let e be the identity element of Z under the operation " + ". Then by definition, $y + e = e + y = y, \forall y \in Z, \Longrightarrow e = 0$. Since $0 \notin Z$, the set Z has no identity element under the operation " + ".

Remark: Some sets may not have identity element under a given binary operation.

Activity 9

The binary operation Θ is defined on the set R of real numbers by $m \,\Theta\, n = m + n + 6, \forall m, n \in R$. Find the identity element of R under the operation Θ

Answer: Let e be the identity element of R under the operation Θ. By definition, $m \,\Theta\, e = e \,\Theta\, m = m, \forall m \in R$ $m + e + 6 = m$ $e = -6$. Hence the identity element of R under the binary operation Θ is -6.

Activity 10

The binary operation $*$ is defined on the set R of real numbers by $p * q = p + q - 4p^2q^2$, where $p, q \in R$. Find the identity element of R under the binary operation $*$.

Answer:

Let e be the identity element, then by definition,

$p * e = e * p = p, \forall p \in R.$

Thus

$p + e - 4p^2e^2 = p$

$e - 4p^2e^2 = 0$

$e(1 - 4p^2e) = 0$

$e = 0$ or $1 - 4p^2e = 0$

$e = 0$ or $e = \frac{1}{4p^2}$

Since e (the identity element) must be unique, the solution $e = \frac{1}{4p^2}$ is inadmissible. It means, it cannot be accepted as the identity element.

Activity 11

The binary operation Θ is defined on the set M of positive numbers by $a \Theta b = \frac{a+b}{3ab}$, $\forall a,b \in M$. Find, if it exists, the identity element of M under the binary operation Θ.

Answer:

Let e be the identity element of M under the binary operation Θ. then $a \Theta e = e \Theta a = a$, $\forall a \in M$

Thus

$$\frac{a+e}{3ae} = a$$

$$a + e = 3a^2e$$

$$3a^2e - e = a$$

$$e(3a^2 - 1) = a$$

$$e = \frac{a}{3a^{2-1}}$$

Since a could be any member of the set M, and e must be unique.

$e = \frac{a}{3a^{2-1}}$ cannot be admitted as the identity element. Hence, the identity element of M is not under the binary operation Θ.

6. **Inverse Property**: Let e be the identity element under a binary operation Θ defined on the set S,(where $x, b \in S$), then the element x, is said to be the inverse of b if and only if $b \Theta x = x \Theta b = e, \forall x, b, e \in S$. The inverse of an element, if it exists, under given binary operation, is unique.

Notation:

The inverse of the element b under a given binary operation is denoted by b^{-1} [NOT E : $b^{-1} \neq \frac{1}{b}$]. The inverse of b, (b^{-1}) is never the same as the inverse presented under the mathematical topic indices. Inverse is simply the opposite of something

For example, the opposite of +2 under the operation of addition is −2.

(+2) + (−2) = 0 [identity element of addition]

In addition, the opposite of +2 under the operation of multiplication is $\frac{1}{2}$ (from indices)

$2 \times \frac{1}{2} = 1$ [identity element of multiplication]

Remark:

The inverse element of a number (in a given set) is the number that makes results in the identity element of the given operation when operated on. That is, $2 + (-2) = 0$, (2 is the number, "+" is the operation, -2 is the inverse and zero(0) is the identity element).

Activity 12

The binary operation Δ is defined on the set R of real numbers by $a \,\Delta\, b = a + b + 2ab$, $\forall\, a, b \in R$. Find

(i) the identity element e of R under Δ

(ii) the inverse p^{-1} of p, where $p \in R$.

(iii) For what value of p is p^{-1} not defined?

Answer:

(i) Let e be the identity element, by definition,

$e\Delta b = b\Delta e = b$
$\Longrightarrow e + b + 2eb = b$
$\Longrightarrow e + 2eb = 0$
Thus, $e(1 + 2b) = 0$

Hence, $e = 0$

(ii) Also by definition of inverse,

$p^{-1} \,\Delta\, p = p \,\Delta\, p^{-1} = e$

$\Longrightarrow p^{-1} + p + 2p^{-1}p = 0$

$\Longrightarrow p^{-1} + 2p^{-1}p = -p$

$\Longrightarrow p^{-1}(1 + 2p) = -p$

$p^{-1} = -\dfrac{p}{1 + 2p}$

(iii) p^{-1} is defined when $1 + 2p \neq 0$. Thus, the value of p for which p^{-1} is not defined is when $p = -\frac{1}{2}$

Activity 13

The binary operation $*$ is defined on the set R of real numbers by $a * b = a + b + 5$

(i) Find the inverse of t under the binary operation $*$ of real numbers.

