# MAT 211: ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

*3 CREDITS*

DISTANCE LEARNING CENTRE,
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# MODULE 1

# FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

# Unit 1: Existence and Uniqueness of Solution

## Introduction

An equation involving derivatives (or differentials) of differential of one or more dependent variables with respect to one or more independent variables is called a Differential Equation (DE). If the dependent variable depends on only one independent variable, then such a differential equation is called ordinary differential equation whereas when two or more independent variables are involved, it has known as a partial differential equation. In this unit, effort shall be directed toward

**Learning Outcomes** At the end of this unit, you should be able to:

1. define a DE, state and distinguish between the types of DE
2. define the order and degree of a DE; and
3. state and prove the existence and uniqueness of solution of a DE

### DIFFERENTIAL EQUATIONS

The first and most basic example of a differential equation is the one we are already familiar with from calculus. That is

$$\frac{dy}{dx} = f(x) \ldots\ldots\ldots \ (i)$$

In this situation, we will eliminate the derivative by integrating $f$. That is,

$$y(x) = \int_a^x f(x)dx + c \ldots\ldots\ldots \ (ii)$$

Recall from the fundamental theorem of calculus, that $\int_a^x f(x)dx$ is an antiderivative for $f(x)$ for any choice of $a$. Note that there is an arbitrary constant $c$ and so we get a family of solutions, one for each choice $c$.

Therefore, this will make us to encounter initial value problems. These are problems where we will be asked to find a solution to an ordinary differential equation that passes through some initial point $(x_0, y_0)$ where $x_0$ is the independent and $y_0$ the dependent variable. To find which solution passes through this point, one simply plugs $x_0$ into the equation for $x$ and $y_0$ for $y(x_0)$. This allows one to make a specific choice for $c$ which normally would be arbitrary.

$y_0 = \int_a^{x_0} f(x)dx + c \,\ldots\ldots\ldots\, (iii)$

$c = y_0 - \int_a^{x_0} f(x)dx \,\ldots\ldots\ldots\, (iv)$

Substituting $(iii)$ into $(ii)$, we have

$y(x) = \int_a^{x} f(x)dx + y_0 - \int_a^{x_0} f(x)dx$

$y(x) = y_0 + \int_{x_0}^{x} f(x)dx \,\ldots\ldots\ldots\, (v)$

The final equation is really a statement of the fundamental theorem calculus.

# Definition I

An equation which expresses a relationship between an independent variable, a dependent variable and one or more differential coefficient of the dependent variable is called an ORDINARY DIFFERENTIAL EQUATION (ODE).

# Examples

$(a)\ \frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0$

$(b)\ xy\frac{d^2y}{dx^2} + y\frac{dy}{dx} + \exp^{3x} = 0$

$(c)\ \frac{dy}{dx} = 2sin^2x$

## Definition II

The Order of an ODE is the order of the highest derivative involved in the equation. Thus,

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 3y = 27x^2$ is a second order ordinary differential equation while $\frac{d^2y}{dx^2} - 4x^{2y} = sinx$ is a first order ODE

However, the degree of an ODE is the degree of the highest derivative after removing the radical sign and fraction. In order words, it is the index (or power) on the highest derivative after eliminating the radical index from the equation

## Examples

$cosx\frac{d^2y}{dx^2} + sinx(\frac{dy}{dx})^2 + 8y = tanx$ ......... $(I)$ $[1 + (\frac{dy}{dx})^2]^3 = (\frac{d^2y}{dx^2})^2$ ......... $(II)$

The degree of equation (1) is 1 and the degree of equation $(II)$ is 2

### Definition III

An ODE is said to be linear if;

$(I)$ Every dependent variable and every derivatives involved in the equation occur to the first degree only.

$(II)$ No product of dependent variable and/or its derivative occur. Otherwise, the differential equation is said to be non-linear.

### Activity

(a) $\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + x^4\frac{dy}{dx} = x\exp 3x.$

(b) $\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = 0.$

(c) $y\frac{d^2y}{dx^2} + y\frac{dy}{dx} + \exp 3x = 0$

(d) $\frac{d^2y}{dx^2} = 2sin^2x$

Examples (b) and (c) are non-linear while (a) and (d) are Linear

## Summary

In this unit, basic definitions of terms used in differential equation were discussed.

## Self Assessment Question

(1) Define Order of a differential equation

(2) When is a differential equation said to be non-linear, give four examples

## Tutor Marked Assignment

1 Classify the following according to the order:

(a). $\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 - 4y = 0.$

(b). $(\frac{d^2y}{dx^2})^2 - y^{\frac{3}{2}} = 2sinx$

(c). $\frac{d^2y}{dx^2} = (4y - \frac{dy}{dx})^{\frac{2}{3}}$

(d). $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0.$

2 What is the degree of each DE in (1)

## Further Readings


Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. *Palgrave Publishers Ltd.* New York.

# Unit 2: Method of Solutions of First Order Ordinary Differential Equation

## Introduction

Solving a first order differential equation is simple but not always straight forward. In this unit, we shall consider techniques for solving some first order ordinary differential equation using variable separable and method of integrating factor methods.

## Learning objectives/outcomes

At the end of this unit, you should be able to:

1. solve first order ordinary differential equations using variable separable and integrating factor methods

2. insert initial conditions to a solution of first order ordinary differential equation and obtain the value of the constant.

## Main Content

A general form of first order ordinary differential equation is the form:
$\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x)$ and $Q(x)$ can be a constant, functions of $x$ and/or functions of $x$ and $y$ respectively. If $Q(x) = 0$, then the equation is said to be homogeneous first order DE.

**Method of Solution**

## Variable Separable

A differential equation of the form $f(x)dx + g(y)dy = 0$ or it is an equation that can be changed to this form is said to belong to the family of variable separable equations. Integrating this equation directly leads us to a given solution.

It is important to know that the constant of integration can be replaced by various constants like $k$, $logk$, $\ln k$, $e^{-k}$, $tan^{-1}k$ and so on.

**Note:** We say that variables are separable if $y$ is put on the left sides and $x$ on the right hand. In other, the two variables can maintain the different sides of the equation.

**Activity 2**

Find the general solution of each of the following differential equations.

(1).

$$y^2 + x^2\frac{dy}{dx} = 0$$

(2).

$$s\frac{ds}{dt} - \frac{t(1+s^2)}{1+t^2} = 0$$

## Solution

(1).

$$y^2 + x^2\frac{dy}{dx} = 0$$
$$y^2 = -x^2\frac{dy}{dx}$$
$$y^2dx = -x^2dy$$
$$\frac{1}{x^2}dx = -\frac{1}{y^2}dy$$

$$\int \frac{1}{y^2}dy + \int \frac{1}{x^2}dx = c \Rightarrow \int y^{-2}dy + \int x^{-2}dx = c$$

$$-y^{-1} + x^{-1} = c \Rightarrow \frac{1}{y} + \frac{1}{x} = \frac{1}{c}.$$

Thus, $x + y = cxy$.

(2).

$$s\frac{ds}{dt} - \frac{t(1+s^2)}{1+t^2} = 0$$

$$s\frac{ds}{dt} - \frac{t(1+s^2)}{1+t^2} = 0$$

$$\left(\frac{s}{1+s^2}\right)ds - \left(\frac{t}{1+t^2}\right)dt = 0$$

$$\frac{1}{2}ln(1+s^2) - \frac{1}{2}(1+t^2)dt = lnk$$

$$ln\left(\frac{1+s^2}{1+t^2}\right)^{\frac{1}{2}} = Ink \Rightarrow \left[\frac{(1+s^2)}{(1+t^2)}\right]^{\frac{1}{2}} = k.$$

Thus,

$$\Rightarrow \left[\frac{(1+s^2)}{(1+t^2)}\right]^{\frac{1}{2}} = k.$$

## Integrating Factor

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \tag{1}$$

is called a Linear differential equation, where P and Q are functions of $x$ (but not of $y$) or constants.

In such a case, multiply both sides of (1) by $e^{\int Pdx}$ to have

$$e^{\int Pdx}[\frac{dy}{dx} + Py] = \int Qe^{\int Pdx} \tag{2}$$

That gives

$$\frac{d}{dx}[ye^{\int Pdx}] = Qe^{\int Pdx} \tag{3}$$

Separating variables to get

$$d[ye^{\int Pdx}] = Qe^{\int Pdx} \tag{4}$$

$$ye^{\int P(x)dx} = Qe^{\int P(x)dx} + c \tag{5}$$

$$y = e^{-\int P(x)dx}\left[\int Qe^{\int P(x)dx}dx\right] + ce^{-\int P(x)dx} \tag{6}$$

This is the required solution

NOTE: $e^{\int Pdx}$ is called the Integrating factor.

$yI.F = \int Q[I.F]dx + c$

**Activity 1**

Solve $\frac{dy}{dx} + 2xy = e^{-x^2}$ given $y(0) = 0$

Solution

$\frac{dy}{dx} + 2xy = e^{-x^2}$

$P = 2x$, $Q = e^{-x^2}$

Integrating factor $I(x) = e^{\int Pdx}$

$I(x) = e^{\int 2xdx} = e^{x^2}$

Multiply the problem by the integrating factor

$e^{x^2}[\frac{dy}{dx} + 2xy] = e^{-x^2}e^{x^2}$

$\frac{d}{dx}[ye^{x^2}] = 1$

$d[ye^{x^2}] = dx$

Integrate both sides to get

$ye^{x^2} = x + c$

$y = xe^{-x^2} + ce^{-x^2}$

using $y(0) = 0$

$0 = 0e^{-0} + ce^{-0}$

$c = 0$

Therefore, $y = xe^{-x^2}$

A solution or an integral of a differential equation is said to be general or complete if the number of arbitrary constants in the solution is equal to order of the differential equation.

For example:

$$\frac{d^2y}{dx^2} + y = 0.$$

The solution then must contain two arbitrary constants because it is a second order.

The solution that is not general such solution is called a particular.

**Activity 1**

Find the general solution and the particular solution

$$\frac{dy}{dx} = 3x^2$$

at point $(1, 2)$

## Solution

$$dy = 3x^2 dx \Rightarrow \int dy = \int 3x^2 dx + c$$

$$y = x^3 + c.$$

For the particular solution at point $(1, 2)$, $y = 2$, $x = 1$.

$$2 = (1)^3 + c \Rightarrow c = 1.$$

$y = x^3 + 1$ is a particular solution.

## Equation of First Order and Degree

If $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ and it is written as $M(x, y)dx + N(x, y)dy = 0$.

E.g

$$x + 3y\frac{dy}{dx} = 0$$

$$3ydy = -xdx \Rightarrow \int 3ydy = -\int xdx$$

$$\frac{3y^2}{2} = \frac{-x^2}{2} + k$$

$$3y^2 + x^2 = 2k^2$$

where $c = 2k^2$

This can also be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a^2 = 2k^2$ and $b^2 = 2k^2$.

## Summary

Two techniques of obtaining solution of first order ordinary differential equations were discussed with examples.

## Self Assessment Questions

Solve the following differential equations.

$(a)$ $(1-x^2)^{\frac{1}{2}}\frac{dy}{dx}+(1+y^2)=0$

$(b)$ $xy(1+x^2)\frac{dy}{dx}-(1+y^2)=0$

$(c)$ $6xydx+(x^2+1)dy=0$

## Tutor Marked Assignment

Solve the following differential equation.

$(i)$ $xy^2\frac{dy}{dx}-x=1$

$(ii)$ $xyy'-lnx=0$

$(iii)$ $(1+x)y'+y=1+x$

$(iv)$ $(xlnx)y'+y=xe^x$

$(v)$ $y'-\frac{x}{y}=0$

# Unit 3: Solution of other forms of first order ordinary differential equation

Introduction: Having laid the foundation for solution of first order ordinary differential equations, we shall be interested in solving some other forms of first order equations. These forms include exact, homogeneous, Bernoulli types etc.

## Learning objectives/outcomes

At the end of this unit, you should be able to:

1. identify and solve an exact equation
2. identify and solve an homogeneous equation
3. identify and solve an homogeneous equation.

## Exact Differential Equation

**Exact equation:** This differential equation can be put in the form:

$A(x)dx + B(y)dy = 0 \ldots \;\; \ldots \;\; \ldots \;\; (I)$

A set of solution can be obtained simply by integration. Such idea can be extended to solve first order differential equation of the form:

$M(x, y)dx + N(x, y)dy = 0$

For which variable separable may not be possible. Suppose we have a function $F(x, y) = c$ in which all its total derivative give the expression $M(x, y)dx + N(x, y)dy$

That is, $dF(x,y) = M(x,y)dx + N(x,y)dy \ldots \ \ldots \ \ldots \ (II)$.
Then, $F(x,y) = c \ldots \ \ldots \ \ldots \ (III)$.

Equation $(III)$ defines explicitly the solution of equation $(I)$. From equation $(III)$, it follows that $dF = 0$ as we have from equation $(II)$

**Note:** Two things are needed:

(i). To find out under what conditions on $M$ and $N$ does function "$F$" exists such that its total derivative is exactly $Mdx + Ndy$.

(ii). If those conditions are satisfied, we are to determine the function "$F$". if such a function $f$ exists such that $Mdx + Ndy$ is its total Then (IV) is called an exact equation: $M(x,y)dx + N(x,y)dy = 0 \ldots$ $(IV)$.

If this equation is exact by definition, it shows that there is a particular function $F$ that exist such that

$$dF = Mdx + Ndy$$

so that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

$$\frac{\partial F}{\partial x} = M; \frac{\partial F}{\partial y} = N$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial F}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial F}{\partial x}\left(\frac{\partial F}{\partial y}\right).$$

From calculus, $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$ provided $Mdx + Ndy = 0$ is continuous.

$$Hence, \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{7}$$

**Activity 2**

Solve this equation $3x(xy-2)dx + (x^3+2y)dy = 0$.

**Solution**

$$3x(xy-2)dx + (x^3+2y)dy = 0$$

$$Mdx + Ndy = 0$$

$$M = 3x(xy-2) = 3x^2y - 6x; N = x^3 + 2y$$

$$\frac{\partial M}{\partial y} = 3x^2, \frac{\partial N}{\partial x} = 3x^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3x^2$, then the equation is exact. Now

$$M = \frac{\partial F}{\partial x} = 3x^2y - 6x$$

$$N = \frac{\partial F}{\partial y} = x^3 + 2y$$

On integration,

$$\int \frac{\partial F}{\partial x} dx = \int Mdx = \int (3x^2y - 6x)dx$$

$$F(x,y) = x^3y - 3x^2 + c(y)$$

$$\frac{\partial F}{\partial y} = x^3 + c'(y) = N = x^3 + 2y.$$

By comparison, $c'(y) = 2y$

$$\int c'(y)dy = \int 2ydy \Rightarrow c(y) = y^2$$

Therefore

$$F(x, y) = x^3y - 3x^2 + c(y)$$

becomes

$$F(x, y) = x^3y - 3x^2 + y^2.$$

Therefore the solution of $F(x, y) = c$ gives

$$F = x^3y + y^2 - 3x^2 = c$$

## Bernoulli Equation

The Bernoulli equation is of the form:

$$\frac{dy}{dx} + Py = Qy^n \tag{8}$$

where $P$ and $Q$ are constants or function of $x$ and can be reduced to the linear form on dividing by $y^n$ and substituting

$$\frac{1}{y^{n-1}} = z \tag{9}$$

Dividing both sides of (8) by $y^n$, we get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P = Q \tag{10}$$

But $\frac{1}{y^{n-1}} = z$, so that $\frac{1-n}{y^n}\frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx} \tag{11}$$

it is seen that (10) becomes

$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

Isolating $\frac{dz}{dx}$ to obtain

$\frac{dz}{dx} + P(1-n)z = Q(1-n)$

which is a linear equation.

