MAT 111: ELEMENTARY SET THEORY AND NUMBERS

University of Ilorin
Centre for Open & Distance Learning
CODL

Published by the Centre for Open and Distance Learning,
University of Ilorin, Nigeria

E-mail: codl@unilorin.edu.ng
Website: https://codl.unilorin.edu.ng

Course Development Team

Subject Matter Expert	Dr. Fadipe-Joseph Olubunmi A., Dr. Odetunde O., Ganiyu Afees B., Oluwayemi M. O., Adeosun Adeshina Taofeek, Aina Ibukun, Opaleye E. O., Akinyele Akinola Yussuff, and Olabode J. O.
Instructional Designers	Mr. O. S. Koledafe Yusuf Shamsudeen O
Content Editors	Mr. A. Mahmud Mrs. Ogechi Bankole

From the Vice Chancellor

Courseware development for instructional use by the Centre for Open and Distance Learning (CODL) has been achieved through the dedication of authors and the team involved in quality assurance based on the core values of the University of Ilorin. The availability, relevance and use of the courseware cannot be timelier than now that the whole world has to bring online education to the front burner. A necessary equipping for addressing some of the weaknesses of regular classroom teaching and learning has thus been achieved in this effort.

This basic course material is available in different electronic modes to ease access and use for the students. They are available on the University's website for download to students and others who have interest in learning from the contents. This is UNILORIN CODL's way of extending knowledge and promoting skills acquisition as open source to those who are interested. As expected, graduates of the University of Ilorin are equipped with requisite skills and competencies for excellence in life. That same expectation applies to all users of these learning materials.

Needless to say, that availability and delivery of the courseware to achieve expected CODL goals are of essence. Ultimate attention is paid to quality and excellence in these complementary processes of teaching and learning. Students are confident that they have the best available to them in every sense.

It is hoped that students will make the best use of these valuable course materials.

Professor S. A. Abdulkareem
Vice Chancellor

Foreword

Courseware remains the nerve centre of Open and Distance Learning. Whereas some institutions and tutors depend entirely on Open Educational Resources (OER), CODL at the University of Ilorin considers it necessary to develop its own materials. Rich as OERs are and widely as they are deployed for supporting online education, adding to them in content and quality by individuals and institutions guarantees progress. Doing it in-house as we have done at the University of Ilorin has brought the best out of the Course Development Team across Faculties in the University. Credit must be given to the team for prompt completion and delivery of assigned tasks in spite of their very busy schedules.

The development of the courseware is similar in many ways to the experience of a pregnant woman eagerly looking forward to the D-day when she will put to bed. It is customary that families waiting for the arrival of a new baby usually do so with high hopes. This is the apt description of the eagerness of the University of Ilorin in seeing that the centre for open and distance learning [CODL] takes off.

The Vice-Chancellor, Prof. Sulyman Age Abdulkareem, deserves every accolade for committing huge financial and material resources to the centre. This commitment, no doubt, boosted the efforts of the team. Careful attention to quality standards, ODL compliance and UNILORIN CODL House Style brought the best out from the course development team. Responses to quality assurance with respect to writing, subject matter content, language and instructional design by authors, reviewers, editors and designers, though painstaking, have yielded the course materials now made available primarily to CODL students as open resources.

Aiming at a parity of standards and esteem with regular university programmes is usually an expectation from students on open and distance education programmes. The reason being that stakeholders hold the view that graduates of face-to-face teaching and learning are superior to those exposed to online education. CODL has the dual-mode mandate. This implies a combination of face-to-face with open and distance education. It is in the light of this that our centre has developed its courseware to combine the strength of both modes to bring out the best from the students. CODL students, other categories of students of the University of Ilorin and similar institutions will find the courseware to be their most dependable companion for the acquisition of knowledge, skills and competences in their respective courses and programmes.

Activities, assessments, assignments, exercises, reports, discussions and projects amongst others at various points in the courseware are targeted at achieving the objectives of teaching and learning. The courseware is interactive and directly points the attention of students and users to key issues helpful to their particular learning. Students' understanding has been viewed as a necessary ingredient at every point. Each course has also been broken into modules and their component units in sequential order.

Courseware for the Bachelor of Science in Computer Science housed primarily in the Faculty of Communication and Information Science provide the foundational model for Open and Distance Learning in the Centre for Open and Distance Learning at the University of Ilorin.

At this juncture, I must commend past directors of this great centre for their painstaking efforts at ensuring that it sees the light of the day. Prof. M. O. Yusuf, Prof. A. A. Fajonyomi and Prof. H. O. Owolabi shall always be remembered for doing their best during their respective tenures. May God continually be pleased with them, Aameen.

Bashiru A. Omipidan
Director, CODL

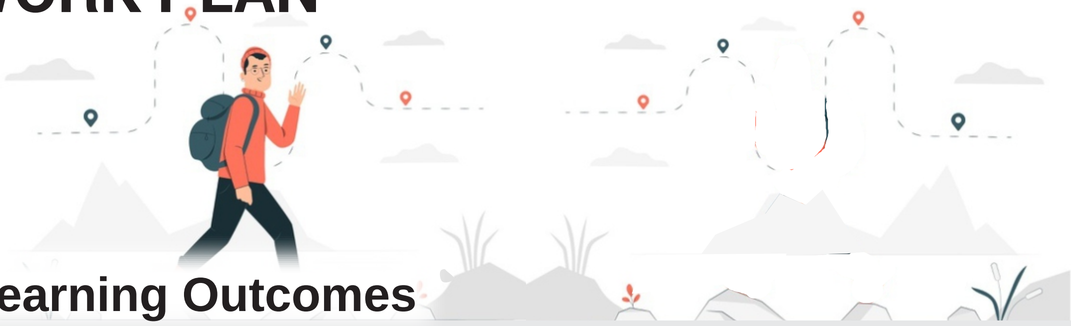
INTRODUCTION

I welcome you to Elementary Set Theory and Numbers, a first-semestercourse. It is a 3-credit course that is available to year one undergraduatestudents in Faculties of Life Sciences, Physical Sciences, Engineering, Edu-cation and allied degrees. This course was designed as a foundation coursefor undergraduate mathematics. It consists of elementary topics from O'levelmathematics and introduction to some of the rudiment topics in advancedmathematics. It was prepared with the aim of introducing undergraduatestudents to some basic theorems and principles that will be useful in ad-vance mathematics.

Course Goal

Your journey through this course will remind you of some basic topics likeset theory, theory of quadratic equation, trigonometry. You will also beintroduced to some elementary advance topics like mathematical induction,complex numbers and circular measures.

WORK PLAN



Learning Outcomes

At the end of this course, you should be able to:

- solve problems on set theory
- prove (or disprove) mathematical assertions using mathematical induction;
- define and perform basic operations on a sequence of numbers;
- solve various problems in complex numbers

Course Guide

Related Courses

Prerequisite for:

MAT 206 Elementary Set Theory And Numbers

MAT 213 Elementary Set Theory And Numbers

- apply sum and product of roots of quadratic equations to solve anygiven problems;
- measure angles in radian and degree;
- apply sum of angles formulae and half angle formulae to obtain a trigonometric ratio of an angle.

Course Requirements

Requirements for success

The CODL Programme is designed for learners who are absent from the lecturer in time and space. Therefore, you should refer to your Student Handbook, available on the website and in hard copy form, to get information on the procedure of distance/e-learning. You can contact the CODL helpdesk which is available 24/7 for every of your enquiry.

Visit CODL virtual classroom on http://codllms.unilorin.edu.ng. Then, log in with your credentials and click on MAT 113. Download and read through the unit of instruction for each week before the scheduled time of interaction with the course tutor/facilitator. You should also download and watch the relevant video and listen to the podcast so that you will understand and follow the course facilitator.

At the scheduled time, you are expected to log in to the classroom for interaction.

Self-assessment component of the courseware is available as exercises to help you learn and master the content you have gone through.

You are to answer the Tutor Marked Assignment (TMA) for each unit and submit for assessment.

Embedded Support Devices

Support menus for guide and references

Throughout your interaction with this course material, you will notice some set of icons used for easier navigation of this course materials. We advise that you familiarize yourself with each of these icons as they will help you in no small ways in achieving success and easy completion of this course. Find in the table below, the complete icon set and their meaning.

 **Introduction**	 **Learning Outcomes**	 **Main Content**

Summary	Tutor Marked Assignment	Self Assessment
Web Resources	Downloadable Resources	Discuss with Colleagues
References	Futher Reading	Self Exploration

SAQ n	SAQ (n) provides you with clues to answering equivalent Self Assessment Question(s). Where "n" serves as the corresponding SAQ number.
T mins	T mins suggests the estimated amount of time it will take you to complete a block of instruction. This is based on the estimation of 300 words per minute for adults (see: https://irisreading.com/what-is-the-average-reading-speed)

Grading and Assessment

20%	20%	60%	100%
TMA	CA	Exam	Total

Module 1

SET THEORY AND MATHEMATICAL INDUCTION

Units

Unit 1 - Elementary Set Theory

Unit 2 - Union, Intersection and Complement of a Set

Unit 3 - Venn Diagrams

Unit 4 - Real Numbers and Mathematical Induction

UNIT 1

Elementary Set Theory

Introduction

In everyday life, we often group certain things together. For instance,we speak of fleet of cars, a football team, the countries of Africa, setof dishes and so on. These are examples of what is called a ”set”.In each of the above examples, there is an obvious connection between theobjects of the set. For example, we may have a set consisting of the followingobjects:a table, a laptop and a biro.

Learning Outcomes

At the end of this unit, you should be able to:

1. define a set
2. define a finite set
3. identify a subset of a set
4. state a proper subset

Main Content

It is important that, given any object, we should be able to tell with-out ambiguity whether or not the object belongs to a given set. Such sets are said to be well-defined. This is the kind of sets that we shall discuss in this unit.

Set

A set is a collection of a distinct-distinguishable objects. Each objectof the set is called a member or an element of the set.

Naming and Listing of Sets

We often use capital letters like A, B, C,... to denote a set and smallletters like a, b, c,... to denote the elements of the set. A set may bedescribed by

(a) Actually listing all the elements of the set, this is called thetabular form;or

(b) stating a rule or properties which decide whether or not a given object is an element of the set. This is called the set–builder form.

In tabular form, the elements of the set are written down separated by commas and enclosed in brackets, {curly brackets like this}. When the set-builder form is used, we denote a typical element by a lower-case followed by colon (:) or a short vertical line (|) and then a description of the typical element. Let us consider some examples.

Activity 1

Write the following sets in the tabular or set-builder form as appropriate

(a) The set consisting of a lion, a tiger and a chair.
(b) All Nigerian mothers.
(c) The numbers 2, 8, 12, 16, and 20.
(d) The vowels of English Alphabet.

Solution

a. Suppose we name the set A, then we may writeA={lion,tiger,table}and we can read it as ”A is the set whose elements are lion, tiger and chair”.

b. Here, the set consists of all Nigerian women who are mothers. There must be millions of them. It will therefore be impractical to list the names of all such women. We therefore use the set-builder form. Suppose wename the set B, then we may write

$$\mathcal{B} = \{x|x \text{ is a Nigerian mother}\} \text{ or } \mathcal{B} = \{x{:}x \text{ is a Nigerian mother}\}.$$

Each of these reads ”B is the set (of elements) x such that x is a Nigerian mother”. The symbol | or : stands for such that.

C. Assume we name the set consisting of the given numbersC, then, sincethere are only a few members of the set, we may writeC={2,8,12,16,20}.(d) Let the set be named D, then we writeD={a,e,i,o,u}.

Finite Set

A set is said to be finite if we can list all its elements. In other words, a finite set consists of a specific number of elements, and if a set is not finite. it is said to be infinite. When a set is finite, it is sometimes convenient to introduce a notation for the number of elements contained in the set. The number is often called the cardinal number. For Instance, if A is a set, we denote the number of elements in A by $n(A)$.

Consider the *set A={a,b,c,d,2,4,5}*,

then *n(A) = 7,* since there are seven elements in this set.

in the school and so on. We would then have the sets :

P = {Mr Mike, Miss Tina, Mrs Gold,...} and

I = *{classroom blocks, Maths books, physics books,...,computers,...}*

These sets have been obtained by removing some of the elements of the set U. A set obtained in this way is called a subset of the original set (U).

Proper Subset

A set A is a subset of a set B if every element of A is also an element of B and we denote

Activity 2

Write all the subsets of{read,eat,sleep}.

If we remove all the elements of all given sets, or we remove some of the elements or remove no element at all, we obtain the following sets:

{},{read},{eat},{sleep},{read, eat},{read, sleep},{eat, sleep}, {read, eat, sleep}. These are the subsets of the given set.

Activity 3

Show that A={2,5,9} is a subset of the set N.

Solution

Each element of A is counting number, and since all natural numbers are counting numbers.

Hence A is a subset of N and we write $A \subseteq N$.

Definition If a set A is a subset of a set B, but set B contains at least one element which is not in set A, then A is called a proper subset of B.

And we write $A \subset B$.

The last expression stands for "A is a proper subset of B" or "A is contained in B", and the symbols $\subset$ and $\subseteq$ may be reversed to give $B \supseteq A$, which means "B contains A" or "A is a subset of B".

Summary

In this unit, we have learnt that a set is a collection of a distinct distinguishable objects. We have also learnt that each object of the set is called a member or an element of the set. A set is also said to be finite if we can list all its elements and if a set is not finite, it is said to be infinite. A set A is a subset of a set B if every element of A is also an element of B and we denote this by $A \subseteq B$, which can be read as "A is a subset of B"

Self Assessment Questions

I. Define a set.

ii. What is a finite set?

iii. {goat, meat, cow} is a set of what?

iv. Let $A=\{1,2,3,4,\cdots\}$ A is finite set. True or False?

v. $A=\{1,2,3,4,\cdots\}$ A is a proper subset of?

a. C　　b. N　　c. R

Tutor Marked Assignment

Give a concise description of each of the following sets.

1.{1, 2, 3,...}

2.{2, 4, 6, 8,...}

3.{1, 3, 5, 7,...}

Tutor Marked Assignment

4.{pot, spoon, knife, frying pan.}

5. List the elements of the following sets:

(a) A; the set of all the prime numbers less than 25.

(b) B; the set of all odd numbers less than 10.

6. The countries of West Africa.

7. The even numbers less than 18.

8. The odd numbers less 17.

9. The prime numbers less 20.

10. The set of writing materials.

References

- Makanjuola, S. O. (2008),Foundation of Abstract Algebra, Evidence Nige-ria Ventures, Ilorin, Kwara State.
- Oluyemi, S. (2004),Language Matematika Fuunamenta, Lautech Press,Ogbomoso.
- Royden H.L.(2005), Introductory to Real Analysis

Further Reading

Algebra and Trigonometry by Ushri Datta, A. S. Muktibodh and S.D. Mohgaonkar.

UNIT 2

Union, Intersection and Complement of a Set

Introduction

In the previous unit, we have learnt what a set is. In this unit, wewill learn how to create new sets from existing ones by combiningelements of both sets (union), collecting common elements (intersec-tion) and collecting elements in the universal set that are not in the chosenset (complement).

Learning Outcomes

At the end of this unit, you should be able to:

1. find union of sets
2. find the intersection of sets
3. express the complement of sets

Main Content

If A and B are two sets, the union of the sets A and B is a third set, whose elements belong to A or to B or to both A and B. (note that there is no repetition of elements). The union of A and B is denoted by A∪B, which is read as "A union B". Hence, if A={1,2,3} and B={a,b,c}, then A∪B={1,2,3,a,b,c}.

The union of sets A and B is the set of all those elements x such that x belongs to at least one of the two sets A and B, that is, the set which consists of elements that are either in A or B or both. It is denoted as A∪B,

$$A \cup B = \{x: x \in A, \text{ or } x \in B \text{ or } x \in \text{ both } A \text{ and } B\}.$$

let

G={h,e,a,p}

and

H={l,a,k,e}

then

$G \cup H = \{h,a,e,l,k,p\}$

Empty Sets

An empty set is a set that has no element. It is denoted by Ø

Disjoint Sets

Two sets A and B are said to be a disjoint set if they do not have anything in common, that is if $A \cap B = \emptyset$.

Intersection of Sets

The intersection of two sets A and B is the third set C, which contains the elements that are common to both A and B. That is, each element of C must be in both A and B, that is, the set of items which are common to the two sets. We denote the intersection of A and B by $A \cap B$,which is read as "A intersection B". Therefore, if $A = \{2,4,a\}$ and $B = \{2,a,c\}$, then $A \cap B = \{2,a\}$.

Equality of sets

Two sets A and B are said to be equal sets if they both have the same elements. The intersection of two sets A and B is the set of all those elements x such that x belongs to both A and B. It is denoted by $A \cap B$. If there is no common element to set A and B, then $A \cap B$ is said to be disjoint.

$$A \cap B = \{x : x \quad A \text{ and } x \quad B\}$$

Let

A x x and **B** x x

then,

A **B**

Complement

If for a given problem ∪ is the universal set, then the complement of a set A is the set of elements which belong to ∪ but does not belong to A. The complement of a set A is usually denoted by A′, thus $A' = \{x \mid x \in \cup, x \notin A\}$.

Summary

In this unit, you have learnt that if A and B are two sets, the union of the sets A and B is a third set, whose elements belong to A or to B or to both A and B and the intersection of two sets A and B is the third set C, which contains the elements that are common to both A and B.

Self Assessment Questions

Let U be a universal set and

A x x is an odd number

B x x and

C x x is a multiple of

Find I A B II A C III A B

Let A B and C

List the following elements

I A B II A B C III A B C

IV A B B C V What is another name for set A

Tutor Marked Assignment

If A B C and U find

I A B C II A B C

III A B C IV A B B C

Further Reading

- H. K. Dass. Advanced engineering Mathematics. TwentyFirst Edition. S.Chand & Company LTD., New Delhi.