(ii) Find the inverse of 6.

Answer

(i) Let t^{-1} be the inverse element of t and e be the identity element. By definition,

$t^{-1} * t = t * t^{-1} = e$

But,

$\Longrightarrow a + e + 5 = a$

$\Longrightarrow e = a - a - 5$

$e = -5$

Thus,

$\Longrightarrow t^{-1} * t = -5$

$\Longrightarrow t^{-1} + t + 5 = -5$

$\Longrightarrow t^{-1} = -5 - 5 - t$

$t^{-1} = -10 - t$

(ii) Let 6^{-1} be the inverse of 6

$\Longrightarrow 6^{-1} * 6 = e$

$\Longrightarrow 6^{-1} + 6 + 5 = -5$

$\Longrightarrow 6^{-1} = -5 - 5 - 6$

$\Longrightarrow 6^{-1} = -16.$

Summary

1. A binary operation: Any operation or sign that combines any two elements of a given set according to some clearly defined rule.

2. Closure property: $a \Delta b \in S$

3. Commutative property: $a * b = b * a, \forall\, a, b \in S$.

4. Associative property: $(a \,\Delta\, b) \,\Delta\, c = a \,\Delta\, (b \,\Delta\, c), \forall\, a, b, c \in S$

5. Distributive property: $a * (b \,\Theta\, c) = (a * b) \,\Theta\, (a * c), \forall\, a, b, c \in S$

6. Identity property: $e * x = x * e, \forall\, e, x \in p$

7. Inverse property: $b * x = x * b = e, \forall\, b, x, e \in S$

Self Assessment Questions

1. List and define the properties of binary operation

2. The binary operation $*$ is defined on the set R of real numbers by

   (a) $a * b = a + b + 3ab$, where $a, b \in R$. Find,

   (i) $3a * 5b$

   (ii) $(x^2 + x + 1) * x$

   (iii) $(a * b) * c$

   (iv) $a * (b * c)$

   (b) $a * b = \frac{a+b}{2ab}$. Find,

   (i) $a * (b * c)$

   (ii) $(a * b) * c$

   (iii) $(2x + 1) * (3x + 2)$

   (iv) $\frac{3}{3a * 5b}$

(c) $m * n = m^2 + m + 2n$. Find,

(i) $-6 * 5$

(ii) $(2 * 3) * 4$

(iii) $(2a + 1) * (3a + 5)$

Tutor Marked Assignment

- The binary operation $*$ and Θ are defined on the set S by the tables below

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Θ	a	b	c
a	a	c	b
b	b	a	c
c	c	b	a

(a) Determine whether or not

(i) the operation $*$ is commutative.

(ii) the operation Θ is commutative.

(b) Determine whether or not

(i) the operation $*$ is associative.

(ii) the operation Θ is associative.

(c) Determine whether or not the operation $*$ is distributive over operation Θ.

- A binary operation Δ is defined on the set $Q = \{2, 3, 4, 5\}$ by the table below

Δ	2	3	4	5
2	2	3	4	5
3	3	4	5	2
4	4	5	1	3
5	5	1	3	4

(a) Determine whether or not

(i) Q is closed with respect to the binary operation Δ

(ii) the operation Δ is commutative.

(b) What is the identity element, e, under the operation Δ?

(c) Find the inverse of each element.

- A binary operation Θ is defined on the set R of real numbers by $a \Theta b = 3a - 4b + + 5ab, \forall a, b \in R$. Find,

  (I) $7\Theta - 4$

  (ii) $(2\Theta 3)\Theta 6$

  (iii) $(2 + x)\Theta(4\Theta - 2)$

- A binary operation Θ is defined on the set R of real numbers by $a\Theta b = a + ab^2 + 3$, $\forall a, b \in R$. Find,

  (I) $(5\Theta 3)\Theta 7$

  (ii) $5\Theta(3\Theta 7)$

  what conclusion can you draw from the results of (i) and (ii)?