**Activity**

Solve $x^2dy + y(x+y)dx = 0$

Solution

$\frac{dy}{dx} + \frac{y}{x} = \frac{-y^2}{x^2}$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$

Put $\frac{1}{y} = z$, so that $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

The given equation reduces to a linear differential equation in $z$

$-\frac{dz}{dx} + \frac{z}{x} = -\frac{1}{x^2}$

Integrating factor $= e^{\int -\frac{1}{x}}dx$

$e^{-\log x} = \frac{1}{x}$

Hence the solution is

$z \times \frac{1}{x} = \int \frac{1}{x^2} \times \frac{1}{x}dx + c$

$\frac{z}{x} = \int x^{-3}dx + c$

$\frac{-1}{xy} = +\frac{x^{-2}}{2} + c$

$\frac{1}{xy} = -\frac{1}{2x^2} + c$

## Homogenous Equation

A differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous expressions of the same degree. Anytime we have homogeneous differential equation in $x$ and $y$, a substitution of the form:

$z = \frac{y}{x}$ or $z = \frac{x}{y}$ immediately reduces the homogeneous differential equation to the variable separable form.

$M(x, y)dx+N(x, y)dy = 0$ is said to be homogeneous if $M(S_x, S_y) = S^n M(x, y)$ and $N(S_x, S_y) = S^n N(x, y)$

that is, if $M$ and $N$ are homogeneous of the same degree $n$.

**Example**

Determine the given equation is homogeneous or not. Hence solve it:

$(x^2 + y^2)dx + 2xydy = 0$

Solution

$$M(x, y) = x^2 + y^2; N(x, y) = 2xy$$

For $x = S_x, y = S_y$

$$S^2 M(x, y) = S^2x^2 + S^2y^2 = S^2(x^2 + y^2)$$

$$S^2 N(x, y) = 2S_xS_y = 2S^2xy$$

Thus, the differential equation is homogeneous since $M$ and $N$ are of the same degree 2

Substitute $z = \frac{x}{y}; x = zy$

$$dx = zdy + ydz$$

$$(x^2 + y^2)dx + 2xydy = 0$$

$$(z^2y^2 + y^2)(zdy + ydz) + 2(zy)ydy = 0$$

$$y^2(z^2 + 1)(zdy + ydz) + 2zy^2dy = 0$$

$$(z^2 + 1)(zdy + ydz) + 2zdy = 0$$

$$z(z^2 + 1)dy + y(z^2 + 1)dz + 2zdy = 0$$

$$(z^3 + 3z)dy + y(z^2 + 1)dz = 0$$

$\frac{z^2+1}{z^3+3z}dz + \frac{dy}{y} = 0$

$\int \frac{z^2+1}{z^3+3z} + \int \frac{1}{y} = 0$

$\frac{1}{3}\ln(z^3+3z) + \ln y = -\ln k$

$\ln[(z^3+3z)^{\frac{1}{3}}y] = -\ln k$

$\ln[(z^3+3z)^{\frac{1}{3}}yk] = 0$

$(z^3+3z)^{\frac{1}{3}}yk = 1$

$(z^3+3z)y^3 = c^3$ where $c = \frac{1}{k}$

$\frac{x^3}{y^3} + \frac{3x}{y}y^3 = c^3$

$\frac{x^3}{y^3} + \frac{3x}{y} = \frac{c^3}{y^3}$

$x^3 + 3xy^2 = A$ where $c^3 = A$

## Summary

Three special forms of first order ordinary differential equations were introduced together with methods of solution.

## Tutor Marked Assignment

1. Solve $\frac{dy}{dx} + 2xy = xe^{-x^2}y^3$

2. Solve $ydx - 2xdy = yx^4dy$

## Tutor Marked Assignment

1. Solve the differential equation $(2xy + x^2)\frac{dy}{dx} = 3y^2 + 2xy$

2. Solve the differential equation $\frac{dy}{dx} = \frac{y}{x} + x\sin\frac{y}{x}$

# References

# Further Readings

# Unit 4: Applications of first-Order Equations

**Introduction:** Some practical applications of first order ordinary differential equations shall be discussed in this unit.

## Learning objectives/outcomes

At the end of this unit, you should be able to:

1. solve orthogonal and oblique trajectories problem ; and

2. solve some problems in mechanics leading to first order ordinary differential equation.

## Main Contents

### (A) Orthogonal and Oblique Trajectories

Let

$$F(x, y, c) = 0 \tag{12}$$

be a given one-parameter family of curves in the xy plane. A curve which intersects the curve of the family (12) at right angles is called an orthogonal trajectory of the given family.

The problem of finding the orthogonal trajectories of a given family of curves arises in many physical situations. For example, in a two-dimensional electric field, the lines of force (flux lines) and the equipotential curves are orthogonal trajectories of each other.

**Activity 1**

Find the orthogonal trajectories of the family of circles
$x^2 + y^2 = c^2$.

**Solution**

Differentiating the equation with respect to $x$ to obtain

$2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

Since an orthogonal trajectory intersects the family of curves at right angle then,

$$\frac{dy}{dx} = \frac{y}{x}$$

($m_2 = -\frac{1}{m_1}$ where $m_1$ and $m_2$ are gradients)

is the orthogonal trajectories to the family of curves $x^2 + y^2 = c$.

$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$

$\Rightarrow lny = lnx + lnk$

$\Rightarrow ln(\frac{y}{x}) = lnk$

$\Rightarrow y = kx$

which represents the family of orthogonal trajectories of the given family of circles (except for a line $x = 0$)

**Figure 3.1**

**Figure 3.1** shows several members of the family of circles and several members of the family of straight lines drawn with dashes.

**B. Oblique Trajectories**

Let

$$F(x, y, c) = 0 \tag{13}$$

be a one-parameter family of curves. A curve which intersects the curves of the family (13) at a constant angle $\alpha \neq 90^o$ is called an oblique trajectory.

Suppose the differential equation of a family is

$$\frac{dy}{dx} = f(x,y) \tag{14}$$

then, the curve of the family (14) through the point (x,y) has the slope $f(x,y)$ at (x,y) and hence, its tangent line has angle of inclination $tan^{-1}f(x,y)$ there. The tangent line of an oblique trajectory which intersects this curve at the angle $\alpha$ will thus have angle of inclination
$tan^{-1}[f(x,y)] + \alpha$
at the point (x,y).
Hence, the slope of this trajectory is given by

$$tan[f(x,y) + \alpha] = \frac{f(x,y) + tan\alpha}{1 - f(x,y)tan\alpha}$$

The differential equation of such a family of oblique trajectory is given by
$\frac{dy}{dx} = \frac{f(x,y)+tan\alpha}{1-f(x,y)tan\alpha}$.

**Problems in Mechanics**

**Newton's Second Law**

The time rate of change of momentum of a body is proportional to the resultant force acting on the body and it is in the direction of this resultant force.
In Mathematical language, this states that
$\frac{d(MV)}{dt} = K_1F$
where M is the mass of the body , V is its velocity F is the resultant force acting upon it and $K_1$ is the constant of proportionality.

$\Rightarrow \frac{MdV}{dt} = k_1 F$ (if M is constant)

$\Rightarrow a = \frac{k_1 F}{M}$

$\Rightarrow F = \frac{1Ma}{k_1}$

$\Rightarrow F = kMa$

(where $k = \frac{1}{k_1}$ and $a = \frac{dv}{dt}$) is the acceleration of the body.

The simplest system of units are those for which $k = 1$, hence

$F = Ma$

**Activity 2**

A body weighing 60 kg fall from rest towards the earth from a great height. As it falls, air resistance acts upon it, and assume that this resistance is numerically equal to 2v, where v is the velocity $(m/s)$. Find the velocity and distance fallen at time $t$ seconds.

**Solution**

Chose the positive x axis vertically downward along the path of the body B and the origin at the point from which it fell. The forces acting on the body are

i. $F_1$, its weight 60 kg, which acts downward and hence it is positive.

ii. $F_2$, the air resistance (2v) which acts upward and hence it is negative.

**Newton's Second Law**

$$F = ma \tag{15}$$

$\Rightarrow \frac{MdV}{dt} = F_1 + F_2$

**DIAGRAM**

**Earth**

Taking $g = 10M/s^2$ and using $W = mg$

$M = \frac{W}{g} = \frac{60}{10} = 6$

$\frac{6dV}{dt} = 60 - 2V$

$\frac{3dV}{dt} = 30 - 2V$

Since the body was initially at rest, the initial condition is

$V(0) = 0$

$\frac{dV}{30-V} = \frac{1}{3dt}$

Integrating

$-ln|30 - V| = \frac{t}{3} + c_1$

$\Rightarrow 30 - V = c_1 e^{-\frac{t}{3}}$

$c_1 = 30$ from the initial condition

$\Rightarrow V = 30 - 30e^{\frac{-t}{3}}$

$$V = 30(1 - e^{-\frac{t}{3}}) \tag{16}$$

Also,

$\frac{dX}{dt} = V$

$dX = Vdt$

$X(t) = 30(t + 3e^{-\frac{t}{3}}) + c_2$

Note that $X(0) = 0$

$\Rightarrow 0 = 90 + c_2 \Rightarrow c_2 = -90$

Hence, the distance fallen is given by

$X = 30(t + 3e^{\frac{-t}{3}}) - 90$

$$X = 30(t + 3e^{\frac{-t}{3}} - 3) \tag{17}$$

**Interpretation**

Equation (16) shows that as $t \to \infty$, the velocity V approaches the limiting velocity $30m/s$. This limiting velocity is approximately attained in a very short time.

Equation (17) shows that as $t \to \infty$, $X \to \infty$. It does not imply that the body will flow through the earth and continue for ever. When the body reaches the earth's surface, equation (17) no longer apply.

**Rate of Change Problem**

In certain problems, the rate at which a quantity changes is a known function of the amount present and/or the time, and it is desired to find the quantity itself. If X denotes the rate at which the quantity present at time $t$, then $\frac{dx}{dt}$ denotes the rate at which the quantity changes and we are at once led to a differential equation.

## Summary

Some real life problems leading to first order ordinary differential equation were solved in this section.

## Self Assessment Questions

1. Find the orthogonal trajectories of each given family of curves. In each case, sketch several members of the family and several of the orthogonal trajectories on the same set of axes.

   (i). $y = cX^3$

   (ii). $y^2 = cX$

   (iii). $cX^2$

   (iv). $y = e^{cX}$

   (v). $y = x - 1 + ce^{-X}$

2. Find the value of k such that the parabolas $y = c_1x^2 + k$ are the orthogonal trajectories of the family of ellipseS $x^2 + 2y^2 - y = c_2$.

3. A ball weighing 1 kg is thrown vertically upwards from a point $10m$ above the surface of the earth with an initial velocity of $15m/s$. As it rises, it is acted upon by air resistance which is numerically equal to $\frac{1}{64}v$, where v is the velocity$(m/s)$. How high will the ball rise?

4. The population of the city of Bingville increases at the rate proportional to the number of its inhabitants present at any time $t$. If the population of Bingville was 30,000 in 1950, and 35,000 in 1960, what will be the population of Bingville in 1970?

## Tutor Marked Assignment

1. Apply the existence and uniqueness theorem to show that each of the following initial value problem has a unique solution defined on some sufficiently small interval $|x - 1| \leq h$ about $x_0 = 1$

(i). $\frac{dy}{dx} = x^2 siny, y(1) = -2$

(ii). $\frac{dy}{dx} = \frac{y^2}{x-2}, y(1) = 0$

2. A chemical reaction converts a certain chemical into another chemical, and the rate at which the first chemical is converted is proportional to the amount of this chemical present at any time. At the end of one hour, 50 grammes of the first chemical remain while at the end of three hour s, only 25 grammes remain. (a). How many grammes of the first chemical were present? (b). How many grammes of the first chemical will remain at the end of five hours? (c). In how many hours will only 2 grammes of the first chemical remain?

# References

# Further Readings

# Unit 5: Second order differential equation

**Introduction:** A second order differential equation is one in which the highest derivative present in it is $\frac{d^2y}{dx^2}$. In this unit, we shall study the general form of second order ordinary differential equation and present method of solution to homogeneous second order ordinary differential equation with constant coefficient.

## Learning objectives/outcomes

At the end of this unit, you should be able to:

(i) identify and solve homogeneous second order differential equation with constant coefficients.

## Main Content

The general form of a second order linear differential equations is:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \tag{18}$$

where $P$, $Q$, $R$ and $G$ are continuous functions. The equation (18) is said to be homogeneous if $G(x) = 0$

That is,

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \tag{19}$$

Otherwise, it is non-homogeneous. In this unit, our interest lies in the equation of the form (19) above where $P(x)$,$Q(x)$ and $R(x)$ are constants.

That is, $P, Q, R \in \Re$ where $P \neq 0$.

Every second order linear differential equation has two linearly independent solutions that satisfy the given equation. Indeed, if $y_1(x)$ and $y_2(x)$ are independent solutions of the linear homogeneous equation (19), then the function $y = k_1y_1(x) + k_2y_2(x)$ is also a solution of the differential equation for all constants $k_1$ and $k_2$.

Given a second order linear differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \tag{20}$$

where $a, b, c \in \Re$. Then, we can assume that $y = e^{mx}$ is a solution (since exponential function has the property that its derivative is a constant multiple of itself).

