STA 121: INTRODUCTION TO PROBABILITY

University of Ilorin
Centre for Open & Distance Learning
CODL

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From the Vice Chancellor

Courseware development for instructional use by the Centre for Open and Distance Learning (CODL) has been achieved through the dedication of authors and the team involved in quality assurance based on the core values of the University of Ilorin. The availability, relevance and use of the courseware cannot be timelier than now that the whole world has to bring online education to the front burner. A necessary equipping for addressing some of the weaknesses of regular classroom teaching and learning has thus been achieved in this effort.

This basic course material is available in different electronic modes to ease access and use for the students. They are available on the University's website for download to students and others who have interest in learning from the contents. This is UNILORIN CODL's way of extending knowledge and promoting skills acquisition as open source to those who are interested. As expected, graduates of the University of Ilorin are equipped with requisite skills and competencies for excellence in life. That same expectation applies to all users of these learning materials.

Needless to say, that availability and delivery of the courseware to achieve expected CODL goals are of essence. Ultimate attention is paid to quality and excellence in these complementary processes of teaching and learning. Students are confident that they have the best available to them in every sense.

It is hoped that students will make the best use of these valuable course materials.

Professor S. A. Abdulkareem
Vice Chancellor

Foreword

Courseware remains the nerve centre of Open and Distance Learning. Whereas some institutions and tutors depend entirely on Open Educational Resources (OER), CODL at the University of Ilorin considers it necessary to develop its own materials. Rich as OERs are and widely as they are deployed for supporting online education, adding to them in content and quality by individuals and institutions guarantees progress. Doing it in-house as we have done at the University of Ilorin has brought the best out of the Course Development Team across Faculties in the University. Credit must be given to the team for prompt completion and delivery of assigned tasks in spite of their very busy schedules.

The development of the courseware is similar in many ways to the experience of a pregnant woman eagerly looking forward to the D-day when she will put to bed. It is customary that families waiting for the arrival of a new baby usually do so with high hopes. This is the apt description of the eagerness of the University of Ilorin in seeing that the centre for open and distance learning [CODL] takes off.

The Vice-Chancellor, Prof. Sulyman Age Abdulkareem, deserves every accolade for committing huge financial and material resources to the centre. This commitment, no doubt, boosted the efforts of the team. Careful attention to quality standards, ODL compliance and UNILORIN CODL House Style brought the best out from the course development team. Responses to quality assurance with respect to writing, subject matter content, language and instructional design by authors, reviewers, editors and designers, though painstaking, have yielded the course materials now made available primarily to CODL students as open resources.

Aiming at a parity of standards and esteem with regular university programmes is usually an expectation from students on open and distance education programmes. The reason being that stakeholders hold the view that graduates of face-to-face teaching and learning are superior to those exposed to online education. CODL has the dual-mode mandate. This implies a combination of face-to-face with open and distance education. It is in the light of this that our centre has developed its courseware to combine the strength of both modes to bring out the best from the students. CODL students, other categories of students of the University of Ilorin and similar institutions will find the courseware to be their most dependable companion for the acquisition of knowledge, skills and competences in their respective courses and programmes.

Activities, assessments, assignments, exercises, reports, discussions and projects amongst others at various points in the courseware are targeted at achieving the objectives of teaching and learning. The courseware is interactive and directly points the attention of students and users to key issues helpful to their particular learning. Students' understanding has been viewed as a necessary ingredient at every point. Each course has also been broken into modules and their component units in sequential order.

At this juncture, I must commend past directors of this great centre for their painstaking efforts at ensuring that it sees the light of the day. Prof. M. O. Yusuf, Prof. A. A. Fajonyomi and Prof. H. O. Owolabi shall always be remembered for doing their best during their respective tenures. May God continually be pleased with them, Amen.

Bashiru, A. Omipidan
Director, CODL

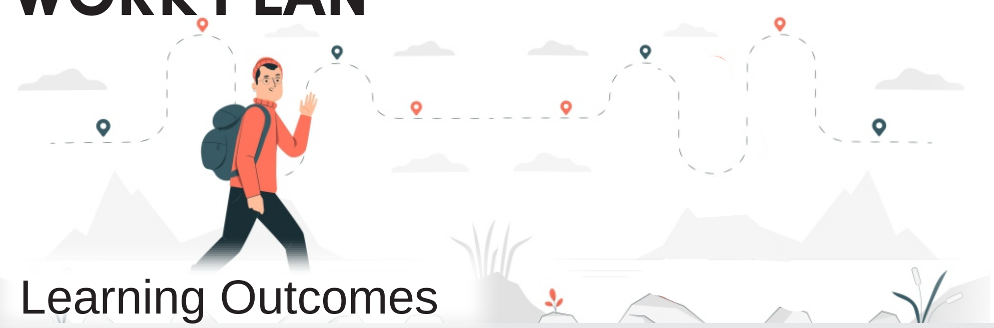
Course Guide

CONGRATULATIONS!!! You are welcome to STA 121 (Introduction to Probability), a two (2) credit unit course of three (3) modules with module one having four (4) study units, module two having six (6) study units and module three having four (4) study units. The main objective of this course is to provide basic knowledge and skills needed for clear understanding of probability and counting techniques required for theoretical and practical application in industrial, commercial, public/private and other human organization. Introduction to probability will equip you with adequate fundamental analytical skills in decision making needed for problem solving steps. The course is organized into three (3) distinct modules with each module addressing at least four study units. The first module introduces you to the set operations by describing and applying the concept of set operations which include; Elementary theory, combination of sets, sample space, sample point and events, dependent, independent and mutually exclusive events. The second module discusses probability and probability measures by explaining the elements of probability with different probability rules that will assist in the application of the probability concepts in practical situations. While the last module addresses the combinatorial analysis which include using the different counting techniques in problem-solving. Generally, the study units in each modules of this course (STA 121) has been put together to give students the comprehensive knowledge that will equip them in both practical and theoretical application of probability in all activities of human endeavor. Therefore, we wish you a very happy and fruitful study as you enjoy the content of this very interesting aspect of statistics.

Course Goal

The goal of this course is to introduce you to the concept of probability as it occurs in day to day activities and also to develop your mind on self confidence in the understanding and application of probability as a tool in decision making process.

WORK PLAN



Learning Outcomes

At the end of this course, you should be able to:

- I. Define and perform any operations in set theory.
- II. Clearly define, distinguish and determine the sample space, sample point and events of a study.
- III. Define random variable.

Week 01

Week 02

- IV. Define and apply the concept of probability and probability rules.
- V. Construct a probability distribution for a random variable
- vi.Distinguish between a discrete variable and a continuous variable

Week 03

Pre-requisite | STA 121 Introduction to Probability

- VII.Understand and be able to apply the different counting techniques.
- VIII. Clearly distinguish between permutation and combination.
- IX. Apply the permutation and combination techniques to relevant real life problems.
- X. Compute the probability of events using counting techniques

Course Guide

Module 1
Set Operations

Unit 1: Elementary Set Theory
Unit 2: Combination of Sets
Unit 3: Sample space, Sample point and Events
Unit 4: Dependent, Independent and Mutually Exclusive events

Module 2
Probability and Probability Measures

Unit 1: Element of Probability
Unit 2: Probability: a measure of uncertainty
Unit 3: Addition and Multiplication Rule of probability
Unit 4: Joint and conditional probabilities
Unit 5: The Bayes' Theorem
Unit 6: Introduction to Probability distribution

Module 3
Combinatorial Analysis

Unit 1: Tree diagram and Multiplication principles of counting
Unit 2: Permutations of Objects
Unit 3: Combination of Objects
Unit 4: Probability and Counting Technique

Requirements for Success

The CODL Programme is designed for learners who are absent from the lecturer in time and space. Therefore, you should refer to your Student Handbook, available on the website and in hard copy form, to get information on the procedure of distance/e-learning. You can contact the CODL helpdesk which is available 24/7 for every of your enquiry.

Visit CODL virtual classroom on http://codllms.unilorin.edu.ng. Then, log in with your credentials and click on STA 121. Download and read through the unit of instruction for each week before the scheduled time of interaction with the course tutor/facilitator. You should also download and watch the relevant video and listen to the podcast so that you will understand and follow the course facilitator.

At the scheduled time, you are expected to log in to the classroom for interaction.

Self-assessment component of the courseware is available as exercises to help you learn and master the content you have gone through.
You are to answer the Tutor Marked Assignment (TMA) for each unit and submit for assessment.

Embedded Support Devices

Throughout your interaction with this course material, you will notice some set of icons used for easier navigation of this course materials. We advise that you familiarize yourself with each of these icons as they will help you in no small ways in achieving success and easy completion of this course. Find in the table below, the complete icon set and their meaning.

Introduction	**Learning Outcomes**	**Main Content**
Summary	**Tutor Marked Assignment**	**Self Assessment**

Web Resources	**Downloadable Resources**	**Discuss with Colleagues**
References	**Futher Reading**	**Self Exploration**

Grading and Assessment

20%	20%	60%	100%
TMA	CA	Exam	Total

Source: towardsdatascience.com

MODULE 1

Set Operations

Unit 1: Elementary Set Theory
Unit 2: Combination of Sets
Unit 3: Sample space, Sample point and Events
Unit 4: Dependent, Independent and Mutually Exclusive events

UNIT 1 ELEMENTARY SET THEORY

Introduction

For you to understand fundamental concept in the study of probability and statistics, the set theory provides a steping stone. Set theory is one of the branches of mathematical sciences that deals with well-defined collection of items. Although, any types of items can be collected and define into a set, set theory is mostly applicable to objects that are relevant the the field of mathematical sciences.

Learning Outcomes

At the end of this unit, you should be able to:

1. Define a set.
2. Distinguish between finite and infinite set
3. Provide practical examples of sets and subsets
4. Describe the elements of a set

Main Content

SET

5mins

SAQ 1

A *set* can be defined as a well-defined collection of objects. For example, there are various departments in the Faculty of Physical Sciences, University of Ilorin constitutes a set, all possible categories of academic staff (Professors, Readers, Senior Lecturers, Lecturer I, Lecturer II, Assistance Lecturers, Graduate Assistance) in the Department of Statistics, University of Ilorin also constitutes another set. with members of this set being the individual staff status. A set may be finite or infinite.

Definitions

Elements of a set

The *elements* of a set are the list of members or objects present in the set. For instance, the individual (distinct) departments in the faculty of Physical Sciences are elements of the set of Faculty of physical sciences. Also, the list categories of academic staff are elements of set of staffs in the department of Statistics.

Finite and Infinite sets

We can say a set is said to be finite if all its elements can be listed or counted. E.g. a set of all vowels of the English alphabets. A set is said to be infinite if all its element cannot be listed or has unlimited elements. E.g. Set of all positive integers.

Notation:
Conventionally, the upper case letters such as **A, B, C, ..., X, Y, Z** are used to denote sets while the lower case letters such as **a, b, c, ... x, y, z** are used, where desirable, to denotes the elements in the set which are separated by comma and are usually enclosed in curly bracket {}. We may write $\boldsymbol{a} \notin \boldsymbol{X}$ means that object $\boldsymbol{a}$ is not an element in set $\boldsymbol{X}$

Example 1 If A is a set of even integers between 1 and 10 inclusive, then we write,

$$A = \{2, 4, 6, 8, 10\}$$

Hence, $4 \in A$ but $5 \notin A$. i.e. 4 is an element of set A but 5 is not an element of set A.

SAQ 2

SAQ 3

Methods of Describing Set

- List the elements

Example: $\boldsymbol{Y}$ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

- Verbal/oral description

Example: "$\boldsymbol{Y}$ is a set of all integers from 1 to 10 inclusive"

- Mathematical/Statistical Notations

Example: $\boldsymbol{Y} = \{integers\ x | 1 \leq x \leq 10\}$
Example: If P = {x: x is a prime number between 1 and 15 inclusive}.
Then, P = {1, 3, 5, 7, 11, 13}.
Example: Suppose that A is the set of all elements y such that y2 - 3y + 2 = 0.
Then, we write A = {y: y^2 - 3y + 2 = 0}.
Thus, A = {1,2}.

Definitions

1. Subsets and Equal Sets
Definition: We say that set **A** is (contained) in another set **B**, if and only if every elements of set **A** is also an element of the set **B**. If this happens, we say that set **A** is a subset of set **B** and write **A**⊂ **B**
In addition, if there is at least one element of **B** that is not an element of **A**, then we say that **A** is a proper subset of **B**.

Example 1 If **A** = {2, 4, 6} and **B** = {1, 2, 3, 4, 5, 6}, then **A**⊂ **B**. That is, set **A** is a proper subset of set **B**.

Example 2 Suppose **X** = {2, 4, 6} and **Y** = {6, 4, 2}, then set **X** is a subset (but not proper subset) of set **Y** and similarly set **Y** is a subset (but not proper subset) of set **X**.

Definition: Two or more sets are said to be equal if and only if they contain exactly the same elements.

Example If **X** = {2, 4, 6} and **Y** = {6, 4, 2}, then **X** = **Y**. Also, if sets **X** and **Y** are equal, then **X** ⊂ **Y** and **Y**⊂ **X**

2. Equivalence of Sets
Definition: Two sets are said to be equivalent if they have the same Cardinality.
The cardinality of a set is the number of distinct elements in that set.

Example Suppose sets **A** = {a, e, i, o, u} and **B** = {2, 4, 6, 8, 10}
The number of distinct elements in set **A**, i.e. $n(\boldsymbol{A}) = 5$
Also, the number of distinct elements in set **B**, i.e. $n(\boldsymbol{B}) = 5$.
Therefore, since $n(\boldsymbol{A}) = n(\boldsymbol{B}) = 5$ then set **A** is said to be equivalent to set **B**

3. Special Sets
Universal Set
Definition: The set that contains all the elements under discussion in a particular problem is called the universal set and is defined by the symbol $\mathfrak{U}$

Example Consider a set $\mathfrak{U}$ = {*x*: *x* is a positive integer}.
Then $\mathfrak{U}$ = {1, 2, 3, ...}
If **A** = {1, 2, 4} and **B** = {3, 5, 7, 9}.
Then we can say that set $\mathfrak{U}$ is a universal set here.

Null or Empty Set
Definition: The set that contains no element is called the null (or empty) set and is denoted by the symbol ∅.
Note that the null set ∅ is a subset of every set and not enclose in curly bracket like a normal set.

Example List all possible subsets of set **A** = {2, 3}.
All possible subsets of set A are as follows: {2}, {3}, {2, 3}, ∅

Compliment of a Set
Definition: Given a set **A** then, the set which contains all elements of the universal set, which are not element of **A**, and is denoted b $\boldsymbol{A}^c$ or $\boldsymbol{A}'$. Thus, we have:

$$\boldsymbol{A}' = \{x: x \in \boldsymbol{A} \text{ and } x \notin \boldsymbol{A}\}$$

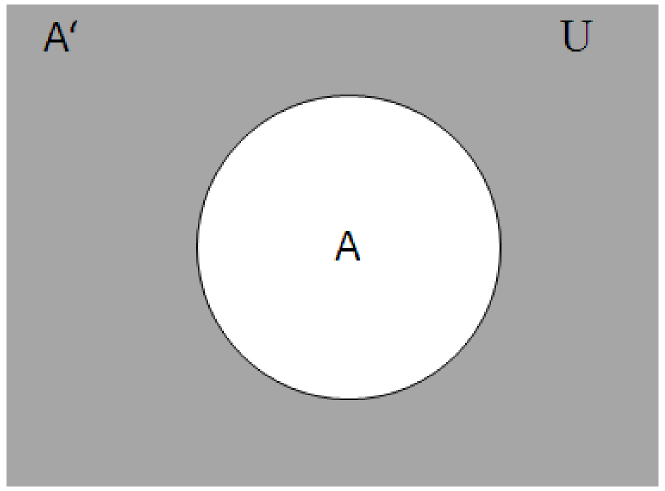


Figure 1: shaded region showing the complement of set **A**. i.e $\boldsymbol{A^c}$

Example Suppose = {1, 2, 3,4,5,6,7,8,9,10} and A = {2, 4, 6, 8}, $\boldsymbol{A'}$. = {1, 3, 5, 7, 9, 10}.

Summary

In this unit, set and set theory has been explained

·A set can be defined as a well-defined collection of objects.

·Conventionally, the upper case letters such as A, B, C, ..., X, Y, Z are used to denote sets while the lower case letters such as a, b, c, ... x, y, z are used, where desirable, to denotes the elements in the set

·A set is said to be finite if all its elements can be listed or counted. E.g. a set of all vowels of the English alphabets.

·A set is said to be infinite if all its element cannot be listed or has unlimited elements. E.g. Set of all positive integers

Self Assessment Questions

1. Define and list elements of any set of your choice.
2. State the methods of describing a set.
3. List the elements of the following sets.

   i. $\boldsymbol{X} = \{integers\ a|1 < a \leq 15\}$

   ii. $\boldsymbol{Y} = \{integers\ b|1 \leq a < 15\}$

   iii. $\boldsymbol{Z} = \{integers\ c|1 \leq a \leq 15\}$

   iv. "***A*** is a set of all even integers from 2 to 20 non-inclusive"

   v. "***B*** is a set of all prime integers from 1 to 20 inclusive"

Tutor Marked Assignment

1. Suppose $\mathfrak{U} = \{integers\ x|1 < x \leq 30\}$, **A** is a set of all prime integers in $\mathfrak{U}$, "**B** is a set of all even integers in $\mathfrak{U}$ and " C is the set of all odd integers in $\mathfrak{U}$. Obtain the following:

   i. Elements of $\mathfrak{U}$
   ii. Elements of **A**
   iii. Elements of **B**
   iv. Elements of **C**
   v. Elements of $\boldsymbol{A'}$
   vi. Elements of $\boldsymbol{B'}$
   vii. Elements of $\boldsymbol{C'}$

2. Using the question in (1) above, state weather the following statements are True or False

   i. $1 \in \mathfrak{U}$
   ii. $23 \notin \boldsymbol{A}$
   iii. $30 \notin \boldsymbol{C}$
   iv. $27 \in \boldsymbol{B}$
   v. $\emptyset \notin \mathfrak{U}$
   vi. **A** and **B** are equivalent
   vii. **B** and **C** are equivalent
   viii. **A** and **C** are equivalent
   ix. $\emptyset'$ and $\mathfrak{U}$ are equal

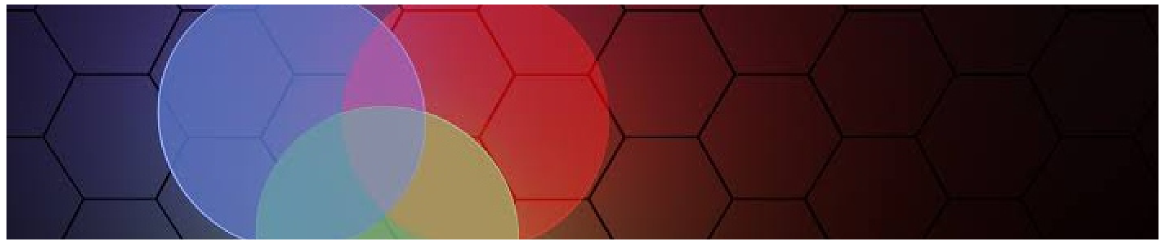
References

·Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.

·Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.

Further Reading

https://www.geeksforgeeks.org/set-theory/
https://www.math-only-math.com/finite-sets-and-infinite-sets.html
https://www.britannica.com/science/set-theory



UNIT 2 COMBINATION OF SETS

Introduction

When you Combine two or more sets means selecting all or part of a set without regard to the order of arrangement in which the sets are selected. For example, suppose we have three sets **X, Y** and **Z,** we might ask what are the possible combinations of the three sets. The possible combination would be; **XY, YZ, XZ** and **XYZ**

Note that **XY** and **YX** are considered one combination because the order of arrangement here does not matter.

Learning Outcomes

At the end of this unit, you should be able to:

1. Combine two or more sets
2. Define and perform the basic operations of combining sets

Main Content

COMBINATION OF SETS

3mins

Union of Sets

Definition Given two sets **A** and **B.** The union of **A** and **B** written as **A ∪ B**, is the set that contains all elements of either A or B or both.

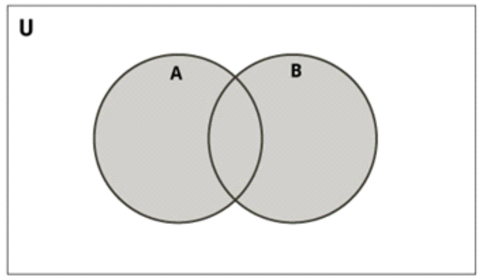


Figure 2.1: the shaded region showing **A ∪ B**

Example If A = {2, 3, 6} and B = {3, 9, 15}, then **A ∪ B** = {2, 3, 6, 9, 15}.

Intersection of Sets

Definition Given two sets A and B. The intersection of A and B written as $A \cap B$, is the set that contains all elements common to both A and B.

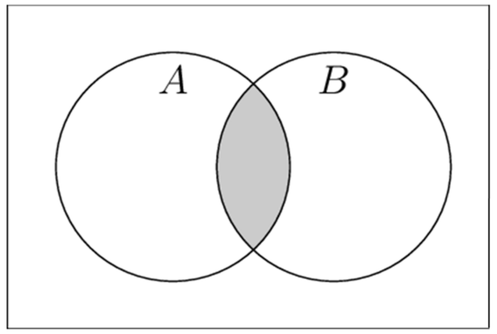


Figure 2.2: the shaded region showing $A \cap B$

Example If A = {2, 3, 6} and B = {3, 9, 15}, then $\mathbf{A \cap B}$ = {3}

1. **Disjoint Sets**

Definition: Two sets A and B are said to be disjoint if they do not have any element in common. In other words, sets A and B are disjoint sets if $\mathbf{A \cap B} = \emptyset$

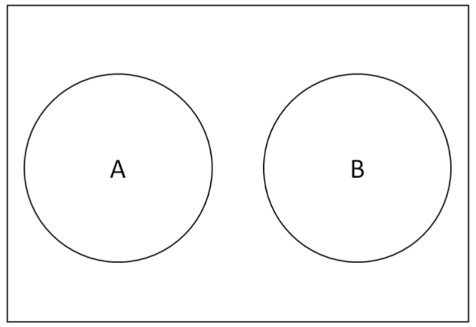


Figure 2.3: diagram showing disjoint sets $\mathbf{A \cap B} = \emptyset$

Example Suppose sets A = {2, 4, 6} and B = {5, 10, 15}, then $\mathbf{A \cap B} = \emptyset$

1. **Difference Between Two Sets**

The difference of sets A and B, written as **A – B** (or $\mathbf{A \cap B'} = \emptyset$} is defined as the set of all elements in **A** that are not in **B.** Similarly, the difference of sets **A** and **B** written as B – A (or $\mathbf{A' \cap B}$) is the set of all elements in **B** that are not in **A**.

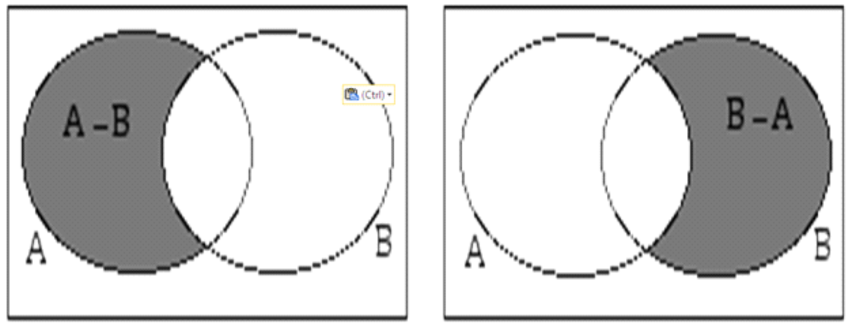


Figure 2.4: the shaded region showing the difference of two sets A and B

Example Given the universal set $\mathfrak{U}$ = {1, 2, 3, ...,10} with **A** = {2, 4, 6, 8} and **B** = {3, 6, 9}.

I.) $\mathbf{A - B = A \cap B'}$ = {2, 4, 6, 8} ∩ {1, 2, 4, 5, 7, 8, 10}
= {2, 4, 8}.

ii.) $\mathbf{B - A = B \cap A'}$ = {3, 6, 9} ∩ {1, 3, 5, 7, 9, 10}
= {3, 9}.

It should be noted that all the above sets combinations can be extended to three set **A**, **B** and **C.**

Basic Laws of Set Operation

2mins

Commutative Laws: For any two sets A and B;
I.) $\mathbf{A \cup B = B \cup A}$
ii.) $\mathbf{A \cap B = B \cap A}$

Associative Laws: For any three sets **A**, **B** and **C**;
I.) $\mathbf{A \cup (B \cup C) = (A \cup B) \cup C}$
ii.) $\mathbf{A \cap (B \cap C) = (A \cap B) \cap C}$

Distributive Laws: For any three sets **A**, **B** and **C**;
I.) $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$
ii.) $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

De Morgan's Laws:
Definition 1: The complement of the union of two sets is equal to the intersection of their complements. That is; $\mathbf{(A \cup B)' = A' \cap B'}$
Definition 2: The complement of the intersection of two sets is equal to the union of their complements. That is; $\mathbf{(A \cap B)' = A' \cup B'}$

Example 1 If $\mathfrak{U}$ = {1, 2, 3, ...,10} with **A** = {2, 4, 6, 8} and **B** = {3, 6, 9}. Show that $\mathbf{(A \cup B)' = A' \cap B'}$

Solution $\mathbf{A \cup B}$ = {2, 3, 4, 6, 8, 9}. Therefore, $\mathbf{(A \cup B)'}$ = {1, 5, 7, 10}. Also, $\mathbf{A'}$ = {1, 3, 5, 7, 9, 10}, $\mathbf{B'}$ = {1, 2, 4, 5, 7, 8, 10}.

Therefore, $\mathbf{A'} \cap \mathbf{B'} = \{1, 5, 7, 10\}$.
Therefore, $(\mathbf{A} \cup \mathbf{B})' = \mathbf{A'} \cap \mathbf{B'}$

Summary

In this unit, combination of sets has been explained.

- Given two sets A and B. The union of A and B written as A ∪ B, is the set that contains all elements of either A or B or both.
- Given two sets A and B. The intersection of A and B written as A ∩ B , is the set that contains all elements common to both A and B.
- Two sets A and B are said to be disjoint if they do not have any element in common.
- The difference of sets A and B, written as A – B (or A ∩ B'), is defined as the set of all elements in A that are not in B.

Self Assessment Questions

Given that $\mathfrak{U} = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathbf{A} = \{1, 2, 3\}$, $\mathbf{B} = \{3, 4, 5, 6\}$ and $\mathbf{C} = \{2, 3, 4\}$. Determine the elements of the following sets:

I.) $\mathbf{A'}$
ii.) $\mathbf{A} \cup \mathbf{B}$
iii.) $\mathbf{B} \cap \mathbf{C}$
iv.) $(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$
v.) $(\mathbf{A} \cap \mathbf{B}) \cup \mathbf{C}$
vi.) $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C})'$
vii.) $(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})'$
viii.) $(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})'$
ix.) $\mathbf{A'} \cup (\mathbf{B'} \cup \mathbf{C'})$
x.) $\mathbf{A'} \cap \mathbf{C}$

Tutor Marked Assignment

Given that $\mathfrak{U} = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathbf{A} = \{1, 2, 3\}$, $\mathbf{B} = \{3, 4, 5, 6\}$ and $\mathbf{C} = \{2, 3, 4\}$. Verify the following laws

i. Commutative laws
ii. Associative laws
iii. Distributive laws
iv. De Morgans laws.

References

- Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.
- Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.

References

https://www.mathwords.com/s/set_subtraction.htm
https://www.probabilitycourse.com/chapter1/1_2_2_set_operations.php
https://stattrek.com/statistics/dictionary.aspx?definition=combination

UNIT 3 SAMPLE SPACE, SAMPLE POINT AND EVENTS

Introduction

In this unit, you will understand sample space, sample point and events are used in probability and how they are closely related to the universal set, elements of a set and subset in set theory respectively.

Learning Outcomes

At the end of this unit, you should be able to:

1. Clearly define and determine the sample space of a study
2. Distinguish between Sample space and event
3. Generate sample points and events from the sample space
4. Combine different events from the sample space

Main Content

SAMPLE SPACE, SAMPLE POINT AND EVENTS

Definition

3mins

Sample space

SAQ 1

SAQ 3

If you Consider an experiment whose outcome cannot be predicted with certainty in advance. Although you will not know the outcome of the experiment in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by S. In other words, a Sample space is the collection or set of all possible results of that experiment. It is often define based on the objective of the study. In probability, the universal set is referred to as the sample space. The following are examples of sample space for a probability experiment.

- Rolling a six-sided die

S= (1, 2, 3, 4, 5, 6). Hence, there are six outcomes in the sample space

Figure showing the sample space for rolling a single six-sided die

·Rolling two fair six-sided dice

$ ◉ □	Die 1					
	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 3)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 4)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 5)

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Hence, there 36 outcomes in the sample space.

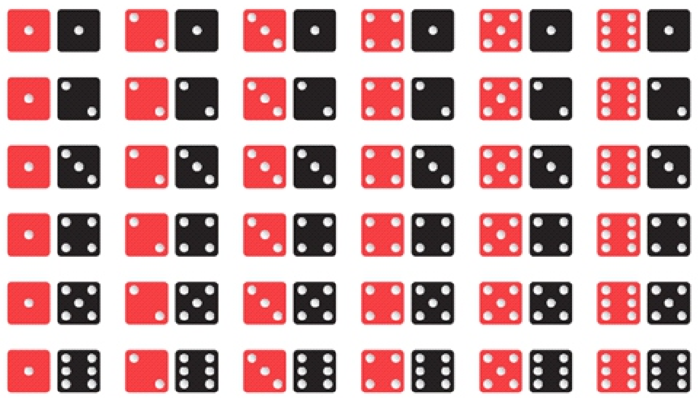

Figure showing the sample space for rolling a pair of six-sided dice

·The sample space for an ordinary deck of cards. There are 4 suits in a deck of cards comprising hearts, clubs, diamonds and spades. Each of the suits has 13 cards, hence, there are 52 outcomes in the sample space.

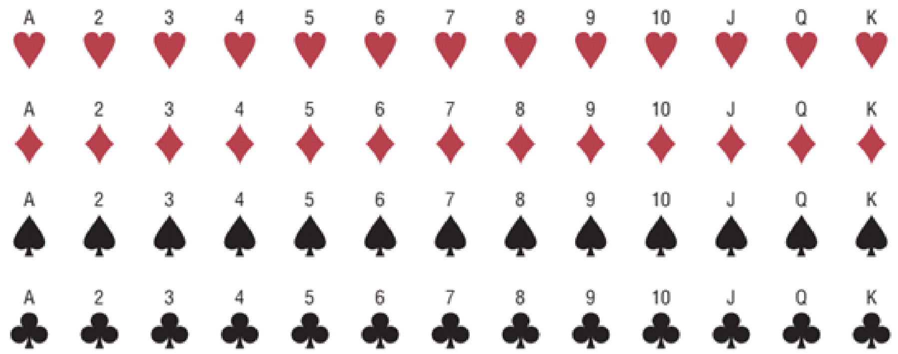

Figure showing sample space for a deck of cards

·Sample space for gender of three children in a family

S = {MMM, MMF, MFM, MFF, FMM, FFM, FMF, FFF}

Where M = *male* and F = *female.* Hence, there are 8 outcomes in the sample space.

·Tossing a coin

S = {T, H}

Where T = *tail* and H = *head*, hence there are 2 outcomes in the sample space.

Sample point

Each unit or item in the sample space is refered to as a sample point. For example, each of the numbers in the sample space of a single roll of a fair six-sided die is a sample point.

Event

An event is any subset of the sample space. That is, an event is a set consisting of possible outcomes of the experiment. If the outcome of the experiment is contained in an event, then we say that the event has occurred. For example, let A be an event consisting of 2 males in the sample space for gender of three children in a family, therefore

A = {MMF, MFM, FMM}

Event A is a subset of S, we therefore say, A has occurred.

Combination of Events

Similar to set theory, it is also possible to combine two or more events for a sample space of an experiment. Therefore, the following relations can be obtained;

·Union of Events: The union of events is the event that consists of all outcomes in all the events. That is, given events $E_1, E_2, \dots, E_n$, the union of the events is $E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n$ which reads $E_1 or\ E_2\ or\ E_3\ or \dots or E_n$.

·Intersection of Events: This the event that consists of all outcomes common to all the events. Given events $E_1, E_2, \dots, E_n$, the intersection of the events is denoted as $E_1 \cap E_2 \cap E_3 \cap \dots \cap E_n$ which reads $E_1 and\ E_2\ and\ E_3\ and \dots and\ E_n$

·Complement of an Event: The complement of an event E is the set of outcomes in the sample space that are not included in the outcomes of event E. The complement of E is denoted as $E'or\ E^c$.

·Exhaustive Events: The events $E_1, E_2, \dots, E_n$, are said to be exhaustive if they cover all possible outcomes of an experiment. That is $E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n$ = S

·Simple and Compound Events: An event E_1 with one outcome is refered to as a simple event. In other word, a simple event is a subset of the sample space whose number of outcome is 1. i.e. $n(E_1) = 1$. On the other hand, a compound event is an event with more than one outcomes. i.e. event E_2 is said to be a compound event if $n(E_2) > 1$

Example

Let S be the sample space of a single roll of a six-sided die. Determine the following;

i. Event E_1, all possible even integers in S
ii. Event E_2, all possible odd integers in S
iii. Event E_3, all possible prime integers in S
iv. $E_1 \cup E_2 \cup E_3$
v. $E_1 \cap E_2 \cap E_3$
vi. $E_1' \cap E_2'$
vii. Show that E_1 and E_2 are exhaustive
viii. Show that E_2 and E_3 are equivalent
ix. All possible simple event in S
x. Which among events E_1, E_2 and E_3 is/are not compound events?

Solution

S = (1, 2, 3, 4, 5, 6)

i. $E_1 = (2, 4, 6)$
ii. $E_2 = (1, 3, 5)$
iii. $E_3 = (1, 3, 5)$
iv. $E_1 \cup E_2 \cup E_3 = (1, 2, 3, 4, 5, 6)$
v. $E_1 \cap E_2 \cap E_3 = \emptyset$ (disjoint event)
vi. $E_1' = (1, 3, 5)$, $E_2' = (2, 4, 6)$, therefore
$E_1' \cap E_2' = \emptyset$
vii. Show that E_1 and E_2 are exhaustive
Recall, for exhaustive events, $E_1 \cup E_2 = S$, therefore $E_1 \cup E_2 = (1, 2, 3, 4, 5, 6)$. We conclude that E_1 and E_2 are exhaustive.
viii. Show that E_2 and E_3 are equivalent
Recall, for equivalent events,
$n(E_2) = n(E_3)$. $n(E_2) = 3$, $n(E_3) = 3$. We conclude that E_2 and E_3 are equivalent.
ix. Recall for a simple event $n(E) = 1$, therefore all possible simple event in S are;
$E_4 = (1), E_5 = (2), E_6 = (3), E_7 = (4), E_8 = (5)$ *and* $E_9 = (6)$
x. Since $n(E_1) = n(E_2) = n(E_3) = 3$ which is greater than 1, then all the three events are compound events.

Summary

In this unit, the basic concepts of sample space, sample point and event as related to set theory are explained.

·Sample space is the totality of all possible outcomes of an experiment. It is similar to the universal set in set theory. Sample space are usually define for a particular study under consideration.

·The sample point is unit in the sample space. It is similar to element of a set in set theory.

·An event is a fractional part of a sample space. It is also similar to a subset in set theory.

·Two or more events can be combine based on set theory operations of union or intersection.

Self Assessment Questions

1. Determine the sample space for an eight-sided die labelled 1, 3, 5, 7, 9, 11, 13, 15 rolled twice.
2. What is the difference between a Sample space, sample point and an event?
3. Obtain the sample space of the following experiment
i.) Gender of four children in a family
ii.) Tossing three coins once

Tutor Marked Assignment

Suppose the sample space S, contains all the sums of the faces that turned up in a single roll of two six-sided dice. Determine the following:

i. Event E_1, all possible even integers in S
ii. Event E_2, all possible odd integers in S
iii. Event E_3, all possible prime integers in S
iv. $E_1 \cup E_2 \cup E_3$
v. $E_1 \cap E_2 \cap E_3$
vi. $E_1' \cap E_2'$
vii. Show that E_1 and E_2 are exhaustive
viii. Show that E_2 and E_3 are equivalent
ix. All possible simple event in S
x. Which among events E_1, E_2 and E_3 is/are not compound events?

References

•Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.

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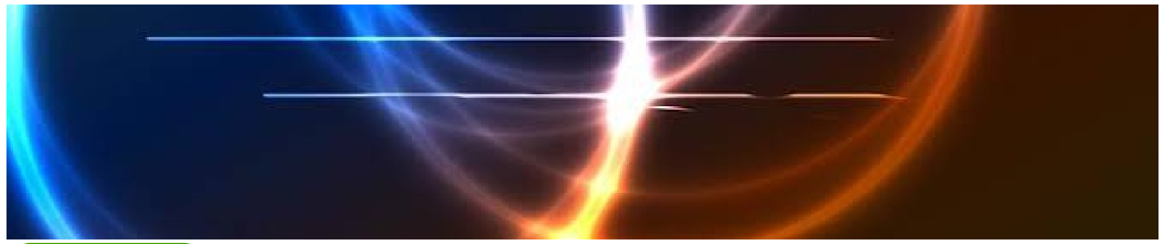

UNIT 4 DEPENDENT, INDEPENDENT AND MUTUALLY EXCLUSIVE EVENTS

Introduction

Sometimes you observe that occurrence or non-occurrence of one event tells us something about other events. That is, information you have on one events may or may not give you a clue about the other events. This scenario is what leads to the concept of dependent and independent events. As a basic rule of thumb, the existence or absence of an event can provide clues about other events.

Learning Outcomes

At the end of this unit, you should be able to:

1. Clearly define Dependent events
2. Clearly define Independent events
3. Clearly define mutually exclusive events
4. Distinguish among dependent, independent and mutually exclusive events

Main Content

DEPENDENT, INDEPENDENT AND MUTUALLY EXCLUSIVE EVENTS

We call events dependent if knowing whether one of them happened tells us something about whether the others happened. Independent events give us no information about one another. The probability of one event occurring does not affect the probability of the other events occurring.

In general, an event is deemed dependent if it provides information about another event. An event is deemed independent if it offers no information about other events.

What are Dependent Events?

1min

SAQ 1

For events to be considered dependent, one must have an influence over how probable another is. In other words, a dependent event can only occur if another event occurs first.
While it is a statistical term, speaking specifically to the subject of probabilities, the same is true of dependent events as they occur in the real world.
For example, say you would like to go on vacation at the end of next month, but that depends on having enough money to cover the trip. You may be counting on a bonus, a commission, or an advance on your

paycheck. It also most likely depends on you being given the last week of the month off to make the trip.
The primary focus when analyzing dependent events is probability. The occurrence of one event exerts an effect on the probability of another event. Consider the following examples:
1. Getting into a traffic accident is dependent upon driving or riding in a vehicle.
2. If you park your vehicle illegally, you're more likely to get a parking ticket.
3. You must buy a lottery ticket to have a chance at winning; your odds of winning are increased if you buy more than one ticket.
4. Committing a serious crime – such as breaking into someone's home – increases your odds of getting caught and going to jail.
5. Having high grades and getting a scholarship.

Statistically Dependent Events

3mins

Two events are dependent if the outcome or occurrence of the first affects the outcome or occurrence of the second so that the probability is changed.

Example

Suppose we have 5 blue marbles and 5 red marbles in a bag. We pull out one marble, which may be blue or red. Now there are 9 marbles left in the bag. What is the probability that the second marble will be red?

Solution

It depends on the marble you pull; If the first marble was red, then the bag is left with 4 red marbles out of 9 so the probability of drawing a red marble on the second draw is $\frac{4}{9}$.
But if the first marble we pull out of the draw is blue, then there are still 5 red marbles in the bag and the probability of pulling a red marble out of the bag is $\frac{5}{9}$.

Therefore, the second draw is a dependent event. It depends upon what happened in the first draw.

Example

A card is chosen at random from a standard deck of 52 playing cards. Without replacing it, a second card is chosen. What is the probability that the first card chosen is a queen and the second card chosen is a jack?

Solution

The probability that the first card is a queen is 4 out of 52. However, if the first card is not replaced, then the second card is chosen from only 51 cards. Accordingly, the probability that the second card is a jack given that the first card is a queen is 4 out of 51.

Therefore, the outcome of choosing the first card has affected the outcome of choosing the second card, making these events dependent.

What are Independent Events?

3mins

SAQ 2

An event is deemed independent when it isn't connected to another event, or its probability of happening, or conversely, of not happening. It is true of events in terms of probability, as well as in real life, which, as mentioned above, is true of dependent events as well.

For example, the color of your hair has absolutely no effect on where you work. The two events of “having black hair” and “working in Allentown” are completely independent of one another.

Independent events don't influence one another or have any effect on how probable another event is.

Other examples of pairs of independent events include:
1. Taking an Uber ride and getting a free meal at your favorite restaurant
2. Winning a card game and running out of bread
3. Finding a dollar on the street and buying a lottery ticket; finding a dollar isn't dictated by buying a lottery ticket, nor does buying the ticket increase your chances of finding a dollar
4. Growing the perfect tomato and owning a cat

Statistically Independent Events

2mins

Two events are independent if the result of the second event is not affected by the result of the first event. If A and B are independent events, the probability of both events occurring is the product of the possibilities of the individual events. In general, to find the probability of two independent events that occur in sequence, you must find the probability of each event occurring separately and then multiply the answers.

Examples
- Drawing a card from a deck and getting a queen, replacing it, and drawing a second card and getting a queen.
- Rolling a die and getting a 6, and then rolling a second die and getting a 3.
- A coin is flipped and a die is rolled.

MUTUALLY EXCLUSIVE EVENTS

SAQ 3

Mutually exclusive events are things that can't happen at the same time. For example, you can't run backwards and forwards at the same time. The events "running forward" and "running backwards" are mutually exclusive. Tossing a coin can also give you this type of event. You can't toss a coin and get both a heads and tails. So "tossing a heads" and "tossing a tails" are mutually exclusive. Some more examples are: your ability to pay your rent if you don't get paid, or watching TV if you don't have a TV.

In general, two events are **mutually exclusive events** if they cannot occur at the same time (i.e., they have no outcomes in common).

Examples
The following are events are mutually exclusive:.
1. Getting an odd number and getting an even when a single die is rolled.
2. Getting a number greater than 4 and getting a number less than 4 in a single roll of a die
3. A day of the week selected at random

Summary

In this unit, Dependent, Independent and Mutually exclusive events has been explained.

•Two events are dependent if the outcome or occurrence of the first affects the outcome or occurrence of the second so that the probability is changed.

•Two events are independent if the result of the second event is not affected by the result of the first event.

•Two events are mutually exclusive events if they cannot occur at the same time.

Self Assessment Questions

1. Define dependent events with examples
2. Define independent events with examples
3. Determine which events are mutually exclusive and which are not;
   a. Getting a 3 and getting an odd number in a single roll of a die.
   b. Getting an odd number and getting a number less than 4 when a die is rolled
   c. Getting a club and getting a king when a single card is picked from a deck
   d. Getting a face card and getting a spade when a single card is picked from a deck.

Tutor Marked Assignment

1.State which events are independent and which are dependent
   a. Eating an excessive amount of ice cream and smoking an excessive amount of cigarettes.
   b. Drawing a ball from an urn, not replacing it, and then drawing a second ball
   c. Smoking excessively and having lung cancer
   d. A father being left-handed and a daughter being left-handed
   e. Tossing a coin and drawing a card from a deck
   f. Getting a raise in salary and purchasing a new car

2.Determine whether these events are mutually exclusive;
   a. Select a registered voter: The voter is a NRC member, and the voter is a SDP member
   b. Roll a die: Get an even number, and get a number less than 3
   c. Select any course: It is a calculus course, and it is an English course
   d. Roll a die: Get a prime number (2, 3, 5), and get an odd number
   e. Select a student in your college: The student is a fresher, and the student is a computer major

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•Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.

•Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.

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MODULE 2

Probability and Probability measures

UNIT 1 ELEMENTS OF PROBABILITY

Introduction

Generally, a person with cynical thinking will say, "The only sure thing in life is death." This philosophy arose because many activities in people's lives are affected by chance. From the time you awake until you go to bed, several decisions regarding the possible events that are governed at least in part by chance are made. For example, should you carry an umbrella to work today? Will your phone battery last until evening? Should you accept the admission offered in a particular institution? These and many other examples are activities demanding decisions that are governed by chance of occurrence based on individual perception or degree of trust.

Learning Outcomes

At the end of this unit, you should be able to:

1. Clearly define the concept of probability
2. Define an experiment, outcome and a trial
3. Distinguish between random and non-random experiments
4. State the Axioms of probability

Main Content

ELEMENTS OF PROBABILITY

5mins

Probability of an event

You see that Probability as a general concept can be defined as the chance of an event occurring. Many people are familiar with probability from observing or playing games of chance, such as card games, slot machines, or lotteries. In addition to being used in games of chance, probability theory is used in the fields of insurance, investments, and weather forecasting and in various other areas.

In statistics, inferences and conclusions are often arrived at based on experiments whose outcomes remain uncertain (or unknown). Therefore, for us to be able to make a valid statement on the outcome of this experiment (and probably play safe by our comments/assertions), efficient use of probability and probability concepts becomes very necessary.

For instance, let us consider the following statements or sentences:

a. Political party XYZ has 70% chance of winning the 2023 presidential election.
b. I may probably be the scholar of my class at the end of this session.
c. I am 65% confident that I may be next scholar in my class.

In each of the above statements, we are expressing an outcome (of an experiment) of which we are not sure/certain, but we have some degree of confidence in the validity of the statement probably based on our past experience or previous information or our understanding of the workings of the experiment.

Basic concepts in Probability

SAQ 2

Experiment

An experiment is a procedure carried out under controlled conditions in order to discover an unknown effect or law. It is also a process by which several chance observations are obtained.

SAQ 3

Random/probability Experiment

When the results of an experiment differs at each time the experiment is performed even under the same condition, it is termed random experiment. A random experiment satisfies the following conditions;

- The experiment can be repeated under the same condition
- The result is unknown in advance
- All possible outcome of the experiment can be predicted

The following are examples of a random experiment

- Toss a coin
- Roll a die
- Number of Samsung phones sold in a shopping mall
- Observe the number of goals scored in a football match
- The colour of a ball selected from a box containing different colour of balls
- The outcome (pass or fail) of STA 121 exam

The following are not examples of a random experiment

- Weight of a new born baby
- The brand of iPhones manufacture in 2019
- The capacity of a lecture theatre at University of Ilorin
- The price of a bag of rice at a local market in Kwara.
- The number of students admitted into the Online Distance Learning (ODL) in Nigeria

When we repeat a random experiment several times, we call each one of them a trial. Thus, a trial is a particular performance of a random experiment.

SAQ 2

Outcome

An outcome of an experiment is one of the possible results obtained when an experiment is performed.

Example 1 Rolling a fair die is a random experiment from which six distinct outcomes (1, 2, 3, 4, 5, 6) can be obtained.

Example 2 Tossing a fair coin is a random experiment from which either of the two chance outcomes (observations) head (H) or tail (T) is obtainable.

Example 3 Observing the gender of a set of twin at birth is an experiment in which four possible outcomes are possible – BB, GG, BG and GB where B = Boy and G = Girl.

SAQ 1

What is Probability?

Probability can be refered to as a study concerning randomness and uncertainty of an event. That is, making statements about what we do not have prior knowledge of until it happens in a situation where there are number of possible outcomes. The language of probability is constantly used in our everyday activities such as; *I will probably buy a new mobile phone on gaining admission into the University, I may probably be the scholar of my class at the end of this session, I have 50:50 chance of passing STA121 course, I am 65% confident that I may be next scholar in my class, etc.*

The concept of probability is usually not a straightforward concept in statistics. This has brought about several definitions offered to describe the word "probability".

i. The classical definition puts probability as the measure of our ignorance about the outcome of an experiment.
ii. In logical reasoning, probability can be defined as a rational and objective degree of belief regarding the outcome of an experiment.
iii. By the frequency approach, probability is defined as the relative frequency with which events occur in the long run.

There is also a mathematical definition of probability.

Mathematical Definition of Probability

Probability as a general concept can be defined as the chance of an event occurring. It can be described as the number of outcomes in an event divided by the total number of possible outcomes in an experiment, provided that each outcome is equally likely.

Basic Properties (Axioms) of Probability

SAQ 4

Property 1: The probability of an event is always between 0 and 1, inclusive. That is, if A is an event, the probability that event A occurs, *Pr(A)* lies between 0 and 1 inclusive. i.e.

$0 \leq P_r(A) \leq 1$

Property 2: The probability of an event that cannot occur is 0. An event that cannot occur is called an **impossible event.** Thus, if event A cannot occur, then *Pr(A)* = 0

Property 3: The probability of an event that must occur is 1. An event that must occur is called a **sure or certain event**. If event A is certain to occur, then *Pr(A)* = 1

Illustrations

• Suppose you have a six-sided die is tossed once, determine the event A that the number on face that turned up is 7. Obviously, this is an impossible event because the die on has six outcomes labelled 1, 2, 3, 4, 5, 6. Therefore number 7 is not included in the outcome. Hence, *Pr(A)* = 1

• Suppose a six-sided die is tossed once, determine the event B that the number on face that turned up is less than 7. Similarly, since it is obvious that all the number on the die are less than 7 and it is certain that one of these number will appear, therefore, the event of getting a number less than 7 is certain. Hence, *Pr(B)* = 1

Summary

In this Unit, the element of probability are explained.
• Probability is referred to as the chance of an event occurring in a sample space.
• The basic concept of probability – Experiment, Random/probability experiment and outcome are explained.
- Experiment is referred to as a process or action that lead to a well-define result
- A random experiment is defined as an experiment with differences in result even if performed under the same condition.

-An outcome is one of the possible result of an experiment.
• The three basic properties also called axioms of probability are;
(i) probability of an event cannot be less than zero or greater than one,
(ii) probability of an impossible event to occur is zero and (iii) probability that an event is sure or certain to occur is one.

Self Assessment Questions

1. Define probability of an event
2. Define experiment, outcome and trial with illustrations
3. List examples of random and non-random experiments
4. State with illustrations, the axioms of probability.

Tutor Marked Assignment

1. State the conditions for a random experiment.
2. An experiment consists of tossing a coin three times. What is the sample space?
of this experiment? Which event corresponds to the experiment resulting in more heads than tails?
3. What is the difference between random and non-random experiment?

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UNIT 2 PROBABILITY: A MEASURE OF UNCERTAINTY

Introduction

Consider statements like; *I am 70% sure that there would be rainfall before the end of the day; there is 50% chance that the next president of Nigeria will be a youth; the chance of graduating from a public university at the age of 18 years old is very slim.* You will findout that the statements are with some degree of uncertainty, but a measure of the uncertainty nature of the events were provided. This unit we shall discuss probability as a measure of uncertainty.

Learning Outcomes

At the end of this unit, you should be able to:

1. Quantify the probability of an event
2. Determine the probability measures on some practical examples
3. Distinguish between Classical, Emperical and Subjective probability

Main Content

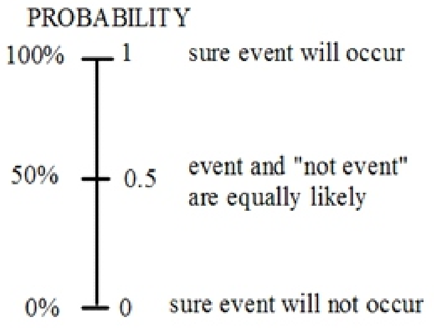
PROBABILITY: A MEASURE OF UNCERTAINTY

SAQ 1

SAQ 2

Quantifying probability

A probability measure is simply a number between 0 and 1 that quantify the uncertainty of a particular event. Many events are uncertain in Nature and we possess different degrees of belief about the truth or falsity of an uncertain event. For example, statement like "the sun will rise tomorrow" is a certainty, "the cloud is made of grass" is impossible and statement like “the foetus may be a girl or a boy” is equally likely. We can think of a probability scale from 0 to 1.



SAQ 3

Approaches towards Assigning Probabilities

There are three approaches we can assigning probabilities to events. These are:

i. Classical Approach
ii. Relative Frequency (or Empirical) Approach
iii. Subjective Approach

The Classical Approach

The classical probability uses sample space to determine the numerical probability that an event will occur. We assumes that all outcomes in the sample space are *equally likely* to occur. Equally likely events are events that have the same probability of occurring.

By classical method, probability that event A would occur, Pr(A) is defined as

$$\Pr(A) = \frac{\textit{Number of outcomes in Event A}}{\textit{Number of outcomes in the sample space S}}$$

$$= \frac{n(A)}{n(S)}$$

Example If a fair die is thrown once and we are interested in the number that comes up, the sample space of all the possible outcomes, S = {1, 2, 3, 4, 5, 6}. What is the probability of an even number of face coming up?
Let event E be defined as the event of obtaining even number of faces when a fair die is tossed once. Hence, E = {2, 4, 6}. Therefore,

$$Pr(E) = \frac{\textit{Number of outcomes in Event E}}{\textit{Number of outcomes in the sample space S}}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2} \; or \; 0.5 \; (50\%)$$

Example Find the probability of getting a black 10 when drawing a card from a deck.
Recall, there are 52 cards in a deck, and there are two black 10s i.e. the 10 of spades and the 10 of clubs. Hence the probability of getting a black 10 is

$$P(black\ 10) = \frac{n(black\ 10)}{n(cards)}$$

$$= \frac{2}{52}$$

$$= \frac{1}{26} \; or \; 0.0385$$

Example If a family has three children, find the probability that two of the three children are females?
Recall, in a family of three

S = {MMM, MMF, MFM, MFF, FMM, FFM, FMF, FFF}

Let Y be the event comprising two female from S, then

Y = {MMF, MFM, FMM}

$$P(Y) = \frac{n(Y)}{n(S)}$$

$$= \frac{3}{8} \; or \; 0.3750$$

Note, the sum of the probabilities of all the outcomes in the sample space is 1. That is, probability of event that at least one of the outcomes will occur. For example, in the roll of a fair die, each outcome in the sample space has a probability of

Outcome (S)	1	2	3	4	5	6
Probability (P)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
Sum of P	$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1$					

In other word,

$$P(S)\frac{n(S)}{n(S)}$$

$$= 1$$

Where *S* is the sample space

Also you should recall that the complement of an event E denoted as E' or E^c is an event containing is those outcome in the sample space that are not present in E. Therefore, the sum of probability of an event E and its complement E' must equal to 1, that is

$$P(E) + P(E') = 1$$

provided E and E' are exhaustive events.

Example Recall the example of children in a family of three

S = {MMM, MMF, MFM, MFF, FMM, FFM, FMF, FFF}

Let Y be the event comprising two females from S and Y' as the complement of Y, then

Y = {MMF, MFM, FMM}

$$P(Y') = \frac{n(Y')}{n(S)}$$

$$= \frac{5}{8} \; or \; 0.6250$$

Therefore,

$$P(Y) + P(Y') = 0.3750 + 0.6250$$
$$= 1$$

Example If the probability that a person lives in an industrialized country of the world is 0.2, find the probability that a person does not live in an industrialized country.

Let event A be the event that a person lives in an industrialized country of the world, A' be the event that a person does not live in an industrialized country of the world. since

$$P(A) + P(A') = 1$$

Therefore,

$$0.2 + P(A') = 1$$
$$P(A') = 1 - 0.2$$
$$= 0.8$$

Note, If the probability of an event or the probability of its complement is known, then the other can be found by subtracting the probability from 1. This rule is important in probability theory because at times the best solution to a problem is to find the probability of the complement of an event and then subtract from 1 to get the probability of the event itself.

The Empirical Approach

The empirical approach which is also known as *Relative Frequency Approach* describes probability as the long-run relative frequency with which an outcome occurs. This method assigns probabilities based on experimentation or historical data. The empirical approach defines probability that event A would occur, $P(A)$ as

$$\mathrm{P}(A) = \frac{frquency\ of\ occurence\ of\ an\ outcome}{total\ frequency\ of\ sample\ in\ the\ data}$$

$$P(A) = \frac{f}{n}$$

The difference between classical and empirical probability is that classical probability we assume that certain outcomes (i.e. each of the sample points) are equally likely (such as the outcomes when a die is rolled), while empirical probability relies on actual experience to determine the likelihood of outcomes.

Example In a sample of 50 students who wish to determine their blood group, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the probability that

a) A selected person has type O blood
b) A selected person has type AB blood
c) A selected person does not have type A blood

Solution

Blood Type	**Frequency *f***
O	21
A	22
B	5
AB	2
Total	50

a) A selected person has type O blood

$$P(O) = \frac{21}{50}$$
$$= 0.42$$

b) A selected person has type AB blood

$$P(AB) = \frac{2}{50}$$
$$= 0.04$$

a) A selected person does not have type A blood

$$P(A) = \frac{22}{50}$$

But $P(A) + P(A') = 1$, therefore

$$P(A') = 1 - P(A)$$

$$= 1 - \frac{22}{50}$$

$$= \frac{28}{50} \; or \; 0.56$$

The Subjective Approach

In the subjective approach to probability, probability is defined as the degree of a person's belief in the occurrence of an event.

Subjective probability is also described as the degree of personal conviction about an event.

When it is not reasonable to use classical approach and there is no historical data to used empirical approach, the use of the subjective approach is inevitable.

In this approach, opinions, experience and personal judgment are employed to express probability of an event.

Example 1 President Muhammad Buhari has 70% chance of winning the 2019 presidential election.

Example 2 I am 65% confident that I will be the next scholar in my class.

Summary

In this Unit, Probability as a measure of uncertainty is explained.

•Probability as a measure of uncertainty take values between 0 and 1 inclusive where the values of 0 implies the event under consideration is impossible to occur, 0.5 implies the occurrence and non-occurrence of the event under consideration are equally likely and 1 implies the event under consideration is sure or certain to occur.

•In the classical approach to probability, probability that event A would occur, *Pr(A)* is defined as:

$$P(A) = \frac{Number\ of\ outcomes\ in\ Event\ A}{Number\ of\ outcomes\ in\ the\ sample\ space\ S}$$

•The empirical approach defines probability that event A would occur, *Pr(A)* as:

$$P(A) = \frac{frquency\ of\ occurence\ of\ an\ outcome}{total\ frequency\ of\ sample\ in\ the\ data}$$

•The difference between classical and empirical probability is that classical probability outcomes are assumed to be equally likely while empirical probability relies on actual experience to determine the likelihood of outcomes.

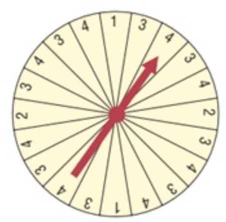
•The subjective approach relies on degree of belief or trust based on some form of prior experience to quantify or measure the probability of an event.

Self Assessment Questions

1. How is probability of an event obtained?
2. What are equally likely events?
3. Classify each statement as an example of classical probability, empirical probability, or subjective probability.
   a) The probability that a person will watch the 9 o'clock evening news is 0.15.
   b) The probability of winning at a Chuck-a-Luck game.
   c) The probability that a bus will be in an accident on a specific run is about 6%.
   d) The probability of getting a royal flush when five cards are selected at random.
   e) The probability that interest rates will rise in the next 6 months is 0.50.

Tutor Marked Assignment

1. If the probability that it will rain tomorrow is 0.20, what is the probability that it won't rain tomorrow?
2. A shopping mall has set up a promotion as follows. With any mall purchase of $50 or more, the customer gets to spin the wheel shown below. If a number 1 comes up, the customer wins $10; if the number 2 comes up, the customer wins $5; and if number 3 or 4 comes up the customer wins a discount coupon. What is the probability that
   i. a customer wins $10
   ii. a customer wins $5
   iii. a customer does not win at all



3. A probability experiment is conducted. Which of the following cannot be considered a probability outcome? State your reasons

a. $\frac{2}{3}$ b. $\frac{3}{2}$ c. 60% d. 1.01 e. 125%

f. – 0.44

References

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•Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.

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Further Reading

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UNIT 3 PROBABILITY THEORY

Introduction

Many problems we face day-day involve obtaining the probability of two or more events occurring at the same time. The probability of two or more events can be determined by the *addition or multiplication* rules of probability.

Learning Outcomes

At the end of this unit, you should be able to:

1. Clearly state the addition rule of probability
2. Clearly state the multiplication rule of probability
3. Determine the probability of dependent, independent and mutually exclusive events

Main Content

PROBABILITY THEORY

3mins

SAQ 5

SAQ 6

In probability theory, it is important for us to understand the meaning of the words ***and*** and ***or*** which also correspond to Intersection (∩) and Union (∪) of two or more events respectively. For example, if you were asked to find the probability of getting a queen ***and*** a heart when you were drawing a single card from a deck, you would be looking for the queen of hearts. Here the word and means "***at the same time***." The word ***or*** has two meanings. For example, if you were asked to find the probability of selecting a queen ***or*** a heart when one card is selected from a deck, you would be looking for one of the 4 queens or one of the 13 hearts. In this case, the queen of hearts would be included in both cases and counted twice. On the other hand, if you were asked to find the probability of getting a queen ***or*** a king, you would be looking for one of the 4 queens or one of the 4 kings. In the first case, both events can occur at the same time; we say that this is an example of the ***inclusive or***. In the second case, both events cannot occur at the same time, and we say that this is an example of the ***exclusive or***.

If you Consider the following event of selecting an individual in an academic gathering. What is the probability that the selected person is a male or a first class degree holder? In this case, there are three possibilities;

- The person is a male
- The person is a first class degree holder
- The person is both male and a first class degree holder.

In this example, the person you select can be a male and a first class degree holder at the same time.
Consider another example of a gathering of university students where there are first class, second class upper and second class lower students. If a person is selected, then the possibilities are

- The person is a first class student
- The person is a second class upper student
- The person is a second class lower student

In this second example, a selected person can only posses on of the attributes. For instance, it is impossible for a selected student to be in both first class and second class. This is an example of *mutually exclusive* events. In the first example, the events are not mutually exclusive.

Recall, two events A and B are said to be mutually exclusive if they cannot occur at the same time. i.e. they have no common outcomes.

In other words, $A \cap B = \emptyset$, therefore, $P(A \cap B) = 0$

Addition rule of probability

4mins

SAQ 1

Addition Rule 1

When two events $E_1$and E_2 are mutually exclusive, the probability that E_1or E_2 will occur $P(E_1 or\ E_2)$ also written as $P(E_1 \cup E_2)$ is defined as $\boldsymbol{P(E_1 \cup E_2) = P(E_1) + P(E_2)}$
More generally, if events $E_1, E_2, E_3, \dots, E_n$, are mutually exclusive, then $\boldsymbol{P(E_1 \cup E_2 \cup E_3 \cup, \dots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n)}$

Example

The faculty of physical science at the university of Ilorin have the following number of academic staffs for three departments:

•Statistics	23
•Physics	26
•Industrial chemistry	17

If an academic staff is selected at random, find the probability that the academic staff is from Statistics or Industrial chemistry department.

Solution

Since a selected academic staff cannot belong to both Statistics and Industrial chemistry departments at the same time, hence, the events are mutually exclusive. Therefore,

$$P(Statistics\ or\ Industrial\ Chemistry) = P(Statistics \cup Industrial\ Chemistry)$$
$$= P(Statistics) + P(Industrial\ Chemistry)$$
$$= \frac{23}{66} + \frac{17}{66}$$
$$= \frac{40}{66}$$
$$= 0.6061$$

Example

A day of the week is selected at random. Find the probability that it is a weekend day.

Solution

Since a day from the weekend can either be a Saturday or a Sunday, then the event is said to be mutually exclusive. Therefore,

$$P(Saturday\ or\ Sunday) = P(Saturday) + P(Sunday)$$
$$= \frac{1}{7} + \frac{1}{7}$$
$$= \frac{2}{7}$$

Addition Rule 2

*When two events E_1 and E_2 are **not** mutually exclusive, the probability that E_1or E_2 will occur $P(E_1 or\ E_2)$ also written as $P(E_1 \cup E_2)$ is defined as*

$$\boldsymbol{P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 and E_2)}$$

Example

On a particular eve in one of the towns in a city, the chance that a person will drive while intoxicated is 0.45, the chance of having an accident while driving in the area of the town is 0.06 and the chance of a person having a driving accident while intoxicated is 0.05. What is the probability of a person driving while intoxicated or having a driving accident?

Solution

Since getting intoxicated and having an accident can both occur simultaneously, then

$$P(intoxicated\ or\ accident) = P(intoxicated) + P(accident) - P(intoxicated\ \ and\ accident)$$
$$P(intoxicated\ or\ accident) = 0.45 + 0.06 - 0.05$$
$$= 0.46$$

Example In a hospital unit there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

Solution

The sample space is shown here.

Staff	Females	Males	Total
Nurses	7	1	8
Physicians	3	2	5
Total	10	3	13

The event of being a nurse and a male is not mutually exclusive. Therefore,

$$P(nurse\ or\ male) = P(nurse) + P(male) - P(nurse\ and\ a\ male)$$
$$= \frac{8}{13} + \frac{3}{13} - \frac{1}{13}$$
$$= \frac{10}{13}$$

MULTIPLICATION RULE OF PROBABILITY

SAQ 3

Multiplication Rule 1

When two events E_1 and E_2 are independent, the probability that both events will occur $P(E_1 and\ E_2)$ also written as $P(E_1 \cap E_2)$ is defined as

$$\boldsymbol{P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)}$$

More generally, if events $E_1, E_2, E_3, \ldots, E_n$ are independent, then

$$\boldsymbol{P(E_1 \cap E_2 \cap E_3 \cap, \ldots \cap E_n) = P(E_1) \cdot P(E_2) \cdot P(E_3) \cdot \ldots \cdot P(E_n)}$$

Example If a coin is flipped and a die is rolled, then find the probability of getting a tail on the coin and a 3 on the die.

Solution The sample space for the coin is H, T while the sample space for the die is 1, 2, 3, 4, 5, 6. Since both events are independent, then.

$$P(tail\ \cap 3) = P(tail) \cdot P(3)$$
$$= \frac{1}{2} \cdot \frac{1}{6}$$
$$= \frac{1}{12}$$

Example A box contains 3 black balls, 2 green balls, and 5 yellow balls. A ball is selected and its colour noted. Then it is replaced. A second ball is selected and its colour noted. Find the probability of

i. Selecting a 2 yellow balls
ii. Selecting 1 black ball and 1 green ball

Solution Since selection is with replacement, then each selection is independent of each other. Therefore, the total number of balls available is not affected at each selection.

i. $P(Yellow\ and\ Yellow) = P(Yellow\ \cap Yellow)$
$$= P(Yellow) \cdot P(Yellow)$$
$$= \frac{5}{10} \cdot \frac{5}{10}$$
$$= \frac{25}{100}$$
$$= \frac{1}{4}\ or\ 0.25$$

ii. $P(black\ and\ green) = P(black\ \cap green)$
$$= P(black) \cdot P(green)$$
$$= \frac{3}{10} \cdot \frac{2}{10}$$
$$= \frac{6}{100}$$
$$= \frac{3}{50}\ or\ 0.06$$

SAQ 4

Multiplication Rule 2

When two events E_1 and E_2 are dependent, such that the outcome of the E_2 depends on the chance of occurrence of E_1 then, the probability that both events will occur $P(E_1 and\ E_2)$ also written as $P(E_1 \cap E_2)$ is defined as

$$\boldsymbol{P(E_1 \cap E_2) = P(E_1)\ \cdot\ P(E_2|E_1)}$$

Where $P(E_2|E_1)$ is read as *"probability of event E_2 occurring given that E_1 has occurred"*

Example Suppose there are 5 reported cases of thefts in a particular community in the moth of June, 3 in July and 4 in August, all in 2019. As a researcher, you are requested to select two thefts cases for further investigation. What is the chance that both cases will have occurred in July?

Solution In this case, the events are dependent since the researcher wishes to investigate two distinct cases. Hence the first case is selected and not replaced and this will affect the total number of cases at each selection.

Let T_1 and T_2 be the theft cases selected in July, then;

$$P(T_1\ and\ T_2) = P(T_1) \cdot P(T_2|T_1)$$
$$= \frac{3}{12} \cdot \frac{2}{11}$$
$$= \frac{6}{132}$$

Example Three cards are drawn from an ordinary deck and not replaced. Find the probability of getting 3 jacks.

Solution Since there are 4 jacks in a deck of card, then the probability is

$$P(J_1 \cap J_2 \cap J_3) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50}$$
$$= \frac{24}{132600}$$

Summary

In this Unit, Probability as a measure of uncertainty is explained.

•When two events E_1 and E_2 are mutually exclusive, the probability that E_1 or E_2 will occur $P(E_1 or\ E_2)$ also written as $P(E_1 \cup E_2)$ is defined as

$$\boldsymbol{P(E_1 \cup E_2) = P(E_1) + P(E_2)}$$

More generally, if events $E_1, E_2, E_3, \dots, E_n$, are mutually exclusive, then

$$\boldsymbol{P(E_1 \cup E_2 \cup E_3 \cup, \dots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n)}$$

•When two events E_1 and E_2 are not mutually exclusive, the probability that E_1 or E_2 will occur $P(E_1 or\ E_2)$ also written as $P(E_1 \cup E_2)$ is defined as

$$\boldsymbol{P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 and E_2)}$$

•Where $P(E_1 and\ E_2)$ is also written as $P(E_1 \cap E_2)$ the empirical approach defines probability that event A would occur, $P(A)$ as

$$\mathrm{P}(A) = \frac{frquency\ of\ occurence\ of\ an\ outcome}{total\ frequency\ of\ sample\ in\ the\ data}$$

•When two events E_1 and E_2 are independent, the probability that both events will occur $P(E_1 and\ E_2)$ also written as $P(E_1 \cap E_2)$ is defined as

$$\boldsymbol{P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)}$$

More generally, if events $E_1, E_2, E_3, \dots, E_n$, are independent, then

$$\boldsymbol{P(E_1 \cap E_2 \cap E_3 \cap, \dots \cap E_n) = P(E_1) \cdot P(E_2) \cdot P(E_3) \cdot \dots \cdot P(E_n)}$$

•When two events E_1 and E_2 are dependent, such that the outcome of the E_2 depends on the chance of occurrence of E_1, then, the probability that both events will occur $P(E_1 and\ E_2)$ also written as $P(E_1 \cap E_2)$ is defined as

$$\boldsymbol{P(E_1 \cap E_2) = P(E_1) \cdot P(E_2|E_1)}$$

Where $P(E_2|E_1)$ is read as *"probability of event E_2 occurring given that E_1 has occurred"*

Self Assessment Questions

1. State the addition rule of probability for mutually exclusive events.
2. State the addition rule of probability for non-mutually exclusive events.
3. State the multiplication rule of probability for independent events.
4. State the multiplication rule of probability for dependent events.
5. Determine whether the following events are mutually exclusive.
   a. Roll a die: get an odd number or an even number.
   b. Select a presidential candidate in an election: the candidate belongs to all political parties.
   c. Roll a die: Get a number greater than 3, and get an even number.
6. Classify the following events as dependent or independent.
   a. Rolling a die and tossing a coin.
   b. Smoking and having lung cancer.
   c. Having big shoe size and high IQ.

Tutor Marked Assignment

1. Define mutually exclusive events, and give an example of two events that are mutually exclusive and two events that are not mutually exclusive.
2. In a fish tank, there are 24 goldfish, 2 angel fish, and 5 guppies. If a fish is selected at random, find the probability that it is a goldfish or an angel fish.
3. On a hospital staff, there are 4 dermatologists, 7 surgeons, 5 general practitioners, 3 psychiatrists, and 3 orthopedic specialists. If a doctor is selected at random, find the probability that the doctor is
   a. A psychiatrist, surgeon, or dermatologist.
   b. An orthopedic specialist, a surgeon, or a dermatologist.
   c. A surgeon or dermatologist
4. Four cards are drawn from a deck without replacement. Find these probabilities.
   a. All are kings
   b. All are diamonds
   c. All are red cards

References

• Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.

• Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.

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UNIT 4 JOINT AND CONDITIONAL PROBABILITIES

Introduction

Suppose we are interested in the probability of two events simultaneously such as the outcomes of two or more different events, the probability of two (or more) events is called the joint probability. Similarly, we may be interested in the probability of an event given the occurrence of another event. The probability of one event given the occurrence of another event is called the conditional probability. The Joint probability of two or more events using the conjunctions ***"and"*** or ***"or"*** has been discussed in the previous unit. This unit therefor explain the conditional probability of an event.

Learning Outcomes

At the end of this unit, you should be able to:

1. Obtain the conditional probability of any even t.

Main Content

JOINT AND CONDITIONAL PROBABILITIES

SAQ 1

As an illustration to understanding the concept of conditional probability, suppose a pair of fair dice is rolled, sample space of the experiment (36 outcomes) is

SAQ 2

Second die (B)

First die (A)	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

We can define the outcome as (A, B) choosing the outcome of the first die as A and the second die as B. The probability of each of the outcome is $\frac{1}{36}$. Suppose we further observed that the first die lands on 5. Then, given this information, what is the probability that the sum of the two dice equals 10?. To calculate this probability, there are only 6 possible outcomes if the first die must land on 5. These are; (5, 1),

$$P(X \cap Y) = \frac{1}{6}$$

$$P(Y|X) = \frac{P(X \text{ and } Y)}{P(X)}$$

$$= \frac{P(X \cap Y)}{P(X)}$$

$$= \frac{1/6}{3/6}$$

$$= \frac{1}{6} \div \frac{3}{6}$$

$$= \frac{1}{6} \times \frac{6}{3}$$

$$= \frac{1}{3} = 0.3333$$

Example

In an electrical and electronic company, a box contains 5 defective (non-acceptable), 25 non-defective (acceptable) and 10 partially defective (fails after hours of use) transistors for use. Suppose a transistor is selected for use, what is the probability that it is acceptable?

Solution

The total number of transistors equals 40.
Since the selected transistor is acceptable, then it must be from among the non-defective transistors. Therefore;

$$P(acceptable|non-defective) = \frac{P(non-defective \text{ and } acceptable)}{P(non-defective)}$$

$$= \frac{25/40}{35/40}$$

$$= \frac{25}{35}$$

$$= 0.7143$$

Example

A recent questions was asked 100 citizens people if they thought women should vie for the position of president in a country. The results of the survey are shown.

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

Determine the following probabilities
a. The respondent answered yes, given that the respondent was a female.
b. The respondent was a male, given that the respondent answered no

Solution

Let M = respondent was a male,
F = respondent was a female
Y = respondent answered yes
N = respondent answered no
Therefore,

$$P(M) = \frac{50}{100}, \quad P(F) = \frac{50}{100}, \quad P(Y) = \frac{40}{100}, \quad P(N) = \frac{60}{100}$$

$$P(F \cap Y) = \frac{8}{100}, \quad P(N \cap M) = \frac{18}{100}$$

Therefore,

a.

$$P(Y|F) = \frac{P(F \cap Y)}{P(F)}$$

$$= \frac{8/100}{50/100}$$

$$= \frac{8}{100} \times \frac{100}{50}$$

$$= \frac{8}{50} = 0.16$$

b.

$$P(M|N) = \frac{P(N \cap M)}{P(N)}$$

$$= \frac{18/100}{60/100}$$

(5, 2), (5, 3), (5, 4), (5, 5) and (5, 6). Therefore, the only outcome that will have its sum equal to 10 if the first die lands on 5 is (5, 5). Note that the condition is that the first die lands on 5. The conditional probability is therefore $\frac{1}{6}$.

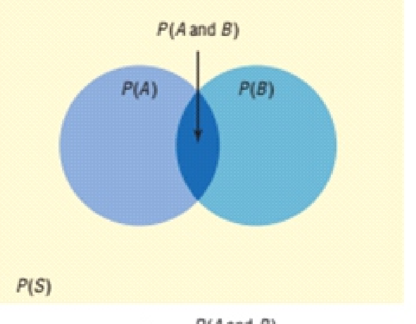
If we let B and A denote the event that the sum of the dice is 10 and the event that the first die lands on 5 respectively, then the probability obtained above is called the conditional probability of event B given that A has occurred.

Formula for Conditional Probability

The probability that the second event ***B*** occurs given that the first event ***A*** has occurred can be found by dividing the probability that both events occurred by the probability that the first event has occurred. That is

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Note that the above formula for conditional probability will be well define (i.e valid) only if $P(A) > 0$. Otherwise, $P(B|A)$ will be undefined.



$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$

Venn Diagram for Conditional Probability

Example A bag contains blue and red balls. Two balls are selected without replacement. If the chance of selecting a blue *and* a red ball is $\frac{5}{36}$ and the probability of selecting a blue ball on the first draw is $\frac{3}{5}$ then find the probability of selecting a red ball on the second draw, given that the first ball selected was a blue ball.

Solution Let B = selecting a blue ball
R = selecting a red ball
Then,

$$P(R|B) = \frac{P(R \text{ and } B)}{P(R)}$$

$$= \frac{5/36}{3/5}$$

$$= \frac{5}{36} \div \frac{3}{5}$$

$$= \frac{5}{36} \times \frac{5}{3}$$

$$= \frac{25}{108}$$

$$= 0.2315$$

Example A single fair die is rolled once. Let X be the event that outcome is an odd number and Y be the event that the outcome is 5. Determine the probability that the outcome is a 5 given that the value is an odd number.

Solution X = event that outcome is an odd number,
Y = event that the outcome is 5
Sample space S = {1, 2, 3, 4, 5, 6}

$$X = \{1, 3, 5\}$$

$$Y = \{5\}$$

$$(X \cap Y) = \{5\}$$

$$P(X) = \frac{3}{6}$$

$$P(Y) = \frac{1}{6}$$

$$= \frac{18}{100} \times \frac{100}{60}$$

$$= \frac{18}{60} = 0.30$$

Summary

In this lesson, conditional probability of event has been explained.

- The probability that the second event B occurs given that the first event A has occurred can be defined as

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

If and only if $P(A) > 0$.

Self Assessment Questions

1. The probability that Sam parks in a no-parking zone and gets a parking ticket is 0.06, and the probability that Sam cannot find a legal parking space and has to park in the noparking zone is 0.20. On Tuesday, Sam arrives at school and has to park in a no-parking zone. Find the probability that he will get a parking ticket.
2. The Gift Basket Store had the following premade gift baskets containing the following combinations in stock.

	Cookies	Mugs	Candy
Coffee	20	13	10
Tea	12	10	12

Choose 1 basket at random. Find the probability that it contains

a. Tea given that it contains mugs
b. Mugs given that it contains coffee

Tutor Marked Assignment

1. In addition to being grouped into four types, human blood is grouped by its Rhesus (Rh) factor. Consider the figures below which show the distributions of these groups

	O	A	B	AB
Rh+	37%	34%	10%	4%
Rh−	6%	6%	2%	1%

Choose 1 person at random. Find the probability that the person

a. Has type O blood given that the person is *Rh +*
b. Has *Rh -* given to the person as type B
c. Has *Rh +* given that the person has type A
d. Has type AB blood given that the person is *Rh -*

1. The medal distribution from the 2008 Summer Olympic Games for the top 23 countries is shown below

	Gold	Silver	Bronze
United States	36	38	36
Russia	23	21	28
China	51	21	28
Great Britain	19	13	15
Others	173	209	246

Choose 1 medal winner at random.

a. Find the probability that the winner won the gold medal, given that the winner was from the United States.
b. Find the probability that the winner was from the United States, given that she or he won a gold medal.

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UNIT 5 THE BAYES' THEOREM

Introduction

Suppose a state in your country is partition into three senatorial district as North (N), South (S) and Central (C) (i.e. the state is partitioned into three disjoint sets N, S and C). Suppose we are interested in the total size of the forest area in the state and it is known that the forest area in N, S and C are $200km^2$, $120km^2$ and $180km^2$ respectively. Therefore, the total size of the forest area in the state would be;

$$200km^2 + 120km^2 + 180km^2 = 500km^2$$

That is, one can simple sum the size of the forest areas in each senatorial district to obtain the forest area in the whole state.

This is an idea behind the total probability rule which is a very important concept in the discussion of Bayes' theorem.

Learning Outcomes

At the end of this unit, you should be able to:

1. State the Total probability rule
2. State the Bayes' theorem
3. Compute the total probability of an event
4. Compute the probability of an event using Bayes' theorem

Main Content

THE BAYES' THEOREM

Firstly, consider the concept of exhaustive, mutually exclusive and complement of an event. The events $E_1, E_2, \ldots, E_n$ are said to be exhaustive if they cover all possible outcomes of an experiment. That is

$$E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S$$

Also, when if events $E_1, E_2, \ldots, E_n$ are mutually exclusive (i.e. they cannot occur at the same time or do not have anything in common), then

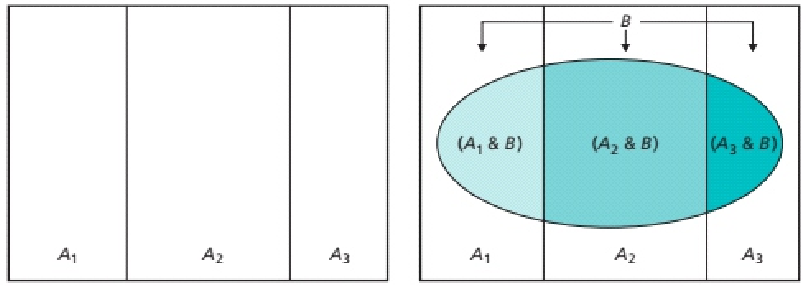
$$P(E_1 \cup E_2 \cup E_3 \cup, \ldots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n)$$

The complement of an event E is the set of outcomes in the sample space that are not included in the outcomes of event E. The complement of E is denoted as E' or E^c.

As an illustration, suppose senior secondary school classes are classified into Science, Commercial and Art classes in a particular school. Suppose that a student is chosen at random let E_1, E_2 and E_3 denote the events that the student chosen is a Science student, Commercial student and Art student respectively. Then the events E_1, E_2 and E_3 are exhaustive because they cover all possible outcomes.

Similarly, we say E_1, E_2 and E_3 are also mutually exclusive. This is because a selected student cannot belong to more than one of the classes at the same time (i.e. can only belong to one class at a time). Therefore, if events are both exhaustive and mutually exclusive, exactly one of them must occur.

Also note that an event and its complement are always exhaustive and mutually exclusive.



(a) (b)

Consider the figure above that portrays three events $A_1, A_2, and\ A_3$ In (a), the events are mutually exclusive and exhaustive. The three events does not overlap which indicates they are mutually exclusive. Also, the three events fill out the entire region enclosed by a rectangle which indicates exhaustive.

Now consider (b) comprising mutually exclusive and exhaustive events $A_1, A_2, and\ A_3$ and another event B. The event B comprises of mutually exclusive and exhaustive events $(A_1 \& B), (A_2 \& B),\ and\ (A_3 \& B)$ shown in colour. This implies that event B can only occur in conjunction with exactly one of the events $A_1, A_2, and\ A_3$. That is

$$B = (A_1 \& B) \cup (A_2 \& B) \cup (A_3 \& B)$$

Therefore,

$$P(B) = P(A_1 \& B) \cup P(A_2 \& B) \cup P(A_3 \& B)$$

$$= P(A_1 \& B) + P(A_2 \& B) + P(A_3 \& B)$$

$$= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

Applying the general multiplication rule, we have

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)$$

The formula above is called the **law of total probability.**

Law of total probability

2mins

SAQ 1

Given $A_1, A_2, A_3, \dots, A_n$ mutually exclusive and exhaustive events; that is, exactly one of the events must occur. For any other event B, the law of total probability states that;

$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)$$

i.e

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \cdots + P(A_n) \cdot P(B|A_n)$$

Example

Suppose there are three bags with 100 balls each as given below

- Bag 1 (B_1) contains 75 red and 25 black balls
- Bag 2 (B_2) contains 60 red and 40 black balls
- Bag 3 (B_3) contains 45 red and 55 black balls

If one bag is selected at random and a ball is also picked at random from the selected bag, what is the probability that the chosen ball is red?

Solution

Let R be the event that a red ball is selected and let (B_i) be the event that bag i is selected. Probability of selecting any bag is

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

Probability of selecting a red ball from B_1 is

$$P(R|B_1) = \frac{75}{100}$$

$$= 0.75$$

Probability of selecting a red ball from B_2 is

$$P(R|B_2) = \frac{60}{100}$$

$$= 0.60$$

Probability of selecting a red ball from B_3 is

$$P(R|B_3) = \frac{45}{100}$$

$$= 0.45$$

Therefore,

$$P(R) = P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2) + P(B_3) \cdot P(R|B_3)$$

$$= \frac{1}{3}(0.75) + \frac{1}{3}(0.60) + \frac{1}{3}(0.45)$$

$$= 0.60$$

Bayes' Theorem

4mins

SAQ 2

The Bayes' theorem was developed by Rev. Thomas Bayes (1702 – 1761).

Similar to the law of total probability, consider three events A_1, A_2, and A_3 that are mutually exclusive and exhaustive and let B be any other event. For Bayes' theorem, it assumes that the probabilities $P(A_1), P(A_2), P(A_3), P(B|A_1), P\ (B|A_2), and\ \ P(B|A_3)$ are known. The problem is to use the six probabilities to determine the conditional probabilities $P(B|A_1), P\ (B|A_2),\ and\ \ P(B|A_3)$.

Recall from conditional probability,

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)}$$

Also, recall from multiplication rule,

$$P(A_1 \cap B) = P(A_1) \cdot P(B|A_1)$$

Similarly, from law of total probability,

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)$$

therefore,

$$P(A_1|B) = \frac{P(A_1) \cdot P(B|A_1)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)}$$

Similarly,

$$P(A_2|B) = \frac{P(A_2) \cdot P(B|A_2)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)}$$

And

$$P(A_3|B) = \frac{P(A_3) \cdot P(B|A_3)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)}$$

The expressions above for $P(A_1|B)$, $P(A_2|B)$ and $P(A_3|B)$ is called **Bayes' theorem.**

In general, Bayes' theorem states that;
Suppose there are $A_1, A_2, A_3, \dots, A_n$ mutually exclusive and exhaustive events, then for any other event B

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \cdots + P(A_n) \cdot P(B|A_n)}$$

$$= \frac{P(A_i) \cdot P(B|A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)}$$

SAQ 3

SAQ 4

Example Suppose it was observed from the last example that a chosen ball is red. What is the probability of chosen bag 1?

Solution Recall that we already know $P(R|B_i)$, that is, the probability of chosen a red ball from any of the bags. But the question here is interested in $P(B_1|R)$. This is a situation where the Bayes' theorem can be used.

$$P(B_1|R) = \frac{P(B_1) \cdot P(R|B_1)}{P(R)}$$

Recall from last example,

$$P(B_1) = \frac{1}{3}$$

Probability of selecting a red ball from B_1 is

$$P(R|B_1) = \frac{75}{100}$$

$$= 0.75$$

and

$$P(R) = P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2) + P(B_3) \cdot P(R|B_3)$$

$$= \frac{1}{3}(0.75) + \frac{1}{3}(0.60) + \frac{1}{3}(0.45)$$

$$= 0.60$$

Therefore

$$P(B_1|R) = \frac{P(B_1) \cdot P(R|B_1)}{P(R)}$$

$$= \frac{\frac{1}{3}(0.75)}{0.60}$$

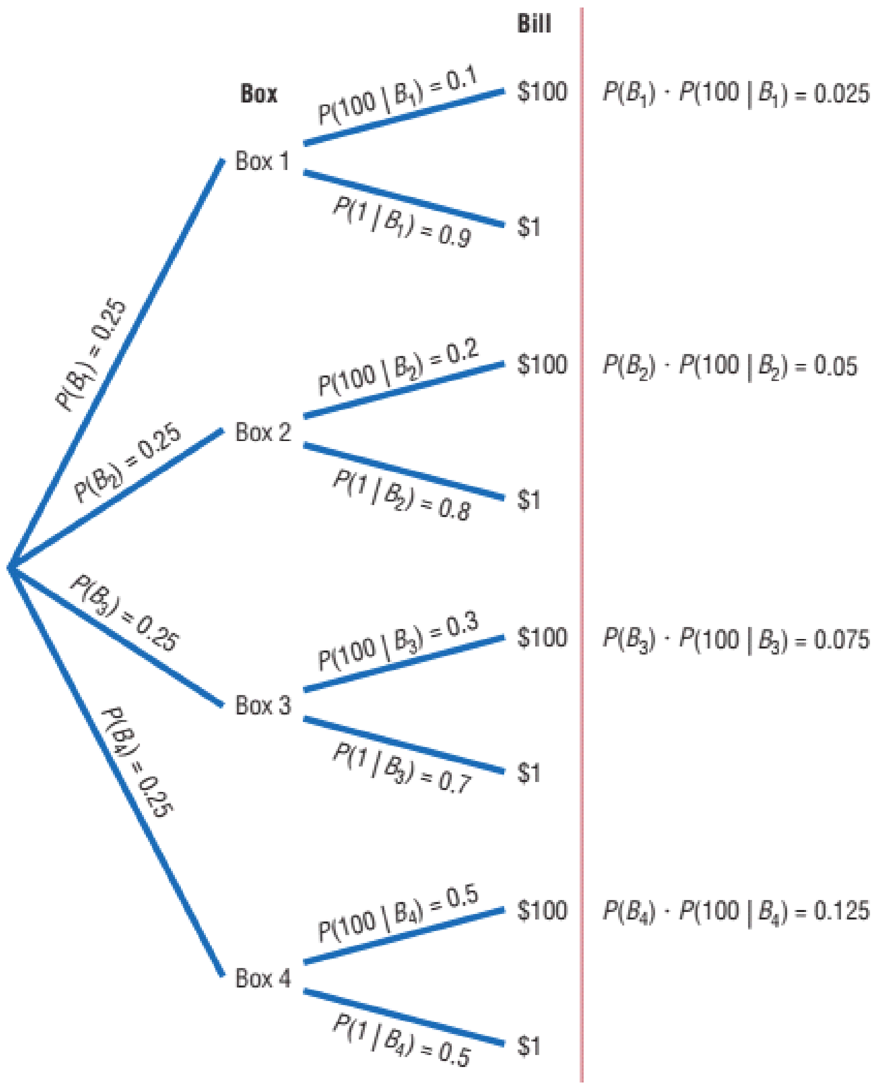
$$= 0.416$$

Example There are four boxes in a competition as follows

- Box 1 (B_1) contains one \$100 bill and nine \$1 bills.
- Box 2 (B_2) contains two \$100 bills and eight \$1 bills.
- Box 3 (B_3) contains three \$100 bills and seven \$1 bills.
- Box 4 (B_4) contains five \$100 bills and five \$1 bills.

A contestant selects a box at random and selects a bill from the box at random. If a \$100 bill is selected, find the probability that it came from box 4.

Solution A tree diagram can also be use to find the corresponding probabilities as follows



Therefore

$$P(B_4|100) = \frac{P(B_4)\cdot P(100|B_4)}{\sum_{i=1}^{4} P(B_i)\cdot P(100|B_i)}$$

$$= \frac{0.125}{0.025 + 0.05 + 0.075 + 0.125}$$

$$= \frac{0.125}{0.275}$$

$$= 0.455$$

Summary

In this Unit, the Bayes' theorem has been explained.

- The Bayes' theorem is applicable for mutually exclusive and exhaustive events.
- Bayes' theorem can be used to revise probabilities of events once additional information becomes known.
- Bayes' theorem is used as the basis for a branch of statistics called Bayesian decision making, which includes the use of subjective probabilities in making statistical inferences.
- Given $A_1, A_2, A_3, \dots, A_n$ mutually exclusive and exhaustive events; that is, exactly one of the events must occur. For any other event B, the law of total probability states that;

$$P(B) = \sum_{i=1}^{n} P(A_i)\cdot P(B|A_i)$$

- Bayes' theorem states that;

Suppose there are $A_1, A_2, A_3, \dots, A_n$ mutually exclusive and exhaustive events, then for any other event B.

$$P(A_i|B) = \frac{P(A_i)\cdot P(B|A_i)}{\sum_{i=1}^{n} P(A_i)\cdot P(B|A_i)}$$

Self Assessment Questions

1. State the law of total probability
2. State the Bayes' theorem
3. Suppose there are three bags with 100 balls each as given below
   - Bag 1 (B_1) contains 75 red and 25 black balls
   - Bag 2 (B_2) contains 60 red and 40 black balls
   - Bag 3 (B_3) contains 45 red and 55 black balls

   If one bag is selected at random and a ball is also picked at random from the selected bag, what is the probability that the chosen ball is black?
4. Suppose it was observed question (iii) above that a chosen ball is black. What is the probability of chosen bag 3?

Tutor Marked Assignment

i.) An appliance store purchases electric ranges from two companies. From company A, 500 ranges are purchased and 2% are defective. From company B, 850 ranges are purchased and 2% are defective. Given that a range is defective, find the probability that it came from company B.

ii.) A store purchases baseball hats from three different manufacturers. In manufacturer A's box, there are 12 blue hats, 6 red

hats, and 6 green hats. In manufacturer B's box, there are 10 blue hats, 10 red hats, and 4 green hats. In manufacturer C's box, there are 8 blue hats, 8 red hats, and 8 green hats. A box is selected at random, and a hat is selected at
random from that box. If the hat is red, find the probability that it came from manufacturer A's box.

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UNIT 6 INTRODUCTION TO PROBABILITY DISTRIBUTION

Introduction

Probabilities are usually assigned to possible outcomes in many real-life situations for the purpose of evaluating the results for decision making. For example, you may want to calculate the probability of making/receiving 0, 1, 2, 3, or more phone calls in a single day. Once these probabilities are assigned, other measures such as mean, variance and standard deviations can be obtained for the events.

Learning Outcomes

At the end of this unit, you should be able to:

1. Construct a probability distribution for a random variable
2. Distinguish between a discrete variable and a continuous variable
3. Find probability of events using binomial probability distribution

Main Content

Types variables

3mins

SAQ 1

Suppose three coins you tossed, the sample space is represented as;

TTT, TTH, THT, HTT, HHT, HTH, THH, HHH

If X denotes the number of heads in the sample space, then X can assumes the value 0, 1, 2 or 3 i.e. when there are no heads ($X = 0$), one head ($X = 1$), two heads ($X = 2$) and three heads ($X = 3$). X is therefore referred to as a random variable.

Definitions

- A variable is define as a characteristics or attributes that can assume different values. Example include, age, weight, height, colour, gender, marital status etc. Variables can be classified as discrete or continuous.
- Discrete variables are variables that can assume specific values. They have finite number of possible values of infinite number of values that can be counted by enumeration using numbers 1, 2, 3 and so on. Examples are; number of phone calls received in a single day, gender of babies born alive, etc.
- Continuous variables are variables that can assume all values in the interval between any two given values. They are obtained from from data that can be measured rather than counted and can assume fractional or decimal values. Examples are age, shoe size in cm, weight

in kg etc.

- A random variable is a variable with quantitative outcomes whose values depend of chance. For example, the chance or probability of no heads ($X = 0$) in the example above is $\frac{1}{8}$. Random variables that assume discrete values are called discrete random variables while random variables that assume continuous values are called continuous random variables.

In the example of tossing a three coins above, probabilities for the values of (X = 0, 1, 2, 3) can be determine as follows;

No heads	One head	Two heads	Three heads
TTT	TTH THT HTT	HHT HTH THH	HHH
$\underbrace{\tfrac{1}{8}}_{\frac{1}{8}}$	$\underbrace{\tfrac{1}{8} \quad \tfrac{1}{8} \quad \tfrac{1}{8}}_{\frac{3}{8}}$	$\underbrace{\tfrac{1}{8} \quad \tfrac{1}{8} \quad \tfrac{1}{8}}_{\frac{3}{8}}$	$\underbrace{\tfrac{1}{8}}_{\frac{1}{8}}$

From these values, a probability distribution can be constructed by listing the outcomes and assigning the probability of each outcome, as shown below.

Number of heads, X	0	1	2	3
Probability $P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example Construct a probability distribution for a single roll of a six-sided die.

Solution Since the sample space is 1, 2, 3, 4, 5, 6 and each outcome will have a probability of $\frac{1}{6}$, therefore, probability distribution is

Number of heads, X	1	2	3	4	5	6
Probability $P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Example Using the data below, construct a probability distribution for the data.

Y	Frequency, f
4	8
5	7
6	9
7	16
	Total = 40

Solution

Y	4	5	6	7
Probability $P(Y)$	$\frac{8}{40}$	$\frac{7}{40}$	$\frac{9}{40}$	$\frac{7}{40}$

Generally, there are two requirements for a probability distribution which are:

1. The sum of all the probabilities of events in the sample space must be equal to one. That is $\sum P(X) = 1$.
2. The probability of each event in the sample space must be between or equal to 0 and 1. That is $0 \leq P(X) \leq 1$.

As an example, determine whether or not the following is a probability distribution.

a.

X	4	6	8	10
$P(X)$	−0.6	0.2	0.7	1.5

b.

X	1	2	3	4
$P(X)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

c.

X	8	9	12
$P(X)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

d.

X	1	3	5	7	9
$P(X)$	0.3	0.1	0.2	0.4	−0.7

a. No, because probability cannot be negative (P(X) = -0.6) or greater than 1 P(X) = -1.5
b. Yes
c. Yes
d. No, because P(X) = -0.7

The Binomial distribution

3mins

SAQ 2

SAQ 3

Many applications of probability and statistics concern the repetition of an experiment and Many types of probability problems have only two outcomes or can be reduced to two outcomes. Each repetition in an experiment is called a trial, and we are particularly interested in cases in which the experiment (each trial) has only two possible outcomes. For example,

- When a coin is tossed, it can land heads or tails.
- When a baby is born, it will be either male or female.
- In a football game, a team either wins or loses.
- A true/false item can be answered in only two ways, true or false.

Other situations can be reduced to two outcomes. For example,

- a medical treatment can be classified as effective or ineffective, depending on the results.
- a person can be classified as having normal or abnormal blood pressure, depending on the measure of the blood

pressure gauge.

- a multiple-choice question, even though there are four or five answer choices, can be classified as correct or incorrect.

Repeated trials of an experiment are called **Bernoulli trials**. Situations like these are called ***binomial experiments.***

In a binomial experiment, the following four conditions must be satisfied;

1. There must be a fixed number of trials.
2. Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as success or failure.
3. The outcome in each trial must be independent of one another.
4. The probability of success must remain the same for each trial. A binomial experiment and its results give rise to a special probability distribution called the binomial distribution. The probability of a success in a binomial experiment can be computed as follows:

Let X be the number of successes in n trials. The probability of X is obtained as

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Where P(X = x) implies the probability that the number of successes X is equal to a value x and X = 0, 1, 2, 3, ... , n.

$$\binom{n}{x} = \frac{n!}{(n-x)!\,x!}$$

n is the number of trials
p is the probability of success in a single trial
q = 1 - p is the probability of failure in a single trial

Example A coin is tossed 3 times. Find the probability of getting exactly two heads.

Solution

$$n = 3 \text{ (fixed)}, \quad p = 0.5, \quad q = 1 - 0.5, \quad x = 2$$

$\therefore$

$$P(X = 2) = \binom{3}{2} 0.5^2 (1 - 0.5)^{3-2}$$

$$= \frac{3!}{(3-2)!\,2!} \times 0.5^2 \times 0.5^1$$

$$= 0.375$$

Recall from the first example in this unit, probability of $X = 2, P(X) = \frac{3}{8}$. The result obtained when binomial distribution was used is the same when the sample space of the experiment was used.

Example A recent study found that one out of five lecturers in UNILORIN say he or she has visited a doctor in any given month. If 10 people are selected at random, find the probability that exactly 3 will have visited a doctor last month.

Solution In this example,

$$n = 10, \quad p = \frac{1}{5}, \quad q = 1 - \frac{1}{5} = \frac{4}{5}, \quad x = 3$$

$$P(X = 3) = \binom{10}{3} \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right)^{10-3}$$

$$= \frac{10!}{(10-3)!\,3!} \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right)^7$$

$$= 0.201$$

Example A survey at university of Ilorin found that 30% of social science students receive part of their spending money from part-time jobs. If 5 students are selected at random, find the probability that at least 3 of them will have part-time jobs.

Solution

$$n = 5, \quad p = 30\% = 0.3, \quad q = 0.7$$

To find the probability that at least 3 have part-time jobs, this implies 3 or more jobs. That is $P(X \geq 3)$, *where* $X = 3$ *or* 4 *or* 5. Therefore,

$$P(X \geq 3) = P(X = 3) \cup P(X = 4) \cup P(X = 5)$$
$$= P(X = 3) + P(X = 4) + P(X = 5)$$

$$P(X = 3) = \frac{5!}{(5-3)!\,3!} \times (0.3)^3 \times (0.7)^2 = 0.132$$

$$P(X = 4) = \frac{5!}{(5-4)!\,4!} \times (0.3)^4 \times (0.7)^1 = 0.028$$

$$P(X = 5) = \frac{5!}{(5-5)!\,5!} \times (0.3)^5 \times (0.7)^0 = 0.002$$

$\therefore$

$$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$$

$$= 0.132 + 0.028 + 0.002$$

$$= 0.162$$

Summary

In this Unit, the probability distribution has been introduced.

- A variable is define as a characteristics or attributes that can assume different values.
- Discrete variables are variables that can assume specific values. They have finite number of possible values of infinite number of values that can be counted by enumeration using numbers 1, 2, 3 and so on.
- Continuous variables are variables that can assume all values in the interval between any two given values.
- A random variable is a variable with quantitative outcomes whose values depend of chance.
- Generally, there are two requirements for a probability distribution which are:
  1. The sum of all the probabilities of events in the sample space must be equal to one. That is $\sum P(X) = 1$.
  2. The probability of each event in the sample space must be between or equal to 0 and 1. That $0 \leq P(X) \leq 1$.
- A binomial experiment and its results give rise to a special probability distribution called the binomial distribution.
- In a binomial experiment, the following four conditions must be satisfied;
  1. There must be a fixed number of trials.
  2. Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as success or failure.
  3. The outcome in each trial must be independent of one another.
  4. The probability of success must remain the same for each trial.
- The probability of success for binomial distribution is

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Self Assessment Questions

1. State whether the variable is discrete or continuous.
   a. The speed of a jet airplane
   b. The number of burgers a fast-food restaurant serves each day
   c. The number of people who play the lottery each day
   d. The weight of an automobile.
   e. The time it takes to have a medical physical exam.
   f. The number of mathematics majors in your school

   g. The blood pressures of all patients admitted to a hospital on a specific day
2. Construct a probability distribution for a family with 4 children. Let X be the number of girls.
3. A report say that 10% of adults in Nigeria are afraid of being alone in a house at night. If a random sample of 20 adults in Nigeria is selected, find these probabilities .
   a. There are exactly 5 people in the sample who are afraid of being alone at night.
   b. There are at most 3 people in the sample who are afraid of being alone at night.
   c. There are at least 3 people in the sample who are afraid of being alone at night.

Tutor Marked Assignment

1. Determine whether the distribution represents a probability distribution. If it does not, state why.

a.

X	3	7	9	12	14
$P(X)$	$\frac{4}{13}$	$\frac{1}{13}$	$\frac{3}{13}$	$\frac{1}{13}$	$\frac{2}{13}$

b.

X	3	6	8	12
$P(X)$	0.3	0.5	0.7	−0.8

c.

X	5	7	9
$P(X)$	0.6	0.8	−0.4

d.

X	1	2	3	4	5
$P(X)$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$

2. Using the sample space for tossing two dice, construct a probability distribution for the sums 2 through 12.
3. A student takes a 20-question, true/false exam and guesses on each question. Find the probability of passing if the lowest passing grade is 15 correct out of 20.
4. Which of the following are binomial experiments or can be reduced to binomial experiments?
   a. Surveying 100 people to determine if they like liquid Soap
   b. Asking 1000 people which brand of cigarettes they smoke
   c. Testing four different brands of aspirin to see which brands are effective
   d. Asking 100 people if they smoke
   e. Surveying 300 prisoners to see whether this is their first offense

References

- Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.
- Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.
- Introductory Statistics, 9th edition by Neil A. Weiss. Published by Library of Congress Cataloging-in-Publication Data, Boston. Page 144 – 157.

Further Reading

- https://www.probabilitycourse.com/chapter1/1_3_1_random_experiments.php
- http://ptrckprry.com/course/langone/lecture/prob-sol.pdf
- http://www.stat.ucdavis.edu/~ntyang/teaching/12SSII/lecture03.pdf
- http://mathandmultimedia.com/2012/08/08/understanding-sample-space-and-sample-points/
- http://www.cse.unsw.edu.au/~cs9417ml/Bayes/Pages/Joint_Probability.html
- https://machinelearningmastery.com/joint-marginal-and-conditional-probability-for-machine-learning/

Source: pexels.com

MODULE 3

Combinational Analysis

Unit 1: Tree diagram and Multiplication principles of counting
Unit 2: Permutations of Objects
Unit 3: Combination of Objects
Unit 4: Probability and Counting Technique

UNIT 1 TREE DIAGRAM AND MULTIPLICATION PRINCIPLES OF COUNTING

Introduction

Many at times we often need to determine the number of ways something can happen – the number of possible outcomes for an experiment, the number of ways an event can occur, the number of ways a certain task can be performed, and so on, For this purpose, we need to use counting techniques.

Learning Outcomes

At the end of this unit, you should be able to:

1. Understand and be able to apply the tree diagram principle
2. Understand and be able to apply the multiplication principle

Main Content

Many problems in probability theory require that we count the number of ways that a particular event can occur or the number of outcomes in the sample space. For this purpose, we need to use counting techniques. Such techniques are called counting rules. Some of these techniques are the tree diagram and the multiplication rule of counting.

THE TREE DIAGRAM PRINCIPLE

2mins

SAQ 1

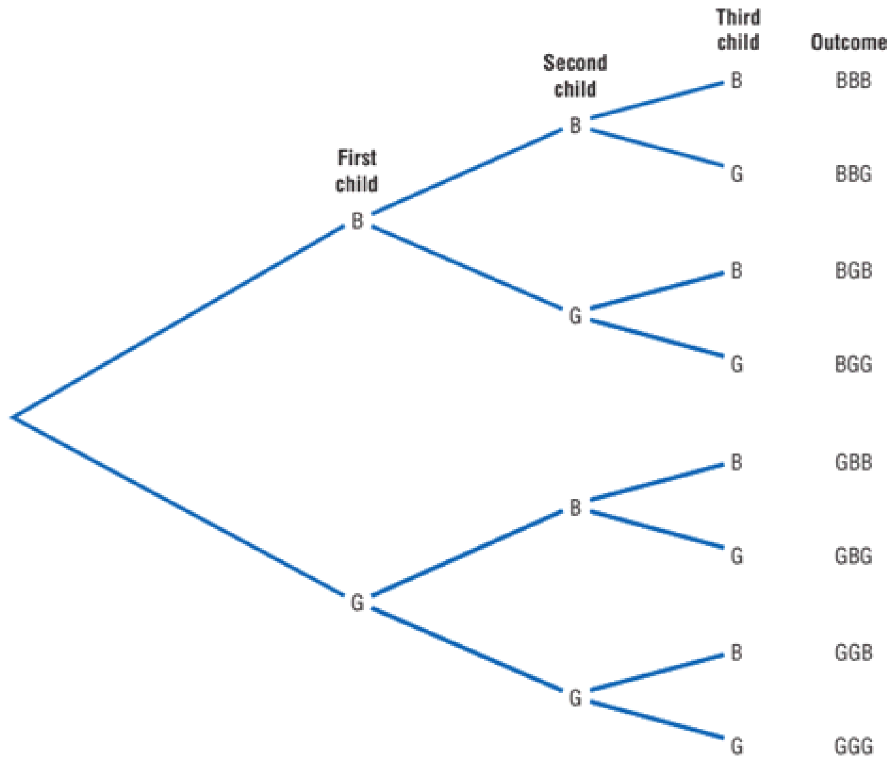
SAQ 2

A tree diagram is a device consisting of line segments emanating from a starting point and also from the outcome point. It is used to determine all possible outcomes of a probability experiment. Tree diagram uses the following principle

- The total number of branches equals the total number of outcomes.
- Each unique outcome is represented by following a branch from start to finish.
- To obtain the probability of the events, we can multiply the probabilities as we work down a particular branch.

