MAT 206:
LINEAR ALGEBRA II

University of Ilorin
Centre for Open & Distance Learning

CODL

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From the Vice Chancellor

Courseware development for instructional use by the Centre for Open and Distance Learning (CODL) has been achieved through the dedication of authors and the team involved in quality assurance based on the core values of the University of Ilorin. The availability, relevance and use of the courseware cannot be timelier than now that the whole world has to bring online education to the front burner. A necessary equipping for addressing some of the weaknesses of regular classroom teaching and learning has thus been achieved in this effort.

This basic course material is available in different electronic modes to ease access and use for the students. They are available on the University's website for download to students and others who have interest in learning from the contents. This is UNILORIN CODL's way of extending knowledge and promoting skills acquisition as open source to those who are interested. As expected, graduates of the University of Ilorin are equipped with requisite skills and competencies for excellence in life. That same expectation applies to all users of these learning materials.

Needless to say, that availability and delivery of the courseware to achieve expected CODL goals are of essence. Ultimate attention is paid to quality and excellence in these complementary processes of teaching and learning. Students are confident that they have the best available to them in every sense.

It is hoped that students will make the best use of these valuable course materials.

Professor S. A. Abdulkareem
Vice Chancellor

Foreword

Courseware remains the nerve centre of Open and Distance Learning. Whereas some institutions and tutors depend entirely on Open Educational Resources (OER), CODL at the University of Ilorin considers it necessary to develop its own materials. Rich as OERs are and widely as they are deployed for supporting online education, adding to them in content and quality by individuals and institutions guarantees progress. Doing it in-house as we have done at the University of Ilorin has brought the best out of the Course Development Team across Faculties in the University. Credit must be given to the team for prompt completion and delivery of assigned tasks in spite of their very busy schedules.

The development of the courseware is similar in many ways to the experience of a pregnant woman eagerly looking forward to the D-day when she will put to bed. It is customary that families waiting for the arrival of a new baby usually do so with high hopes. This is the apt description of the eagerness of the University of Ilorin in seeing that the centre for open and distance learning [CODL] takes off.

The Vice-Chancellor, Prof. Sulyman Age Abdulkareem, deserves every accolade for committing huge financial and material resources to the centre. This commitment, no doubt, boosted the efforts of the team. Careful attention to quality standards, ODL compliance and UNILORIN CODL House Style brought the best out from the course development team. Responses to quality assurance with respect to writing, subject matter content, language and instructional design by authors, reviewers, editors and designers, though painstaking, have yielded the course materials now made available primarily to CODL students as open resources.

Aiming at a parity of standards and esteem with regular university programmes is usually an expectation from students on open and distance education programmes. The reason being that stakeholders hold the view that graduates of face-to-face teaching and learning are superior to those exposed to online education. CODL has the dual-mode mandate. This implies a combination of face-to-face with open and distance education. It is in the light of this that our centre has developed its courseware to combine the strength of both modes to bring out the best from the students. CODL students, other categories of students of the University of Ilorin and similar institutions will find the courseware to be their most dependable companion for the acquisition of knowledge, skills and competences in their respective courses and programmes.

Activities, assessments, assignments, exercises, reports, discussions and projects amongst others at various points in the courseware are targeted at achieving the objectives of teaching and learning. The courseware is interactive and directly points the attention of students and users to key issues helpful to their particular learning. Students' understanding has been viewed as a necessary ingredient at every point. Each course has also been broken into modules and their component units in sequential order.

Courseware for the Bachelor of Science in Computer Science housed primarily in the Faculty of Communication and Information Science provide the foundational model for Open and Distance Learning in the Centre for Open and Distance Learning at the University of Ilorin.

At this juncture, I must commend past directors of this great centre for their painstaking efforts at ensuring that it sees the light of the day. Prof. M. O. Yusuf, Prof. A. A. Fajonyomi and Prof. H. O. Owolabi shall always be remembered for doing their best during their respective tenures. May God continually be pleased with them, Aameen.

Bashiru, A. Omipidan
Director, CODL

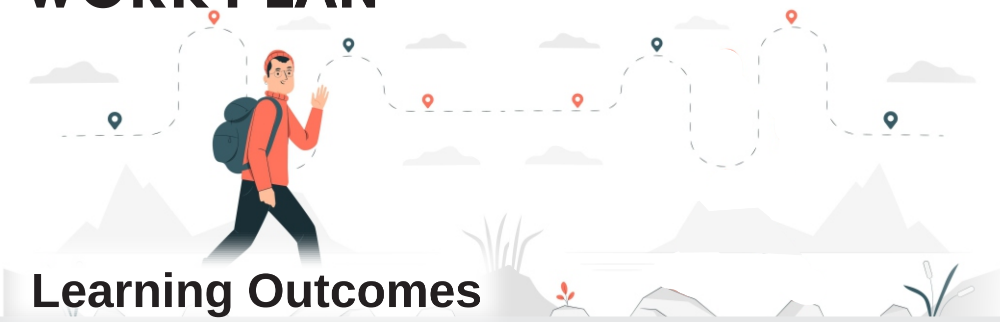
INTRODUCTION

I welcome you to Linear Algebra II, a second-semester course. It is a 2-credit course that is available to year two undergraduate students in Faculties of Physical Sciences, Communication and Information Sciences, Engineering, Education and allied degrees. This course was designed as an intermediate course for undergraduate mathematics. It consists of basic topics from algebra and its application. It was prepared with the aim of introducing undergraduate students to some basic theorems, rules and principles that will be useful in advance mathematics.

Course Goal

Your journey through this course will introduce you to some basic topics like system of linear equations, change of basis, Eigenvalues and eigenvectors problem, orthogonal and canonical forms of system of equations. You will also be introduced to some practical applications of some of the topics to real life issue.

WORK PLAN



Learning Outcomes

At the end of this course, you should be able to:

- solve problems on system of linear equations;
- accurately apply change of basis;
- solve eigenvalues and eigenvectors problem;
- obtain an equivalence and similarity equations of system of linear equations;

Course Guide

Module 1

System of Linear Equations

Unit 1 - Definition of System of Linear Equations

Unit 2 - Homogeneous System of Linear Equations

Unit 3- Solution of Homogeneous System of Linear Equations

Unit 4 - Non-Homogeneous System of Linear Equations

Module 2

Change of Basis

Unit 1 - Scalars and Vectors

Unit 2 - Vector Space

Unit 3 - Basis of a Vector Space

Related Courses

Prerequisite: MAT 111 & MAT 113

Required for: MAT 306

MAT 206
LINEAR ALGEBRA II

- solve minimal and characteristics polynomial of a linear transformation; and
- obtain orthogonal, diagonalization and canonical form of linear equations

Module 3
Equivalence and Similarity Relation

Unit 1 - Relation and Properties

Unit 2 - Equivalence Relation

Unit 3 - Similarity

Module 4
Eigenvalues and Eigenvectors

Unit 1 - Eigenvalues

Unit 2 - Minimal Polynomial

Unit 3 - Characteristics Polynomial

Unit 4 - Eigenvector

Module 5
Eigenvalues and Eigenvectors

Unit 1 - Orthogonal Diagonalization, Canonical form

Course Requirements

Requirements for success

The CODL Programme is designed for learners who are absent from the lecturer in time and space. Therefore, you should refer to your Student Handbook, available on the website and in hard copy form, to get information on the procedure of distance/e-learning. You can contact the CODL helpdesk which is available 24/7 for every of your enquiry.

Visit CODL virtual classroom on http://codllms.unilorin.edu.ng. Then, log in with your credentials and click on MAT 206. Download and read through the unit of instruction for each week before the scheduled time of interaction with the course tutor/facilitator. You should also download and watch the relevant video and listen to the podcast so that you will understand and follow the course facilitator.

At the scheduled time, you are expected to log in to the classroom for interaction.

Self-assessment component of the courseware is available as exercises to help you learn and master the content you have gone through.

You are to answer the Tutor Marked Assignment (TMA) for each unit and submit for assessment.

Embedded Support Devices

Support menus for guide and references

Throughout your interaction with this course material, you will notice some set of icons used for easier navigation of this course materials. We advise that you familiarize yourself with each of these icons as they will help you in no small ways in achieving success and easy completion of this course. Find in the table below, the complete icon set and their meaning.

Introduction	Learning Outcomes	Main Content
Summary	Tutor Marked Assignment	Self Assessment
Web Resources	Downloadable Resources	Discuss with Colleagues
References	Futher Reading	Self Exploration

Grading and Assessment

TMA	CA	Exam	Total
20%	20%	60%	100%

Module 1

System of Linear Equations

Units

UNIT 1

Definition of System of Linear Equations

Introduction

System of linear equations occur in several forms and has application in several fields of Mathematics including, Algebra, Calculus and some Engineering problems.

Learning Outcomes

At the end of this unit, you should be able to:

1. recognize at first glance a system of equations; and
2. to solve a system of linear equations.

Main Content

A linear equation is not always of the form $y = 35 - 5x$, it can also be like $y = 5(7 - x)$ or $y + 5x = 35$ or $y + 5x - 35 = 0$, $\frac{4x}{5y-2} = 10$, $\frac{y+2}{x} = 5$ e.t.c. A system of linear equations is when we have two or more linear equations working together. For example; here are two linear equations:

$$2x + y = 5$$
$$-x + y = 2$$

Together they are a system of linear equations.

Can you discover the values of x and y yourself? have a go at it!

More general form of system linear equations is given as

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\ . \quad . \quad \ldots. \qquad . \\ . \quad . \quad \ldots. \qquad . \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_n \end{pmatrix}$$

Summary

A system of linear equation consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously. The solution to a system of linear equation in two variables is any ordered pair that satisfies each equation independently. System of equations are classified as independent with one solution, dependent with an infinite number of solutions or inconsistent with no solution.

Self Assessment Questions

Which of the following equations are not a linear equation?

$(A) x_1 + 3x_2x_3 = -5$ $\quad (B) x_1 + 3x_2 + x_3 = 0$ $\quad (C) \sqrt{6}x_1 + \frac{3}{2}x_2 + x_3 = 0$

$(D) \sqrt{5}x_1 + \sqrt{3x_2} = -2x_3$ $\quad (E) x_1^2 - x_2 + 4x_3 = 0$.

Tutor Marked Assignment

Classify each of the following equations as a linear or non-linear equation

$(A) 2x_1 + 3x_2^2 = -5$ $\quad (B) x_1 - 4x_2 = 0$ $\quad (C) \sqrt{7}x_1^2 + \frac{2}{3}x_2 - x_3 = 0$.

$(D) \sqrt{5}x_1 + \sqrt{3x_2} = -2x_3$ $\quad (E) x_1^2 - x_2 + 4x_3 = 0$.