- A binary operation Θ is defined on the set R of real numbers by $a\Theta b = \frac{a}{b} + \frac{b}{a} + 3, \forall a, b \in R$, $a \neq 0, b \neq 0$. Find,

  (I) $(m\Theta m)\Theta 2$

  (ii) $(m\Theta 2)\Theta m$

- The binary operations $*$ and Δ are defined on the set R of real numbers by $a * b = a + 2b + 4, \forall a, b \in R, a \Delta b = a + 3b - ab, \forall a, b \in R$.

  What conclusions can you draw from the result of (i) and (ii)?

- A binary operation $*$ is defined on the set R of real numbers by $p * q = p + q - 3pq$, where $p, q \in R$. Find,

  (a) The identity element e under the operation

  (b) The inverse of an element $x \in R$, stating the value for which no inverse exists

- A binary operation $*$ is defined on the set R of real numbers by $p * q = 2p + 2q + pq - 3\sqrt{3}$, where $p, q \in R$. Find,

  (a) $(\sqrt{2} - 1) * (\sqrt{2} + 1)$

  (b) $\left(\frac{1}{1+\sqrt{2}}\right) * \left(\frac{1}{1-\sqrt{2}}\right)$

- The binary operation $*$ is defined on the set R of real numbers by $a * b = 2a + 3b + 2ab$, $\forall a, b \in R$. Find,

  (I) $2a * 3b$
  (ii) $a * b$
  (iii) $(a + 2) * (a + 2)$
  (iv) $(a + ab + 3) * c$
  (v) $m * (2a + 3b + 2ab)$
  (vi) $3 * (1 + b + \mathrm{a})$

- A binary operation Φ is defined on the set R of real numbers by $x \Phi y = x - 5xy + y$. Find, under the operation Φ, the

  (a) Identity element
  (b) Inverse of 1
  (c) Inverse of 2.

- A binary operation Θ is defined on the set R of real numbers by $x\Theta y = x + y^2 + 2xy$, where $x, y \in R$. Determine whether or not the operation Θ is

(a) commutative

(b) associative.

- The binary operation $*$ is defined on the set R of real numbers by $a * b = a + b + 4a^2b^2$ where $a, b \in R$.

(I) Is $*$ commutative?

(ii) Is $*$ associative?

(iii) Is $*$ distributive over addition, " + " of real numbers?

Δ	2	3	4	5
2	2	3	4	5
3	3	4	5	2
4	4	5	1	3
5	5	1	3	4

$a\Delta e = a$

$2\Delta 2 = 2$

$3\Delta 2 = 3$

$4\Delta 2 = 4$

$5\Delta 2 = 5$

$2\Delta 3 = 3$

$3\Delta 3 = 4$

$4\Delta 3 = 5$

$5\Delta 3 = 2$

$2\Delta 4 = 4$

$3\Delta 4 = 5$

$4\Delta 4 = 1$

$5\Delta 4 = 3$

$2\Delta 5 = 5$

$3\Delta 5 = 1$

$4\Delta 5 = 3$

$5\Delta 5 = 4$

If,

$$a\Delta e = a$$
$$e\Delta a = a$$

Then,

$$2\Delta 2 = 2$$
$$2\Delta 3 = 3$$
$$2\Delta 4 = 4$$
$$2\Delta 5 = 5$$

Therefore,

$$e = 2$$

For,

$$a\Delta a^{-1} = e$$
$$a\Delta a^{-1} = 2$$

Then,

$$2\Delta 2 = 2$$
$$5\Delta 3 = 2$$
$$3\Delta 5 = 1 \neq \mathrm{e}$$

The inverse of, 2 is 2 and 5 is 3. The inverse of 3 does not exist since 1 is not the identity element, *e*.

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Further Reading

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UNIT 2

Cartesian Product

Introduction

In mathematics, any binary operation on n independent sets, is a subset of Cartesian product of the n sets. The relation is homogeneous when it is formed with one set. For example, any curve in the Cartesian plane is a subset of the Cartesian product of real numbers, $R \times R$.