Thus, $y' = me^{mx}, y'' = m^2e^{mx}$

Substituting $y, y', y''$ into equation (20), we have

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c)e^{mx} = 0$$

Since $e^{mx}$ can never be zero, then

$$am^2 + bm + c = 0 \tag{21}$$

Equation (21) above is called the characteristics equation (or auxiliary equation) of the differential equation (20). The nature of roots of the quadratic equation (21) determines the nature of the two independent solutions of the differential equation. Let us take time to study the three possible form of roots of the quadratic equation (21). The two possible roots of (21) are obtained by using the quadratic formula $m_{1,2} = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

**Case 1**: when $m_1$ and $m_2$ are real and unequal, that is: $m_1 \neq m_2, m_1, m_2 \in \Re$. In this case, $b^2 - 4ac > 0$, the general solution of differential equation

(20) under this scenario is

$$y(x) = k_1 e^{m_1 x} + k_2 x e^{m_2 x}$$

**Case 2**: when $m_1$ and $m_2$ are real and equal, that is: $m_1 = m_2, m_1, m_2 \in \Re$. In this case $b^2 - 4ac = 0$, the general solution of differential equation (24) under this scenario is

$$y_1(x) = k_1 e^{mx} + k_2 e^{mx}$$

**Case 3**: when $m_1$ and $m_2$ are complex roots and conjugate of each other. In this scenario, $b^2 - 4ac \leq 0$ and

$m_1 = \frac{-b+i\sqrt{b^2-4ac}}{2a}$ and $m_2 = \frac{-b-i\sqrt{b^2-4ac}}{2a}$

or $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

Then, the general solution is given as:

$$y(x) = k_1 e^{(\alpha+i\beta)x} + k_2 e^{(\alpha-i\beta)x}$$

$$y(x) = e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x)$$

**Activity 1:**

Solve the differential equation $4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

Solution

The characteristics equation is:

$$4m^2 + 3m - 1 = 0$$

$$4m^2 + 4m - m - 1 = 0$$

$$4m(m+1) - 1(m+1) = 0$$

$$(4m-1)(m+1) = 0$$

$$4m - 1 = 0 \text{ or } m + 1 = 0$$

$$m = \frac{1}{4} \text{ or } -1$$

The solution of the differential equation is

$$y(x) = k_1 e^{\frac{1}{4}x} + k_2 e^{-x}$$

**Activity 2:**

Solve the differential equation $y'' - 6y' + 9y = 0$

Solution

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m = 3$ twice

$y = c_1 e^{3x} + c_2 x e^{3x}$

**Activity 3:** Solve $2y'' - 2y' + 5y = 0$

Solution

$$2m^2 - 2m + 5 = 0$$

$$m_{1,2} = \frac{2 \pm \sqrt{4 - 4(2)(5)}}{4}$$

$$m_{1,2} = \frac{2 \pm \sqrt{4 - 40}}{4}$$

$$m_{1,2} = \frac{2 \pm \sqrt{-36}}{4}$$

$$m_{1,2} = \frac{2 \pm 6i}{4}$$

$$m_{1,2} = \frac{1 \pm 3i}{2}$$

$$m_1 = \frac{1+3i}{2}, m_2 = \frac{1-3i}{2}$$

$$y(x) = e^{\frac{1}{2}x}(c_1 cos\frac{3}{2}x + c_2 sin\frac{3}{2}x)$$

## Initial Value Problems

An initial value problem of second order differential equation with constant coefficients consists of finding a solution that satisfies the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0$

**Activity 3:**

Solve the initial value problem

$y'' + 4y' - 5y = 0$

Given that $y(0) = 1, y'(0) = 0$

Solution

The characteristics equation is

$m^2 + 4m - 5 = 0$

$m^2 + 5m - m - 5 = 0$

$m(m + 5) - 1(m + 5) = 0$

$(m - 1)(m + 5) = 0$

$m = 1$ or $m = -5$

The general solution of the differential equation is

$y(x) = c_1 e^x + c_2 e^{-5x}$

Using $y(0) = 1$

$$1 = c_1 + c_2 \tag{22}$$

$y'(x) = c_1 e^x - 5c_2 e^{-5x}$

$y'(0) = 0$

$$c_1 - 5c_2 = 0 \tag{23}$$

Solving the equation (22) and (23) simultaneously to obtain

$c_1 = \frac{5}{6}, c_2 = \frac{1}{6}$

$y(x) = \frac{5}{6}e^x + \frac{1}{6}e^{-5x}$

**Activity 4:** Solve the initial value problem $y'' - y = 0$

$y(0) = 3, y'(0) = 2$

Solution

The auxilliary equation is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$y(x) = k_1e^x + k_2e^{-x}$

$y(0) = 3$

$$k_1 + k_2 = 3 \tag{24}$$

$y'(x) = k_1e^x - k_2e^{-x}$

$y'(0) = 2$

$$k_1 - k_2 = 2 \tag{25}$$

Adding equation (24) and (25) to obtain

$2k_1 = 5$

$k_1 = \frac{5}{2}$

Also, $k_2 = \frac{1}{2}$

$y(x) = \frac{5}{2}e^x + \frac{1}{2}e^{-x}$

## Boundary Value Problem

A boundary value problem of second order Ordinary differential equation with constant coefficients satisfies the boundary condition of the form $y(x_0) =$

$y_0, y(x_1) = y_1$

The difference between an initial value problem and a boundary value problem is that while an initial value problem has one fixed point $x_0$, a boundary value problem has two ends $x_0$ and $x_1$.

**Activity 5:**

Solve the boundary value problem

$y'' - 6y' + 25y = 0$ where $y(0) = 1, y(\frac{\Pi}{8}) = 2$

Solution

The characteristics equation is

$m^2 - 6m + 25 = 0$

$m_{1,2} = \frac{6 \pm \sqrt{36-4(1)(25)}}{2}$

$= 3 \pm 4i$

$m_1 = 3 + 4i, m_2 = 3 - 4i$

$y(x) = e^{3x}(c_1 cos4x + c_2 sin4x)$

$y(0) = 1$ gives

$c_1 = 1$

$y(\frac{\pi}{8}) = 2$ gives

$2 = e^{\frac{3\pi}{8}}(cos\frac{\pi}{2} + c_2 sin\frac{\pi}{2})$

$2 = e^{\frac{3\pi}{8}} c_2$

$c_2 = 2e^{\frac{-3\pi}{8}}$

$y(x) = e^{3x}(cos4x + 2e^{\frac{-3\pi}{8}} sin4x)$

## Summary

Solutions of second order homogeneous differential equation with constant coefficients was considered in this unit. Initial and boundary value problems

were also presented and solved.

### Self Assessment Questions

Solve the following ordinary differential equations

(i) $4y'' + y' = 0$

(ii) $16y'' + 24y' + 9y = 0$

(iii) $y'' + 12y' + 36y = 0, y(1) = 0, y'(1) = 1$

(iv) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

(v) $y'' - 2y' + 5y = 0, y(0) = 1, y(3) = 0$

### Tutor Marked Assignment

Solve the following ordinary differential equation

(a) $y'' - 6y' + 8y = 0$

(b) $y'' - 4y' + 8y = 0$

(c) $y'' - 2y' + y = 0$

(d) $y'' + 16y = 0, y(\frac{\pi}{4}) = -3, y'(\frac{\pi}{4}) = 4$

(e) $y'' + 100y = 0, y(0) = 1, y'(\pi) = -4$

## References

James Stewart (2011)

Calculus 8th Edition

Early Transcendents

# Further Reading

# Unit 6: General Theory of $n^{th}$ order linear differential equation

**Introduction:** The subject of ordinary differential equations is one great theoretical and practical importance because of its variety applications in Sciences and Engineering.

## Learning outcomes/Objectives

At the end of this unit, you should be able to:

(i) identify and classify an $n^{th}$ order linear differential equation as homogeneous or non-homogeneous

(ii) solve a $n^{th}$ order linear differential equation with constant coefficients.

## Main Content

### Basic Theory of Linear Differential Equations

A linear differential equation of order n is an equation of the form:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \qquad (26)$$

where $a_0$ is not identically zero. Assume that $a_0, a_1, \ldots, a_n$ and F are continuous real functions on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$. The right hand side is called the **non-homogeneous term**. If F is identically zero, the equation reduces to:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (27)$$

and it is called **homogeneous**.

**Examples**

$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3y = e^x$ is a linear differential equation of second order.

while

$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx} - 5y = sinx$ is a linear differential equation of the third order.

Now, there is need to state the basic existence theorem for the initial value problems associated with an $n^{th}$ order linear differential equation.

**Theorem 4.01**

Consider

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (28)$$

where $a_0, a_1, \ldots, a_n$ and F are continuous real functions on a real interval $a \le x \le b$.

Let $x_0$ be any point of the interval $a \le x \le b$ and $c_0, c_1, \ldots, c_{n-1}$ be n arbitrary real constants. Then, there exists a unique solution of (28) such that $f(x_0) = c_0, f'(x_0) = c_1, \ldots, f^{n-1}(x_0) = c_{n-1}$ and this solution is defined over the entire interval $a \le x \le b$.

Consider the initial value problem

$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3y = e^x$

$y(1) = 2$

$y'(1) = -5$

The coefficients 1, $3x$ and $x^3$ as well as the non-homogeneous term $e^x$, in this second order differential equation are all continuous for all values of x,

$-\infty \leq x \leq \infty$. The point $x_0$ here is the point 1, which certainly belongs to this interval; and the real numbers $c_0$ and $c_1$ are 2 and -5 respectively. This tells us that a solution of the given problem exists is unique, and is defined for all x, $-\infty \leq x \leq \infty$.

Also, consider the initial value problem

$2\frac{d^3y}{dx^2} + x\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx} - 5y = sinx$

$y(4) = 3$

$y'(4) = 5$

$y''(4) = -\frac{7}{2}$

Here, the coefficients $2, x, 3x^2$ and $-5$ as well as the non-homogeneous term $sinx$ are all continuous fr all x, $-\infty \leq x \leq \infty$. The point $x_0 = 4$ belongs to the interval.; the real numbers $c_0, c_1$ and $c_2$ are 3,5 and $-\frac{7}{2}$ respectively. Therefore, this problem also has a unique solution which is defined for all x, $-\infty \leq x \leq \infty$.

**Theorem**

Let $f_1, f_2 \ldots, f_m$ be any m solutions of the homogeneous linear differential equation

$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0.$

Then,

$c_1f_1 + c_2f_2 + \cdots + c_mf_m$ (a linear combination of $f_1, f_2, \ldots, f_m$) is a solution of the differential equation where $c_1, c_2, \ldots, c_m$ are m arbitrary constants.

**Definition**

The n functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on $a \leq x \leq b$ if there exists constants $c_1, c_2, \ldots, c_n$, **not all zero**, such that $a \leq x \leq b$ such that $c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$ for all x such that

$a \leq x \leq b$.

As an example, x and 2x are are linearly dependent on the interval $0 \leq x \leq 1$ for there exist constants $c_1$ and $c_2$, not both zero such that

$c_1(x) + c_2(2x) = 0$ for all x on the interval $0 \leq x \leq 1$. For example, let $c_1 = 2, c_2 = -1$

**Definition**

The n functions $f_1, f_2, \ldots, f_n$ are said to be **linearly independent** on the interval $a \leq x \leq b$ if they are not linearly dependent. That is, the functions $f_1, f_2, \ldots, f_n$ are linearly independent on $a \leq x \leq b$ if the relation $c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$ for all x such that $a \leq x \leq b$ implies that $c_1 = c_2 = \cdots = c_n = 0$.

**Definition**

If $f_1, f_2, \ldots, f_n$ are linearly independent solutions of the nth order homogeneous linear differential equation (28) on $a \leq x \leq b$, then the function f defined by $c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x)$, $a \leq x \leq b$ where $c_1, c_2, \ldots, c_n$ are arbitrary constants is called a **general solution** of (28) on $a \leq x \leq b$.

For example, $sinx$ and $cosx$ are solutions of $\frac{d^2y}{dx^2} + y = 0$ for all x; $-\infty \leq x \leq \infty$.

One can show that the two solutions are linearly independent. Thus, the general solution may be expressed as the linear combination

$y = c_1sinx + c_2sinx$ where $c_1$ and $c_2$ are arbitrary constants.

**Definition**

Let $f_1, f_2, \ldots, f_n$ be real functions each of which has an $(n-1)$ derivatives on a real interval $a \leq x \leq b$, the determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

in which primes denote derivatives is called the **Wronskian** of these n functions.

**Theorem**

The n solution $f_1, f_2, \ldots, f_n$ of the nth order homogeneous differential equation (28) are linearly independent on $a \leq x \leq b$ if and only if the Wronskian of $f_1, f_2, \ldots, f_n$ is different from zero for some x on the interval $a \leq x \leq b$.

**Activity 1**

Show that the solutions $sinx$ and $cosx$ of $\frac{d^2y}{dx^2} + y = 0$ are linearly independent.

$$W(sinx, cosx) = \begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix} = -sin^2x - cos^2x = -1 \neq 0$$

Since $W(sinx, cosx) \neq 0$, the solutions of $sinx$ and $cosx$ are linearly independent.

**Activity 2**

The solutions $e^x, e^{-x}$ and $e^{2x}$ of $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$ are linearly independent for

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0$$

**Theorem**

Let f be a nontrivial solution of the $n^{th}$ homogeneous linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{29}$$

This reduces the equation (29) to an $(n-1)^{th}$ order homogeneous linear differential equation in the dependent variable $w = \frac{du}{dx}$.

Consider the nontrivial solution of the second order homogeneous linear equation

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{30}$$

Let the transformation be

$$y = fu \tag{31}$$

then,

$$\frac{dy}{dx} = \frac{f du}{dx} + \frac{u df}{dx} \tag{32}$$

$$\frac{d^2 y}{dx^2} = f\frac{d^2 u}{dx^2} + 2\frac{df}{dx}\frac{du}{dx} + u\frac{d^2 f}{dx^2} \tag{33}$$

Substitute equations (31), (32) and (33) into (30) to obtain

$a_0\left[f\frac{d^2u}{dx^2}+2\frac{df}{dx}\frac{du}{dx}+u\frac{d^2f}{dx^2}\right]+a_1\left[f\frac{du}{dx}+u\frac{df}{dx}\right]+a_2fu=0$
$=a_0f\frac{d^2u}{dx^2}+\left[2a_0\frac{df}{dx}+a_1f\right]\frac{du}{dx}+\left[a_0\frac{d^2f}{dx^2}+a_1\frac{df}{dx}+a_2f\right]U=0$

Since f is a solution of (13), the coefficient of u is zero.

$\Rightarrow a_0f\frac{d^2u}{dx^2}+\left[2a_0\frac{df}{dx}+a_1f\right]\frac{dy}{dx}=0$

Let

$$w=\frac{du}{dx}$$

$$\Rightarrow a_0f\frac{dw}{dx}+\left[2a_0\frac{df}{dx}+a_1f\right]w=0$$

$$\frac{dw}{w}=-\left[\frac{2f'}{f}+\frac{a_1}{a_0}\right]dx$$

$$\ln|w|=-\ln nf^2+-\int\frac{a_1}{a_0}dx+\ln c$$

$$w=\frac{ce^{-\int\frac{a_1dx}{a_0}}}{f^2}$$

Since

$$\frac{du}{dx}=w$$

,

$$U=\int\frac{e^{-\int\frac{a_1dx}{a_0}}dx}{f^2}$$

$$y=fU=f\int\frac{e^{-\int\frac{a_1dx}{a_0}}dx}{f^2}$$

which is the solution of the original second order differential equation (30).

**Activity 3**

Given that $y = x$ is a solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \tag{34}$$

find a linearly independent solution by reducing the order.

**Solution**

Let

$$y = xu \tag{35}$$

$$\frac{dy}{dx} = x\frac{du}{dx} + u \tag{36}$$

$$\frac{d^2y}{dx^2} = x\frac{d^2u}{dx^2} + \frac{du}{dx}\frac{du}{dx} = x\frac{d^2u}{dx^2} + 2\frac{du}{dx} \tag{37}$$

Substitute (35), (36) and (37) into (34) to obtain

$(x^2+1)(x\frac{d^2u}{dx^2} + 2\frac{du}{dx}) - 2x(x\frac{du}{dx} + u) + 2xu = 0$

$\Rightarrow x(x^2+1)\frac{d^2u}{dx^2} + (x^2+1)\frac{du}{dx}) - 2x^2\frac{du}{dx} - 2xu + 2xu = 0$

$\Rightarrow x(x^2+1)\frac{d^2u}{dx^2} + 2\frac{du}{dx} = 0$

Let $W = \frac{du}{dx}$, then,

$x(x^2+1)\frac{dw}{dx} + 2W = 0$

$\Rightarrow \frac{dW}{W} = \frac{dx}{x(x^2+1)}$

$\Rightarrow \frac{dW}{W} = \left[\frac{-2}{x} + \frac{2x}{x^2+1}\right] dx$

Integrating both sides to obtain

$\ln|W| = -2\ln|x| + I_n + (x^2+1) + \ln c$

$\ln|W| = \ln|x^2| + \ln(x^2+1) + \ln c$ (Where c is a constant)

$\ln|W| = I_n|x^{-2} + (x^2+1)c|$

$W = c\frac{(x^2+1)}{x^2}$

Now, recall that $\frac{du}{dx} = W$

$\frac{du}{dx} = c\frac{(x^2+1)}{x^2}$

$u = cx - x^{-2+1} + c_2$

$u = cx - \frac{1}{x} + c_2$ (where $c_2$ is a constant)

$g(x) = y = xu = cx^2 - 1 + xc_2$

The general solution is the linear combination of the solutions

$y_G = a_1 x + a_2 \left[cx^2 - 1 + xc_2\right]$

**Definition**

Consider the nth-order (non-homogeneous) linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \qquad (38)$$

the corresponding equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (39)$$

The general solution of (39) is called the **complementary equation** of equation (38). Let this be denoted by $y_c$. Any particular equation of (38) involving no arbitrary constants is called a **particular integral** of (38). This can be denoted as $y_p$.