References

- K. A. Stroud and Dexter J. Booth (2007). Engineering Mathematics. Palgrave Macmillan. Houndsmill, Basingstoke, Hampshire RG21 6XS and 175 Fifth Avenue, New York.
- S. A. Ilori and O. Akinyele. (2009). Elementary Abstract and Linear Algebra, Ibadan University Press, Ibadan. https://web.cortland.edu/jubrani/272ch5.pdf

Further Reading

- Steven Roman(2005). Advanced Linear Algebra, Springer, USA.
- Derek J.S. Robinson(2006). A course in Linear Algebra with application, World Scientific Pub Co. Inc. USA.

UNIT 2

Homogeneous System of Linear Equations

Introduction

Classification of system of linear equations based on their characteristics is important in order to be able to solve the given equations easily, thus, there is a need to classify any system of linear equations so that it can be readily analyzed.

Learning Outcomes

At the end of this unit, you should be able to:

1. identify and classify a system of linear equation; and
2. state the properties of homogeneous of linear system of equations.

Main Content

Any system of linear equation of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= c_2 \\ . \quad . \quad \ldots . \quad & \quad . \\ . \quad . \quad \ldots . \quad & \quad . \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= c_n \end{aligned}$$

is said to be homogeneous if $c_1 = c_2 = \cdots = c_n = 0$

That is the system of equation above

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ . \quad . \quad \ldots . \quad & \quad . \\ . \quad . \quad \ldots . \quad & \quad . \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

For example, given a system of linear equation

$$2x_1 + 3x_2 - 4x_3 = 0$$

$$x_1 - x_2 + 3x_3 = 0$$

$$-x_1 + 2x_2 + 5x_3 = 0$$

Any homogenous system of linear equation is always consistent since it has at least one solution, namely

$$x_1 = x_2 = \cdots = x_n = 0$$

This solution is called the trivial solution. Any other solution of the homogeneous system is called a non-trivial solution. Any system of linear equation can be represented in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$Ax = 0$$

where A is called the coefficient matrix and x the matrix of unknown variables.

Summary

System of equation is homogeneous linear equation if the constant term in all the equations are zero. Every homogeneous system of equation has a trivial solution, that is $x_1 = x_2 = \cdots = x_n = 0$.

Self Assessment Questions

When is a system of linear equations said to be homogenous?

Tutor Marked Assignment

Which of these system of equations is not an homogenous linear equation?

1.

$$2x_1 + 3x_2 = 4x_3$$

$$x_1 = - x_2 + 3x_3$$

$$-x_1 + 2x_2 + 5x_3 = 0$$

2.

$$x_1^2 + 3x_2 + 4x_3 = 0$$

$$x_1 - x_2 + 3x_3 = 0$$

$$- x_1 + 2x_2 + 5x_3 = 0$$

References

- K. A. Stroud and Dexter J. Booth (2007). Engineering Mathematics. Palgrave Macmillan. Houndsmill, Basingstoke, Hampshire RG216XS and 175 Fifth Avenue, New York.
- S. A. Ilori and O. Akinyele (2009). Elementary Abstract and Linear Algebra, Ibadan University Press, Ibadan.
- https://web.cortland.edu/jubrani/272ch5.pdf

Further Reading

- Steven Roman(2005). Advanced Linear Algebra, Springer, USA.
- Derek J.S. Robinson(2006). A course in Linear Algebra with application, World Scientific Pub Co. Inc. USA.

UNIT 3

Solution of Homogeneous System of Linear Equations

Introduction

To obtain solutions to a set of linear equations, many approaches exist. We shall be interested in obtaining solution to a system of homogeneous equation by using row echelon form and the ranks of the coefficient and augmented matrices.

Learning Outcomes

At the end of this unit, you should be able to:

1. state the rank of a matrix;
2. perform row echelon (canonical) form on a matrix; and
3. state consistency and inconsistency of a system of equations.

Main Content

Consistent

A system of equations is said to be consistent if they have one or more solution, for example, the system of equations

$$a + b + c = 0$$

$$a - b - 3c = 0$$

$$3a + b - c = 0$$

has two sets of solutions which are (0, 0, 0) and (1, −2, 1)

Inconsistent

If a system of equations has no solution, it is said to inconsistent. Example,

$$x + 2y = 3$$

$$3x + 6y = -7$$

has no solutions.

Rank:

The rank of a matrix is the number of non zero row in the upper triangular matrix

Canonical form (row echelon form)

Any non zero matrix A can be reduced to an echelon form by performing elementary transformation on them. The resulting matrix is called canonical form and the number of non zero rows in the canonical form gives the rank of the matrix

Associated with any $(m \times n)$ matrix $A \in M_{m,n}(F)$, over a field F is a linear transformation $T_A : V_n(F) \to V_m(F)$ defined by $T_A(x) = A_x$, for any column vector $x \in V_n(F)$. Thus the set of solutions of the homogeneous system of linear equations $Ax = 0$, constitutes the kernel of T_A, which is a subspace of $V_n(F)$. This subspace is called the solution space of the homogeneous system of linear equations or the row null space of the matrix A. Similarly, the column null space of the matrix A is a solution space of the homogeneous system of the linear equation $yA = 0$ where $y \in V_m(F)$ is a row vector. The $i - th$ equation of the system $Ax = 0$ is $a_{i1}x_1 + ... + a_{1n}x_n = 0$, i.e. $R_i x = 0$, where $R_i = (a_{i1}, ..., a_{in})$ is the i – th row of A. Therefore, $x \in V_n(F)$ is a solution of the homogeneous system of linear equations, $Ax = 0$, if and only if x is orthogonal to each of the row vectors R_i of A for $i = 1, ..., m$, in other words x is a solution of $Ax = 0$ if and only if x is orthogonal to the row space R_A of A. Note that:

1. The dimension of the column null space of A is equal to $m-r$, where $A \in M_{m,n}(F)$ and r is the rank of A.

2. A homogenous system $Ax = 0$ of m linear equations in n unknowns has non-trivial solutions, if and only if rank $(A) < n$. In particular, this is the case if the number of equations is less than the number of unknowns, i.e. if $m < n$.

3. If $A \in M_{m,n}(F)$ and if $Ax = y$ is a homogenous system of linear equations such that the coefficients of the matrix has rank r, and if $v_1, ..., v_n-r$ is a linearly independent solution of the corresponding homogenous system $Ax = 0$ and if be is a solution of $Ax = y$ then the vector $v \in V_n(F)$ is a solution of the non homogenous system $Ax = y$, if and only if v is of the form $v = b + a_1v_1 + ... + a_n -_r v_n -r, a_i \in F$. We call v the general solution of the non-homogenous system of linear equations, $Ax = y$ and b the particular solution of the system.

4. If $A \in GL_n(F)$ is an $(n \times n)$ non-singular matrix over a field F and $Ax = y$ is a non-homogenous system of linear equations, then we have shown in that the solution of the system $Ax = y$ is unique and is given by Crammer's rule.

Activity 1

Solve the following system of homogenous system of linear equations over the field R of real numbers.

$$x_1 + 3x_2 - 2x_3 = 0$$
$$x_1 - 8x_2 + 8x_3 = 0$$
$$3x_1 - 2x_2 + 4x_3 = 0$$

Solution

First, we shall find the rank of the matrix of coefficients,

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 1 & -8 & 8 \\ 3 & -2 & 4 \end{pmatrix} \in M_{3,3}(R)$$

$$\begin{pmatrix} 1 & 3 & -2 \\ 1 & -8 & 8 \\ 3 & -2 & 4 \end{pmatrix} \begin{matrix} R_2(-1R_1) \\ R_3(-3R_1) \end{matrix} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -11 & 10 \\ 0 & -11 & 10 \end{pmatrix} \begin{matrix} R_2(\frac{-1}{11}) \\ R_3(-R_2) \end{matrix} \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & \frac{-10}{11} \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are only two non-zero rows in the row reduced Echelon form, it follows that the rank of the matrix of coefficients is 2, hence, the homogeneous system has $(3 - 2) = 1$ linearly independent solution and every other solution is a linear combination of it. From the row reduced Echelon form of A, the system of linear equations becomes

$$x_1 + 3x_2 - 2x_3 = 0$$
$$x_2 - \frac{10}{11}x_3 = 0$$

put $x_3 = 11k$, then $x_2 = 10k$ and $x_1 = 22k - 30k = -8k$. Hence the solution space of the homogenous system is $\{(x_1, x_2, x_3) = k(-8, 10, 11) | k \in R\}$ i.e. a basis for the solution space of the system is $\{(-8, 10, 11)\}$.

Summary

System of linear equation is said to be consistent if it has one or infinite set of solutions while it is inconsistent if it has no solution. The rank of a matrix is the number of non-zero rows in an echelon form of the matrix. The solution set of any homogeneous linear equation is unique if a single set of solution satisfies all the linear equations.

Self Assessment Questions

1. Define the consistency of system of linear equations.
2. Define the rank of a matrix.

Tutor Marked Assignment

Given a system of linear equations

$$\begin{aligned} x - 3y - z &= 0 \\ -2x + 3y - z &= 0 \\ x + y + 3z &= 0 \end{aligned}$$

Is the system consistent? If yes, how many unique solution(s) does it have? Obtain all the solution(s).

References

- Steven Roman(2005). Advanced Linear Algebra, Springer, USA.
- Derek J.S. Robinson(2006). A course in Linear Algebra with application, World Scientific Pub Co. Inc.
- K. A. Stroud and Dexter J. Booth (2007). Engineering Mathematics. Palgrave Macmillan. Houndsmill, Basingstoke, Hampshire RG21 6XS and 175 Fifth Avenue, New York.
- S. A. Ilori and O. Akinyele(2009). Elementary Abstract and Linear Algebra, Ibadan University Press, Ibadan.

Further Reading

- H. K. Dass(2009). Advanced Engineering Mathematics. S. Chand & Co. New Delhi.

UNIT 4

Non-Homogeneous System of Linear Equations and its Solution

Introduction

Having understood the concept of homogeneity, it is important to analyze what makes a system of linear equation to be non-homogenous. Effort shall be directed towards that in this unit together with how to obtain the solution of non-homogenous linear equation.

Learning Outcomes

At the end of this unit, you should be able to:

1. identify and classify a system of linear equation as non-homogenous;
2. state the properties of non-homogeneous linear system of equations; and
3. obtain solution to a system of non-homogenous linear equation.

Main Content

A system of linear equation of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= c_2 \\
. \quad . \quad \ldots \quad & \quad . \\
. \quad . \quad \ldots \quad & \quad . \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= c_n
\end{aligned}$$

is said to be non-homogeneous if not all all the c_i' sare equal to zero. If we put $A = (a_{i,j})_{m,n}$,

$$A = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} c_1 \\ . \\ . \\ . \\ c_m \end{pmatrix}$$

We can write the non-homogenous system in the form

$$AX = C$$

Matrix A is called the matrix of coefficients, and X the matrix of variables and C is the augmented matrix. Thus a homogeneous system can be written in the form $Ax = 0$ (where the augmented matrix of the non-homogeneous system of linear equations is 0).

The system is said to be consistent or solvable over a field F if it has a solution in F.

The matrix of the coefficient as well as augment can be matched together reduced into echelon form by elementary row transformation, the value of the unknown is calculated from the last equation by backward substitution. For consistency of a non homogeneous system of linear equation, if

1. Rank A = Rank C=n a unique solution (only one solution) exist.

2. Rank A = Rank C=r ($r < n$) an indefinite solution exist (more than one solution).

3. Rank $A \neq$ Rank C=r ($r < n$) no solution exist.

Activity 1

Prove the following non-homogenous system of linear equations is consistent and solve it over the field R of real numbers.

$$x + y = -1$$
$$2x - 3z = -1$$
$$y + 4z = 4$$

Solution

To test for consistency, we shall find the ranks of the coefficient matrix and the augmented matrix as follows:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 2 & 0 & -3 & -1 \\ 0 & 1 & 4 & 4 \end{array}\right) \quad R_2 - 2R_1 \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -2 & -3 & 1 \\ 0 & 1 & 4 & 4 \end{array}\right)$$

$$\begin{array}{l} (\frac{-1}{2})R_2 \\ R_3 - 1R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & 0 & \frac{5}{2} & \frac{9}{2} \end{array}\right) \quad (\frac{2}{5})R_3 \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & \frac{9}{5} \end{array}\right)$$

It follows that Rank of the coefficient matrix = Rank of augmented matrix = 3. hence the system is consistent. Also it follows that the coefficient matrix is non-singular and so we can obtain the only solution by backward substitution from:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & \frac{9}{5} \end{array}\right)$$

$z = \frac{9}{5}$, $y + \frac{3}{2}z = \frac{-1}{2}$, $x + y = -1$ which gives $(x, y, z) = (\frac{11}{5}, \frac{-16}{5}, \frac{9}{5})$

Alternatively, by Crammer's rule as follows:

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -3 \\ 0 & 1 & 4 \end{vmatrix} = 1(0+3) - 1(8-0) + 0(2-0) = 3 - 8 + 0 = -5$$

$$x = \frac{\Delta_x}{\Delta} \text{ but } \Delta_x = \begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & -3 \\ 4 & 1 & 4 \end{vmatrix} = 1(0+3) - 1(-4+12) + 0(-1-0) = -3 - 8 = -11, \quad x = \frac{11}{5}$$

$$x = \frac{\Delta_y}{\Delta} \text{ but } \Delta_y = \begin{vmatrix} 1 & -1 & 0 \\ 2 & -1 & -3 \\ 0 & 4 & 4 \end{vmatrix} = 1(-4 - -12) + 1(8-0) + 0(8-0) = 8 + 8 + 0 = 16, \quad y = \frac{-16}{5}$$

$$z = \frac{\Delta_z}{\Delta} \text{ but } \Delta_z = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{vmatrix} = 1(0 - -1) - 1(8-0) - 1(2-0) = 1 - 8 - 2 = -9, \quad z = \frac{9}{5}$$

Hence the solution is $(x, y, z) = (\frac{11}{5}, \frac{-16}{5}, \frac{9}{5})$.

Activity 2

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 4 \\ 3 & 2 & 2 \\ 5 & 1 & 2 \end{bmatrix}$$

by reducing it to canonical form

$$\begin{array}{c} R_1 \to \\ R_2 \to \\ R_3 \to \\ R_4 \to \end{array} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 4 \\ 3 & 2 & 2 \\ 5 & 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 9 & 3 \end{bmatrix} \begin{array}{l} \\ 3R_1 - R_2 \\ 3R_1 - R_3 \\ 5R_1 - R_4 \end{array}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & -12 \end{bmatrix} \begin{array}{l} \\ \\ 4R_2 - R_3 \\ 9R_2 - R_4 \end{array}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \frac{12}{5}R_3 - R_4 \end{array}$$

$$\text{Rank} = 3$$

Summary

A system of linear equation consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.

The solution to a system of linear equation in two variables is any ordered pair that satisfies each equation independently. System of equations are classified as independent with one solution, dependent with an infinite number of solutions or inconsistent with no solution.

Self Assessment Questions

Solve each of the following system of linear equations over R show that they are consistent.

1.
$$x + 3y - 3z = 0$$
$$2x - 3y + z = 0$$
$$3x - 2y + 2z = 0$$

2.
$$x - y - z = -4$$
$$2x + 3y - 12z = 7$$
$$3x - 4y + z = -15$$

3.
$$x + y + z = -2$$
$$2x + 3y - 4z = -3$$
$$y - 6z = 1$$

Tutor Marked Assignment

1. Define a system of linear equations.

2. When is a solution said to be (i) Inconsistent. (ii) Dependent.

3. What is the difference between a homogenous system and a nonhomogenous system of linear equations?

4. Using Crammers rule, solve

$$x_1 + 2x_2 + x_3 = 4$$

$$3x_1 - 4x_2 - 2x_3 = 2$$

$$5x_1 + 3x_2 + 5x_3 = -1$$

References

- Steven Roman. (2005). Advanced Linear Algebra, Springer, USA.
- Derek J.S. Robinson(2006). A course in Linear Algebra with application, World Scientific Pub Co. Inc.
- S. A. Ilori and O. Akinyele(2009). Elementary Abstract and Linear Algebra, Ibadan University Press, Ibadan.

Further Reading

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Module 2

Change of Basis

Units

Unit 1 - Scalars and Vectors
Unit 2 - Vector Space
Unit 3 - Basis of a Vector Space

UNIT 1

Scalars and Vectors

Introduction

Description of quantities depends on two factors which are magnitude and direction. However, some quantities are majorly described by their magnitude while others depend on both. In this unit, scalars and vectors shall be described using their characteristics.

Learning Outcomes

At the end of this unit, you should be able to:

1. identify vectors and scalars; and
2. give examples of scalar and vector quantities.

Main Content

Scalars:

Any quantities that are completely described by their magnitude are termed scalar quantity. Examples of this include room temperature, speed of a vehicle, volume of a solid, pressure exerted etc.

Vectors:

Any quantity that depends on magnitude as well as its direction before it can be completely described is known as vector quantity. Examples of this include velocity of a body, acceleration etc. The magnitude of the velocity is the speed while its direction is the sense of movement. Alternatively, a vector quantity B is an ordered pairs ($\underline{b}_1, \underline{b}_2$) in two dimension or ordered number triple ($\underline{b}_1, \underline{b}_2$, $\underline{b}_3$) in three dimensions where all the bi are real numbers (scalars). Two vectors are equal if they have the same magnitude and direction. The negative of a vector is a vector with same magnitude but opposite direction.

Activity 1

Given that vectors $A = (2x + y, 4, 3 - z)$ and $B = (7, y, -5)$, if $A = B$ what is the value of x, y and z respectively?

Solution

From equality of vectors, $2x + y = 7$, $4 = y$ and $3 - z = -5$, hence,

$(x, y, z) = \frac{3}{2}, 4, 8$.

Summary

Scalar quantities are completely described by magnitude while vectors are described by magnitude and directions.

Self Assessment Questions

1. Mention some types of vector quantities.
2. Define equal vectors.

Tutor Marked Assignment

1. Differentiate between mass and weight of a body in term of scalar and vector.

2. Categorize the following as scalar or vector quantities: (a) Force (b) Momentum (c) Speed (d) Acceleration (e) Velocity.

References

- Gilbert Strang (2016). Linear Algebra and its Applications, WellesleyCambridge Press.

Further Reading

- Morton L. Curtis(2012). Abstract Linear Algebra, Springer, Switzerland.

UNIT 2

Vector Space

Introduction

Notion of abstract vector space with some examples shall be discussed here. A vector space is a set of objects called vectors with possibilities of performing some vector algebra like addition, scalar multiplication and so on subject to reasonable rules. This occurs in numerous branches of Mathematics as well as in many applications.

Learning Outcomes

At the end of this unit, you should be able to:

1. give a concise definition of vector space;
2. state examples of vector space; and
3. solve simple vector algebra.

Main Content

Vector Space:

A non-empty set W of vectors is called a vector space if two operations of additions and scalar multiplication is defined on it and the following axioms are satisfied

(I) $\underline{a} + \underline{b}$ is in W (additive closure property)

(ii) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ (Commutative property)

(iii) $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c}$ (associative property)

(iv) $\underline{a} + 0 = \underline{a}$ (existence of additive identity)

(v) $\underline{a} + \underline{-a} = 0$ (existence of additive inverse)

(vi) $k\underline{a} \in W$ (closure of scalar multiplication of vector)

(vii) $k(\underline{a} + \underline{b}) = k\underline{a} + k\underline{b}$ (scalar multiplication over addition)

(viii) $(k + q)\underline{a} = k\underline{a} + q\underline{a}$

(ix) $(kq)\underline{a} = k(q\underline{a}) = q(k\underline{a})$

(x) $1 \ldots \underline{a} = \underline{a}$ (Multiplicative identity)

for every vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ and scalars k, q

Examples of Vector Space

(1) Consider P_n as the set of all polynomials of degree n, $n \geq 0$), members of P_n have the form $P_n(y) = b_n y^n + b_n - 1y^{n-1} + b_n - 2y^{n-2} + \cdots + b_2 y^2 + b_1 y + b_0$. The set P_n is a vector space.

(2) The set $M(m, n)$ of all $m \times n$ matrices is a vector space.

(3) The Euclidean space R^n is a vector space.

(4) The set $c[a, b]$ of all continuous functions on the closed interval $[a, b]$ is a vector space.

Note that all the examples mentioned above satisfied all axioms highlighted earlier.

Activity 1

Given two matrices $A = \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 0 & -3 \end{bmatrix}$, verify that M_{22} is a vector space.

Answer:

This is left for the reader to establish that all the axioms are satisfied.

Summary

The notion of vector space was discovered under ordinary addition and scalar multiplication.

Self Assessment Questions

1. What do you understand by vector space?
2. Verify that the Euclidean space R^n is a vector space.

Tutor Marked Assignment

Given the following vectors: $u = (0, 1, 2)$, $v = (3, 2, 4)$ and $w = (2, -1, 2)$, Compute (i)u-v (ii) $2u$+v (iii)$3u$+v-w (iv) Find the Euclidean distance between u, v.