Learning Outcomes

At the end of this unit, you should be able to:

1. define Cartesian product
2. calculate the Cartesian product of two sets

Main Content

Cartesian Product: Let X and Y be two non-empty sets. The Cartesian product $X \times Y$ of the two sets is the set of the ordered pairs $\{(x_i, y_i) : x_i \in X, y_i \in Y\}$ We write $\{x, y\}$ to mean a set containing just the two elements x and y. More generally, $\{x_1, x_2, \ldots, xn\}$ is a set containing just the n elements $x_1, x_2, \ldots, xn$ We write the ordered pair with first element x and second element y as (x, y); this is not the same as (y, x) unless x and y are equal.

The point with coordinates $(2, 3)$ is not the same as the point with coordinates $(3, 2)$. The rule for equality of ordered pair is:

$$(x, y) = (u, v) \text{ } iff \text{ } x = u \text{ and } y = v$$

This notation can be extended to ordered $n-$tuples for large n. For example, a point in three-dimensional space is given by an ordered triple (x, y, z) coordinates $Y = X$, we write $X * Y$ more briefly is X^2. Similarly, if we have sets $X_1, \ldots, X_n$, we let $X_1 * \ldots * X_n$ be the set of all ordered $n-$tuples $(x_1, x_2, \ldots, x_n)$ such that $x_1 \in X_1, \ldots, x_n \in X_n$. If $X_1 = X_2 = \ldots = X_n = X$, say, we write this set as X^n

.If the sets are finite, we can do some counting. We use the notation $|X|$ for the number of elements of the set X. Composition: Let X and Y be sets with $|X| = p$ and $|Y| = q$.

Then

(a) $|X * Y| = pq$;

(b) $|X|^n = p^n$.

Proof:

(a) In how many ways can we choose an ordered pair (x, y) with $x \in X$ and $y \in Y$? There are p choices for x, and q choices for y, each choice of x can be combined with each choice for y, so we multiply the numbers.

(b) This is an exercise for you. The idea of coordinating the plane of 2 or 3-dimensional space by ordered pair or triples of real numbers was invented by Descartes. In his honor, we call the system "Cartesian Coordinates." This great idea of Descartes allows us to use algebraic methods to solve geometric problems.

By means of Cartesian coordinates, the set of all points in the plane is matched up with the set of all ordered pairs (x, y), where x and y are real numbers. We call this set $R * R$, or R^2.

This notation works much more generally, as we now explain.

Let X and Y be any two sets. We define the Cartesian product $X * Y$ to be the set of all ordered pairs (x, y), with $x \in X$ and $y \in Y$; that is, all ordered pairs which can be made using an element of X as first coordinate and and an element of Y as second coordinate. We write this as follows:

$$X * Y = \{(x, y) : x \in X, y \in Y\}$$

Activity 1

Let $A = 1, 2, 3, 4, 5$ and B = a, e, i, o, u, obtain the Cartesian product (i) A × B (ii) B × A

Solution

(i) A×B = $(1, a),(1, e),(1, i),(1, o),(1, u),(2, a),(2, e),(2, i),(2, o),(2, u),(3, a),(3, e),(3, i),(3, o),(3, u),(4, a),(4, e),(4, i),(4, o),(4, u),(5, a),(5, e),(5, i),(5, o),(5, u)$

(ii) B×A = $(a, 1),(a, 2),(a, 3),(a, 4),(a, 5),(e, 1),(e, 2),(e, 3),(e, 4),(e, 5),(i, 1),(i, 2),(i, 3),(i, 4),(i, 5),(o,1),(o,2),(o,3),(o,4),(o,5),(u,1),(u,2),(u,3),(u,4),(u,5)$

Summary

Cartesian product of two sets is an ordered pair of each elements of one set with respect to all the elements of the other set while a relation is a subset of the Cartesian product.

Self Assessment Questions

Define (i) Cartesian Product of set X and Y (ii) Relation of set X and Y

Tutor Marked Assignment

- Find the Cartesian product $A \times B$ if $A = \{a, e, i, o, u\}$ and $B = \{x : 0 \leq x \leq 10$, x is even$\}$
- Given sets $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Obtain the Cartesian Product $B \times A$

References

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Further Reading

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UNIT 3

Relation

Introduction

In mathematics, an n–ary relation on n sets, is any subset of Cartesian product of the n sets. The relation is homogeneous when it is formed with one set. For example, any curve in the Cartesian plane is a subset of the Cartesian product of real numbers, $R \times R$.

