MAT 113: ELEMENTARY VECTORS, GEOMETRY AND MECHANICS

University of Ilorin
Centre for Open & Distance Learning
CODL

Published by the Centre for Open and Distance Learning,
University of Ilorin, Nigeria

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From the Vice Chancellor

Courseware development for instructional use by the Centre for Open and Distance Learning (CODL) has been achieved through the dedication of authors and the team involved in quality assurance based on the core values of the University of Ilorin. The availability, relevance and use of the courseware cannot be timelier than now that the whole world has to bring online education to the front burner. A necessary equipping for addressing some of the weaknesses of regular classroom teaching and learning has thus been achieved in this effort.

This basic course material is available in different electronic modes to ease access and use for the students. They are available on the University's website for download to students and others who have interest in learning from the contents. This is UNILORIN CODL's way of extending knowledge and promoting skills acquisition as open source to those who are interested. As expected, graduates of the University of Ilorin are equipped with requisite skills and competencies for excellence in life. That same expectation applies to all users of these learning materials.

Needless to say, that availability and delivery of the courseware to achieve expected CODL goals are of essence. Ultimate attention is paid to quality and excellence in these complementary processes of teaching and learning. Students are confident that they have the best available to them in every sense.

It is hoped that students will make the best use of these valuable course materials.

Professor S. A. Abdulkareem
Vice Chancellor

Foreword

Courseware remains the nerve centre of Open and Distance Learning. Whereas some institutions and tutors depend entirely on Open Educational Resources (OER), CODL at the University of Ilorin considers it necessary to develop its own materials. Rich as OERs are and widely as they are deployed for supporting online education, adding to them in content and quality by individuals and institutions guarantees progress. Doing it in-house as we have done at the University of Ilorin has brought the best out of the Course Development Team across Faculties in the University. Credit must be given to the team for prompt completion and delivery of assigned tasks in spite of their very busy schedules.

The development of the courseware is similar in many ways to the experience of a pregnant woman eagerly looking forward to the D-day when she will put to bed. It is customary that families waiting for the arrival of a new baby usually do so with high hopes. This is the apt description of the eagerness of the University of Ilorin in seeing that the centre for open and distance learning [CODL] takes off.

The Vice-Chancellor, Prof. Sulyman Age Abdulkareem, deserves every accolade for committing huge financial and material resources to the centre. This commitment, no doubt, boosted the efforts of the team. Careful attention to quality standards, ODL compliance and UNILORIN CODL House Style brought the best out from the course development team. Responses to quality assurance with respect to writing, subject matter content, language and instructional design by authors, reviewers, editors and designers, though painstaking, have yielded the course materials now made available primarily to CODL students as open resources.

Aiming at a parity of standards and esteem with regular university programmes is usually an expectation from students on open and distance education programmes. The reason being that stakeholders hold the view that graduates of face-to-face teaching and learning are superior to those exposed to online education. CODL has the dual-mode mandate. This implies a combination of face-to-face with open and distance education. It is in the light of this that our centre has developed its courseware to combine the strength of both modes to bring out the best from the students. CODL students, other categories of students of the University of Ilorin and similar institutions will find the courseware to be their most dependable companion for the acquisition of knowledge, skills and competences in their respective courses and programmes.

Activities, assessments, assignments, exercises, reports, discussions and projects amongst others at various points in the courseware are targeted at achieving the objectives of teaching and learning. The courseware is interactive and directly points the attention of students and users to key issues helpful to their particular learning. Students' understanding has been viewed as a necessary ingredient at every point. Each course has also been broken into modules and their component units in sequential order.

Courseware for the Bachelor of Science in Computer Science housed primarily in the Faculty of Communication and Information Science provide the foundational model for Open and Distance Learning in the Centre for Open and Distance Learning at the University of Ilorin.

At this juncture, I must commend past directors of this great centre for their painstaking efforts at ensuring that it sees the light of the day. Prof. M. O. Yusuf, Prof. A. A. Fajonyomi and Prof. H. O. Owolabi shall always be remembered for doing their best during their respective tenures. May God continually be pleased with them, Aameen.

Bashiru, A. Omipidan
Director, CODL

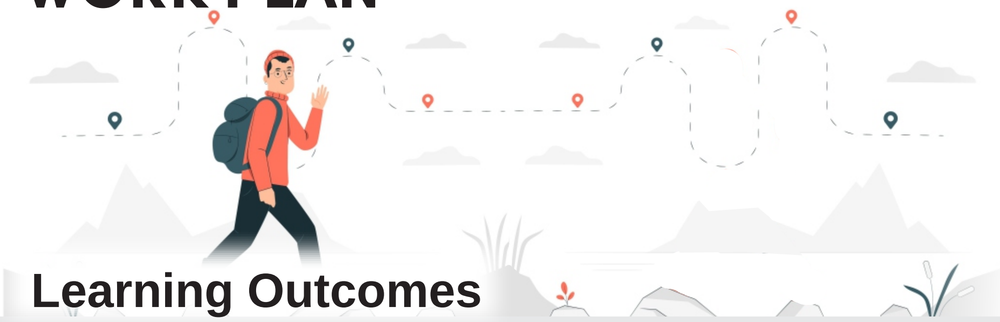
INTRODUCTION

I welcome you to Elementary Differential and Integral Calculus, a second-semester course. It is a 3-credit course that is available to year one undergraduate students in Faculties of Life Sciences, Physical Sciences, Engineering, Education and allied degrees. This course was designed as a foundation course for undergraduate mathematics. It consists of elementary topics from O'level mathematics and introduction to some of the rudiment topics in advanced mathematics. It was prepared with the aim of introducing undergraduate students to some basic theorems and principles that will be useful in advance mathematics.

Course Goal

Your journey through this course will make you understand how to perform some basic operations on vectors of one to three dimensions. You will also be introduced to mechanics.

WORK PLAN



Learning Outcomes

At the end of this course, you should be able to:

- represent vectors in one to three dimensions geometrically;
- perform addition and scalar, multiplication of vectors;
- obtain scalar and vector products of two vectors;

Course Guide

Module 1

Theory of Vector

Unit 1 - Geometric representation of vector

Unit 2 - Component of a vector in three dimensions

Unit 3 - Direction Cosines

Module 2

Addition and Scalar Multiplication of Vector

Unit 1 - Addition and Multiplication of Vectors

Unit 2 - Scalar Products of Two Vectors

Unit 3 - Vector Products of Two Vectors

Module 3

Differentiation and Integration of Vectors

Unit 1 - Differentiation of Vector-valued Functions

Unit 2 - Partial Derivatives of Vectors

Related Courses

Prerequisite: MAT 111

Required for: MAT 206

MAT 213

MAT 113
ELEMENTARY VECTORS, GEOMETRY AND MECHANICS

- differentiate vectors;
- integrate vectors;
- solve problems in mechanics; and
- solve problems in coordinate geometry.

Unit 3 - Different types of derivatives of vector-valued functions and their consequences

Unit 4 - Integration of Vector function

Module 4
Force, Momentum, Laws of Motion Under Gravity, Projectile, Vertical Motion And Impact of Two Smooth Spheres

Unit 1 - Force and Momentum

Unit 2 - Laws of Motion under gravity

Unit 3 - Kinematics and Acceleration of Particle Moving in a Plane

Unit 4 - Projectiles and Vertical Motion

Unit 5 - The Trajectory of a projectile

Unit 6 - Impact of two smooth spheres

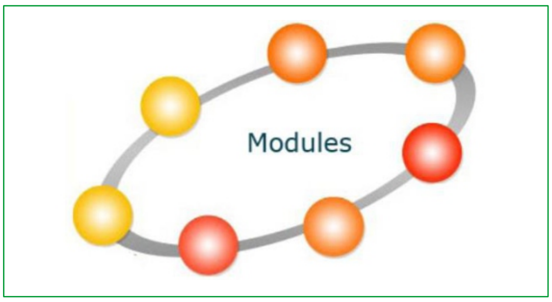
Module 5
Coordinate GEOMETRY

Unit 1 - Two Dimensional Coordinate Geometry

Unit 2 - Straight Lines

Unit 3 - Conics, Parabola, Hyperbola and Ellipse

Unit 4 - Parabola, Hyperbola and Ellipse



Course Requirements

Requirements for success

The CODL Programme is designed for learners who are absent from the lecturer in time and space. Therefore, you should refer to your Student Handbook, available on the website and in hard copy form, to get information on the procedure of distance/e-learning. You can contact the CODL helpdesk which is available 24/7 for every of your enquiry.

Visit CODL virtual classroom on http://codllms.unilorin.edu.ng. Then, log in with your credentials and click on MAT 113. Download and read through the unit of instruction for each week before the scheduled time of interaction with the course tutor/facilitator. You should also download and watch the relevant video and listen to the podcast so that you will understand and follow the course facilitator.

At the scheduled time, you are expected to log in to the classroom for interaction.

Self-assessment component of the courseware is available as exercises to help you learn and master the content you have gone through.

You are to answer the Tutor Marked Assignment (TMA) for each unit and submit for assessment.

Embedded Support Devices

Support menus for guide and references

Throughout your interaction with this course material, you will notice some set of icons used for easier navigation of this course materials. We advise that you familiarize yourself with each of these icons as they will help you in no small ways in achieving success and easy completion of this course. Find in the table below, the complete icon set and their meaning.

Introduction	Learning Outcomes	Main Content
Summary	Tutor Marked Assignment	Self Assessment
Web Resources	Downloadable Resources	Discuss with Colleagues
References	Futher Reading	Self Exploration

Grading and Assessment

TMA	CA	Exam	Total
20%	20%	60%	100%

Module 1

THEORY OF VECTOR

Units

Unit 1 - Geometric representation of vectors

Unit 2 - Component of a vector in three dimensions

Unit 3 - Direction Cosines

UNIT 1

Geometric representation of vectors

Introduction

Many measurable physical quantities we encounter have only numerical values attached to them. We describe such quantities completely when only their magnitudes or sizes are known. Such quantities are regarded as scalar quantities. Examples of such quantities are mass, length, volume, density, time, speed. Scalars are added by ordinary algebraic methods.

On the other hand, there are some other measurable physical quantities with both magnitudes (or numerical values) and directions. These quantities are not completely defined with only numerical values, but also with directions. These are known as vector quantities. Examples of such quantities are displacement, velocity, acceleration, weight, force, momentum. Vectors are added by geometrical methods.

Learning Outcomes

At the end of this unit, you should be able to:

1. define scalar and vector quantities
2. represent a vector by a straight line
3. solve a vector component in two dimensions

Main Content

Physical quantities can be classified into two main quantities, vector and scalar.

(a) Scalar quantity can be defined as a quantity with only magnitude, e.g. area, volume, mass, time, length, etc.

(b) vector quantity is defined when both its magnitude and direction are known, e.g. force, velocity, acceleration, etc.

Therefore,

(a) A speed of 10km/h is regarded as a scalar quantity since it is referring to the speed magnitude only.

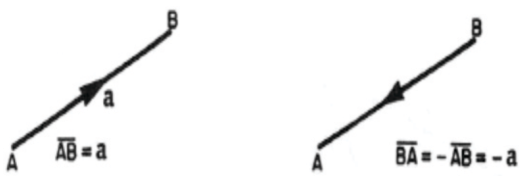
(b) A velocity of 10km/h due North is a vector quantity since it is referring to both magnitude (10km/h) and direction (North).

Vector representation by straight line

A vector quantity can be represented by a drawn line where the length of the line denotes the magnitude, and the direction of the line denotes the direction in which vector quantity acts.

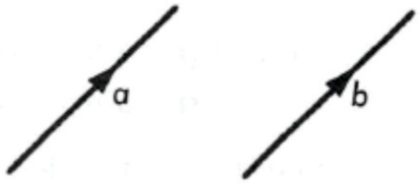
For example, the vector quantity AB is represented as $\overline{AB}$ or a

The magnitude of vector AB is written as $\overline{AB}$ or/a/. It should be noted that BA represents a vector quantity of the same magnitude but opposite direction

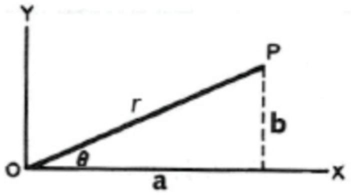


Equal vector

Vectors A and B are said to be equal if they have the same magnitude and direction



Component of a vector in two dimensions



The vector $\overline{OP}$ is defined by its magnitude (r) and direction (θ). It can also be defined in terms of the components of OX and OY directions. That is, vector $\overline{OP}$ is equivalent to a vector **a** in the OX direction plus a vector **b** in the OY direction.

$$\overline{OP} = \mathbf{a}\,(along\ OX) + \mathbf{b}\,(along\ OY)$$

If a unit vector in OX direction is represented by $\boldsymbol{i}$, then $\mathbf{a} = \boldsymbol{i}a$, and a unit vector in OY direction is represented by $\boldsymbol{j}$, then $\boldsymbol{b} = \boldsymbol{j}b$. Vector $\overline{OP}$ can then be written as

$$\boldsymbol{r} = ai + bj$$

where **i** and **j** are unit vectors in the OX and OY directions.

For vectors $z_1 = a_1\boldsymbol{i} + b_1\boldsymbol{j}$ and $z_2 = a_2\boldsymbol{i} + b_2\boldsymbol{j}$, the sum of vectors

$z_1 + z_2 = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}$

and the difference between the vectors

$z_1 - z_2 = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j}$

Activity 1

If vectors $z_1 = 2\boldsymbol{i} + 2\boldsymbol{j}$, $z_2 = 3\boldsymbol{i} - 6\boldsymbol{j}$ and $z_3 = 4\boldsymbol{i} - 6\boldsymbol{j}$ find.

(I) $z_1 + z_2$ (ii) $z_1 - z_2$ (iii) $z_3 + z_2$

Solution

(I) vectors $z_1 + z_2$

$= (2\mathbf{i} + 2\mathbf{j}) + (3\mathbf{i} - 6\mathbf{j})$

$= (2 + 3)\mathbf{i} + (2 - 6)\mathbf{j}$

$= 5\mathbf{i} - 4\mathbf{j}$

(ii) vectors $z_1 - z_2$

$= (2\mathbf{i} + 2\mathbf{j}) - (3\mathbf{i} - 6\mathbf{j})$

$= (2 - 3)\mathbf{i} + (2 + 6)\mathbf{j}$

$= -\mathbf{i} + 8\mathbf{j}$

(iii) vectors $z_3 + z_2$

$= (4\mathbf{i} - 6\mathbf{j}) + (3\mathbf{i} - 6\mathbf{j})$

$= (4 + 3)\mathbf{i} + (-6 - 6)\mathbf{j}$

$= 7\mathbf{i} - 12\mathbf{j}$

Summary

In this unit, you have learnt vector and scalar quantities. You have also learnt how to represent a vector by a straight line and how to solve a vector component in two dimensions.

Self Assessment Questions

(1) Which of these quantities are scalar and vector quantities

(I) a temperature of 100 degree census

(ii) the weight of a10kg mass

(iii) A north wind of 20 knots

(2) Given vectors $z_1 = 3\boldsymbol{i} + 2\boldsymbol{j}$, $z_2 = 2\boldsymbol{i} + \boldsymbol{j}$ and $z_3 = \boldsymbol{i} - 2\boldsymbol{j}$

Find (i) $z_1 - z_2$ (ii) $|z_1 + z_2|$ (iii) $|z_3 - z_1|$

Tutor Marked Assignment

- Which of these quantities are scalar and vector quantities

  (I) an acceleration of 3m/s2 vertically upward

  (ii) the sum of 200 naira

  (iii) A north wind of 20 knots

- Given vectors $z_1 = 3\boldsymbol{i} - 2\boldsymbol{j}$, $z_2 = 2\boldsymbol{i} + \boldsymbol{j}$ and $z_3 = \boldsymbol{i} + 2\boldsymbol{j}$

  Find (i) $z_3 - z_2$ (ii) $|z_3 + z_2|$ (iii) $|z_1 + z_2|$

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

  Tuttuh A., Sivasubramaniam M. and Adegoke S. (2012). Further Mathematics Project
- Third Edition. NPS Educational. Iqra Books, Nigeria.

Further Reading

- H. K. Dass. Advanced engineering Mathematics. Twenty First Edition. S.Chand & Company LTD., New Delhi.

UNIT 2

Component of a vector in three dimensions

Introduction

The component of a vector can also be implemented in three dimensions in a similar way to component of a vector in two dimension. This is discussed in this unit.

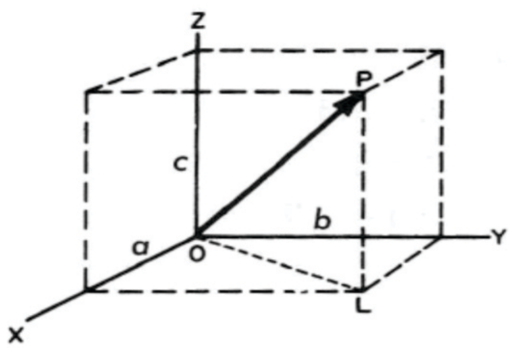
Learning Outcomes

At the end of this unit, you should be able to:

1. draw vectors in three dimensions
2. add vectors in three dimensions
3. solve a vector component in three dimensions

Main Content

Let vector $\overline{OP}$ be dened in its components (a along OX, b along OY andcalong OZ) where i is the unit vector in OX direction, j is the unit vector in OY direction and k is the unit vector in OZ direction.



Therefore,

$\overline{OP}$=a$\boldsymbol{i}$+b$\boldsymbol{j}$+c$\boldsymbol{k}$.

Also, $OL^2 = a^2 + b^2$ and $OP^2 = OL^2 + c^2$.

Hence, $OP^2 = a^2 + b^2 + c^2$.