Solution $y = y_c + y_p$ of (38) is called the general solution of (38).

1. Find the general solution of the corresponding homogeneous equation.

2. Make sure that the function $g(t)$ belongs to the class of functions discussed in this section, that is, it involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use the method of variation of parameters (discussed in the next section).

3. If $g_1(t)+\cdots+g_n(t)$, that is, if $g(t)$ is a sum of $n$ terms, then form $n$ sub problems, each of which contains only one of the terms $g_1(t)+\cdots+g_n(t)$. The $i^{th}$ sub problem consists of the equation

$$ay'' + by'' + cy = g_i(t), \tag{40}$$

where i runs from 1 to n.

4. For the $i^{th}$ sub problem assume a particular solution $Y_i(t)$ consisting of the appropriate exponential function, sine, cosine, polynomial, or combination thereof. If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the homogeneous equation (found in step 1), then multiply $Y_i(t)$ by t, or (if necessary) by $t^2$, so as to remove the duplication. See Table 3.6.1.

5. Find a particular solution $Y_i(t)$ for each of the sub problems. Then the sum $Y_1(t)+\cdots+Y_n(t)$ is a particular solution of the non homogeneous equation.

6. Form the sum of the general solution of the homogeneous equation (step 1) and the particular solution of the non homogeneous equation (step 5). This is the general solution of the non homogeneous equation.

7. Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

For some problems this entire procedure is easy to carry out by hand, but in many cases it requires considerable algebra. Once you understand clearly how the method works, a computer algebra system can be of great assistance in executing the details. The method of undetermined coefficients is self correcting in the sense that if one assumes too little for $Y(t)$, then a contradiction is soon reached that usually points the way to the modification that is needed in the assumed form. On the other hand, if one assumes too many terms, then some necessary work is done and some coefficients turn out to be zero, but at least the correct answer is obtained.

Notes. Here $s$ is the smallest non negative integer ($s = 0, 1, 2$) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, $s$ is the number of times 0 is a root of the characteristic equation, $\alpha$ is a root of the characteristic equation respectively, and $\alpha + i\beta$ is a root of the characteristic equation respectively.

If $g(t)$ is a sum of terms, it is usually easier in practice to compute separately the particular solution corresponding to each term in $g(t)$. From the principle of superposition (since the differential equation is linear), the particular solution of the complete problem is the sum of the particular solutions of the individual problems. This is illustrated in the following example.

We shall be concerned with the equation

$$a_o\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0 \tag{41}$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real constants.

We shall seek solutions of (41) of the form $y = e^{mx}$ where m will be chosen such that $e^{mx}$ does satisfy the equation

$$\begin{aligned} y &= e^{mx} \\ y^{'} &= me^{mx} \\ y^{''} &= m^2e^{mx} \\ &\vdots \\ y^n &= m^ne^{mx} \end{aligned} \tag{42}$$

substituting (42) in (41), we obtain

$$\begin{aligned} a_0m^ne^{mx} + a_1m^{n-1}e^{mx} + \cdots + a_{n-1}me^{mx} + a_ne^{mx} &= 0 \\ \Rightarrow e^{mx}(a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n) &= 0 \\ \Rightarrow a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n &= 0 \end{aligned} \tag{43}$$

since $e^{mx} \neq 0$

This equation (43) is called auxiliary equation or characteristic equation of the given differential equation (41). We write the auxiliary equation (43) and solve for $m$. Observe that (43) is formally obtained from (41) by merely replacing the $k^{th}$ derivative in (41) by $m^k, (k = 0, 1, 2, ..., n)$. Three cases arise, according as the roots of (43) are real and distinct,real and repeated, or complex.

Consider the nth -order homogeneous linear differential equation (41) with constant coefficients.If the auxiliary equation (2) has:

1. the n distinct real roots $m_1, m_2, \cdots, m_n$,then the general solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}$$

2. the real root m occurring k times,then the general solution of (1) corresponding to the K-fold repeated roots is $e^{mx}(c_1+c_2x+c_3x^2+\cdots+c_kx^{k-1})$ and remaining roots are the distinct real numbers $m_{k+1}, \cdots, m_n$then the general solution of (41)

3. The conjugate complex roots $(a + bi)$ and $(a - bi)$, neither repeated, then the corresponding part of the general solution of may be written

$$e^{ax}(c_1 \sin bx + c_2 \cos bx)$$

and if $(a+bi)$ and $(a-bi)$ are k-fold repeated roots, then the corresponding part of the general solution is $y = e^{ax}(c_1 + c_2x + c_3x^3 + \cdots + c_kx^{k-1})\sin bx + c_{k+1} + c_{k+2}x + \cdots + c_{2k}x^{k-1}\cos bx$

**Activity 1**

Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

solution

let $y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2e^{mx}$

The auxiliary equation is

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m_1 = 1, m_2 = 2$

The roots are real and distinct

Thus $e^x$ and $e^{2x}$ are solutions and the general solution may be written as

$y = c_1e^x + c_2e^{2x}$

We verify that $e^x$ and $e^{2x}$ are indeed linearly independent. Their wronskian is $w(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$

Therefore, we are assured of their linear independence

**Activity 2**

Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Solution

let $y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

The auxiliary equation is

$m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

The roots of this equation are

$m_1 = 3, m_2 = 3$ (not distinct)

$y = (c_1 + c_2 x)e^{3x}$

**Activity 3**

Solve $\frac{d^2y}{dx^2} + y = 0$

The auxiliary equation is $m^2 + 1 = 0$

$m_1 = i, m_2 = -i$

The general solution is $c_1 e^{ix} + c_2 e^{-ix}$

Using Euler's formula

$e^{i\theta} = cos\theta + isin\theta$ and $e^{-i\theta} = cos\theta - isin\theta$

$y = c_1(cosx + isinx) + c_2(cosx - isinx) = (c_1 + c_2)cosx + i(c_1 - c_2)sinx$

$y = Acosx + Bsinx$

where $A$ and $B$ are arbitrary constants.

**Activity 4**

Find the general solution of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$

The characteristics (auxiliary) equation is

$m^2 - 6m + 25 = 0$

$m = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6+8i}{2} = 3 + 4i$

$m_1 = 3 + 4i, m_2 = 3 - 4i$

The general solution is

$y = c_1 e^{(3+4i)x} + c_2 e^{(3-4i)x}$

$y = c_1 e^{3x} e^{4ix} + c_2 e^{3x} e^{-4ix}$

$y = e^{3x}(c_1 e^{4ix} + c_2 e^{-4ix})$

$y = e^{3x}[c_1(cos4x + isin4x) + c_2(cos4x - isin4x)]$

$y = e^{3x}[(c_1 + c_2)cos4x + (c_1 - c_2)isin4x]$

$y = e^{3x}(k_1 cos4x + k_2 sin4x)$

**Activity 5**

Solve the initial value problem $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$

$y(0) = -3, y'(0) = -1$

From activity (4), the solution is

$$y = e^{3x}(k_1 cos4x + k_2 sin4x) \tag{44}$$

$\frac{dy}{dx} = e^{3x}(k_1(4)sin4x + k_2(4)cos4x) + (k_1 cos4x + k_2 sin4x)3e^{3x}$

$$\frac{dy}{dx} = e^{3x}[(3k_1 - 4k_2)sin4x + (4k_1 + 3k_2)cos4x] \tag{45}$$

Applying the condition $y(0) = -3$ to equation (44)

$-3 = e^0(k_1 sin0 + k_2 cos0)$

$$k_2 = -3 \tag{46}$$

Applying the condition $y'(0) = -3$ t o equation (45)

$-1 = e^0[(3k_1 - 4k_2)sin0 + (4k_1 + 3k_2)cos0]$

$$4k_1 + 3k_2 = -1 \tag{47}$$

Substituting equation (46) into (47), we have

$k_1 = 2$

Therefore, $k_1 = 2, k_2 = -3$

The general equation (44) becomes

$y = e^{3x}[2cos4x - 3sin4x]$

**The Method of Undetermined Coefficients**

We now consider the (non homogeneous) differential equation.

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} = F(x) \tag{48}$$

where the coefficients $a_0, a_1, \cdots, a_n$ are constants but where the non homogeneous term $F$ is (in general) a non constant function of x. The general solution may be written as

$y = y_c + y_p$

where $y_c$ is the complementary function, that is the general solution of the corresponding homogeneous equation (48) with $F$ replaced by 0 and $y_p$ is a particular integral, that is, any solution of (48) containing no arbitrary constants.

The method of Undetermined Coefficients applies when the on homogeneous function $F$ in the differential equation is a finite linear combination of Undetermined Coefficients (U.C) functions.

A function is called a U.C function if it is either (44) a function defined by one of the following

(i) $x^n$ where $n$ is a positive integer or zero

(ii) $e^{ax}$ where $a$ is a constant $\neq 0$

(iii) sin(b x + c) where $b$ and $c$ are constants $b \neq 0$

(iv) cos(b x + c) where $b$ and $c$ are constants $b \neq 0$

or (45) is a function defined as a finite product of two or more functions of these four types.

**Illustrative Examples**

(1) Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10sinx \tag{49}$$

solution

The homogeneous equation is

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0 \tag{50}$$

The characteristic equation is $m_2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m_1 = 3, m_2 = -1$

The complementary function is

$$y_c = c_1e^{3x} + c_2e^-x \tag{51}$$

We now form the U.C set for each of the two functions $2e^x$ and $-10sinx$

$S_1 = \{e^x\}$

$S_2 = \{sinx, cosx\}$

Note that neither of these sets is identical with non included in the other and also by examining the complementary function, none of the functions $e^x$, $sinx$, $cosx$ of $S_1$ and $S_2$ with the undetermined coefficients $A$, $B$, $C$.

We determine these unknown coefficients by substituting the linear combination formed into equation (49)

$$y_p = Ae^x + Bsinx + Ccosx \tag{52}$$

$$y_p' = Ae^x + Bcosx - Csinx \tag{53}$$

$$y_p'' = Ae^x - Bsinx - Ccosx \tag{54}$$

Substituting (52), (53) and (54) int o equation (49)

$y = [Ae^x - Bsinx - Ccosx] - 2[Ae^x + Bcosx - Csinx] - 3[Ae^x + Bsinx + Ccosx] = 2e^x and - 10sinx$

$= -4Ae^x + [-4B + 2C]sinx + [-4C - 2B]cosx = 2e^x - 10sinx$

Equating coefficients of like terms

$-4A = -2$

$-4B + 2C = -10$

$-4C - 2B = 0$

$A = -\frac{1}{2}, B = 2, C = -1$

The particular integral

$y_p = -\frac{1}{2}e^x + 2sinx - cosx$

The general solution is

$y = y_c + y_p = C_1e^{3x} + C_2e^x - \frac{1}{2}e^x + 2sinx - cosx$

(2) Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

Solution

Consider $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

The complementary function is

$y_c = C_1e^x + C_2e^{2x}$

The non-homogeneous term $2x^2 + e^x + 2xe^x + 4e^{3x}$ form the U.C set

$S_1 = \{x^2, x, 1\}$

$S_2 = \{e^x\}$

$S_3 = \{xe^x, e^x\}$

$S_4 = \{e^{3x}\}$

These sets reduced to $S_1$, $S_3$, $S_4$ since $S_2$ is contained in $S_3$.

Note that $S_3 = \{xe^x, e^x\}$ includes $e^x$ which is included in the complementary function. Each member of $S_3$ was multiplied by $x$ to obtain $S_3'$

$S_3' = \{x^2e^x, xe^x\}$

The linear combinations are

$y_p = Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x$

$y_p' = 2Ax + B + 3De^{3x} + Ex^2e^x + 2Exe^x + Fxe^x + Fe^x$

$y_p'' = 2A + 9De^{3x} + Ex^2e^x + 4Exe^x + 2Ee^x + Fxe^x + 2Fe^x$

Substituting $y_p$, $y_p'$, $y_p''$ into the original differential equation, we have

$(2A - 3B + 2C) + (2B - 6A)x + 2Ax^2 + 2De^{3x} + (-2E)xe^x + (2E - F)e^x = 2x^2 + e^x + 2xe^x + 4e^{3x}$

Equating the coefficients

$2A - 3B + 2C = 0$

$2B - 6A = 0$

$2A = 2$

$2D = 4$

$-2E = 2$

$2E - F = 1$

$A = 1, B = 3, C = \frac{7}{2}, D = 2, E = -1, F = 3$

Th general solution is

$y = y_c + y_p = C_1e^x + C_2e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x$

(3) $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 3x^2 + 4sinx - 2cosx$

Solution

$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 0$

The complementary function is

$y_c = C_1 + C_2x + C_3sinx + C_4cosx$

The U.C sets for each of the three functions in the non homogeneous term are

$S_1 = \{x^2, x, 1\}$

$S_2 = \{sinx, cosx\}$

Since $S_1 = \{x^2, x, 1\}$ includes 1 and $x$ which are found in the complementary function. We multiply each member of this set $S_1$ by $x^2$ since multiplying it by $x$ will still include a member or term in the complementary function. Also, each member of set $x^2$ is multiplied by $x$. then, we obtain

$S_1 = \{x^4, x^3, x^2\}$

$S_2 = \{xsinx, xcosx\}$

$y_p = Ax^4 + Bx^3 + Cx^2 + Dxsinx + Excosx$

Substituting $y_p$, $y_p'$, $y_p''$ into the original differential equation, we have

$24A + Dxsinx - 4Dcosx + Excosx + 4Esinx + 12Ax^2 + 6Bx + 2C - D.xsinx + 2Dcosx - Excosx - 2Esinx = 3x^2 + 4sinx - 2cosx$

Equating the coefficients, we have

$24A + 2C = 0$

$6B = 0$

$12A = 3$

$-2D = -2$

$2E = 4$

$A = \frac{1}{4}, B = 0, C = -3, D = 1, E = 2$

The particular integral is

$$y_p = \frac{1}{4}x^4 - 3x^2 + xsinx + 2xcosx$$

The general solution is

$$y = y_c + y_p = C_1 + C_2x + C_3sinx + C_4cosx + \frac{1}{4}x^4 - 3x^2 + xsinx + 2xcosx$$

(4) Solve the initial value problem $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10sinx$ $y(0) = 2$ $y'(0) = 4$

Solution

The general solution (following the normal procedures) is

$$y = C_1e^{3x} + C_2e^{-x} - \frac{1}{2}e^x + 2sinx - cosx$$

$$y' = 3C_1e^{3x} - C_2e^{-x} - \frac{1}{2}e^x + 2cosx + sinx$$

Applying the initial conditions $y(0) = 2$ and $y'(0) = 4$

$$2 = C_1e^0 + C_2e^0 - \frac{1}{2}e^0 + 2sin0 - cos0$$

$$4 = 3C_1e^0 - C_2e^0 + 2cos0 + sin0$$

$$C_1 + C_2 = \frac{7}{2}$$

$$3C_1 - C_2 = \frac{5}{2}$$

$$C_1 = \frac{3}{2}, C_2 = 2$$

The final solution is

$$y = \frac{3}{2}e^{3x} + 2e^{-x} - \frac{1}{2}e^x + 2sinx - cosx$$

**Variation of Parameters**

While the process of carrying out the method of undetermined coefficients is actually straight forward, the method applied is general to a rather class of problems. For example, it would not apply to the apparently simple equation

$\frac{d^2y}{dx^2} + y = tanx$

We thus seek a method of finding a particular integral which applies in all cases (including variable coefficients) in which the complementary function is known. Such method is the method of variation of parameters. We now develop this method in connection with the general second order linear differential equation with variable coefficients.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x) \tag{55}$$

Suppose that $y_1$ and $y_2$ are linearly independent solution of the corresponding homogeneous equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{56}$$

The complementary function of equation (55) is

$c_1y_1 + c_2y_2$

where $c_1$ and $c_2$ are arbitrary constants. The procedure in the method of variation of parameters is to replace the arbitrary constants $c_1$ and $c_2$ by respective functions $v_1$ and $v_2$ which will be determined. Now

$v_1y_1 + v_2y_2$

will be a particular integral of equation (55). We thus assume

$$y_p = v_1y_1 + v_2y_2 \tag{57}$$

$$y_p' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2 \tag{58}$$

At this point we impose equation (57), we then simplify $y_p'$ by demanding that

$$v_1'y_1 + v_2'y_2 = 0$$

With this condition imposed.