References

- Makanjuola, S. O. (2008),Foundation of Abstract Algebra, Evidence Nige-ria Ventures, Ilorin, Kwara State.
- Oluyemi, S. (2004),Language Matematika Fuunamenta, Lautech Press,Ogbomoso.
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Further Reading

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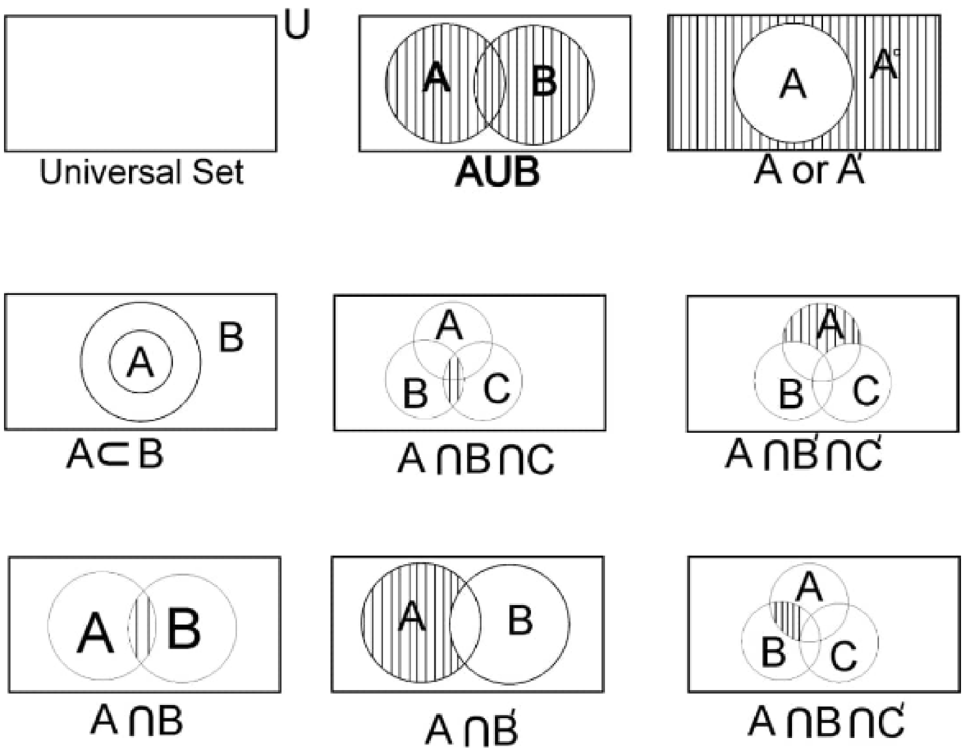
UNIT 3

Venn Diagrams

Introduction

A pictorial representation of sets is referred to as Venn Diagram



Learning Outcomes

At the end of this unit, you should be able to:

1. draw venn diagrams
2. explain laws of algebra on sets
3. find union of sets using Venn diagram
4. find the intersection of sets with Venn diagram
5. determine the complement of sets using Venn diagram

Main Content

It is sometimes found useful to illustrate the relationships betweensets by the use of a diagram. Such a diagram is called a Venn diagram.

Figure 1

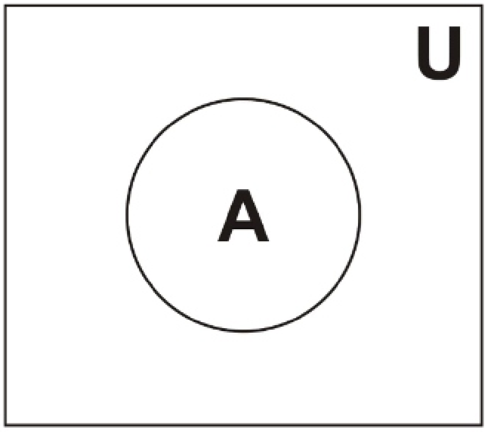
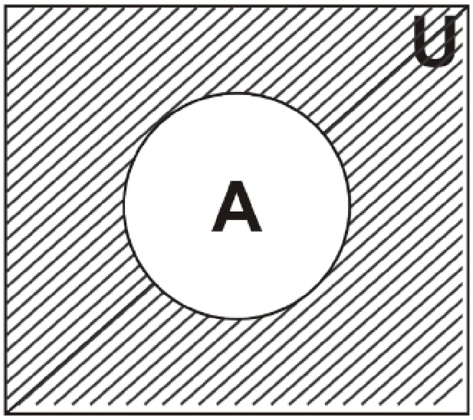


Figure 2



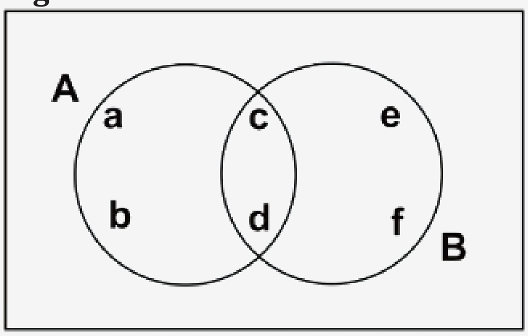
In this diagram, the shaded portion indicates all elements of U which are not in A. Thus the shaded portion represents the complement of A which is denoted by A′

Activity 1

Illustrate the set A = {a,b,c,d} and B = {c,d,e,f} with venn diagram.

Solution

Figure 3

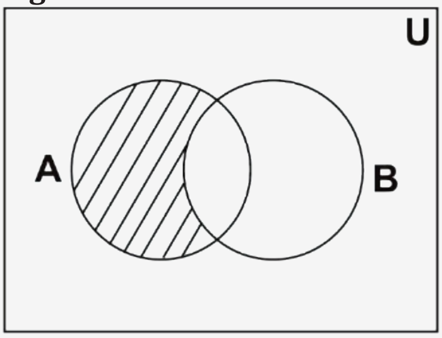


Activity 2

Figure 4 below shows two sets A and B together with the universal set U:

What statement in terms of set operations does the shadedportion represents?

Figure 4



Solution

The shaded portion shows element in A that are not in B.

Laws of algebra of sets

These refer to some identities in sets. If A, B and C are subsets of the universal set U, the following laws of operation are satisfied.

i Commutative laws

$A \cup B = B \cup A$

$A \cap B = B \cap A$

ii Associative laws

$A \cap (B \cap C) = (A \cap B) \cap C)$

$A \cup (B \cup C) = (A \cup B) \cup C)$

iii Distribution

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv Law of set Absorption

$A \cap (A \cup B) = A$

$A \cup (A \cap B) = A$

v Idempotent law

A A = A

A A =A

vi De Morgan's law

(A B)'=A' B'

(A B)'=A' B'

vii law of complement

$A \cap A' = \emptyset$

$A \cup A' = U$

$U' = \emptyset \quad \emptyset' = U$

Others $\emptyset \cap A = \emptyset$; $\emptyset \cup A = A$; $U \cap A = A$; $U \cup A = U$

viii Law of Inclusion

If $A \subset B$ and $B \subset C$ then $A \subset C$

If $A \subset B$ and $A \subset C$ then $A \subset B \cap C$

Activity 3

All pupils of JS II play games. 15 of them play Baseball, while 10play Hockey and there is no pupil who plays both games. Illustratethis information by a venn diagram and hence find the number ofpupils in JS II.

Solution

Let B represent the set of pupils who play baseball and H the set of pupilswho play hockey. It follows that n(B) = 15 and n(H) = 10

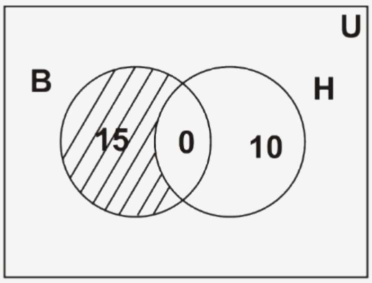
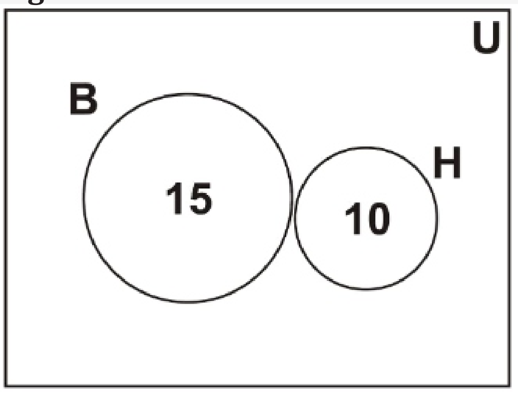


Figure 5

The number of pupils in the class is therefore given by 0 + 15 + 0 + 10 = 25.

Figure 6



It follows that there are twenty five pupils in the class

Application of set theory

Activity 4

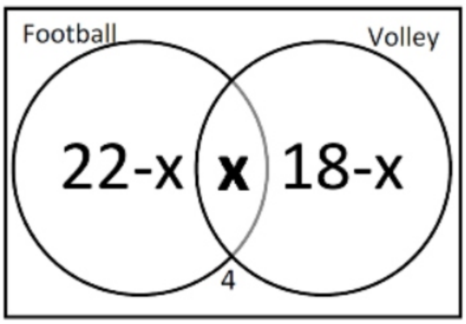
In a class of 35 students, 22 play football and 18 play volley ball.Find the number of students that play both games if 4 students donot play any of the two games.

Solution A

By Venn diagram.

Step 1 - here we draw a suitable Venn diagram and interpret the problem in the relevant cells.

Step 2 - add all the cells together and equate to the universal value.
Step 3 - solve the resulting equation.



∴

x x x x

x x

x x

x x

Solution B

By formula.

Note

N A B N A N B N A B N A B

substituting from the problem

x

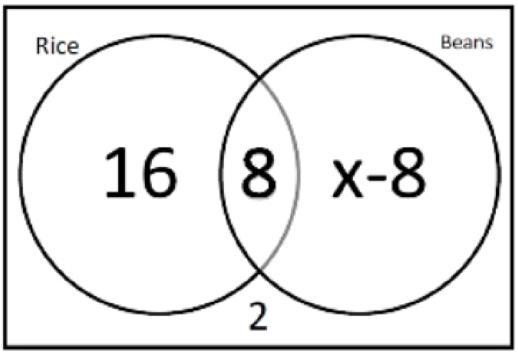
x x x

Activity 5

In an extended family of 40, it was discovered that between Rice an Beans, 24 of them like Rice, 8 like both while 2 do not like either. Find how manymembers of the family that like Beans only.

Solution A

By Venn diagram.



x

x x

x

Solution B

By formula.

$n(A \cup B) = n(A) + n(B) + n(A \cup B)' - n(A \cap B)$

$40 = 24 + x + 2 - 8$

$40 = 18 + x$

$x = 40 - 18$

$x = 22$

Hint we denote the unknown cell by x

Activity 6

All the 50 Science students of Ilorin College were asked their subject combinations. 18 of them offered further Mathematics, 21 offered Chemistry while16 offered Biology. 7 students offered Further Mathematics and Chemistry, 8 offered Further Mathematics and Biology, where 9 offered Chemistry and Biology.

(a) Find the number of students that offered the three subjects

(b) Find the number that offered

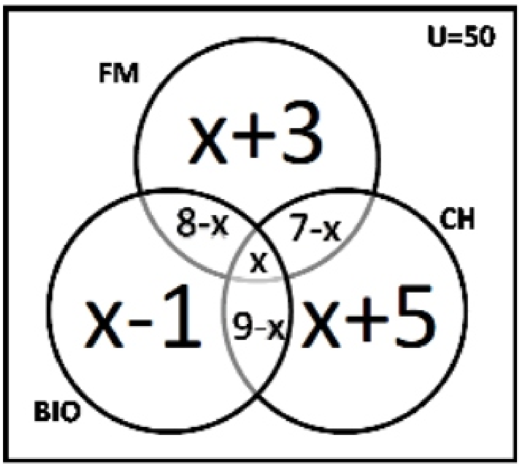
(i) Further Mathematics only,

(ii) Chemistry only,

(iii) Biology only.

Solution A

By Venn diagram.



(a) Further Mathematics only

x x x

$18 \quad 7 \quad 8 + x = 3 + x$

Chemistry only

$21 \quad (7 \quad x + x + 9 \quad x)$

$21 \quad 7 \quad 9 + x = 5 + x$

Biology only

$16 \quad (8 \quad x + x + 9 \quad x)$

$16 \quad 8 \quad 9 + x = \quad 1 + x$

$x + 3 + x + 5 + x \quad 1 + x + 8 \quad x + 7 \quad x + 9 \quad x = 50$

$4x \quad 3x + 31 = 50$

$x + 31 = 50$

$x = 50 \quad 31$

$\boxed{x = 19}$

Hence,

(I) that offered Further Mathematics

$3 + x + 8 \quad x + 7 \quad x + x$

$3 + 19 + 8 \quad 19 + 7 \quad 19 + 19$

$22 \quad 11 \quad 12 + 19$

$41 \quad 23 = 18$

agreed with question.

i. Further Mathematics only

$3 + x = 3 + 19 = 22$

ii. Chemistry only

$x + 5 \quad 19 + 5 = 24$

iii. Biology only

$19 \quad 1 = 18$

Solution B

By formula.

$n(\mathcal{F}\ \ \mathcal{C}\ \ \mathcal{B}) = n(\mathcal{F})+n(\mathcal{C})+n(\mathcal{B})\ \ n(\mathcal{F}\ \ \mathcal{C})\ \ n(\mathcal{F}\ \ \mathcal{B})\ \ n(\mathcal{C}\ \ \mathcal{B})+n(\mathcal{F}\ \ \mathcal{C}\ \ \mathcal{B})$

$50 = 18 + 21 + 16\ \ 7\ \ 9\ \ 8 + n(\mathcal{F}\ \ \mathcal{C}\ \ \mathcal{B})$

$n(\mathcal{F}\ \ \mathcal{C}\ \ \mathcal{B}) = 50\ \ 55 + 24$

$= 74\ \ 55 = 19$

Summary

In a Venn diagram, a set is represented by a simple example planeregion, usually a circle. The universal set is usually represented by aclosed region different in shape which is used to represent the othersets. We usually use the rectangle to represent the universal set by usingseveral circles to represent the set, and operation between sets is illustratedby shading.

Self Assessment Questions

1. After examining 300 items, a factory quality controller came up withthe following reports: Defects in finishing = 50, defects in hardness= 76, and defects in dimension = 67. Defects in both finishing anddimension = 14 and defects in both hardness and dimension = 23. All three defects = 5.

By using a venn diagram to illustrate this report, find

(i)the number of items with just two defects

(ii)the number of items with just one defect

(iii)the number of items with at least one defect and

(iv)the number of items without any defects.

2. If $\mathcal{U}=\{1,2,3,\ldots,40\}$, list the numbers of elements of the following subsets of U.

I. A ={x: 3x <27} ii. B ={x: 3<2x <12} iii. $\mathcal{C} = \{x : x \text{ is a prime number } 39\}$

3 A sport club has facilities for football (F), basket ball (B) and Hockey (H). An inquiry into the use of these facilities by 140 members re-vealed the following results;

$n(F)=86, n(B)=34, n(H)=41, n(F\quad B)=14, n(B\quad H)=13$ and $n(F\quad H)=18$.

If 18 members do not use any of these facilities at all

(a) How many members use all the three?

(b) Determine the following:

(i) $n(F\quad B\quad H\quad)$ (ii) $n(F\quad B\quad\quad H\quad)$ (iii) $n(F\quad H\quad B)+n(F\quad H\quad B\quad)$

Tutor Marked Assignment

1. Find the possible value of x and yin each of following:

$\{4,9\}=\{2^x,3^y\}$

2. Write the following sets using a statement to designate each.

$A=\{5,6,7,8,9\}$ $B=\{a,d,g,m,r,s\}$ $C=\{1,3,5,7,\ldots,2^n\quad 1\}$

3. All the 40 candidates in an examination centre were asked their subject combination. 17 chose Mathematics, 18 chose Physics, 16 chose Chemistry, 7 chose Mathematics and Physics, 10 chose Physics and Chemistry, 6 chose Mathematics and Chemistry, while 4 chose the three sub-jects.

Find

(I) the number of candidates that chose Mathematics only

(ii)the number of candidates that chose just two subjects

(iii)the number of candidates that chose just one subject

(iv)the number of candidates that chose at least one of the subjects and

(v)the number of candidates that did not chose any of the three subjects

4. In a class of 40 students, 30 take Account and 20 take Physics. If 8students take neither Account nor Physics, find the number of students who take Account but not Physics

5. A vendor sells three Nigeria dailies, the Times, the Punch, and the Vanguard. 70 customers buy the Times, 60 buy the Punch and 50buy the Vanguard. 17 buy both the Times and the Punch. 15 thePunch and Vanguard and 16 the Vanguard and the Times, while 3 customers buy all the three papers. How many customers buy only one paper.

6. In a survey of 70 people to find out the colour that they like, it wasdiscovered that 45 liked white, 65 liked Blue and 50 liked Green. 25liked white and Blue, 20 liked white and Green and 30 liked Blue andGreen, Find the number of people that liked the three colours.

7. In a certain government office, there are 400 employees, 150 are grad-uates, 160 are NCE holders, 180 are school certificate holders. If 50of them do not hold any certificate, find the number of employeesthat hold the three certificates.

References

- Makanjuola, S. O. (2008),Foundation of Abstract Algebra, Evidence Nige-ria Ventures, Ilorin, Kwara State.
- Oluyemi, S. (2004),Language Matematika Fuunamenta, Lautech Press,Ogbomoso.
- Royden H.L.(2005), Introductory to Real Analysis

Further Reading

Algebra and Trigonometry by Ushri Datta, A. S. Muktibodh and S.D. Mohgaonkar.