So, vector $\overline{OP}$ = a$\boldsymbol{i}$+b$\boldsymbol{j}$+c$\boldsymbol{k}$

with magnitude $|\overline{Op}| = \sqrt{a^2 + b^2 + c^2}$

Activity 1

Given vectors z_1 = 3i+4j+k; z_2 = 2i - 3j+ 2k and z_3=i+ j+ 3k

Find (i) z_1+z_2 (ii) $|z_1+z_2|$ (iii) $|z_3 - z_2|$

Solution

(i) $z_1 + z_2 = (3i + 4j + k) + (2i - 3j + 2k)$

$= (3 + 2)I + (4 - 3)j + (1 + 2)k$

$= 5i + j + 3k$

(ii) From (i), $z_1 + z_2 = 5i + j + 3k$

$|z_1 + z_2| = \sqrt{5^2 + 1^2 + 3^2} = \sqrt{35}$

$= 5.9161$

(iii) We first find $z_3 - z_2 = (i + j + 3k) - (2i - 3j + 2k)$

$z_3 - z_2 = (1 - 2)I + (1 + 3)j + (3 - 2)k$

$= -i + 4j + k$

Then,

$|z_3 - z_2| = \sqrt{-1^2 + 4^2 + 1^2} = \sqrt{18}$

$= 4.2426$

Summary

You have been taught how to solve problems involving component of vectors in three dimensions in this unit. In the next unit, you will learn how the direction of a vector in three dimensions is determined.

Self Assessment Questions

1. Draw vector components in three dimensions

2. Given vectors $z_1 = 3i+2j-3k$, $z_2 = 2i-j+k$ and $z_3 = i+2j-5k$

   Find (i) $z_1 + z_2$ (ii) $|z_1 + z_2|$ (iii) $|z_3 - z_1|$

Tutor Marked Assignment

- Given vectors $z_1 = 3i + 2j - 3k$, $z_2 = 2i - j + k$ and $z_3 = i + 2j - 5k$

  Find (i) $z_3 + z_2$ (ii) $|z_3 - z_2|$ (iii) $|z_1 + z_2 + z_3|$

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

- Tuttuh A., Sivasubramaniam M. and Adegoke S. (2012). Further Mathematics Project 2. Third Edition. NPS Educational. Iqra Books, Nigeria.

Further Reading

- H. K. Dass. Advanced engineering Mathematics. TwentyFirst Edition. S.Chand & Company LTD., New Delhi.

UNIT 3

Direction cosines

Introduction

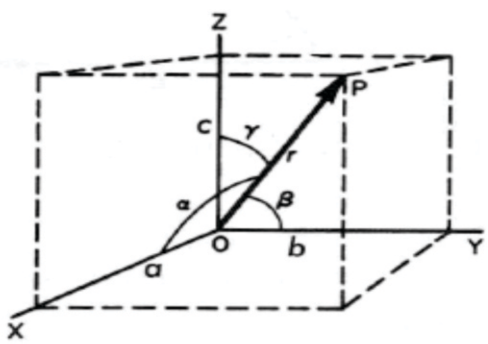
The direction of a vector in three dimensions is determined by the cosines of the angles between the vector and the three coordinate axes.

Learning Outcomes

At the end of this unit, you should be able to:

1. sketch a direction cosines of three dimensions in coordinate form
2. obtain a direction cosines of a vector
3. solve problems involving direction cosines

Main Content



For vector $\overline{OP}$ =ai+bj+ck, we have

$\frac{a}{r}$ = cos$\Rightarrow a = r$ cos α,

$\frac{b}{r}$ = cos$\Rightarrow b = r$ cos β and

$\frac{c}{r}$ = cos γ $\Rightarrow c = r$ cos γ

It is also known that $a^2+b^2+c^2=r^2$ then,

$r^2\cos^2\alpha + r^2\cos^2\beta + r^2\cos^2\gamma = r^2$

We divide through by r^2 to have

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Let $l = \cos\alpha$; $m = \cos\beta$ and $n = \cos\gamma$

then, $l^2 + m^2 + n^2 = 1$

where l; m and n are called direction cosines of the vector $\overline{OP}$

Therefore, for vector $\overline{OP} = ai+bj+ck$,

$l = \frac{a}{r}, m = \frac{b}{r}$ and $n = \frac{c}{r}$ where $r = \sqrt{a^2+b^2+c^2}$

Activity 1

Find the direction cosines of the vectors

(I) $z_1 = 3i + 4j + 3k$ (ii) $z_2 = 2i + 3j + 5k$

Solution

(I) a= 3; b =4; c =3; $r = \sqrt{9 + 16 + 9} = \sqrt{34}$

The direction cosines of vector z_1 is

$l = \frac{3}{\sqrt{34}}$; $m = \frac{4}{\sqrt{34}}$; $n = \frac{3}{\sqrt{34}}$

(ii) a=2; b=3; c=5; $r = \sqrt{4 + 9 + 25} = \sqrt{38}$

The directional cosines of vector z_2 is given as

$l = \frac{2}{\sqrt{38}}$; $m = \frac{3}{\sqrt{38}}$; $n = \frac{5}{\sqrt{38}}$

Activity 2

Find the vector $\overline{AB}$ if its length is 26, and direction cosines is

$l = \frac{5}{13}$ and $m = \frac{12}{13}$

Solution

It should be noted that

$a = r\cos\alpha = 26 \cdot \frac{5}{13} = 10$

and $b = r\cos\beta = 26 \cdot -\frac{12}{13} = -24$

thus vector $\overline{AB} = 10i - 24j$

Summary

In this unit, you have learnt how to sketch a direction cosines of three dimensions in coordinate form and how to obtain a direction cosines of a vector.

Self Assessment Questions

1. Sketch a direction cosines of a three dimension vector in coordinate form

2. Find the direction cosines of vectors $z_1 = 3i + 2j - 3k$, $z_2 = 2i + 2j - 5k$ and $z_3 = 6i + 3j - 5k$

Tutor Marked Assignment

- Find the direction cosines of vectors $z_1 = 3i - 2j + 3k$, $z_2 = 2i - j + k$ and $z_3 = i + 2j - 5k$

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

- Tuttuh A., Sivasubramaniam M. and Adegoke S. (2012). Further Mathematics Project 2. Third Edition. NPS Educational. Iqra Books, Nigeria.

Further Reading

- H. K. Dass. Advanced engineering Mathematics. TwentyFirst Edition. S.Chand & Company LTD., New Delhi.

Module 2

ADDITION AND SCALAR MULTIPLICATION OF VECTORS

Units

UNIT 1

Addition and Multiplication of Vectors

Introduction

A vector quantity has a direction and magnitude, while scalar quantity has only magnitude. Example of a scalar quantity is speed while velocity is an example of a vector quantity. Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and an arrow indicating the direction.

Learning Outcomes

At the end of this unit, you should be able to:

1. differentiate between a scalar quantity and a scalar quantity;
2. add vector addition problems;
3. perform dot and cross product;
4. prove the associative and commutative properties of a vector; and
5. add vector at right angle.

Main Content

Two vectors of the same size can be added together by adding the corresponding elements to form another vector of the same size, called the sum of the vectors. Vector addition is denoted by the symbol +. (Thus, the symbol + is overloaded to mean scalar addition when scalars appear on its left and right hand side, and vector addition when vectors appear on its left and right hand sides)

Activity 1

$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

vector subtraction is similar.

As an example

$$\begin{bmatrix}1\\9\end{bmatrix} \quad \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\8\end{bmatrix}$$

The result of vector subtraction is called the difference of the two vectors properties. Several properties of vectors are easily verified. For any vectors a, b, and c of the same size, we have the following:

(1) vector addition is commutative: $a + b = b + a$

(2) vector addition is associative: $(a + b) + c = a + (b + c)$.

(3) $a + 0 = 0 + a = a$. Adding the zero vector to a vector has no effect. (This is an example where the size of the zero vector follows from the text: It must be the same as the size of a)

(4) $a - a = 0$. Subtracting a vector from itself yields the zero vector. (the size of 0 is the size of a)

The property above can also be written in the form below.

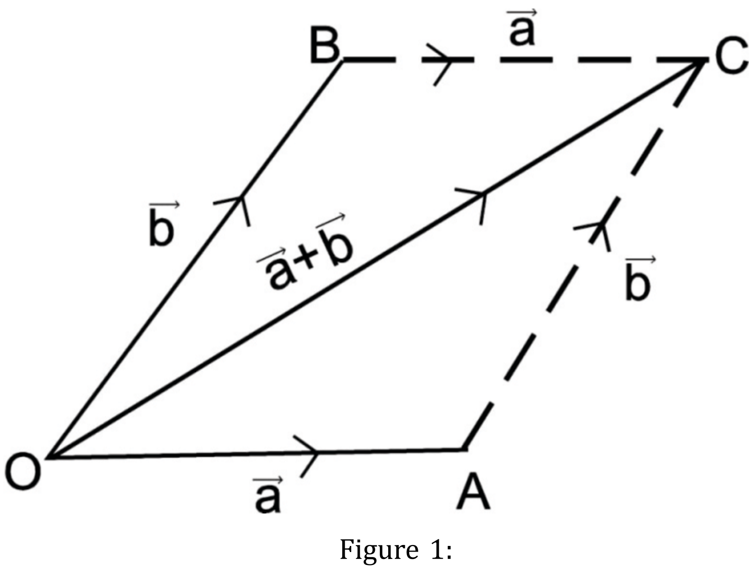


Figure 1:

From figure 1, using triangular law, one may note that

$$\overline{OA} + \overline{AC} = \overline{OC}$$

or

$$\overline{OA} + \overline{OB} = \overline{OC} \quad (\text{since } \overline{AC} = \overline{OB})$$

which is the parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

Activity 2

Commutative property

For any two vectors $\vec{a}$ and $\vec{b}$,

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

Proof:

Consider the parallelogram ABCD given as

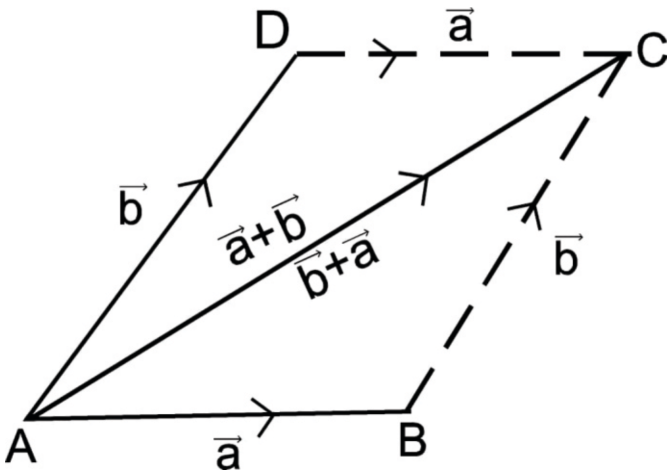


Figure 2

Let $\vec{AB}$ be $\vec{a}$ and $\vec{BC}$ be $\vec{b}$, then using the triangle law, from triangle ABC, we have $\vec{AC} = \vec{a} + \vec{b}$. Now since the opposite side of a parallelogram are equal and parallel, from figure 2, we have $\vec{AD} = \vec{BC} = \vec{b}$ and $\vec{DC} = \vec{a}$. Again triangle law, from ADC, we have $\vec{AC} = \vec{AD} + \vec{DC} = \vec{b} + \vec{a}$

Hence, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Activity 3

Associative Property

For any three *vector* $\vec{a}$, $\vec{b}$ and $\vec{c}$

Proof:

Let the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RS}$, respectively as shown below

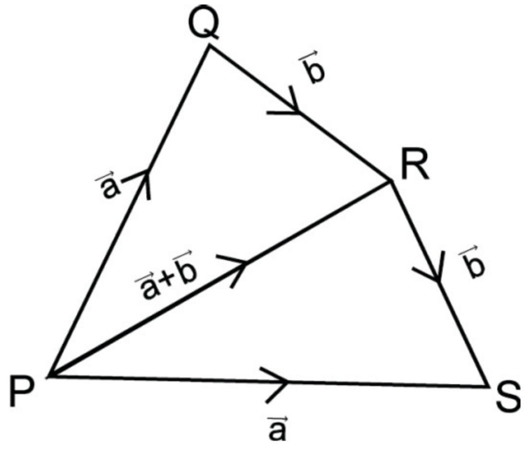


Figure 3

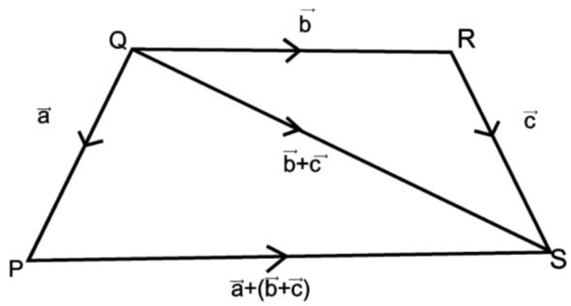


Figure 4

Then $\vec{a} + \vec{b} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

and $\vec{b} + \vec{c} = \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{QS}$

so $(\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{PR} + \overrightarrow{RS} = PS$

and $\vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$

Hence, $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

Remark: The associativity property of vector addition enables us to write the sum of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ as $\vec{a} + \vec{b} + \vec{c}$ without the brackets.

note that for any vector $\vec{a}$, we have $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$ Here, the zero vector $\vec{0}$ is called additive identity for the vector addition

Activity 4

Given that $\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\overrightarrow{QR} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, find the sum of the vectors.

Solution

$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

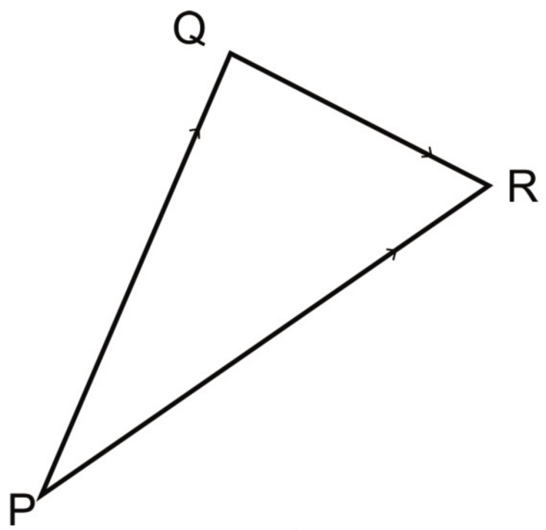


Figure 5

Activity 5

ABCD is a quadrilateral, simplify the following:

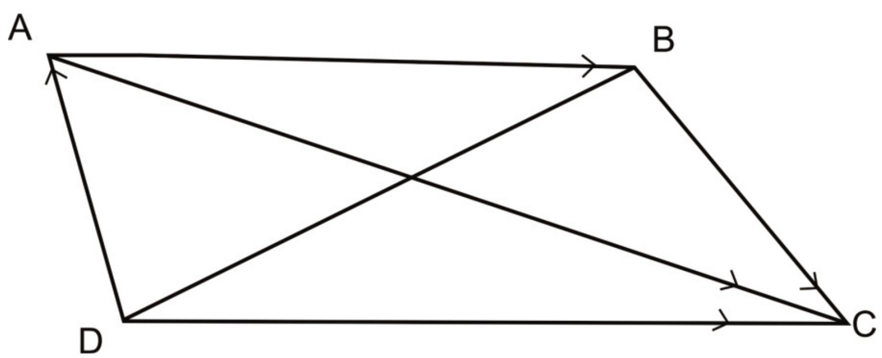


Figure 6

(a) $\overrightarrow{DC} + \overrightarrow{CA}$

(b) $\overrightarrow{BD} + \overrightarrow{DC} + \overrightarrow{CA}$

Solution

(a) $\overrightarrow{DC} + \overrightarrow{CA} = \overrightarrow{DA}$

(b) $\overrightarrow{BD} + \overrightarrow{DC} + \overrightarrow{CA} = (\overrightarrow{BD} + \overrightarrow{DC}) + \overrightarrow{CA}$

$= \overrightarrow{BC} + \overrightarrow{CA}$

$= \overrightarrow{BA}$

Alternatively it can be solved as:

$\overrightarrow{BD} + \overrightarrow{DC} + \overrightarrow{CA} = \overrightarrow{BD} + (\overrightarrow{DC} + \overrightarrow{CA})$

$= \overrightarrow{BD} + \overrightarrow{DA}$

$= \overrightarrow{BA}$

Adding vector at right angles.

Following are the tips for adding vectors at right angles:

a. The vectors are being added at right angles to each other.

b. In this case, the vectors are treated as sides of a right triangle. The resultant would be the hypotenuse of the triangle. The angle of the resultant is referred as the positive x-axis.

c. We use the Pythagoras theorem to find the hypotenuse and the tangent function to find the angle.

$X^2+Y^2=R^2$

$tan\ \theta = \frac{Y}{X}$

Activity 6

A river is flowing due east (0^0) at a speed of 5.00m/s. A boat travelling due north (90^0) across the river at speed of 8.00m/s. Find the resultant velocity vector of the boat.

Solution

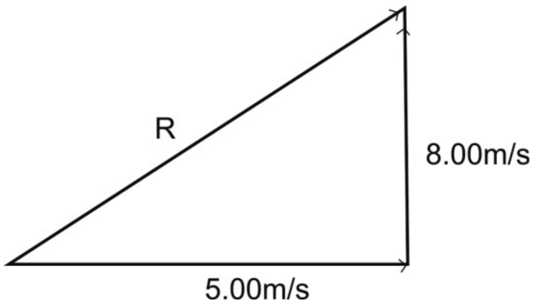


Figure 7

To find R:

$R^2 = 8^2 + 5^2$

$R^2 = 89$

$R = 9.43m/s$

To find θ

$\tan\theta = \frac{8}{5}$

$= \tan\theta^1(1{:}60) = 58.0^0$

Therefore, the resultant velocity is 9.43m/s, 58.0^0 or 9.43m/s at 58.0^0

Note:

If the resultant is supposed to be in the second or third quadrants, then using tangent to find the angle will give you 180^0 off from the angle you want. In case like that, you add 180^0 to your angle to get the correct one.