References

- Morton L. Curtis,(2012). Abstract Linear Algebra. Springer. Switzerland.
- Steven Roman, (2005). Advanced Linear Algebra, Springer, USA.
- Gilbert Strang,(2016). Linear Algebra and its Applications, Wellesley-Cambridge Press.

Further Reading

- Math 2331- Linear Algebra by Jiwen He, page 1-21 Available online at: www.math.uh.edu/jiwenhe.

UNIT 3

Basis of a Vector Space

Introduction

Concepts of linear combination, linear dependency and independence of a subset of a vector space is important to the knowledge of basis of a vector space. Those terms shall be considered and used to introduce basis of a vector space.

Learning Outcomes

At the end of this unit, you should be able to:

1. define linearly dependent and independent vectors; and
2. obtain the basis of a vector space.

Main Content

Suppose that V is vector space and $v_1, ..., v_n$ is an ordered list of vectors from $V \neq 0$. If a vector x is a linear combination of $v_1, ..., v_n$, we have

$$\mathbf{x} = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \quad (2.1)$$

for some scalars $c_1, ..., c_n$. The equation (2.1) can be written in matrix notation as

$$x = [\, v_1 \;\; v_2 \;\; . \;\; . \;\; . \;\; v_n \,]. \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix} \quad (2.2)$$

The vectors $[v_1\ v_2 ...\ v_n]$ might seem a bit strange, since it is a row vector whose entries are vectors from V, rather than scalars. Nonetheless the matrix product in (2.2) makes sense. If you think about it, it makes sense to multiply a matrix of vectors with a (compatibly sized) matrix of scalars, since the entries in the product are linear combinations of vectors. It would not make sense to multiply two matrices of vectors together, unless you have some way to multiply two vectors and get another vector. You don't have that in a general vector space.

Linearly Dependent Vectors: If the vector x in (2.1) is a zero vector, but the scalars c_1, c_2, ..., c_n are not all zeros, then we say that the vectors v_1, v_2, ..., v_n are linearly dependent vectors.

Linearly Independent Vectors: f the vector x in (2.1) is a zero vector implies that all the scalars c_1, c_2, ..., c_n are zeros, then we say that the vectors v_1, v_2, ..., v_n are linearly independent vectors.

The two definitions above follows, since the set of vectors v_1, v_2, ..., v_n are non. zero. As we go further, we will understand better. Now, suppose that V is vector space over a field F with $V \neq 0$, then a maximal linearly in dependent subset of V is called a basis of V. The vectors in a basis are called basis vectors. Suppose that we have two ordered bases U and V of an n-dimensional vector space V. There is a unique matrix A so that $U = VA$. We will denote this matrix by $S_V U$. Thus, the defining equation of S_{VU} is $U = VS_{VU}$.

Proposition 2.1. Let V be an n-dimensional vector space. Let U, V and W be ordered bases of V. Then, we have the following.

(1) $S_{UU} = I$.

(2) $S_{UW} = S_{UV} S_{VW}$.

(3) $S_{UV} = [S_{VU}]^{-1}$.

Proof.

For the first statement, note that U = UI so, by uniqueness, SUU = I. For the second statement, recall the defining equations W = USUW , V = USUV , W = V SV W .

Compute as follows

$W = V S_{VW}$

$= (US_{UV})S_{VW}$

$= U(S_{UV}S_{VW})$

and so, by uniqueness, $S_{UW} = S_{UV} S_{VW}$.

For the last statement, set $W = U$ in the second statement. This yields $S_{UV} S_{VU} = S_{UU} = I$,

so S_{UV} and S_{VU} are inverses of each other.

The matrices S_{UV} tell you how to change coordinates from one basis to another, as detailed in the following Proposition.

Proposition 2.2. Let V be a vector space of dimension n and let U and V be ordered bases of V. Then we have

$[v]_U = S_{UV} [v]_V$, for all vectors $v \in V$.

Proof.

The defining equations are

$v = U[v]_U, v = V[v]_V$.

Thus, we have

$v = V[v]_V$

$= (US_{UV})[v]_V$

$= U(S_{UV} [v]_V)$,

and so, by uniqueness, $[v]_U = S_{UV} [v]_V$.

Thus, S_{UV} tells you how to compute the U-coordinates of a vector from the V-coordinates.

Activity 1

Consider the space P_3 of polynomials of degree less than 3. Of course, $[P = 1 \;\; x \, x^2\,]$ is an ordered basis of P_3. Consider

$$V = [2 + 3x + 2x^2 \quad 2x + x^2 \quad 2 + 2x + 2x^2].$$

Show that V is a basis of P_3 and find the transition matrices S_{PV} and S_{VP}. Let $p(x) = 1+x+5x^2$. Use the transition matrix to calculate the coordinates of $p(x)$ with respect to V and verify the computation.

Solution

By reading off the coefficients, we have

$$2 + 3x + 2x^2, 2x + x^2, 2 + 2x + 2x^2 = [1, x, x^2]. \begin{bmatrix} 2 & 0 & 2 \\ 3 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} \qquad (2.3)$$

Thus, $V = P\,A$, where A is the 3×3 matrix in the equation (2.3). A quick check with a calculator shows that A is invertible, so V is an ordered basis.

Equation (3×3) shows that $S_{PV} = A$. But then we have $S_{VP} = S_{PV}^{-1}$.

A calculation shows

$$S_{VP} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}$$

Since $p(x) = 1 + x + 5x^2$, we have

$$p(x) = [1 \;\; x \;\; x^2]. \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

So we have

What we want is $[\,p(x)]V$. From Proposition 2.2 we have

$$[p(x)]_V = S_{VP}[p(x)]_P = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \\ \frac{19}{2} \end{bmatrix}.$$

The significance of this calculation is that we are claiming that

$$1 + x + 5x^2 = p(x)$$

$$= V[p(x)]_V$$

$$= [2 + 3x + 2x^2, \quad 2x + x^2, \quad 2 + 2x + 2x^2]. \begin{bmatrix} -8 \\ 4 \\ \frac{19}{2} \end{bmatrix}.$$

$$= \quad 8(2 + 3x + 2x^2) + 4(2x + x^2) + \frac{19}{2}(2 + 2x + 2x^2).$$

The reader is invited to carry out the algebra to show this is correct.

Activity 2

Let U be the ordered basis of R^2 given by

$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(1) Show that U is a basis and find S_{EU} and S_{UE}.

(2) Let $y = [3\ 5]^T$. Find $[y]_U$ and express y as a linear combination of U.

Solution

Let A be the matrix such that $U = EA$. Then, as above,

$$A = mat(U) = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

The determinant of A is 2, so A is invertible. Then, since E is a basis, U is a basis.

We have $S_E U = A$, and so

$$S_{UE} = [S_{EU}]^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix}.$$

For the last part, we note that $[y]_E = y = [3\ 5]^T$ and we have

$$[y]_U = S_{UE}[y]_E = \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \end{bmatrix}.$$

By saying that

$$[y]_U = \begin{bmatrix} \frac{-1}{2} \\ 2 \end{bmatrix}.$$

We're saying that

$$\begin{aligned} \begin{bmatrix} 3 \\ 5 \end{bmatrix} &= y \\ &= U[y]_U \\ &= [u_1 \quad u_2] \begin{bmatrix} \frac{-1}{2} \\ 2 \end{bmatrix} \\ &= \frac{-1}{2} u_1 + 2u_2 \\ &= \frac{-1}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \end{aligned}$$

Summary

Linear combination of set of vectors was used to define linear dependence and independence of the vectors.

Self Assessment Questions

Let U be the basis $[x^2\ x\ 1]$ and let $V = [x^2 + x + 2, 4x^2 + 5x + 6, x^2 + x + 1]$ Show that V is a basis of P_3. Find the matrix of D with respect to V. Let $p(x)$ be the polynomial such that $[p(x)]_v = [2\ 1 - 1]^T$. Find the coordinates with respect to V of $p'(x)$.

Tutor Marked Assignment

1. Consider the ordered basis U of R^3 given by

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let V be the ordered basis of R^3 given by

$$v_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(1) Find S_{UV} and S_{VU}.

(2) Suppose that $[x]U = [1\ 2\ -1]^T$. Find [x]V and express x as a linear combination of the basis V. Verify.

References

- Steven Roman(2005). Advanced Linear Algebra, Springer, USA.
- Derek J.S. Robinson. (2006). A course in Linear Algebra with application, World Scientific Pub Co Inc.
- S. A. Ilori and O. Akinyele(2009). Elementary Abstract and Linear Algebra, Ibadan University Press, Ibadan.

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Module 3

Equivalence and Similarity Relation

Units

Unit 1 - Relation and Properties
Unit 2 - Equivalence Relation
Unit 3 - Similarity

UNIT 1

Relation and Properties

Introduction

Relation in algebra is a very important aspect that allows comparison and/or give specific property(ies) of two sets. In this unit, we shall be looking at the idea of relation from different angle of definitions.

Learning Outcomes

At the end of this unit, you should be able to:

1. give a concise definition of relation;
2. state properties of relation; and
3. list examples of related and unrelated sets.

Main Content

Cartesian Product: Let A and B be two non-empty sets. The Cartesian product $A \times B$ of the two sets is the set of the ordered pairs $\{(a_i, b_i) : a_i \in A, b_i \in B\}$.

Relation: A relation R between two sets A and B denoted as aRb is a subset of the Cartesian product $A \times B$, that is, a collection of ordered pairs (a, b).

Example: Let $X = c, d, e$ and $Y = 4, 5$. Obtain the Cartesian product (i) $X \times Y$ (ii) $Y \times X$ (iii) is $Q = (c, 4),(c, 5),(e, 5)$ a relation from X to Y ?

Solution

(i) $X \times Y = (c, 4),(c, 5),(d, 4),(d, 5),(e, 4),(e, 5)$

(ii) $Y \times X = (4, c),(4, d),(4, e),(5, c),(5, d),(5, e)$

(iii) Yes (because all members of $Q \in X \times Y$

Properties of Relation

Let A and B be two sets and R denote a relation on the set A to itself of to B, then the following properties exist:

1. **Reflexivity:** For a relation R on set A to A, *ie* $R \subset A \times A$, for all $a \in A, (a, a) \in R$.
2. **Symmetry:** for all $a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$.
3. **Transitivity:** for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Activity 1

Let $X = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and let R be a relation on set X defined by $R_{fact} = \{(x, y) : y$ is a multiple of $x, \forall x, y \in X\}$ (a) list all the elements of R_{fact} (b) is R_{fact} reflexive? (c) is R_{fact} symmetric? (d) is R_{fact} transitive?

Solution

(a) $R_{fact} = (2, 4),(2, 6),(2, 8),(3, 6),(3, 9)$

Activities (b),(c) and (d) are left for you to answer

Summary

Cartesian product of two sets is an ordered pair of each elements of one set with respect to all the elements of the other set while a relation is a subset of the Cartesian product.

Self Assessment Questions

Define (i) Cartesian Product of set X and Y (ii) Relation of set X and Y.

Tutor Marked Assignment

Find the Cartesian product $A \times B$ if $A = \{a, e, i, o, u\}$ and $B = \{x : 0 \leq x \leq 10, x \text{ is even}\}$.

References

- Morton L. Curtis(2012). Abstract Linear Algebra, Springer, Switzerland.
- Steven Roman(2005). Advanced Linear Algebra, Springer, USA.
- Gilbert Strang (2016). Linear Algebra and its Applications, Wellesley-Cambridge Press.

Further Reading

- Jiwen He, Linear Algebra. Available online at: www.math.uh.edu/jiwenhe.

UNIT 2

Equivalence Relation

Introduction

Relations are one of the basic building blocks of Mathematics. It can be said that for every $a \in A$, and $b \in B$, a is related to b under a relation R if the pair (a, b) belongs to the subset and we write aRb. In this unit, special consideration shall be given to equivalence relation.

Learning Outcomes

At the end of this unit, you should be able to:

1. define explicitly an equivalence relation; and
2. give some examples of an equivalence relation.

Main Content

Definition: An equivalence relation on any set S is a relation on S which satisfies reflexive, symmetry and transitive axioms. Equivalence relation is often denoted with the symbol $\sim$(tilde), i.e. $x \sim y \, \forall x, y \in S$.

Activity 1

Given set $S = <$ {set of all real numbers} and defines a relation R as"*square*", that is, $R = (u, v) | u^2 = v^2$. The square relation is an equivalence relation since all the axioms are satisfied, i.e.

(i) for all $x \in \Re, x^2 = x^2$, so $(x, x) \in \Re$.

(ii) If $(x, y) \in \Re, x^2 = y^2$, so $y^2 = x^2$ and $(y, x) \in \Re$.

(iii) If $(x, y) \in \Re$ and $(y, z) \in$, then $x^2 = y^2 = z^2 \in$, so $(x, z) \in \Re$.

Activity 2

Suppose set $X = Z$ (set of all integers) and R be a relation "*parity*", i.e. $R = \{(a, b)|a, b$ *have the same parity*$\}$. Then R is an equivalence relation since

(a) for any $a \in Z$, a has the same parity as itself.

(b) for $(a, b) \in Z$, a and b have the same parity, so $(b, a) \in Z$.

(c) if $(a, b) \in Z$ and $(b, c) \in Z$, then a, c have the same parity as y, thus $(a, c) \in Z$.

Equivalence Class: Let R be an equivalence relation and set S, then equivalence class is defined as the class containing and element x of S by: $[x]_R = \{\ y : (x,y) \in R\}$ or $\{\ y : xRy\}$.

Example, if $\sim$ is the relation "having the same parity" on Z, then $[2] = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$etc.

Summary

Any relation that satisfies all the three axioms of reflexivity, symmetry and transitivity is an equivalence relation.

Self Assessment Questions

1. Define an equivalence relation.
2. Give two examples of equivalence relation.
3. What do you understand by equivalence class?

Tutor Marked Assignment

Show whether the set $A = \{1, 2, 3, 4\}$ is equivalence on the field $\Re <$ or not.

References

- Morton L. Curtis(2012). Abstract Linear Algebra, Springer, Switzerland.
- Gilbert Strang(2016). Linear Algebra and its Applications, WellesleyCambridge Press.

Further Reading

- Steven Roman(2005). Advanced Linear Algebra, Springer, USA.

UNIT 3

Similarity

Introduction

Similarity can be formalized within an algebra of binary relations as well as in matrices. In this unit, we shall be considering similarity in the two context listed above.

Learning Outcomes

At the end of this unit, you should be able to:

1. define equality, similarity relation; and
2. List the properties of similar relation.

Main Content

Equality Relation: A relation E on a non-empty set S is an equality relation if $E = \{(x, y) : x = y \forall x, y \in S\}$. Thus equality relations is a special case of equivalence relation (the least equivalence relation on set S).

Equivalence relation are defined as those relation which are reflexive, symmetry and transitive. It corresponds in a bijective manner to partitions of the set S on which they are defined. Thus $E \leq E^* = E^2$(that is, equality $\leq$ symmetry $\leq$ Reflexive $\leq$ transitive). Similarity distinguish non-trivial types in contrast to the familiar type of equivalence. Relation of similarity (otherwise known as tolerance relation) is the basic tool of analogy that permits mutual comparative analysis which in turn gives an insight into the comparison between abstraction and analogy. Thus, for a binary relation R on a given set S, a relation T_R (tolerance relation) can be defined as follows: $T_R = RR^*$

1. T_R is a symmetric relation in S.
2. $\forall x, y \in S; xT_R y$ *iff* $R(x) \cup R(y) = \phi$.
3. T_R is a tolerance *iff* R is defined everywhere (equivalent to $E \leq RR^*$).

4. T_R is a weak tolerance *iff* R is weakly reflexive.

5. R is a function implies T_R is an equivalence relation.

6. $\forall$ relation R which is a function and $T = T_R$, then T is an equivalence relation.

7. T is an equivalence relation *iff* $T = T_T$ for $E \leq T$.

In the context of matrices, the relationship between the matrix representation A_T of a linear transformation $T: V_n(F) \to V_n(F)$ with respect to a basis $\{v_1, ..., v_n\}$ and its matrix representation B_T with respect to another basis $\{v'_1, ..., v'_n\}$ is that

$$B_T = P^{-1} A_T P$$

where P is the non-singular matrix associated with the change of coordinates from the basis $\{v_1, ..., v_n\}$ to the basis $\{v'_1, ..., v'_n\}$, i.e

$$P = (p_{ij})_{n,n}, \quad \text{where } v'_v = \sum_{i=1}^{n} P_{ij} v_i.$$

Two square matrices A and B in $M_n(F)$ are similar if there is a non-singular matrix P in $Gl_n(F)$ such that $B = P^{-1}AP$.

Example

Given

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$$

Calculation shows that

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

The relation of similarity among matrices in $M_n(F)$ is an equivalence relation as can be shown as follows:

Reflexive Law:

Since a square matrix A in $M_n(F)$ can be written as $A = I^{-1}AI$ (where $I^{-1} = I$ is the identity matrix of order n), It follows that $A \sim A$.

Symmetric Law:

Suppose $A \sim B$, where A and B are matrices in $M_n(F)$. It follows that there exists a non-singular matrix $P \in Gl_n(F)$ such that $B = P^{-1}AP$. Now the last relation can be written as

$$A = PBP^{-1} \text{ or } A = (P^{-1})^{-1} . B(P^{-1}).$$

This implies that $B \sim A$.

Transitive Law:

Suppose $A \sim B$ and $B \sim C$, where A, B, C are matrices in $M_n(F)$. Then there exists $P, Q \in Gl_n(F)$ such that since $P, Q \in Gl_n(F)$ implies $PQ \in Gl_n(F)$, it follows that $A \sim C$. It follows that two similar matrices represent the same linear transformation with respect to two bases which may be different. We then have the following facts to state about similar matrices.

1. Similar matrices in $M_n(F)$ have the same determinant.
2. Similar matrices in $M_n(F)$ have the same characteristic roots.
3. Similar matrices in $M_n(F)$ have the same characteristic polynomial.
4. Similar matrices in $M_n(F)$ have the same minimal polynomial in $F(x)$; and
5. Similar matrices in $M_n(F)$ have the same rank.

Summary

Equality, equivalence and similarity relations were presented and their characteristics together with relationship were established.

Self Assessment Questions

check that these matrices are similar or not

$$\begin{pmatrix} 1 & 8 & 0 \\ 1 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 & 0 & 2 \\ 0 & 3 & 1 \\ 3 & 2 & 4 \end{pmatrix}$$

Tutor Marked Assignment

Show that these matrices are not similar.

$$\begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & 1 & 7 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

For $S = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix}$ $T = \begin{pmatrix} 0 & 0 \\ -\frac{11}{2} & -5 \end{pmatrix}$ $P = \begin{pmatrix} 4 & 2 \\ -3 & 2 \end{pmatrix}$

Show that $T = PSP^{-1}$.

References

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- Derek J.S. Robinson(2006). A course in Linear Algebra with Applications, World Scientific Pub Co Inc.
- S. A. Ilori and O. Akinyele (2009). Elementary Abstract and Linear Algebra, Ibadan University Press, Ibadan.

Further Reading

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Module 4

Eigenvalues and Eigenvectors

Units

UNIT 1

Eigenvalues

Introduction

For a square matrix A, an eigenvalues are the scalars λ_i that form the roots of the characteristics polynomial obtained when the determinant of equation $|A-\lambda I| = 0$ is solved. In this, unit, some basic definitions of eigenvalues shall be presented with examples.

Learning Outcomes

At the end of this unit, you should be able to:

1. define an eigenvalue; and
2. compute eigenvalues for $n \times n$ matrix

Main Content

The Eigen value problem is the following: Given a linear transformation:

$T: V_n(F) \to V_n(F)$

Determine the scalars $\lambda \in F$ and those non zero vectors $v \in V_n(F)$ which satisfy the equation

$$T(v) = \lambda v.$$

λ is called an eigenvalue or characteristic value of T and the vector v is called an eigenvector or characteristic vector of T corresponding to the eigenvalue λ. If A is an $n \times n$ matrix, a scalar $\lambda \in F$ for which there is some non zero column n vector x (with entries from F) such that

$$Ax = \lambda x.$$

is said to be an eigenvalue of A and x is said to be an eigenvector of A. If matrix $A \in M_n(F)$ corresponds to the linear transformation T, then their eigenvalues and eigenvectors coincide. Note that the eigenvalues are the fixed direction of the linear transformation T.

Also of x is an eigenvector than all the vectors in the 1–dimensional vector space over F generated by x are also eigenvectors. Since $T(v) = \lambda v$ is equivalent to $(T - \lambda I)v = 0$, where I represents the identity linear transformation on Vn(F), it follows that an eigenvector v of T belongs to the nullspace of the linear transformation T corresponding to the eigenvalue λ is called the eigenspace of T corresponding to the eigenvalue. it is therefore , the nullspace of $T - \lambda I$ and so a subspace of $V_n(F)$. Similarly, $Ax = \lambda x$ is equivalent to $(A - \lambda I)x = 0$, where I is the identity matrix of order n and so the set of all eigenvector of matrix A corresponding to the eigenvalue λ is called the eigenspace or row null space of A corresponding to the eigenvalue λ. $(A - \lambda I)x = 0$ corresponds to a homogeneous system of linear equations, Besides the trivial solution $x = 0$ non trivial solutions exist if

$$|A - \lambda I| = 0$$

i.e if matrix $A - \lambda I$ is singular. This last equation in λ is called the characteristic equation of the matrix A and the n-roots $\lambda_1, \lambda_2, ..., \lambda_n$ are the characteristic roots or eigenvalues of A. We refer to the polynomial in λ as the characteristic polynomial of A. Note that

$$CA(\lambda) = |A - \lambda I| = (-1)^n \lambda^n + b_n - 1\lambda^n - 1 + ... + b_1\lambda + b_0.$$

Activity 1

Find the eigenvalue of the matrix $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution

We solve for $|A - \lambda I| = 0$ where λ is the root of the polynomial obtained and I is an identity matrix (of the same order as matrix A). Thus,

$$|A - \lambda I| = \begin{vmatrix} -1 - \lambda & 2 \\ 3 & 4 - \lambda \end{vmatrix} = (-1 - \lambda)(4 - \lambda) - 6 = 0$$

$$\Longrightarrow \quad -4 + \lambda - 4\lambda + \lambda^2 - 6 = 0.$$

so that $\lambda^2 - 3\lambda - 10 = 0$. The roots of the quadratic equation give the eigenvalues of the matric, thus $\lambda 1 = -2$ and $\lambda^2 = 5$.

Summary

The eigenvalues of a square matrix are the roots of the characteristic polynomial obtained from solving the determinant equation of $|A-\lambda I|$.

Self Assessment Questions

1. Given that $A = \begin{bmatrix} 4 & -1 & 2 \\ 5 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$, obtain the eigenvalues of A.

2. Is it possible to obtain the eigenvalues of $B = \begin{bmatrix} 4 & -5 & -1 & 2 \\ 2 & -7 & 3 & -2 \\ 1 & 2 & 4 & -3 \end{bmatrix}$?

   Give your reason(s).

Tutor Marked Assignment

Find the eigenvalues of the following matrices:

(i) $A = \begin{bmatrix} 0 & -3 & 7 \\ 6 & -3 & -4 \\ -2 & 6 & 4 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & -2 \\ 5 & 4 & 2 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

References

- Morton L. Curtis(2012) Abstract Linear Algebra, Springer, Switzerland.
- Steven Roman(2005). Advanced Linear Algebra, Springer, USA.

Further Reading

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UNIT 2

Minimal Polynomial

Introduction

Minimal polynomial of an operator is a product of distinct irreducible monic polynomials. Thus, in this unit, extensive definition of minimal polynomial with its characteristics shall be discussed.

Learning Outcomes

At the end of this unit, you should be able to:

1. define minimal polynomial; and
2. state some characteristics of minimal polynomial.

Main Content

Every $n{\times}n$ matrix $A \in M_n(F)$ yields polynomials $F(A) = a_0I + a_1A + ... + a_kA^k$, which are again members of $M_n(F)$. Since there are exactly n^2 linearly independent $n \times n$ matrices over F i.e $dimM_n(F) = n^2$, the $n^2 + 1$ matrices $I, A, ..., A^{n2}$ are linearly dependent and the dependence relation provides a non zero polynomial $f(x)$ of degree at most n^2 with $F(A) = 0$. Since there is an F-isomorphism between $M_n(F)$ and the vector space $\psi(V_n(F), V_n(F))$ of all linear transformations from $V_n(F)$ to $V_n(F)$ we have that for every linear transformation T of $V_n(F)$ there exists a non-zero polynomial $f(x)$ with $f(T) = 0$. Now, lets consider the set M of all polynomials $f(x)$ over F such that $f(T) = 0$ or $f(A) = 0$. Then M is an ideal of $f(x)$ i.e M is an abelian group under addition and M is closed under multiplication by any polynomial in $F(x)$. thus $M = \langle m(x) \rangle$ is an ideal generated by a unique monic polynomial $m(x)$ (since $F(x)$ is a principal ideal domain where every ideal is principal).

The minimal polynomial of a linear transformation T, denoted by $m(x)$, is the generator of the ideal of all polynomial $f(x)$ over F satisfying $f(T) = 0$. Thus $m(x)$ satisfies the properties

$$m(T) = 0 \quad \text{and} \quad f(T) = 0 \Rightarrow m(x) | f(x)$$

The minimal polynomial of a matrix is defined in a similar way. It is identical with the minimal polynomial of the corresponding linear transformation T_A of $V_n(F)$. For each monic polynomial $f(x) = c_0 + c_1x + ... + c_n - 1x^{n^1}$ of degree n, we can construct an $n \times n$ matrix with minimal polynomial $f(x)$. This matrix is called the companion matrix of $f(x)$ and is given by:

$$C_f = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & \\ . & . & . & 1 & . & . & \\ . & . & . & . & 1 & . & \\ . & . & . & . & . & 1 & \\ c_0 & c_1 & . & . & . & c_n & 1 \end{pmatrix}$$

with zero entries except for entries 1 in the diagonal above the main diagonal and entries $-c_0, ..., -c_{n-1}$ in the last row. we then have the following fact: For each monic polynomial $f(x)$, the companion matrix Cf has minimal polynomial $f(x)$ and characteristic polynomial $(-1)^n f(\lambda)$. Note that $Cf - (\lambda)I = (-1)^n f(\lambda)$, with $(-1)^n f \lambda^n$ as its leading term.

Summary

1. Minimal polynomial is a product of distinct real linear and real quadratic factors.

2. Minimal polynomial is a unique monic order space over $T(V_T)$ that generates annihilator $(a_{nn}(V_T))$.

3. It is the unique monic polynomial $M_T(x)$ of smallest degree for which $M_T(T) = 0$.

4. If A is a square matrix over a field F, the minimal polynomial $M_A(x)$ of A is defined as the unique monic polynomial $P(x) \in F(x)$ of smallest degree for which $P(A) = 0$

5. If A and B are similar matrices, then $M_A(x) = M_B(x)$. Thus, the minimal polynomial is an invariant under similarity.

Self Assessment Questions

1. List the characteristics of minimal polynomial.

Tutor Marked Assignment

- Prove that Minimal polynomial is a product of distinct real linear and real quadratic factors.

References

- Derek J.S. Robinson (2006). A course in Linear Algebra with application. World Scientific Pub Co Inc.
- Gilbert Strang(2016). Linear Algebra and its Applications, WellesleyCambridge Press.

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UNIT 3

Characteristics Polynomial

Introduction

The Characteristics Polynomial is an important polynomial, the roots of which gives the eigenvalues and hence their corresponding eigenvectors. In this unit, the concept of characteristics polynomial shall be discussed with some fundamental principles.

Learning Outcomes

At the end of this unit, you should be able to:

1. define a Characteristics Polynomial;
2. distinguish between Characteristics Polynomial and minimal polynomial;
3. obtain Characteristics Polynomial for n × n matrix.

Main Content

Definition 1: Let τ be a linear operator on a finite dimensional vector space V, the Characteristics Polynomial $C_T(x)$ of τ is defined to be the product of the elementary divisors of τ. If M is any square matrix that represent τ, then $C_T(x) = C_M(x) = det(xI - M)$.

Definition 2: The characteristics polynomial of the $n \times n$ matrix B equals $\sum_i^n a_i(-x)^{n-i}$ where a_i is the sum of all the $i \times i$ sub determinants of $det(B)$ whose principal diagonals are part of the principal diagonal of B.

Definition 3: Similar matrices have the same characteristics polynomial hence they have the eigenvalues, trace and determinants.

Definition 4: The minimal polynomial and the characteristic polynomial of a linear operator $\tau \in L(V)$ have the same set of prime factors.

Activity 1

Find the characteristic polynomial for the matrix A over R

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

and show by direct substitution, that this matrix satisfies its characteristic equation. Furthermore, find its characteristic roots, characteristics vectors and minimal polynomial of A.

Solution

The characteristic polynomial is given by

$$C_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(2-\lambda)^2 - (1)(2-\lambda-1) + (1)(1-(2-\lambda))$$

$$= (2-\lambda)^3 - (2-\lambda) + 1 + 1 - (2-\lambda)$$

simplifying further, we have

$$= \lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$$= (\lambda - 1)^2 \ (4 - \lambda)$$

The characteristic roots are the roots of the characteristic equation. Therefore,

$$= (\lambda - 1)^2(4 - \lambda) \Rightarrow \lambda = 1,1,4.$$

i.e the characteristic roots of A are $\lambda = 1, 1, 4$.

Next we find the characteristic vectors, which are basis for the eigenspaces corresponding to $\lambda = 1, 1, 4$.

When $\lambda = 1$

$$(A - \lambda I) = (A - I) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

First, we obtain the row-reduced Echelon form of $(A - I)$.

$$(A - I) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the row-reduced Echelon form of $A - I$, we have $x + y + z = 0$ put $z = a, y = b$, then $x = -a - b$. Hence, the solution space of the homogenous system $(A - I)x = 0$ is $\{(x, y, z) = a(-1, 0, 1) + b(-1, 1, 0) | a, b \in R\}$ i.e a basis for the eigenspace of A corresponding to $\lambda = 1$ is $\{(-1, 0, 1), (-1, 1, 0)\}$.

When $\lambda = 4$

$$(A - \lambda I) = (A - 4I) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

First, we obtain the row-reduced Echelon form of $(A - \mathrm{I})$.

$$(A - 4I) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{matrix} R_1(\frac{-1}{2}) \\ R_2 - R_1(\frac{-1}{2}) \\ R_3 - R_1(\frac{-1}{2}) \end{matrix} \begin{pmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{-3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & (\frac{-3}{2}) \end{pmatrix} \begin{matrix} R_2(\frac{-2}{3}) \\ R_3 + R_2 \end{matrix} \begin{pmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

From the row-reduced Echelon form of $A - 4I$, we have

$x - \frac{1}{2}y - \frac{1}{2}z = 0$

$\mathrm{y} - \mathrm{z} = 0.$

put $z = a$.Then $y = a$ and $x = \frac{1}{2}a + \frac{1}{2}a = a$

hence the solution space of the homogenous system $(A-4I)x = 0$ is $\{(x, y, z) = a(1, 1, 1) | a \in R\}$. i.e a basis for the eigenspace of A corresponding to $\lambda = 4$ is $\{(1, 1, 1)\}$. Thus a basis for the characteristic vectors of A is $\{(-1, 0, 1), (-1, 1, 0), (1, 1, 1)\}$. The minimal polynomial $m(x)$ of A must divide the characteristic polynomial $C_A(x)$ of A. Since $C_A(x) = (x - 1)2\ (4 - x)$, it follows that the minimal polynomial $m(x)$ of A is one of the divisor of $(x - 1)2\ (4 - x)$, namely $(x - 1), (x - 4), (x - 1)2, (x - 1)2\ (x - 4), -C_A(x)$.

Note that $m(x)$ must be a monic polynomial i.e its leading coefficient must be equal to +1. By testing each of the above divisors to know which one gives zero when $x = A$, we find that the minimal polynomial is $(x-1),(x-4) = x^2 - 5x + 4$. Since

$$(A - I)(A - 4I) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that the minimal polynomial may also be obtained as follows: First try to solve $A = \alpha_0 I$. There is no solution.

Next try to solve $A^2 = \alpha_1 A + \alpha_0 I \Rightarrow \alpha_1 = 5, \alpha_0 = -4$. Hence, the minimal polynomial is $m(x) = x^2 - 5x + 4$.

Summary

The algebraic multiplicity of λ(eigenvalues) is the multiplicity of λ as a root of the characteristic polynomial.

Self Assessment Questions

1. Differentiate between minimal polynomial and characteristics polynomial.

2. Given matrix $A = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 4 & 2 \\ 3 & 2 & 4 \end{bmatrix}$, obtain the trace, determinant and characteristic polynomial of the matrix.

3. Find the minimal and characteristic polynomial for each of the following matrices over R

$$M = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

Tutor Marked Assignment

1. Find the minimal and characteristic polynomial for each of the following matrices over R

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 2 & -10 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 2 & -10 & 2 \end{pmatrix}$$

References

- Steven Roman(2005). Advanced Linear Algebra, Springer, USA.
- Derek J.S. Robinson (2006). A course in Linear Algebra with application. World Scientific Pub Co Inc.
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UNIT 4

Eigenvectors

Introduction

An eigenvector of an $n \times n$ matrix A is a non-zero n−column vector X such that $AX = \lambda X$ for some scalar λ (called the eigenvalues). In this unit, we shall introduce eigenvector concept using worked examples.

Learning Outcomes

At the end of this unit, you should be able to:

1. define an eigenvector for a square matrix; and
2. calculate eigenvectors of a square matrix.

Main Content

Definition 1: The eigenvectors of a square matrix A associated with an eigenvalue λ_i are non-zero vectors in the null space of the matrix $A - \lambda_i I_n$. The set of all eigenvectors associated with a given eigenvalue λ_i, together with the zero vector forms a subspace of V (which is referred to as the eigenspace of λ_i and denoted as $E\lambda$). The above definition applies to both linear operator and matrices.

Activity 1

Obtain the eigenvector of the matrix $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

Solution

The characteristic polynomial is obtained from $|B - \lambda I| = 0$ i.e. $\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 0-\lambda \end{vmatrix}$

$$= 0 \Rightarrow -\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

$\therefore \lambda = -1, 2, 3$ are the roots of the characteristic polynomial and hence the eigenvalue. To find the eigenvectors associated to each eigenvalue, we have: For $\lambda = -1$,

$$BX = \lambda X, \Longrightarrow (B - \lambda I)X = 0$$

where I is a 3×3 identity matrix. On solving, we have

$$X = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Similarly for $\lambda = 2, X = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and for $\lambda = 3, X = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$.

Summary

The eigenvector for an eigenvalue λ corresponds to the column vector solution of $AX = \lambda X$ for any square matrix A. Also,

1. The eigenvalue of a matrix A are the roots of the characteristic equation of A.

2. If $v_1, \ldots, v_n$ are nonzero eigenvectors of a linear transformation T, corresponding to the distinct eigenvalues then they are linearly independent. Hence, if $T: V_n(F) \to V_n(F)$ has n distinct eigenvalues, $\lambda_1, \ldots, \lambda_n$ belonging to eigenvectors, $v_1, \ldots, v_n$ then $\{v_1, \ldots, v_n\}$ is a basis for $V_n(F)$. The matrix of T w.r.t this basis is the diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ . & . & . & . \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

3. The characteristic polynomial of a triangular matrix A with diagonal entries $d_1, \ldots, d_n$ is $|A - \lambda I| = (d_1 - \lambda)(d_2 - \lambda)\ldots(d_n - \lambda)$.

4. The minimal polynomial may also be found as follows: try to solve the equation (beginning with $i = 1$) $A_i = \alpha_{i-1}A^{i-1} + \ldots + \alpha_i A + \alpha_0 I$, $i = 1, 2, \ldots$ for the coefficients $\alpha_{i-1}, \alpha_i, \alpha_0$. The first i for which solutions exist provides the minimal polynomial which is then $m(x) = x^i - \alpha_{i-1} x^{i-1} \ldots - \alpha_i x - \alpha_0$.

Self Assessment Questions

1. Given matrix $A = \begin{bmatrix} 2 & 4 & -3 \\ 3 & -4 & 2 \\ -3 & 2 & -4 \end{bmatrix}$, obtain the eigenvector.

2. Find the eigenvector for each of the following matrices over R.

$$M = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

Tutor Marked Assignment

1. Find the eigenvectors for each of the following matrices over R,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 2 & -10 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 2 & -10 & 2 \end{pmatrix}$$

References

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Module 5

Orthogonal Diagonalization, Canonical form

Units

Unit 1 - Orthogonal Diagonalization, Canonical form

UNIT 1

Orthogonal Diagonalization

Introduction

The algebra of diagonal matrices is very easy. To add or multiply two diagonal matrices, one just adds or multiplies corresponding diagonal matrices. Thus it is of interest to know which matrices are similar to diagonal matrices and which pairs of diagonal matrices are similar to each other. A very important properties in algebra of matrices involving diagonal matrix shall be discussed in this section. A diagonal matrix is a square matrix with non-zero elements only in the diagonal running from the upper left to the lower right.

Learning Outcomes

At the end of this unit, you should be able to:

1. define a diagonal matrix;
2. perform diagonalization of any square matrix;
3. add or multiply two diagonal matrices; and
4. show that two matrices are similar.

Main Content

A matrix A in $M_n(F)$ is similar to a diagonal matrix D if and only if the characteristic vectors of A span $V_n(F)$; and if this is the case, the characteristic roots of A are the diagonal entries in D. In particular, it follows that the characteristic roots of a diagonal matrix are the entries on the diagonal.

If a matrix A and B in $M_n(F)$ have the same set of n distinct characteristic roots, then they are similar. If a matrix P in $M_n(F)$ is a matrix whose columns are n linearly independent characteristic vectors of a matrix A in $M_n(F)$ then P is non-singular and $P^{-1}AP$ is a diagonal matrix.

1. To construct, (If it exists) a diagonal matrix similar to a given matrix, one computes the characteristic roots and vectors.

2. It can also be shown that a matrix is similar to a diagonal matrix if and only if the linear factors of its minimal polynomial are distinct.

3. There are matrices which are NOT similar to any diagonal matrix. e.g

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \alpha \neq 0$$

Activity 1

Find a matrix P over the field R of real numbers such that $P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{pmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix} \in M_3(R)$$

and exhibits a diagonal form.

Solution

First, we obtain the characteristic polynomial of A

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix}$$

$(8-\lambda)[(-3-\lambda)(1-\lambda)-8]-(-8)[4(1-\lambda)-(-6)]+(-2)[-16-3(-3-\lambda)] = (8-\lambda)(3+\lambda)(1-\lambda)-8(8-\lambda)+(-8)[(4-4\lambda)+6]-2[-16+9+3\lambda)] = (\lambda-8)(3-2\lambda-\lambda^2)-64+8\lambda+32-32\lambda+48+14-6\lambda = 3\lambda-2\lambda^2-\lambda^3-24+16\lambda+8\lambda^2-64+8\lambda+32-32\lambda+62-6\lambda$

$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$

$= (\lambda - 1)(-\lambda^2 + 5\lambda - 6).$

$= (\lambda - 1)(-\lambda^2 + 3\lambda + 2\lambda - 6).$

$= (\lambda - 1)\ [(-\lambda(\lambda - 3) + 2(\lambda - 3)]\ .$

$= (\lambda - 1)(2 - \lambda)(\lambda - 3).$

$\lambda = 1, 2, 3.$

This implies that the characteristic roots of A are the roots of the characteristic equation, $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$. Hence, the characteristic roots of A are 1, 2, 3. Next, find the characteristic vectors which form bases for the eigenspace corresponding to $\lambda = 1$ is equal to the row null space of $A - I$, it is therefore equal to the solution space of the homogeneous system of linear equations $(A - I)x = 0$.

Obtain the row-reduced Echelon form of $A - I$.

$$(A - I) = \begin{pmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{pmatrix} \quad R_1(\frac{1}{7}) \begin{pmatrix} 1 & \frac{-8}{7} & \frac{-2}{7} \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{pmatrix}$$

$$\begin{matrix} R_2 - 4R_1 \\ R_3 - 3R_1 \end{matrix} \begin{pmatrix} 1 & \frac{-8}{7} & \frac{-2}{7} \\ 0 & \frac{4}{7} & \frac{-6}{7} \\ 0 & \frac{-4}{7} & \frac{6}{7} \end{pmatrix} \quad \begin{matrix} R_2(\frac{7}{4}) \\ R_3 + R_1 \end{matrix} \begin{pmatrix} 1 & \frac{-8}{7} & \frac{-2}{7} \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

From the row reduced Echelon form of $A - I$, we have $x - \frac{8}{7}y - \frac{2}{7}z = 0$

$y - \frac{3}{2}z = 0$.

put $z = 2a$. Then $y = 3a$ and $x = \frac{24a}{7} + \frac{4a}{7} = 4a$

hence the solution space of the homogenous system $(A-I)x = 0$ is $\{(x, y, z) = a(4, 3, 2) | a \in R\}$ i.e a basis for the eigenspace of A corresponding to $\lambda = 1$ is $\{(4, 3, 2)\}$.

For $\lambda = 2$: The eigenspace of A corresponding to the homogeneous system o linear equations $(A-2I)x = 0$ Obtain the row reduced Echelon form $A-2I$.

$$(A - 2I) = \begin{pmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{pmatrix} \quad R_1(\frac{1}{6}) \begin{pmatrix} 1 & \frac{-4}{3} & \frac{-1}{3} \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{pmatrix}$$

$$\begin{matrix} R_2 - 4R_1 \\ R_3 - 3R_1 \end{matrix} \begin{pmatrix} 1 & \frac{-4}{3} & \frac{-1}{3} \\ 0 & \frac{1}{3} & \frac{-2}{3} \\ 0 & 0 & 0 \end{pmatrix} \quad R_2(3) \begin{pmatrix} 1 & \frac{-4}{3} & \frac{-1}{3} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

From the row reduced Echelon form of $A - 2I$, we have $x - \frac{4}{3}y - \frac{1}{3}z = 0$

$y - 2z = 0$.

put $z{=}a$. Then $y = 2a$ and $x = \frac{8a}{3} + \frac{1a}{3} = 3a$

hence the solution space of the homogenous system $(A-2I)x = 0$ is $\{(x, y, z) = a(3, 2, 1) | a \in R\}$ i.e a basis for the eigenspace of A corresponding to $\lambda = 2$ is $\{(3, 2, 1)\}$.

For $\lambda = 3$: The eigenspace of A corresponding to the homogeneous system of linear equations $(A-3I)x = 0$ Obtain the row reduced Echelon form $A-3I$.

$$(A - 3I) = \begin{pmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{pmatrix} \quad R_1(\frac{1}{5}) \begin{pmatrix} 1 & \frac{-8}{5} & \frac{-2}{5} \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{pmatrix}$$

$$\begin{matrix} R_2 - 4R_1 \\ R_3 - 3R_1 \end{matrix} \begin{pmatrix} 1 & \frac{-8}{5} & \frac{-2}{5} \\ 0 & \frac{2}{5} & \frac{-2}{5} \\ 0 & \frac{4}{5} & \frac{-4}{5} \end{pmatrix} \quad \begin{matrix} R_2(\frac{5}{2}) \\ R_3 - 2R_2 \end{matrix} \begin{pmatrix} 1 & \frac{-8}{5} & \frac{-2}{5} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

From the row reduced Echelon form of $A - 3I$, we have $x - \frac{8}{5}y - \frac{2}{5}z = 0$

y-2z=0.

put $z{=}a$. Then $y = a$ and $x = \frac{8a}{5} + \frac{2a}{5} = 2a$

Hence, the solution space of the homogenous system $(A - 3I)x = 0$ is $\{(x, y, z) = a(2, 1, 1) | a \in R\}$

i.e a basis for the eigenspace of A corresponding to $\lambda = 2$ is $\{(2, 1, 1)\}$. The required real matrix is

$$P = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Note that $|p| = -1$ which implies that P^{-1} exists. Hence, A is reducible to a diagonal form given by

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Canonical Forms

A homogeneous quadratic form in n-variables $x_1, ..., x_n$ is defined as a polynomial

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i c_{ij} x_j .$$

In which each term is of second degree. This form can be written in matrix notation as

$$\sum_{i} \sum_{j} x_i c_{ij} x_j$$

Where $C = (c_{ij})_{n,n}$ and x = column $(x_1, ..., x_n)$.

If as above, $C = A + B$, such that $A^T = A$, $B^T = -B$. Then,

$x^T Cx = x^T(A + B)x = x^TAx + x^TBx = x^TAx$

Since, $x^TBx = 0$.

Thus, any homogeneous quadratic form in n variables any be expressed uniquely as x^TAx, where $A = (a_{ij})_{n,n}$ is a symmetric matrix.

The problem which we shall be concerned with is how to reduce a homogenous quadratic form to a sum of squares. This may be done by a change of coordinates by a non-singular matrix A in the quadratic form x^TAx into a diagonal matrix D in which case, $D = P^TAP$, (since $x–Py \Rightarrow x^TAx = (Py)^T(Py) = y^T(P^TAP)y$). The diagonal quadratic form to which the original quadratic form is reduced can be expressed as

$$d_1 y_1^2 + d_2 y_2^2 + d_r y_r^2$$

each $d_i \neq 0$.

The number r of non-zero diagonal terms is an invariant called the rank of the given quadratic form. Any real quadratic form can be further reduced by non-singular linear transformation of the variables to a form

$$z_1^2 + ... + z_p^2 - z_{p+1}^2 - ... - z_r^2$$

which is called the CANONICAL FORM of the real quadratic form. There is another invariant, called the SIGNATURE 's' of the real quadratic form which is given by a = number of positive squares - number of negative squares, in the canonical form.

If a real quadratic form x^TAx is reduced to a diagonal form by an orthogonal transformation, then the diagonal matrix D of this new form is given by $D = P^TAP$, where P is the orthonormal matrix, i.e. $D = P^{-1}AP$, since $P^T = P^{-1}$ for P orthonormal. Hence, the new matrix D and the orthogonal matrix A are similar. We can then apply the techniques which we have developed under similar matrices to compute P and D for any real quadratic form. Note that D is obtained by computing the characteristic roots of A (in which case D is essentially unique except for the order of the diagonal elements) and P is obtained by computing the normalized characteristic vectors of A.

A quadratic form x^TAx in n-variables $x_1, ..., x_n$ is said to be non-singular if its rank is n, since matrix A is then non-singular. A real quadratic form x^TAx is said to be positive-definite (i.e. its canonical form consists of n positive squares) if $x^TAx > 0$, unless $x_1 = ... = x_n = 0$.

Activity 2

Consider the real quadratic form

$$= 3\,x_1^2 + 3x_2^2 + 5x_3^2 + 2x_1x_2 - 2x_1x3 - 2x_2x_3.$$

i. Reduce ϕ to a sum of squares and determine the corresponding linear transformation.

ii. Find the rank and signature of ϕ.

iii. Determine whether ϕ is positive definite or not.

Solution

i. First, we write down asymmetric matrix A such that $\phi = x^TAx$. It is given by

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}$$

Note that to obtain A, we split terms $a_{ij}x_ix_j$ where $i \neq j$ into $b_{ij}x_ix_j + c_{ij}x_ix_j$ that

$b_{ij} = c_{ij} = \frac{1}{2}a_{ij}$ Next, we diagonalize A by obtaining the characteristic polynomial as follows:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)(3-\lambda)(5-\lambda) + 1 + 1 - (3-\lambda) - (3-\lambda) - (5-\lambda)$$

$$= \lambda^3 + 11\lambda^2 - 36\lambda + 36$$

$$= (2-\lambda)(\lambda-3)(\lambda-6).$$

This implies that the characteristic roots of A are the roots of the characteristic equation, $(2-\lambda)(\lambda-3)(\lambda-6)=0$. Hence, the characteristic roots of A are $2, 3, 6$. A sum of squares to which ϕ reduces is $2y_1^2 + 3y_2^2 + 6y_3^2$.

To determine the corresponding linear transformation, we find the characteristic vector which are basis of the eigenspaces corresponding to $\lambda = 2, 3, 6$.

For $\lambda = 2$: The eigenspace of A corresponding to the homogeneous system of linear equations $(A-2I)x=0$

Activity 3

Obtain the row reduced Echelon form $A - 2I$.

$$(A - 2I) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \end{matrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$R_3 + R_2 \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad R_3(\frac{1}{2}) \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the row reduced Echelon form of $A - 2I$, we have:

$x_1 + x_2 - x_3 = 0$

if $x_3 = 0$.

put $x_2 = a$. Then $x_1 = -a$.

Hence, the solution space of the homogenous system $(A-2I)x=0$ is $\{(x_1, x_2, x_3) = a(-1,1,0) | a \in R\}$.

i.e a basis for the eigenspace of A corresponding to $\lambda = 2$ is $\{(-1,1,0)\}$.

For $\lambda = 3$: The eigenspace of A corresponding to the homogeneous system of linear equations $(A-3I)x=0$.

For $\lambda = 6$: The eigenspace of A corresponding to the homogeneous system of linear equations $(A-6I)x=0$.

Activity 4

Obtain the row reduced Echelon form $A-6I$.

$$(A - 6I) = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -3 & -1 \\ -1 & -1 & -1 \end{pmatrix} \quad R_1(-\frac{1}{3}) \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 1 & -3 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\begin{matrix} R_2 - R_1 \\ R_3 + R_1 \end{matrix} \begin{pmatrix} 1 & \frac{-1}{3} & \frac{1}{3} \\ 0 & \frac{-8}{3} & \frac{-4}{3} \\ 0 & \frac{-4}{3} & \frac{-2}{3} \end{pmatrix} \quad \begin{matrix} R_2(\frac{-3}{8}) \\ R_3 - \frac{-1}{2}R_2 \end{matrix} \begin{pmatrix} 1 & \frac{-1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

From the row reduced Echelon form of $A-6I$, we have:

$x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3 = 0$

$x_2 + \frac{1}{2}x_3 = 0.$

put $x_3 = 2a$. Then $x_2 = -a$ and $x_1 = -\frac{1}{3}a - \frac{2}{3}a = -a$

Hence the solution space of the homogenous system $(A-6I)x=0$ is $\{(x_1, x_2, x_3) = a(-1,-1,2) | a \in R\}$ i.e a basis for the eigenspace of A corresponding to $\lambda = 3$ is $\{(-1, -1, 2)\}$.

Hence the matrix P which determines the orthogonal linear transformation reducing ϕ to a sum of squares is

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Where the columns of P are the normalized characteristic vectors of A found above. Hence the required linear transformation $T: R^3 \to R^3$ is defined by

$$T.\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

ii. Since the diagonal form of ϕ consists of 3 squares, it follows that the rank of $\phi = 3$. The canonical form of ϕ is equal to $z_1^2 + z_2^2 + z_3^2$.

Hence the signature s of ϕ is given by s = number of positive squares number of negative squares(in the canonical form) = 3 – 0 = 3

iii. Since ϕ is a quadratic form in 3 variables x_1, x_2, x_3 and the canonical form consists of 3 positive squares, it follows that ϕ is positive definite.

Summary

This unit focused on the algebra of diagonal matrix together with technique of diagonalizing a matrix and obtaining the canonical form.

Self Assessment Questions

1. $\phi = x_1^2 - x_2^2 + x_3^2 + 2x_1x_2 - 4x_1x_3 + 4x_2x_3$
2. $\phi = x_1^2 - 2x_2^2 - 4x_2x3 + 6x_3x_1$
3. $\phi = 2x_1^2 + 2x_2^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$
4. $\phi = x^2 + x^2 + x^2 + 2x_1x_3 + 2x_2x_3$

reduce ϕ to a sum of squares and determine the corresponding linear transformation; Find the rank and signature of ϕ; and determine whether φ is positive definite or not.

Tutor Marked Assignment

Find the rank of each of the following real homogenous quadratic forms.

1. $\phi = x_1^2 - x_2^2 + x_3^2 + 2x_1x_2 - 4x_1x_3 + 4x_2x_3$
2. $\phi = x_1^2 - 2x_2^2 - 4x_2x3 + 6x_3x_1$
3. $\phi = 2x_1^2 + 2x_2^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$
4. For $\phi = x_1^2 + x_2^2 + x_3^2 + 2x_1x_3 + 2x_2x_3$ reduce ϕ to a sum of squares and determine the corresponding linear transformation; Find the rank and signature of ϕ; and determine whether ϕ is positive definite or not.

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