UNIT 4

Real Numbers and Mathematical Induction

Introduction

In Mathematics, a real number is a value of a continuous quantity thatcan represent a distance along a line.Real numbers can be thoughtof as points on an infinitely long line called Number line or Real lineand it can mainly be classified into rational and irrational numbers.

Learning Outcomes

At the end of this unit, you should be able to:

1. define real numbers
2. classify a real numbers
3. describe properties of a real number

Main Content

A real number is a value that represents any quantity along a number line.

Forms of a Real Numbers System

The types of numbers that form the set of real numbers are:

1. The natural or counting numbers: 1,2,3,4,5,6....

A natural number is a prime number if it is greater than one and its only factors are one and itself; 2,3,5,7,100....

A natural number is a composite number if it is greater than one and is not prime; 8,24,33....

2. Whole numbers consist of the natural numbers and zero; 0.1.2.3.4...

3. Integers consist of the natural numbers together with the negatives and zero:... 4. 3. 2. 1.0.1.2.3.4...

An integer is even if it can be written in the form [2n],where n is aninteger; 2.4.6. 8...

An integer is odd if it can be written in form 2n−1 ,where n is an integer(if 2 is not a factor); 1.3.5.7.9.11...

4. Rational numbers are numbers that can be written as an integer divided by an integer (or a ratio of integers) and it comprises of allintegers and fractions. Examples are: $\frac{1}{2}$, $\frac{3}{5}$,0.19,2.14.3.1,...

5. Irrational numbers are numbers that cannot be written as an integer divided by an integer. Examples are: 3. . ^{3}5...

6. A real number is positive if it is greater than 0, negative if it is less than 0.

7. Undefined numbers are numbers in the form no, where n is an integer

Number Classification

Activity 1

Classify each of the following numbers:

5....24....3.7... $\frac{7}{12}$ ...$\sqrt{17}$... $\frac{32}{0}$...... $4\frac{1}{5}$

Answer (Prime number, Even number, rational number, irrational number,Undefined number)

Properties of a real number

On the set {R} of real numbers, there are two binary operations, denoted by + and called addition and multiplication respectively. These operations satisfy the following properties:

1. $a+b=b+a$ for all a, b in $\mathcal{R}$ (commutative properties of addition)

2. $(a+b)+c=a+(b+c)$ for all a, b, c in $\mathcal{R}$ (associative property of addition);

3. there exists an element 0 in $\mathcal{R}$ such that $0+a=a$ and $a+0=a$ for all a in R (existence of a zero element);

4. for each a in $\mathcal{R}$ there exists an element $-a$ in $\mathcal{R}$ such that $a+(-a)=0$ and $(-a)+a=0$ (existence of negative element s);

5. $ab=ba$ for all a, b in $\mathcal{R}$ (commutative property of multiplication);

6. $(ab)c=a(bc)$ for all a, b, c in (associative property of multiplication);

7. there exists an element 1 in R distinct from 0 such that $1.a=a$ and $a.1=a$ for all a in R (existence of a unit element);

8. for each $a \neq 0$ in R there exists an element $\frac{1}{a}$ in R such that $a.\frac{1}{a}=1$ and $\frac{1}{a}a=1$ (existence of reciprocals);

9. $a(b+c)=(a.b)+(a.c)$ and $(b+c).a=(b.a)+(c.a)$ for all a, b, c in $\mathcal{R}$ (distributive property of multiplication over addition).

Summary

You should be familiar with these properties. The first four are con-cerned with addition, the next four with multiplication, and the last one connects the two operations. Note that the techniques of algebra can be derived from these nine properties.

Self Assessment Questions

1. Define a real number.
2. Classify each of the following numbers:

−12....	9....	0.2347...	$\frac{4}{19}$...
$\sqrt{7}$...	310...	4411...	−2000....

3. Given the $\{-10.3, -8, -5, 0, \frac{7}{5}, \sqrt{11}, 23, 25\frac{1}{4}, 0.12, 10, 33, 4, 9, 13, 50\}$

4. Give the list of all

Rational numbers	Undefined numbers
Even numbers	Whole numbers
Composite number	Negative real numbers.

Tutor Marked Assignment

1. List two classes of a Real number
2. Define and give five(5) examples for each real number:

(i) irrationalnumber (ii) whole number (iii)fraction

References

- Oluyemi, S. (2004),Language Matematika Fuunamenta, Lautech Press,Ogbomoso.
- Royden H.L.(2005), Introductory to Real Analysis

Further Reading

Algebra and Trigonometry by Ushri Datta, A. S. Muktibodh and S.D. Mohgaonkar.

UNIT 5

Mathematical Induction

Introduction

The principle of mathematical induction is used to establish some concepts or results which are given in terms of the natural numbers.

At the end of this unit, you should be able to:

Learning Outcomes

1. explain the concept of Mathematical Induction
2. proof by induction, some mathematical problems for natural numbers
3. state the principle of solving by mathematical induction
4. solve the related problems in mathematical induction.

Main Content

The method of mathematical induction, MI, enjoys a popular appli-cation as proof of theorems, equations, inequalities and for solving problems in geometry, theory of divisibility and others.This method, among others, enables us to write precise expression for thegeneral term of a sequence or series.

Principle of Solving by Mathematical Induction

In this context, the principle of mathematical induction can be for-mulated as follows. For each $n \quad \mathcal{N}$, let $\mathcal{T}(n)$ be a statement about n. Suppose that :

(I) $\mathcal{T}_n$ is true .

(ii) For every $\mathcal{K} \quad \mathcal{N}$. if $\mathcal{T}(k)$ is assumed to be True, then $\mathcal{T}(k+\ 1)$ is proved true. Then $\mathcal{T}(n)$ is true for all $n \quad \mathcal{N}$.

$\left(\frac{(k+1)(k+2)}{2}\right)^2$=RHS of $T(k+1)$

Hence, the statement is true for $n=1$,$n=k+1$ when $n=k$,is true for any natural n.

Activity 3

Show that $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+nthterm=\frac{n}{n+1}$ by Mathematical induction.

Proof:

Let $T(n)$ be the proposition ,then

$T(n)=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Step 1:let $n=1$ then

LHS=$\frac{1}{1(1+1)}=\frac{1}{2}$ and

RHS=$\frac{1}{1+1}=\frac{1}{2}$

$T(1)$ is true.

Step 2:let $n=k$,then $T(k)$ is assume to be true: $T(k)=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true.

step 3:let $n=k+1$

We need to show that $n=k+1$ is true.

if we add (k+1) to both sides of the assumed quality, we obtain

$T(k+1)$=$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$

putting the value $T(k)$

LHS $=\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)}$

$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)(k+1)}{(k+1)(k+2)}=\frac{(k+1)}{(k+2)}$ =RHS of $T(k+1)$

LHS $=\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)}$

$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)(k+1)}{(k+1)(k+2)}=\frac{(k+1)}{(k+2)}$ =RHS of $T(k+1)$

Conclusion, since $n=1$ is true, $n=k$ is assumed to be true and $n=K+1$ is true. Hence $T(n)$ is true for all natural number.

Activity 1

Show that $7^{2n+1}+1$ is an integer multiple of 8.

Solution

Let $T(n)$ be the proposition ,then

$T(n)=7^{2n+1}+1$

Step 1: let $n=1$ and $T(1)=7^{3+1}=344=43\times 8.$ $T(1)$ is True.

Step 2: let $n=k.$ $T(k)$ is assume to be true: $T(k)=7^{2k+1}+1$ is true.

step 3: let n = k + 1 We show that n=k+ 1 is true.

$T(k+1) = 7^{2(k+1)+1}+1 = 7^{2k+3}+1 = 7^2 7^{2k+1}+1 = 7^{2(72k+1)} - 48 = 7^{2T(k)} - 48$

Since the statement is true for $T(k)$ and 48 is a multiple of $8.$ then $T(k+1)$ is true.

Hence, the statement is true for $n=1.$ $n=k+1$ when n=k,is true for any natural $n.$

Activity 2

Prove by Mathematical induction

$$1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Proof:

Let $T(n)$ be the proposition

Step 1:let $n=1$ and $T(1)=1^3=\left(\frac{1(1+1)}{2}\right)^2$

LHS=1 and RHS=1,clearly LHS=RHS.$T(1)$ is true.

Step 2:let $n=k$,then $T(k)$ is assume to be true:$T(k)=1^3+2^3+\cdots+k^3=\left(\frac{k(k+1)}{2}\right)^2$ is true.

step 3:let $n=k+1$

We need to show that $n=k+1$ is true.

if we add (k+1) to both sides of the assumed quality,we obtain

$T(k+1)=1^3+2^3+\cdots+k^3+(k+1)^3=\left(\frac{(k+1)(k+2)}{2}\right)^2.$

putting the value of $T(k)$ in $T(k+1)$

LHS=$\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3$

$(k+1)^2\left(\frac{k^2}{4}+K+1\right)$

$\frac{(k+1)^2}{4}\left(k^2+4k+4\right)$

$\frac{(k+1)^2}{4}(k+2)^2$

Summary

Mathematical induction is a method of proving that is commonlyused to establish some concepts or results which are given in terms ofthe natural numbers. It also enables us to write adequate expressionsfor the general term of a sequence or series

Self Assessment Questions

1. What is a Mathematical Induction?
2. prove that $5^{2n}\ ^{1}$ is divisible by 8 for all n n
3. Prove that $\sum_{k=1}^{n} n^3 = n^2 (n+1)^2 \frac{1}{4}$ by Mathematical induction.

Tutor Marked Assignment

1. Show that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in N$
2. Prove that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2$ for all $n \in N$
3. Show by induction, that $\frac{1}{3} + \frac{1}{2.4} + \frac{1}{n(n+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$
4. Prove by induction, that $9(9^n - 1) - 8n$ is an integer multiple of 64.
5. Prove by induction, that $\sum_{k=1}^{n} n^3 = \left(\frac{n(n+1)}{2}\right)^2$.
6. Show that $1^3 + 3^3 + \cdots + (2n+1)^2 = \frac{n(4n^2-1)}{3}$
7. Prove by Mathematical induction, that $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}$ for all $n \in N$
8. If $n \in N$, Prove by mathematical induction, that $a + arar^2 + \cdots + ar^{n+1} = a\frac{1-r^n}{1-r}$ for real numbers a and r,$r \neq 1$

References

Bartle R.G. Sherbert D.R, (2000). Introductory to Real Analysis,John Willey and Son,Inc,University of Illinois, Urbana Champaign.

Blitzer, R. (2009)Introductory and Intermediate Algebra For College Students, Pearson Prentice Hall, Upper Saddle River, New Jersey.

Makanjuola, S. O. (2008),Foundation of Abstract Algebra, Evidence Nigeria Ventures, Ilorin, Kwara State.

Further Reading

John k. Hunter.(2012), Introductory to Real Analysis, 1-7

Module 2

SERIES AND SEQUENCE

Units

UNIT 2

Sequence

Introduction

The word sequence denotes certain objects or events occurring in the same order. In itself, a sequence is a set of numbers arranged in a specific manner. The applications of sequences can be found in many areas such as the process of analyzing economic data and certain areas of Physics. Sequences are also called progressions, and they are used to represent ordered list of numbers.

Learning Outcomes

At the end of this unit, you should be able to:

1. Discuss sequences of a real number
2. Give examples of sequences
3. Solve problems on sequences

Main Content

A sequence is simply an ordered list $u_1, u_2, u_3, ..., u_n$ of numbers (or terms). This is often abbreviated as U_n.

A sequence can also be defined as a set of numbers which are written in some particular order

For example

$$1,2,5,7,9, ...$$

This is called a sequence of odd numbers.

Here is another sequence

$$2,4,6,8,10, ...$$

This is the sequence of an even numbers.

Also,

$$1, -1, 1, -1, 1, -1, ...$$

is a sequence of numbers alternating 1 and −1.

We can call the first term in a sequence u1, the second term u2, and so on. With this same notation, we would write un to represent the nth term in the sequence. So

$$u_1, u_2, u_3, ..., u_n$$

would represent a finite sequence containing n terms. We could also use this notation to represent the rule for the Fibonacci sequence. We would write

$$u_n = u_{n-1} + u_{n-2}$$

to say that each term was the sum of the two proceeding terms.A sequence is an endless array of numbers that follows acertain order so that there is first term, second term and so on. Thecommon ones are AP, Arithmetic progression (linear sequence) andGeometric progression, GP(exponential sequence). For example, the array

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$$

This is a sequence where nth member is obtained by the inverse of n.

Summary

Sequences are certain objects or events that occur in the same order. This can be written in a general form as $u_1, u_2, u_3, \ldots, u_n$.

Self Assessment Questions

1. Define the word sequence of a real number.
2. Give an example of a sequence.
3. Give two examples of odd sequences.

Tutor Marked Assignment

1. Write the Fibonacci sequence.
2. 2,2,3,4.... is a sequence of real numbers. True or False?
3. Give two examples of even sequences.

References

1. Analysisby Malcolm R.,(2)Adams Sequences and Series: An Introduction to Mathematical.
2. Straud K.A.(747) Engineering Mathematics.

Further Reading

John Bird, Higher Engineering Mathematics, fifth edition.

UNIT 2

Series

Introduction

Series is the sum of terms in a sequence. Series is also the way of adding sequence of a number up.

Learning Outcomes

At the end of this unit, you should be able to:

1. Define series of a real number
2. Give example of a series
3. Solve problems on series

Main Content

A series is the summation of terms in a sequence.For example, suppose we have the sequence

$$u_1,u_2,u_3,...,un.$$

The series of the sequence is

$$u_1+u_2+u_3+...+un$$

and we have Sn for the sum of these n terms. We can indicate the term of a series as follows:

$u_1$will be the first term, u_2 the second term, u_3 the third term, so that

ur will represent the rth term and ur+ 1 the (r+ 1)thterm etc. Also, the sum of the 5th term will be indicated as S_5, so the sum of the first n terms will be stated as Sn.

Activity 1

The series of-1, 1 ,$-1,\cdots$, (-1) 1,... is 0

Activity 2

The series of $1, 2, 3, \cdots, n$ is $1 + 2 + 3 + \cdots + n = \sum_{j=1}^{n} k$

Activity 2

-

The series of $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots$ is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} \cdots$

Summary

Series is the addition of infinite numbers. Every member of each set is called a term and the first term is always referred to as a. The three dots after each set show that the set continues indefinitely. The nth term of the sequence is usually a function of n.

Self Assessment Questions

1. Define series of a real number.
2. Write $x_1, x_2, x_3, \ldots xn$ in series form.
3. What is the difference between a series and a sequence?

Tutor Marked Assignment

1. Give an example of a series.
2. What is the series of 2,5,6,8?
3. Write $t_1, t_2, t_3, \cdots tn$ in series form.

References

1. Analysis by Malcolm R.,(13)Adams Sequences and Series: An Introduction to Mathematical.

2. Straud K.A.(749) Engineering Mathematics

Further Reading

John Bird, Higher Engineering Mathematics, fifth edition.

UNIT 3

Arithmetic Progression

Introduction

Arithmetic Progression(AP) is a sequence of numbers such that the difference between the consecutive terms is constant.

Learning Outcomes

At the end of this unit, you should be able to:

1. define arithmetic progression
2. identify arithmetic progression problems
3. solve arithmetic progression problems

Main Content

When a sequence has a constant difference between successive terms. it is called an arithmetic progression (AP). The first term is denoted as a and the common difference is denoted as d

Example include:

(I) 1, 3, 5, 7, 9, 11, has the common difference of 2 and

(ii) a,a+d,a+ 2d, a+ 3d,...has the nth term asa+ (n−1)d

Activity 1

Find the 7th term of the series

1, 3, 5, 7, 9, 11,...

Solution

un=a+ (n−1)d

where a is the first term, d is the common difference therefore

$u_7 = 1 + (7-1)2 = 13$

Activity 2

Find the nth term of the sequence

T_1	T_2	T_3	T_4	T_5
5	9	13	17	21

I.	5	9	13	17	21
	T1	=a	= 5	d= 4	
	T2	= a +d×1	= 5 + 4 ×1	= 9	
	T3	= a+d×2	= 5 + 4 ×2	= 13	
	T4	= a+d×3	= 5 + 4 ×3	= 17	
*	T5	=a+d×4	= 5 + 4 ×4	= 21	
	Tn	=a+d×(n−1)	= 5 + 4×(n−1)	= 5 + 4(n−1)	

Summary

Arithmetic Progression (AP) is a sequence that has a constant difference between successive terms in which the general form can be written as

$u_n = a + (n-1)d$, where a is the first term, d is the common difference and n is the number of terms.

Self Assessment Questions

1. Find the 8th and 10th term of this sequence 1,4,7,10,...
2. If the 8th and 9th term of an AP are 70and 90 respectively. Find the first term and the common difference.
3. Find the 5th and 12th term of this sequence -1,-5,-9,-13,...

Tutor Marked Assignment

1. Given the following information, find the unknown variable of AP.

i $a=6, \quad d=4 \quad n=7 \quad T_n=?$

ii $a=3, \quad d=5 \quad T_n=43 \quad n=?$

iii $a=8, \quad n=6 \quad T_n = 32 \quad d=?$

iv $d=5, \quad n=9 \quad T_n=50 \quad a=?$

2. Find the 39th term of an AP whose first term is 30 and 66th term is 100

References

- Straud K.A. Engineering Mathematics.
- John Bird, Higher Engineering Mathematics, fifth edition.

Further Reading

- Introduction to scalars and vectors. www.siyavula.com

UNIT 4

Sum of Arithmetic Progression

Introduction

Arithmetic progression or arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms is constant.

Learning Outcomes

At the end of this unit, you should be able to:

1. identify arithmetic progression problems
2. derive the formula for the sum of an arithmetic progression
3. obtain the sum of an arithmetic progression.