Finding X and Y components , every vector has what we call an X component and Y component. If you think of a vector as the hypotenuse of a triangle, then the X and Y components would be the sides of the triangle.

For instance,

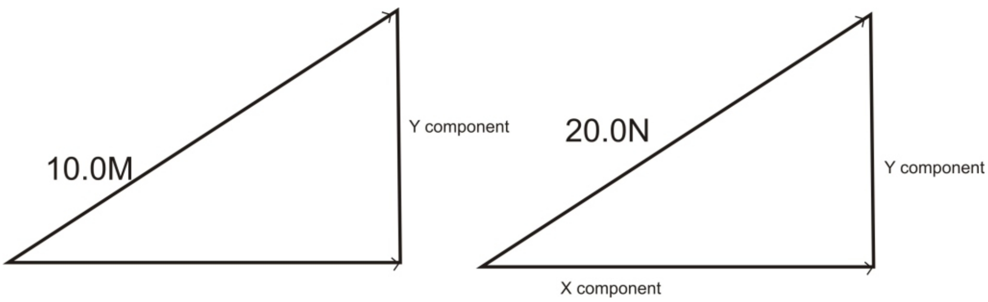


Figure 8

To find the X and Y component, we use $X = R\cos\theta$ and $Y = R\sin\theta$ where θ is always measured from the positive x axis.

Activity 7

Find the X and Y components of the vector 44.5m/s, 138^0

Solution

$X = 44.5\cos\ 138 = -33.1\text{m/s}$

$Y = 44.5\sin\ 138 = 29.8\text{m/s}$

Activity 8

Find the X and Y component of 22.7N at -65^0

Solution

$X = 22.7\cos\ (-65) = 9.6\text{N}$

$Y = 22.7\sin\ (-65) = -21\text{N}$

Summary

In this unit, you have learnt that vector quantity has magnitude and direction while scalar quantity has only magnitude. Vector quantity has two properties which are commutative and associative property.

Self Assessment Questions

1. A river is flowing due east(0^0) at a speed of 8.00 m/s. A boat is traveling due north (90^0) across the river at a speed of 10.00m/s. Find the resultant velocity vector of the boat.

2. Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3i + j + 4k$ and $b = i - j + k$

Tutor Marked Assignment

- Find the X and Y components of the vector 85.50m/s,120^0.
- Find the X and Y components of 30N at -40^0
- Given, that $\overrightarrow{PQ} =$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

and

$$\overrightarrow{QR} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Find the sum of the vectors.

References

- Straud K.A. Engineering Mathematics.
- John Bird, Higher Engineering Mathematics, fifth edition.

Further Reading

- Introduction to scalars and vectors. www.siyavula.com

UNIT 2

Scalar Products of Two Vectors

Introduction

Vectors can be multiplied in two ways, scalar or dot product where the result is a scalar, vector or cross product where the result is a vector. In this unit, we will look at the scalar or dot product of two vectors.

Learning Outcomes

At the end of this unit, you should be able to:

1. Obtain the scalar product of two vectors.
2. solve problem on direction cosines and
3. find angle between two vectors.

Main Content

The first type of vectors multiplication is called Dot product. This type of multiplication (written as $A.B$) multiplies one vector by another vector and gives a scalar result.

The dot product of two vectors A and B is the product of their magnitude times the cosine of the angle between them: $A.B = AB \, Cos \, \theta$.

In terms of rectangular components, this is equal to the transpose of column vector A times column vector B, which gives a 1×1 matrix (i.e a scalar)

$$A{:}B = A^{T}B = \begin{pmatrix} A_x & A_y & A_z \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} B_z = A_xB_x + A_yB_y + A_zB_z$$

The dot product is commutative ($A.B = B.A$). It can also be defined as $\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$ if θ is the angle between the two vectors. A vector is said to be orthogonal if the dot product of the vector is zero

Scalar products can be found by taking the component of on evector in the direction of the other vector and multiplying it with the magnitude of the other vector.

Activity 1

If i, j, k are unit vectors in the directions OX, OY, OZ respectively, then any position vector $\overline{OP}$ (=r)can be represented in the form $\overline{OP} = r = a_x i + a_y j + a_z k$.

Then, $|r| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

The direction of OP is denoted by starting the direction cosines of the angles made by OP and the three coordinate axes.

$$l = \cos\alpha = \frac{OL}{OP} = \frac{a_x}{|r|}$$
$$m = \cos\beta = \frac{OM}{OP} = \frac{a_y}{|r|}$$
$$n = \cos\gamma = \frac{ON}{OP} = \frac{a_z}{|r|}$$

$\therefore l, m, n = \cos\alpha, \cos\beta, \cos\gamma$.

So, if P is the point (3, 2, 6), then

$(/r/)^2 = 9 + 4 + 36 = 49 \quad \therefore |r| = 7$

$l = \cos\alpha = \frac{3}{7} = 0.4286$

$m = \cos\beta = \frac{2}{7} = 0.2857$

$n = \cos\gamma = \frac{6}{7} = 0.8571$

Activity 2

If the direction cosines of A are l_1, m_1, n_1 and those of B are l_2, m_2, n_2. then the angle between the vectors is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

If A = $2i + 3j + 4k$ and B = $i - 2j + 3k$, then, find the direction cosines of each A and B and hence find θ.

Solution

For $A: |r_1| = \sqrt{4+9+16} = \sqrt{29}$

$\therefore\ l_1 = \frac{2}{\sqrt{29}}\ ;\ m_1 = \frac{3}{\sqrt{29}}\ ;\ n_1 = \frac{4}{\sqrt{29}}$

For $B: /r_2/ = \sqrt{1+4+9} = \sqrt{14}$

$\therefore\ l_2 = \frac{1}{\sqrt{14}}\ ;\ m_{14} = \frac{-2}{\sqrt{14}}\ ;\ n_2 = \frac{3}{\sqrt{14}}$.

Then, $\cos\theta = \frac{1}{\sqrt{14+29}}(2-6+12) = 0.3970$

$\therefore \theta = 66^0 36'$.

Activity 3

If A and B are two vectors, the scalar product of A and B is defined as

$A.B = AB \cos\theta$

where θ is the angle between the two vectors. If $A.B = 0$, then A is perpendicular to B i.e. $A \perp B$. If we consider the scalar products of the unit vectors i, j, k which are mutually perpendicular, then,

Solution

$i.j = (1)(1)\cos 90^0 = 0 \therefore i.j = j.k = k.i = 0$ and

$i.i = (1)(1)\cos 90^0 = 1 \therefore i.i = j.j = k.k = 1$.

In general, if $A = a_x i + a_y j + a_z k$ and $B = b_x i + b_y j + b_z k$, so

that, if $A = 2i - 3j + 4k$ and $B = i + 2j + 5k$, then

$A = 2i - 3j + 4k \therefore A = /A/ = \sqrt{4+9+16} = \sqrt{29}$

$B = i + 2j + 5k \therefore B = /B/ = \sqrt{1+4+25} = \sqrt{30}$.

Since $A.B = 2 - 6 + 20 = 16$ and $A.B = AB \cos\theta$

$\therefore 16 = \sqrt{29}\sqrt{30}\cos\theta \quad \therefore \cos\theta = 0.5425 \quad \therefore \theta = 57^0 9'$. So, the scaler product of $A = a_x i + a_y j + a_z k$ and $B = b_x i + b_y j + b_z k$ is $A{:}B = a_x b_x + a_y b_y + a_z b_z$ and $A{:}B = AB\ \cos\theta$, where θ is the angle between th vectors.

Summary

In this unit, you have been taught how to obtain the scalar product of two vectors. You have also been taught how to solve problem on direction cosines and angle between two vectors.

Self Assessment Questions

Evaluate each of the following:

1. $2(i.i)$
2. $3(j.k)$
3. $5(k.j)$
4. $j.(2i - 3j + k)$
5. $(2i - j).(3i + k)$.

Tutor Marked Assignment

- If $A = A_1 i + A_2 j + A_3 k$ and $B = B_1 i + B_2 j + B_3 k$, prove that $A.B = A_1 B_1 + A_2 B_2 + A_3 B_3$.

References

- Stroud, K. A. and Booth, D. J. (2001), Engineering Mathematics, Fifth Edition, Palgrave, New York.
- Kreyszig, E. (2011), Advanced Engineering Mathematics, Tenth Edition, John Wiley and Sons, Inc. New York.
- Blitzer, R. (2009) Introductory and Intermediate Algebra For College Students, Pearson Prentice Hall, Upper Saddle River, New Jersey.

Further Reading

- Check Schaum's Outlines, Advanced Calculus, Second Edition, by Robert Wrede and Murray R. Spiegel, (2002).

UNIT 3

Vector Products of Two Vectors

Introduction

The magnitude vector product of two given vectors can be found by taking the product of the magnitudes of the vectors times the sine of the angle between them.

Learning Outcomes

At the end of this unit, you should be able to:

1. obtain the vector product of two vectors.
2. solve problem on sines and cosines and
3. find angle between two vectors.

Main Content

The second type of vector multiplication is called cross product. This type of multiplication (written as $A \times B$) multiplies one vector by another and gives another vector as a result. The result of the cross product operation is a vector whose magnitude is $|A \times B| = AB \, Sin \, \theta$, where θ is the magnitude between the two vectors. The direction of $A \times B$ is perpendicular to the plane containing vectors $A \times B$, in a right sense. Cross product is defined for three dimensional vectors only.

A convenient mnemonic for finding the regular component of the cross product is through a matrix.

$$AxB = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z - A_zB_y)i - (A_xB_z - A_zB_x)j + (A_xB_y - A_yB_x)k$$

Another way to represent the components of the cross product is to write the component of vector A into a 3 × 3 matrix, then multiply that martix by the column vector B.

$$AB = \begin{pmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x A_z \\ A_x B_y - A_y B_x \end{pmatrix}$$

The cross product is anti-commutative ($A \times B = -B \times A$) and nonassociative ($A \times (B \times C) \neq (A \times B) \times C$)

If $A = a_x i + a_y j + a_z k$ and $B = b_x i + b_y j + b_z k$, then, the vector product $A \times B$ has a magnitude $|A \times B| = AB \, \text{Sin}\theta$ in the direction perpendicular to A and B such that A, B and $(A \times B)$ form a right-handed set.

Thus, we can write this as $A \times B = (AB \, \text{Sin}\, \theta)n$, where n is defined as a unit vector in the positive normal direction to the plane of A and B, i.e. forming a right-handed set. Also

$$A \text{ x } B = \begin{pmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} .$$

Activity 1

If we consider the vector product of the unit vectors i, j, k, then

$i \times j = (1)(1)\text{Sin } 90^0 k = k$

$j \times k = i, k \times i = j.$

Note that

$j \times i = -(i \times j) = -k$

$k \times j = -i, i \times k = -j.$

Also,

$i \times i = (1)(1) Sin\, 0^0 \text{n} = 0$

$j \times j = k \times k = 0.$

Activity 2

If $A = 3i - 2j + 4k$ and $B = 2i - 3j - 2k$, then find $A \text{ X } B$.

Solution

We simply evaluate the determinant

$$AB = \begin{pmatrix} i & j & k \\ 3 & -2 & 4 \\ 2 & -3 & -2 \end{pmatrix}.$$

$= i(4 + 12)\ -j(-6 - 8) + (-9 + 4) = 16i + 14j - 5k.$

Since, the scalar product of two vectors is scalar and the vector of two vectors is a vector and we know that

$$/A \times B/ = AB \, Sin\,\theta,$$

the angle between the vectors A and B is

$A = 3i - 2j + 4k, B = 2i - 3j - 2k$ and $A \text{ X } B = 16i + 14j - 5k,$

$\therefore |AB| = \sqrt{16^2 + 14^2 + 5^2} = \sqrt{477} = 21.84$

$A = |A| = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{29} = 5.385$

$B = |B| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{17} = 4.123$

$\therefore 21.84 = (5.385)(4.123)\, Sin\,\theta$

$\therefore Sin\,\theta = 0.983 \therefore \theta = 79^0 40'.$

So, to recapitulate:

If $A = a_x i + a_y j + a_z k$ and $B = b_x i + b_y j + b_z k$ and θ is the angle between them, then,

(i) Scalar Product, $A.B = a_x b_x + a_y b_y + a_z b_z = AB\,Cos\,\theta$.

(ii) Vector Product is

$$AB = \begin{pmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}.$$

and $|A \times B| = AB\,Sin\,\theta$.

Direct Product

The third type of vector multiplication is called the direct product, and is written as AB. Multiplying one vector by another under the direct product gives a tensor result. A tensor is a 3×3 matrix that is used to represent certain quantities as stress and pressure.

The rectangular component of the direct product may be found by matrix multiplication: one multiplies the column vector A by the transpose of B, which gives a 3×3 matrix

$$AB = AB^T = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$(B_x B_y B_z) = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

Vector Product Identities

A few vector identities are of interest:

$A.B \times C = A \times B.C = B.C \times A = C.A \times B = C \times A.B$

$A \times (B \times C) = B(A.C) - C(A.B)$

$(A \times B) \times C = B(A.C) - A(B.C)$

$(A \times B).(C \times B) = (A.C)(B.D) - (A.D)(B.C)$

$(A \times B) \times (C \times D) = (A \times B \cdot D)C - (A \times B \cdot C)D$

Unit Vector

The unit vector of a vector in the direction of vector a is given by

$$\hat{a} = \frac{1}{\vec{a}}\vec{a}$$

Activity 9

Let $u = (1, 2, 3)$ and $v = (4, 5, 6)$, find $u \cdot v$

Solution

$u \cdot v = (1)(4) + (2)(5) + (3)(6) = 4 + 10 + 18 = 32$

Activity 10

Let u=(1,2,1), v=(3,2,4) and w=(1,-1,3), find

(I) $u \cdot v$ (ii) $u \cdot (v + w)$

Solution

(I) $u.v=(1)(3)+(2)(2)+(1)(4)=3+4+4=11$

(ii) $u.(v+w)$

$v+w=(3+1),(2-1),(4+3)$ $v+w=(4,1,7)$ or

$$v+w = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} = (4;1;7)$$

$u.(v+w)=(1,2,1).(4,1,7)$

$u.(v+w)=(1)(4)+(2)(1)+(1)(7)$

$u.(v+w)=4+2+7$

$u.(v+w)=13$

Activity 11

For given vectors $\vec{a}=2i+j+3k$ and $\vec{b}=3i+5i-2k$, find

(i) $\vec{a}\times\vec{b}$ (ii) $|\vec{a}\times\vec{b}|$

Solution

(i) $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$=i(-2-15)-(-4-9)j+(10-3)k=-17i+13j+7k$

(ii) $\vec{a} \times \vec{b} = \sqrt{(17)^2+(13)^2+(7)^2} = \sqrt{507}$

Summary

In this unit, you have learnt how to obtain the vector product and solve problems on angle between two vectors.

Self Assessment Questions

Prove that |A| of the vector $A = A_1i+A_2j+A_3k$ is $A \sqrt{A_1^2 i + A_2^2 j + A_3^2 k}$.

Tutor Marked Assignment

- Determine the vector having initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ and find its magnitude.
- Find the cross product of the following

  (i) Vector $A=3i+4j-5k$, $B=-2i+3j-k$

  (ii) $A=6i+3k$, $B= 3j-3k$

References

- Stroud, K. A. and Booth, D. J. (2001), Engineering Mathematics, Fifth Edition, Palgrave, New York.
- Kreyszig, E. (2011), Advanced Engineering Mathematics, Tenth Edition, John Wiley and Sons, Inc. New York.
- Blitzer, R. (2009) Introductory and Intermediate Algebra For College Students, Pearson Prentice Hall, Upper Saddle River, New Jersey.

Further Reading

- Check Schaum's Outlines, Advanced Calculus, Second Edition, by Robert Wrede and Murray R. Spiegel, (2002).

Module 3

DIFFERENTIATION AND INTEGRATION OF VECTORS

Units

UNIT 1

Differentiation of Vector-Valued Functions

Introduction

In one of the previous courses you learnt in details, the concept of differentiation and integration of real valued functions $y = f(x)$ with $x, y \in R$. Similarly, the concept of vector field and vector quantities was introduced to you in the previous units of this course. Both the knowledge of calculus (differentiation and integration) and that of vector field (as earlier discussed in the course) are important in understanding the concept of the unit. It is therefore important that you understand the concept of differentiation and integration of real-valued functionsand that of the vector-valued functions before studying differentiation and integration of vector-valued functions.

Learning Outcomes

At the end of this unit, you should be able to:

1. define and identify a vector-valued function of one (scalar) variable ;
2. determine the limit of a vector-valued function;
3. find the continuity of a vector-valued function; and
4. find the derivatives of a vector-valued function.

Main Content

The real-valued functions are scalar-valued functions because their range and image are real numbers (or scalars) and are defined as $f : D \to R$ where $D \subset R^n$ and n is the number of variables.

In this unit however, we consider vector-valued functions $f : D \to \mathbb{R}^n$ where n is a positive integer usually considered for n = 2(*plane*) and n = 3(*space*). Unlike the real-valued functions whose result of applying it is real-valued functions (or scalar) quantity, the result of applying vector-valued functions is either a 2 dimensional vector which is plane vector or a 3 dimensional vector which is space vector.

A curve in 2 *dim* (or 3 *dim*) can be represented as the image of a real-valued function on an interval I, the position vector of a point on the curve can be given as

$$r = f(t),\ t \in I$$

and the curve defined by $r = r(t)$, a vector-valued function of one (scalar) variable.