# Summary

The general $n^{th}$ order linear differential equation was presented with some methods of solution which include Wronskian and variation of parameter etc.

## Self Assessment Question

Solve:

1. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

2. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$

3. $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 5y = 0$

4. $3\frac{d^2y}{dx^2} - 14\frac{dy}{dx} - 5y = 0$

5. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

6. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$

7. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$

## Tutor Marked Assignment

1. Given that $e^{-x}, e^{3x}$ and $e^{4x}$ are all solutions of

$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 12y = 0$

Show that they are linearly independent on the interval $-\infty < x < \infty$. Write the general solution.

2. Show that $y = e^{2x}$ is a solution of $(2x+1)\frac{d^2y}{dx^2} - 4(x+1)\frac{dy}{dx} + 4y = 0$ is linearly independent solution by reducing the order. Write the general solution.

3. Given that $y = x^2$ is a solution of $(x^3 - x^2)\frac{d^2y}{dx^2} - (x^3 + 2x^2 - 2x)\frac{dy}{dx} + (2x^2 + 2x - 2)y = 0$, find a linearly independent solution by reducing the order.

(4) Write the general solution of the following:

(a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2$

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 4e^{2x} - 21e^{-3x}$

(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 6sin2x + 7cos7x$

(d) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 10sin4x$

(e) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = cos4x$

(5) Solve the initial value problems:

(a) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 9x^2 + 4$

$y(0) = 6, y'(0) = 8$

(b) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 5sin2x$

$y(0) = 1, y'(0) = -2$

(c) $\frac{d^2y}{dx^2} + y = 3x^2 - 4sinx$

$y(0) = 1, y'(0) = 1$

(6) Solve

(a) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$

(b) $\frac{d^2y}{dx^2} + 9y = 0$

(c) $4\frac{d^2y}{dx^2} + y = 0$

(d) $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = 0$

(7) Solve the initial value problems

(a) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$

$y(0) = 3, y'(0) = 5$

(b) $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0$

$y(0) = 3, y'(0) = -1$

(c) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 29y = 0$

$y(0) = 0, y'(0) = 5$

(d) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 37y = 0$

$y(0) = 2, y'(0) = -4$

## References

## Further Readings

# MODULE 2

## LAPLACE TRANSFORM

# Unit 1: Laplace Transforms

# Introduction

The standard methods of solving second-order differential equations with constant coefficients i.e $ay'' + by' + cy = f(x)$ are either by substitution of an assumed solution or by using operator D methods. In each case ,the general solution is first obtained and the arbitrary constants evaluated by using the initial conditions.

A much neater and less tedious method is by the use of Laplace transform, in which the solution of the differential equation is obtained largely by algebraic processes.

## Learning outcomes/objectives

At the end of this unit, you should be able to:

(i) define a Laplace transform; and

(ii) solve problem on Laplace transform

## Main content

The Laplace transform of a function $F(t)$ is denoted by $\ell\{F(t)\}$ and is defined by

$$F(s) = \ell\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,$$

$t > 0$ where constant parameter $s$ is assumed to be positive and large enough to ensure that the product $F(t)e^{-st}$ converges to zero as $t \to \infty$. The result will be a function of $s$ .Since the limits are substituted for t. $\therefore$

$$F(s) = \ell\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

## Activity 1

Find the Laplace transform of $f(t) = a$ (constant)

Solution:

$$\ell(a) = \int_0^\infty ae^{-st}dt$$

$$= a\left[\frac{e^{st}}{-s}\right]_0^\infty$$

$$= \frac{-a}{s}(0-1)$$

$$= \frac{a}{s}$$

## Activity 2

Find the Laplace transform of $f(t)$=1

Solution:

$$\ell(1) = \int_0^\infty e^{-st}dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_0^\infty$$

$$= \frac{-1}{s}(0-1)$$

$$= \frac{1}{s}$$

## Activity 3

Find the Laplace transform of $f(t) = e^{at}$ (a constant)

Solution:

$$\ell(e^{at}) = \int_0^{\infty} e^{at} e^{-st} dt$$

$$= \int_0^{\infty} e^{(a-s)t} dt$$

$$= \left| \frac{e^{(a-s)t}}{(a-s)} \right|_0^{\infty}$$

$$= \frac{1}{a-s} \left[ e^{-}(s-a)t \right]_0^{\infty}$$

$$= \frac{1}{a-s}(0-1)$$

$$= \frac{-1}{a-s}$$

$$= \frac{1}{s-a}$$

## Activity 4

If $F(t) = Sinbt$ for $t > 0$

$$\ell(Sinbt) = \int_0^{\infty} e^{-st} Sinbtdt$$

$$= \left| \frac{-e^{-st}}{s^2+b^2}(sSinbt + bCosbt) \right|_0^{\infty}$$

$$= \frac{b}{s^2 + b^2}$$

for all $s > 0$

## Activity 5

If $F(t) = Cosbt$

$$\ell(Cosbt) = \int_0^\infty Cosbte^{-st}dt$$

$$= \left| \frac{-e^{-st}}{s^2 + b^2}(-sCosbt + bSinbt) \right|_0^\infty$$

$$= \frac{s}{s^2 + b^2}$$

for all $s > 0$

## Activity 6

If $f(t) = t^n$ where n is a positive integer.

$$\ell(t^n) = \int_0^\infty t^n e^{-st}dt$$

$$\left| t^n \frac{e^{-st}}{-s} \right|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$(0 - 0) + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

Similarly

$$\int_0^\infty t^{n-1}e^{-st}dt = \frac{n-1}{s}\int_0^\infty t^{n-2}e^{-st}dt$$

Therefore

$$\ell(t^n) = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}\cdots.\frac{1}{s}\ell(1)$$

$$= \frac{n(n-1)(n-2)\cdots 2.1}{s^n}.\frac{1}{s}$$

$$= \frac{n!}{s^{n+1}}$$

## Activity 7

If $f(t) = Sinhat$

$$\ell(Sinhat) = \ell\left\{\frac{1}{2}(e^{at} - e^{-at})\right\}$$
$$= \frac{1}{2}\int_0^\infty (e^{at} - e^{-at})e^{-st}dt$$
$$= \frac{1}{2}\int_0^\infty (e^{-(s-a)t} - e^{-(s+a)t}dt$$
$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$
$$= \frac{a}{s^2 - a^2}$$

Similarly

If $f(t) = Coshat$

$$\ell(Coshat) = \ell\left\{\frac{1}{2}(e^{at} + e^{-at})\right\}$$

$$= \frac{1}{2}\int_0^\infty (e^{at} + e^{-at})e^{-st}dt$$

$$= \frac{1}{2}\int_0^\infty (e^{(a-s)t} + e^{-(a+s)t}dt$$

$$= \frac{1}{2}\int_0^\infty (e^{-(s-a)t} + e^{-(s+a)t}dt$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{s}{s^2 - a^2}$$

## Activity 8

Evaluate $\ell\left\{2Sin3t\right\} + \ell\left\{4Sinh3t\right\}$

$$= \ell\left\{2Sin3t\right\} + \ell\left\{4Sinh3t\right\}$$

$$= \frac{2.3}{s^2+3^2} + \frac{4.3}{s^2-3^2}$$

$$= \frac{6}{s^2+3^2} + \frac{12}{s^2-3^2}$$

$$= \frac{6(s^2-9)+12(s^2+9)}{(S^4-81)}$$

$$= \frac{18s^2+54}{S^4-81}$$

$$= \frac{18(s^2+3)}{(S^4-81)}$$

**Standard Transform**

| $f(t) = \ell^{-1}\{F(s)\}$ | $F(s) = \ell\{f(t)\}$ | |
|---|---|---|
| $1$ | $\frac{1}{s}$ | $s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}$ | $s > a$ |
| $Sinbt$ | $\frac{b}{s^2-b^2}$ | $s > b$ |
| $Cosbt$ | $\frac{s}{s^2-b^2}$ | $s > b$ |

**Activity 9**

$$\begin{aligned}
\ell(tSin3t) &= \frac{-d}{ds}(\ell\{Sin3t\}) \\
&= \frac{-d}{ds}\left\{\frac{3}{s^2+3^2}\right\} \\
&= 3(s^2+3^2)^{-2}.2s \\
&= \frac{6s}{(s^2+9)^2}
\end{aligned}$$

# Summary

Definition of Laplace transform with derivation of some basic results and tables of standard Laplace transform were presented in this unit.

# Student Marked Questions

1. Define a Laplace Transform

2. Find the Laplace transform of $F(t) = e^{4t}$

3. Determine the following:

   (i)$F(t) = t^3$

   (ii)$F(t) = Cos4t$

4. Find the Laplace transform of $F(t) = 2sSin3t + 4Sinh3t$

### Tutor Marked Assignment

1. Find the Laplace transforms of the following functions: (i) $tCosh4t$ (ii) $t^2Cost$ (iii) $e^{3t}Cos5t$

2. Find the Laplace Transform of $F(t) = (t-3)^3$

## References

## Further Readings

# Unit 2: Important theorem/ properties of laplace transform

Introduction: Our attention in this unit shall be directed towards some important theorems on Laplace transform that makes the problem a lot easier. Having understood the basics of laplace transform we shall be dealing with some properties which include shifting property etc. that makes a seemingly complicated function to be easily solved.

## Learning Outcomes/Objectives

At the end of this unit, you should be able to:

(i) state shifting properties of Laplace transform; and

(ii) apply the properties to solve Laplace transform problems

## Theorem

- If $\ell\left\{F(t)\right\} = f(s)$,then $\ell\left\{e^{-at}F(t)\right\} = f(s+a)$

- If $\ell\left\{F(t)\right\} = f(s)$,then $\ell\left\{tF(t)\right\} = \frac{-d}{ds}\left\{f(s)\right\}$

- If $\ell\left\{F(t)\right\} = f(s)$,then $\ell\left\{\frac{F(t)}{t}\right\} = \int_0^\infty f(s)ds$
  provided $\lim_{t\to 0}\left\{\frac{F(t)}{t}\right\}$ exists

In general, $\ell\left\{t^nF(t)\right\} = (-1)^n\frac{d^n}{ds^n}\left\{f(s)\right\}$

## Main Content:

## Property 1:

Laplace transform is linear, given $c, k \in \Re$ (scalars) and $f(t)$ and $g(t)$ are two functions of t, then

$$\ell\{cf(t) + kg(t)\} = c\ell(f(t)) + k\ell(g(t))$$

The proof of this is trivial

## Property 2:

First shifting property. Multiplying f(t) by $e^{at}$ replaces $s$ by $s - a$

That is, $\ell\{f(t)\} = F(s)$, then

$$\ell\{e^{at}.f(t)\} = F(s - a)$$

**Activity 1:** Find the laplace transform of $f(t) = e^{-3t}t^3$

solution

Note that

$\ell\{t^n\} = \frac{n!}{s^{n+1}}$

$\ell\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$

$\ell\{e^{-3t}t^3\} = \frac{3!}{(s-(-3))^4} = \frac{6}{(s+3)^4}$

**Property 3:** The s-differentiation rule:

If $\ell\{f(t)\} = F(s)$, then

$\ell\{tf(t)\} = -\frac{d}{ds}F(s)$

Indeed, $\ell\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$

**Activity 2:** Find the Laplace transform of $\ell\{f(t)\} = tsin3t$

Solution

$$\ell\{sin3t\} = \frac{3}{s^2+9}$$

$\ell\{tsin3t\} = -\frac{d}{ds}[\frac{3}{s^2+9}] = -\frac{-6s}{(s^2+9)^2}$

$= \frac{6s}{(s^2+9)^2}$

**Activity 3:**

Find the Laplace transform of $f(t) = t^2e^{4t}$

solution

We shall attempt this example using the properties 2 and 3

**Property 2:**

$f(t) = t^2e^{4t}$

$\ell\{t^2\} = \frac{2!}{s^3}$

$\ell\{t^2e^4t\} = \frac{2!}{(s-4)^3}$

**Property 3:**

$\ell\{e^{4t}\} = \frac{1}{s-4}$

$\ell\{t^2e^4t\} = (-1)^2\frac{d^2}{ds^2}[\frac{1}{s-4}]$

$= \frac{d^2}{ds^2}[\frac{1}{s-4}] = \frac{d}{ds}[\frac{-1}{(s-4)^2}]$

$\frac{2}{(s-4)^3}$

**Property 4:**

Second Shifting Rule

(a) $\ell\{f(t-a)H(t-a)\} = e^{-as}\ell\{f(t)\}$

(b) $\ell\{g(t)H(t-a)\} = e^{-as}\ell\{f(t+a)\}$

**Activity 4:** Find the Laplace transform of $f(t) = sintH(t-\pi)$

Solution

$\ell\{sint\} = \frac{1}{s^2+1}$

$\ell\{sintH(t-\pi)\} = e^{-\pi s}\ell\{sin(t+\pi)\}$

$= e^{-\pi s}\ell\{-sint\}$

$e^{-\pi s}[\frac{-1}{s^2+1}]$

**Property 5:** Laplace transform of a derivative

Let $f(t) = y'(t)$, then

$\ell\{f(t)\} = \ell\{y'(t)\} = \int_0^\infty y'(t)e^{-st}dt$

$= S\ell\{y(t)\} - y(0)$

Indeed, if $f(t) = y^n(t)$, then

$\ell\{y^n(t)\} = S^n\ell\{y(t)\} - S^{n-1}y(0) - S^{n-2}y'(0) - \cdots - y^{n-1}(0)$