Main Content

The sum (S) of an AP can be obtained by multiplying the average of all the terms by the number of terms. The average of all the term , $\frac{s=(a+l)}{2}$ where 'a' is the first term and l is the last term, i.e

l = a+ (n−1)d for n terms

Hence, the sum of n terms,

$$S_n = \frac{n(a+l)}{2}$$

$$= \frac{n}{2}[a + [a + (n-1)d]]$$

$$= \frac{n}{2}[a + a + (nd - d)]$$

$$S_n = \frac{n}{2}[2a + d(n-1)]$$

Activity 1

Find the sum of the first 7 terms of the series 1, 3, 5, 9, 11,...

$$a = 1 \quad d = 2 \quad n = 7$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_7 = \frac{7}{2}[2(1) + (7-1)2]$$

$$= \frac{7}{2}[2 + (6)2]$$
$$= \frac{7}{2}[2 + 12]$$
$$= \frac{7}{2}[14]$$
$$= 49$$

Activity 2

If the 15th term of a series is 30 and the first term is 2, find the common difference.

Solution

$u_n = a + (n-1)d$

$u_{15} = a + (15-1)d$

$30 = 2 + (15-1)d$

$30 = 2 + (14)d$

$30 = 2 + 14d$

$30-2 = 14d$

$28 = 14d$

$\boxed{d = 2}$

Therefore, the common difference is 2.

Activity 3

Find the sum of the first 20 terms of the series

3.5, 4.1, 4.7, 5.3,...

Solution

$a = 3.5, d = 0.6, n = 20$

$S_{20} = \frac{20}{2}\ [2(3.5) + (20-1)0.6]$

$= 10[7 + 11.4]$

$= 10[18.4]$

$= 184$

Activity 4

The sum of 4 terms of AP is 70 and the common difference is 10. Determine the first term.

Solution

$Sn= \frac{n}{2}[2a+(n-1)d]$

$S_4= \frac{4}{2}[2a+(4-1)10]$

$70 = 2[2a+(3)10]$

$70 = [4a+60]$

$4a= 70-60$

$4a= 10$

$a= \frac{10}{4}$

$a = \frac{5}{2}$

Activity 5

There are three numbers in an arithmetic progression. Their sum is 18 and the product is 162. Determine the three numbers.

Solution

Let the number be $(a-d), a, (a+d)$

Then $(a-d) + a + (a+d) = 18$

$a-d+a+a+d= 18$

$3a= 18$

Therefore, $a= 6$

Also

$(a-d), (a), (a+d)$

$(6-3), 6, (6+3)$

3 , 6 , 9

The numbers are 3, 6, 9

Activity 6

The first, twelfth and last terms ofan arithmetic progression are 4, 31.5, 376.5 respectively. Determine

a. The number of terms in the series

b. The sum of terms and

c. The 80th term

Solution

(a) Let AP be a , $a + d$, $a+ 2d,...,a+ (n-1) d$
where $a= 4$

The 12th term is $a + (12-1) \quad d= 31.5$

i.e $4 + (12-1) d= 31.5$

from which $11d= 31.5-4$

$11d= 27.5$

Hence, $d= 2.5$

The last term is $a+ (n-1)d$

i.e $4 + (n-1)(2.5) = 376.5$

$(n-1) = (376.5-4)/2.5$

$= 149(n-1)$

$= 149n$

$= 150$

(b) sum of the terms

$Sn=n/2[2a+ (n-1)d]$

$S150= 150/2[2(4) + (150-1)(2.5)]$

$= 75[8 + (149)(2.5)]$

$= 75[8 + 372.5]$

$= 28537.5$

C. The 80th term is

$u_{80}= 4 + (80-1)2.5$

$= 4 + (79)(2.5)$

$= 4 + 197.5$

$=201.5$

Summary

The sum (S) of an AP is the multiplication of the average of all the terms by the number of terms. The average of all the terms

$\frac{Sn=(a+l)}{2}$, in which 'a' is the first term and list the last term and and it can be generalized as $Sn=n_2[2a+d(n-1)]$

Self Assessment Questions

I Find the sum of the 7th and 11th term of the sequence 1,4,7,10,....

ii If the 8th and 9th terms of an AP are 70and 90 respectively. Find the first term and the common difference

Tutor Marked Assignment

Find the sum of each of the following AP:

I. 3 + 5 + 7 +...+ 99 + 101

ii. 2 + 5 + 8 +...up to the 18th term

iii. a= 5 andT8= 70

References

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Further Reading

Algebra and Trigonometry by Ushri Datta, A. S. Muktibodh and S.D. Mohgaonkar.

UNIT 5

Geometric Progression

Introduction

Geometric Progression(GP) also known as geometric sequenceis a sequence of numbers where each term after the first is foundby multiplying the previous by a fixed, non zero number calledcommon ratio.

Learning Outcomes

At the end of this unit, you should be able to:

1. define geometric progression
2. identify geometric progression problems
3. solve geometric progression problems

Main Content

A geometric progression (GP) is a sequence where each new termafter the first is obtained by multiplying the proceeding term by aconstant r called the common ratio. If the first of the sequence isa,then the geometric progression is

$$a, ar, ar^2, ar^3, \ldots$$

where the nth term isar^{n-1}.Let a GP be a, ar, ar^2,ar^3,...,ar^{n-1}then the sum of n terms is given as

$$Sn = a+ar+ar^2+ar^3+\ldots+ar^{n-1} \quad (1)$$

multiplying equation 1 by r gives

$$rSn = ar+ar^2+ar^3+ar^4+\ldots+ar^{n-1}+arn+\ldots \quad (2)$$

subtracting (1) from (2) gives:

$Sn-rSn=a-ar^n$

i.e

$Sn(1-r) = a(1-r^n)$

thus the sum of n terms is given as

$Sn = a(1-r^n)/(1-r)$ which is valid for $r < 1$ also
$Sn = a(r^n-1)/(r-1)$ which is valid when $r > 1$

when the common ratio of a GP is less than unity, the sum of n terms,
$Sn = a(1-r^n)/(1-r)$, which may be written as $Sn = a/(1-r) - ar^n/(1-r^n)$ since $r < 1$, r1, r^n becomes less as n increase i.e $r \to 0$ as $n \to \infty$

hence $ar^n/(1-r) \to 0$ as $n \to \infty$. Thus $Sn \to a/(1-r)$ as $n \to \infty$.The quantity $a/(1-r)$ is called the sum of infinity. $S\infty$, is the limiting value of the sum of an infinite number of terms. i.e $S\infty = a/(1-r)$ which is valid when $-1 < r < 1$

Activity 1

Find the 7th term of the series 3, 6, 12, 24, ...

Solution

$a = 3$, $r = 2$

$un = ar^{n-1}$

$u_7 = 3(2)^{7-1}$

$= 3(64)$

$u_7 = 192$

Activity 2

Find the sum of the 7 terms of the series 2, 8, 32, 128,...

$a = 2$, $r = 4$ r¿1

$Sn = a(r^n-1)/r-1$

$S_7 = 2(4^7-1)/(4-1)$

$= 2(16383)/3$

$= 10922$

Activity 3

The first term of a geometric progression is 12and the fifth term is 55. Determine the 8th term and the 11th term,

Solution

The 5th term is given as $ar^4 = 55$

$a = 12$

hence $r^4 = 55/a = 55/12$

$r4 = 55/12 = 4.58$

$r = 1.463$

The 8th term is

$ar^7 = (12)(1.463)^7$

$= 172.063 \equiv 172$

The 11th term is

$ar^{10} = 12(1.463)^{10} = 539$

Activity 3

Find the sum of the first 8 term of the series 8,4,2,1/2,...

Solution

$a = 8,\ r = 4/8 = 1/2$

$S_n = a(1-r^n)/1-r$

$= 8(1-(^1/_2)^8)/(1-^1/_2)$

$= 8(1-^1/_{256})/(1-^1/_2)$

$= ^{1515}/_{16}$

$= 15.9375$

Activity 3

If the 5th term of a GP is 162 and the 8th term is 4374, find the first term and the common ratio.

Solution

5th term = 162

i.e

$u_5 = 162$

$\boxed{162 = ar^4}$ 3

also, the 8th term = 4374

$u_8 = ar^{8-1}$

$\boxed{4374 = ar^7}$ 4

from equations (3) and (4)

$162 = ar^4$

$4374 = ar^7$

$ar^7 = 4374$

$ar^4 = 162$

$ar^7/ar^4 = 4374/162$

$r(7-4) = 27$

$r^3 = 27$

$r = 3$

Also $ar^7 = 4374$

$a(3)^7 = 4374$

$a = 2$

Activity 6

Find the sum to infinity of the series 3,1,1/3,...

$r = 1/3$ the sum to infinity

$S\infty = a/1-r$

$= 3/(1-1/3)$

$= 9/2$

$= 4.5$

Summary

A geometric progression (GP) is a sequence where each newterm after the first is obtained by multiplying the proceeding termby a constant r called the common ratio instead of the commondifference in AP.

Self Assessment Questions

1.Find the 15th and 18th term of the series 4,8,16,32,...

2. If the common ratio of a GP is 4 and the first termis 4 , also the sum of the nt term is 1364, find the number of terms.

3. Find the sum to infinity of the series 2, 1, 1/2,...

Tutor Marked Assignment

1) If the common ratio of a GP is 4 and the first term is 4 , also the sum of the nth term is 1364. What is n?(2) Find the sum of the first 15 term of the series generated in Question 1.

References

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Further Reading

John Bird, Higher Engineering Mathematics, fifth edition.

Module 3

QUADRATIC EQUATIONS AND METHOD OF SOLUTION

Units

Unit 1 - Quadratic Equation

Unit 2 - Methods of Solving Quadratic Equation

UNIT 1

Quadratic Equations

Introduction

In this unit, you are going to learn the form of quadratic equation,zero of the equation and its roots as the solution to the equation.Quadratic equations are used in everyday life, as when calculatingareas,

Learning Outcomes

At the end of this unit, you should be able to:

1. define a quadratic equation
2. determine zeros or roots of a quadratic equations
3. use different methods of solving quadratic equations
4. establish relationship between roots and coefficients

Main Content

A quadratic equation is an algebraic equation of the form

EX FX G E 16

where E F and G are real numbers. E is the coefficient of X , F is the coefficient of X and G is a constant while X is to be determined. Thus, the solution of X is regarded as the roots of the quadratic equation.

Activity 1

The following equations are quadratic.

I X X II Y Y III T T— —

Zeroes and Roots of a Quadratic equation

The value of x for which the equation becomes zero is called zero of a quadratic equation. For instance, 3 is zero of the polynomial **x x** because it becomes zero at **x**

Roots of a Quadratic Equation

Suppose and are the roots of quadratic equation (1). Then,

E F G

Therefore, the values of independent variable x for which the entire equation will be zero is regarded as the roots of the quadratic equation.

Activity 2

If x is one of the roots of quadratic equation **x x** validate that the root satisfies the equation.

Solution

Given: **x x**

Substitute the root **x** into the equation, we have

Hence, the root x= 3 satisfies the equation.

Now, it is convenient, having known the roots, to construct a quadratic equation by multiplying the factors **x** or **x**

x OR **x**

x x

x x x

x x

Summary

Quadratic equation has been discussed as an algebraic equationwhose roots are the value of independent variablex. Also, quadraticequation can be constructed by multiplying the factors of the valuesofx.

Self Assessment Questions

1. Define a quadratic equation
2. The roots of equation x   RX S   are   and . Find the equation whose roots are α-1 β, β-1 α
3. Find the roots of i. x   x   ii. x   iii. x   x

Tutor Marked Assignment

1. What is a quadratic equation?
2. Determine the roots of the followings:

   (I) x   x   (ii) x   (iii)x   x

References

Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition.Palgrave Publishers Ltd.New York

Further Reading

Peter B., Michael E., Davis H., Janine M., Bill P. and Jacqui R.(2011). The Improving Mathematics Education in Schools (TIMES)Project.Australian Mathematical Sciences Institute. Australia.

UNIT 2

Methods of Solving Quadratic Equations

Introduction

In this unit, various methods of solving quadratic equations shall be discussed. Among which are factorization, completing the square methods and quadratic formula.

Learning Outcomes

At the end of this unit, you should be able to:

1. know how to solve quadratic equations
2. determine the nature of roots of the equations; and
3. establish relationships between roots and coefficients.

Main Content

Factorization Method

Recall from unit 1, equation (5),

(I) the sum of the terms, say R S F

(ii) the product of the terms, say

RS GX R and X S

are therefore taken as the two factors of the quadratic equation.

Such that, equation EX FX G can be written as follow;

X R X S (6)

therefore, the root or solution of the quadratic equation

EX FX G E is R or S

$$x = -\frac{f}{2e} \pm \frac{\sqrt{f^2 - 4eg}}{2e}$$

$$x = -\frac{f \pm \sqrt{f^2 - 4eg}}{2e}$$

Which is the roots of the quadratic equation.

Let D F EG

$$x = -\frac{f \pm \sqrt{D}}{2e}$$

$$x = -\frac{f + \sqrt{D}}{2e} \quad \text{or} \quad -\frac{f - \sqrt{D}}{2e}$$

For a quadratic equation EX FX G E if

1. D F EG , the equation has two real and unequal (distinct) roots.
2. D , the equation will have two real and equal roots and both roots are equal to $\frac{F}{E}$
3. D , the equation will have complex or imaginary roots. D is the discriminant of the equation.

Activity 2

Find the roots of X X by completing the square method

X X

Following through the steps 1 to 5, we have

$$x^2 + 3x = \frac{3}{2}$$

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = \frac{3}{2} + \left(\frac{3}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{3}{2} + \frac{9}{4}$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{15}{4}$$

$$x + \frac{3}{2} = \pm\sqrt{\frac{15}{4}}$$

$$x = -\frac{3}{2} \pm \sqrt{\frac{15}{4}}$$

$$x = -\frac{3}{2} \pm \frac{\sqrt{15}}{2}$$

$$x = -\frac{3}{2} + \frac{\sqrt{15}}{2} \quad \text{or} \quad -\frac{3}{2} - \frac{\sqrt{15}}{2}$$

$$x = -\frac{3}{2} + \frac{1}{2}\sqrt{15} \quad \text{or} \quad -\frac{3}{2} - \frac{1}{2}\sqrt{15}$$

Activity 3

1. By method of completing the square, find the roots of the equation

X X

I Y Y II X X

Solution

X X X X

divide through by √2

X X

$$x^2 - \frac{3}{\sqrt{2}}x + (\frac{-3}{2\sqrt{2}})^2 = -3 + +(\frac{-3}{2\sqrt{2}})^2$$

$$(x + \frac{3}{2\sqrt{2}})^2 = -3 + \frac{9}{8} = -\frac{15}{8}$$

take the square root of both sides

$$x + \frac{3}{2\sqrt{2}} = \sqrt{-\frac{15}{8}} = \frac{\sqrt{-15}}{2\sqrt{2}} \implies x = -\frac{3}{2\sqrt{2}}\frac{\sqrt{-15}}{2\sqrt{2}} = \frac{-3\sqrt{-15}}{2\sqrt{2}}$$

$$x = \frac{-3\sqrt{15}}{2\sqrt{2}}, \quad (i^2 = -1)$$

$$x = \frac{-3+i\sqrt{15}}{2\sqrt{2}} \quad or \quad \frac{-3-i\sqrt{15}}{2\sqrt{2}}$$

(2i) Given quadratic equation $4y^2 - 12y + 7 = 0$, the discriminant

D F EG

is used to determine the nature of the roots of the equation. From the equation,

A B AND C D

This means the roots are real and distinct.

Quadratic Formula

Let EX FX G E the quadratic formula is given as

$$x = -\frac{f \pm \sqrt{f^2 - 4eg}}{2e}$$

Establishing Relationships between Roots and Co-efficients

Considering a quadratic equation (1) with roots γ and λ, equation (1) can further be written as

X $\frac{F}{E}$XGE so that we have

X

the sum of the roots $\frac{F}{E}$

the product of the roots $\frac{G}{E}$

Also, the roots of quadratic equation (1) by quadratic formula are

$$\frac{-f+\sqrt{f^2-4eg}}{2e} \text{ and } \frac{-f-\sqrt{f^2-4eg}}{2e}.$$

And this implies,

$$\lambda=\frac{-f+\sqrt{f^2-4eg}}{2e} \qquad \boxed{\text{i}}$$

$$\gamma=\frac{-f-\sqrt{f^2-4eg}}{2e} \qquad \boxed{\text{ii}}$$

By adding $\boxed{\text{i}}$ and $\boxed{\text{ii}}$

$\frac{F}{E}$

$$\text{Sum of the roots} = -\frac{\text{coefficient of x}}{\text{coefficient of x}^2} \quad \frac{F}{E} \qquad \boxed{\text{iii}}$$

By taking the product of $\boxed{\text{i}}$ and $\boxed{\text{ii}}$

$\frac{G}{E}$

$$\text{Product of roots} = \frac{\text{constant term}}{\text{coefficient of x}^2} \quad \frac{G}{E} \qquad \boxed{\text{iv}}$$

Thus, the relationship between roots in $\boxed{\text{iii}}$ and $\boxed{\text{iv}}$ are useful to form quadratic equation when the roots are known

NOTE: A quadratic equation is a perfect square when discriminantD= 0.

Activity 4

To generate a quadratic formula using equation AX BX C A
divide both side by coefficient of x^2, say a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

subtract $\frac{c}{a}$ from both side

$x^2 + \frac{b}{a}x = -\frac{c}{a}$

Take the half square of the coefficient of x, then add it to both sides

$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$

left hand side is factored, so we have

$(x + (\frac{b}{2a}))^2 \;=\; -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2-4ac}{4a^2}$

Take the square root of both sides

$x + \frac{b}{2a} \;=\; \sqrt{\frac{b^2-4ac}{4a^2}} \;=\; \frac{\sqrt{b^2-4ac}}{2a}$

$x \;=\; \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \;=$

$\frac{-b\pm\sqrt{b^2-4ac}}{2a} \implies x \;=\; \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $\frac{-b-\sqrt{b^2-4ac}}{2a}$

are the root or solution of the quadratic equation.