Definition (Vector-valued functions of a single scalar variable). A function $r(t)$ of the form

$$r(t) = x(t)i + y(t)j \text{ plane (2 dimensional)} \quad (1)$$

or

$$r(t) = x(t)i + y(t)j + z(t)k \text{ space (3 dimensional)} \quad (2)$$

is a vector-valued function t, if to each t of the range of values of t, there corresponds a vector r, such that

$$r = r(t).$$

The vector r can be written in components form as:

$$r = (x(t), y(t), z(t)) = x(t)i + y(t)j + z(t)k$$

where x, y, z are components of $r(t)$ defined for the range of values of t and i, j, k are unit vectors in the direction of the principal axes.

We begin with the limit and continuity of a vector-valued function $r(t)$.

Limit of a vector-valued function

Let $r(t)$ be a vector-valued function. The limit of $r(t) = x(t)i + y(t)j + z(t)k$ as t tends to k is defined as

$$\lim_{t\to k} r(t) = \left[\lim_{t\to k} x(t)\right] i + \left[\lim_{t\to k} y(t)\right] j + \left[\lim_{t\to k} z(t)\right] k$$

provided the limit of each components of $x(t), y(t)$ and $z(t)$ exist as $t \to k$.

Continuity of a vector-valued function

Let r(t) be a vector-valued function. The function of $r(t) = x(t)i + y(t)j + z(t)k$ is said to be continuous at a point $t = k$ if:

(I) $r(t)$ is defined at $t = k$;

(ii) $\lim_{t\to k} r(t)$ exists; and

(iii) $r(k) = \lim_{t\to k} r(t)$.

Activity 1

(1) Let $r(t) = (t^2 - 1)i + \sin 2tj + \cos 2tk$, find the limit of $r(t)$ as t tends to 0.

(2) Discuss the continuity of each of the following:

(i) $r(t) = e^{-3t}i - \sqrt{t^2 + 4}\,j + \cos t\, k$ at $t = 0$; and

(ii) $r(t) = \frac{27}{t^2}\, i + (t - 1)^2 j + 5k$ at $t = 3$.

Solution

(1) Let $\lim_{t\to k} \left((t^2 - 1)i + \sin 2tj + \cos 2tk \right) = -i + \sin 0j + \cos 0k = -i + k$

(2) (i) Let $r(t) = e^{-3t}\, i - \sqrt{t^2 + 4j} + \cos tk$. Then, $r(0) = e^0 i - 2j + \cos 0k = i - 2j + k$ and $\lim_{t\to 0} (e^{-3t}\, i - \sqrt{t^2 + 4j} + \cos tk = i - 2j + k)$. Thus, $r(t)$ is continuous at $t = 0$ since $r(0) = \lim_{t\to 0} r(t)$

(ii) Let $r(t) = \frac{27}{t^2}\, i + (t - 1)^2 j + 5k$ at $t = 3$. Then, $r(3) = \frac{27}{t^2}\, i + (3 - 1)^2 j + 5k$.

$\Rightarrow r(3) = \lim_{t\to 3} r(t) = 3i + 4j + 5k$. Hence, $r(t)$ is continuous at $t = 3$.

Derivative of vector-valued functions

Suppose that a curve C represents the path taken by a body with t being the time, then the vector $r(t)$ is the position vector of the body at time t and $r(t + \delta t)$ is the position vector at a later time $t+\delta t$ so that the average velocity of the body on the time interval $[t, t + \delta t]$ is given as

$$\frac{\text{displacement vector}}{\text{length of time interval}} = \frac{r(t+\delta t) - r(t)}{\delta t}.$$

Since the components of r are x, y, z, then, we have

$$\left(\frac{x(t+\delta t) - x(t)}{\delta t}, \frac{y(t+\delta t) - y(t)}{\delta t}, \frac{z(t+\delta t) - z(t)}{\delta t}\right) = \frac{x(t+\delta t) - x(t)}{\delta t}\, i + \frac{y(t+\delta t) - y(t)}{\delta t}\, j + \frac{z(t+\delta t) - z(t)}{\delta t}\, k.$$

in terms of the components of r. Suppose each of the scalar functions x, y, z is differentiable, then the vector r has a limit written as

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (x', y', z') = x'(t)i + y'(t)j + z'(t)$$

as the velocity of the body $v(t)$ so that

$$v(t) = \lim_{\delta t \to 0} \frac{r(t+\delta t) - r(t)}{\delta t} = \frac{d}{dt} r(t) = r'(t).$$

The length of $v(t) = |v(t)|$ is the speed of the body. Similarly,

$$a(t) = \lim_{\delta t \to 0} \frac{v(t+\delta t) - v(t)}{\delta t} = \frac{d}{dt} v(t) = \frac{d^2}{dt^2} r(t) = r^{n}(t).$$

Some Properties of Vector Differentiation

Let u, v and ω be vector-valued functions of t, then the following properties hold:

(1) $\frac{d}{dt}(u \pm v) = \frac{du}{d(t)} \pm \frac{dv}{d(t)}$

(2) $\frac{d}{dt}(u \times v) = \frac{du}{d(t)} \times \frac{dv}{d(t)}$

(3) $\frac{d}{dt}(u \cdot v) = \frac{du}{d(t)} \cdot \frac{dv}{d(t)}$

(4) $\frac{d}{dt}(c\omega) = c\frac{d\omega}{d(t)}$ where c is a scalar.

(5) $\frac{d}{dt}(u \times (v \times \omega)) = u \times \left(v \times \frac{d\omega}{dt}\right) + u \times \left(\frac{dv}{dt} \times \omega\right) + \frac{du}{dt} \times (v \times \omega)$

(6) $\frac{d}{dt}(u\,.\,(v \times \omega)) = u\,.\,v \times \frac{d\omega}{dt} + u\,.\,\frac{dv}{dt} \times \omega + \frac{du}{dt}\,.\,v \times \omega$

Theorem

Let $r(t)$ be a vector-valued function:

(I) Plane (2 dimensional vector): If $r(t) = x(t)i + y(t)j$ then

$$\frac{dr}{dt} = r'(t) = x'(t)i + y'(t)j; \text{ and}$$

(ii) Space (3 dimensional vector): If $r(t) = x(t)i + y(t)j + z(t)k$ then

$$\frac{dr}{dt} = r'(t) = x'(t)i + y'(t)j + z'(t).$$

Activity 2

(1) Let $r(t) = x(t)i + y(t)j + z(t)k$ where x, y, z are differentiable functions of a scalar variable t, then differentiate $r(t)$.

(2) Differentiate each of the following:

(a) $costi - 3e^{-2t}j$ **(b).** $log_e ti - sin(5-2t)j - e^{-2t}k$

(c) $\frac{3}{t^2}i + sin3tj - cos2tk$ **(d)** $sin2ti + cos5tj + (t^3 - lnt)k$

(3) A body moves along a curve with parametric equations $x = 2sin5t$, $y = 4cos5t$ and $z = 2e^{3t}$, where t is the time. Find:

(a) its velocity and acceleration; and

(b) the magnitudes of velocity and acceleration at $t = 0$.

Solution

(1). Let $r(t) = x(t)i + y(t)j + z(t)k$ and suppose x, y, z are differentiable functions of a scalar variable t we define differential coefficient of $r(t)$ as follow:

$$\frac{dr}{dt} = \lim_{\delta t \to 0} \frac{r(t+\delta t) - r(t)}{\delta t}$$

$$\lim_{\delta t \to 0} \left(\frac{x\,[(t+\delta t)i + y(t+\delta t)j + z(t+\delta t)k] - [x(t)i + y(t)j + z(t)k]}{\delta t} \right)$$

$$= \lim_{\delta t \to 0} \left[\frac{x(t+\delta t) - x(t)}{\delta t} i + \frac{y(t+\delta t) - y(t)}{\delta t} j + \frac{z(t+\delta t) - z(t)}{\delta t} k \right]$$

$$= \lim_{\delta t \to 0} \left(\frac{\delta x}{\delta t} i + \frac{\delta y}{\delta t} j + \frac{\delta z}{\delta t} k \right) = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k = r'(t).$$

Hence, $\frac{dr}{dt} = r'(t) = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k.$

(2). (a). Let $r(t) = costi - 3e^{-2t} j$, **then** $\frac{dr}{dt} = r'(t) = -sinti + 6e^{-2t} j$

(b). Let $r(t) = log_e ti - sin(5-2t)j - e^{-2t} k$

Then,

$\frac{dr}{dt} = r'(t) = \frac{1}{t} i + 2cos(5-2t)j + 2e^{-2t} k$

(c). Let $r(t) = \frac{3}{t^2} i + sin3tj - cos2tk$, **then** $\frac{dr}{dt} = r'(t) = -\frac{6}{t^3} i + 3cos3tj + 2sin2tk$

(d). Let $r(t) = sin2ti + cos5tj + (t^3 - lnt)k$, **then** $r'(t) = 2cos2ti - 5sin5tj - (\frac{1}{t} - 3t^2)k$

(3). Let $r(t) = x(t)i + y(t)j + z(t)k$. **Then,**

(a). the velocity is given as

$$v(t) = \frac{dr}{dt} = 10cos5ti - 20sin5tj + 6e^{3t} k$$

and the acceleration,

$$a(t) = \frac{dv}{dt} = \frac{d^2 r}{dt^2} = -50sin5ti - 100cos5tj + 18e^{3t} k.$$

(b). If $t = 0$, **then** $\frac{dr}{dt} = 10i + 6k = v(t)$ **and** $\frac{dv}{dt} = -100j + 18k = a(t)$.

The magnitude of velocity, $v(t) = |v(t)| = \sqrt{10^2 + 6^2} = \sqrt{136} = 2\sqrt{34} \;\; or \;\; 11.66.$

The magnitude of acceleration, $a(t) = |a(t)| = \sqrt{100^2 + 18^2} = \sqrt{10324} = 4\sqrt{949} \;\; or \;\; 101.61.$

Velocity and Acceleration of Vectors

Let $r(t) = x(t)i + y(t)j$ where $x(t)$ and $y(t)$ are both differentiable with respect to t. Then, the velocity vector, the speed at any time t and the acceleration vector of r(t) is given as:

$$Velocity = r'(t) = x'(t)i + y'(t)j = v(t);$$

$$Speed = |r'(t)| = \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2} = |v(t)|; \quad and$$

$$Acceleration = r''(t) = x''(t)i + y''(t)j = a(t);$$

Activity 3

(1). Let $r(t) = (1 + t^2)i + 5tj$ be the position vector of an object that moves along a plane curve C. Find its velocity vector, speed and acceleration vector after 4seconds.

(2). Let $r(t) = e^{-2t}i + (t^2 - 2t + 1)j + \frac{1}{1+t^2}k$. **Find** $r'(t)$, $r''(t)$ **and** $r'(t).r''(t)$.

Solution

(1). Let $r(t) = (1 + t^2)i + 5tj$, **then:**

(i). $v(t) = 2ti + 5j = r'(t)$; **and** $v(4) = 2(4)i + 5j = 8i + 5j$ **which is the velocity vector of the object after 4seconds.**

(ii). The speed at time $t = 4secs.$ **is given as**

$$|v(t)| = r'(t) = \sqrt{8^2 + 5^2} = \sqrt{64 + 25} = \sqrt{89} = 9.43m/s.$$

(iii). $|v'(t)| = r''(t) = 2i = a(t)$, **the acceleration vector of the object after 4seconds.**

(2). Let $r(t) = e^{-2t}i + (t^2 - 2t + 1)j + \frac{1}{1+t^2}k$. **Then,**

(i). Let $r(t) = e^{-2t}i + (t^2 - 2t + 1)j + \frac{1}{1+t^2}k = x(t)i + y(t)j + z(t)k$. **Then,**

$x(t) = e^{-2t}$, $x'(t) = -2e^{-2t}$

$y(t) = t^2 - 2t + 1$, $y'(t) = 2t - 2$

For $z(t) = \frac{1}{(1+t^2)}k$, **let** $u(t) = 1$ **and** $v(t) = 1 + t^2$. **Then by quotient rule,**

$$z'(t) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2} = -\frac{2t}{(1+t^2)^2}k.$$

Hence, $r'(t) = -2e^{-2t}i + (2t - 2)j - \frac{2t}{(1+t^2)^2}k = -2e^{-2t}i + (2t - 2)j - \frac{2t}{t^4+2t^2+1}k.$

(ii). Similarly, $r''(t) = 4e^{-2t}i + 2j - \frac{2-6t^2}{(1+t^2)^3}k$.

(iii). $r'(t).r''(t) = \left(-2e^{-2t}i + (2t-2)j - \frac{2t}{(1+t^2)^2}k\right).\left(4e^{-2t}i + 2j - \frac{2-6t^2}{(1+t^2)^3}k\right)$

$= -8e^{-4t} + (4t-4) + \frac{4t-12t^3}{(1+t^2)^5}$.

Summary

In this unit, you have learnt a vector-valued function of one (scalar) variable. You have equally been taught how to determine the limit of a vector-valued function. You also learnt how to find the continuity and the derivatives of a vector-valued function.

Self Assessment Questions

(1). Find the derivative of each of the following vector-valued functions.

(a). $r(t) = a^t i + t^{t^2} j + t^t k$; **and**

(b). $r(t) = e^{-cost}i - cos^2tj + sin^2tk$.

(2). Find the limit of $r(t) = e^{-cost}i - cos^2tj + sin^2tk$ **as** t **approaches** 0.

(3). Find the domain of t **in which** $r(t)$ **is not continuous and hence discuss the continuity of** $r(t) = \frac{t-3}{t^2-3t-4}i + \frac{1}{\sqrt{t^3+1}}j - \frac{1}{t+1}k$

Tutor Marked Assignment

- **Let** $u = 3ti + t^2j + (t^2-5)k$ **and** $v = (2-t)i + 2j + tk$**, find:**
  (a). $\frac{d}{dt}(u \times v)$ **(b).** $\frac{d}{dt}(u+v)$ **(c).** $\frac{d}{dt}(u \cdot v)$
- **An object of** $3kg$ **moving with initial velocity** $(4i - 3j + 5k)$**m/s is acted upon by a certain force** $(5i - 3j + k)$**N. What is the:**
  (a). distance and velocity after $10s$**, and**
  (b). time in which the object reaches the $xy-$ **plane.**

- **A body moves along the curve $x = t^3 - 5$, $y = 2t + 3$, $z = t - 4$; Find the component of its velocity and acceleration at time $t = 2s$ in the direction of vector $4i + j - k$.**
- **Let $r(t) = e^{2t}i - sintj + costk$, evaluate: (a). $r'(t)$ (b). $r''(t)$ (c). $r'(t) \cdot r''(t)$ (d). $r'(t) \times r''(t)$.**

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UNIT 2

Partial Derivative of Vectors

Introduction

In the previous unit, you learnt a vector-valued function, its limit, continuity and derivatives. If however a vector-valued function say, $r(t)$, has more than one independent variables, then the derivative involved in such a case will be a partial derivative and intances like this are the focus of this unit.

Learning Outcomes

At the end of this unit, you should be able to:

1. define partial derivatives of a vector-valued function;
2. find partial derivatives of a vector-valued function; and
3. apply the knowledge of the partial derivatives of vector valued functions in solving grad., div. and curl of any given vector-valued function(s).

Main Content

Suppose r is a vector valued-function depending on multiple (more than one) scalar variables, say x, y, z. Then, $r = r(x, y, z)$. In this case, we can find the derivatives of r with respect to x, y and z as follows:

$$\frac{\partial r}{\partial x} = \lim_{\delta t \to 0} \frac{r(x + \delta x, y, z) - r(x, y, z)}{\delta t}$$

Similarly,

$$\frac{\partial r}{\partial y} = \lim_{\delta t \to 0} \frac{r(x, y + \delta y, z) - r(x, y, z)}{\delta t}$$

and

$$\frac{\partial r}{\partial x} = \lim_{\delta t \to 0} \frac{r(x, y, z + \delta z) - r(x, y, z)}{\delta t}$$

if the limits exist.

Similarly, if $F = u(x, y, z, t)i + v(x, y, z, t)j + w(x, y, z, t)k$, the partial derivatives of F with respect to x, y, z, t respectively can be written as:

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}i + \frac{\partial v}{\partial x}j + \frac{\partial w}{\partial x}k$$
$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y}i + \frac{\partial v}{\partial y}j + \frac{\partial w}{\partial y}k$$
$$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z}i + \frac{\partial v}{\partial z}j + \frac{\partial w}{\partial z}k$$
$$\frac{\partial F}{\partial t} = \frac{\partial u}{\partial t}i + \frac{\partial v}{\partial t}j + \frac{\partial w}{\partial t}k$$

Activity 1

Let $A = 2x^3zi + xy^2z^3j - x^2z^3k$, find the partial derivatives of A withrespect to x, y and z.

Solution

$$\frac{\partial A}{\partial x} = 6x^2zi + y^2z^3j - 2xz^3k$$
$$\frac{\partial A}{\partial y} = 2xyz^3j$$
$$\frac{\partial A}{\partial z} = 2x^3i + 3xy^2z^2j - 3x^2z^2k$$

Activity 2

Given that
$A = \sin(xyz)i + e^{xy}j + \ln(2xz)\,k$
Obtain $\frac{\partial^2 A}{\partial x \partial y}$

Solution

$$\frac{\partial A}{\partial x} = yx\cos(xyz)i + ye^{xy}j + \frac{1}{Z}k$$

$$\frac{\partial^2 A}{\partial x \partial y} = (z\cos(xyz) - xyz^2 \sin(xyz))i + (e^{xy}(1+x))j$$

Summary

In this unit, you have been taught how to find the partial derivatives of a vector-valued function and how to apply your knowledge of the partial derivatives of vector-valued functions in solving grad., div. and curl of any given vector-valued function(s).