**Activity 5:** Find the laplace transform of $f(t) = \frac{d^2y}{dt^2}$

Solution

$\ell\{y'(t)\} = \int_0^\infty y''(t)e^{-st}dt$

Using integration by parts

Choosing $u = e^{-st}$ ; $\frac{du}{dt} = -se^{-st}$

$dv = y''(t)$ ; $v = y'(t)$

$= |y'(t)e^{-st}|_0^\infty - \int y'(t)e^{-st}dt$

$= 0 - y'(0) + S\int y'(t)e^{-st}dt$

$= -y'(0) + S\,|y(t)e^{-st}|_0^\infty - \int y(t)e^{-st}dt$

$= -y'(0) + S[0 - y(0) + S\int y(t)e^{-st}dt]$

$= -y'(0) - Sy(0) + S^2\int y(t)e^{-st}dt$

$= -y'(0) - Sy(0) + S^2\ell\{y(t)\}dt$

$= S^2\ell\{y(t)\}dt - Sy(0) - y'(0)$

**Property 6:**

Time scaling property

Let $\ell\{f(t)\} = F(s)$, then

$\ell\{f(at)\} = \frac{1}{a}F[\frac{s}{a}]$

**Activity 6:**

Find $\ell\{cos3t\}$

Solution

Since $\ell\{cost\} = \frac{s}{s^2+1}$

Then, $\ell\{cos3t\} = \frac{1}{3}[\frac{\frac{s}{3}}{\frac{s}{3}^2+1}]$

$= \frac{1}{3}[\frac{\frac{s}{3}}{\frac{s^2+9}{9}}] = \frac{s}{s^2+3^2}$

**Property 7:**

Let $f(t) = \int_0^t g(\tau)d\tau$

Then,

$\ell\{f(t)\} = \frac{F(s)}{s} = \frac{1}{s}\ell\{g(t)\}$

**Activity 7:**

Show that

$\ell\{\int_0^t g(\tau)d\tau\} = \frac{F(s)}{s}$

Solution (or Proof)

$\ell\{\int_0^t g(\tau)d\tau\} = \int_0^\infty [\int_0^t g(\tau)d\tau]e^{-st}dt$

$= \int_0^\infty [\int_0^t g(\tau)d\tau]\frac{d}{dt}[\frac{e^{-st}}{-s}]$

Using integration by parts

$\int udv = uv - \int vdu$

$u = \int_0^t g(\tau)d\tau$ and $dv = \frac{d}{dt}[\frac{e^{-st}}{-s}]$

We have

$$\ell\{\int_0^t g(\tau)d\tau\} = [\left|\frac{e^{-st}\int_0^t g(\tau)d\tau}{-s}]\right|_0^\infty + \frac{1}{s}[\int_0^\infty e^{-st}d[\int_0^t g(\tau)d\tau]]$$

By fundamental theorem of calculus,

$\frac{d}{dx}[\int_0^x f(u)du] = f(x)$

Also, $d[\int_0^x f(u)du] = f(x)dx$

Thus,

$\ell\{\int_0^t g(\tau)d\tau\} = \frac{1}{s}\int_0^\infty e^{-st}g(t)dt$

$= \frac{1}{s}F(s)$

Other properties include:

(i) t-division frequency integration $\ell[\{\frac{f(t)}{t}\}] = \int_s^\infty F(s)ds$

(ii) Convolution rule: $\ell[\{(fog)(t)\}] = \int_0^t f(x)g(t-x)dx$

## Summary:

Properties of Laplace transform were considered in this unit with worked examples

## Self Assessment Questions

Find the Laplace transform of the following using the appropriate property

(i) $t^3e^{6t}$

(ii) $cos4t$

(iii) $tsin2t$

## Tutor Marked Assignment

Find the Laplace transform of the following

(i) $cosh4t$

(ii) $e^{2t}sin3t$

(iii) $t^2e^{-3t}$

(v) $e^{-2t}[cos2t + \frac{5}{2}sin2t]$

## References:

C.K. Alexander, M.N.O Sadiku(2012); Fundamentals of electric circuits, Available online: angms.science—doc—math—transforms—math_trans_3Lproperty.pdf

Alexei Vyssotski (2016); Basic of instrumentation, measurement and analysis. Available online at https:——www.vyssotskioch—Basics of instrumentation—Laplace transform.pdf

## Further Reading:

# Unit 3: Inverse Transforms

**Introduction:** The reverse process of obtaining the initial function from its transform shall be discussed in this unit. It is expected that you are familiar with transform functions as well as techniques of partial fraction in some cases.

## Learning Outcomes/Objectives:

At the end of this unit, you should be able to:
(i) find the laplace inverse of $F(s)$

## Main content:

Now consider the reverse process. That is given a Laplace transform $F(s)$, to find a function $f(t)$ whose Laplace transform is the given $f(s)$.We introduce the notation $\ell^{-1}\{F(s)\}$ to denote such a function $f(t)$ . i.e

$$f(t) = \ell^{-1}\{F(s)\}$$

Suppose that $\ell\{f(t)\} = F(s)$
Then, $f(t) = \ell^{-1}\{F(s)\}$

$$= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - i\tau}^{\gamma + i\tau} e^{st} F(s) ds$$

The algorithm for finding the Laplace inverse is not by using the above formula but by using the Laplace transform table.

## Activity 1

$$\ell^{-1}\left\{\frac{a}{s^2+a^2}\right\} = Sinat$$

since

$$\ell(Sinat) = \frac{a}{s^2 + a^2}$$

## Activity 2

Determine $\ell^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$

**Solution:** We shall adopt partial fraction method to simplify the expression

$$\frac{5s+1}{s^2-s-12} = \frac{5s+1}{(s-4)(s+3)}$$

$$\frac{5s+1}{(s-4)(s+3)} = \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

$$5s+1 \equiv A(s+3) + B(s-4)$$

Put $s = 4$ we have

$5(4) + 1 = 7A$

A=3

Put $s = -3$,we have

$5(-3) + 1 = B(-3-4)$

$-14 = -7B$

$B = 2$

Hence ,

$$\ell^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$$

$$= \ell^{-1}\left\{\frac{3}{(s-4)} + \frac{2}{(s+3)}\right\}$$

$$= 3e^{4t} + 2e^{-3t}$$

## Activity 3

Determine $\ell^{-1}\left\{\frac{1}{s^2+6s+13}\right\}$

Now,

$$\frac{1}{s^2+6s+13} = \frac{1}{(s+3)^2+4} = \frac{1}{(s+3)^2+2^2}$$
$$= \ell^{-1}\left\{\frac{1}{s^2+6s+13}\right\}$$
$$= \frac{1}{2}\ell^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\}$$
$$= \frac{1}{2}e^{-3t}Sin2t$$

## Summary

Techniques of finding the inverse Laplace transform were discussed in this unit.

## Self Assessment Questions

Find the function $f(t)$ whose laplace transform is given by:

(A)

$$F(s) = \frac{s}{s^2+9} + \frac{4}{s-3} + \frac{s+1}{s^3}$$

(B)

$$F(s) = \sum_{n=0}^{6} \frac{s}{s^2+n^2} + \frac{n}{s^2+n^2}$$

## Tutor Marked Assignment

Find the function $f(t)$ whose laplace transform is given by:

1.
$$F(s) = \frac{2}{(s+3)^3}$$

2.
$$F(s) = \frac{e^{-\pi s}}{s^2+1}$$

3.
$$F(s) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

## References:

C.K. Alexander, M.N.O Sadiku(2012); Fundamentals of electric circuits,
Available online: angms.science—doc—math—transforms—math_trans_3Lproperty.pdf
Alexei Vyssotski (2016); Basic of instrumentation, measurement and analysis. Available online at https:——www.vyssotskioch—Basics of instrumentation—Laplace transform.pdf

## Further Reading:

# Unit 4: Solution of Differential Equations by Laplace Transforms Method

**Introduction**: Application of Laplace transform to solving ordinary differential equation shall be discuss in this unit. We shall be using the knowledge of Laplace transform as well as its inverse to solve differential equation of any order, initial value problem and boundary value problem.

## Learning Outcome/Object

At the end of this unit, you should be able to solve any order differential equation using Laplace transform.

## Main Content

Let $F'(t)$ be the first derivative of $F(t)$ with respect to $t$ and $F''(t)$ be the second derivative of $F(t)$ with respect to $t$

Then

$$\ell\{F'(t)\} = \int_0^\infty e^{-st}F'(t)dt$$

Integrating by parts, we have

$$= |e^{-st}F(t)|_0^\infty - \int_0^\infty F(t)(-se^{-st})dt$$

$$= 0 - F(0) + s\int_0^\infty F(t)(e^{-st})dt$$

$$= -F(0) + s\int_0^\infty F(t)(e^{-st})dt$$

$$= -F(0) + s\ell\{F(t)\}$$

Similarly

$$\ell\left\{F''(t)\right\} = -F'(0) + s\ell\left\{F'(t)\right\}$$

$$= -F'(0) + s(-F(0) + s\ell\left\{F(t)\right\}$$

$$= -F'(0) + -sF(0) + s^2\ell\left\{F(t)\right\}$$

$$= s^2 f(s) - sf(0) - F'(0)$$

Also

$$F'''(t) = s^3 f(s) - s^2 f(0) - sf'(0) - f"(0)$$

## Theorem

Let $F$ be a real function having a continuous $(n-1)st$ derivative $F^{n-1}$ and hence $F, F', \cdots, F^{n-1}$ are all of exponential order $e^{at}$. Suppose $F^n$ is piecewise continuous in every finite closed interval $0 \leq t \leq b$, then $\ell\left\{F^n\right\}$ existS for $s > \alpha$ and

$$\ell\left\{F^n(t)\right\} = s^n\ell\left\{F(t)\right\} - s^{n-1}F(0) - s^{n-2}F'(0) - s^{n-3}F''(0) - \cdots - F^{n-1}(0)$$

## Activity 1

Solve the initial value problem $\frac{dy}{dt} - 2y = e^{5t}$, $y(0) = 3$

Solution

Taking the Laplace transform of both sides

$$\ell\left\{\frac{dy}{dt}\right\} - 2\ell\left\{y\right\} = \ell\left\{e^{5t}\right\}$$

$$\Rightarrow sy(s) - y(0) - 2y(s) = \frac{1}{s-5}$$

where $y(s) = \ell(y)$

$$(s-2)y(s) - 3 = \frac{1}{s-5}$$

since y(0)=3.

$$(s-2)y(s) = \frac{1}{s-5} + 3 = \frac{1+3(s-5)}{(s-5)}$$
$$(s-2)y(s) = \frac{3s-14}{(s-5)}$$
$$y(s) = \frac{3s-14}{(s-5)(s-2}$$
$$\ell(y) = \frac{3s-14}{(s-5)(s-2}$$
$$y = \ell^{-1}\left\{\frac{3s-14}{(s-5)(s-2}\right\}$$

This implies

$$\frac{3s-14}{(s-5)(s-2} \equiv \frac{A}{s-2} + \frac{B}{s-5}$$
$$3s - 14 \equiv A(s-5) + B(s-2)$$

$A = \frac{8}{3}$ and $B = \frac{1}{3}$

Therefore

$$\ell^{-1}\left\{\frac{3s-14}{(s-5)(s-2}\right\} = \frac{8}{3}\ell^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{3}\ell^{-1}\left\{\frac{1}{s-5}\right\}$$
$$= \frac{8}{3}e^{2t} + \frac{1}{3}e^{5t}$$

## Activity 11

Solve the initial value problem $\frac{d^2y}{dt^2} - \frac{2dy}{dt} - 8y = 0$ at $y(0) = 3, y'(0) = 6$

Solution:

Taking the Laplace transform of both sides

$$\ell\{y''\} - 2\ell\{y'\} - 8\ell\{y\} = \ell\{0\}$$

$$\overline{s^2}y(\overline{s}) - \overline{s}y(0) - y'(0) - 2(\overline{s}y(\overline{s}) - y(0) - 8y(\overline{s}))$$

$$\overline{s^2}y(\overline{s}) - 3\overline{s} - 6 - 2\overline{s}y(\overline{s}) + 6 - 8y(\overline{s}) = 0$$

where $\ell\{y\} = y\overline{s}$

$$\begin{aligned}(\overline{s^2} - 2\overline{s} - 8)y(\overline{s}) - 3\overline{s} = 0 \\ y(\overline{s}) &= \frac{3\overline{s}}{(\overline{s^2} - 2\overline{s} - 8)y(\overline{s})} \\ &= \frac{3\overline{s}}{(\overline{s} - 4)(\overline{s} + 2)}\end{aligned}$$

Resolving it to partial fractions

$$\left\{\frac{3\overline{s}}{(\overline{s} - 4)(\overline{s} + 2)}\right\} = \frac{A}{\overline{s} - 4} + \frac{B}{\overline{s} + 2}$$

$$\Rightarrow A = 2, \quad B = 1$$

$$\left\{\frac{3\overline{s}}{(\overline{s} - 4)(\overline{s} + 2)}\right\} = 2\ell^{-1}\frac{1}{\overline{s} - 4} + \ell^{-1}\frac{1}{\overline{s} + 2}$$

$\therefore$

$$y = 2e^{4t} + e^{-2t}$$

### Activity 2

Solve $y'' = 3 + 2t \quad y(0) = y'(0) = 0$

<u>solution</u> Taking the Laplace transform of both sides

$$s^2\overline{Y}(s) - sy(0) - y'(0) = \frac{3}{s} + \frac{2}{s^2}$$

Using the initial conditions, we have

$$s^2\overline{Y}(s) = \frac{3}{s} + \frac{2}{s^2}$$

$$\overline{Y}(s) = \frac{3}{s^3} + \frac{2}{s^4}$$

$$y(t) = L^{-1}\overline{Y}(s) = L^{-1}(\frac{3}{S^3} + \frac{2}{s^4})$$

$$= 3t^2 + 2t^3$$

### Summary

Laplace transform solution of differential equation present an easy method of solving DE . We were ask to solve some DEs (most especially initial value problem) using Laplace transform and its inverse in this unit.

## Student Marked Questions

1. $\frac{dy}{dt} - y = e^{3t}$ at $y(0) = 2$

2. $\frac{dy}{dt} + y = 2Sint$ at $y(0) = -1$

3. $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 18e^{-t} + Sin3t$ at $y(0) = 0, y'(0) = 3$

## Tutor Marked Assignment

1. Find the Laplace transforms of the following functions: (i)$tCosh4t$ (ii)$t^2Cost$ (iii) $e^{3t}Cos5t$

2. Find the Laplace Transform of $F(t) = (t-3)^3$

3. Solve the following initial value problem

(a)

$$y'' - e^{-2t}(cos2t + \frac{5}{2}sin2t) = 0 \quad y(0) = y'(0) = 0$$

(b)

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 5y = sinx \;\; y(0) = y'(0) = y''(0) = 0$$

# References

# Further Readings

# Unit 5: Solving systems of ordinary DE with constant coefficient by Laplace transform

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**Introduction:** We shall be discussing solution to system of ordinary differential equations using Laplace transform method. Majorly, the steps involved are the same as when it is only one equation; the difference is the simple manipulation invovled.

## Learning Outcome/Objective

At the end of this unit, you should be able to solve system of differential equations using Laplace transform.

## Main Content

A system of equations consists several equations in some unknown variables. A typical example of system of differential equations with constant coefficients is of the form

$$\frac{dy_1}{dt} = c_{11}y_1 + c_{12}y_2 + \cdots + c_{1n}y_n + f_1(t)$$
$$\frac{dy_2}{dt} = c_{21}y_1 + c_{22}y_2 + \cdots + c_{2n}y_n + f_2(t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{dy_n}{dt} = c_{n1}y_1 + c_{n2}y_2 + \cdots + c_{nn}y_n + f_n(t) \tag{1}$$

The system of (1) is homogenous if all $f_i(t) = 0$ while it is non-homogenous when $f_i(t) \neq 0$.

The simplest method to solving system of (1) is by reducing it to an $nth$ order DE. We shall illustrate the approach using examples.