Thus, $x \;=\; \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is regarded as the quadratic formula

Activity 5

(1) λ and γ are the roots of $2x^2 - x - 4 = 0$. Find (a) λγ (b) λ+γ (c) $\lambda^2+\gamma^2$

Solution

(1) Recall that,

$(\lambda + \gamma) = -\frac{f}{e}$ and $\lambda\gamma = \frac{g}{e}$. Here, f=−1, e=2 and g=−4. Therefore,

(a) $\lambda\gamma = \frac{-4}{2} \implies x \;=\; -2$

(b) $(\lambda + \gamma) = -(\frac{-1}{2}) = \frac{1}{2}$

(c) $\lambda^2 + \gamma^2 = (\lambda + \gamma)^2$ - $2\lambda\gamma = (\frac{1}{2})^2$-$2(-2) = \frac{1}{4} + 4 = \frac{17}{4}$

Summary

In this unit, you have learnt methods of solving quadratic equationsvia factorization and completing the square methods, nature of rootsof the equation and relationships between roots and coefficients.

Self Assessment Questions

(1) Using factorization method, solve the following quadratic equations.

(a) T T (b) Y Y

(2) By method of completing the square, find the roots of the equation

X X

(3) Determine the nature of roots in the following quadratic equations:

(a) Y Y (b) X X

Tutor Marked Assignment

(1) Determine the quadratic formula for the quadratic equation

AX BX C A

(2) What will be the value of t if T Y Y T has a real root?

(3) Suppose a root of AY BY C A is five times other, show that

B AC

References

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Further Reading

K. A. Stroud and D. J. Booth (2007). Engineering Mathematics, NewYork: Palgrave Macmillan, 6th.

Module 4

COMPLEX NUMBERS

Units

Unit 1 - Set of Complex Numbers and Its Basic Operations
Unit 2 - Differentiation with Product Rule
Unit 3 - Differentiation with Quotient Rule
Unit 4 - Differentiation of Composite Function
Unit 5 - Differentiation of Trigonometric and Exponential Functions

UNIT 1

Set of Complex Numbers and Its Basic Operations

Introduction

There are six sets of numbers all of which evolved to meet somespecific needs. The development of number system started with theset of natural numbersNwhich is the smallest set of number andends with the introduction of the set of complex numbersCbeing thelargest set of numbers. A brief introduction of the sets of numbers will helpus have a deeper understanding of this chapter. I am going to

Learning Outcomes

At the end of this unit, you should be able to:

1. identify the different types of sets of numbers
2. and identify the complex number as the largest set of numbers.
3. perform basic operations on the set of complex numbers.

find the modulus and argument of a complex number.

find the conjugate of any given complex number.

Main Content

For a clear understandingof this unit, following are the definitions of the set of numbers.

Set of natural numbers N

These are the counting numbers otherwise known as whole numbers i.e. 1,2, 3, ... The deficiency ofthe set of number to accommodate solution of problems such as $x+5=1$ and $x^2+2x=0$ leads to the extension of number system to include integers.

Integers $\mathbb{Z}$:

The integers are the positive (natural) numbers, the negative numbers and zero. That is, $\mathbb{Z}$. The limitation to solve problems such as x and x gave birth to another set of numbers.

Set of rational numbers $\mathbb{Q}$

Let p q $\mathbb{Q}$, the set of number $\frac{p}{q}$ $\mathbb{Q}$ / is called rational number. A rational number is a terminating (or repeating) decimals such as $\frac{2}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}$ and E ... Note that 0 and 1 are also rational numbers.

For instance, for p and p $\mathbb{Q}$,we have the additive identity 0 and the multiplicative identity 1 respectively. This set of number however can not accommodate problems such as determining the ratio of circumference of a circle and its diameter and solving x or x among others. Hence, the need to furtherextend the number system to accommodate a wide range of numbers.

Set of irrational numbers $\mathbb{Q}$

Is the set of non-terminating and non-repeating numbers in decimal such as etc. In general, surd (square roots of prime numbers) are irrational numbers.

Set of Real numbers $\mathbb{R}$

Real number is the union of rational and irrational numbers and hence more involving with a wide range of properties. The set of numbers also accommodates many mathematical problems but not problems such as

x 0 and x x 0.

In general, a quadratic equation x c 0 c 0 has no solution in R. Hence, the need for the largest set of numbers called complex numbers which is the focus of this study. We now consider the last set of numbers as follows.

Set of complex number C:

Let x y $\mathbb{R}$, the set of complex numbers denoted by $\mathbb{C}$ is define for any z $\mathbb{C}$ as z x iy where the iota i or i .

The deficiency of real numbers in solving problems such as x or x x warrants the extension of real numbers system to the largest field of number known as the complex numbers. An Italian mathematician, Giralamo Cardano (1501 -1576), was the first to apply the concept of complex number as a formular for solving cubic equation.

However, German mathematician by name Carl Friedrich Gauss was the first to introduce the term "complex number", while Euler introduced the symbol i with the property i

Gauss defined the complex number as z x iy with x and y being real numbers. A complex number denoted by z is an ordered pair of real number with the property

$$z \quad x \quad y \quad x \; iy \qquad (7)$$

$$z \quad x \; y \quad \begin{cases} x & \mathrm{Re}\, z \\ y & \mathrm{Im}\, z \end{cases} \qquad (7)$$

The complex number z has two parts as x Re z and y Im z as the real and the imaginary part respectively.

Modulus of a complex number:

The modulus of a complex number z otherwise known as the absolute value of z is the length from origin which is defined as

$$z \quad \sqrt{\quad x \quad y} \quad \mathbb{R}$$

ϱ
= X Re z z
; Y Im z z

Given z x iy therefore, we have that z x y zz. Then,

arg z tan ! Y ! / X

Note: In finding arg z, you must be able to identify and locate the quadrant in which the complex number z lies on the Cartesian plane noting that tan has period

Basic Operations of Complex Numbers

Let z a ib and z c id

1. z z a ib c id a c i b d

2. z z a ib c id a c i b d

3. z z c id a ib c a i d b

4. z z a ib c id ac bd i ad bc

5. $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac-iad+ibc+bd}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$

Activity 1:

Let z= 2 + 3i. Find modulus and argument of z.

(2). Let z= 5−I and ω= 1 + 4i. Find:

(a).z (b).z (c). z (d).z

(3). Let z i, evaluate |z / z|

Solutions

(1). The modulus $|z| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$.

Argument $tan\theta = \frac{3}{2}$ which implies that $\theta = tan^{-1}\left(\frac{3}{2}\right) = 56.3^o$.

(2). Let $z = 5 - i$ and $\omega = 1 + 4i$.

(a). $z\omega = (5-i)(1+4i) = 5 + 20i - i - 4i^2 = 5 + 20i - i + 4 = 9 + 19i$

Conjugate of a complex number:

noted that if z and z are complex numbers, then

1. $\overline{z \quad z} \quad z^{-} \quad z^{-}$

2. $\overline{z\,z} \quad \overline{z}\,\overline{z}$

3. $z\bar{z} \quad x \ iy \quad x \ iy \quad x \quad y \quad z$

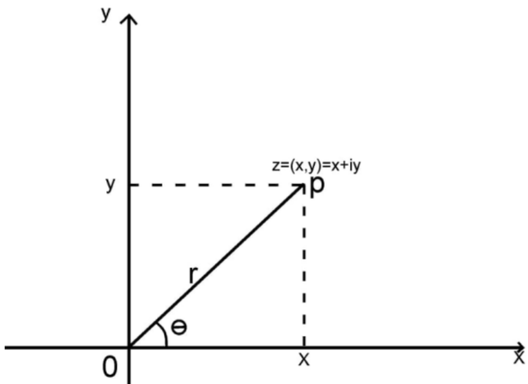
4. $z \ \bar{z} \quad x \ iy \quad x \ iy \quad x \quad \mathrm{Re}\, z \quad x \quad y \quad z$

5 $z \ \bar{z} \quad x \ iy \quad x \ iy \quad yi \quad \mathrm{Im}\, z$

6. $\frac{z}{\bar{z}} \quad \frac{x \ iy}{x \ iy} \quad \frac{x \ iy \ \ x \ iy}{x \ iy \ \ x \ iy} \quad \frac{x \quad y \quad xyi}{x \quad y} \quad \frac{x \quad y}{x \quad y} \ i \ \frac{xy}{x \quad y}$

Argand Diagram

The Argand diagram is named after a mathematician, Argand, whorepresented a complex number z by a pointP whose coordinates are(x,y). The axis ofxis the real axis while the axis of theyis calledthe imaginary axis as shown in the diagrams below:



OP is the complex number $z \ x \ iy \ \ r$ the length or absolute value of z and

arg z is the angle. The absolute value (or magnitude or length)of a complex number z is given as $r \quad z$ which is the point on the circle $x \quad y \quad r$. If z ,we have a unit circle $x \quad y$ Let $z \quad x \quad y$ then

$$\begin{cases} |x| & = |Re(z)| \leq |z|, \\ |y| & = |Im(z)| \leq |z|. \end{cases}$$

Given

z = x + iy therefore, we have that $|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$. Then,

Note: In finding $\theta = argz$, you must be able to identify and locate the quadrant in which the complex number z lies on the Cartesian plane noting that $tan\theta$ has period π.

Basic Operations of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$

(1). $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$

(2). $z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d)$

$z_2 - z_1 = (c + id) - (a + ib) = (c - a) + i(d - b)$

(3). $z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$

(4). $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac-iad+ibc+bd}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$

Activity 1

(1). Let $z = 2 + 3i$. Find modulus and argument of z.

(2). Let $z = 5 - i$ and $\omega = 1 + 4i$. Find:

(a). $z\omega$ (b). $z - \omega$ (c). $\omega - z$ (d). $\overline{z} + \overline{\omega}$

(3). Let $z = 2 + 10i$, evaluate $\left|\frac{z}{\bar{z}}\right|$

Solutions

(1). The modulus $|z| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$.

Argument $tan\theta = \frac{3}{2}$ which implies that $\theta = tan^{-1}\left(\frac{3}{2}\right) = 56.3^o$.

(2). Let $z = 5 - i$ and $\omega = 1 + 4i$.

(a). $z\omega = (5 - i)(1 + 4i) = 5 + 20i - i - 4i^2 = 5 + 20i - i + 4 = 9 + 19i$

(b). $z - \omega = (5 - i) - (1 + 4i) = 4 - 5i$

(c). $\omega - z = (1 + 4i) - (5 - i) = -4 + 5i$

(d). $\overline{z} + \overline{\omega} = (5 + i) + (1 - 4i) = 6 - 3i$

(3). Let $z = 2 + 10i$,

$$\left|\frac{z}{\overline{z}}\right| = \frac{2+10i}{2-10i} = \frac{(2+10i)(2+10i)}{(2-10i)(2+10i)} = \frac{4+40i-100}{4+100} = \frac{-96+40i}{104} = \frac{-24}{21} + i\frac{10}{21}$$

$$\left|\frac{z}{\overline{z}}\right| = \left|\frac{-24}{21} + i\frac{10}{21}\right| = \sqrt{\left(\frac{-24}{21}\right)^2 + \left(\frac{10}{21}\right)^2} = \sqrt{\left(\frac{-8}{7}\right)^2 + \left(\frac{10}{21}\right)^2} =$$

$$\sqrt{(1.1429)^2 + (0.4762)^2} = \sqrt{1.306 + 0.2268} = 1.2381$$

Activity 2

(1). Find the conjugate of the following complex numbers: $3-2i, -5+4i$ and 8.

Solutions

$3 + 2i, \; -5 - 4i$ and 8

Summary

In this unit, you have learnt the basic definitions and development of set of numbers, the conjugate of complex numbers, modulus and argument as well the basic operations of complex numbers.

Self Assessment Questions

1. Identify the various sets of numbers.

2. Which of the set of numbers is the largest set?

3. Let z I z I and z I

(a). Find z z z

(b). Find z z

(c). Evaluate z z

(4). Find the modulus and argument of z I

Tutor Marked Assignment

1. Find the conjugate, modulus and argument of z z, given that z I

2.Findthe conjugate of each of the following: I and cos $\frac{\pi}{3}$ISIN . $\frac{\pi}{3}$

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UNIT 2

Concepts of Complex Numbers

Introduction

Having defined complex functions and some of its basic operationsin the previous unit, I will now teah you the basic properties ofcomplex numbers as well as the various forms of complex numbers.By the end of this unit therefore, you should be well familiar with the basicconcepts and properties of complex numbers

Learning Outcomes

At the end of this unit, you should be able to:

1. identify some basic properties of complex numbers
2. state and prove the properties of complex numbers as a field.
3. identify the various forms of complex numbers.
4. find complex conjugates
5. proof that a complex number is an Abelian group under addition.

Main Content

Some Properties of Complex Numbers

The following are the properties of complex numbers

(i) **Closure property of $\mathbb{C}$** For any complex numbers z z $\mathbb{C}$ the set of complex numbers $\mathbb{C}$ is closed under the four (4) basic arithmetic operations of addition (+), subtraction (−), multiplication (×), and division (÷).

(ii) **For any complex number**

z, the modulus (or absolute value) of z is given by z p $\overline{X \quad Y}$.

(iii) **For any complex number Z X IY**

there exists a conjugate complex number $z = x - iy$ such that:

$$z\overline{z} = |z|^2 = |\overline{z}|^2\,;\ z + \overline{z} = 2Re(z);\ \text{and}\ z - \overline{z} = 2iIm(z).$$

(iv) **Let**

z be a complex number, then z has the following representations:

(1). **Cartesian form:** $z = x + iy$;

(2). **Polar form:** $z = rcos\theta + irsin\theta$; and

(3). **Exponential form:** $z = re^{i\theta}$

where $e^{i\theta} = cos\theta + isin\theta$ and $\frac{\partial}{\partial\theta} = iz\frac{\partial}{\partial z}$.

(v) **De Moivre's theorem** $z^n = (rcos\theta + irsin\theta)^n = r^n\,(cosn\theta + isinn\theta)$.

(vi) Let Z be the centre of a circle of radius R, then Z Z0 = R or Z Z0 RE'.
In addition,

(vii) Let z be a complex number, then ARG Z IM LOGZ AND LOG Z RE LOGZ

Proof:

Let $z = re^{i\theta}$

$$log_e z = log_e r + log_e e^{i\theta} = log_e r + i\theta log_e e$$

$$\Rightarrow logz = logr + i\theta = Re(z) + iIm(z).$$

Thus,

$$Arg\ z = Im\ logz\ \ and\ \ log\,|z| = Re\ logz$$

which completes the proof.

Activity 1:

LET Z1 I AND Z2 I EVALUATE EACH OF THE FOLLOWING

A Z1 Z2 B Z1Z2 C Z1/Z2 D Z1 Z2

SIMPLIFY A I 4 B I 7 C I 24

FIND THE MODULUS AND PRINCIPAL ARGUMENT OF

A I B I C I

Let z_1 i and z_2 i evaluate

$$\text{(a). } Re\left\{\frac{z_1 z_2}{z_1}\right\} \text{ (b). } Im\left\{\frac{z_1 z_2}{z_1}\right\}$$

Solutions

(1). (a). $z_1 + z_2 = (3+4i) + (5+2i) = 8+6i.$

(b). $z_1 z_2 = (3+4i)(5+2i) = 15+6i+20i+8i^2 = 7+26i.$

(c). $\frac{z_2}{z_1} = \frac{5+2i}{3+4i} = \frac{(5+2i)(3-4i)}{(3+4i)(3-4i)} = \frac{15-20i+6i-8i^2}{9+25} = \frac{23-14i}{25} = \frac{23}{25} - \frac{14}{25}i.$

(d). $z_1 - z_2 = (3+4i) - (5+2i) = -2+2i.$

(2). (a). If $i^2 = -1$, then $i^4 = i^2 \times i^2 = -1 \times -1 = 1$

(b). Let $i^7 = i^4 \times i^3$. From (a) above, $i^4 = 1$ and $i^3 = i^2 \times i = -i.$

Thus, $i^7 = i^4 \times i^3 = (1)(-i) = -i.$

(c). Let $i^{13} = i^7 \times i^6$ where $i^6 = i^3 \times i^3 = (-i)(-i) = i^2 = -1.$

Therefore, $i^{13} = i^7 \times i^6 = (-i)(-1) = i.$

(3). (a). Modulus z: $|z| = \sqrt{x^2+y^2} = \sqrt{3^2+(-4)^2} = 5$

Argument of z: $tan\theta = \frac{y}{x} \Rightarrow \theta = tan^{-1}\left(\frac{-4}{3}\right) = 53.12^0$

(b). Modulus z: $|z| = \sqrt{x^2+y^2} = \sqrt{2^2+1^2} = \sqrt{5}$

Argument of z: $tan\theta = \frac{y}{x}$ and $\theta = tan^{-1}\left(\frac{1}{2}\right) = 26.56^0$

(c). Modulus z: $|z| = \sqrt{x^2+y^2} = \sqrt{\sqrt{3}^2+1^2} = \sqrt{4} = 2$

Argument of z: $tan\theta = \frac{y}{x}$ and $\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^0$

(4). $\frac{z_1 z_2}{\overline{z_2}} = \frac{(5-3i)(-5+3i)}{5+3i} = \frac{-25+15i+5i+9}{5+3i} = \frac{-16+30i}{5+3i} = \frac{(-16+30i)(5-3i)}{(5+3i)(5-3i)}$

$= \frac{-80+48i+150i-90}{25+9} = \frac{-170+198i}{34} = \frac{-170}{34} + i\frac{198i}{34} = Re(z) + Im(z).$

Therefore, $Re(z) = -\frac{85}{17}$ and $Im(z) = \frac{99}{17}$.