Learning Outcomes

At the end of this unit, you should be able to:

1. define relation
2. state the properties of relation of two sets

Main Content

A relation R between two sets X and Y is a subset of the Cartesian product $X \times Y$ i.e, a collection of ordered pairs(x,y). When we are given a relation, we can consider the set of all ordered pairs that satisfy the relation. Every relation on a set R determines a certain subset of $R * R$. Infact, this point of view is used as the abstract definition of relation: The homogeneous binary relations are studied for properties like reflexiveness, symmetry, and transitivity which determine different kinds of orderings on the set. Heterogeneous n – ary relations, are used in the semantics of predicate calculus, and in relational databases. In relational databases jargon, the relations are called tables. There is a relational algebra consisting in the operations on sets, because relations are sets, extended with operators like projection, which forms a new relation selecting a subset of the columns (*tuple entries*) in a table, the selection operator, which selects just the rows (*tuples*),according to some conditions, and join which works like a composition operator.

The use of the term ”relation” is often used as shorthand to refer to binary relations, where the set of all the starting points is called the domain and the set of the ending points is the co-domain.

Subset of a Relation

A relation on a set R is a subset of $R * R$. If R is any relation on a set, we write a R b if the pair (a, b) belongs to the relation. Hence, **Properties of Relation**

Let X and Y be two sets and R be a relation on the set X to itself or to set Y , then the following properties exist:

1. **Reflexivity:** For a relation R on set X to X, i.e. $R \subset X \times X$, for all $x \in X,(x, x) \in R$, that is, if x R x for every x

2. Symmetry :for all x, y ∈ X, if (x, y) ∈ R, then (y, X) ∈ R, that is, if x R y implies y R x

3. **Transitivity:** for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$, that is, if x R y and y R z implies x R z

Activity 1

Let X = 2, 3, 4, 5, 6, 7, 8, 9 and let R be a relation on set X defined by Rfact = (x, y) : y is a multiple of x, ∀x, y ∈ X

(a) list all the elements of Rfact (b) is Rfact reflexive? (c) is Rfact symmetric? (d) is Rfact transitive?

Solution

(a) Rfact = (2, 4),(2, 6),(2, 8),(3, 6),(3, 9) Activities (b),(c) and (d) are left for you to answer

Activity 2

The relation '≤' is reflexive and transitive, but not symmetric. The relation '=' is symmetric, reflexive and transitive. The "multiplicative principle" used in part (a) of the above proof is very important. For example, if $X = \{1, 2\}$ and $Y = \{a, b, c\}$, then we can arrange the elements of $x * y$ in a table with two rows and three columns as follows;

$(1, a)$ $(1, b)$ $(1, c)$

$(2, a)$ $(2, b)$ $(2, c)$

Moreover, if we are given a set S, we shall be interested in relations between the elements of the set. Let us consider the set of integers, to see some examples. The following are relations on the set of integers;

"a is less than or equal to b"

"a divides b"

"a is equal to b"

"a has the same sign as b"

Here, we consider negative numbers to have a minus sign, positive numbers to have a plus sign, and the number 0 to have a neutral sign.

Partition

We define a <u>partition</u> of a set S to be a collection of nonempty pairwise disjoint subsets of S whose union is S. Each subset in a partition is called a <u>cell</u>. If we are given a partition of a set S, we can consider the following relation: "a and b are in the same cell". Thus every partition determines a relation. If a is an element, we write $cl(a)$ for the cell that contains a.

Equivalence relations

An equivalence relation is a relation determined by a partition, by the rule "a and b and are in the same cell. For example, consider the partition of the integers into the three sets of negative numbers, positive numbers, and the set {0}. A relation is an equivalence relation if and only if it is symmetric, reflexive and transitive.

Summary

In this unit, you have learnt the concept of relation. You also gained insights to different types of relations and the definition of a subset of a relation. For example, a relation is said to be symmetric if $a\,R\,b$ implies $b\,R\,a$, reflexive if $a\,R\,a$ for every a and transitive if $a\,R\,b$ and $b\,R\,c$ implies $a\,R\,c$. You also learnt about partition as well as equivalence relation.