Example Determine all possible outcomes for the gender of the children if a family has three children. Use B for boy and G for girl.

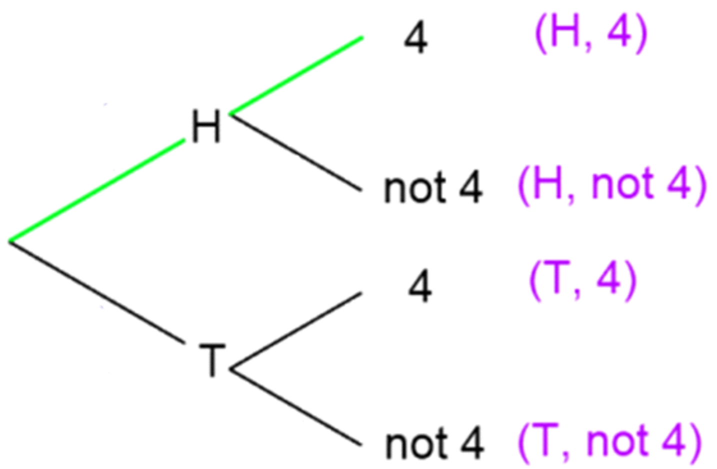
Solution



All possible outcomes are therefore; BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG.
Therefore, the total number of all possible outcome is 8

Example List all possible outcome when a coin is throw and obtaining a 4 in a single roll of a die.

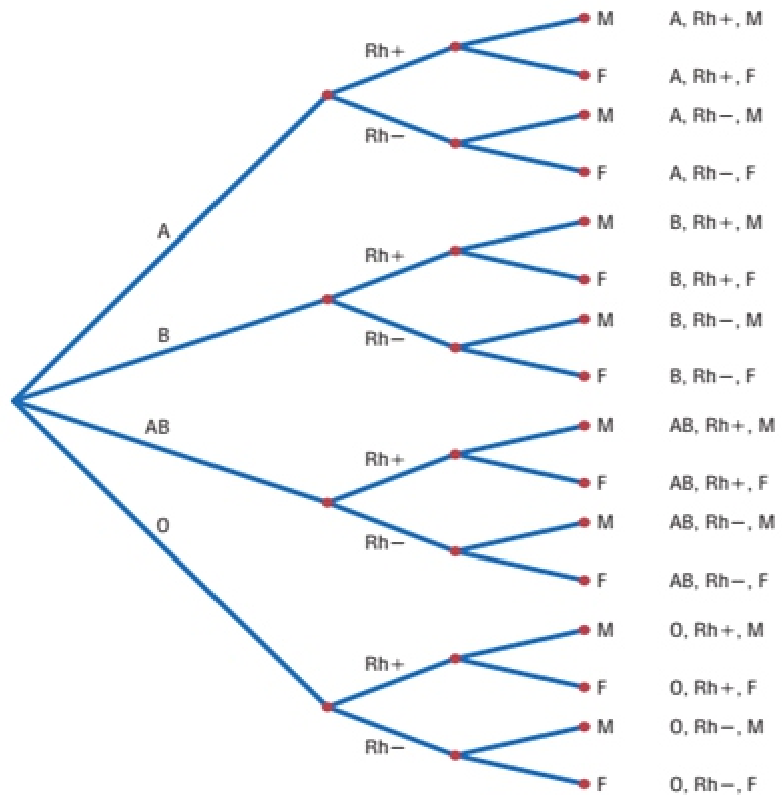
Solution



All possible outcome is (H, 4), (H, not 4), (T, 4), (T, not 4). The total number of outcome is 4.

Example There are four blood types, A, B, AB, and O. Blood can also be Rh and Rh. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labelled?

Solution



Therefore, there are 16 possible way a donor can have his or her blood labelled.

The multiplication principle

3mins

SAQ 3

Sometimes, we can list the possibilities and count them, however, doing so is impractical. Therefore we need to develop techniques that do not rely on a direct listing for determining the number of ways something can happen. The multiplication principle works as follows;

In a sequence of different events $E_1, E_2, \ldots, E_k$, in which E_1 has n_1 possible outcomes, E_2 has n_2 possible outcomes and so forth. The total possible outcomes for $E_1, E_2, \ldots, E_n$ is $n_1 \times n_2 \times \ldots \times n_k$. Where $n_1, n_2, \ldots, n_k$ are possible outcomes for events $E_1, E_2, \ldots, E_k$, respectively.

Example The manager of a department store chain wishes to make four-digit identification cards for her employees. How many different cards can be made if she uses the digits 1, 2, 3, 4, 5, and 6 and repetitions are permitted?

Solution Since there are 4 spaces to fill on each card and there are 6 choices for each space, the total number of cards that can be made is $6 \times 6 \times 6 \times 6 = 1296$

Suppose repetitions are not permitted in the example above, therefore, the first digits to be selected can be chosen in 6 ways, the second digits will have 5 ways since there are only five digits left. Similarly, the third and forth digits will have 4 and 3 ways respectively. Hence, the solution is

$$6 \times 5 \times 4 \times 3 = 360 \ ways$$

Example Until a few years ago, a three-digit area code was designed as follows:

- The first could be any digit from 2 through 9.
- The second digits could be only a 0 or 1
- The last could be any digit.

How many different such area codes were possible?

Solution
- The first digit will have 8 ways of selection
- The second digit will have 2 ways
- The last digit will have 10 ways

Therefore, the total number of different area codes is

$$8 \times 2 \times 10 = 160$$

Example How many multiples of 5 are there from 10 to 95?

Solution Note that multiples of 5 are integers that have 0 or 5 as the last digit to the extreme right. Therefore,
The last digit can be chosen in 2 ways (0 or 5).
The first digit can be any of 1, 2, 3, 4, 5, 6, 7, 8, 9
The total multiples of 5 from 10 to 95 are $2 \times 9 = 18$

Example In a city, the bus route numbers consist of natural (whole) number less than 100. These numbers are usually followed by one of the first six alphabets. How many different bus routes are possible?

Solution Natural whole numbers less than 100 are from 1 to 99. Therefore, the number can be chosen in 99 ways.
The first six alphabets are A, B, C, D, E and F. Also, the alphabet can be chosen in 6 ways.
The total number of possible routes are $99 \times 6 = 594$

Example How many 3-digit number can be formed from 1, 3, 5, 7 and 9 if repetitions are not allowed?

Solution The three digits will have hundred's, ten's and unit's place. Out of the five numbers given, the hundred's place can be filled with any of the given numbers in 5 ways. The ten's place can be filled in 4 ways from the remaining numbers after the hundred's place has been filled, and finally, the unit's place can be chosen from the remaining in 3 ways from the three numbers.
Therefore, the number of 3-digit numbers that can be formed are $5 \times 4 \times 3 = 60$

Summary

In this Unit, Tree diagram and Multiplication principles of counting has been introduced.

- A tree diagram is a device consisting of line segments emanating from a starting point and also from the outcome point and it uses the following principle
  - -The total number of branches equals the total number of outcomes.
  - -Each unique outcome is represented by following a branch from start to finish.
  - -To obtain the probability of the events, we can multiply the probabilities as we work down a particular branch.
- The multiplication principle works as follows;

In a sequence of different events $E_1, E_2, \ldots, E_k$, in which E_1 has n_1 possible outcomes, E_2 has n_2 possible outcomes and so forth. The total possible outcomes for $E_1, E_2, \ldots, E_n$ is $n_1 \times n_2 \times \ldots \times n_k$. Where $n_1, n_2, \ldots, n_k$ are possible outcomes for events $E_1, E_2, \ldots, E_k$, respectively.

Self Assessment Questions

1. Determine all possible outcomes for the gender of the children if a family has five children. Use M for male and F for female.
2. Mr. Ajayi has in his closet 3 shirts, 2 pants and 2 pair of shoes. Using the tree diagram, list the different ways can he dress every morning?
3. How many 5-digit zip codes are possible if digits can be repeated? If there cannot be repetitions?

Tutor Marked Assignment

1. With the use of a tree diagram, determine all possible outcomes when four coins are tossed simultaneously.
2. An inspector must select 3 tests to perform in a certain order on a manufactured part. He has a choice of 7 tests. How many ways can he perform 3 different tests?
3. A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.
4. The license plates of a state consist of three letters followed by three digits. How many different license plates are possible?

References

• Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.

• Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.

• Introductory Statistics, 9th edition by Neil A. Weiss. Published by Library of Congress Cataloging-in-Publication Data, Boston. Page 144 – 157.

Further Reading

- https://www.webassign.net/userimages/Stat281%20Ch.%204.3%20Trees%20and%20Counting%20Techniques.pdf?id=203881&db=v4net
- http://www.mathspadilla.com/macsII/Unit8-Combinatorics/tree_diagrams.html
- https://nrich.maths.org/tree-diagram-intro
- https://online.stat.psu.edu/stat414/node/9/

Permutations

UNIT 2 PERMUTATIONS

Introduction

There are some problems in probability theory that will require you counting number of ways a particular event occurs which makes learning topics of permutations and combinations pertinent. Before we fully into permutations and combinations are discussed, it is essential to discuss counting techniques that will enable you to solve a variety of counting problems, including the problems of counting the number of possible permutations of '***n***' objects taking k or all at a time as well as problems that could arise in reality.

Learning Outcomes

At the end of this unit, you should be able to:

1. Use the permutation rule to count the number of possible arrangements of n objects taken k at a time.
2. Count the number of possible distinguishable permutations of n objects when the objects are of distinctive types.
3. Apply the techniques learned in this unit to relevant real life problems.

Main Content

PERMUTATIONS

3mins

SAQ 1

Suppose that in an arrangement of $\boldsymbol{k}$ objects selected from a set of $\boldsymbol{n}$ distinct objects in which the order of arrangements is important so that different orderings of the same set of $\boldsymbol{k}$ objects are regarded as **different**. In this case, we call such arrangements is called a permutation.

Definition

The permutation of $\boldsymbol{n}$ objects taking $\boldsymbol{k}$ at a time and denoted by ${}^{\mathbf{n}}\mathbf{P}_{\mathbf{k}}$ or $\mathbf{P(n, k)}$ is defined as

$$\boldsymbol{P(n,k)} = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{k})!}$$

Note that for any counting of

- $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$
- $0! = 1$
- $1! = 1$

Illustration

Suppose we want to determine how many possible distinct arrangements are there in the letters **A**, **B** and **C** taking all at a time.

By definition,

$$\boldsymbol{P(n,k)} = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{k})!}$$

Here, $\boldsymbol{n}$ = 3 and $\boldsymbol{k}$ = 3, then

$$P(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 6$$

The distinct arrangements of the letters **A**, **B** and **C** taking all at a time are **ABC**, **ACB**, **BAC**, **BCA**, **CAB** and **CBA**

Example How many distinct arrangements can be formed from digits **1**, **2**, **3** and **4** taking three digits at a time? Hence, list all the distinct arrangements.

Solution Here, $\boldsymbol{n}$ = 4 and $\boldsymbol{k}$ = 3, then $^{4}\mathbf{P}_3$ =

$$P(4,3) = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$$

The distinct arrangements of the digits **1**, **2**, **3**, and **4** taking three digits at a time are

123,	132,	213,	231,	312,	321	(suspending digit 4)
124,	142,	214,	241,	412,	421	(suspending digit 3)
134,	143,	314,	341,	413,	431	(suspending digit 2)
234,	243,	324,	342,	423,	432	(suspending digit 1)

Example How many 2-letter arrangements can be made from the letters in **ARMTI**? Hence, list all the arrangements.

Solution Here, $\boldsymbol{n}$ = 5 and $\boldsymbol{k}$ = 2, then $^{5}\mathbf{P}_2$ =

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20$$

The distinct 2-letter arrangements that can be made from the letters in **ARMTI** are

ARRA	AM	MA	AT	TA	AI	IA	RM	MR
RTTR	RI	IR	MT	TM	MI	IM	TI	IT

Example Compute the following

a. $^{6}\mathbf{P}_1$ b. $^{8}\mathbf{P}_3$ c. $^{11}\mathbf{P}_5$ d. $^{8}\mathbf{P}_0$ e. $^{5}\mathbf{P}_5$

Solution

a. $^{6}\mathbf{P}_1$ =

$$P(6,1) = \frac{6!}{(6-1)!} = \frac{6!}{5!} = 6$$

b. $^{8}\mathbf{P}_3$ =

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336$$

c. $^{11}\mathbf{P}_5$ =

$$P(11,5) = \frac{11!}{(11-5)!} = \frac{11!}{6!} = 55{,}440$$

d. $^{8}\mathbf{P}_0$ =

$$P(8,0) = \frac{8!}{(8-0)!} = \frac{8!}{8!} = 1$$

e. $^{5}\mathbf{P}_5$ =

$$P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 120$$

Example How many ways can **3** out of **7** different history textbooks be arranged on a shelf in the library?

Solution Here, $\boldsymbol{n}$ = 7 and $\boldsymbol{k}$ = 3, then $^{7}\mathbf{P}_3$ =

$$P(7,3) = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$$

Example How many different four-letter permutations can be formed from the letters of the word **DRAGON**?

Solution For this example, $\boldsymbol{n}$ = 6 and $\boldsymbol{k}$ = 4, then $^{6}\mathbf{P}_4$ =

$$P(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$$

Example An investigative agency has 8 cases and investigative 5 officers. How many different ways can the cases be assigned if only one case is assigned to each officer?

Solution In this example, $\boldsymbol{n}$ = 8 and $\boldsymbol{k}$ = 5, then $^{8}\mathbf{P}_5$ =

$$P(8,5) = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6{,}720$$

PERMUTATIONS OF *n* OBJECTS, NOT ALL DISTINCT

3mins

Given a set of n objects in which n_1 objects are identical and of one kind, n_2 objects are also identical and of another kind, . . . , n_k objects are as well identical and yet of another kind such that $n_1 + n_2 + n_3 + . . . + n_k = n$; then the number of permutations of these n objects taken all at a time is given by

$$\frac{n!}{n_1!\ n_2!\ n_3!\ .\ .\ .\ n_k!}$$

SAQ 3

Examples

How many different permutations can be made from the letters in each of the following words?

a. **BOOK**
b. **ATLANTA**
c. **CALCULATOR**
d. **MISSISSIPPI**
e. **INDEPENDENCE**

Solution

a. In the word "**BOOK**", there are one letter **B**, two letter **O** and one letter **K**. Therefore, the number of permutations that can be made from the letters of the word "**BOOK**" is

$$\frac{4!}{1!\,2!\,1!} = 12$$

b. In the word "**ATLANTA**", there are three letter **A**, two letter **T**, one letter **L** and one letter **N.** Hence, the number of permutations that can be made from the letters of the word "**ATLANTA**" is

$$\frac{7!}{3!\,2!\,1!\,1!} = 420$$

c. The word "**CALCULATOR**" has two letter **C**, two letter **A**, two letter **L**, one letter **U**, one letter **T**, one letter **O** and one letter **R.** So, the number of permutations that can be made from the letters of the word "**CALCULATOR**" is

$$\frac{10!}{2!\,2!\,2!\,1!\,1!\,1!\,1!} = 453{,}600$$

d. The word "MISSISSIPPI" has one letter M, four letter I, four letter S and two letter P. Hence, the number of permutations which can be made from the letters of the word "MISSISSIPPI" is

$$\frac{11!}{1!\,4!\,4!\,2!} = 34{,}650$$

e. There are one letter **I**, three letter **N**, two letter **D**, four letter **E**, one letter **P** and one letter **C** in the word "**INDEPENDENCE**", Therefore, the number of permutations that can be made from the letters of the word "**INDEPENDENCE**" is

$$\frac{12!}{1!\,3!\,2!\,4!\,1!\,1!} = 1{,}663{,}200$$

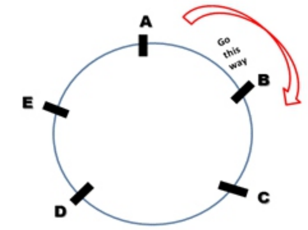
CIRCULAR PERMUTATIONS

3mins

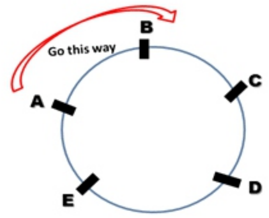
If **n** distinct objects are arranged in a circular form, the number of different circular arrangements of the **n** objects is **(n – 1)!**

Illustration

Consider the arrangements of five objects labelled **A**, **B**, **C**, **D** and **E** on a circular form as demonstrated below

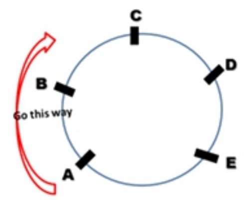


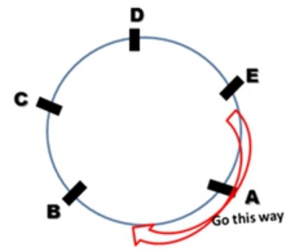
We have **ABCDE**

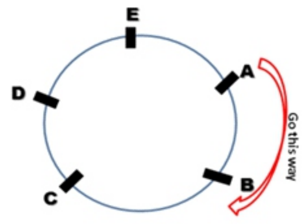


We also have **ABCDE**

Ditto for others as depicted below







Remarks

You will observed that the permutations **ABCDE** as demonstrated are not distinguishable. Hence, it is pertinent to use one of the **n** objects to mark the starting point (origin) so that the remaining **n – 1** objects could then be arranged in **(n – 1)!** ways if there are no other restrictions.

$$\boxed{\text{That is, } {}^{(n-1)}\mathbf{P}_{(n-1)} = (n-1)! \text{ ways}}$$

Summary

In this unit, Permutation rule has been explained.