Activity 2:

(1). Prove that C is an Abelian group under addition.

(2). Show that the set of complex number C is indeed a field

(3). Let z = 2 + 3i, evaluate z^2 and $\frac{1}{z^2}$

(4). Let z1 = 2 + 5i

and $z_2 = i - 1$, express each of the following in terms of $z = x + iy$:

(a). $\frac{2z_2}{z_1}$ (b). $\frac{1}{z_2^2}$ (c). $\frac{z_1}{z_2}$ (d). $(z_1 - z_2)^2$

(5). Let $z = x - iy$, find $Re\left(z^2\right)$, $Im\left(z^2\right)$ and $Im\left(\frac{z}{\bar{z}}\right)$.

(6). Find the complex conjugate of $Re\left(\frac{8-3i}{5+2i}\right)$.

(7). Let $z = 3 - 2i$, evaluate each of the following:

(a). $\frac{1}{z}$ (b). $Re\left(\frac{1}{z}\right)$ (c). $\left|\frac{1}{z}\right|$ (d). z^2.

Solutions

(1). We need to prove that the set of

complex number $\mathbb{C}$ satisfies the five properties of an Abelian group:

Let $z_1 = a + ib$, $z_2 = c + id$, and $z_3 = e + if$ where $a, b, c, d, e, f \in \mathbb{R}$.

(i.) **Closure property:**

$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d) \in \mathbb{C}$.

(ii.) **Associative law:**

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) = (a + c + e) + i(b + d + f)$.

(iii.) **Existence of identity element:** For all $z_1 \in \mathbb{C}$, we need to show

that there exists z_e such that $z_1 + z_e = z_1$

(iv.) **Existence of inverse element:** For any $z_1 \in \mathbb{C}$, we need to show

that there exists z_i such that $z_1 + z_i = z_e$.

$z_1 + z_i = (a + ib) + z_i = z_e = 0 + 0i$. Thus, $z_i = -a - ib$.

(v.) **Commutative law:** For all $z_1, z_2 \in \mathbb{C}$,

$z_1 + z_2 = z_2 + z_1 = (a + c) + i(b + d)$.

Since (i) - (v) holds, $\mathbb{C}$, the set of complex number is an Abelian group.

(2). It suffices to show

that the set of complex number $\mathbb{C}$ satisfies the 5 axioms of a field:

Let $z_1 = a + ib$, $z_2 = c + id$, and $z_3 = e + if$ where $a, b, c, d, e, f \in \mathbb{R}$.

(i.) **Commutative law:** For all $z_1, z_2 \in \mathbb{C}$, we verify **addition** and **multiplication** as follows:

$z_1 + z_2 = z_2 + z_1 = (a + c) + i(b + d)$ and

$z_1 z_2 = z_2 z_1 = (ac - bd) + i(ad + bc)$

(ii.) **Associative law:** For all $z_1, z_2, z_3 \in \mathbb{C}$,

Addition and Multiplication:

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) = (a + c + e) + i(b + d + f)$.

Similarly, $(z_1 z_2)z_3 = z_1(z_2 z_3)$ holds.

(iii.) **Distributive law:** For any $z, z_1, z_2 \in \mathbb{C}$, clearly

$z(z_1 + z_2) = zz_1 + zz_2$ holds

(iv.) **Existence of identity elements:** For all $z_1 \in \mathbb{C}$, there exists the following identity elements: Additive identity: $z_e = 0 + 0i$ such that $z_1 + z_e = z_1$. See 1(iii) above.

Multiplicative identity: $z_1 z_p = z_1 \Rightarrow z_p = \frac{z_1}{z_1} = 1 + 0i = 1$.

(v.) **Existence of inverse elements:** For each $z = x + iy \in \mathbb{C}$, we need to show that there exist complex numbers z_u and z_v called additive inverse and multiplicative inverse respective:

Additive inverse: Let $z_u + z = z_e$ so that $z_u + (x + iy) = z_e$ where $z_e = 0 + 0i$

$\Rightarrow z_u = z_e - z = -(x + iy) = -x - iy$ is the additive inverse element of z.

Multiplicative inverse:

Let $z \times z_v = z_p$ so that $z_v = \frac{z_p}{z} = \frac{1}{z}$ or $z_u = \frac{1}{(x+iy)}$ since $z_p = 1$.

That is, for

any $z = x + iy$, $\frac{1}{z} = \frac{1}{x+iy} = \left\{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right\}$ is the inverse element.

Note that $z \times \frac{1}{z} = (x + iy)\left\{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right\} = \frac{x^2+y^2}{x^2+y^2} = 1 = z_e$.

Thus, $\mathbb{C}$ is a field since it satisfies all the conditions (i)-(v).

Remark 1

(1.) A set in which two operations $\oplus$ and $\otimes$ are defined so as to have properties (i)-(v) above is called a field.

(2.) The two major fields of numbers are the real numbers $\mathbb{R}$ and the complex number $\mathbb{C}$.

(3.) The largest field of numbers is the complex field $\mathbb{C}$.

(4.) The major limitation of $\mathbb{C}$ is that the set of complex numbers is not an ordered field without $\mathbb{R}$.

Read further on real and complex fields

Activity 3

(1). let $z = 2 + 3i$, then

(a). $z^2 = (2 + 3i)^2 = 4 + 12i - 9 = -5 + 12i$

(b). $\frac{1}{z^2} = \frac{1}{-5+12i} = \frac{-5-12i}{(-5+12i)(-5-12i)} = \frac{-5}{169} + \frac{12}{169}i$

(2). Let $z_1 = 2 + 5i$ and $z_2 = -1 + i$, then

(a). $\frac{2z_2}{z_1} = \frac{-2+2i}{(2+5i)} = \frac{(-2+2i)(2-5i)}{(2+5i)(2-5i)} = \frac{-4+10i+4i+10}{4+25} = -\frac{6}{29} + \frac{14}{29}i$

(b). $\frac{1}{z_2^2} = \frac{1}{(-1+i)^2} = \frac{1}{1-2i-1} = \frac{1}{-2i} = \frac{-2i}{(-2i)(2i)} = \frac{2i}{4} = \frac{1}{2}i$

(c). $\frac{z_1}{z_2} = \frac{2+5i}{-1+i} = \frac{(2+5i)(-1-i)}{(-1+i)(-1-i)} = \frac{-2-2i-5i+5}{1+1} = \frac{3-7i}{1+1} = \frac{3}{2} - \frac{7}{2}i$

(d). $(z_1 - z_2)^2 = (3 + 4i)^2 = 9 + 24i - 16 = -7 + 24i$

Activity 4

(1). Let $z = x - iy$,

(a). $z^2 = (x - iy)(x - iy) = (x^2 - y^2) - 2ixy.$

Thus $Re\,(z^2) = x^2 - y^2$ and $Im\,(z^2) = -2xy$

(b). $\frac{z}{\bar{z}} = \frac{x-iy}{x+iy} = \frac{(x-iy)(x-iy)}{(x+iy)(x-iy)} = \frac{x^2-y^2-2ixy}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} - i\frac{2xy}{x^2+y^2}$
and $Im\left(\frac{z}{\bar{z}}\right) = -\frac{2xy}{x^2+y^2}$.

Activity 5

Let $z = 3 - 2i$. Then, $z = \frac{8-3i}{5+2i} = \frac{(8-3i)(5-2i)}{(5+2i)(5-2i)} = \frac{40-31i-6}{25+4} = \frac{34}{29} - i\frac{31}{29}$.

Then the conjugate $\bar{z} = \frac{34}{29} + \frac{31}{29}i$.

Activity 6

(a). $\frac{1}{z} = \frac{1}{3-2i} = \frac{3+2i}{(3-2i)(3+2i)} = \frac{3}{13} + \frac{2}{13}i$

(b). $Re\left(\frac{1}{z}\right) = \frac{3}{13}$ and $Im\left(\frac{1}{z}\right) = \frac{2}{13}$.

(c). $\left|\frac{1}{z}\right| = \sqrt{\left(\frac{3}{13}\right)^2 + \left(\frac{2}{13}\right)^2} = \sqrt{\frac{13}{169}} = \frac{1}{13}$.

(d). $z^2 = (3-2i)^2 = 9 - 12i - 4 = 5 - 12i$.

Summary

In this unit, you have learnt the basic properties of complex numbers, forms of complex numbers, modulus and principal arguments of complex numbers. You have also learnt how to find the real part and the complex part of complex numbers.

Self Assessment Questions

(1). Discuss the field properties of complex numbers.

(2). When is a field said to be order

(3). Discuss the relationships between real and complex numbers and their shortcoming(s), if any.

(4). State the various forms of complex numbers.

(5). Evaluate $\frac{1+3i}{2-i}$ and hence find its real and imaginary parts

Tutor Marked Assignment

(1). Find the real and imaginary parts of $\frac{z}{z}$

(2). Express $z = 4 - i3$ in polar and exponential forms.

(3). Define logarithmic of a complex number.

(4). Using series functions, show that $e^{i\theta} = cos\theta + isin\theta$.

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UNIT 3

Relationship Between Polar and Exponential Forms of Complex Numbers

Introduction

In the last unit, you learnt the Cartesian, polar and exponential forms of complex number. In this unit, you will learn the relationships between polar and exponential forms of complex numbers.

Learning Outcomes

At the end of this unit, you should be able to:

1. Express complex numbers in polar form
2. Express complex numbers in exponential form
3. Verify triangle inequalities of complex numbers

Main Content

The polar form is used to express the complex numbers $z = x + iy$ in terms of polar coordinates r, θ where r = radius of circle, absolute value (or modulus) of z and θ = the argument (or amplitude) of z denoted by *argz.*

These are defined by

$$x = r\cos\theta, y = r\sin\theta$$

so that $z = x + iy$ becomes

$$z = r(\cos\theta + i\sin\theta) \qquad (10)$$

where $\cos\theta + i\sin\theta = e^{i\theta}$ *is the Euler's formula. Substituting the Euler's formula in the above equation,*

$$z = re^{i\theta} \qquad (11)$$

referred to as the **exponential form.**

Geometrically, $|z| = r$ is the distance of the point from the origin or radius of a circle centered at 0 while $argz = \theta$ is the directed angle from positive x − axis to OP as shown in the figure above. All angles are measured in **radians** and positive in the counterclockwise sense as in the case in calculus.

If $r = \sqrt{x^2+y^2}$,

then $x = \left(\sqrt{x^2+y^2}\right) cos\theta$ and $y = \left(\sqrt{x^2+y^2}\right) sin\theta$. So that

$$cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \quad sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad and \quad tan\theta = \frac{y}{x}.$$

For $z = 0$, the angle θ is undefined and for $z \neq 0$, it is determined only up to integer multiple 2π. The value of θ lies in the range $-\pi < \theta \leq \pi$ is theprincipal value of the argument of z denoted by $Argz$ such that

$$-\pi < Argz \leq \theta$$

Activity 1

(1). Find the modulus and principal argument of $\frac{3+i}{1-2i}$. Hence express in polar and exponential forms.

(2). Express $\frac{(1+i)^2}{1-i}$ in both polar and exponential forms.

(3). Express $2 + i$ in polar form.

Solutions

(1). $\frac{3+i}{1-2i} = \frac{(3+i)(1+2i)}{(1-2i)(1-2i)} = \frac{3+7i-2}{1+4} = \frac{1}{5} + \frac{7}{5}i$ and hence $|z| = r = \left|\frac{3+i}{1-2i}\right| = \sqrt{\frac{1}{25}+\frac{49}{25}} = \sqrt{2}$. In addition, $cos\theta = \frac{1}{5\sqrt{2}}$ and $sin\theta = \frac{7}{5\sqrt{2}}$. In polar form, $z = r(cos\theta + isin\theta) = \sqrt{2}\left\{\frac{1}{5\sqrt{2}} + \frac{7}{5\sqrt{2}i}\right\}$.

The exponential form is given by $z = re^{i\theta} = \sqrt{2}e^{i\theta}$

$tan\theta = \frac{y}{x} = \frac{7}{5} \times \frac{5}{1} = 7$

In radian $\theta = tan^{-1}(7)$.

$\theta = \frac{9}{20}\pi$ or $\frac{\pi}{2}$

Therefore, $z = \sqrt{2}e^{i\frac{9}{20}\pi}$ or $\sqrt{2}e^{i\frac{\pi}{2}}$.

(2). $\frac{(1+i)^2}{1-i} = \frac{(1+i)^2(1+i)}{(1-i)(1+i)} = \frac{(1+2i-1)(1+i)}{1+1} = \frac{2i(1+i)}{2} = -1+i$

$\left|\frac{(1+i)^2}{1-i}\right| = \sqrt{-1^2+1^2} = \sqrt{2}$ and

$\theta = argz = tan^{-1}\frac{y}{x} = tan^{-1}(-1) = \frac{3\pi}{4} + 2n\pi$ for $(n = 0, 1, 2, \ldots)$. In polar form, $z = r(cos\theta + sin\theta)$. Thus, $z = \sqrt{2}\left(cos\frac{3\pi}{4} + isin\frac{3\pi}{4}\right)$ or $\sqrt{2}\left\{-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right\}$.

In exponential form, $z = re^{i\theta} = \sqrt{2}e^{i\frac{3\pi}{4}}$.

(3). Let

$z = x + iy = r(cos\theta + sin\theta)$. Then, $2 = rcos\theta$ and $1 = rsin\theta$

$|z^2| = (rcos\theta)^2 + (rsin\theta)^2 = x^2 + y^2 = r^2(cos^2\theta + sin^2\theta) \Rightarrow x^2 + y^2 = r^2$.

Then, $r^2 = 2^2 + 1^2 = 5$ and $r = \sqrt{5}$. Hence, $cos\theta = \frac{y}{r} = \frac{2}{\sqrt{5}}$ and $sin\theta = \frac{y}{r} = \frac{1}{\sqrt{5}}$. Therefore, $z = r(cos\theta + sin\theta) = \sqrt{5}\left\{\frac{2}{\sqrt{5}} + i\frac{1}{\sqrt{5}}\right\}$ is the required polar form.

Triangle inequalities: Let $z_1, z_2 \in \mathbb{C}$, then the triangle inequalities are given as

$$\begin{cases} |z_1 + z_2| & \leq |z_1| + |z_2|, \\ |z_1 - z_2| & \geq |z_1| - |z_2|. \end{cases} \tag{12}$$

This means that geometrically, no side of a triangle is greater in length than the sum of the lengths of the other two sides or less than the difference of the lengths of the other two sides.

We verify the triangle inequalities in the following Activity:

Activity 2

(1). Let $z_1 = 1 + 2i$ and $z_2 = -4 - 5i$. Verify the triangle inequalities.

Solutions

$z_1 + z_2| = |(1 + 2i) + (-4 - 5i)| = | - 3 - 3i| \Rightarrow |z_1 + z_2| = \sqrt{9 + 9} = 3\sqrt{2}$

and $|z_1| + |z_2| = \sqrt{1 + 4} + \sqrt{16 + 25} = \sqrt{5} + \sqrt{41}$

$z_1 + z_2| = \sqrt{18} = 4.243 < \sqrt{5} + \sqrt{41} = 8.639 = |z_1| + |z_2$

$|z_1 - z_2| = \sqrt{74} = 8.6 > 4.2 = |z1| - |z2|$ which confirm that the inequalities hold true.

We can however extend the triangle inequalities to arbitrary sum using mathematical induction as follows:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n| .$$

Summary

In this unit, you have been taught how to convert complex numbers from exponential nth root to polar form and vice versa. You also learnt the triangle inequalities for complex numbers.

Self Assessment Questions

(1). State and prove the triangle inequalities.

(2). Find the square root of $5 - 10i$ in polar form.

(3). What is the relationship between the three forms of complex numbers?

(4). Solve $x^5 - 1 + i = 0$.

Tutor Marked Assignment

(1). Let $z = 2 + 2i$, express z in polar and exponential forms and find the cube root.

Solve $x^7 - i = 0$.

(2). Express $z = -3 + 3i$ in polar and exponential forms and find its fifth root.

(3). Find the square root of $3 + 4i$

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UNIT 2

Further Operations On Complex Numbers

Introduction

In the previous unit, you were taught the basic operations of complex numbers. In this unit, you will learn multiplication and division of complex numbers.

Learning Outcomes

At the end of this unit, you should be able to:

1. Find the square root of a complex number
2. Determine multiplication and division of complex numbers in polar form
3. Express a complex number in Cartesian form and polar form

Main Content

Let $z = x + iy$, the real part of z, $\text{Re}z = x$ and the imaginary part of z, $im z = y$ where $I = \sqrt{-1}$.

Multiplication and Division in Polar Form

Suppose $z_1 = r_1(\cos\alpha + i\sin\alpha)$ and $z_2 = r_2(\cos\beta + i\sin\beta)$, then

Multiplication: Let $z = z_1z_2$. Then,

$$z_1z_2 = r_1r_2\{\cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta - \sin\alpha\sin\beta\}$$

$$= r_1r_2\{(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\}$$

$$= r_1r_2\{\cos(\alpha + \beta) + i\sin(\alpha + \beta)\}$$

$$z_1z_2 = r_1r_2[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]. \qquad (13)$$

and

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}z_1 + \mathrm{Arg}z_2 \qquad (15)$$

Division: Suppose $z = \frac{z_1}{z_2}$ is the number satisfying $zz_2 = z_1$ so that $|zz_2| = |z||z_2| = |z_1|$,

$$\mathrm{Arg}(zz_2) = \mathrm{Arg}z + \mathrm{Arg}z_2 = \mathrm{Arg}z_1$$

which yields

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad (provided \quad z_2 \neq 0) \qquad (16)$$

and thus

$$Arg\frac{z_1}{z_2} = Argz_1 - Argz_2. \qquad (17)$$

Furthermore,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{cos(\alpha - \beta) + isin(\alpha - \beta)\}. \qquad (18)$$

Integer Power of of a Complex Number z

Following 13 and 18, we have that if $z_1 = z_2$ and $\alpha = \beta$. Then, $z^2 = r^2(\cos 2\alpha + i\sin 2\alpha)$ and $z^{-2} = r^{-2}[\cos(-2\alpha) + i\sin(-2\alpha)] = \frac{1}{z^2}$

Thus, $z^{-2} = r^{-2}(\cos^2\alpha - i\sin^2\alpha)$. Generally therefore, we have that for any integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta). \qquad (19)$$

Similarly,

$$z^{\frac{1}{n}} = \{cos\theta + i\sin\theta\}^{\frac{1}{n}} = r^{\frac{1}{n}}\left\{cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right\}$$

where $k = 0, \pm 1, \pm 2, \ldots$ so that $\frac{\theta_0 + 2k\pi}{n}$ and $n\theta = \theta_0 + 2k\pi$.