Self Assessment Questions

(1). Let $r = xsinyi + e^2zcosyj - 3e^{-}yk$. **Find the following partial derivatives of** r :

$$\frac{\partial r}{\partial x};\ \frac{\partial^2 r}{\partial x^2};\ \frac{\partial r}{\partial y};\ \frac{\partial r}{\partial z};\ \frac{\partial^2 r}{\partial y^2};\ \frac{\partial^2 r}{\partial z^2};\ \frac{\partial^2 r}{\partial x \partial y};\ \frac{\partial^2 r}{\partial y \partial x};\ \frac{\partial^2 r}{\partial z \partial y}\ and\ \frac{\partial^2 r}{\partial y \partial z}$$

(2). Let $A = xyz^3 i + 3yzj - x^2yzk$ **and** $B = 3yi - xzj + y^2zk$, **find:** $\frac{\partial^2 (A \times B)}{\partial x \partial y}$ **at the point** $(-1, 1, -1)$.

(3). Suppose $F(x, y, z) = 2xyz^3$ **and** $r = 2xyi - y^3zj + x^3yk$, **find:** **(a).** $\frac{\partial (Fr)}{\partial z}$ (b). $\frac{\partial^2 (Fr)}{\partial x \partial z}$ (c). $\frac{\partial^2 (Fr)}{\partial y \partial z}$ (d). $\frac{\partial^2 (Fr)}{\partial x^2}$ **at the point** $(-1, 1, 2)$.

Tutor Marked Assignment

- **Let** $F = cos(x + 3y)i - e^c osyz^2 j + yz^3 k$, **evaluate** $\frac{\partial F}{\partial x}$, $\frac{\partial^2 F}{\partial x \partial y}$, $\frac{\partial^2 F}{\partial y \partial x}$ **and** $\frac{\partial^2 F}{\partial^2 x}$.
- **Given** $\Phi = 3x^2 sinxy^3 i + 5xyz2j - \frac{y}{z}k$, **find** $\frac{\partial \Phi}{\partial z}$.

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UNIT 3

Different types of derivatives of vector-valued functions and their consequences

Introduction

In this unit, you will be taught three major types of derivatives of vector-valued functions and their consequences which are gradient, divergence and curl.

Learning Outcomes

At the end of this unit, you should be able to:

1. find the gradient of a scalar field;
2. determine the divergence of a vector field; and
3. obtain the curl of the two vector points.

Main Content

Let V be a vector field, X a scalar and $u, v, w \in V$ with $\alpha, \beta \in X$. Then, we have the following products:

(1). Scalar multiplication: $\alpha u = u\alpha \in V$ (vector).

(2). Dot or Scalar product: $u \cdot v = v \cdot u = \beta \in X$ (scalar).

(3). Cross or Vector product: $u \times v = v \times u = w \in V$ (vector).

The three major different types of derivatives of vectorvalued functions are highlighted above. You will now learn their consequences (grad., div. and curl).

Gradient of a vector: Del or nabla denoted as ∇ is a vector differential operator defined as

$$\nabla = \left(\frac{\partial}{\partial x}i, \frac{\partial}{\partial y}j, \frac{\partial}{\partial z}k\right) = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$

so that

$$\nabla F = \left(\frac{\partial}{\partial x}i, \frac{\partial}{\partial y}j, \frac{\partial}{\partial z}k\right) F = \frac{\partial F}{\partial x}i + \frac{\partial F}{\partial y}j + \frac{\partial F}{\partial z}k$$

known as gradient (scalar multiplication). Thus,

$$Grad\ f = \nabla f = \left(\frac{\partial f}{\partial x}i, \frac{\partial f}{\partial y}j, \frac{\partial f}{\partial z}k\right) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

which is a vector. It will be observed that in case of gradient of function f defined as ∇f, the function f is a scalar quantity while the differential operator ∇ a vector quantity which makes ∇f, the gradient of a function a scalar multiplication.

Activity 1

Find the gradient of the scalar field $f(x, y, z) = 2x^2 \cos z - xe{-}^{3y}$ at point (1, 0, 0).

Solution

By definition, grad. f is defined as

$$Grad\ f = \nabla f = \left(\frac{\partial f}{\partial x}i, \frac{\partial f}{\partial y}j, \frac{\partial f}{\partial z}k\right)$$

$$= \frac{\partial}{\partial x}\left(2x^2cosz - xe^{-3y}\right)i + \frac{\partial}{\partial y}\left(2x^2cosz - xe^{-3y}\right)j + \frac{\partial}{\partial z}\left(2x^2cosz - xe^{-3y}\right)k$$

$$= (4xcosz - e^{-3y})i + 3xe^{-3y}j - 2x^2sinzk.$$

At point (1, 0, 0) therefore, Grad $f = \nabla f = 3i + 3j$.

Divergence of a vector field **(Scalar Product):** The divergence of a vector $F = (F^1, F^2, F^3)$ is defined

$$Div.F = \nabla \cdot F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(F_1 i + F_2 j + F_3 k) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar. Hence, divergence of a vector field is a scalar product.

Activity 2

Given $F = y^2 log_e xi - zsinyj + xe{-}^{2z}k$, find div. F at point (1, 0,0).

$$Div.F = \nabla \cdot F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \cdot F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \cdot (y^2 log_e xi - zsinyj + xe^{-2z}k)$$

$$= \frac{\partial\left(y^2 log_e x\right)}{\partial x} - \frac{\partial\left(zsiny\right)}{\partial y} + \frac{\partial\left(xe^{-2z}\right)}{\partial z} = \frac{y^2}{x} - zcosy - 2xe^{-2z}$$

Curl of the product of two vector point functions (Cross Product): The curl of a vector field $F = (F_1, F_2, F_3)$ is defined as cross product of ∇ and the vector F as follow:

$$Curl\ F = \nabla \times F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \times (F_1 i + F_2 j + F_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k.$$

The curl of a vector field F is a measure of the tendency of the vector F to rotate.

Remark:- We remark that a vector field F is:

(1). rotational if its curl exists (i.e if curl F 6≠ 0); and

(2). irrotational if curl $F = 0$.

Activity 3

Let $F = x^2 zi - 2xyj + 3yzk$, evaluate curl F at point $(1, -1, 2)$.

Solution

By definition,

$$Curl\ F = \nabla \times F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \times (x^2 zi - 2xyj + 3yzk) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & -2xy & 3yz \end{vmatrix}$$

$$= \left(\frac{\partial 3yz}{\partial y} - \frac{\partial(-2xy)}{\partial z}\right) i + \left(\frac{\partial x^2 z}{\partial z} - \frac{\partial 3yz}{\partial x}\right) j + \left(\frac{\partial(-2xy)}{\partial x} - \frac{\partial x^2 z}{\partial y}\right) k = 3zi + x^2 j - 2yk.$$

At point $(1, -1, 2)$, curl $F = 6i + j + 2k$.

Summary

After studying this unit, there is no doubt that you can now find the gradient of a scalar field, determine the divergence of a vector field and obtain the curl of two vector points.

Self Assessment Questions

(1). Let $\Phi = x^2 y sin(1-z^2) + xy^3 e^{-3z^2}$**, evaluate** $\nabla\Phi$ **and** $|\nabla\Phi|$ **at point** $(1, -1, 0)$**.**

(2). Find the directional derivative of $\Phi = 3xy^2 + 5x^3yz^2$ **at** $(1, 1, 1)$ **in the direction** $2i + j - 3k$**.**

(3). Let $F = x^3 zi + xy^2 z^2 j - 3yzk$**, find** $\nabla.F$ **at** $(-1, 1, 1)$**.**

Tutor Marked Assignment

- Let $F = 3x^2 z + xy^3$, determine:

  (a). ∇F (b). $|\nabla F|$ (c). a unit vector perpendicular to the surface at the point $(2, 1, 1)$.

  (d). the angle between the surface $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 - z^2 = 1$ at the point $(-1, 1, -1)$.

- Suppose $A = 2x^3 yi + y^2 zj - 3xyz^2 zk$ and $F = 3xy + xz^3$, find:

  (a). $A \times \nabla F$ (b). $A \cdot \nabla F$ at the point $(1, 1, 1)$.

- Suppose $F = xy^2zi + xyj - 3xz^2k$, find:

  (a). $A\nabla \cdot F$ (Div. F) atthe point $(1, 1, 1)$.

- Let $F = (2xy^3, xz^2, xy^2 - 2z^3)$, find:

  (a). grad $\cdot$ div F (b). curl F (c). $|$curl curl $F|$

References

- H. K. Dass (2013). Advanced Engineering Mathematics, New Delhi: S. Chand and Company PVT Ltd, pg. 383-420.
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UNIT 4

Integration of vector function

Introduction

Building on your knowledge of the derivatives of vectorvalued functions and their consequences in the previous units, you will now learn the anti-derivative of a vector-valued function in this unit.

Learning Outcomes

At the end of this unit, you should be able to:

1. integrate a vector-valued function;
2. evaluate definite and indefinite integrals of vector-valued functions; and
3. apply the knowledge of anti-derivative of a vector-valued function to practical problems.

Main Content

Let $r(t) = \dfrac{dF}{dt}$, then its reverse process (anti differentiation) known as integration is given as

$$\int r dt = F + C$$

where C is a vector constant.

Activity 1

(1). Evaluate each of the following indefinite integrals:

(a). $\int (3t^2 i - 4tj + 8e^{-2t}k)dt$ **(b).** $\int 12(cos3ti + sin4tj - e^{-3t}k)dt$

(2). Determine each of the following definite integrals:

$$\textbf{(a).}\ \int_0^{\frac{\pi}{2}}(sinti - costj)dt\ \ \textbf{(b).}\ \int_0^2(ti - 2tj + 5k)dt$$

(3). Given $r'(t) = 9t^2i + (2t-1)j + 5k$, find the anti derivative of $r'(t)$ that satisfies the initial condition $r(0) = 2i + j - 3k$.

(1). (a). $\int(3t^2i - 4tj + 8e^{-2t}k)dt = t^3i - 2t^2j - 4e^{-2t}k$

(b). $\int 12(cos3ti + sin4tj - e^{-3t}k)dt = 4sin3ti - 3cos4tj + 4e^{-3t}k$

(2). (a). $\int_0^{\frac{\pi}{2}}(sinti - costj)dt = |-costi - sintj|_0^{\frac{\pi}{2}} = (-cos\frac{\pi}{2}i - sin\frac{\pi}{2}j) - (-cos0i - sin0j) = i - j$

(b). $\int_0^2(ti - 2tj + 5k)dt = |\frac{t^2}{2}i - t^2j + 5tk|_0^2 = (2i - 4j + 10k) - (0 - 0 + 0) = 2i - 4j + 10k$

(3). Let $r'(t) = \frac{dr}{dt} = 9t^2i + (2t-1)j + 5k$. **Then,** $dr = [9t^2i + (2t-1)j + 5k]dt$ **and**

$r(t) = \int(9t^2i + (2t-1)j + 5k)dt = (3t^3 + c_1)i + (t^2 - t + c_2)j + (5t + c_3)k$

where $r(0) = 2i + j - 3k$ **so that** $r(0) = c_1i + c_2j + c_3k = 2i + j - 3k$ **which implies that** $c_1 = 2\ \ c_2 = 1,\ \ and\ \ c_3 = -3$. **Thus,** $r(t) = (3t^3 + 2)i + (t^2 - t + 1)j + (5t - 3)k$.

Summary

In this unit, you have been taught how to integrate a vectorvalued function and apply your knowledge of anti-derivative of a vector-valued function to practical problems.

Self Assessment Questions

(1). Evaluate each of the following indefinite integrals:

$$\textbf{(a).}\ \int\left(\frac{1}{t^2+8}i + log_etj - \frac{1}{t^3}k\right)dt\ \ \textbf{(b).}\ \int(i - 2j + t^{\frac{3}{2}}k)dt$$

$$\textbf{(c).}\ \int(3t^2i - 4tj + 5k)dt\ \ \textbf{(d).}\ \int[(t^2 + 2t)i - sec^2tj + 4sin2tk]dt$$

(2). Determine each of the following definite integrals

(a). $\int_0^{\frac{\pi}{6}}(sin3ti - costj)dt$ **(b).** $\int_0^2(4t^3i - 6tj + k)dt$

(3). Determine $r(t)$ using the given initial conditions in each of the following

(a). $\frac{dr}{dt} = 4sin2ti + 8costj + sec^2t$; **with** $r(\frac{\pi}{2}) = 2i + 3j - k$.
(b). $r'(t) = 2ti + j - k$; **with** $r(0) = i + j + k$.

Tutor Marked Assignment

- **Find the anti-derivative of the vector-valued function** $r'(t) = \frac{cost}{1-sint}i + \frac{1}{t^2+1} - e^{-3t}k$ **with the initial condition** $r(0) = i + j + k$.
- **Determine** $r(t)$ **using the given initial conditions in each of the following**

  (a). $r'(t) = sinti - costj + e^{-3t}k)dt$; **with** $r(0) = i - 3j + k$.

  (b). $\int(8ti + 3j - e^{-t}k)dt$; **with** $r(0) = 2i + 3j + k$.
- **Determine each of the following definite integrals:**

  (a). $\int(e^{-2t}sinti - e^{-2t}costj + e^{-2t}k)dt$ **(b).** $\int(ln3ti + tantj - e^{-3t}k)dt$

References

- H. K. Dass (2013). Advanced Engineering Mathematics, New Delhi: S. Chand and Company PVT Ltd, pg. 383-420.
- B. D. Gupta (2010). Mathematical Physics, India: Vikas Publishing House, 4th Ed, pg. 1.50-1.120.

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Module 4

FORCE, MOMENTUM, LAWS OF MOTION UNDER GRAVITY, PROJECTILE, VERTICAL MOTION AND IMPACT OF TWO SMOOTH SPHERES

Units

UNIT 1

Force and Momentum

Introduction

In one way or the other you might have exerted or experienced a form of force. For instance, if you kick a ball, fetch water from the well, throw a Javelin, strike a hammer to drive a nail to the wood or a wall etc., a force has been applied to achieve these tasks.

Learning Outcomes

At the end of this unit, you should be able to:

1. define force;
2. define momemtum;
3. discuss types of forces; and
4. calculate magnitude of force.

Main Content

Force

Force can be defined as anything that causes changes to an object or a mass (amount of substance or matter contained in or made up a body). It can also be defined as the product of mass and acceleration. This is because, when a force acted on a body, it causes the body to change position or cause the body to accelerate. A force is the action which tends to change the state of rest or uniform motion of a body in a straight line. The effect of a force on a body depends on: the magnitude of the force; the direction in which the force acts; and the point of application of the force. On these facts, force is a vector quantity since it has magnitude and direction. The unit of force thus depends on the unit of mass and the unit of acceleration. $F = ma$. where m is the mass and a is the acceleration The unit of mass is kg while the unit of acceleration is ms^{-2}. Therefore, F unit is $kgms^{-2}$ but the generalized unit is Newton, denoted by N.

Types of force

- Magnetic Force
- Gravitational Force
- Mechanical Force

When a magnet is in contact with magnetic and nonmagnetic mixture like iron fillings and rice, the force of attraction that makes iron fillings to be separated from nonmagnetic object is called magnetic force.

Suppose a Mango was plucked from its tree, a force makes it falls down. This kind of force is known has gravitational force.

The categories of action such as when you start a car, the force that enables its movement is known as mechanical force.

Momentum

The product of mass of a particle or an object and its velocity at an instance is regarded as its momentum. Momentum Q is a vector since velocity is a vector quantity. It is mathematically written as $Q = mV$.

Summary

In this unit, you have learnt how to define force, momemtum and types of forces. You also learnt how to calculate magnitude of force.

Self Assessment Questions

(1) Define force.

(2) Define momemtum.

(3) How many types of force do we have? Discuss.

Tutor Marked Assignment

- Let the mass of a stone be $10kg$ and the stone has an acceleration $5ms^{-2}$ when it is thrown up. Find the magnitude of the force.

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

Further Reading

- Tuttuh-Adegun M. R., Sivasubrama S., and Adegoke R., (2001). Further Mathematics Project. NPS Educational Publishers Limited.

UNIT 2

Laws of Motion under Gravity

Introduction

In this unit, you will be taught three types of motion.

At the end of this unit, you should be able to:

Learning Outcomes

1. state Newton's first law of motion;
2. explain Newton's second law of motion;
3. state Newton's third law of motion; and
4. solve problems using Newton's laws of motion.

Main Content

Newton's first law of motion

Newton's first law of motion states that, every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by external impressed force(s).

Newton's second law of motion

Newton's second law of motion states that, the rate of change of momentum of a body is proportional to the applied force and is in the direction of the force(s).

$F \quad \frac{d}{dt}(mv)$

$F = km\,\frac{dv}{dt}$ (where k is a constant and can be 1)

$F = ma$(since $\frac{dv}{dt} = a$).

Newton's third law of motion

Newton's third law of motion states that, action and reaction are equal and opposite. When two bodies are in contact, the force of action and reaction are equal in magnitude and opposite in direction.

Activity 1

A ball of mass $4kg$ on a football pitch was kicked by a playerat penalty shoot out. If the ball has an acceleration of $3ms^{-2}$. Find the magnitude of force F.

Solution

By Newton's law of motion

$F = m \times a$

here, $m = 4kg$, $a = 3ms^{-2}$

$F = (4 \times 3)N$

$F = 12N$.

Activity 2

Find the momentum of an object which has a velocity $15ms^{-1}$ and mass $100kg$

Solution

The momentum of an object is mathematically express as $Q = mV$ Here, the mass $m = 100kg$ and the velocity is $V = 15ms^{-1}$

substitute the given values of m and V to the formula, we have

$Q = 100 \times 15$

$Q = 1500kg/ms$

Summary

Now that you have studied this unit, you should be able to explain Newton's laws of motion and apply them to solve problems.