Self Assessment Questions

1. What is a relation?
2. Briefly discuss partition.
3. Discuss different types of relations definitions.

Tutor Marked Assignment

For $A1$-$A8$, define a relation R on N by aRb if $(a-b)$ is an even integer. Are the following statements true or false?

A1 $(5,9) \in R$

A2 R is reflexive

A3 $(10,1) \notin R$

A4 R is symmetric

A5 $7R4$

A6 R is transitive

A7 $(2, 2) \in R$

A8 R is equivalence relation.

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Further Reading

- Relations and Functions. -Purplemath
- https://simple.wikipedia.org/wiki/Relation(mathematics)

Module 4

FUNCTION THEORY

Units

UNIT 1

Functions

Introduction

A function is a relation between sets that associates to every element of a first set exactly one element of the second set. Typical examples are functions from integers to integers or from the real numbers to real numbers. Functions were originally the idealization of how a varying quantity depends on another quantity. For example, the position of a planet is a function of time.

A function is a process or a relation that associates each element x of a set X, the domain of the function, to a single element y of another set Y (possibly the same set), the co-domain of the function. If the function is called f, this relation is denoted $y = f(x)$ (which is spoken aloud as f of x), the element x is the argument or input of the function, and y is the value of the function, the output, or the image of x by f. The symbol that is used for representing the input is the variable of the function (one of ten says that f is a function of the variable x).

Learning Outcomes

At the end of this unit, you should be able to:

1. define a function
2. explain injective, surjective and bijective functions

Main Content

A function from a set A to a set B is a subset f of $A * B$ such that for every a $\in A$ there is a unique $b \in B$ such that $(a, b) \in f$. Usually we write $f(a) = b$ instead of $(a, b) \in f$.

If f is a function from A to B we write $f : A \to B$. We call A the domain of f and B the co-domain of f. The set $Im(f) = \{b \in B | b = f(a)$ for some $a \in A\}$ is called the image of f. If an element 'a'is mapped to an element 'b' we write $a \to b$. The most common way of specifying a function is illustrated by the following example, in which we take $A = B = N$.

$f : N \to N$

$x \to x^2$

This function, which sends each natural member to its square, can also be described by the formula

$$f(x) = x^2$$

Let A and B be sets. A function f from A to B can be thought of as some kind of rule, or machine, that assigns one element of B to each element of A. For example, if $A = B = N$, there is a function which to each element $n \in N$ assigns the square $n^2 \in N$.

A function is sometimes called a map or mapping. If an element $b \in B$ is assigned to $a \in A$ we say that a is set to b, or that a maps to b, and we write $f(a) = b$.

Now, let f be a function from A to B. We can then consider the set of all pair $(a, b) \in A * B$ such that $f(a) = b$. This is a subset of $A * B$, called the graph of f. The graph of a function has the following property: for every $a \in A$ there is a unique element $b \in B$ such that (a, b) is in the graph. This leads to the abstract definition of a function:

Let f $: A \to B$ be a function. We say that

1. f is injective (or one-to-one) if $f(x) = f(y)$ implies $x = y$.

2. f is surjective (or onto) if for every $b \in B$, there is some $a \in A$ such that $f(a) = b$.

3. f is bijective if it is both surjective and injective.

4. if $f : A \to B$ and $g : B \to C$ are functions, we can define a function h from A to C by the rule $h(a) = g(f(a))$. This function is called composite and it is denoted by $g\ o\ f$.

Summary

In this unit, you have learnt that a function is a process or a relation that associates each element x of a set X, the domain of the function, to a single element y of another set Y (*possibly the same set*), the co-domain of the function.

Self Assessment Questions

1. Define a function
2. Define an injective and a bijective functions

Tutor Marked Assignment

1. Given the functions $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$, find $fog(x)$

2. If $g : N \to N, x \to x^2 - x + 1$ and let $f : N \to N$ be defined by $f(x) = x + 2$

1. Is f injective?

2. Is f surjective?

3. Is f bijective?

4. What is the domain of f?

5. What is the codomain of f?

6. What is the image of f?

7. Is g injective?

8. Is g surjective?

9. Is g bijective?

10. What is the codomain of g?

11. Write down some elements of $Im(g)$.

12. Compute $g \; o \; f(7)$

13. Compute $f \; o \; g(7)$

14. Is $f \; o \; g$ injective?

15. Is $g \; o \; f$ injective?

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