Activity 1

(1). Express $2 + 3i$ in polar and exponential forms

Solutions

Let $z = 2 + 3i$. So, $x = r\cos\theta$ and $y = r\sin\theta$. Hence, $2 = r\cos\theta$ and $3 = r\sin\theta$.

$$r^2cos\theta + r^2sin\theta = 4 + 9 \Rightarrow \quad r = \sqrt{13}.$$

Thus, $cos\theta = \frac{x}{r} = \frac{2}{\sqrt{13}}$ *and* $sin\theta = \frac{y}{r} = \frac{3}{\sqrt{13}}$.

Therefore, $z = r(cos\theta + isin\theta) = \sqrt{13}\left(\frac{2}{\sqrt{13}} + i\frac{3}{\sqrt{13}}\right)$.

Square Root of a Complex Number

Let $z_1 = \alpha + i\beta$

be a complex number whose square root is given as $z_2 = x + iy$.

Then, $\sqrt{z_1} = z_2$ which implies that $\sqrt{\alpha + i\beta} = x + iy$ and

$$\alpha + i\beta = (x + iy)^2 = (x^2 - y^2) + i2xy.$$

Thus, $\alpha + i\beta = (x^2 - y^2) + i2xy$ so that $\alpha = x^2 - y^2$ and $\beta = 2xy$.

And $x^2 - y^2 = \alpha$ and $2xy = \beta$. So that $\alpha^2 + \beta^2 = (x^2 - y^2)^2 + (2xy)^2$

Hence, $x^2 - y^2 = \sqrt{\alpha^2 + \beta^2}$ and $2x^2 = \alpha + \sqrt{\alpha^2 + \beta^2}$ which implies that

$$x = \sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}.$$

Summary

In this unit, you have learnt the nth root of complex numbers, complex numbers in polar and exponential forms as well as the relationship between the two forms. You equally learnt the problems on the nth root of complex numbers

Self Assessment Questions

(1). Express $1 - 4i$ in polar and exponential forms

(2). Find the square root of $5 - 10i$.

(3). Express $1 - 2i$ in polar form

Tutor Marked Assignment

(1). Let $z = 2 + 2i$, express z in polar form and hence find the cube root.
(2). Express $z = -3 + 3i$ in polar form and exponential form. Hence find its fifth root.
(3). Express $z = 5 + 2i$ in polar and exponential forms.
(4). Find the square root of $1 - 3i$.
(5). What are the circular functions of complex numbers?
(6). State the relationship between circular and hyperbolic functions.
(7). Express the of each of the following complex numbers in Cartesian form.

$$e^{\frac{i3\pi}{4}}, \text{ (b). } e^{\frac{i\pi}{3}} \text{ (c). and } cos\tfrac{\pi}{4} + isin\tfrac{\pi}{4}$$

(8). Let $z = re^{i\theta}$ and $\omega = \rho e^{i\alpha}$,
find $\frac{z}{\omega}$ and z^{-2}. Given that $\theta = \frac{3\pi}{4}$, $\alpha = \frac{\pi}{2}$, $r = 3$ and $\omega = 5$.

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UNIT 5

De Moivre's Formula and Its Applications

Introduction

De Moivre's formula has a wide range of applications in Mathematics and other fields of science. In this unit, I will teach you the applications of De Moivre's formula in complex analysis in finding the nth root of complex numbers

Learning Outcomes

At the end of this unit, you should be able to:

1. Find the square root of a complex number
2. Find the cube root of a complex number
3. Find the nth root of a complex number

Main Content

Suppose $|z| = r = 1$, we
have a unit circle and (19) yields the famous formula of De Moivre

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

This formula helps in expressing $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$. For instance if

(1). If $n = 2$, $\cos 2\theta = \cos^2\theta - \sin^2\theta = Re(z)$ and $\sin 2\theta = 2\cos\theta\sin\theta = Im(z)$.

(2). If $n = 3$, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = Re(z)$
and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = Im(z)$.

Nth root of a complex number

By De Moivre's theorem,

$$z^k = [r(\cos\alpha + i\sin\alpha)]^k = r^k(\cos k\alpha + i\sin k\alpha).$$

From the knowledge of polar form using De Moivre's theorem, *nth*−root of a complex number in a polar form is established for $n \in \mathbb{Q}$ being a fractional power (i.e. $n = \frac{1}{k}$)such as:

(1). Let $n = \frac{1}{2}$, we have square root.
(2). If $n = \frac{1}{3}$, we have cube root.
(3). For $n = \frac{1}{4}$, we have fourth root.
(4). If $n = \frac{1}{5}$, we have fifth root.
⋮
(n). If $n = \frac{1}{k}$, we have *kth* root.
In general, nth root of z is derived from the expression $z^n = \sqrt[n]{z}$

Activity 1

(1). Find the square root of $z = 27(\cos 120^0 + i\sin 120^0)$.
(2). Express
$4 + 2i$ in polar and exponential forms and hence find the cube root.

Solutions

(1). The square root
of z is $z^{\frac{1}{2}} = \{27\,(cos120^0 + isin120^0)\}^{\frac{1}{2}} = \sqrt{27}\left\{cos\frac{120^0}{2} + isin\frac{120^0}{2}\right\}$
$= 3\sqrt{3}(cos60^0 + isin60^0)$ or $3\sqrt{3}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$.

(2). Let $z = 4 + 2i$, $z = r = \sqrt{4^2 + 2^2} = 2\sqrt{5}$ and $\theta = 26.56^0$.
In polar form, $z = 2\sqrt{5}(cos26.56^0 + isin26.56^0)$
and in exponential forms, $z = 2\sqrt{5}e^{i26.56^0}$.

Suppose θ is "1 rev. $+26.56^0$", "2 revs. $+26.56^0$", "3 revs. $+26.56^0$", ..., we have 26.56^0, 386.56^0, 746.56^0, 1106.56^0, ... which all yield $\frac{1}{2}$ or 0.5 in the same position. Using De Moivre's theorem in each case therefore, we have that:

$$z^{\frac{1}{3}} = \left\{20^{\frac{1}{3}}, \frac{26.56^0}{3}\right\};\ \left\{20^{\frac{1}{3}}, \frac{386.56^0}{3}\right\};\ \left\{20^{\frac{1}{3}}, \frac{746.56^0}{3}\right\};\ \left\{20^{\frac{1}{3}}, \frac{1106.56^0}{3}\right\};\ \ldots$$

Remark 2

Note that $2k\pi + \theta$ is an n revolutions plus θ, the amplitude (or argand) of the complex number z. Hence we have that

(1). When $k = 1$, then $2\pi + \theta = 360^0 + \theta$ which is $(1\text{rev.} + \theta)$.

(2). When $k = 2$, then $4\pi + \theta = 720^0 + \theta$ which is $(2\text{revs.} + \theta)$.

(3). When $k = 3$, then $6\pi + \theta = 1080^0 + \theta$ which is $(3\text{revs.} + \theta)$.

(4). When $k = 4$, then $6\pi + \theta = 1440^0 + \theta$ which is $(4\text{revs.} + \theta)$

(m). When $k = m - 1$,

then $2(m - 1)\pi + \theta = 10800 + \theta$ which is $[(m - 1)\text{revs.} + \theta]$.

It is usually convenient

to express a given complex number z in polar form and use

De Moivre's theorem to find its *nth* root with $k = 0, 1, 2, \ldots, m - 1$.

Activitv 2

(1). Solve $x^3 + 1 - i = 0$

Solution

(1). Let $x^3 + 1 - i = 0$, then $x^3 = -1 + i \Rightarrow z = -1 + i$, $|z| = \sqrt{2}$ and $\theta = tan^{-1}\left(\frac{1}{-1}\right) = tan^{-1}(-1) = 135^0 = \frac{3\pi}{4}$ or $315^0 = \frac{7\pi}{4}$. But z lies in the fourth quadrant, hence $\theta = 315^0$ or $\frac{7\pi}{4}$ is the principal argument of z. Thus,

$$z = r(cos\theta + isin\theta) = \sqrt{2}\left(cos\frac{7\pi}{4} + isin\frac{7\pi}{4}\right)$$

$$or \quad x^3 = \sqrt{2}cos\left(2k\pi + \frac{7\pi}{4}\right) + i\sqrt{2}sin\left(2k\pi + \frac{7\pi}{4}\right)$$

$$x = \left[\sqrt{2}\left\{cos\left(2k\pi + \frac{7\pi}{4}\right) + isin\left(2k\pi + \frac{7\pi}{4}\right)\right\}\right]^{\frac{1}{3}}$$

$$= \sqrt{2}\left\{cos\left(2k\pi + \frac{7\pi}{4}\right)\frac{1}{3} + isin\left(2k\pi + \frac{7\pi}{4}\right)\frac{1}{3}\right\}$$

$= \left\{2\sqrt{5}, 8.85^0\right\};\ \left\{2\sqrt{5}, 128.85^0\right\};\ \left\{2\sqrt{5}, 248.85^0\right\};\ \left\{2\sqrt{5}, 368.85^0\right\} = \left\{2\sqrt{5}, 8.85^0\right\};\ \ldots.$

The fourth value is a repetition of the first value.

Hence we have that $z_1 = (2\sqrt{5}, 8.85^0)$, $z_2 = (2\sqrt{5}, 128.85^0)$ and $z_3 = (2\sqrt{5}, 248.85^0)$.

In general, $cos\theta + isin\theta = cos(2k\pi + \theta) + isin(2k\pi + \theta)$ where $k \in \mathbb{Z}$. By De Moivre's theorem therefore,

$$[cos\theta + isin\theta]^{\frac{1}{n}} = [cos(2k\pi + \theta) + isin(2k\pi + \theta)]^{\frac{1}{n}} = cos\frac{2k\pi + \theta}{n} + isin\frac{2k\pi + \theta}{n} \tag{20}$$

where $k = 0, 1, 2, \ldots, m-1$ successively. Hence we have:

(1). For $k = 0$; we have $cos\frac{\theta}{m} + isin\frac{\theta}{m}$.

(2). If $k = 1$; we have $cos\left(\frac{2\pi+\theta}{m}\right) + isin\left(\frac{2\pi+\theta}{m}\right)$.

(3). For $k = 2$; we have $cos\left(\frac{4\pi+\theta}{m}\right) + isin\left(\frac{4\pi+\theta}{m}\right)$.

(4). If $k = 3$; we have $cos\left(\frac{6\pi+\theta}{m}\right) + isin\left(\frac{6\pi+\theta}{m}\right)$.

$\vdots$

(m-1). For $k = m-1$; we have $cos\left(\frac{2(m-1)\pi+\theta}{m}\right) + isin\left(\frac{2(m-1)\pi+\theta}{m}\right)$.

(m). For $k = m$; we have $cos\left(\frac{2m\pi+\theta}{m}\right) + isin\left(\frac{2k\pi+\theta}{m}\right) =$
$cos\left(2k\pi + \frac{\theta}{m}\right) + isin\left(2k\pi + \frac{2\theta}{m}\right) = cos\frac{\theta}{m} + isin\frac{\theta}{m})$
which yields same value as for $k = 0$ above.

The value of $(cos\theta + isin\theta)^{\frac{1}{k}}$ for $k = m, m+1, m+2, \ldots$ are just a repetition of the first n value as obtained above.

$= \sqrt{2}\left[cos\,(8k+7)\,\frac{\pi}{12} + isin\,(8k+7)\,\frac{\pi}{12}\right]$ where $k = 0, 1,$ and 2 which yields the following results:

$$x_1 = \sqrt{2}\left[cos\frac{7\pi}{12} + isin\frac{7\pi}{12}\right]\ ;\ \ x_2 = \sqrt{2}\left[cos\frac{15\pi}{12} + isin\frac{15\pi}{12}\right]\quad and$$

$$x_3 = \sqrt{2}\left[cos\frac{23\pi}{12} + isin\frac{23\pi}{12}\right].$$

Summary

In this unit, you have learnt the square root of complex numbers, cube root of complex numbers up to the nth root of complex numbers.

Self Assessment Questions

(1). State and prove the triangle inequalities.

(2). Find the square root of 5 – 10i in polar form.

(3). Solve $x^7 - i = 0$

(4). Solve x5 – 1 + i = 0.

(5). Express z = −3 + 3i in polar and exponential forms. Hence find its fifth root.

(6). Express z = −3 + 3i in polar form. Hence find its cube root.

Tutor Marked Assignment

(1). Let $z = 2 + 2i$, express z in polar form and hence find the fifth root.
(2). Express $z = -3 + 3i$ in polar and exponential form. Hence find its fifth root.

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Module 5

CIRCULAR MEASURE

Units

Unit 1 - Radian measure of an angle, conversion from degree to radian and radian to degree

Unit 2 - Definite Integral

Unit 3 - Integration by Substitution

UNIT 1

Application III: Volume of solids of revolution

Introduction

The measurement of angles in degrees goes back to antiquity. It may have arisen from the idea that there were roughly 360 days in a year. In any event, both the Greeks and the Indians divided the angle in a circle into 360 equal parts, which we now call degrees. They further divided each degree into 60 equal parts called minutes and divided each minute into 60 seconds. This way of measuring angles is very inconvenient and it was realised in the 16th century (or even before) that it was better to measure angles via arc length. On this note, one radian written as 1^c (where the c refers to circular measure) is defined to be the angle subtended at the centre of a unit circle by a unit arc length on the circumference.

Learning Outcomes

At the end of this unit, you should be able to:

1. Define angles in both radian and degrees
2. Convert from degrees to radians
3. Convert from radians to degrees

Main Content

Apart from the unit of angle measure known as degree, there is another more relevant unit of angle measure called **Radian**. A Radian is a unit of circular measure whose angle is subtended at the center by an arc equal to its radius R

For instance, consider the circle below with an arc AB, radius R and center 0. If the length of the arc AB=radius r, then the angle subtended at the center is called **Radian.**

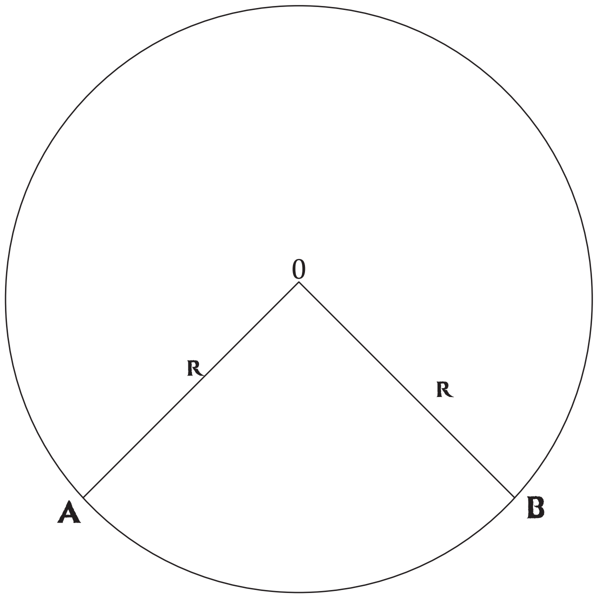

fig 138

Now, if the angle subtended at the center of the circle is 180^0, then this implies that 180^0 = _ (length of the arc)

$\Rightarrow$ 180^0 = — ($2\pi r$)

$\Rightarrow$ $180^0 = \pi r$ (or $\pi r = 180^0$)

So, if r = 1, then $1\pi = 180^0$

$\Rightarrow$ 1radian = $180^0\pi = 57.3^0$ (1d.p.)

or π radian = 180^0

So that $1^0 = \frac{\pi}{180^0}$ radian $\approx$ 0.0175 radian (3s.f.)

Conversion from Degrees to Radians

To convert angles in degrees to angles in radians, multiply the angle by the factor $\frac{\pi}{180^0}$ radian

Activity 1

Convert each of the following angles in degrees to their radians form:

(a.) 45^0 (b.) 90^0 (c.) 360^0 (d.) 15^0 (e.) 480^0

Solution

(a) $45^0 = 45^0 \times \frac{\pi}{180^0}$ radians$= \frac{\pi}{4}$ radians

(b) $90^0 = 90^0 \times \frac{\pi}{180^0}$ radians$= \frac{\pi}{2}$ radians

The rests are left as exercise.

Conversion from Radians to Degrees

To convert angles in radians to angles in degrees, multiply the angle by the factor $\frac{180^0}{\pi}$

Activity 2

Convert each of the following angles in radians to their degrees form

(a.) 12π radian (b.) — π radian (c.) —π radian (d.) 1.5π rad.

Solution ! !

a) 12π radians$= 12\pi \times \frac{180^0}{\pi} = 2160^0$

! !

(b) 23π radians$= 23\pi \times \frac{180^0}{\pi} = 120^0$

The rests are left as exercise for the students

Summary

In this unit, you have learnt angles in radians and degrees and how to convert from radians to degrees and vice versa.

Self Assessment Questions

1. Change the following angles in degrees to radians

   (a.) 21^0 (b.) 80^0 (c.) 130^0 (d.) 210^0 (e.) 300^0 (d.) 450^0

2. Change the following angles in radians to degrees

   (a.) — π radian (b.) —π radian (c.) —π radian (d.) —π rad (e.) —π rad.