Self Assessment Questions

In a sport activity, one of the participants threw a javelin of mass $6.5kg$ and the javelin traveled some reasonable distance with an acceleration of $10ms^{-2}$. Find the magnitude of force F.

Tutor Marked Assignment

- A constant force F of magnitude $54N$ is applied to a bag of mass $9kg$. Find the acceleration produced in the bag. Assuming the bag starts from rest, what velocity does it attain after 4 seconds?

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

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- Tuttuh-Adegun M. R., Sivasubrama S., and Adegoke R., (2001). Further Mathematics Project. NPS Educational Publishers Limited.

UNIT 3

Kinematics and Acceleration of Particle Moving in a Plane

Introduction

Kinematics is the motion of an object without any explanation of the cause of the motion. That is, kinematics is lagged pure motion of physical bodies.

Learning Outcomes

At the end of this unit, you should be able to:

1. velocity of a particle;
2. acceleration or deceleration of a particle; and
3. distance of a particle moving in a plane.

Main Content

A particle in a plane can be described as position vector r and $V = \frac{dr}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$

If V is the velocity of the body which has traveled a distance x in time t, we have $V = \frac{dx}{dt}$ and hence $x = \int vdt$.

Also, if a is the acceleration of the body at time t, $a = \frac{dv}{dt}$ and $v = \int adt$

Activity 1

A body moves along a straight line so that at time $t(s)$ after passing a formed point A on the line its velocity is $(3t^2 + 2t + 4)m/s$. If it arrives at point B on the same line $4sec$ after passing point A. Calculate the distance AB.

Solution

Recall that $x = \int vdt$ and v is given as $(3t^2+2t+4)m/s$. Therefore,

$x = \int(3t^2 + 2t + 4)dt$ when $t = 4sec$

$\frac{3t^3}{3} + \frac{2t^2}{2} + 4t = t^3 + t^2 + 4t$, by substituting $t = 4$, we have

$4^3 + 4^2 + 4(4) = 64 + 16 + 16 = 96$

Therefore, the distance $|AB|$ is $96m$

Summary

Now, you should be able to find the velocity of a particle, the acceleration or deceleration of a particle and the distance of a particle moving in a plane.

Self Assessment Questions

A particle is projected in a straight line from a point O with a speed of $12ms^{-1}$. At time t seconds later, its acceleration is $(2 + 4t)ms^{-2}$. For the time when $t = 6$. Calculate for the particle:

(i) its velocity.

(ii) its distance from O.

Tutor Marked Assignment

- A particle moves in a straight line from the origin O with initial velocity $4ms^{-1}$. Its acceleration at t seconds later is $(4t - 6)ms^{-2}$. Calculate:

(i) its velocity after $4s$.

(ii) its distance from O when it is momentarily at rest.

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

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UNIT 4

Projectiles and Vertical Motion

Introduction

In unit 2, you were taught the laws of motion under gravity. In this unit, motion of object under gravity in two dimensions. Projectile motion can be treated as two rectilinear motions, one in the horizontal direction experiencing zero acceleration and the other in vertical direction.

Learning Outcomes

At the end of this unit, you should be able to:

1. solve problems on motion of projectile under gravity using Newton's laws of motion.

Main Content

Corresponding equations of motion in respect of projectile

Suppose a table tennis ball is projected with an initial velocity u at an angle θ to the table (horizontal), both vertical and horizontal components of velocity shall be considered as shown in the figure below and the equations of motion in this respect follow suit.

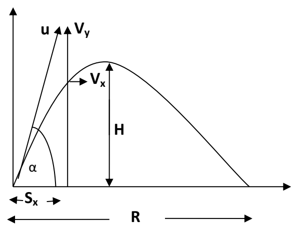


Figure 9

$(i)\ v_x = ucos\theta\ \ (ii)\ v_y = usin\theta - gt\ \ (iii)\ S_x = utcos\theta\ \ (iv)\ S_y = utsin\theta - \frac{1}{2}gt^2\ \ (v)\ v_x{}^2 = u^2cos^2\theta\ \ (vi)\ v_y{}^2 = (usin\theta - gt)^2 = u^2sin^2\theta - 2gS_y$

where S_x and S_y are horizontal and vertical distance respectively, v_x and v_y are horizontal and vertical components of the velocity respectively.

Now, it is worth mentioning that the force due to gravity acts against the motion of the particle, therefore, g becomes negative in the equations. Likewise, also note that, the horizontal component of g is zero while otherwise stated, g shall be taken as $9.8ms^{-2}$

Activity 1

A ball is projected with an initial velocity of $21ms^{-1}$ at an angle of 30^0 to the horizontal. Find, after 1 seconds:

(i) the vertical component of the velocity;

(ii) the horizontal component of the velocity;

(iii) the magnitude of the velocity;

(iv) the vertical distance traveled; and

(v) the horizontal distance traveled.

Solution

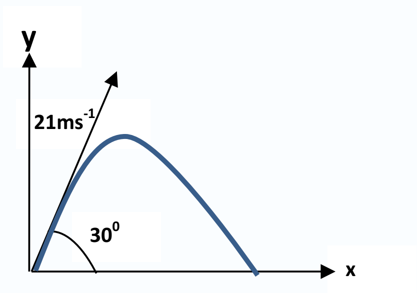


Figure 10

$(i) \quad v_y = usin\theta - gt$

$v_y = 21sin30^0 - 9.8 \times 1 = 10.5 - 9.8$

$v_y = 0.7ms^{-1}$

$(ii) \quad v_x = ucos\theta$

$v_x = 21cos30^0 = 3.24ms^{-1}(2decimalplaces)$

(iii) **Let v be the magnitude of the velocity, then**

$v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{0.7^2 + 3.24^2}$

$v = \sqrt{0.49 + 10.4976} = \sqrt{10.9876}$

$v = 3.32ms^{-1}$

$(iv) \quad S_y = utsin\theta - \frac{1}{2}gt^2 = 21 \times 1 \times sin30^0 - \frac{1}{2} \times 9.8 \times 1^2$

$S_y = 21 \times \frac{1}{2} - 4.9 = 10.5 - 4.9$

$S_y = 5.6m$

$(v) \quad S_x = utcos\theta = 21 \times 1 \times cos30^0$

$S_x = 3.24m$

The Greatest Height Reached

The vertical component of the velocity of an object projected becomes zero when it reaches its greatest height. Thus,

$${v_y}^2 = u^2sin^2\theta - 2gS_y$$

$$0 = u^2sin^2\theta - 2gS_y$$

Therefore,

$$2gS_y = u^2sin^2\theta$$

$$S_y = \frac{u^2sin^2\theta}{2g}$$

$$H = \frac{u^2sin^2\theta}{2g}$$

Time taken to reach the greatest height

Suppose t is the time taken to reach the greatest height then, vertical component velocity is been made use of. $v_y = usin\,\theta - gt$, which becomes

$$0 = usin\theta - gt$$

$$gt = usin\theta$$

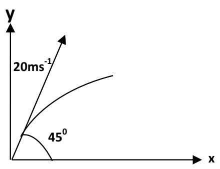
$$t = \frac{usin\theta}{g}$$

Activity 1

At a shooting range, a Police man lying on the floor fired a bullet from a point making angle 45^0 to the horizontal to hit the target. Find, if the initial velocity of the bullet is $20ms{-}1$

:(I) the greatest height reached; and

(ii) the time taken to reach the greatest height.



(I) Let h be the greatest height, then

$$h = \frac{u^2 sin^2\theta}{2g} = \left(\frac{20^2 \times (sin45)^2}{2 \times 9.8}\right)m$$

$$h = \frac{400 \times 0.7240}{19.6} = \frac{289.6}{19.6}$$

$$h = 14.78m$$

(ii) Let t be the time taken to reach the greatest height,

then

$$t = \frac{usin\theta}{g}$$

$$t = \left(\frac{20 \times sin45}{9.8}\right)S$$

$$t = 1.74S$$

Time of Flight

As usual saying, "what goes up, must come down". In this manner, the time taken for an object which was projected, to return to its original level is called the time of flight. Hint: When a projected object returns to its original level, the vertical distance S_y becomes zero. Hence, by using the formula $S_y = ut\sin\theta - \frac{1}{2}gt^2$ therefore,

$utsin\theta - \frac{1}{2}gt^2 = 0$

$t(usin\theta - \frac{1}{2}gt) = 0$ **which implies**

$t = 0$ **or**

$usin\theta - \frac{1}{2}gt = 0$ **so that**

$t = \frac{2usin\theta}{g}$

$t = 0$ **means the time the object was projected initially. Therefore, the time of flight, denoted by** T**, is given as**

$T = \frac{2usin\theta}{g}$.

Range

This is the horizontal distance covered when the object returns to its original level. Note, the horizontal component of the velocity is constant. Therefore, the range will be a product of the horizontal component of the velocity and the time of flight. This is denoted as R and given as

$$R = ucos\theta \times T$$

$$R = ucos\theta \times \frac{2usin\theta}{g} = \frac{u^2cos\theta sin\theta}{g}$$

$$R = \frac{u^2sin2\theta}{g}$$

Maximum Range

When objects are projected at a particular velocity, the range differs with various angle of projection. We recall that

$$R = \frac{u^2sin2\theta}{g}$$

For R to be maximum, $sin2\theta$ must be maximized. Here $sin2\theta$ is maximum when

$sin2\theta = 1$

$2\theta = 90^0$

$\theta = 45^0$

Therefore, a projectile will have a maximum range if it is projected at an angle of 45^0 to the horizontal.

Suppose R_{max} represents the maximum range, then

$$R_{max} = \frac{u^2}{g}$$

Summary

In this unit,you have learnt motions of projectile under gravity.

Self Assessment Questions

(1) Prove that the greatest height reached by a projectile is

$$\boxed{H = \frac{u^2 sin^2\theta}{2g}}$$

(2) Prove that the time taken to reach the greatest height by the projectile is

$$\boxed{t = \frac{usin\theta}{g}}$$

Tutor Marked Assignment

- A Soldier lying on the floor fired a bullet with an initial velocity of $60ms^{-1}$. If the vertical distance moved by the bullet, after $2seconds$ is $30m$. Find:

(i) the angle θ to the horizontal, at which the bullet is projected;

(ii) the horizontal distance traveled by the bullet after $4sec$.

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

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UNIT 5

The Trajectory of a Projectile

Introduction

In this unit, you will learn how to derive the equation of the path of a projectile. y is taken as the vertical distance moved and x as the horizontal distance moved by the projected object.

Learning Outcomes

At the end of this unit, you should be able to:

1. derive the equation of the path of a projectile.

Main Content

Now, suppose a particle at an angle θ to the horizontal with an initial velocity u is projected. The following equation of motion are needed.

$x = utcos\theta \ldots (1)$

$y = utsin\theta - \frac{1}{2}gt^2 \ldots (2)$

from (1) $t = \frac{x}{ucos\theta} \ldots (3)$

substitute the value of $t = \frac{x}{ucos\theta}$ into equation (2), it becomes

$y = u \times \frac{x}{ucos\theta} \times sin\theta - \frac{1}{2}g(\frac{x}{ucos\theta})^2$

$y = xtan\theta - \frac{1}{2}\frac{gx^2}{u^2cos^2\theta}$

$y = xtan\theta - \frac{1}{2}\frac{gx^2sec^2\theta}{u^2}$

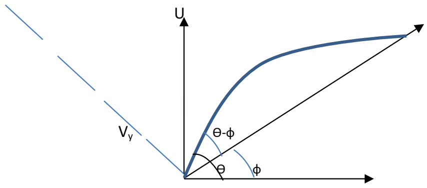
$y = bx - ax^2$, where

$b = tan\theta$

$a = \frac{gsec^2\theta}{2u^2}$

Therefore, the trajectory or path of a projectile is a parabola.

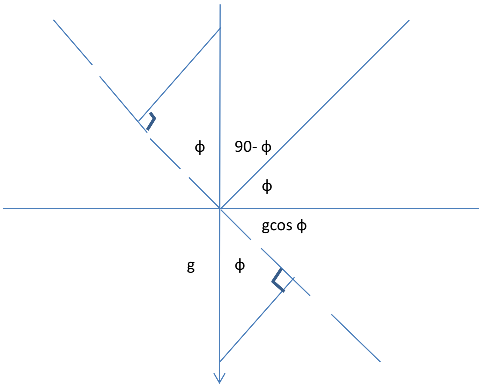
Projection along an inclined plane



From the above figure, you can see that an object projected with an initial velocity *u* at angle θ to the horizontal from a point on the plane which is inclined at an angle ϕ to the horizontal such that, θ is greater than ϕ Let V_x be the component of the velocity along the plane and V_y be the component of the velocity in a direction perpendicular to the plane. Then

$V_x = Ucos(\theta - \phi)$

$V_y = Usin(\theta - \phi)$



Component of acceleration

The retardation along the axis perpendicular to the plane is *gcos* ϕ while the distance covered along the direction perpendicular to the plane is zero.

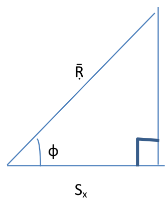
By making use of $S = ut + \frac{1}{2}at^2$**, we have**
$usin(\theta - \phi) \times t - \frac{1}{2}gcos\phi \times t^2$**. Therefore,**
$\frac{1}{2}gt^2cos\phi = utsin(\theta - \phi)$
$t = \frac{2usin(\theta - \phi)}{gcos\phi}$

Let S_x **be the horizontal distance traveled by the object in time** t**, then**

$S_x = (u cos\theta) \times t = u cos\theta \times \frac{2u sin(\theta-\phi)}{g cos\phi}$

$S_x = \frac{2u^2 cos\theta sin(\theta-\phi)}{g cos\phi}$

Range of an inclined plane



$cos\phi = \frac{S_x}{\bar{R}}$

Therefore, $\bar{R} = S_x sec\phi$

$\bar{R} = \frac{2u^2 cos\theta sin(\theta-\phi)}{g cos\phi} \times sec\phi$

$\bar{R} = \frac{2u^2}{g} cos\theta sec^2\phi sin(\theta-\phi)$

Summary

You have been taught the vertical motion via projectile in this unit.

Self Assessment Questions

Prove that the path of a projectile is a parabola

Tutor Marked Assignment

In a sport competition, an athlete threw a shut-put at point *P*, with an initial velocity of $56ms^{-1}$, at an angle of 60^0 to the horizontal. Find its vertical displacement when its horizontal displacement is *70m*.

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

Further Reading

- Tuttuh-Adegun M. R., Sivasubrama S., and Adegoke R., (2001). Further Mathematics Project. NPS Educational Publishers Limited.

UNIT 6

Impact of two smooth spheres

Introduction

There are several occasions when two bodies or objects collide or have an impact. In this unit, you will learn about the impact of smooth spheres. One of the examples can be found in a sport such as snooker.

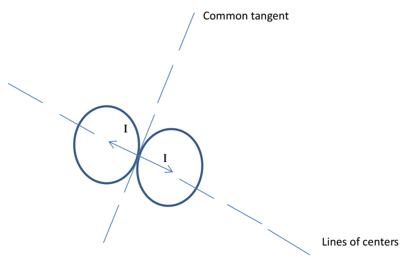
Learning Outcomes

At the end of this unit, you should be able to:

1. solve problems relating to impact of two smooth spheres
2. apply momemtum, impulse and Newton's law of restitution to investigate collisions involving objects free to move in two dimensions.

Main Content

There is always a common tangent at the moment of impact of two smooth spheres, which is perpendicular to the line through the centers of the two spheres.



From the above figure, it can be deduced that:

(1) the impulse affecting each sphere acts along the line of centers because the reaction between the two spheres acts along the line of centers

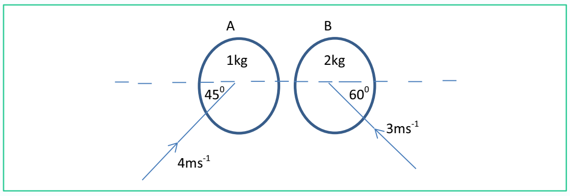
(2) Newton's law of restitution applies to the components of the velocities of the spheres parallel to the line of centers;

(3) the law of conservation of momentum applies parallel to the line of centers; and the components of the velocities of the spheres perpendicular to the line of centers are unchanged in the impact. Although the law of conservation of momentum also applies perpendicular to the line of centers, but since the components of velocity in this direction are unchanged, it becomes negligible.

Newton's law of restitution shall be made use of when it is perpendicular to the center, that is, $V \sin \beta = {}_e U \sin \alpha$ while e is the coefficient of restitution

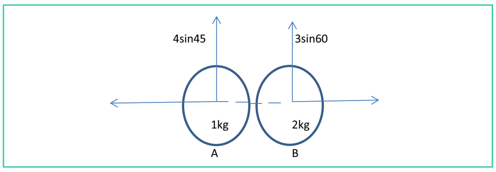
By dividing, $\tan \beta = e \tan\alpha$ and since $e \leq 1$ then $\tan \beta \leq \tan \alpha$ so, $\beta \leq \alpha$

Activity 1



Two smooth spheres A and B of mass $1kg$ and $2kg$ collides. A moves with $4ms^{-1}$ speed in a 45^0 direction to the line of center while B moves with $3ms{-}1$ at 60^0 to the line of centers before collision as shown in the figure below. Find: (a) the kinetic energy lost in the impact (b) the magnitude of the impulse exerted by A on B.

Solution



From the diagram, you can see that there is no change in the components perpendicular to the line of centers. Therefore, changes in the velocity components parallel to the line of centers shall be obtained as a tool to find the loss in Kinetic Energy.

The conservation of momentum is given as

$1 \times 4cos45^0 - 2 \times 3cos60^0 = 2W - V$. When simplified, it becomes

$2\sqrt{2} - 3 = 2W - V \ \ldots \ (i)$

By Newton's law of restitution, we have

$V + W = \frac{3}{4}(4cos45^0 + 3cos60^0)$. On simplifying

$V + W = \frac{3\sqrt{2}}{2} + \frac{9}{8} \ \ldots \ (ii)$

Make V the subject of the formula in equation (ii) then substitute it into equation (i). $3W = \frac{7\sqrt{2}}{2} - \frac{15}{8} \ \ldots \ (iii)$

$W = \frac{7\sqrt{2}}{3} - \frac{5}{8} \approx 1.0249$

Then substitute the value of $W = 1.0249$ into equation (iii) to obtain the value of V. Recall that

$V = \frac{3\sqrt{2}}{2} + \frac{9}{8} - W$. Then V becomes

$V = \frac{3\sqrt{2}}{2} + \frac{9}{8} - 1.0249 \approx 2.2214$

Kinetic energy lost by $A = \frac{1}{2} \times 1 \times ((2\sqrt{2})^2 - 2.2214^2) = 1.53J$

Kinetic energy lost by $B = \frac{1}{2} \times 2 \times (1.5^2 - 1.0249^2) = 1.20J$

Therefore, the loss in kinetic energy is due to its changes in the velocities component parallel to the line of centers

$Total \quad K.E \quad lost = 2.73J$

(b) Impulse on $B = 2(1.0249 + 1.5) = 5.05Ns$

Summary

In this unit, you have learnt about an impact between two smooth spheres and found that: The reaction between the spheres acts along the line of centers, so the impulse affecting each sphere also acts along the line of centers. The components of the velocities of the spheres perpendicular to the line of centers are unchanged in the impact. Newton's law of restitution applies to the components of the velocities of the spheres parallel to the line of centers. The law of conservation of momentum applies parallel to the line of centers.

Self Assessment Questions

Suppose two small smooth spheres A and B have equal radii. The mass of A is $4mkg$ and the mass of Q is $5mkg$. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision, the velocity of P is $(3j)ms^{-1}$ and the velocity of Q is $(5i - j)ms^{-1}$. Immediately after the collision, the velocity of P is $(2i + j)ms^{-1}$. Find the speed of B immediately after the collision.

Tutor Marked Assignment

Two small smooth spheres P and Q have equal radii. The mass of P is $2mkg$ and the mass of Q is $3mkg$. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision, the velocity of P is $(5j)ms^{-1}$ and the velocity of Q is $(3i - j)ms^{-1}$. Immediately after the collision, the velocity of P is $(3i + 2j)ms^{-1}$. Find

(i) the speed of Q immediately after the collision

(ii) a unit vector parallel to the line of centers of the spheres at the instant of the collision.

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

Further Reading

- Tuttuh-Adegun M. R., Sivasubrama S., and Adegoke R., (2001). Further Mathematics Project. NPS Educational Publishers Ltd.

Module 5

COORDINATE GEOMETRY

Units

Unit 1 - Two Dimensional Coordinate Geometry
Unit 2 - Straight Lines
Unit 3 - Conics, Parabola, Hyperbola and Ellipse
Unit 4 - Parabola, Hyperbola and Ellipse

UNIT 1

Two Dimensional Coordinate Geometry

Introduction

In a 100m relay race, participants start the race from one end and stop at another. The path which they follow is seen to be a straight line. Meanwhile, when lines intercept or drawn by each side, there are relationships that come in place which we shall be considering in this unit.

Learning Outcomes

At the end of this unit, you should be able to:

1. explain the relationship between Cartesian and polar coordinates;
2. determine the Cartesian and Polar coordinates of a point; and
3. obtain the length of a straight line between two points.

Main Content

By considering *fig.1* below, the line along *y*-direction is known as ordinate, one along *x*-direction is known as abscesses and *P*(*x,y*) is a point on the plain.

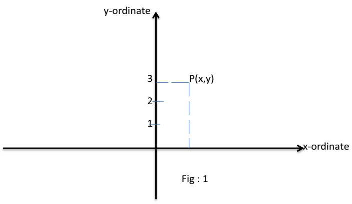


Fig : 1

Relationship between Cartesian and Polar Coordinates

Cartesian and Polar coordinates of point A in a plain are determined by A(x, y) and A(r, θ). Where x and y are points along x and y directions respectively, r is radius to the center and θ is the angle between the radius and the tangent.

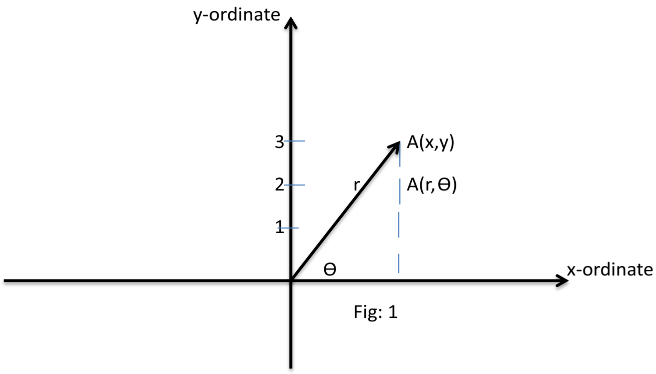


Fig: 1

From $fig.2$**,**

$Cos\theta = \frac{x}{r}$

$Sin\theta = \frac{y}{r}$

therefore, $tan\theta = \frac{y}{x}$

$Cos^2\theta = \frac{x^2}{r^2}$

$Sin^2\theta = \frac{y^2}{r^2}$

$Cos^2\theta + Sin^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2}$

while $Cos^2\theta + Sin^2\theta = 1$ **then,**

$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

By simplification, $r^2 = x^2 + y^2$ **which means**

$r = \sqrt{x^2 + y^2}$

$\theta = tan^{-1}\frac{y}{x}$

Activity 1

Find the Cartesian Coordinate of the point $B(3, \frac{\pi}{2})$

Solution

Here, the radius *r* and the tangential angle has been given. *r* = *3*, *θ*=90 (since *π* is 180) Following the relationship above,

$$x = rcos\theta$$
$$x = 3cos90$$
$$x = 3 \times 0, \textbf{ so } x = 0$$
$$\textbf{also,}$$
$$y = rsin\theta$$
$$y = 3sin90$$
$$y = 3 \times 1, \quad y = 3 \textbf{ Therefore, the coordinate is } (0, 3)$$

Length of a Straight Line between Two Points

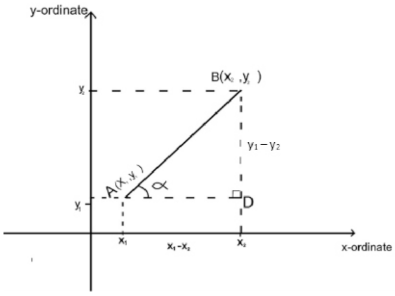


fig. 3

By Pythagoras Theorem:

$$|AB|^2 = |AD|^2 + |BD|^2$$
$$|AB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is the formula for finding the length or distance of a straight line between two points.

Activity 1

Find the distance between the following pairs of points

(I) $A(1,3)$ and $B(4,7)$ (ii) $P(-3,-4)$ and $Q(2,8)$

Solution

(I) Recall that, the formula for finding the distance between two points on a straight line is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore, for (i), $x_2 = 4$, $x_1 = 1$, $y_2 = 7$, $y_1 = 3$

which implies that,

$AB = \sqrt{(4-1)^2 + (7-3)^2}$

$AB = \sqrt{(3)^2 + (4)^2}$

$AB = \sqrt{9 + 16} = \sqrt{25}$

$AB = 5$

(ii) $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

So, $x_2 = 2$, $x_1 = -3$, $y_2 = 8$, $y_1 = -4$

which implies that,

$PQ = \sqrt{(2-(-3))^2 + (8-(-4))^2}$

$PQ = \sqrt{(5)^2 + (12)^2}$

$PQ = \sqrt{25 + 144} = \sqrt{169}$

$PQ = 13$

Summary

In this unit, you have learnt the relationship between Cartesian and polar coordinates. You have also learnt how to determine the Cartesian and Polar coordinates of a point and obtain the length of a straight line between two points.

Self Assessment Question

Find the distance between the points $A(2, 1.5)$ and $B(3, 3.5)$

Tutor Marked Assignment

- Find the distance between the points $A(5, 3)$ and $B(9, 7)$

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

Further Reading

- Tuttuh-Adegun M. R., Sivasubrama S., and Adegoke R., (2001). Further Mathematics Project 2. NPS Educational Publishers Limited

UNIT 2

Straight Lines

Introduction

In this unit, you will be learning about straight lines. For example, how to find the gradient of a line, division of a line into ratios and equation of a straight line.

Learning Outcomes

At the end of this unit, you should be able to:

1. find the gradient of a line;
2. determine the angle between two straight lines; and
3. obtain equation of a straight line.

Main Content

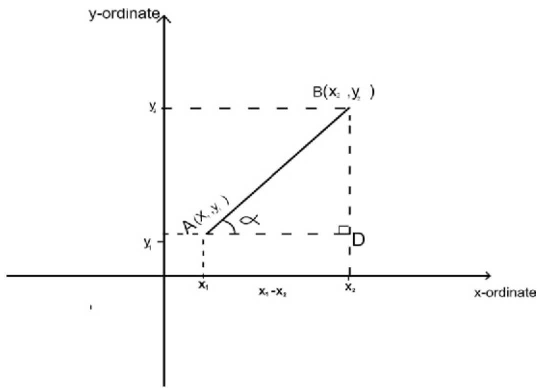


fig. 4

$$tan\alpha = \frac{BD}{AD} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$Gradient = m = Slope$$

$$tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

Gradient is the tangent of the angle made with the straight line.

Activity 1

Find the gradient and the angle with x-axis of points $A(3,5)$ and $B(5,9)$.

Solution

As stated above, gradient is denoted as m, such that

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-5}{5-3}$$
$$m = \frac{4}{2} = 2$$

So the gradient is 2.

To obtain the angle, we have that $tan\alpha = m$ Therefore, you substitute for m as 2. So that we have, $tan\alpha = 2$ By simplification

$$\alpha = tan^{-1}2$$
$$\alpha = 63.43^0.$$

Division of a Line to a Given Ratio

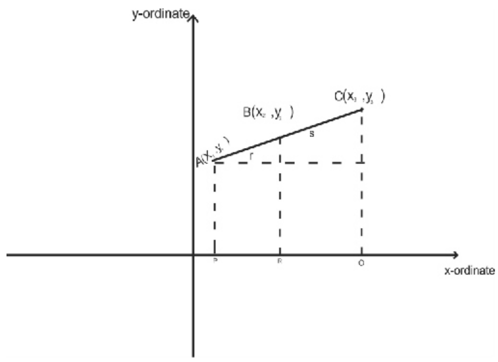


fig. 5

$$AC : CB = r : s$$

$$\frac{AC}{CB} = \frac{r}{s} = \frac{x - x_1}{x_2 - x}$$

$\frac{r}{s} = \frac{x-x_1}{x_2-x}$

$s(x - x_1) = r(x_2 - x)$

$sx - sx_1 = rx_2 - rx$

$sx + rx = rx_2 + sx_1$

$x(s + r) = rx_2 + sx_1$

$x = \frac{rx_2+sx_1}{s+r}$

Similarly,

$y = \frac{ry_2+sy_1}{s+r}$

Now, if $r = s$, we have

$x = \frac{r(x_2+x_1)}{2r}$.

Therefore, $x = \frac{(x_2+x_1)}{2}$

External Division

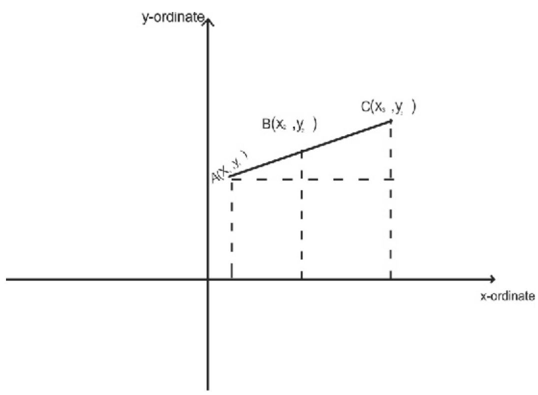


fig. 6

$$AC : CB = r : s$$

$$\frac{AC}{CB} = \frac{r}{s}$$

$$\frac{x-x_1}{x-x_2} = \frac{r}{s}$$

$$s(x - x_1) = r(x - x_2)$$

$$sx - sx_1 = rx - rx_2$$

$$sx - rx = sx_1 - rx_2$$

$$x = \frac{sx_1 - rx_2}{s - r}$$

Similarly,

$$y = \frac{sy_1 - ry_2}{s - r}$$

Angle Between Two Straight Lines

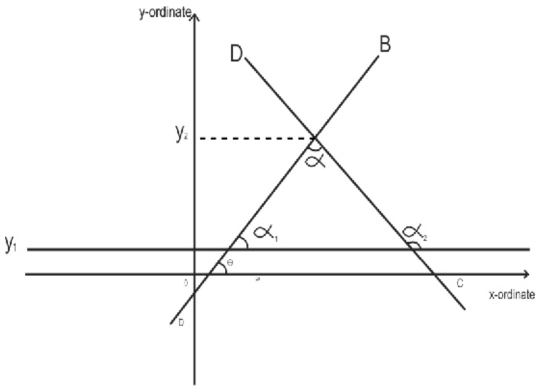


fig. 7

$\alpha_2 = \alpha_1 + \alpha$

$\alpha = \alpha_2 - \alpha_1$

$Tan\alpha = Tan(\alpha_2 - \alpha_1) = \frac{Tan\alpha_2 - Tan\alpha_1}{1 + Tan\alpha_2 Tan\alpha_1}$

$Tan\alpha = \frac{m_2 - m_1}{1 + m_2 m_1}$

m_2, m_1 **are the gradient of** $DCandAB$ **respectively.**

Note. When the two lines are perpendicular, the angle

$\alpha = 90^0$

$\alpha = tan^{-1}(\frac{m_2 - m_1}{1 + m_2 m_1})$

That is, $1 + m_2 m_1 = 0$

$m_2 m_1 = -1$ when the lines are perpendicular and

$m_2 = m_1$ when the lines are parallel.

Activity 1

Find if AB is parallel or perpendicular to PQ for $A(4, 2)$ and $B(8, 4)$, $P(4, 4)$ and $Q(6, 5)$.

Solution

Let m_1 be the gradient of AB such that

$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$ then

$y_2 = 4$, $y_1 = 2$, $x_2 = 8$ and $x_1 = 4$

This implies that,

$m_1 = \frac{4-2}{8-4}$

$m_1 = \frac{1}{2}$ then

Likewise, let m_2 be the gradient of PQ such that

$m_2 = \frac{5-4}{6-4}$

$m_2 = \frac{1}{2}$

Therefore, since $m_1 = m_2$

The lines $/AB/$ and $/PQ/$ are parallel

Equation of a Straight Line

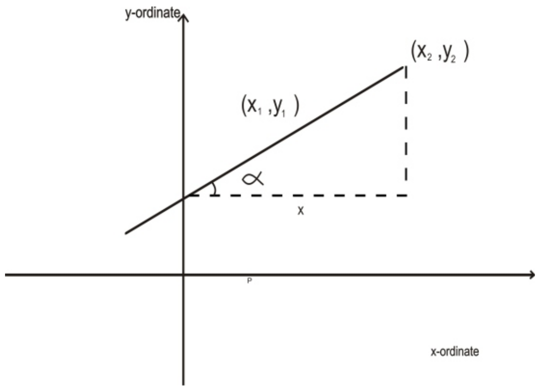


fig. 8

$m = \frac{y-c}{x}$

$mx = y - c$

$y = mx + c$

Given a straight line $\left|AB\right|$ **in the figure below**

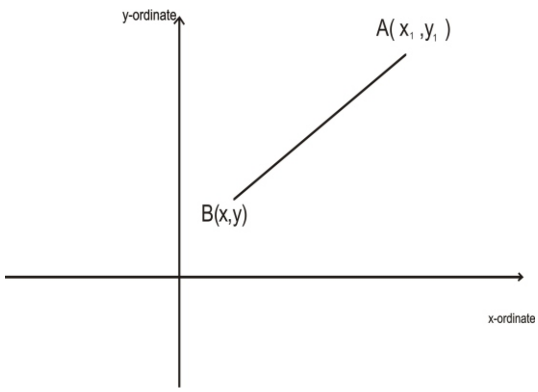


fig. 9

Remark: When two points are given:

It implies that, $y - y_1 = m(x - x_1)$.

where $m = \frac{y_2 - y_1}{x_2 - x_1}$

Activity 2

Find the equation of the line that passes through the points $A(-1, 2)$ and $B(2, 1)$

Solution

Recall that the gradient is

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Therefore, $y_2 = 1$, $y_1 = 2$, $x_2 = 2$ and $x_1 = -1$

By substitution,

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-2}{2--1} = \frac{-1}{3}$

Hence, $m = \frac{-1}{3}$

To obtain the equation of the line, we make use of the gradient of a straight line formula. That is

$m = \frac{y - y_1}{x - x_1} = \frac{y-2}{x-(-1)} = \frac{-1}{3}$

which can be expressed as

$3y - 6 = -x - 1$

$3y = -x + 5$

$y = \frac{-x}{3} + \frac{5}{3}$ (From $y = mx + c$)

$y = \frac{-1}{3}x + \frac{5}{3}$ is the required equation of the line

where $m = \frac{-1}{3}$ is the gradient and $c = \frac{5}{3}$ is the constant.

Summary

This unit has focused on teaching you how to find the gradient of a line, determine the angle between two straight lines and how to obtain equation of a straight line. Hope you understood the unit well and can now attempt the questions that follow.

Self Assessment Questions

Determine if AB is parallel or perpendicular to PQ such that $A(-2,-4)$, $B(4,-6)$, $P(10,8)$ and $Q(12,14)$.

Tutor Marked Assignment

- Find the equation of the straight line which passes through the points $M(4,-6)$ and $N(-8,4)$.

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

Further Reading

- Tuttuh-Adegun M. R., Sivasubrama S., and Adegoke R., (2001). Further Mathematics Project. NPS Educational Publishers Limited.

UNIT 3

Conics: Circle, Parabola, Hyperbola and Ellipse

Introduction

In our daily activities, by one way or the other, we might have come across with what form a circle and cone. For instance, the wheel of cars or motorcycle and a funnel. These always have relationships with planes that are unnoticed to us. You are going to learn about these relationships and their properties in this unit.

Learning Outcomes

At the end of this unit, you should be able to:

1. explain circle, parabola, ellipse and hyperbola geometry;
2. obtain the equations of circle, parabola, ellipse and hyperbola; and
3. obtain the general equations of circle, parabola, ellipse and hyperbola.

Main Content

Definition

Conic is the locus of a point that moves in a plane in such a way that its distance from a fixed point (focus) bears a constant ratio (eccentricity) to its distance from a fixed straight line (directrix) or simply, the point intersection of a plane and a right circular cone. Its the magnitude of eccentricity (e) that determines the type of curve, thus

(I) If $e = 1$, the conic is called a parabola.

(ii) If e is between 0 and 1, the conic is called an ellipse.

(iii) If $e > 1$, the conic is called hyperbola.

Meanwhile, all these conics can be obtained from a generalized equation

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Circle

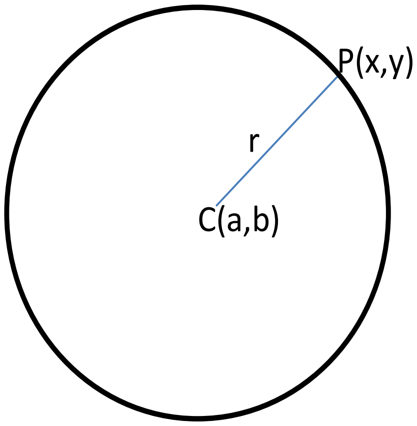


fig. 1

Let $CP = r$

$r^2 = (x-a)^2 + (y-b)^2 \ \ldots \ (1)$

Meanwhile, if c is the origin, then $(a, b) = (0, 0)$

Therefore, $r^2 = x^2 + y^2$

Otherwise, by expanding the bracket, we have

$r^2 = x^2 - 2ax + a^2 + y^2 - 2by + b^2$

now, bring r^2 to right-hand side and rearrange, we have

$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$

Conventionally, you replace $-a$ by g and $-b$ by f so that we have

$x^2 + y^2 + 2gx + 2fy + g^2 + f^2 - r^2 = 0$

If you look at the last three terms, you see that they do not have x or y. So they can be represented with c, thus

$x^2 + y^2 + 2gx + 2fy + c = 0$ which is the generalized equation of circle

We can recognize it, because it is quadratic in both x and y, and it has two additional properties. First, there is no term in xy. Secondly, the coefficient of x^2 is the same as the coefficient of y^2. From the figure 1, it is seen that the center of the circle is at $(a, b) = (-g, -f)$ and, recall that $c = g^2 + f^2 - r^2$ then,

The radius of the circle can be written as

$r = \sqrt{g^2 + f^2 - c}$

Activity 1

Find the center and radius of

(a) $(x - 2)^2 + y^2 = 25$

(b) $x^2 + y^2 - 6x + 4y - 12 = 0$

(c) $x^2 + y^2 - 4x - 2y = 4$

Solution

(a) $(x - 2)^2 + (y - 0)^2 = 5^2$

Center = (2, 0)

Radius = 5

(b) Here, you can see that the expression on the left-hand side is in a quadratic form and that the coefficients of $x2$ and $y2$ are equal. This is an equation of a circle. Now, we compare it with the standard equation of circle. $x2 + y2 + 2gx + 2fy + c = 0$ and it is seen that $g = -3$ and $f = 2$, this means center is $(-g, -f) = (3, -2)$ and $c = -12$ then,

$$r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{-3^2 + 2^2 - -12}$$

$$r = \sqrt{9 + 4 + 12}$$

$$r = \sqrt{25}$$

$$r = 5$$

(c) $x^2 + y^2 - 4x - 2y = 4$

$x^2 - 4x + y^2 - 2y = 4$

Divide each coefficient of y and x by 2 and add it to theequation.

$x^2 - 4x + (-2)^2 + y^2 - 2y + (-1)^2 = 4 + (-2)^2 + (-1)^2$

$(x - 2)^2 + (y - 1)^2 = 4 + 4 + 1$

$(x - 2)^2 + (y - 1)^2 = 9$

$a = 2, b = 1, r = 3$

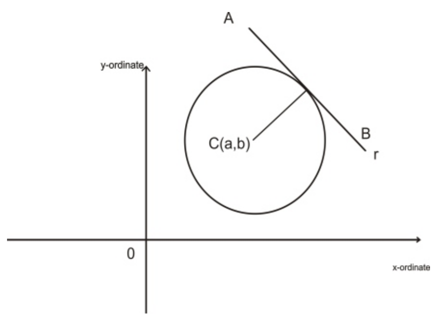


fig. 2

General equation of a circle

$x^2 + y^2 + 2ax + 2by + c = 0$

$2x + 2y\frac{dy}{dx} + 2a + 2b\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-(x+a)}{y+b}$ ...gradient function

at point $(x_1, y_1, \frac{dy}{dx} = \frac{-(x_1+a)}{y_1+b}))$

Normal

Normal is a line drawn from the center to a tangent through point of contact.

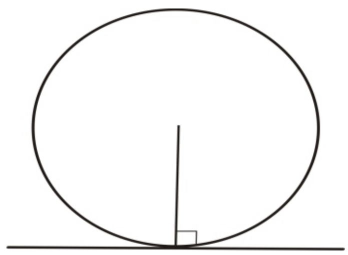


fig. 3

When two lines are perpendicular to each other, the angle between them is 90^0 If gradient $AB = m1$ and $CD = m2$ then $m_1 m_2 = -1$ If two lines are parallel then, $m_1 = m_2$

m_1=Gradient of tangent= $\frac{-(x+a)}{y+b}$

m_2=Gradient of tangent= $\frac{(y+b)}{x+a}$

$m_1 m_2 = \frac{-(x+a)}{y+b} \times \frac{(y+b)}{x+a}$

$m_1 m_2 = -1$

Activity 1

Find equation of the tangent to the circle $2x^2 + 2y^2 + x - 11y - 1 = 0$ at the point $(-2, 5)$.

Solution

Divide both sides by 2

$x^2 + y^2 + \frac{x}{2} - \frac{-11y}{2} - \frac{1}{2} = 0$

Complete the square with respect to x and y

This implies, $2a = \frac{1}{2}, \quad 2b = -\frac{11}{2}$

$a = \frac{1}{4}$ and $b = \frac{-11}{4}$

Equation of the tangent at point $(-2, 5)$

$\frac{dy}{dx} = \frac{-(-2+\frac{1}{4})}{5+(-\frac{11}{4})}$

$\frac{y-5}{x-(-2)} = \frac{-(-2+\frac{1}{4})}{5+(-\frac{11}{4})} = \frac{7}{9}$

$\frac{y-5}{x+2} = \frac{7}{9}$

This implies,

$9y - 7x = 59$ is the equation of the tangent to the circle.

Summary

In this unit, you have learnt about the following, circle, parabola, ellipse and hyperbola geometry. You also learnt how to obtain the general equations of circle, parabola, ellipse and hyperbola.

Self Assessment Questions

(1) Find the general equation of a circle center $(1, 4)$ and has radius 4units.

(2) Determine the equation of the tangent to the circle at the point $(2, 3)$. $x^2 + y^2 - 3x + 4y - 19 = 0$.

Tutor Marked Assignment

- Find the center and radius of the circle $9x^2 + 9y^2 - 6x - 9y - 23 = 0$

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

Further Reading

- Eduard O. Introduction to conic sections. Tuttuh-Adegun M. R., Sivasubrama S., and Adegoke R., (2001). Further Mathematics Project 3. NPS Educational Publishers Limited.

UNIT 4

Parabola, Hyperbola and Ellipse

Introduction

In this unit, you will learn about parabola, hyperbola and ellipse.

Learning Outcomes

At the end of this unit, you should be able to:

1. understand the properties of parabola, ellipse and hyperbola;
2. determine the eccentricity of parabola, ellipse and hyperbola;
3. find the parametric equation of parabola;
4. obtain the tangent of a parabola, ellipse and hyperbola; and
5. solve for the normal of a parabola, ellipse and hyperbola.

Main Content

Assuming point P is a moving point and it is always perpendicular to the line tracing a path (locus of a point) as shown in fig. 1 below. Point S is fixed. The ratio of $PS : PM$ is always constant, that is

$$\frac{PS}{PM} = e$$

e is called Eccentricity of the point. The vertical line is called Directrix.

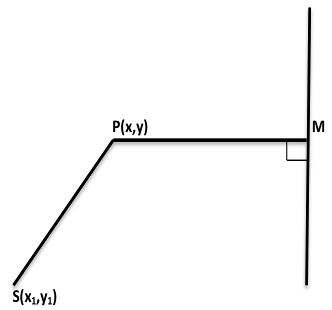


fig. 1

Therefore, when

$e = 0 \Rightarrow$ Circle

$e = 1 \Rightarrow$ Parabol

$e < 1 \Rightarrow$ Ellipse

$e > 1 \Rightarrow$ Hyperbola

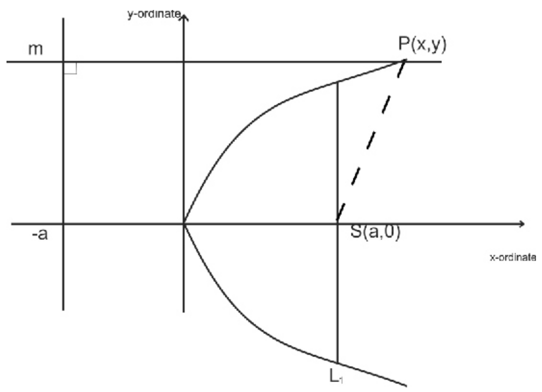


fig. 2

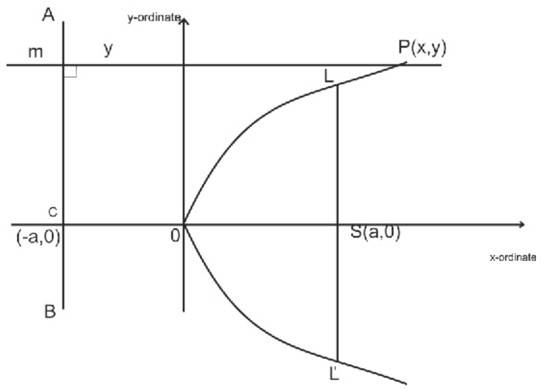


fig. 3

$x = -a$

$x + a = 0$

Parabola

$e = 1$

$\frac{PS}{PM} = 1$

$/PS/^2 = /PM/^2$

since $PM = PS$, $(x-a)^2 + y^2 = (x+a)^2$ which by expansion becomes $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2 y^2 = 4ax$. This is the general equation of a parabola.Point S(a, 0) is the focus and directrix the line $PM = x + a$, the line through the locus perpendicular to the line of symmetry is know as lactus rectum

Some Properties of a Parabola

(1) The value of the constant a determined the position of the curve, if a is positive, the curve lies entirely to the right of OY.

(2) The equation of the parabola in standard form (length of lactus rectum) is y 2 = $4ax$.

(3) The equation of the tangent at the point x_1, y_1 on the parabola $y^2 = 4ax$ is defined as $yy1 = 2a(x + x_1)$.

(4) The equation of the normal at any point (x_1,y_1) to the parabola is $y - y_1 = \frac{-y_1(x - x_1)}{2a}$

Parametric Equation of a Parabola

Consider the point ($at2$, $2at$) which always lies on the parabola;
$y^2 = 4ax$, since $x = at^2$
then $y = 2at \ldots (1)$
Therefore, for all values of t, equation (1) is called the parametric equation of a parabola.

Activity 1

Find the point of intersection of the line $2y = x + 6$ and the parabola $y^2 = 8x$ and the equation of the tangent and normal to the parabola at the point of intersection.

Solution

Given equations
$2y = x + 6 \ldots (1)$
$y^2 = 8x \ldots (2)$
from equation (1), we have
$x = 2y - 6 \ldots (3)$

substitute for x as $2y - 6$ in equation (2)

$y^2 = 8(2y - 6)$

$y^2 = 16y - 48$ and this implies $(y - 12)(y - 4)$

then, $y = 12, y = 4$

By substituting the values of y into equation (3), we have

$x = 2(12) - 6 = 18$ or

$x = 2(4) - 6 = 2$

Therefore, $y = 12, x = 18, y = 4, x = 2$

Hence, points of intersection are (18, 12) and (2, 4)

Some Properties of Ellipse

(i) **The general equation of an ellipse is** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

where $b^2 = a^2(1 - e^2)$

(ii) **The length of lactus rectum of the ellipse is** $b^2 = a^2(1-e^2)$

(iii) **The equation of the tangent at point** (x_1, y_1) **on the ellipse is** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **or** $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(iv) **The equation of the normal at point** (x_1, y_1) **is** $\frac{x-x_1}{\frac{x_1}{a^2}} = \frac{y-y_1}{\frac{y_1}{b^2}}$

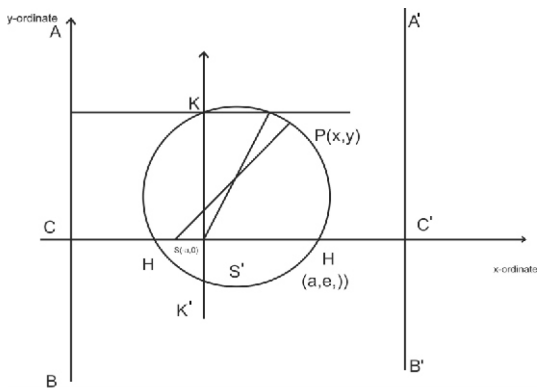


fig. 4

Activity 2

Find the eccentricity and the distance between the foci of the ellipse $3x^2 + 4y^2 = 12$.

Solution

Given $3x^2 + 4y^2 = 12$, by comparing this with the general equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

it becomes,

$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$

Here $a^2 = 4 \implies a = 2$ and $b^2 = 3 \implies b = \sqrt{3}$.

To obtain e,

recall that $b^2 = a^2(1 - e^2)$ then

$e = \sqrt{1 - \frac{b^2}{a^2}}$

By substitution, $e = \sqrt{1 - \frac{3}{4}} = \sqrt{\frac{1}{4}}$

Therefore, $e = \frac{1}{2}$

Distance between the foci is $2ae$, that is, $2 \times 2 \times \frac{1}{2} = 2$.

Properties of Hyperbola

(i) The general equation of an hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Where $b^2 = a^2(e^2 - 1)$

(ii) The equation of the tangent at the point (x, y) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(iii) The equation of the normal at any point (x_1, y_1) is $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{-(y - y_1)}{\frac{y_1}{b^2}}$

or $\frac{x - x_1}{\frac{x_1}{a^2}} + \frac{(y - y_1)}{\frac{y_1}{b^2}} = 0$

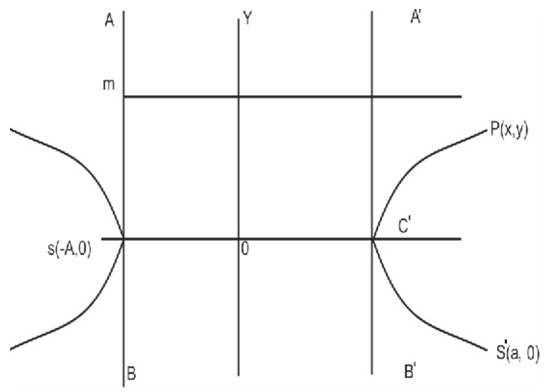


fig. 5

Activity 3

Prove that the point $x = asec\theta$, $y = btan\theta$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all value of θ

Solution

We recall that the equation of hyperbola is given as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and point $(x, y) = (asec\theta, btan\theta)$

then, $x^2 = a^2sec^2\theta$ and $y^2 = b^2tan^2\theta$

Therefore, $\frac{a^2sec^2\theta}{a^2} - \frac{b^2tan^2\theta}{b^2} = 1$

Since $sec^2\theta - tan^2\theta = 1$, for all values of θ on the hyperbola

Summary

In this unit, you have learnt about circle, parabola, ellipse and hyperbola geometry and their general equations. The eccentricity, properties, tangent and normal of circle, parabola, ellipse and hyperbola have also been explained in this unit to enable you solve related problems.

Self Assessment Questions

Find the equation of the tangent from the point (4, 4) to the hyperbola $9x^2 - 9y^2 = 16$

Tutor Marked Assignment

- Find the eccentricity and the coordinate of the foci of the hyperbola $4x^2 - 9y^2 = 36$
- Determine the equation of line through the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ parallel to the line $5x - 12y = 7$

References

- Stroud K. A., and Dexter J. B., (2001). Engineering Mathematics. Fifth Edition. Palgrave Publishers Ltd. New York.

Further Reading

- Eduard O. Introduction to conic sections. Tuttuh-Adegun M. R., Sivasubrama S., and Adegoke R., (2001). Further Mathematics Project 3. NPS Educational Publishers Limited.