## Activity 1

Solve the equation

$$\frac{dy_1}{dt} = 4y_1 - y_2 + e^t \quad y_1(0) = y_1'(0) = 0 \qquad 1(a)$$

$$\frac{dy_2}{dt} = 2y_1 + y_2 + (t+1) \quad y_2(0) = y_2'(0) = 0 \qquad 1(b)$$

Soln

From (1a), we have

$$y_2 = -\frac{dy_1}{dt} + 4y_1 + e^t$$

$$\frac{dy_2}{dt} = -\frac{d^2y_1}{dt^2} + 4\frac{dy_1}{dt} + e^t \qquad (2)$$

Substituting (2) and (1b) into (1a)

$$-\frac{d^2y_1}{dt^2} + 4\frac{dy_1}{dt} + e^t = 2y_1 + (-\frac{dy_1}{dt} + 4y_1 + e^t) + (t+1) = \frac{d^2y_1}{dt^2} + 5\frac{dy_1}{dt} - 6y_1 = t+1 \qquad (3)$$

Applying Laplace transform to both sides of (3)

$$S^2\overline{Y_1}(s) - sy_1(0) - y_1'(0) + 5(S\overline{Y}(s) - y_1(0)) - 6\overline{Y_1}(s) = \frac{1}{s^2} + \frac{1}{s}$$

$$\overline{Y}_1(s)(s^2 + 5s - 6) = \frac{s+1}{s^2}$$

$$\overline{Y}_1(s) = \frac{s+1}{s^2(s+6)(s-1)}$$

$$\overline{Y}_1(s) = -\frac{31}{252}(\frac{1}{s}) - \frac{1}{6}(\frac{1}{s^2}) + \frac{5}{252}(\frac{1}{s+6}) + \frac{2}{7}(\frac{1}{s-1})$$

Taking Laplace inverse of both sides, we have

$$y_1(t) = -\frac{31}{252} - \frac{1}{6}t + \frac{5}{252}e^{-6t} + \frac{2}{7}e^t$$

But

$$y_2 = -\frac{dy_1}{dt} + 4y_1 + e^t$$

$$= -[-\frac{1}{6} - \frac{5}{42}e^{-6t} + \frac{2}{7}e^t] + 4[-\frac{31}{252} - \frac{1}{6}t + \frac{5}{252}e^{-6t} + \frac{2}{7}e^t] + e^t$$

$$= \frac{1}{6} + \frac{5}{42}e^{-6t} - \frac{2}{7}e^t - \frac{31}{63} - \frac{2}{3}t + \frac{5}{63}e^{-6t} + \frac{8}{7}e^t + e^t$$

## Activity 2

Solve the initial value problem

$$2y' - 6y + 3x = 0,\ 3x' - 3x - 2y = 0,\ y(0) = 3, x(0) = 1$$

Solution

Taking the Laplace transform of both sides

$$2(sy(s) - y(0) - 6y(s) + 3x(s) = 0$$

$$3(sx(s) - x(0) - 3x(s) - 2y(s) = 0$$

Using the initial conditions

$$2sy(s) - 6 - 6y(s) + 3x(s) = 0$$

$$3sx(s) - 3 - 3x(s) - 2y(s) = 0$$

$$3x(s) + (2s - 6)y(s) = 6 \tag{59}$$

$$(3s - 3)x(s) - 2y(s) = 3 \tag{60}$$

we now solve the equation simultaneously, multiply (60) by $(s - 3)$

$$3x(s) + (2s - 6)y(s) = 6 \tag{61}$$

$$(s-3)(3s-3)x(s)-(2s-6)y(s)=3(s-3) \tag{62}$$

Adding (61) and (62), we have

$$3x(s)+(3s-3)(s-3)x(s)=6+3(s-3)$$

$$(3+3s^2-9s-3s+9)x(s)=3s-3$$

$$(3s^2-12s+12)=3s-3 \tag{63}$$

$$x(s)=\frac{s-1}{s^2-4s+4}$$

$$\frac{s-1}{(s-2)^2}=\frac{A}{s-2}+\frac{B}{(s-2)^2}$$

$$=\frac{A(s-2)+B}{(s-2)^2}$$

$$s-1=A(s-2)+B$$

Put $s=2$

$B=1, s=0$

$-1=-2A+B$

$2A=2$

$A=1$

$$x(s)=)=\frac{s-1}{s^2-4s+4}=\frac{1}{s-2}+\frac{1}{(s-2)^2}$$

$$x(s)=e^{2t}+te^{2t}$$

Multiply (61) by $-(s-1)$ to have

$$-(3s-3)x(s)-(2s-6)(s-1)y(s)=-6(s-1) \tag{64}$$

Adding (60) and (64) gives

$$-((2s-6)(s-1)+2)y(s)=-6(s-1)+3$$

$$-(2s^2 - 2s - 6s + 6 + 2)y(s) = -6s + 6 + 3$$

$$y(s) = \frac{-3(2s-3)}{(2s^2 - 8s + 8)}$$

$$y(s) = \frac{-3(2s-3)}{-2(s-2)^2}$$

$$\frac{3}{2}\left\{\frac{A}{s-2} + \frac{B}{(s-2)^2}\right\}$$

$$2s - 3 = A(s-2) + B$$

$\therefore A = 2$ and $B = 1$

$$y(S) = \frac{3}{2}\left\{\frac{2}{s-2} + \frac{1}{(s-2)^2}\right\}$$

$$y = \frac{3}{2}\ell^{-1}\left\{\frac{2}{s-2} + \frac{1}{(s-2)^2}\right\}$$

$$y = \frac{3}{2}\{2e^{2t} + te^{2t}$$

$$y = 3e^{2t} + \frac{3t}{2}e^{2t}$$

## Summary

Laplace transform method was used to solve system of ordinary differential equations with constant coefficients.

## Self Assessment Questions

Solve the system of equations by Laplace transform method:

$\frac{dx_1}{dt} = 2x_1 + x_2 + (t-1)$

$\frac{dx_2}{dt} = 4x_1 - x_2 + (t-1)$

## Tutor Marked Assignment

Find the general solution by using Laplace transform

$\frac{dy}{dt} = 3y + x + t^3, y(0) = 0$

$\frac{dx}{dt} = x - y + t^3, x(0) = 1$

## References:

Carloi Enrique Frasser (2019); Laplace transform and systems of ordinary differential equation. Available online at Research gate.

## Further Readings:

# MODULE 3

## PARTIAL DIFFERENTIAL EQUATIONS

# Unit 1: Basic Treatment of PDE in Two Independent variables

**Introduction:** Partial differential equations are equations satisfied by derivatives of functions of two or more independent variables. They describe all types of physical phenomenon in engineering and science, ranging from transient heat conduction through vibrations of strings and plates. In this unit, we shall be introducing the basics of Partial differential equations.

## Learning outcomes/Objectives:

At the end of this unit, you should be able to:

(i) define a partial differential equation; and

(i) identify and classify partial differential equations.

## Main Content:

**Definitions**: A Partial Differential Equation (PDE) is an equation for same quantity $v$ (dependent variable) which depends on the independent variables $x_1, x_2, x_3, \cdots, x_n \geq 2$, and involves derivatives of $v$ with respect to at least some of the independent variables.

$F(x_1, \cdots, x_n, \partial x_1 v, \cdots, \partial x_n v, \partial^2 x_1 v, \partial^2 x_1 x_2 v, \cdots, \partial^n x_1, \cdots x_n v) = 0$

$$v = f(x_1, x_2, x_3, \cdots, x_n)$$

Thus,given a first order PDE is of the form:

$$c_1(x_1,x_2,x_3,\cdots,x_n)\frac{\partial v}{\partial x_1}+c_2(x_1,x_2,x_3,\cdots,x_n)\frac{\partial v}{\partial x_2}+\cdots+c_n(x_1,x_2,x_3,\cdots,x_n)\frac{\partial v}{\partial x_n}+kv(x_1,x_2,x_3,\cdots \tag{65}$$

which can compactly be written as:

$$c_1(x_1,x_2,\cdots,x_n,v,v_{x_1},v_{x_2},\cdots,v_{x_n})=\alpha(x_1,x_2,\cdots,x_n) \tag{66}$$

Equation (65) above is a first order non-homogeneous partial differential equation while it will be homogeneous if $\alpha(x_1,x_2,\cdots,x_n)=0$. The expression in (65) are too general to be directly useful, so only some important special cases will be examined. We consider the two special cases of (i) linear (ii) Semilinear and quasilinear type of partial differential equation.

**Definition**

Linear first order partial differential equation for $v(x,y)$ is denoted as:

$$c_1(x,y)U_x+c_2(x,y)U_y+c_3(x,y)U=c_4(x,y) \tag{67}$$

where all $c_i(x,y)$ are arbitrary functions of $x$ and $y$ (which can be ordinary constant) and the term $c_4(x,y)$ is the non-homogeneous term if $c_4(x,y)=0$, then equation (66) is homogeneous. When $c_1$, $c_2$ and $c_3(x,y)$ are all constants, the partial differential equation (66) becomes a constant coefficient partial differential equation. Examples of linear equations includes:

$\frac{\partial v}{\partial t}+c\frac{\partial v}{\partial x}=0$

$U_x+xU_y=U+2$

NOTE: (i) In applications, $x_i$ are often space variables (e.g $x$, $y$, $z$) and a solution may be required in some region $\Omega$ of space. In this case, there will be some conditions to be satisfied on the boundary $\partial\Omega$; there are called boundary conditions

(ii) Also. in application, one of the independent variables can be time ($t$ say) then these will be some initial conditions to be satisfied (i.e v is given at $t = 0$ everywhere in $\Omega$).

(iii) Again, in applications, system of PDEs can arise involving the independent variables $v_1, v_2, v_3, \cdots, v_m, m \geq 1$

The order of the PDE is the order of the highest (partial) differential coefficient in the equation

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0;\ First\ order\ linear\ PDE\ \ [simplest\ wave\ equation]$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = f(x, y);\ Second\ order\ linear\ PDE\,[Poison]$$

**Definition ii**: A non-linear equation is semilinear if the coefficients of the highest derivative are functions of the independent variables only.

$$(x + 3)\frac{\partial v}{\partial x} + xy^2\frac{\partial v}{\partial y} = v^3$$

**Definition iii**: A non-linear PDE of order m is Quasilinear if it is linear in the derivative of order m with coefficients depending only on $x, y, \cdots$ and derivatives of order less that m

$$\left[1 + \left(\frac{\partial v}{\partial y}\right)^2\right]\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x \partial y} + \left[1 + \left(\frac{\partial v}{\partial x}\right)^2\right]\frac{\partial^2 v}{\partial y^2} = 0$$

**Wave Equations**

Waves in a string, sound waves, waves on stretch membrane

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}\quad or\quad \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \nabla^2 v$$

**Heat Conduction Equations**

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$

or

$$\frac{\partial v}{\partial t} = (k\nabla v)\cdot\nabla$$

where k is a constant (diffusion coefficient or thermometric conductivity)

**Laplace's Equation**

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \;\; [\textit{Second order linear equation}]$$

or more generally

$$\nabla^2 v = 0$$

# Existence and Uniqueness

Before attempting to solve a problem involving PDE, it is needed to know if a solution exists, and if it exists is the solution unique. Also, in a problem wih respect to time, whether a solution exists for $t > 0$ (global existence) or only up to a given value of t. That is, only for $0 < t < t_0$ (finte time blow up, shock formation).

However, we say that the PDE with boundary or initial condition is well-formed (or well-posed) if its solution exits (globally), is unique and depends continuously on the assigned data. If any of the properties (existence, uniqueness and stability) is not satisfied, the problem (PDE, boundary conditions and initial conditions) is said to be ill-posed.

Example: A simple example of showing uniqueness is provided by:

$$\nabla^2 v = F \ in \ \Omega \ (Poisson's \ equation)$$

with $v = 0$ on $\partial\Omega$, the boundary of $\Omega$, and F in some given function of x.

Solution

Suppose $v_1$ and $v_2$ are two solutions satisfying the equation and the boundary conditions. Then consider $w = v_1 - v_2$; $\nabla^2 w = 0$ in $\Omega$ and $w = 0$ on $\partial\Omega$. Now the divergence theorem gives,

$$\begin{aligned} \int_{\partial\Omega} w\nabla w \cdot n ds &= \int_{\Omega} \nabla \cdot (w\nabla w) dv, \\ &= \int_{\Omega} \left( w\nabla^2 w + (\nabla w)^2 \right) dv \end{aligned}$$

where n is a unit normal outwards from $\Omega$

$$\int_{\Omega} (\nabla w)^2 dv = \int_{\partial} \Omega w \frac{\partial w}{\partial n} ds = 0$$

Now the integrand $(\nabla w)^2$ is non-negative in $\Omega$ and hence for the equality to hold we must have $\nabla w = 0$; that is, $w$=constant in $\Omega$. Since $w = 0$ on $\partial\Omega$ and the solution is smooth, we must have $w = 0$ in $\Omega$; i.e $w_1 = w_2$. The same proof works if $\frac{\partial v}{\partial n}$ is given on $\partial\Omega$ or for mixed conditions.

# Method of Forming Partial Differential Equation

A PDE is formed by two methods.

(i) By eliminating arbitrary constants

(ii) By eliminating arbitrary functions

## Method of Eliminating Arbitrary Constants

Example

Form a partial differential equation from $4x^2 + y^2 + (z-c)^2 = a^2$

Solution

$$x^2 + y^2 + (z-c)^2 = a^2 \tag{68}$$

The above equation contains two arbitrary constants $a$ and $c$.

Differentiating equation (68) with respect to $x$, we obtain $2x+2(z-c)\frac{\partial z}{\partial x} = 0$

$$2x + 2(z-c)p = 0 \tag{69}$$

where $p = \frac{\partial z}{\partial x}$.

Also, differentiating equation (68) with respect to $y$, we obtain $2y + (z-c)\frac{\partial z}{\partial y} = 0$

$$2y + (z-c)q = 0, \tag{70}$$

where $q = \frac{\partial z}{\partial y}$.

Let us eliminate c from equations (69) and (70), we have from equation (69), $(z-c) = -\frac{x}{p}$.

Putting this value of $z-c$ in equation (70), we have

$y - \frac{x}{p}q = 0$ or $yp - xq = 0$

Substituting for p and q, we have our PDE as

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

# Method of Elimination of Arbitrary Function

Example: Form the partial differential equations from $z = f(x^2 - y^2)$

solution

$$z = f(x^2 - y^2) \tag{71}$$

Differentiating equation (71) with respect to $x$ and $y$ we have

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2) \cdot 2x \tag{72}$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2) \cdot -2y \tag{73}$$

Dividing equation (72) by (73), we have

$\frac{p}{q} = -\frac{x}{y}$ or $py = -qx$

# Tutor Marked Assessment

Form the partial differential equations from

(i) $ax^2 + by^2 + z^2 = 1$

(ii) $(x - h)^2 + (y - k)^2 + z^2 = a^2$

(iii) $2z = (ax + y)^2 + b$

(iv) $f(x + y + z, x^2 + y^2 + z^2) = 0$

(v) $f(x^2 + y^2) = z$

**References:**

**Further Reading:**

# Lagrange's Linear Equation

Consider an equation in the form: $Pp + Qq = R$

where P, Q and R are the functions of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

Consider equation

$$P_p + Q_q = R \tag{74}$$

This form of equation is obtained by eliminating an arbitrary function f

$$f(u, v) = 0 \tag{75}$$

where u and v are functions of x, y and z.

Differentiating equation (75) partially with respect to $x$ and $y$, we have

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0 \tag{76}$$

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] = 0 \tag{77}$$

Let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from equations (76) and (77)

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot p\right] = -\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot p\right] \tag{78}$$

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot q\right] = -\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot q\right] \tag{79}$$

Dividing equation (78) by equation (79), we get

$$\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot p\right]\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot q\right] = \left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot q\right]\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot p\right] \tag{80}$$

$$\left[\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right]p+\left[\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right]q=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} \tag{81}$$

If equation (78) and (79) are the same then the coefficients of p and q are equal.

$$\begin{aligned} P &= \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} \\ Q &= \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} \\ R &= \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} \end{aligned} \tag{82}$$

Now suppose $u = c_1$ and $v = c_2$ are two solutions, where $a$ and $b$ are constants.