Tutor Marked Assignment

Express in radians

1. 135^0

2. 270^0

3. 100^0

Express in degrees

1. $\frac{\pi}{6}$
2. $\frac{\pi}{4}$
3. $\frac{\pi}{3}$

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UNIT 2

Concept of Circular Measure

Introduction

In this unit, I will teach you the methods of measuring angles. This is done by treating them as the angle formed by a sector of a circle.

Learning Outcomes

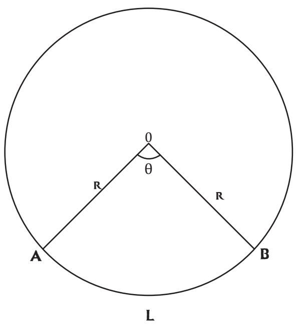
At the end of this unit, you should be able to:

1. find the length of an arc
2. determine the area of a sector of a circle
3. calculate the area of a segment of a circle
4. determine the length of a chord

Main Content

In the circle below, 0 is the center of the circle, θ is the angle subtended at the center and AB is an arc with length L. It is obvious that the length L of the arc AB varies directly as the angle θ subtended at the center of the circle i.e **L**

So when L = the circumference of the circle, then $\theta = 360^0 = 2\pi$radians, to calculate the length L of an AB, $L = r\theta$ (where L is the length of the arc AB, r is the radius, θ is the angle subtended at the center in radian)



Area of a Sector of a Circle

In a similar manner, we can deduce the formula to finding area of a sector of a circle given as:

A R (θ in radians)

AREA of △OAB R SIN (θ in radians) 16

Area of a Segment

To calculate the area of a segment of a circle, the formula to be applied is

A R SIN (θ in radians)

Length of a Chord

To calculate the length of a chord of a circle, the formula to be applied is

L RSIN — (θ in radians)

Activity 1

Given that the arc length of a sector with radius 4cm is 3cm, find
(I) The angle subtended at the centre of the sector
(ii)The area of the sector

Solution

$l = r\theta$

$3 = 4\theta$

$\theta =$ —

A -R

A — —

$A = 6cm^2$

Summary

In this unit, you have learnt that radian is the angle subtended at the centre of a unit circle by an arc of unit length.

Self Assessment Questions

1. A sector of a circle

is bounded by radii of length 10cm and arc of length 6cm. Calculate

(a) in radian, the angle subtended at the center

(b) the area of the sector.

Tutor Marked Assignment

1. AB is an arc with center O and radius 24cm. If $\angle AOB = \frac{5\pi}{6}$, calculate correct to two decimal places

(a) the perimeter of the sector AOB

(b) the area of sector AOB (take = 3.142)

2. A chord P Q of a circle of radius 18cm subtends and angle of $\_\pi$ radians at the center O of the circle, calculate to 3 significant figures

References

A guide for teachers - Years 11 and 12 Functions: Module 7. Pure Mathematics for Advanced level, Heinemann Educational Books Schools Publishing, 90-106.

Peter Brown and Michael Evans(2013). College Mathematics, Trigonometric functions and circular measure, Supporting Australian Mathematics Project, 28-33.

Further Reading

Peter Brown and Michael Evans(2013). College Mathematics, Trigonometric functions and circular measure, Supporting Australian Mathematics Project, 28-33.

UNIT 3

Trigonometric Function of Angles

Introduction

If we fix an acute angle θ, then all right-angled triangles that have θ as one of their angles are similar. So, in all such triangles, corresponding pairs of sides are in the same ratio.

The side opposite the right angle is called the hypotenuse. We label the side opposite θ as the opposite and the remaining side as the adjacent.

Learning Outcomes

At the end of this unit, you should be able to:

1. find the trigonometric ratio of acute angles
2. find the inverse of trigonometric ratios
3. evaluate the trigonometric ratio of any angle
4. determine the trigonometric ratio of special angle
5. express correctly the addition formulae
6. express trigonometric expression in surd form
7. solve trigonometric expression using factor formulae

Main Content

For an acute angle θ, the trigonometric function are illustrated as follows. (See Figure 7 below)

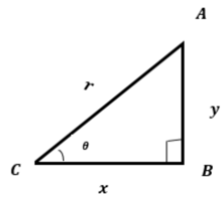


Figure 7

$$
\left.\begin{aligned}
\sin\theta &= \frac{\text{Opposite}}{\text{Hypotenus}} = \frac{y}{r} \\
\cos\theta &= \frac{\text{Adjacent}}{\text{Hypotenus}} = \frac{x}{r} \\
\tan\theta &= \frac{\text{Opposite}}{\text{Adjacent}} = \frac{y}{x}
\end{aligned}\right\}
$$

Inverse of Trigonometric Ratios

The corresponding inverse

functions of the basic trigonometric ratios are defined as follows:

$$
\left.\begin{aligned}
&\textbf{Cosecant Functions} \\
&\qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} \\
&\textbf{Secant Functions} \\
&\qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} \\
&\textbf{Cotangent Functions} \\
&\qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}
\end{aligned}\right\}
$$

Activity 1

If $\cos\theta = \frac{4}{5}$ and θ is acute. Find the following: (a) $\tan\theta$ (b) $\csc\theta$

Solution

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{4}{5}$$

Using Fig. 7 r = 5, x = 4 and by Pythagoras theorem

X Y R

Y

Y p

Y p

Y

Let OP be aby line in the plane through O and let ⟨POX =   and x and y be the co-ordinates of P

Let OP = r which is measured positive for all positions of P.

If P is located in the 1st quadrant as seen in Figure 9,

x, y, r are positive so that sine, cosine and tangent are all positive .

If P is in the 2nd quadrant, y is positive and x

is negative so that sine is positive , cosine and tangent are negative

If P is located in the 3rd quadrant, x and y are negative, therefore, sine and cosine are negative leaving tangent to be positive.

Lastly, If P is in the 4th quadrant, x is positive, y is negative so that sine and tangent are negative, then cosine is positive.

With the aid of this diagram, it can easily be shown that each trigonometric ratio remains positive.

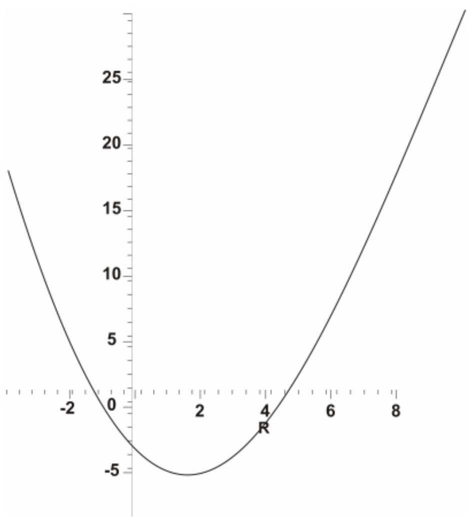


Figure 10

From Fig. 8

$$\sin\theta = \frac{y}{r}; \quad \cos\theta = \frac{x}{r}; \quad \tan\theta = \frac{y}{x} \tag{26}$$

Hence, x = 4, r = 5 and y = 3.

$\tan\theta \quad \dfrac{\text{Opposite}}{\text{Adjacent}} \quad \dfrac{y}{x} \quad —$

$\csc\theta \quad \dfrac{1}{\sin\theta} \quad \dfrac{r}{y}$

$\sin\theta \quad \dfrac{\text{Opposite}}{\text{Hypotenuse}} \quad \dfrac{y}{r} \quad —$

$\csc\theta \quad \dfrac{1}{—} \quad 1 \div — \quad —$

Trigonometric Ratios of any Angle

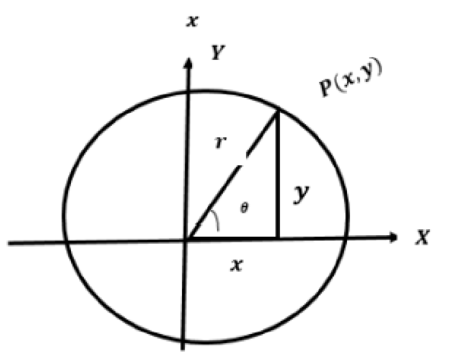


Figure 8

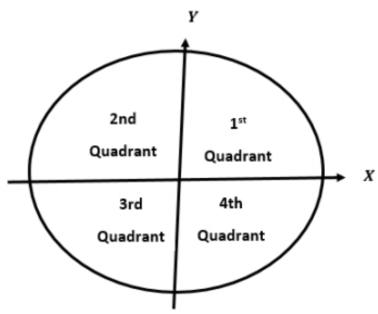


Figure 9

Also,

$$\left.\begin{array}{lll} \sin\ \theta & \frac{Y}{R} & \sin\theta \\ \cos\ \theta & \frac{X}{R} & \cos\theta \\ \tan\ \theta & \frac{Y}{X} & \tan\theta \end{array}\right\} \qquad (27)$$

180 + θ which is on the third quadrant implies

$$\left.\begin{array}{lll} \sin\ {}^{0}\ \theta & \frac{Y}{R} & \sin\theta \\ \cos\ {}^{0}\ \theta & \frac{X}{R} & \cos\theta \\ \tan\ {}^{0}\ \theta & \frac{Y}{X} & \tan\theta \end{array}\right\} \qquad (28)$$

180 - θ which is located on the second quadrant implies

$$\left.\begin{array}{lll} \sin\ {}^{0}\ \theta & \frac{Y}{R} & \sin\theta \\ \cos\ {}^{0}\ \theta & \frac{R}{X} & \cos\theta \\ \tan\ {}^{0}\ \theta & \frac{Y}{X} & \tan\theta \end{array}\right\} \qquad (29)$$

Alternatively, replace θ *by* - θ

$$\begin{array}{lll} \sin\ {}^{0}\ \theta & \sin(\ \theta) & \sin\theta \\ \cos\ {}^{0}\ \theta & \cos(\ \theta) & \cos\theta \\ \tan\ {}^{0}\ \theta & \tan(\ \theta) & \tan\theta \end{array}$$

LASTLY AT THE FOURTH QUADRANT

REPLACING BY

$$\begin{array}{lll} \sin\ {}^{0} & \sin & \sin \\ \cos\ {}^{0} & \cos & \cos \\ \tan\ {}^{0} & \tan & \tan \end{array}$$

Activity 2

Evaluate the following

(a) sin 210^0

(b)cos 240^0

(c) tan 150^0

Solution

(a) sin 210^0

〈210^0 is located in the 3rd quadrant and where sin θ is negative.

$$\sin 210^0 = \sin(180^0 + \theta)$$

$$= \sin(180^0 + 30^0)$$

$$= -\sin 30^0$$

Using tables to evaluate

$$-\sin 30^0 = -0.5$$

(b) cos 240^0

〈240^0 is located in the 3rd quadrant where cos θ is negative.

$$\cos 240^0 = \cos(180^0 + \theta)$$

$= \cos(180^0 + 60)$

$= -\cos 60^0$

Using tables to evaluate

$$-\cos 60^0 = -0.5$$

Activity 3

Find the value of θ between 0^0 and 360^0 that satisfies sin θ = -0.8660^0

Solution

(a) sin θ is negative at the 3rd quadrant and 4th quadrant.

From the table, the acute angle θ that has its sine to be 0.86660 is 60^0

$\sin 60^0 = 0.8660$

In the 3rd quadrant,we have

$$\theta = 180^0 + 60^0 = 240^0$$

In the 4th quadrant

$$\theta = 360^0 - 60^0 = 300^0$$

Hence, $\theta = 240^0$ or 300^0

$$= -\sin 30^0$$

Trigonometric Ratios of Special Angles (30^0 , 45^0) and 60^0

We shall make use of the elementary geometry to compute the value of trigonometric ratios for some special angles. Consider an Isosceles triangle right angled at C, with $\langle A = \langle B = 45^0$ and $|AC| = |BC| = 1$.

By Pythagoras theorem $AB = \sqrt{2}$

$$\sin 45^0 = \frac{\text{Opposite}}{\text{Hypotenus}} = \frac{AC}{AB}$$

$$\cos 45^0 = \frac{\text{Adjacent}}{\text{Hypotenus}} = \frac{BC}{AB}$$

$$\tan 45^0 = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{AC}{BC}$$

Hence $\sin 45^0 = \cos 45^0 = \dfrac{1}{\sqrt{2}}$.$\tan 45^0 = 1$

Consider an equilateral triangle *ABC*

$\sin 60^0 = \frac{\sqrt{3}}{2}$ $\tan 60^0 = \sqrt{3}$ $\cos 60^0 = \frac{1}{2}$

$\csc 60^0 = \frac{2}{\sqrt{3}}$ $\sec 60^0 = 2$ $\cot 60^0 = \frac{1}{\sqrt{3}}$

Trigonometric ratios for 30^0can be obtained for $\langle ABD$.
Using AD as its bases and *BD* as height

$\sin 30^0 = \dfrac{1}{2}$: $\cos 30^0 = \dfrac{\sqrt{3}}{2}$: $\tan 30^0 = \dfrac{1}{\sqrt{3}}$

Activity 4

Evaluate each of the following and leave your answer in surd form:

(a) sin 75^0

(b) tan 15^0

Solution

A SIN ^{0} SIN ^{0} ^{0}

SIN A B SIN A COS B COS A SIN B

SIN ^{0} SIN ^{0} COS ^{0} COS ^{0} SIN ^{0}

$$\frac{\sqrt{2}}{2} \text{ x } \frac{\sqrt{3}}{2} \quad \frac{\sqrt{2}}{2} \text{ x } \frac{1}{2}$$

$$\frac{\sqrt{6}}{4} \quad \frac{\sqrt{2}}{4} \quad \frac{\sqrt{6} \quad \sqrt{2}}{4}$$

B TAN ^{0} TAN ^{0} ^{0}

TAN A B $\dfrac{\text{TAN A} \quad \text{TAN B}}{\text{TAN A TAN B}}$

$\dfrac{\text{TAN}^{\;0} \quad \text{TAN}^{\;0} \quad \text{TAN}^{\;0}}{\text{TAN}^{\;0} \text{ TAN}^{\;0}}$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \left(1 \times \frac{1}{\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Activity 5

Express the following as a product of two trigonometric ratios:

(a) sin 8θ + sin 6θ (b) cos 8θ + cos 4θ .

Solution

A SIN SIN SIN M SIN N WHERE M N

SIN M SIN N SIN M N COS M N

SIN SIN SIN

________ COS ________

SIN _____ COS ___

SIN COS

The Addition and Factor Formulae

Some fundamental identities involving trigonometric functions shall be studied

The addition formulae is summarized below:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

The Factor Formula

The sums and differences of sines and cosines can be expressed as products of sines and cosines and vice-versa from the addition theorem.

$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$$

$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$$

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$$

Sines and Cosines as a Sum or a Difference

$$A \pm B = M, N$$

$$A = \frac{M + N}{2}$$

$$B = \frac{M - N}{2}$$

$$\sin M + \sin N = 2\sin\frac{M+N}{2}\cos\frac{M-N}{2}$$

$$\sin M - \sin N = 2\cos\frac{M+N}{2}\sin\frac{M-N}{2}$$

$$\cos M + \cos N = 2\cos\frac{M+N}{2}\cos\frac{M-N}{2}$$

$$\cos M - \cos N = -2\sin\frac{M+N}{2}\sin\frac{M-N}{2}$$

B COS COS COS M COS N WHERE M N

COS M COS N COS M N COS M N

COS _______ COS _______

COS COS

Summary

In this unit, you have learnt trigonometry functions of angles.

Note: SOHCAHTOA means

$$\sin = \frac{\text{Opposite}}{\text{Hypotenus}}$$

$$\cos = \frac{\text{Adjacent}}{\text{Hypotenus}}$$

$$\tan = \frac{\text{Opposite}}{\text{Adjacent}}$$

Self Assessment Questions

1. Evaluate the following: (i) $\tan 150^0$ (ii) $\cos 300^0$
2. Find the value of θ between 0^0 and 360^0 that satisfies $\cos\theta = 0.7071$
3. Show that $\tan\theta + \cot\theta = \sec\theta \csc\theta$
4. Evaluate each of the following leaving your answers in surd form.
(i) $\cos 105^0$ (ii) $\tan 75^0$
5. Show that $\sin 2\alpha \cos 4\alpha + \sin 3\alpha \cos 9\alpha = \frac{1}{2}(\sin 12\alpha - \sin 2\alpha)$
6. Express $\cos 10\theta - \cos 5\theta$ as a product of two trigonometric ratio.
7. Show that $2\cos 5\theta \cos\theta = \cos 6\theta + \cos 4\theta$
8. Show that $\sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$
9. verify that $1 - \dfrac{\sin\theta \tan\theta}{1+\sec\theta} = \cos\theta$
10. Express in terms of trigonometric ratios of positive acute angles
(i) $\sin -500^0$ (ii) $\cos -170^0$

Tutor Marked Assignment

1. If $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$, Evaluate $\tan(\alpha + \beta)$.
2. If $\sin\alpha = \frac{5}{13}$, $\cos\beta = \frac{1}{5}$, Find the possible values of $\sin(\alpha + \beta)$.

3. Express in terms of the trigonometric ratios of positive acute angles

(i) $\cos -120^\circ$ (ii) $\tan 260^\circ$ (iii) $\sin -400^\circ$

4. Show that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$

5. If $\tan(A + B) = \frac{4}{3}$ and $\tan A = \frac{1}{2}$, Evaluate tan B

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Bunday, B.D. and Mulholland, H. (1988). Pure Mathematics for Advanced level, *Heinemann Educational Books Schools Publishing*, 90-106.

Further Reading

Thong, H.S., Chiang, T.G., Meng, K.K. and Ezekute, G.O. (2002). College Mathematics, *Aficana-Fep Publishers limited Nigeria*, 1