· The permutation of n objects taking k at a time and denoted by ${}^{n}\mathbf{P}_{k}$ or **P(n, k)** is defined as

$$P(n,k) = \frac{n!}{(n-k)!}$$

Note that for any counting of

- $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$
- $0! = 1$
- $1! = 1$

· Given a set of n objects in which n_1 objects are identical and of one kind, n_2 objects are also identical and of another kind, n_k, . . . , objects are as well identical and yet of another kind such that $\mathbf{n_1} + \mathbf{n_2} + \mathbf{n_3} + \ldots + \mathbf{n_k} = \mathbf{n}$; then the number of permutations of these n objects taken all at a time is given by

$$\frac{n!}{\mathbf{n_1}!\ \mathbf{n_2}!\ \mathbf{n_3}!\ \ldots\ \mathbf{n_k}!}$$

· If n distinct objects are arranged in a circular form, the number of different circular arrangements of the n objects is (n – 1)!

Self Assessment Questions

1. Evaluate the following

a) ${}^{7}\mathbf{P}_{5}$ b) ${}^{12}\mathbf{P}_{4}$ c) ${}^{5}\mathbf{P}_{3}$ d) ${}^{6}\mathbf{P}_{0}$ e) ${}^{5}\mathbf{P}_{5}$ f) ${}^{6}\mathbf{P}_{2}$

2. A school musical director can select 2 musical plays to present next year. One will be presented in the fall, and one will be presented in the spring. If she has 9 to pick from, how many different possibilities are there?

3. In how many ways can the word **ANTAGONIST** be arranged

Tutor Marked Assignment

1. Find the value of

i.) ${}^{n}\mathbf{P}_{4} = 72$

ii.) ${}^{n}\mathbf{P}_{4} = 42 \times {}^{n}\mathbf{P}_{2}$

References

· Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.

· Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.

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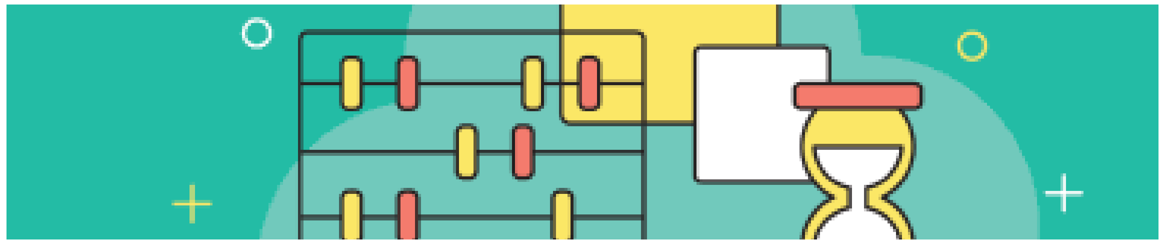

UNIT 3 COMBINATIONS

Introduction

Suppose you as a fine artist wishes to select two colours of material to design a new photo, and you have four colours. How many different possibilities can there be in this situation? This type of problem differs from permutation in that the order of selection is not important. That is, if you selects yellow and red, this selection is the same as the selection red and yellow. This type of selection is called a *combination*.

Learning Outcomes

At the end of this unit, you should be able to:

1. Use the combination rule to count the number of possible selection of k objects from n objects.
2. Count the number of possible distinguishable combinations of n objects when the objects are of distinctive types.
3. Apply the techniques learned in this unit to relevant real life problems.

Main Content

COMBINATIONS

4mins

The difference between a permutation and a combination is that in a combination, the order or arrangement of the objects is not important and by contrast, order is important in a permutation. Suppose that in the selection of k objects from a set of n objects in which the order of arrangements is not important so that different orderings of the same set of k objects are regarded as **the same**. In this case, each selection is called a combination.

Definition

A selection of distinct objects without regard to order is called a combination.

For instance, given the letters A, B, C, and D, list the permutations and combinations for selecting two letters.

The permutations are

AB	BA	CA	DA
AC	BC	CB	DB
AD	BD	CD	DC

In permutations, AB is different from BA.
But in combinations, AB is the same as BA since the order of the objects does not matter in combinations. Therefore, if duplicates are removed from a list of permutations, what is left is a list of combinations as

AB	~~BA~~	~~CA~~	~~DA~~
AC	BC	~~CB~~	~~DB~~
AD	BD	CD	~~DC~~

Hence the combinations of A, B, C, and D are AB, AC, AD, BC, BD, and CD.

COMBINATION RULE

SAQ 1

SAQ 2

The number of combinations of k objects selected from n objects is denoted by ${}^{n}C_{k}$ or $\binom{n}{k}$ is defined as

${}^{n}C_{k}$=

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Where $k \leq n$

Note that for any counting of n objects,

- $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$
- $0! = 1$
- $1! = 1$

Illustration

Suppose we want to determine how many possible distinct selections of two letters are there in the letters **A**, **B**, **C** and **D.**

By definition,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Here, n = 4 and k = 2, then

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!}$$

$$= \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1}$$

$$= 6$$

The distinct selection have been listed above earlier.

Example How many distinct selections can be formed from digits **1**, **2**, **3** and **4** taking three digits at a time? Hence, list all the distinct selections.

Solution Here, n = 4 and k = 3, then ${}^{4}C_{3}$ =

$$\binom{4}{3} = \frac{4!}{3!\,(4-3)!}$$

$$= \frac{4 \times 3!}{3!\,1!}$$

$$= 4$$

The distinct selection of the digits 1, 2, 3, and 4 taking three digits at a time are

123 (suspending digit 4)

124 (suspending digit 3)

134 (suspending digit 2)

234 (suspending digit 1)

Example How many 2-letter selections can be made from the letters in **ARMTI**? Hence, list all the selections.

Solution Here, n = 5 and k = 2, then ${}^{5}C_{2}$ =

$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!}$$

$$= \frac{5!}{2!\,3!}$$

$$= \frac{5 \times 4 \times 3!}{2 \times 1 \times 3!}$$

$$= 10$$

The distinct 2-letter selections that can be made from the letters in **ARMTI** are

AR AM AT AI RM RT RI MT MI TI

Example Compute the following

i.) ${}^{6}C_{1}$ ii.) ${}^{8}C_{3}$ iii.) ${}^{11}C_{5}$ iv.) ${}^{8}C_{0}$ v.) ${}^{5}C_{5}$

Solution i.) ${}^{6}C_{1}$ =

$$\binom{6}{1} = \frac{6!}{1!\,(6-1)!}$$

$$= \frac{6 \times 5!}{1 \times 5!}$$

$$= 6$$

ii.) $^8C_3$

$$\binom{8}{3} = \frac{8!}{3!\,(8-3)!}$$
$$= \frac{8!}{3!\,5!}$$
$$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!}$$
$$= 56$$

iii.) $^{11}C_5$

$$\binom{11}{5} = \frac{11!}{5!\,(11-5)!}$$
$$= \frac{11!}{5!\,6!}$$
$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{5 \times 4 \times 3 \times 2 \times 1 \times 6!}$$
$$= 462$$

iv.) $^8C_0 =$

$$\binom{8}{0} = \frac{8!}{0!\,(8-0)!}$$
$$= \frac{8!}{0!\,8!}$$
$$= 1$$

v.) $^5C_5 =$

$$\binom{5}{5} = \frac{5!}{5!\,(5-5)!}$$
$$= \frac{5!}{5!\,0!}$$
$$= 1$$

Example A journal editor has received 8 articles to review. He decides that he can use 3 reviews in his journals. How many different ways can these 3 reviews be selected?

Solution

$$\binom{8}{3} = \frac{8!}{3!\,(8-3)!}$$
$$= \frac{8!}{3!\,5!}$$
$$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!}$$
$$= 56 \; ways$$

Example To recruit new members, a compact-disc (CD) company advertises a special introductory offer: A new buyer agrees to buy 1 CD at regular company prices and receives free any 4 CDs of his or her choice from a collection of 69 CDs. How many possibilities does a new member have for the selection of the 4 free CDs?

Solution

$$\binom{69}{4} = \frac{69!}{4!\,(69-4)!}$$
$$= \frac{69!}{4!\,65!}$$
$$= \frac{69 \times 68 \times 67 \times 66 \times 65!}{4 \times 3 \times 2 \times 1 \times 65!}$$
$$= 864501 \text{ possibilities}$$

Example A committee of 2 women and 3 men is to be chosen out of 6 women and 7 men.

a) How may different committees can be formed?

b) If two are husband and wife, in how many ways can the committee be formed if the two must:

i.) Be on the committee?

ii.) Not be on the committee?

iii.) Not to be chosen together on the committee?

Solution a) The committee to be formed implies;

$$\binom{6}{2} \; and \; \binom{7}{3}$$
$$= \binom{6}{2} \times \binom{7}{3}$$
$$= 15 \times 35$$
$$= 525 \; committees$$

b) (i). If the two must be on the committee, then both the number of committee to be formed and the number of available men and women is reduced by 1. i.e.

$$\binom{6-1}{2-1} \; and \; \binom{7-1}{3-1}$$
$$= \binom{5}{1} \times \binom{6}{2}$$
$$= 5 \times 15$$
$$= 75$$

(iii) If the two are not to be chosen together on the committee, then the husband may be chosen but not the wife or the wife is chosen without the husband.

with Husband chosen but not wife **with Wife chosen but not husband**

$$\binom{7-1}{3-1} \text{ and } \binom{6-1}{2} \quad or \quad \binom{7-1}{3} \text{ and } \binom{6-1}{2-1}$$

$$= \binom{6}{2} \times \binom{5}{2} \quad + \quad \binom{6}{3} \times \binom{5}{1}$$

$$= 15 \times 10 \quad + \quad 20 \times 5$$

$$= 150 \quad + \quad 100$$

$$= 250$$

Summary

In this unit, the combination rule has been explained.

- A selection of distinct objects without regard to order is called a combination.
- In a combination, the order or arrangement of the objects is not important.
- The number of combinations of k objects selected from n objects is denoted by $^n\mathbf{C}_k$ or $\binom{n}{k}$ is defined as

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Where $k \leq n$

- Note that for any counting of
  - $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$
  - $0! = 1$
  - $1! = 1$

Self Assessment Questions

1. Evaluate each of the following expressions.

a. $\binom{5}{2}$ b. $\binom{8}{3}$ c. $\binom{7}{4}$ d. $\binom{6}{2}$ e. $\binom{6}{4}$

f. $\binom{3}{0}$ g. $\binom{3}{3}$ h. $\binom{9}{7}$ i. $\binom{12}{2}$ j. $\binom{4}{3}$

2. An economics professor is using a new method to teach a junior-level course with an enrollment of 42 students. The professor wants to conduct in-depth interviews with the students to get feedback on the new teaching method but does not want to interview all 42 of them. The professor decides to interview a sample of 5 students from the class. How many different samples are possible?

Tutor Marked Assignment

1. Verify the following relations

a. $\binom{n}{k} = \binom{n}{n-k}$

b. $\binom{n}{k} + \binom{n}{n-k} = \binom{n+1}{r}$

2. In a club there are 5 women and 4 men. A committee of 3 is to be chosen. How many different possibilities are there if

a) Without restriction

b) At least one of each sex must be a member

c) If a particular lady must be a member.

References

- Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.
- Probability and Statistics with R by María Dolores Ugarte, Ana F. Militino and Alan T. Arnholt. Published by CRC Press, Taylor and Francis group, USA. Page 80 – 81.
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Further Reading

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UNIT 4 PROBABILITY AND COUNTING TECHNIQUE

Introduction

If you used the multiplication counting principle, the permutation rules, and the combination rule, one can compute the probability of outcomes of many experiments, such as getting a full house when 5 cards are dealt or selecting a committee of 3 women and 2 men from a club consisting of 10 women and 10 men.

Learning Outcomes

At the end of this unit, you should be able to:

1. Compute the probability of events using counting techniques

Main Content

You should recall from multiplication principle, permutation rule and combination rule as follows;

MULTIPLICATION PRINCIPLE

4mins

In a sequence of different events $E_1, E_2, \ldots, E_k$ in which E_1 has n_1 possible outcomes, E_2 has n_2 possible outcomes and so forth. The total possible outcomes for $E_1, E_2, \ldots, E_n$ is $n_1 \times n_2 \times \ldots \times n_k$. Where $n_1, n_2, \ldots, n_k$ are possible outcomes for events $E_1, E_2, \ldots, E_k$, respectively.

PERMUTATION RULE

Generally, The permutation of ***n*** objects taking ***k*** at a time and denoted by $^{n}P_{k}$ or **P(n, k)** is defined as

$^{n}P_{k}=$

$$P(n,k) = \frac{n!}{(n-k)!}$$

Permutations of *n* objects, not all distinct

Given a set of ***n*** objects in which n_1 objects are identical and of one kind, n_2 objects are also identical and of another kind, . . . , n_k objects are as well identical and yet of another kind such that $n_1 + n_2 + n_3 + \ldots + n_k = n$; then the number of permutations of these n objects taken all at a time is given by

$$\frac{n!}{n_1!\ n_2!\ n_3!\ \ldots\ n_k!}$$

Circular permutations

If ***n*** distinct objects are arranged in a circular form, the number of different circular arrangements of the ***n*** objects is ***(n – 1)!***

COMBINATION RULE

SAQ 1

SAQ 2

The number of combinations of k objects selected from n objects is denoted by ^nC_k or $\binom{n}{k}$ is defined as

$^nC_k =$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Where $k \leq n$

The counting rules above (multiplication principle, permutation rule and combination rule) can be combined with the probability rules to solve many types of probability problems.

Example Find the probability of getting 4 aces when 5 cards are drawn from an ordinary deck of cards.

Solution Recall that there are 52 cards in a deck of cards. The number of ways to select 5 cards is

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!}$$

$$= 2,598,960 \text{ ways}$$

The 4 aces can only be selected in 1 ways, i.e. $^4C_4 = 1$ way

If the 4 aces can be selected in 1 way, then the last card of the 5 cards can be selected from the remaining 48 cards in $^{48}C_1 = 48$ ways

Therefore,

$$P(4\ aces) = \frac{\binom{4}{4} \times \binom{48}{1}}{\binom{52}{5}}$$

$$= \frac{1}{54,145}$$

$$= 0.000018$$

Example A box contains 24 transistors, 4 of which are defective. If 4 are sold at random, find the following probabilities.

a. Exactly 2 are defective.
b. None is defective.
c. All are defective.
d. At least 1 is defective.

Solution There are $^{24}C_4 = 10,626$ ways to sell the 4 defective transistors

a. Two defective transistors can be selected as $^4C_2$ and two non-defective ones as $^{20}C_2$. Hence,

$$P(exactly\ 2\ defectives) = \frac{\binom{4}{2} \times \binom{20}{2}}{10,626}$$

$$= \frac{190}{1771}$$

$$= 0.1073$$

b. The number of ways to choose no defectives is $^{20}C_4$. Hence,

$$P(no\ defectives) = \frac{\binom{20}{4}}{10,626}$$

$$= \frac{1615}{3542}$$

$$= 0.4560$$

c. The number of ways to choose 4 defectives from 4 is $^4C_4$, or 1. Hence.

$$P(all\ are\ defectives) = \frac{1}{10,626}$$

$$= 0.00009$$

d. To find the probability of at least 1 defective transistor, find the probability that there are no defective transistors, and then subtract that probability from 1. Hence

$$P(at\ least\ 1\ defective) = 1 - P(no\ defectives)$$

$$= 1 - 0.4560$$

$$= 0.5440$$

Example A store has 6 TV Graphic magazines and 8 Newstime magazines on the counter. If two customers purchased a magazine each, find the probability that one of each magazine was purchased.

Solution The total number of magazines available in the store is 14 and two customers can purchase a magazine in $^{14}C_2 = 91$.

The 2 purchased magazines can be chosen from the two types of magazines as $^6C_1$ and $^8C_1$.

$$\therefore P(1\ TV\ graphic\ and\ 1\ newstime) = \frac{\binom{6}{1}\times\binom{8}{1}}{\binom{14}{2}}$$

$$= \frac{48}{91}$$

$$= 0.5275$$

Example There are 8 married couples in a tennis club. If 1 man and 1 woman are selected at random to plan the summer tournament, find the probability that they are married to each other.

Solution There are 8C1 or 8 ways to select the man and also 8C1 or 8 ways to select the woman. Therefore there are 8 × 8 or 64, ways to select 1 man and 1 woman. Since there are 8 married couples, the solution is

$$\frac{8}{64}$$

$$= 0.125$$

As earlier indicated, the counting rules and the probability rules can be used to solve a large variety of probability problems found in business, gambling, economics, biology, and other fields.

Summary

In this unit, probability and counting techniques has been explained.
I.) Depending on the counting techniques, the probability can be use to solve variety of problems in many disciplines.

Self Assessment Questions

1. A combination lock consists of the 26 letters of the alphabet. If a 3-letter combination is needed, find the probability that the combination will consist of the letters ABC in that order. The same letter can be used more than once. (Note: A combination lock is really a permutation lock.)
2. There are 7 men and 7 women that are related in a compound. If 1 man and 1 woman are selected at random to plan the environmental sanitation, find the probability that they are related to each other.

Tutor Marked Assignment

1. A box contains 25 computer chips, 5 of which are defective. If 5 are sold at random, find the following probabilities.
   a. Exactly 3 are defective.
   b. None is defective.
   c. All are defective.
   d. At least 1 is defective.

References

- Introduction to Probability and Statistics for Engineers and Scientists, 3rd edition by Sheldon M. Ross. Published by Elsevier Academic Press, USA. Page 55 – 59.
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