Differentiating $u = c_1$ and $v = c_2$, we obtain

$$\frac{\partial u}{\partial x}\cdot dx+\frac{\partial u}{\partial y}\cdot dy+\frac{\partial u}{\partial z}\cdot dz=0 \tag{83}$$

$$\frac{\partial v}{\partial x}\cdot dx+\frac{\partial v}{\partial y}\cdot dy+\frac{\partial v}{\partial z}\cdot dz=0 \tag{84}$$

Solving equations (83) and (84), we get

$$\frac{dx}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}} \tag{85}$$

From equations (85), we have

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Solutions of these equations are $u = c_1$ and $v = c_2$

Therefore, $f(u, v) = 0$ is the required solution.

## WORKING RULE

First step: Write down the auxiliary equations

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Second step: Solve the above auxiliary equations

Let the two solutions be $u = c_1$ and $v = c_2$

Third step: Then $f(u, v) = 0$ or $u = \phi(v)$ is the required solution of

$P_p + Q_q = R$

**Examples**

(1) Solve the following differential equation $yq - xp = z$ where $p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$

Solution

$yq - xp = z$

Here are the auxiliary equations are:

$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$

$-\ln x = \ln y - \ln a$

$$xy = a \tag{86}$$

$\ln y = \ln z + \ln b$

$$\frac{y}{z} = b \tag{87}$$

From equation (86) and (87), the solution is $f(xy, \frac{y}{z})$

(2) Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

The auxiliary equations are:

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy} = \frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

$$\frac{dx-dy}{(x-y)} = \frac{dy-dz}{(y-z)} = \frac{dz-dx}{(z-x)}$$

By integrating, we have

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log \frac{x-y}{y-z} = \log c_1 \text{ or } \frac{x-y}{y-z} = c_1$$

Similarly, from the above expression, we have

$$\frac{y-z}{z-x} = c_2$$

The required solution is

$$f\left[\frac{x-y}{y-z}, \frac{y-z}{z-x}\right] = 0$$

## METHOD OF MULTIPLIERS

Let the auxiliary equations be:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Let $l$, $m$, $n$ may be constants or functions of $x$, $y$ and $z$, then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx+mdy+ndz}{lP+mQ+nR}$$

where $l$, $m$ and $n$ are chosen in such a way that

$$lP + mQ + nR = 0$$

$$ldx + mdy + ndz = 0$$

Solving this differential equation, a choice is made by taking $u = c_1$ and $v = c_2$

Therefore, required solution is $f(u, v) = 0$

**Examples**

1. Solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$

   Here, the auxiliary equations are:

   $$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

   Using multipliers $x$, $y$ and $z$,we get

   Each fraction $= \frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{xdx+ydy+zdz}{0}$

   $$xdx + ydy + zdz = 0$$

   which on integration gives $x^2 + y^2 + z^2 = c_1$

   Again using multipliers; $l$, $m$ and $n$, we get

   Each fraction $= \frac{xdx+ydy+zdz}{l(mz-ny)+m(nx-lz)+n(ly-mx)} = \frac{ldx+mdy+ndz}{0}$

   $$ldx + mdy + ndz = 0$$

   which on integration gives

   $$lx + my + nz = c_2$$

   Hence, the required solution is:

   $$x^2 + y^2 + z^2 = f(lx + my + nz)$$

2. Find the general solution of $x(z^2 - y^2)\frac{\partial z}{\partial x} + y(x^2 - z^2)\frac{\partial z}{\partial x} = z(y^2 - x^2)$

   Solution

   The auxiliary simultaneous equations are:

   $$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

   Using multipliers $x$, $y$ and $z$, we get

Each term will give

$$\frac{xdx+ydy+zdz}{x^2(z^2-y^2)+y^2(x^2-z^2)+z^2(y^2-x^2)} = \frac{xdx+ydy+zdz}{0}$$

$$xdx + ydy + zdz = 0$$

On integration, $x^2 + y^2 + z^2 = c_1$

The above expression can be written as

$\frac{dx/x}{(z^2-y^2)} = \frac{dy/y}{(x^2-z^2)} = \frac{dz/z}{(y^2-x^2)} = \frac{dx/x+dy/y+dz/z}{(z^2-y^2)+(x^2-z^2)+(y^2-x^2)} = \frac{dx/x+dy/y+dz/z}{0} =$
$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2 = xyz = c_2$$

The general solution is $xyz = f(x^2 + y^2 + z^2)$

3. Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

Solution

$$(x^2 - y^2 - z^2)p + 2xyq = 2xz \tag{88}$$

Here the auxiliary equations are:

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Then, we have

$$\frac{dy}{y} = \frac{dz}{z}$$

which on integration gives

$$\log y = \log z + \log a \text{ or } \log \frac{y}{z} = \log a$$

$$\frac{y}{z} = a \tag{89}$$

Using multipliers $x$, $y$ and $z$; we have

$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$

$\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} = \frac{dz}{z}$

which on integration gives

$$\log(x^2+y^2+z^2) = \log z + \log b$$

The required solution is

$$x^2+y^2+z^2 = zf\frac{y}{z}$$

4. Solve $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$

Solution

$$px(z-2y^2) = (z-qy)(z-y^2-2x^3)$$

$$px(z-2y^2) + qy(z-y^2-2x^3) = z(z-y^2-2x^3)$$

Here, the auxiliary equations are: $\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$

Considering the first and last expressions, we have

$\frac{dx}{x(z-2y^2)} = \frac{dz}{z(z-y^2-2x^3)}$

Put $y = az$

$$\frac{dx}{x(z-2a^2z)} = \frac{dz}{z(z-a^2z^2-2x^3)}$$

$zdx - a^2z^2dx - 2x^3dx = xdz - 2a^2xzdz$

$xdz - zdx - a^2(2xzdz - z^2dx) + 2x^3dx = 0$

On integrating, we have

$$\frac{z}{x} - a^2\frac{z^2}{x} + x^2 = b$$

The required solution is

$\frac{y}{z} = f(\frac{z}{x} - \frac{a^2z^2}{x} + x^2)$

## Self Assessment Questions

Solve the following partial differential equations

(1) $x^2p + y^2q + z^2 = 0$

(2) $zx\frac{\partial z}{\partial x} - zy\frac{\partial z}{\partial y} = y^2 - x^2$

(3) $(y - z)p + (x - y)q = z - x$

(4) $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

## Tutor Marked Assignment

(1) $y\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z^2 + 1$

(2) $zp + yq = x$

(3) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$

(4) $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x + y)z$

# LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS OF A $N^{TH}$ ORDER WITH CONSTANTS COEFFICIENTS

An equation of the type

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdot + a_n\frac{\partial^n z}{\partial y^n} = F(x, y) \tag{90}$$

is called a Homogeneous linear P.D.E of $n^{th}$ order with constant coefficients.

Putting $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial x} = D'$

The equation (90) becomes

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_n D'^n)z = F(x, y) \tag{91}$$

## RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation:

$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$

or $(a_0 D^2 + a_1 DD' + a_2 D'^2)z = 0$

First step: Put $D = m$ and $D' = 1$

$a_0 m^2 + a_1 m + a_2 = 0$

This is the auxiliary equation

Second step: Solve the auxiliary equation

Case 1: If the roots of the auxiliary equation are real and different, say $m_1$ and $m_2$

The the complementary function $= f_1(y + m_1 x) + f_2(y + m_2 x)$

Case 2: If the roots are equal, say $m$

The the complementary function $= f_1(y + mx) + x f_2(y + mx)$

## RULES FOR FINDING THE PARTICULAR INTEGRAL

Given the PDE as:

$f(D, D')z = F(x, y)$

Particular integral $= \frac{1}{f(D,D')}F(x,y)$

(i) When $F(x,y) = e^{px+qy}$; Particular integral $= \frac{1}{f(D,D')}e^{px+qy} = \frac{e^{px+qy}}{f(p,q)}$ Put $D = p, D' = q$

(ii) When $F(x,y) = \sin(px+qy)$ *or* $\cos(px+qy)$
Particular Integral $= \frac{1}{f(D^2,DD',D'^2)}\sin(px+qy)$ or $\cos(px+qy)$
$= \frac{\sin(px+qy) \;\; or \;\; \cos(px+qy)}{f(-p^2,-pq,-q^2)}$
Put $D = -p_2, DD' = -pq$ and $D'^2 = -q^2$

(iii) When $F(x,y) = x^m y^n$
Particular integral $= \frac{1}{f(D,D')}x^m y^n = [f(D,D')]^{-1}x^m y^n$

# SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION

This section illustrates worked examples on second order linear PDE as given below **Examples**

Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = sinxcos2y$

<u>Solution</u>

$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = sinxcos2y$

This can be written in the form: $[D^2 - DD']z = sinxcos2y$

where $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$

Writing $D = m$ and $D' = 1$, the auxiliary equation is

$m^2 - m = 0 = m(m-1) = 0$

$m = 0, 1$

Complementary function $= f_1(y) + f_2(y+x)$

Particular integral $= \frac{1}{D^2-DD'} sinxcos2y = \frac{1}{D^2-DD'} \frac{1}{2}[sin(x+2y)+sin(x-2y)]$

$\frac{1}{2}\frac{1}{D^2-DD'} sin(x+2y) + \frac{1}{2}\frac{1}{D^2-DD'} sin(x-2y)$

Put $D^2 = -1, DD' = -2$ in the first integral and $D^2 = -1, DD' = -2$ in the second integral

Particular integral $= \frac{1}{2}\frac{sin(x+2y)}{-1-(-2)} + \frac{1}{2}\frac{sin(x-2y)}{-1-(2)}$

$= \frac{1}{2} sin(x+2y) - \frac{1}{6} sin(x-2y)$

Hence, the complete solution is $z = f_1(y) + f_2(y+x) + \frac{1}{2} sin(x+2y) - \frac{1}{6} sin(x-2y)$

## Self Assessment Questions

(1) Solve: $(D^2 + DD' - 6D'^2)z = cos(2x+y)$

(2) Solve: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = sinx$

## Tutor Marked Assignment

(1) Solve: $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + sin(x+2y)$

(2) Solve: $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

# APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS TO PHYSICAL, LIFE AND SOCIAL SCIENCES

To solve a physical problem mathematically, it is commonly necessary to resolve a differential equation. The differential equation expresses the relevant physical law and the particular integral applies to the specific situation.
The following are the applications of Ordinary Differential Equation in real life situation:

(a) Through electric circuits

(b) Heat Conduction

(c) Vertical motion

(d) Chemical action

**Examples** The rate at which ice melts is proportional to the amount of ice at the instant. Find the amount of ice left after two hours if half of the quantity melts in 30 minutes.
Solution
Let $m$ be the amounts of ice at any time $t$
$\frac{dm}{dt} = km$
Using variable separable, we have
$\frac{dm}{m} = kdt$

$\int \frac{dm}{m} = k \int dt + c$

$$\log m = kt + c, t = 0, m = M \tag{92}$$

$\log M = 0 + c = \log M = c$

On putting the value of $c$, (92) becomes

$$\log m = kt + \log M \tag{93}$$

$m = \frac{M}{2}$; when $t = \frac{1}{2} hour$

$\log \frac{M}{2} = \frac{K}{2} + \log M = \log \frac{1}{2} = \frac{k}{2}$

$k = 2 \log \frac{1}{2}$

On putting the value of k in (93), we have

$$\log m = (2 \log \frac{1}{2})t + \log M \tag{94}$$

On putting $t = 2 hours$ in (94), we have

$4 \log \frac{1}{2} + \log M$

$\log \frac{m}{M} = \log[\frac{1}{2}]^4 = \frac{m}{M} = \frac{1}{16} m = \frac{M}{16}$

Therefore after two hours, amount of ice left $= \frac{1}{16}$ of the amount of ice at the beginning.

## APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

(a) Conduction of heat in bars and solids

(b) Slow motion in hydrodynamics

(c) Diffusion of vorticity in viscous fluid flow

(d) Diffusion of neutrons in atomic piles

(e) Wave equations

(f) Vibrating strings.

**EXAMPLES** Consider an elastic string tightly stretched between two points $O$ and $A$. Let O be the origin and $OA$ as $x$ axis. Let y be the displacement at the point $P(x, y)$ at any time. The wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Let $y = XT$ where X is a function of $x$ only and $T$ is a function of $t$ only.

$$\frac{\partial y}{\partial t} = X \frac{\partial T}{\partial t}$$

and $\frac{\partial y}{\partial x} = T \frac{\partial X}{\partial x}$

Since $T$ and $X$ are functions of a single variable only.

$$\frac{\partial^2 y}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2}$$

and $\frac{\partial^2 y}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$

Substituting these values in the given equation, we get

$$X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

By separating the variables, we get

$$\frac{\partial^2 T}{\partial t^2} / c^2 T = \frac{\partial^2 X}{\partial x^2} / X = k$$

$$\frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = \frac{1}{X} \frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial^2 T}{\partial t^2} - kc^2 T = 0 \text{ and } \frac{\partial^2 X}{\partial x^2} - k = 0$$

Auxiliary equations are: $m^2 - kc^2 = 0 = m = \pm c\sqrt{k}$

$m^2 - k = 0 = m = \pm\sqrt{k}$

**case 1**

If $k > 0; T = c_1 e^{c\sqrt{k}t} + c_2 e^{-c\sqrt{k}t}$

$X = c_3 e^{c\sqrt{k}x} + c_4 e^{-c\sqrt{k}x}$

**case 2**

If $k < 0; T = c_5 \cos c\sqrt{k}t + c_6 \sin c\sqrt{k}t$

$X = c_7 \cos c\sqrt{k}x + c_8 \sin c\sqrt{k}x$

**case 3**

If $k = 0; T = c_9 t + c_{10}$

$X = c_{11}x + c_{12}$

These are 3 cases depending upon the particular problems. Here, we are dealing with wave motion $(k < 0)$

$y = TX$

$y = [c_5 \cos c\sqrt{kt} + c_6 \sin c\sqrt{kt}][c_7 \cos c\sqrt{kx} + c_8 \sin c\sqrt{kx}]$

## Self Assessment Questions

(a) A tightly stretched string with fixed end points $x = 0$ and $x = \pi$ is initially at rest in its equilibrum position. If it is set vibrating by giving each point at a velocity $[\frac{\partial u}{\partial t}]_{t=0} = 0.03 \sin x - 0.04 \sin 3x$, then find the displacement $y(x, t)$ at any point of the string at any time $t$

(b) A string of length $L$ is initially at rest in equilibrum position at each points given velocity, $[\frac{\partial y}{\partial t}]_{t=0} = b \sin^3 \frac{\pi x}{L}$. Find the displacement $y(x, t)$.

## Tutor Marked Assignment

(a) The acceleration and velocity of a body falling in the air approximately satisfy the equation. $Acceleration = g - kv^2$, where $v$ is the velocity of the body at any time $t$ and $g, t$ are constants. Find the distance travelled as a function of the time, if the body falls from rest. Show that the value of $v$ will never extended $\sqrt{\frac{g}{k}}$

## References:

H.K Dass (2008); Advanced Engineering Mathematics. S.Chand Publishing Limited: Univesity of Hull, England.

## Further Readings: