MAT 213:
LINEAR ALGEBRA I

University of Ilorin
Centre for Open & Distance Learning

CODL

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From the Vice Chancellor

Courseware development for instructional use by the Centre for Open and Distance Learning (CODL) has been achieved through the dedication of authors and the team involved in quality assurance based on the core values of the University of Ilorin. The availability, relevance and use of the courseware cannot be timelier than now that the whole world has to bring online education to the front burner. A necessary equipping for addressing some of the weaknesses of regular classroom teaching and learning has thus been achieved in this effort.

This basic course material is available in different electronic modes to ease access and use for the students. They are available on the University's website for download to students and others who have interest in learning from the contents. This is UNILORIN CODL's way of extending knowledge and promoting skills acquisition as open source to those who are interested. As expected, graduates of the University of Ilorin are equipped with requisite skills and competencies for excellence in life. That same expectation applies to all users of these learning materials.

Needless to say, that availability and delivery of the courseware to achieve expected CODL goals are of essence. Ultimate attention is paid to quality and excellence in these complementary processes of teaching and learning. Students are confident that they have the best available to them in every sense.

It is hoped that students will make the best use of these valuable course materials.

Professor S. A. Abdulkareem
Vice Chancellor

Foreword

Courseware remains the nerve centre of Open and Distance Learning. Whereas some institutions and tutors depend entirely on Open Educational Resources (OER), CODL at the University of Ilorin considers it necessary to develop its own materials. Rich as OERs are and widely as they are deployed for supporting online education, adding to them in content and quality by individuals and institutions guarantees progress. Doing it in-house as we have done at the University of Ilorin has brought the best out of the Course Development Team across Faculties in the University. Credit must be given to the team for prompt completion and delivery of assigned tasks in spite of their very busy schedules.

The development of the courseware is similar in many ways to the experience of a pregnant woman eagerly looking forward to the D-day when she will put to bed. It is customary that families waiting for the arrival of a new baby usually do so with high hopes. This is the apt description of the eagerness of the University of Ilorin in seeing that the centre for open and distance learning [CODL] takes off.

The Vice-Chancellor, Prof. Sulyman Age Abdulkareem, deserves every accolade for committing huge financial and material resources to the centre. This commitment, no doubt, boosted the efforts of the team. Careful attention to quality standards, ODL compliance and UNILORIN CODL House Style brought the best out from the course development team. Responses to quality assurance with respect to writing, subject matter content, language and instructional design by authors, reviewers, editors and designers, though painstaking, have yielded the course materials now made available primarily to CODL students as open resources.

Aiming at a parity of standards and esteem with regular university programmes is usually an expectation from students on open and distance education programmes. The reason being that stakeholders hold the view that graduates of face-to-face teaching and learning are superior to those exposed to online education. CODL has the dual-mode mandate. This implies a combination of face-to-face with open and distance education. It is in the light of this that our centre has developed its courseware to combine the strength of both modes to bring out the best from the students. CODL students, other categories of students of the University of Ilorin and similar institutions will find the courseware to be their most dependable companion for the acquisition of knowledge, skills and competences in their respective courses and programmes.

Activities, assessments, assignments, exercises, reports, discussions and projects amongst others at various points in the courseware are targeted at achieving the objectives of teaching and learning. The courseware is interactive and directly points the attention of students and users to key issues helpful to their particular learning. Students' understanding has been viewed as a necessary ingredient at every point. Each course has also been broken into modules and their component units in sequential order.

Courseware for the Bachelor of Science in Computer Science housed primarily in the Faculty of Communication and Information Science provide the foundational model for Open and Distance Learning in the Centre for Open and Distance Learning at the University of Ilorin.

At this juncture, I must commend past directors of this great centre for their painstaking efforts at ensuring that it sees the light of the day. Prof. M. O. Yusuf, Prof. A. A. Fajonyomi and Prof. H. O. Owolabi shall always be remembered for doing their best during their respective tenures. May God continually be pleased with them, Aameen.

Bashiru, A. Omipidan
Director, CODL

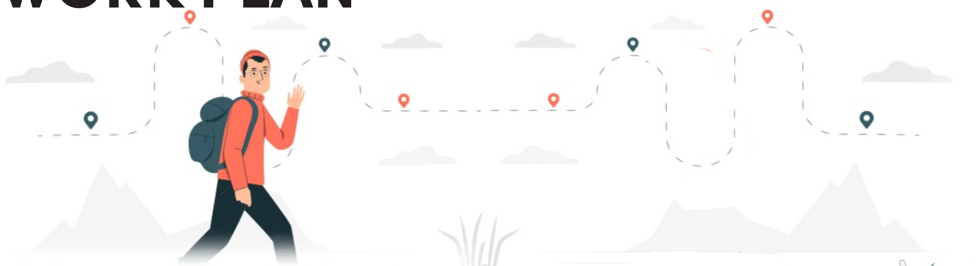
INTRODUCTION

I welcome you to Linear Algebra I. It is a 2-credit course that is available to year two undergraduate students in Faculties of Physical Sciences, Communication and Information Sciences, Engineering, Education and allied degrees. This course was designed as an intermediate course for undergraduate mathematics. It consists of basic topics from algebra and its application. It was prepared with the aim of introducing undergraduate students to some basic theorems, rules and principles that will be useful in advance mathematics.

Course Goal

Your journey through this course will introduce you to some basic topics like vector space over a real field, sub-spaces, linear dependence and independence of vectors, linear transformations and their representation by matrices, null space, rank, singular and non-singular transformation. You will also be introduced to algebra of matrices.

WORK PLAN



Learning Outcomes

At the end of this course, you should be able to:

- solve problems on vector space, sub-space, dimension and change of basis;
- solve problem on linear transformation;
- obtain the rank, range and null space of vectors;

Course Guide

Module 1

VECTOR SPACE OVER THE REAL FIELD

Unit 1 - Vector Space

Unit 2 - Vector Subspace

Module 2

LINEAR INDEPENDENCE, DEPENDENCE, BASIS AND DIMENSION OF VECTORS

Unit 1 - Linearly Independence of Vectors

Unit 2 - Linearly Dependence of Vectors

Unit 3 - Basis of Vector

Unit 4 - Dimension of a Vector

Related Courses

Prerequisite: MAT 111 & MAT 113

Required for: MAT 306, CSC 430

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- transform a vector space into its singular and non-singular form;
- solve problem on linear dependence and independence of vectors; and
- solve problems on algebra of vectors

Module 3

LINEAR TRANSFORMATIONS AND THEIR REPRESENTATION BY MATRIX

Unit 1 - Linear Transformation

Unit 2 - Composition of Linear Transformation

Unit 3 - Null Space

Unit 4 - Rank

Module 4

ALGEBRA OF MATRICES

Unit 1 - Matrices Definition and Types

Unit 2 - Algebra of Matrix

Unit 3 - Multiplication of Matrices

Unit 4 - Inverse of Matrices

Unit 5 - Solving System of Linear Equations

Course Requirements

Requirements for success

The CODL Programme is designed for learners who are absent from the lecturer in time and space. Therefore, you should refer to your Student Handbook, available on the website and in hard copy form, to get information on the procedure of distance/e-learning. You can contact the CODL helpdesk which is available 24/7 for every of your enquiry.

Visit CODL virtual classroom on http://codllms.unilorin.edu.ng. Then, log in with your credentials and click on MAT 213. Download and read through the unit of instruction for each week before the scheduled time of interaction with the course tutor/facilitator. You should also download and watch the relevant video and listen to the podcast so that you will understand and follow the course facilitator.

At the scheduled time, you are expected to log in to the classroom for interaction.

Self-assessment component of the courseware is available as exercises to help you learn and master the content you have gone through.

You are to answer the Tutor Marked Assignment (TMA) for each unit and submit for assessment.

Embedded Support Devices

Support menus for guide and references

Throughout your interaction with this course material, you will notice some set of icons used for easier navigation of this course materials. We advise that you familiarize yourself with each of these icons as they will help you in no small ways in achieving success and easy completion of this course. Find in the table below, the complete icon set and their meaning.

Introduction	Learning Outcomes	Main Content
Summary	Tutor Marked Assignment	Self Assessment
Web Resources	Downloadable Resources	Discuss with Colleagues
References	Futher Reading	Self Exploration

Grading and Assessment

TMA	CA	Exam	Total
20%	20%	60%	100%

Module 1

VECTOR SPACE OVER THE REAL FIELD

Units

UNIT 1

Vector Space

Introduction

A vector space V is a space with some certain characteristics. In this unit, we shall consider the basic definition of vector space, its representation and its characteristics/properties.

Learning Outcomes

At the end of this unit, you should be able to:

1. define vector space over the field;
2. add two or more vectors;
3. multiply scalars by vectors;
4. find the sum of the product of scalars and vectors; and
5. find the sum of matrix space.

Main Content

One of the properties of Mathematics problems is that when there are two solutions, the sum of the solutions is also a solution. Such problems are called linear problems. Such a problem are more often easier to solve than the general problems. Many problems that arise in applications must also be linear for such problems to be of practical use in our day to day activities. For instance, the principle of superposition in Physics is an expression that the differential equations that is satisfied by heat, light, electricity and other phenomena are linear.

In Geometry, vector is any physical quantity which has both magnitude and direction. It can be represented by a line segment $\overline{PO}$ of appropriate length and direction; O is the origin while P is the end point. If the end point P has coordinates (v_1, v_2, v_3), the vector $V = \overline{OP}$ is then specified and written as $V = (v_1, v_2, v_3)$.

Suppose vectors $V = (v_1, v_2, v_3)$ and $U = (u_1, u_2, u_3)$, the sum of V and U is formed by adding the corresponding vectors V and U, i.e., $V + U = (v_1 + u_1, v_2 + u_2, v_3 + u_3)$. This corresponds with the well known parallelogram rule.

On the other hand, a scalar is any quantity that possesses only magnitude. It is in a numerical value. When multiplying vector U by a real number α (scalar), another vector, $\alpha V = (\alpha v_1, \alpha v_2, \alpha v_3)$ is formed. Thus, αV is on the same straight line as vector V. Also, given αU and αV as vectors, α and β as scalars, $\alpha U + \beta V$ represents a vector in the same plane as U and V. A vector in such a plane can be written as a linear combination of U and V, hence U and V span the plane. In this work, a vector space shall be represented by V while a field shall be represented by F. Again, the real line R has "dimension" one (R), Cartesian plane R^2 has "dimension" two while the space R^3 has "dimension" three.

Given that $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ are n-tuples of real numbers, their sum is written as $(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n)$. The sum itself is an n-tuple of real numbers, i.e., $(x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$.

Suppose $\lambda \in R$ and $(x_1, x_2, ..., x_n) \in R^n$, the product $\lambda(x_1, x_2, ..., x_n)$ results in $(\lambda x_1, \lambda x_2, ..., \lambda x_n)$. The product itself is an n-tuple of real numbers, i.e., $(\lambda x_1, \lambda x_2, ..., \lambda x_n) \in R^n$.

Vector Space

Let F be a field and V be a non-empty set with operations of addition and multiplication defined on them, i.e., for any element u, $v \in V$, $u+v \in V$ and for the scalar $\lambda \in F$, $\lambda u \in V$. The following axioms or rules hold:

1. $(u + v) + w = u + (v + w)\ \forall\ u, v, w \in V$. (Associativity with respect to addition)
2. There exists zero vector, 0, such that $u + 0 = u$. (Additive identity)
3. There exists additive inverse $-u \in V : u + -u = 0$.
4. $u + v = v + u$ for any $u, v \in V$. (Commutativity)
5. For any scalar $k \in F$ and $x, y \in R^n$, $k(u + v) = ku + kv$
6. For any scalar $\lambda\mu \in F$ and $u \in R^n$, $(\lambda + \mu)u = \lambda u + \mu u$. (Distributivity)
7. For any scalars λ, $\mu \in F$ and $u \in V$, $(\lambda\mu)u = \lambda(\mu u)$. (Associativity with respect to multiplication)
8. $1u = u$ for the unit scalar $1 \in F$ and $u \in V$, i.e., $\forall\ u \in R^n$, $1u = u$

Naturally, the axioms are of two sets. The first four involve the additive structure of V. Therefore, V is a commutative group under addition. The other four axioms are concerned with "action" of the field F of scalars on the vector space V. The elements of a real vector space R^n called vectors are the n-tuples of real numbers while the elements of a field are called scalars.

Matrix Space M

Let $M_{m,n}$ be the set of all $m \times n$ matrices with entries in a field F. The sum of two vectors A and B in F is defined by $(A+B)_{i,j}=A_{i,j}+B_{i,j}$. The product of the scalar α and vector A is defined by $\alpha(A_{i,j}) = \alpha A_{i,j}$

Activity 1

Let F be an arbitrary field and F^n denotes the set of all n-tuples of all elements in F; i.e.$(f_1, f_2, \ldots, f_n)$. If u,v are vector spaces in V; with $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$, show that V is a vector space over the field F.

Solution

V is a vector space over F with the following operations:

(i) Addition in Vector Space:

$(u_1, u_2, \ldots, u_n)+(v_1, v_2, \ldots, v_n)=(v_1+u_1, v_2+u_2, \ldots, v_n+u_n)$

(ii) Scalar Multiplication in Vector Space:

$f_1(u_1, u_2, \ldots, u_n)=(f_1u_1, f_1u_2, \ldots, f_1u_n)$

(iii) The Zero Vector Space Let zero vector in R^n be the n-tuple zeros then, $0 = (0, 0, \ldots, 0)$

(iv) The Negative of a Vector Space Let the negative of u be defined by $-(u_1, u_2, \ldots, u_n)=(-u_1, -u_2, \ldots, -u_n)$

Then,

u+(-u)=$(u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n)$ =0

(v) Commutativity of a Vector Space

Suppose $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$,$u + v$=$(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)$ = $(u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$ $v + u$=$(v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n)$

(vi) Associativity of Vector Space

Let $u, v, w \in V$ where u and v are as defined in (v) above while $w = (w_1, w_2, \ldots, w_n)$ then, $u + v + w$=$(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n)$ $u + (v + w)$=u_1 + (v1 + w_1), $u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)$ $(u + v) + w$=$(u_1 + v_1) + w_1$,$(u_2 + v_2) + w_2$,...,$(u_n + v_n)$ + w_n)

(vii) Identity Additive in Vector Space

If u is as defined in (v), then, $u + 0 = u$

(viii) Multiplicative Identity in Vector Space

If $u \in V$ and $1 \in F$ then, $1u = u$

Activity 2

In a vector space, show that there exists only one zero vector (i.e. zero vector is unique)

Proof:

Let 0 be a zero vector, and let 00 also be a zero vector. Then

$0 = 0 + 0'$ and $00 = 0' + 0$

But $0 + 0' = 0' + 0$ (commutativity)

$\therefore 0 = 0'$

Hence, zero vector is unique

Activity 3

If x is a vector space, for each $x \in V$, show that there is a unique $-x \in V$.

Proof:

Let a, $0 \in V$

Then,

$x + a = 0$ (1)

$x + b = 0$ (2)

Now,

$a = a + 0$ (Zero vector)

$a = a + (x + b)$ (From (2))

$a = (a + x) + b$ (Associative)

$a = 0 + b$ (From (1))

$a = b + 0$ (Commutative)

$a = b$

Hence, for each $x \in V$, $\exists$ a unique $-x \in V$

Activity 4

Theorem 1:

If V is a vector space over the field F and zero $\in V$ then, $\forall \alpha, 0 \in F$ and $0, u, v \in V$;

(i) $\alpha 0 = 0$

(ii) $0u = 0$

(iii) $(-\alpha)v = -(\alpha v)$

(iv) $\alpha u=0 \Rightarrow \alpha = 0$ or $u = 0$

(v) $\alpha(u - v) = \alpha u - \alpha v$

Proof of Activity 4:

For all $0, u, v \in V$ and $\alpha \in F$

(i) Let

$0 = 0 + 0$

$\alpha 0 = \alpha 0 + \alpha 0$

$\alpha 0 - \alpha 0 = \alpha 0 + \alpha 0 - \alpha 0$

$0 = \alpha 0$ (Right cancellation law)

$\alpha 0 \in V$

(ii) $0 = 0 + 0$

$0u = 0u + 0u$

$0u = 0u + 0u - 0u$

$0u = 0u \in V$ (Right cancellation $\in V$)

(iii) $0 = [\alpha v + (-\alpha)v]$

$0 = [\alpha v + (-\alpha u)]v$

$-\alpha v$ is the additive inverse of αv hence,

$0 - \alpha v = \alpha v + (-\alpha v) - \alpha v$

$0 - \alpha v = -\alpha v$

$0 - (\alpha v) = (-\alpha v)$

$(-\alpha v) = -(\alpha v)$

(iv) Let there exists $\alpha \neq 0$ such that $\alpha\alpha^{-1} = 1$

Let $\alpha u = 0$

$\alpha^{-1}(\alpha u) = \alpha^{-1}0$

$\alpha^{-1}(\alpha u) = 0$

$\alpha^{-1}\alpha u = 0$

$1u = 0$

$u = 0$

Also,

Let $u \neq 0$, u + 0 = u

$\alpha(u + 0) = \alpha u$

$\alpha(u + 0) = 0$ [since $\alpha u = 0$]

$\alpha u + \alpha 0 = 0$

$\alpha u = 0$

Summary

A vector space over a field F is a non-empty set V together with two operations of addition and scalar multiplication that satisfy the commutativity, associativity, identity , inverse and distributivity axioms.

Self Assessment Questions

1. Define a vector space V over a field F.
2. Show that zero vector is unique.
3. If V is a vector space over the field F and zero $\in V$ then, $\forall \alpha, 0 \in F$ and $0, u, v \in V$. Prove that V is a vector space.

Tutor Marked Assignment

Question 1

Let V be a vector space and $x, y \in V$. Show that there exists a unique u in V such that $x + u = y$

Question 2

Let V be the set of x-y plane whose elements are the ordered pair $(x_1, y_1) \in R$ of real numbers while c is the field of real numbers defined by $(x, y) + (x + x_1) = (x + x_1, y + y_1)$, $c(x, y) = (cx, y)$ Show whether or not V is a vector space over the field of real numbers.

Question 3

Given that R be the field of real numbers and let P_n be the set of polynomials of degree n over the field R. Show that P_n is a vector space over the field F

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Further Reading

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UNIT 2

Vector Subspace

Introduction

To most algebraic structures, there exist substructures and vector spaces are not exempted. In this unit, subspaces of a vector space shall be discussed.

Learning Outcomes

At the end of this unit, you should be able to:

1. define a vector subspace;
2. state the properties of a vector subspace;
3. find the union and intersection of subspaces; and
4. define and find the linear combinations of vector spaces.

Main Content

A subspace of a vector space V is a subset S of V that is a vector space in its own right under the operations obtained by restricting the operation V and S

Subspace

Suppose V is a vector space over the field F and T a non-empty subset of V (i.e. $T \subseteq V$). Then T is a subspace of V if T is a vector space over F. Moreover;

(i) T contains OV (i.e. zero vector belongs to T);

(ii) T is closed under addition ($\forall\, x, y \in T \Rightarrow x + y \in T$);

(iii) T is closed under scalar multiplication (i.e., $\forall x \in T, \alpha \in R, \alpha x \in T$). It is a space whose elements are all in another space. It is to be noted that

(I) above is always a subspace itself and

(ii) is a space of V under than V itself is called a subspace.

Properties of Subspace

Given a vector space V under a field F, the subspace:

(i) are non-empty and closed under vector addition and scalar multiplication;

(ii) can be characterized by being closed under linear combinations; and

(iii) is a flat in an n-space that passes through the origin over the field of real numbers and sub fields.

The Intersection of Subspaces of Vectors

The intersection of two or more subspaces of a vector is a subspace. For instance, given $u_1, u_2, \ldots u_n$ subspaces, $0 \in u_1 \cap u_2 \cap \cdots \cap u_n$ is a subspace. More so, suppose $x, y \in u_i$; $i = 1, 2$ then, $x + y$ and αx belongs to u_i, and $x + y$ and αx both belongs to $u_1 \cap u_2$.

The Union of Subspaces of Vectors

In general, the union of two or more subspaces may not be a subspace.

Activity 1

(i) Let V be a vector space over F then V itself is a subspace of itself (i.e. $V \subset V$) and the subset consisting zero vector alone is a subspace of V called zero subspace of V.

(ii) In R^3, the set T of the form $T = \{a, b, 0 | a, b \in R\}$ is a subspace of R^3.

(iii) If V is the space of all $n \times n$ square matrices, the set T that consists of $A = a_{i,j} \; \forall \; a_{i,j} = a_{j,i}$. All symmetric matrices is a subspace of V.

Activity 2

Theorem:

The necessary and sufficient condition for a non-empty set T of a vector space $V(f)$ to be a vector space of V is $a, b \in F$ and $\alpha, \beta \in T$.

Proof:

The Necessary:

If T is a subspace of V, then T must be closed under scalar multiplication and vector addition.

$a \in F, \alpha \in T \Rightarrow a\alpha \in T$

and

$b \in F, \beta \in T \Rightarrow b\beta \in T$

$a\alpha \in T, b\beta \in T \Rightarrow a\alpha + b\beta \in T$

Therefore, the condition is necessary

The Sufficient:

Let T be a non-empty subset of V that satisfies the condition $a, b \in F$ and $\alpha, \beta \in T \Rightarrow a\alpha + b\beta \in T$.

Suppose $a = 1 = b$ and $\alpha, \beta \in T$, then, $1\alpha + 1\beta \in T$

$\Rightarrow \alpha + \beta \in T$. Thus, T is closed under vector addition.

If $a = -1, b = 0$ and $\alpha \in T$ then,

$(-1)\alpha + 0\beta \in T$

$(-1)\alpha + 0 \in T$

$\Rightarrow -\alpha \in T$

Then, the additive inverse of each element of T also belongs to T.

If $\alpha = 0, \beta = 0$ and $\alpha \in T$,

$0\alpha + 0\beta \in T \Rightarrow 0 = T$

Hence, zero vector of $V \in T$ which is also zero vector of T.

Therefore, vector addition is commutative and associative since the elements of T are also in V.

Let $\beta = 0, a, b \in F$ and $\alpha \in T$ then,

$a\alpha + b0 \in T$. Then T is closed under scalar multiplication. The remaining postulates of a vector space hold and these could be completed by the students.

Therefore, $T(f)$ is a subspace of $V(f)$

Activity 3

Suppose $W = \{(x, y) : x = 3y\}$, show that W is a subspace of R^2.

Solution

Let $t_1, t_2 \in T$

and

$\forall t \in T, \alpha \in R$ then, $\alpha t \in T$

Let $a, b \in W$ and

$a = (x, y) : x = 3y$

$b = (h, \mathrm{k}) : h = 3k$

Then,

$a = (x, y) : x = 3y$

$a + b = (x + h, y + k)$

$a + b = (x + k, y + h) \in W$.

Also,

Let $\alpha \in R$ and $a \in W$

$\alpha a = \alpha(x, y); x = 3y$

$\alpha a = (\alpha x, \alpha y); x = 3y$

$\Rightarrow (\alpha x, \alpha y) \in W$

Activity 4

Express $U = (4, 2)$ as a linear combination of $u_1 = (1, -2)$, $u_2 = (-3, 1)$

Solution

Let $a, b \in F$ then,

$(4, 2) = au_1 + bu_2$

$(4, 2) = a(1, -2) + b(3, 1)$

$(4, 2) = (a - 3b, -2a + b)$

Equating the corresponding coefficients

$a = -2, b = -2$

Hence,

$(4, 2) = -2(1, -2) - 2(3, 1)$

Summary

A non-empty subset S of a vector space V is a subspace of V if and only if S is closed under addition and multiplication operations. Equivalently, S is closed under linear combinations, that is, $\forall\, a, b \in F$ and $u, v \in S \Rightarrow au + bv \in S$

Self Assessment Questions

1. Prove the necessary and sufficient conditions for a non-empty set of a subspace to be to be a vector space.

2. Express $U = (4, 2)$ as a linear combination of $u_1 = (1, -2), u_2 = (3, 1)$

Tutor Marked Assignment

(i) If $u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1) \in R^3$, show whetheror not the unit vectors u_1, u_2, u_3 span R_3

(ii) Express $U = (8, 5)$ as a linear combination of $u_1 = (2, -1), u_2 = (1, 4)$

(iii) Express $U = (6, -3, 4)$ as a linear combination of $u_1 = (1, -2, 3), u_2 = (6, -1, -1)$ and $u_3 = (2, 6, 1)$

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Further Reading

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Module 2

LINEAR INDEPENDENCE, DEPENDENCE, BASIS AND DIMENSION OF VECTORS

Units

UNIT 1

Linearly Independence of Vectors

Introduction

In this unit, the idea of linear combination of vectors shall be used to establish the criteria for linearly independent set of vectors.

Learning Outcomes

At the end of this unit, you should be able to:

1. define linear independence of vectors; and
2. state whether a set of vectors are linearly independent or not.

Main Content

Suppose V is a vector space over the field F while $u_1, u_2, \dots, u_n \in V$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$, any vector that can be expressed in the form $\alpha_1 u_1, \alpha_2 u_2, \dots, \alpha_n u_n$ is called the linear combination of the vectors u_i.

Let V be a vector space and let $S = v_1, v_2, \dots, v_n$ be a subset of V. We say that S spans V if every vector v in V and scalars c_i, $i = 1, 2, \dots, n$ can be written as a linear combination of vectors in S, i.e., $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$

In general, let X be a non-empty subset of a vector space V and denote by $\langle X \rangle$, the set of all linear combinations of vectors in X. Thus, a typical element of $\langle X \rangle$ is a vector of the form $a \rangle x^1 + a^2x^2 + \cdots + a_nx_n$ where x_i; $i = 1, 2, \dots, n$ are vectors belonging to X and a_i; $i = 1, 2, \dots, n$ are scalars. If X is a non-empty subset of a vector space V, then $\langle X \rangle$, the set of all linear combinations of element X is a subspace of V.

Let V be a vector space and S a finite set $\{u_1, u_2, \dots, u_n\}$ of V and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$. S is said to be linearly independent if there exists scalars $a_i \in F$; $1 \leq i \leq n$ with all α_i equal zero such that $u_1\alpha_1 + u_2\alpha_2 + \cdots + u_n\alpha_n = 0$.

Activity 1

Determine whether or not the matrices

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},$$

span the space 2 × 2 matrices

Solution

Let Q be the space of 2 × 2 matrices

Now, let

$$\begin{pmatrix} m & a \\ t & p \end{pmatrix} \in Q$$

and $\alpha_1, \alpha_2, \alpha_2 \in R$

$$\begin{pmatrix} m & a \\ t & p \end{pmatrix} = \alpha 1 \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} + \alpha 2 \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + \alpha 3 \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} m & a \\ t & p \end{pmatrix} = \alpha 1 \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} + \alpha 2 \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + \alpha 3 \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} m & a \\ t & p \end{pmatrix} = \begin{pmatrix} 2\alpha 1 & \alpha 1 \\ 0 & -\alpha 1 \end{pmatrix} + \begin{pmatrix} \alpha 2 & -\alpha 2 \\ \alpha 1 & 0 \end{pmatrix} + \begin{pmatrix} \alpha 3 & 0 \\ -2\alpha 3 & \alpha 3 \end{pmatrix}$$

$$\begin{pmatrix} m & a \\ t & p \end{pmatrix} = \begin{pmatrix} 2\alpha 1 + \alpha 2 + \alpha 3 & \alpha 1 - \alpha 2 \\ \alpha 2 - 2\alpha 3 & -\alpha 1 + \alpha 3 \end{pmatrix}$$

Equate the correspondent coefficients to obtain

$2\alpha_1 + \alpha_2 + \alpha_3 = m$ . . .(i)

$\alpha_1 - \alpha_2 = a$ . . .(ii)

$\alpha_2 - 2\alpha_3 = T$ . . .(iii)

$-\alpha_1 + \alpha_3 = P$ . . . (iv)

Since solutions of (i) to (iv) are inconsistent, the matrices do not span the space of 2×2 matrices.

Activity 2

Given the vectors $x^2 + x - 1$, $2x^2 - 3x + 5$ and $-2x^2 + x + 4$ of R^3. Show that the vectors are linearly independent.

If $\alpha, \beta, \delta \in F$,

$\alpha(x^2 + x - 1) + \beta(x^2 - 3x + 5) + \delta(-2x^2 + x + 4) = 0$; α, β, δ are zeros

$(\alpha x^2 + 2\beta x^2 - 2\delta x^2, -\alpha x - 3\beta x + \delta x, -\alpha + 5\beta + 4\delta) = 0$

Equating coefficients of x,

$\alpha + 2\beta - 2\delta = 0 \ldots$(a)

$-\alpha - 3\beta + \delta \ldots$(b)

$-\alpha + 5\beta + 4\delta = 0 \ldots$(c)

Add (a) and (c) to obtain

$7\beta + 2\delta = 0 \ldots$(d)

Subtract (b) from (c) to obtain

$8\beta + 3\delta = 0 \ldots$(e)

Solving (d) and (e) simultaneously to obtain $\beta = 0$ and $\delta = 0$ Substitute for β and δ in (a) to obtain

$\alpha = 0$

Since $\alpha = \beta = \delta = 0$ the given vectors are linearly independent over the field F.

Activity 1

Given the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}; b > 1$$

and

$$W = (\,w_1\, w_2\,)$$

where $(w_1, w_2)^T$ is the transpose of W. Show that the set of matrices in R^2 are linearly independent.

Solution

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0+0 & 0+1 \\ b+0 & 0+0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$$

$$ABw = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$ABw = \begin{pmatrix} 0w_1 & w_2 \\ bw_1 & 0w_2 \end{pmatrix}$$

$$ABw = \begin{pmatrix} w_2 \\ bw_1 \end{pmatrix}$$

$\therefore AB_w = cw$. Hence, the set of matrices are linearly independent.

Summary

Set of vectors $v_1, v_2, \ldots, v_n \in X$ (X is a subspace of veector V) are said to be linearly independence if for every scalar $a_1, a_2, \ldots, a_n \neq 0$ (at least one of them is not equal to zero), the linear combination $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$

Self Assessment Questions

(1) Given the vectors $x^2 + x - 1$, $2x^2 - 3x + 5$ and $-2x^2 + x + 4$ of R^3 . Show that the vectors are linearly independent.

(2) Given the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}; \ b > 1$$

and

$$W = \begin{pmatrix} w_1 & w_2 \end{pmatrix}$$

where $(w_1, w_2)^T$ is the transpose of W, show that the set of matrices in R^2 are linearly independent.

Tutor Marked Assignment

(1) Given that $u_1, u_2, \ldots, u_n \in V$ are linearly independent, show that every element in their linear span has a unique representation.

(2) Given the matrices

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \ B = \begin{pmatrix} -4 & 6 \\ 3 & -1 \end{pmatrix},$$

show that the set of matrices in R^2 are linearly dependent.

(3) If x, y, z are linearly independent vectors, prove that $x + y$, $y + z$ and $z + x$ are linearly independent.

(4) Test the following sets of vectors in R^3 for linear dependence or independence. A=(1,0,-1), B=(0,2,1), C=(1,3,0)

(5) Determine if the set of vectors A=(1,2,1), B=(1,-1,1), C=(2,0,1) are

linearly dependent or not.

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UNIT 2

Linearly Dependence of Vectors

Introduction

In this unit, the idea of linear combinations of vectors to establish the criteria for linearly dependent set of vectors shall be investigated.

Learning Outcomes

At the end of this unit, you should be able to:

1. define linear dependency of vectors; and
2. state whether or not a set of vectors are linearly dependent or not.

Main Content

Let V be a vector space and S a finite set $\{u_1, u_2, \ldots, u_n\}$ of V and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$. S is said to be linearly dependent if for all scalars $\alpha_i \in F$; $1 \le i \le n$ with some or all α_i not equal to zero, the linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$.

In other words, let V be a vector space, X be a non-empty subset of V and a_i, $i = 1, \ldots, k \in R$. Then, X is said to be linearly dependent if there are distinct vectors V_1, $V_2, \ldots, V_k \in X$ such that $a_1 V_1 + a_2 V_2 + \cdots + a_k V_k = 0$.

That is , if the linear combination of the vectors $V_k \in X$ equals zero such that the scalars a_k are non-zeros (at least one of the $a'_k s$ not zero).

Activity 1

Show that the vector A=(5, 23, 203) is linearly dependent on vectors B=(1,1,1) and C=(1,10,100)

Solution

To show that vector A is linearly dependent on vectors B and C, it is sufficient to show that the linear combinations of vector B and C will give vector A. That is,

$A = \alpha_1 B + \alpha_2 C$ for some $\alpha_1, \alpha_2 \neq 0 \in R$

$(5, 23, 203) = \alpha_1(1, 1, 1) + \alpha_2(1, 10, 100)$

which gives the simultaneous equation

$5 = \alpha_1 + \alpha_2$

$23 = \alpha_1 + 10\alpha_2$

$203 = \alpha_1 + 100\alpha$

Solving the above simultaneous equation to obtain: $\alpha_1 = 3, \alpha_2 = 2$. Since $\alpha_i, i = 1, 2$ are non-zeros, then the sets of vectors A, B and C are linearly dependent with vector $A = 3B + 2C$

Summary

Set of vectors $V_1, V_2, \ldots, V_n \in X$ (X a subset of vector space V) are said to be linearly dependent if for every scalars $a_1, a_2, \ldots, a_n \neq 0$ (at least one of them is not equal to zero), the linear combination $a_1 v_1 + a_2 + \cdots + a_n v_n \neq 0$ is true.

Self Assessment Questions

(i) Differentiate between independent and dependent vectors

(ii) Explain the difference between Span of a set of vectors and Linear dependence of set of vectors

(iii) Show that the vector A=(-1, 1, 3) is linearly dependent on vectors $v_1 = (0, 1, 0)$, $v_2 = (1, 2, -2)$ and $v_3 = (4, 2, 0)$

Tutor Marked Assignment

(a) Show whether the vectors $U = (1, 1, 2, 4)$, $u_1 = (2, -1, -5, 2)$, $u_2 = (1, -1, -4, 0)$ and $u_3 = (2, 1, 1, 6)$ are linearly dependent or linearly independent.

(b) Show that the vector $B=(6, 2, 13)$ is linearly dependent on vectors $u_1 = (-1, 1, 5)$, $u_2 = (1, 2, 1)$ and $u_3 = (3, 1, 4)$

(c) Determine whether the following set of vectors is linearly dependent or independent.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

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UNIT 3

Basis of a Vector

Introduction

The concept of linear independence gave rise to basis of a vector. In this unit, bases (plural form of basis) shall be discussed.

Learning Outcomes

At the end of this unit, you should be able to:

1. define basis of a vector and
2. obtain basis for a vector.

Main Content

Given any V, its basis is a linearly independent spanning set of V. In basis $v_1, v_2, \ldots, v_n$ every vector of V can be written as a linear combination of $v_1, v_2, \ldots, v_n$ with uniquely determined scalar coefficient. Let V be a vector space. A subset S of V is a basis of V if

(i) S is a linearly independent set of vector space; and

(ii) S generates or spans the entire space of the vector space. i.e. $L(S) = V$ For instance,

(I) Vectors $e_1, e_2, \ldots, e_n$ form a basis of F^n since every vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ can uniquely be written as $\sum \alpha_i e_i$. This basis is known as standard basis of F^n.

(ii) Vectors $\{(1,3),(2,-1)\}$ is a basis of R^2 In a nutshell, if S is a non-empty subset of a vector V, then S is called a basis of V if the following are true:

(i) S is linearly dependent.

(ii) S generates V.

Activity 1

Show that $\{(1, 2),(3, -4)\}$ is a basis of R^2.

Solution

Test for Linearly Independence

$\alpha(1, 2) + \beta(3, -4) = (0, 0)$

$(\alpha + 3\beta, 2\alpha - 4\beta) = (0, 0)$

$\alpha + 3\beta = 0 \ldots(a)$

$2\alpha - 4\beta = 0 \ldots(b)$

Solving (a) and (b),

$\alpha = 0 = \beta$

Hence, it is linearly independent.

Test for Span

Let $(a, b) \in R^2\ \forall\ t, T \in R^2$

$(a, b) = t(1, 2) + T(3, -4)$

$(a, b) = (t, 2t) + (3T, -4T) \ldots(e)$

$t + 3T = a$

$2t - 4T = b \ldots(f)$

From (e),

$T = \frac{a - t}{3}$

Substitute for T in (f) to obtain

$2t - 4\frac{a - t}{3} = b$

$t = \frac{3b+4a}{10}$

Substitute for t in (e) to get

$\frac{3b+4a}{10} + 3T = a$

$30T = 10a - 4a + 3b$

$T = \frac{6a+3b}{30} = \frac{2a+b}{10}$

$(a, b) = \frac{3b+4a}{10}(1,2) + \frac{2a+b}{10}(3, -4)\forall a, b \in R$

Since each a, b is again a scalar from R, we have expressed the vector sum of the given vectors as a linear combination of the vectors from R^2, and therefore by definition of Span of set, $v_1 + v_2 \in R^2$. Hence, $\{(1,2),(3,-4)\}$ spans R

Activity 2

If V is the vector space of 2 × 2 matrices over the F, determine whether or not the vectors

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis.

Solution

For linearly independence, let $\alpha_1, \alpha_2, \alpha_3 \in F$

$${}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + {}_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + {}_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} {}_{\alpha 1} + 0_{\alpha 2} + 0_{\alpha 3} & 0_{\alpha 1} + {}_{\alpha 2} + 0_{\alpha 3} \\ 0_{\alpha 1} + 0_{\alpha 2} + 0_{\alpha 3} & 0_{\alpha 1} + 0_{\alpha 2} + 0_{\alpha 3} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$${}_{\alpha 1} = {}_{\alpha 2} = {}_{\alpha 3} = 0.$$

Therefore, the vectors form a basis for V since there are three vectors that form the basis V, dim V is 3.

Summary

One important corollary of a basis is that every non-zero finitely generated vector space V has a basis. Also, let V be a non-zero finitely generated vector space, then, any two bases of V have equal number of elements.

Self Assessment Questions

(i) Show that $\{(-1,-2),(-3,4)\}$ is a basis of R^2.

(ii) Let $V=\{v_1,v_2,v_3\}$ be a basis for a vector space S over a scalar field K. Then show that any vector $v \in V$ can be written uniquely as a linear combination $v=c_1v_1+c_2v_2+c_3v_3$, where $c_1,c_2,c_3 \in K$ are scalars.

(iii) Show that the set $A=\{1,1-u,3+4u+u^2\}$ is a basis of the vector space P_2 of all polynomials of degree 2 or less.

Tutor Marked Assignment

(a) Prove that if $S=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ is a basis of a finite dimensional vector space of dimension n, every element α of V can uniquely be expressed as $\alpha=a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n$

(b) Let V be the vector space of all 2×2 matrices, and let the subset S of V be defined by $S=\{A_1,A_2,A_3,A_4\}$, where

$$A1=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}, A_2=\begin{bmatrix}0 & 1\\ 1 & 4\end{bmatrix}, A_3=\begin{bmatrix}1 & 0\\ 1 & 10\end{bmatrix}, A_4=\begin{bmatrix}3 & 7\\ 2 & 6\end{bmatrix}.$$

Find a basis of the span Span(S) consisting of vectors in S and find the dimension of Span(S).

(c) Let P_3 be the vector space of all polynomials of degree 3 or less. Let $S=\{p_1(x),p_2(x),p_3(x),p_4(x)\}$, where $p_1(x)=1+3x+2x^2-x^3$, $p_2(x)=x+x^3$, $p_3(x)=x+x^2-x^3$, $p_4(x)=3+8x+8x^3$

(i) Find a basis Q of the Span(S) consisting of polynomials in S.

(ii) For each polynomial in S that is not in Q, find the coordinate vector with respect to the basis Q

(d) Show that the set $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, $u_3 = (1, 0, 1)$ is a basis of R^3 where R is the field of real numbers and determine the coordinates of the vector (a,b,c) with respect to the basis.

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UNIT 4

Dimension of a Vector

Introduction

This is a classical result in linear algebra. If a vector space V has a finite spanning set S, then the size of any linearly independent set cannot exceed the size of S. Hence, the concept of dimension stems from the result of basis. In this unit, the basic definitions of dimension of a vector shall be explored.

Learning Outcomes

At the end of this unit, you should be able to:

1. define dimension of vector and
2. solve exercise on dimension of a vector.

Main Content

Let V be a finitely generated vector space. If V is non-zero, the dimension of V is defined to be the number of elements in a basis V. In other-words, the number of vectors in a basis for V is called the dimension of V. In terms of matrix, the dimension of the range $R(A)$ of a matrix A is called the rank of A.

Let V be a vector space. V is a finite dimensional vector (or finitely generator) if there exists a finite subset S of V, $\{u_1 u_2, \ldots, u_n\}$ which spans V. On the other hand, the vector space V is called an infinite dimensional space if every finite subset of V fails to span V. For instance, vectors R^2 and R^3 are finite dimensional space.

Vector space F of all real value functions defined on the real number line is an infinite dimensional vector. Similarly, the vector P_∞ of all polynomials are infinite dimensional vector space. In essence, the dimension of a vector space is the maximum number of linearly independent vectors that form a basis of V and is denoted by dim V.

Suppose the dimension of V is n, it means there are n vectors that form the basis for the vector space and the maximum number of linearly independent vectors in that vector space must be n.

Coordinates of Vector Space

Suppose e_i; $i = 1, 2, \ldots, n$ is a basis of a finite dimensional vector space V over the field F, u be any vector in V and $\{e_i\}$ generates V, then u is a linear combination of the $e'_i s$, thus, $u = \alpha_1 e_1 + \alpha_2 e_2 + \ldots, +\alpha_n e_n$; $\forall\ \alpha_i \in K$

The $e'_i s$ are independent while the $\alpha'_i s$ representation are unique, and are determined by u and the basis $\{e_i\}$. The scalars are called the n-tuple coordinates of $u \in e_i$ while the n-tuple α_i ;$i = 1(1)n$ are called the coordinates vector of u relative to the basis $\{e_i\}$.. It is denoted by $[U]e = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Note that for the same basis set of e_i, the coordinates of the vector e_i are unique with respect to a particular ordering of e_i. Also, the basis set e_i can be ordered in several ways.

Activity 1

The dimension of n-Euclidean space R^n is n, since the column of the identity matrix I_n form a basis of R^n

.

Activity 2

The dimension of polynomial $P_n(R)$ is n since $(1, x, x_2, \ldots, x^{n-1})$ form the standard basis of $P_n(R)$.

Activity 3

Suppose $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, = \quad \{x_1, x_2\}$ and $y = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

Find the change of coordinates matrix P_α from α to the standard basis R^2 and change of coordinates matrix P_α^{-1} from the standard basis to R^2 to α.

Solution

$$P_\alpha = [x_1,\ x_2] = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

and so

$$P_\alpha^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$[y]_\alpha = P_\alpha^{-1} y = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Self Assessment Questions

Let V be a vector space and assume that the vectors $V_1, V_2, \ldots, V_n$ are linearly independent, also, the vectors $S_1, S_2, \ldots, S_m$ span V. Prove that $m \geq n$

Tutor Marked Assignment

Let V be a vector space and assume that the vectors $V_1, V_2, \ldots, V_n$ are linearly independent, also, the vectors $S_1, S_2, \ldots, S_m$ span V. Prove that $m \geq n$

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Module 3

LINEAR TRANSFORMATIONS AND THEIR REPRESENTATION BY MATRIX

Units

UNIT 1

Linear Transformation

Introduction

Functions from one vector space to another that preserve the vector space operations are called linear transformation. Linear transformation on vector space as well as matrices shall be discussed here.

Learning Outcomes

At the end of this unit, you should be able to:

Solve problems on addition and scalar multiplication on the following to form vector spaces:

1. Matrices;
2. functions; and
3. transformations.

Main Content

Let A and B be any non-empty vector spaces. If to each $a \in A$ is assigned a unique $b \in B$ then such assignment is called mapping or transformation or operator of A into B. The mapping is denoted by a capital letter say, F and written as $F : A \rightarrow B$

The vector $b \in B$ assigned to a vector $a \in A$ is called the image of a under F denoted by $F(a)$. $F : A \rightarrow B$ is a linear mapping if for all vectors $u, v \in A$ and scalars $a, b \in R$:

(i) $F(au+bv) = aF(u) + bF(v)$;

(ii) $F(cu) = cF(u)$

F is one-to-one (or injective) if $\forall u \neq v \in A$ implies $F(u) \neq F(v)$. F is onto (or surjective) if, for each $b \in B$, there exists $u \in A$ such that $F(u) = b$.

The range (or image) of F is the set range $F = \{b \in B | \exists u;\ b = F(u)\}$. If $F{:}A \to B$ then, A is called the domain, B is called the co-domain and $F(au) = aF(u)$ is called the range of F. To each map $F : A \to B$, if there corresponds the subset of $A \times B$ given by $\{(a, f(a)) : a \in A\}$, it is called the graph of the mapping. If $F : X \to Y$ and $G : X \to Y$ be two maps, then they are said to be equal if $f(x) = g(x)\ \forall\, x \in X$ and written as $X = Y$.

Activity 1

Let B be an $m\times n$ matrix over a field F. The function $T_A : F^n \to F^n$ defined by $T_A(v) = Av$ where all vectors are written as column vectors is a linear transformation from F^n to F^m. This is just a multiplication function on A.

Activity 2

The coordinate map $\phi{:}V \to F^n$ of an n-dimensional vector space is a linear transformation from V to F^n.

Activity 3

Suppose $F : X \to Y$ is a transformation define by $F(x) = mx + cw$ here $m, c \in R$ and $m \neq 0$. Show that F is not a linear transformation.

Solution

We check the two properties of linear transformation.

(i) $\forall\ u, v \in X,\ F(u+v) = F(u) + F(v)$

$F(u + v) = m(u + v) + c = mu + mv + c$ (i)

But

$F(u) + F(v) = mu + c + mv + c = mu + mv + 2c$ (ii)

The only time eqs. (i) and (ii) are equal is when $c = 0$, but $c \in R$, hence, the transformation is not a linear transformation.

A linear transformation $F : A \to A$ is called a linear operator on A. The set of all linear transformation from A to B is denoted by $L(A, B)$ and the set of all linear operators on A is denoted by $L(A)$.

Some Basic Terms and Definitions

(1.) Monomorphism or Embedding: This is used for injective linear transformation.

(2.) Epimorphism: It is used for surjective linear transformation.

(3.) Isomorphism: Used for bijective linear transformation.

(4.) Antomorphism: Used for bijective linear operator.

Summary

For two non-empty sets A and B, a linear transformation $F : A \to B$ is a rule that assign to each $a \in A$, a unique $b \in B$.

Self Assessment Questions

1. Define linear transformation, Give examples.
2. What do you understand by endomorphism?

Tutor Marked Assignment

(1) Differentiate between

(i) Homomorphism and monomorphism.

(ii) Endomorphism and Isomorphism.

(2) Explain in details what you understand by epimorphism.

(3) Define a linear transformation $T : P_3 \to M_{22}$ by:

$$T(a+bx+cx^2+dx^3) = \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix}$$

Show that T is a linear transformation.

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UNIT 2

Composition of Linear Transformation

Introduction

With the knowledge of linear transformation, it will be interesting to combine two linear transformations to ascertain the possible outcome. Thus, in this unit, composition of two linear transformations shall be dealt with.

Learning Outcomes

At the end of this unit, you should be able to:

1. define composition as a linear transformation in vector space setting;
2. show that the composition of linear transformation is linear; and
3. relate composition of linear transformation to matrix multiplication.

Main Content

Let f : $A \to B$ and $g : B \to C$ be two mappings then, (gof) : $A \to C$ defined by (gof) = go($f(a)$) $\forall$ '$a \in A$ is called the **composition** of the mappings of f and g.

Let X, Y, $W \in V$ and F be a linear transformation from X to Y. Let G be a linear transformation from W to X. The composition of F,G and H is given by $H = F oG = F G = F(G)$ For better understanding,

Let $w \in W$ then, $G(W)$ is a vector in X and $F(G(W))$ is a vector in Y.

Thus, H maps W to Y. Hence,

$H(W) = (F oG)(w) = (F G)(w) = F(G(w)) \ldots$ (a)

For H to be linear, the following must hold:

(i) $F(v + x) = F(v) + F(x)$

(ii) $F(cx) = cF(x)$

Let $w_1, w_2 \in W$

$H(w_1 + w_2) = (F\,oG)(w_1 + w_2)$

$= F(G(w_1 + w_2))$

$= F(G(w_1) + G(w_2))$ (By linearity of G)

$= F(G(w_1)) + F(G(w_2))$ (By linearity of F)

$= (F\,oG)(w_1) + (F\,oG)(w_2)$ But $(H(w)) = (F\,oG)w$

$= H(w_1) + H(w_2)$ (By definition of H)

Similarly,

$H(cw_2) = (F\,oG)(cw_2)$

$= F(G(w_2))$

$= F(c(G(w_2)))$

$= cF\,G(w_2))$

$= c(F\,oG)(w_2)$

$= cH(w_2)$

Composition of Linear Transformation and Matrix Multiplication

Let $X = R^n$, $Y = R^m$ and $W = R^p$. Let F be a real $m \times n$ matrix $A = [a_{i,j}]$ and G be $n \times p$ matrix $B = [b_{j,k}]$ then, with column vectors x and n-entries, vector Y with m-entries can be written as $y = Ax \ldots$(a)

Also, with column vector w and p-entries (b) can be written as $x = Bw \ldots$(b)

Substitute (b) into (a) to obtain

$y = A(Bw) = AB(w) = ABw = Cw$; where $C = AB$

This is a matrix setting. Therefore, the composition of linear transformations in Euclidean space is multiplication by matrices.

Activity 1

Given that $F: A \to B$ and $A = \{1, 2, 3, 4, 5\}$ and $B = \{p, q, r, s, t\}$, what is the

(i.) domain (ii.) co-domain (iii.) range of the mapping?

Solution

Solution to Activity 1

(i). Domain of F is $\{1, 2, 3, 4, 5\} = A$

(ii). Co-domain of F is $\{ p, q, r, s, t\} = B$

(iii). Range of F is $\{ p, q, r, t\}$

Activity 2

Let A be a 2×3 matrix given as

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 3 & -5 \end{pmatrix}$$

the vector is written as R^3 and R^2 as column vector, show that A determines the mapping $T: R^3 \rightarrow R^2$ defined by $T(V) = AV$ if

$$A = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

Solution

Solution to Activity 2

Let

$$T\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 3 & -3 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -13 \\ 18 \end{pmatrix}$$

It shows that every $m \times n$ matrix over a field determines the mapping $T : K^n \times ^1 \rightarrow K^m \times ^1$ defined by $T(V) = AV \forall v \in K^n$

Summary

A basic result asserts that a function is unchanged when it is composed with an elementary function .

Self Assessment Questions

(i) Let

$$A= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}; b > 1$$

and

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Show that ABW = CW where $C = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$

(ii). Show that $T: F^{mxn} \to F$ is a linear transformation where $T(A) = tr(A)$

Tutor Marked Assignment

(a). Find the linear the linear transformation that maps (x_1, x_2) onto $(3x_1 + 2x_2, 7x_1 - 5x_2)$

(b). If $A = F^{mxn}$, show that $T: F^n \to F^m$ is a linear transformation where $T(v) = Av$.

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UNIT 3

Null Space, Singular and Nonsingular Mapping

Introduction

Nullity or null space of a linear transformation has to do with the dimension of the kernel of the transformation . Thus, in this unit, kernel of transformation as well as its null space shall be discussed.

Learning Outcomes

At the end of this unit, you should be able to:

1. explain the kernel of linear transformation;
2. represent null space as matrix multiplication; and
3. define singular and non-singular transformations

Main Content

Kernel (Null Space) and Range of Linear Transformation Let $T : V \to W$ be linear transformation of two vector spaces V and W. The set of all elements $v \in V$ for which $T(V) = 0$ where zero denotes zero vector in W, is called Kernel or null space, i.e., $ker(T) = \{v \in V : T(V) = 0\}$ Note that $ker(T) = \{0\}$ since $T(0) = 0$ and $ker(T)$ is a subspace of the domain V called null space. For any linear transformation $T : V \to W$, the subspace $\ker(T) = \{v \in V \,|\, T(v) = 0\}$ is called the kernel (null space). The dimension of $ker(T)$ is called the nullity of T and denoted by $\text{null}(T)$, i.e., dim $(ker(T) = null(T))$.

Summary

For a linear transformation $T \in L\ (u,\ v)$, the null space (or kernel) is the subspace $ker(T) = v \in V \,|\, T(v) = 0$.

Self Assessment Questions

(1.) Define nullity of a transformation.

(2.) What do you understand by kernel of a linear transformation.

(3.) Define Cokernel (left null space).

Tutor Marked Assignment

(1.) Let $T \in L(V, W)$ be a linear transformation, show that T is an injective if and only if $ker(T) = 0$

(2.) Let $T(v) = w$. Then show that $\{u \in V \mid T(u) = w\} = v + kerT$.

(3.) Find $kerT$ where $T : E^3 \to E^2$ is defined by $\mathrm{T}(x_1, x_2, x_3) = (x_1+x_2, x_2-x_3)$

(4.) Calculate the nullity $Null(T)$ for the linear transformation $T : E^3 \to E^2$ defined by $T((a, b, c)) = (a+2b+c, -a+3b+c)$. Also find a basis for $ker(T)$.

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UNIT 4

Rank

Introduction

The image of a linear transformation has been defined earlier in this unit. The definition shall be used to obtain the rank of a transformation. In addition, the knowledge of dimension of a transformation is required to establish the singularity or non-singularity of two transformations. The two terms are briefly defined in the unit.

Learning Outcomes

At the end of this unit, you should be able to:

1. explain the rank of linear transformation;
2. solve system of linear equations using rank;
3. explain the kernel of linear transformation;
4. represent null space as matrix multiplication; and
5. define singular transformation and non-singular transformation.

Main Content

Suppose $T \in L(V, W)$ is a linear transformation, the subspace $T(v)|n \in V$ is called the image of the transformation denoted as $Im(T)$. Suppose $T \in L(V, W)$ is a linear transformation, the subspace $T(v)|v \in V$ is called the image of the transformation and denoted by $Im(T)$. The dimension of $\mathrm{Im}(T)$ is called the rank of T.

Let $T: V \rightarrow W$ be a linear transformation of two vectors. The range of T denoted by R_T is $R_T = \{w \in W : T(u) = w\}\ \forall u \in V$ It is a subset of W.

In $L : V \rightarrow W$, two elements of V have the same image in W if and only if the difference between them lies in the kernel of $L : L(v_1) = L(v_2) \Leftrightarrow L(v_1 - v_2) = 0$

This shows that the image of L is isomorphic to the quotient of the vector space V by the kernel such that $im(L) \cong v/ker(L)$. Hence $dim(kerL) + dim(imL) = dim(V)$. This is called **Rank-Nullity Theorem** where rank is the dimension of the image of L and by nullity, it implies the kernel of L.

Let V and W be two vector spaces over the field F, and T be a linear transformation from V into W. If V is finite dimensional, the rank of T is the dimension of the range space while the nullity of T is the null space of T.

Singular Transformation

A linear mapping (or transformation) $T \in L(V, W)$ is said to be singular if there exist some vectors $v \in V : v \neq 0$ and $T(v) = 0$. A linear mapping (or transformation or linear operator from a set to itself) is a non-singular transformation if the determinant vanishes.

Non-Singular Transformation

A linear transformation $T \in L(V, W)$ is said to be non-singular $T(v) = 0 \Rightarrow v = 0$ i.e. nullity $Null(t) = 0$.

Finite Dimensional Theorem

Let U and V be two vector spaces of the same field F and let T be a linear transformation from V to W. If V is finite dimensional then, $dimV = RankT + NullityT$

Representation as Matrix Multiplication

Suppose a linear transformation is represented by $m \times n$ matrix A whose coefficients are in the real number field of a R^2 and operating on column vectors x with n components over F. The kernel of the transformation is the set of solutions to the equation $Ax = 0$; zero is the zero vector. Then

$$N(A) = Null(A) = ker(A) = \{x \in F^n : Ax = 0\}.$$

The resulting matrix is equivalent to a homogeneous system of equations $Ax = 0$

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$

.

.

.

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$

Hence,

$Ax = 0 \Leftrightarrow$

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$

.

.

.

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$

Cokernel (Left Null Space):

Let T denote the transpose of a column vector. Cokernel of a matrix A consists of all vectors $x : x^TA = 0^T$. Cokernel is the same as kernel of A^T. The four fundamental subspaces associated with A are the kernel, row space, column space and cokernel.

Non-Homogeneous System of Equations

For a non-homogeneous system of linear equations $Ax = b$

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

.

.

.

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

Suppose two solutions u and v are possible in the above, then the difference between them can be expressed as the sum of a fixed solution v and an arbitrary element of the kernel, i.e.,

$A(u - v) = b - b$

$Au - Av = b - b$

$Au - Av = 0$

This can be expressed as

$\{v \in X : Av = b \Lambda x \in Null(A)\}$

Activity 1

If $T : R^3 \rightarrow R^3$ is a projection mapping into x-y plane, show that the range space is the entire x-y plane.

Summary

In this unit, rank of linear transformation and multiplication of matrices were discussed in detail.

Self Assessment Questions

(1.) What is Cokernel space?

(2.) Define linear operator.

(3.) What do you understand by non-singular transformation?

Tutor Marked Assignment

Given that

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 2 & 3 \end{pmatrix},$$

find

(i). R_A ;

(ii). Nullity of A; and

(iii). Dimension of A.

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Module 4

ALGEBRA OF MATRICES

Units

UNIT 1

Matrices

Introduction

In this unit, the basic terms, definition, and types of matrices shall be discussed.

Learning Outcomes

At the end of this unit, you should be able to:

1. define matrix
2. state the order of matrix, square; row and column matrices
3. solve problems on equality of matrices

Main Content

A rectangular array of elements enclosed in brackets is known as matrix. e.g.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A matrix is denoted by upper case letters $A, B, \ldots, Z$ while the elements of the matrix are denoted by lower case letters $a, b, \ldots, z$. The plural of matrix is matrices.

An $m \times n$ matrix A can be written as (a_{ij}), $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The horizontal arrangement of matrix is called its rows while the vertical arrangement is its column. Matrix A with m rows and n columns is called $m \times n$ (or m by n) matrix and such a matrix is said to be of order $m \times n$.

The entries that appear with rows and columns designated a_{ij} entry of the matrix. The first subscript denotes the row while the second denotes the column in which the entry stands.

Order of a Matrix

The number of rows and columns in a matrix are called the order (or size) of the matrix. If a matrix has m rows and n columns, it is called m by n (or $m \times n$ matrix). The matrix below is a 3×2 matrix.

$$K = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Row Vector

A row vector, V with n components $x_1, x_2, \ldots, x_n$ is of the form:

$$V = \begin{pmatrix} v_{11} & v_{12} & \ldots & v_{1n} \end{pmatrix}$$

Column Vector

A column vector U with n components is of the form:

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Equality of Matrices

Two or more matrices are said to be equal if they have the same row(s) and column(s) and their corresponding entries are the same.

Product of Matrices

Let $\alpha \in R$, the scalar multiple of A by α, αA is the matrix D for which $\alpha a_{ij} = d_{ij}$;$i = 1, 2, \ldots, m$, $j = 1, 2, \ldots n$.

Square Matrix

These are matrices that have the same number of rows and columns

For Example,

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 7 & 2 \\ 9 & 5 & 1 \end{pmatrix}$$

is a square matrix 3x3 having three rows and three columns. Also,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is an $m \times n$ matrix.

Some Properties of Matrix Operation

Let P, Q, R be $m \times n$ matrices, then, the following hold:

$P + Q = Q + P$ (Commutative Law)

$(P + Q) + R = P + (Q + R)$ (Associative Law)

$PQ \neq QP$

$(PQ)R = P(QR)$

$(P + Q)R = PR + QR$

Zero Matrix

A matrix whose elements or entries are all Zero is called null (or zero) matrix

Identity Matrix

Any matrix which has one in every element of the diagonal and zero elsewhere is called an identity (or unit matrix). All unit matrices are square matrices.

Transpose of Matrix

When the rows of a matrix A are interchanged with columns of the same matrix, the resulting matrix is called transpose of the matrix and denoted by A^T or A'.

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

its transpose is

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

A square matrix A of order n is called a non-singular matrix if the determinant is not equal to zero.

A square matrix A of order n is called a singular matrix if its determinant is zero.

Activity 1

Classify the following matrices according to their properties as either row matrix, column matrix, square matrix etc.,:

$$A=\begin{pmatrix} 2 & 6 & -1 \\ 3 & 5 & 0 \\ -4 & 1 & 8 \end{pmatrix}, B=\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}, C=\begin{pmatrix} 2 & 6 & -13 & 5 & 0 \end{pmatrix}, D=\begin{pmatrix} 4 & 9 & -1 \\ 7 & 5 & 9 \end{pmatrix}$$

Solution

1. Matrix A is a 3 × 3 square matrix
2. Matrix B is a 3 × 1 column matrix
3. Matrix C is a 1 × 5 row matrix
4. Matrix D is a 2 × 3 matrix (non-square matrix)

Activity 2

Find the transpose of the matrix $A = \begin{pmatrix} 0 & 4 & 1 \\ 5 & 2 & 10 \\ 4 & -2 & 3 \end{pmatrix}$

Solution

The transpose of the matrix A denoted as A^T is given as:

$$A^T = \begin{pmatrix} 0 & 5 & 4 \\ 4 & 2 & -2 \\ 1 & 10 & 3 \end{pmatrix}$$

Summary

Basic definition of terms , types of matrices were discussed in this unit.

Self Assessment Questions

(1.) What is a matrix?

(2.) Define equality of matrix.

(3.) Explain the following: (i). Determinant (ii). identity matrix

Tutor Marked Assignment

Find the transpose of the following square matrices:

(1.)

$$P = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 7 & 2 \\ 9 & 5 & 1 \end{pmatrix}$$

(2.)

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

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UNIT 2

Algebra of Matrix

Introduction

In this unit, addition, subtraction and scalar multiplication of matrices shall be discussed.

Learning Outcomes

At the end of this unit, you should be able to:

1. addition, multiplication of matrices; and
2. scalar multiplication of matrices.

Main Content

Addition of Matrices and Scalar Multiplication of Matrix

Let A and B be two matrices of order $m \times n$. The sum of A and B, written as $A + B$ is matrix C obtained by adding the corresponding entries of the matrices, i.e., $a_{ij} + b_{ij} = c_{ij}$; $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. That is, addition (or subtraction) of two matrices are possible if they are of the same order.

The scalar multiplication of matrices is obtained by multiplying the scalar by each entry of the matrix.

Activity 1

Given that

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -7 \\ 5 & 19 \end{pmatrix}$$

are 2x2 matrices.

$$A+B = \begin{pmatrix} 3+a_{11} & -7+a_{12} \\ 5+a_{21} & 19+a_{22} \end{pmatrix}$$

If A is a matrix of any order and α be any scalar, then αA is obtained by multiplying each element (or entry) of A by the scalar α

Activity 2

Let

$$A= \begin{pmatrix} 1 & 2 & 6 \\ 0 & 4 & 0 \end{pmatrix}, B= \begin{pmatrix} 1 & 2 & -5 \\ 3 & 4 & -1 \end{pmatrix}$$

(i). Find 3A (ii). $4A - 3B$

Solution (i)

$$3A= 3 \begin{pmatrix} 1 & 2 & 6 \\ 0 & 4 & 0 \end{pmatrix}$$

$$3A= \begin{pmatrix} 3 & 6 & 18 \\ 0 & 12 & 0 \end{pmatrix}$$

Solution (ii)

$$4A - 3B= 4 \begin{pmatrix} 1 & 2 & 6 \\ 0 & 4 & 0 \end{pmatrix} - 3 \begin{pmatrix} 1 & 2 & -5 \\ 3 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4\times 1 & 4\times 2 & 4\times 6 \\ 4\times 0 & 4\times 4 & 4\times 0 \end{pmatrix} - \begin{pmatrix} 3\times 1 & 3\times 2 & 3\times(-5) \\ 3\times 3 & 3\times 4 & 3\times(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 8 & 24 \\ 0 & 16 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 6 & -15) \\ 9 & 12 & -3 \end{pmatrix}$$

$$4A - 3B = \begin{pmatrix} 1 & 2 & 39 \\ -9 & 4 & 3 \end{pmatrix}$$

Summary

Algebra of matrices has been discussed in this unit with given examples.

Self Assessment Questions

(1.) Given

$$A=\begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}, B=\begin{pmatrix} -2 & 3 \\ 4 & 4 \\ 0 & 6 \end{pmatrix},$$

find (i) $3B$ (ii) $-2A$ (iii) $3A - 4B$ (iv) $4A - 3B$

(2.) Let

$$A=\begin{pmatrix} 1 & 2 & 6 \\ 0 & 4 & 0 \end{pmatrix}, B=\begin{pmatrix} 1 & 2 & -5 \\ 3 & 4 & -1 \end{pmatrix}$$

Find (i). $A - 2B$ (ii) $3A + 4B$

(3.) Given

$$A=\begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}, B=\begin{pmatrix} -2 & 3 \\ 4 & 4 \\ 0 & 6 \end{pmatrix},$$

find

(i) $3B$

(ii) $-2A$

(iii) $3A - 4B$

(iv) $4A - 3B$

Tutor Marked Assignment

If

$$X=\begin{pmatrix} 2 & 4 & 1 \\ 3 & 5 & 1 \end{pmatrix}, Y=\begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 1 \end{pmatrix}$$

find $X - Y$

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UNIT 3

Multiplication of Matrices

Introduction

In this unit, multiplication of two or more matrices shall be focused on

Learning Outcomes

At the end of this unit, you should be able to:

1. perform matrix multiplication.

Main Content

Let A and B be two matrices. If the number of columns in A is the same as the number of rows in B, multiplication of matrices A and B is possible. The product of A and B is the matrix AB obtained by multiplying i^{th} row of A by j^{th} column of B.

Activity 1

Find the product of the matrices. Is the product SR of the matrices possible?

$$R = \begin{pmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{pmatrix}$$

and

$$S = \begin{pmatrix} 1 & 2 & -2 & 3 \\ -1 & 0 & 3 & -4 \end{pmatrix}$$

Solution

$$RS = \begin{pmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 & 3 \\ -1 & 0 & 3 & -4 \end{pmatrix}$$

$$RS = \begin{pmatrix} 2 \times 1 + 3 \times (-1) & 2 \times 2 + 3 \times 0 & 2 \times (-2) + 3 \times 3 & 2 \times 3 + 3 \times (-4) \\ -3 \times 1 + 1 \times (-1) & -3 \times 2 + 1 \times 0 & -3 \times (-2) + 1 \times 3 & -3 \times 3 + 1 \times (-4) \\ 1 \times 1 + (-3) \times (-1) & 1 \times 2 + (-3) \times 0 & 1 \times (-2) + (-3) \times 3 & 1 \times 3 + (-3) \times (-4) \end{pmatrix}$$

$$RS == \begin{pmatrix} -1 & 4 & 5 & -6 \\ -4 & -6 & 9 & -13 \\ 4 & 2 & -11 & 15 \end{pmatrix}$$

The product *SR* of the two matrices is not possible because the number of columns in *S* is greater than the number of rows in *R*.

Activity 2

Given the matrices:

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 3 & -2 \\ 7 & 9 & -8 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 6 \\ 5 & -7 & 4 \end{pmatrix}, C = \begin{pmatrix} -2 & 3 \\ 1 & 3 \\ -4 & 5 \end{pmatrix}$$

Find the product (i) *AB* (ii) *AC* (iii) *BA* (iv) Is product of two matrices commutative? (Hint use the products *AB* and *BA* to justify your answer.

Solution

$$(i) \qquad AB = \begin{pmatrix} 13 & -18 & 22 \\ -4 & 5 & 10 \\ -15 & 22 & 22 \end{pmatrix}$$

$$(ii) \qquad AC = \begin{pmatrix} -5 & 20 \\ 11 & -1 \\ 37 & 8 \end{pmatrix}$$

$$(iii) \qquad BA = \begin{pmatrix} 2 & 0 & 3 \\ 46 & 30 & 13 \\ 38 & 30 & -13 \end{pmatrix}$$

(iv) Matrices product is not commutative because $AB \neq BA$.

Summary

Matrix product is discussed in this unit with given examples.

Self Assessment Questions

(1) If

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 3 \\ -3 & -5 & 2 \\ 2 & 4 & 1 \end{pmatrix}$$

(i). Find AB

(ii). Find BA. Is AB=BA?

Tutor Marked Assignment

If

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & 4 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

find (i) AB (ii) $2BA$ (iii) A^TB

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UNIT 4

Inverse of Matrices

Introduction

The inverse of any matrix cannot be obtained if the matrix is singular (i.e., when the determinant is zero). Hence, any Square matrix is invertible if and only if it is not a singular matrix. In this unit, attention shall be paid to the techniques of obtaining the inverse of a non-singular square matrix.

Learning Outcomes

At the end of this unit, you should be able to:

1. calculate the determinant of a square matrix
2. determine inverse of a matrix.

Main Content

The determinant of a matrix A is a scalar value that can be computed from the elements of a square matrix and is denoted as $det(A)$ or $|A|$. It gives some underlying properties of the square matrix itself. If the determinant of a matrix is zero, the matrix is said to be singular, otherwise it is non-singular. Any non-singular matrix is invertible (i.e., the inverse can be computed). The inverse of any matrix, when multiplied by itself gives a unit matrix.

Activity 1

Show whether or not the matrix

$$A= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has an inverse.

Solution

Let B be inverse of A;

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Then,

$AB = I_2$

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ b_{11} + b_{21} & b_{21} + b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equate the corresponding entries to obtain

$b_{11} + b_{21} = 1 \ldots$(a)

$b_{12} + b_{22} = 0 \ldots$(b)

$b_{11} + b_{21} = 0 \ldots$(c)

$b_{12} + b_{22} = 1 \ldots$(d)

From (a) and (c)

$b_{11} + b_{21} = 1$

$b_{11} + b_{21} = 0$

Also, from (b) and (d),

$b_{12} + b_{22} = 0$

$b_{12} + b_{22} = 1$

The above system is inconsistent. Hence, A does not have an inverse.

Note

Though the concept of an inverse of a matrix is defined for square matrices, there may be instances where two matrices of order $m \times n$ and $n \times m$; $m \neq n$ satisfy the equation $AB = I_m$. It should not be concluded that A is a non-singular matrix or that B is its inverse.

Activity 2

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then,

$$AB= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Since A is 2×3 and B is 3×2 matrix, A is not a singular matrix and B is not its inverse since A and B are not square matrix. Any matrix which has an inverse is said to be **invertible**.

Activity 3

Uniqueness of Matrix Inverse

Theorem: Let P be a non singular $n \times n$ matrix while Q and R are inverses of P. Then $Q = R$

Proof:

Let P be a matrix while Q and R be its inverses.

Then,

$P\,Q = QP \ldots$(a)

$P\,Q = I_n \ldots$(b)

$P\,R = RP = I_n$

$(QP)R = (RP)Q \ldots$(c)

$I_nR = I_nQ \ldots$(d)

Therefore, $Q = R$

The inverse of matrix P is denoted by P−1 and it satisfies the equations $PP^{-1} = P^{-1}P = I$

To calculate the inverse of a matrix, there are some terminologies you have to understand. These shall be discussed below:

Determinant of a Matrix

Suppose

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} ..(a)$$

is a square matrix, the determinant of A denoted by $|A|$ is

$$|A|= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

In a 2 × 2 matrix

$$B= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the determinant of B is the product of the leading diagonal minus the product of the minor diagonal. i.e.

$$|B|= a_{11}.a_{22} - a_{12}.a_{21}$$

For example, if

$$A= \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix},$$

$$|A| = 5 \times 2 - (-1 \times 3) = 10 + 3 = 13$$

To find the determinant of a matrix other than 2 × 2, the idea of minor and co-factor of a matrix are required.

Minor of a Matrix

Given that

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

the minors of A are obtained by crossing out the rows and columns of the elements whose minor are to be found. Hence, minor of

$$a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$

$$a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$a_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$a_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Co-factor of an Entry

The minor of an element together with its associated sign is called its co-factor. The associated sign for a matrix is obtained by multiplying each entry by the corresponding +1 or -1, beginning from a_{11}. For a 3×3 matrix, the associated signs are obtained by the formula $(-1)^{i+j}$ where i and j are the row and column number associated with the element:

$$\begin{vmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{vmatrix}$$

Adjoint of a matrix

This is the transpose of the cofactor matrix. Thus, for any square matrix A, its adjoint is denoted as $Adj(A)$ and given as: $Adj(A) = [Cof\,(A)]^T$. The inverse of the matrix is then obtained from:

$$A^{-1} = \frac{Adj\,(A)}{|A|}$$

Activity 3: Example on Inverse of Matrix

Find the inverse of the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & -1 \\ 0 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Solution

(i) $A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$

Obtain the determinant of matrix A

$$|A| = \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 2 \times (-1) - 1 \times 0 = -2$$

Obtain the minor matrix of A,

$M_{11} = -1, M_{12} = 0, M_{21} = 1, M_{22} = 2$

The minor matrix is given as:

$$Min\,(A) = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$$

The cofactor matrix is given as:

$$Cof\,(A) = \begin{pmatrix} -1 & 0 \\ -1 & 2 \end{pmatrix}$$

The Adjoint matrix is the transpose of cofactor matrix and is given as:

$$Adj\,(A) = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$$

The inverse of the given matrix A is obtained from $A^{-1} = \frac{Adj\,(A)}{|A|}$ and given as:

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{pmatrix}$$

$$\text{(ii)}\ \ B = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$|B| = \begin{vmatrix} 1 & 0 & 2 \\ -1 & 2 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 1(2 - (-2)) + 0(-1 - 0) + 2(-2 - 0) = 4 - 4 = 0$$

Since the determinant of matrix B is zero, the matrix is singular and does not have an inverse (not invertible).

$$\text{(iii)}.\ \ C = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$|C| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 1(-1 - 2) - 2(0 - 0) + 2(0 - 0) = -3$$

Obtain the minor matrix of C,

$M_{11} = \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -3, M_{12} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0, M_{13} = \begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} = 0, M_{21} = \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -2, M_{22} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, M_{23} = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2, M_{31} = \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -2, M_{32} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1$

The minor matrix is given as:

$$Min\,(C) = \begin{pmatrix} -3 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 1 & -1 \end{pmatrix}$$

The cofactor matrix is obtained from the minor matrix with the associated sign as:

$$Cof\,(C) = \begin{pmatrix} -3 & 0 & 0 \\ 2 & 1 & -2 \\ -2 & -1 & -1 \end{pmatrix}$$

The adjoint matrix is the transpose of cofactor matrix as was obtained as:

$$Adj\,(C) = [Cof(C)]^T = \begin{pmatrix} -3 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & -2 & -1 \end{pmatrix}$$

The inverse of the given matrix C is obtained from $C^{-1} = \frac{Adj\,(C)}{|C|}$ and given as

$$C^{-1} = \frac{1}{-3}\begin{pmatrix} -3 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & -2 & -1 \end{pmatrix}$$

Summary

In this unit, the following tasks have been achieved:

1. Finding determinant of a matrix;
2. Finding minor of a matrix; and
3. Finding minor, co-factor and inverse of matrices.

Self Assessment Questions

1. If

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 6 \\ 1 & 1 & 2 \end{pmatrix}$$

and

$$B=\begin{pmatrix} 2 & 1 & -4 \\ 0 & -\frac{1}{3} & -1 \\ -1 & -\frac{1}{3} & 2 \end{pmatrix},$$

determine whether or not A is the inverse of B.

2. Find the inverse of the following matrices:

$$A=\begin{pmatrix} -4 & 5 \\ 1 & -2 \end{pmatrix}, \quad B=\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad C=\begin{pmatrix} 1 & -2 & -2 \\ 1 & -1 & -2 \\ 0 & -2 & 1 \end{pmatrix}$$

Tutor Marked Assignment

1. Find the inverse of these matrices

$$A=\begin{pmatrix} -1 & 4 & 6 \\ 2 & -1 & 3 \\ 0 & 1 & -2 \end{pmatrix}, \quad = B \begin{pmatrix} 10 & -2 & -1 \\ 3 & -4 & 1 \\ -1 & -1 & -2 \end{pmatrix}$$

2. Find the determinant of

$$P = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

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UNIT 5

Solving System of Linear Equations

Introduction

In this unit, our focus shall be on solving system of linear equations with some techniques.

Learning Outcomes

At the end of this unit, you should be able to:

1. matrix inversion method
2. Gauss-Jordan method.

Main Content

Solving system of Equations by Matrix Inversion Method:Theorem

If A is a non singular matrix of order n, the system of linear equation $Ax = b$ of n equations with n unknowns possesses a unique solution denoted by

$$x = A^{-1}b$$

Proof:

Let

$$Ax = b \ldots\text{(a)}$$

Where A is the coefficient matrix, x is the matrix of the unknown and b is the matrix of the right hand side of the equations. For the system of equations to have a solution, the coefficient matrix A must not be singular. Since A is non-singular, that implies A is invertible. Hence, pre-multiplying both sides of eq. (a) with A^{-1} to obtain:

$$A^{-1}Ax = A^{-1}b$$

By associativity property of multiplication, the equation becomes:

$$(A^{-1}A)x = A^{-1}b$$

Since $A^{-1}A = I$ (an identity matrix of same order as A);

$$\therefore \mathrm{x} = A^{-1}b$$

For uniqueness of the solution, assuming there exists scalars y and z then,

$Ay = b$; $Az = b$

$Ay = Az$

$A^{-1}(Ay) = A^{-1}(Az)$

$(A^{-1}A)y = (A^{-1}A)z$

$y = z$

Hence, the solution is unique.

Activity 1

Systems of Equation

Solve the system of equations

$x_1 - x_2 = 3, x_1 + x_2 = 5$

Solution

The system of simultaneous equation can be re-written as:

$$\begin{matrix} x_1 - x_2 = 3 \\ x_1 + x_2 = 5 \end{matrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

By matrix inversion formula, the solution set x_1, x_2 can be obtained from:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Thus, using the normal approach to obtain the inverse of the coefficient matrix and compute the unknown as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + \frac{5}{2} \\ -\frac{3}{2} + \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Solving System of Equations by Gauss-Jordan Elimination Method

It is method used to solve system of linear equation by series of iteration to reduce the coefficients of entries in the matrix to zero (or unity) such that values of other variables can easily be obtained from the resulting matrix after the iterations.

Activity 1

Gauss Elimination Method

Solve the system of equations

$4x + 2y = 1$, $x + 3y = 0$

Solution

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ldots (a)$$

$$\left[\begin{array}{cc:c} 4 & 2 & 1 \\ 1 & 3 & 0 \end{array}\right]$$

To eliminate X, put R_2 to $4R_2 - R_1$

$$\left[\begin{array}{cc:c} 4 & 2 & 1 \\ 0 & 10 & -1 \end{array}\right]$$

Hence,

$4x + 2y = 1 \ldots$(b)

$10y = -1 \ldots$(c)

$y = \frac{-1}{10}$

Substitute for y in (b)

$x = \frac{6}{20} = \frac{3}{10}$

Summary

In this unit, solving system of linear equations using matrix inversion and Gauss-Jordan methods were discussed.

Self Assessment Questions

Use matrix inversion and Gauss-Jordan elimination methods to solve the following equations

(a) $x + y - 3z = 1, 3x - 4y + 7z = 19, 5x - 3y - 8z = 12$

(b) $x + 3y + 5z = 2, x + 2y + 10z = 7, 4x + y + z = -4$

Tutor Marked Assignment

Use matrix inversion and Gauss-Jordan elimination methods to solve the following equations

(a) $2x + y - 3z = 5, 3x - 2y + 2z = 5, 5x - 3y - z = 16$

(b) $x + 2y + 7z = 10, x - 5y + 10z = 6, 4x + y + z = 7$

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