MAT 201:
MATHEMATICAL METHODS 1

University of Ilorin
Centre for Open &
Distance Learning

CODL

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From the Vice Chancellor

Courseware development for instructional use by the Centre for Open and Distance Learning (CODL) has been achieved through the dedication of authors and the team involved in quality assurance based on the core values of the University of Ilorin. The availability, relevance and use of the courseware cannot be timelier than now that the whole world has to bring online education to the front burner. A necessary equipping for addressing some of the weaknesses of regular classroom teaching and learning has thus been achieved in this effort.

This basic course material is available in different electronic modes to ease access and use for the students. They are available on the University's website for download to students and others who have interest in learning from the contents. This is UNILORIN CODL's way of extending knowledge and promoting skills acquisition as open source to those who are interested. As expected, graduates of the University of Ilorin are equipped with requisite skills and competencies for excellence in life. That same expectation applies to all users of these learning materials.

Needless to say, that availability and delivery of the courseware to achieve expected CODL goals are of essence. Ultimate attention is paid to quality and excellence in these complementary processes of teaching and learning. Students are confident that they have the best available to them in every sense.

It is hoped that students will make the best use of these valuable course materials.

Professor S. A. Abdulkareem
Vice Chancellor

Foreword

Courseware remains the nerve centre of Open and Distance Learning. Whereas some institutions and tutors depend entirely on Open Educational Resources (OER), CODL at the University of Ilorin considers it necessary to develop its own materials. Rich as OERs are and widely as they are deployed for supporting online education, adding to them in content and quality by individuals and institutions guarantees progress. Doing it in-house as we have done at the University of Ilorin has brought the best out of the Course Development Team across Faculties in the University. Credit must be given to the team for prompt completion and delivery of assigned tasks in spite of their very busy schedules.

The development of the courseware is similar in many ways to the experience of a pregnant woman eagerly looking forward to the D-day when she will put to bed. It is customary that families waiting for the arrival of a new baby usually do so with high hopes. This is the apt description of the eagerness of the University of Ilorin in seeing that the centre for open and distance learning [CODL] takes off.

The Vice-Chancellor, Prof. Sulyman Age Abdulkareem, deserves every accolade for committing huge financial and material resources to the centre. This commitment, no doubt, boosted the efforts of the team. Careful attention to quality standards, ODL compliance and UNILORIN CODL House Style brought the best out from the course development team. Responses to quality assurance with respect to writing, subject matter content, language and instructional design by authors, reviewers, editors and designers, though painstaking, have yielded the course materials now made available primarily to CODL students as open resources.

Aiming at a parity of standards and esteem with regular university programmes is usually an expectation from students on open and distance education programmes. The reason being that stakeholders hold the view that graduates of face-to-face teaching and learning are superior to those exposed to online education. CODL has the dual-mode mandate. This implies a combination of face-to-face with open and distance education. It is in the light of this that our centre has developed its courseware to combine the strength of both modes to bring out the best from the students. CODL students, other categories of students of the University of Ilorin and similar institutions will find the courseware to be their most dependable companion for the acquisition of knowledge, skills and competences in their respective courses and programmes.

Activities, assessments, assignments, exercises, reports, discussions and projects amongst others at various points in the courseware are targeted at achieving the objectives of teaching and learning. The courseware is interactive and directly points the attention of students and users to key issues helpful to their particular learning. Students' understanding has been viewed as a necessary ingredient at every point. Each course has also been broken into modules and their component units in sequential order.

Courseware for the Bachelor of Science in Computer Science housed primarily in the Faculty of Communication and Information Science provide the foundational model for Open and Distance Learning in the Centre for Open and Distance Learning at the University of Ilorin.

At this juncture, I must commend past directors of this great centre for their painstaking efforts at ensuring that it sees the light of the day. Prof. M. O. Yusuf, Prof. A. A. Fajonyomi and Prof. H. O. Owolabi shall always be remembered for doing their best during their respective tenures. May God continually be pleased with them, Aameen.

Bashiru, A. Omipidan
Director, CODL

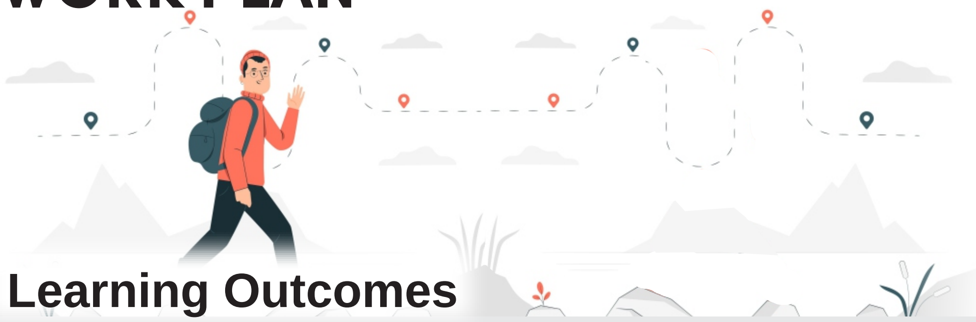
INTRODUCTION

I welcome you to Mathematical Methods 1. It is a 3-credit course that is available to year two undergraduate students in Faculties of Life Sciences, Physical Sciences, Engineering, Education and allied degrees. This course was designed to introduce you to different mathematical methods.

Course Goal

Your journey through this course will remind you of some basic topics like differentiation and integration. You will be introduced to different methods of solving mathematical problems.

WORK PLAN



Learning Outcomes

At the end of this course, you should be able to:

- differentiate functions of one or more variables;
- integrate functions of one or more variables;
- apply Mean Value Theorem;
- expand functions and express in Taylor series form;

Course Guide

Module 1

Functions of a Real Variable

Unit 1 - Real-valued functions of a real-variable

Unit 2 - Graphs and Graphing

Unit 3 - Even and Odd functions

Unit 4 - Interior Exterior and Boundary Points

Module 2

Review of Differentiation

Unit 1 - Differentiation

Unit 2 - Differentiation of Trigonometric Functions

Unit 3 - Applications of Differentiation

Unit 4 - Maximum & Minimum values

Related Courses

Prerequisite: MAT 112

Required for: CSC 430

MAT 201
Mathematical Methods 1

- apply Langranges multiplier;
- evaluate integrals; and
- apply the mathematical methods to solve problems.

Module 3
A Review of Integration

Unit 1 - Integration

Unit 2 - Forms of Integration

Unit 3 - Definite Integral

Unit 4 - Applications of Integration

Unit 5 - Initial value Problems

Unit 6 - Extrema, Mean Value Theorem

Unit 7 - Taylor's Series, Partial Derivatives and Maclaurin's Series

Module 4
Partial Derivatives

Unit 1 - Differentiation of Functions of more than one variable

Unit 2 - Second Order Partial Differential Equations

Unit 3 - Chain Rule

Unit 4 - Lagrange Multipliers

Module 5
Linearisation and Differentials

Unit 1 - Linearisation

Unit 2 - Multiple integrals

Course Requirements

Requirements for success

The CODL Programme is designed for learners who are absent from the lecturer in time and space. Therefore, you should refer to your Student Handbook, available on the website and in hard copy form, to get information on the procedure of distance/e-learning. You can contact the CODL helpdesk which is available 24/7 for every of your enquiry.

Visit CODL virtual classroom on http://codllms.unilorin.edu.ng. Then, log in with your credentials and click on MAT 201. Download and read through the unit of instruction for each week before the scheduled time of interaction with the course tutor/facilitator. You should also download and watch the relevant video and listen to the podcast so that you will understand and follow the course facilitator.

At the scheduled time, you are expected to log in to the classroom for interaction.

Self-assessment component of the courseware is available as exercises to help you learn and master the content you have gone through.

You are to answer the Tutor Marked Assignment (TMA) for each unit and submit for assessment.

Embedded Support Devices

Support menus for guide and references

Throughout your interaction with this course material, you will notice some set of icons used for easier navigation of this course materials. We advise that you familiarize yourself with each of these icons as they will help you in no small ways in achieving success and easy completion of this course. Find in the table below, the complete icon set and their meaning.

Introduction	Learning Outcomes	Main Content
Summary	Tutor Marked Assignment	Self Assessment
Web Resources	Downloadable Resources	Discuss with Colleagues
References	Futher Reading	Self Exploration

Grading and Assessment

TMA	CA	Exam	Total
20%	20%	60%	100%

Module 1

FUNCTIONS OF A REAL VARIABLE

Units

Unit 1 - Real-valued functions of a real-variable

Unit 2 - Graphs and Graphing

Unit 3 - Even and Odd functions

Unit 4 - Interior, Exterior and Boundary Points

UNIT 1

Real-valued functions of a real-variable

Introduction

Functions have domains and ranges that are sets of real numbers. Such functions are called real-valued functions of a real variable and are usually defined by formulae or equations. For example, the function A defined by the rule $A(r) = \pi r^2$ gives the area of a circle as a function of its radius r. The area of a circle of radius 2 is $A(2) = \pi(2)^2 = 4\pi$. Geometrically, the domain of the function $A = \pi r^2$ is the set of all positive real numbers.

Learning Outcomes

At the end of this unit, you should be able to:

1. define a real-valued function;
2. give examples of a closed interval;
3. give examples of an open interval;
4. find the domain of a given function; and
5. find the range of a given function.

Main Content

Let $D \subseteq R$. A function f from D into R is a rule which associates with each $x \in D$ one and only one $y \in R$. D is called the domain of the function. If $x \in D$, then the element $y \in R$ is called the value of f at x or the image of x under f; y is denoted by $f(x)$. If $U \subseteq D$, then $f(U) = \{y \in R | y = f(x) \text{ for some } x \in U\}$. If $U = D$, then $f(D)$ is called the range of f. If $y \in R$, then $f^{-1}(y) = \{x \in D | f(x) = y\}$. Note that

1. $f^{-1}(y)$ might be ϕ. (y is not in the range of f)
2. $f^{-1}(y)$ might have more than one element.
3. f has an inverse function if for each $y \in f(D)$, there is one and only one $x \in f^{-1}(y)$.

Activity 1

The function A defined by the rule $A(r) = \pi r^2$ gives the area of a circle as a function of it's radius r. The area of a circle of radius 3 is $A(3) = \pi(3)^2 = 9\pi$.

Geometrically, the domain of the function $A = \pi r^2$ is the set of all positive real numbers.

Activity 2

The function $y = x^2$. The formula $y = x^2$ defines the number y to be thesquare of the number x. If $x = 5$, then $y = 5^2 = 25$.

The domain is the set of allowable $x - values$. In this case, the set of all real numbers is the domain of the function. The range which consists of the resulting $y - values$, is the set of non-negative real numbers.

Intervals

The domain and range of many functions are intervals of real numbers. The set of all real numbers that is strictly between two fixed numbers a and b is an open interval. The interval is "open" at each end because it contains neither of its end points.

Intervals that contain both end points are closed. Intervals that contain one end point, but not both, are half-open. The domain and range of functions can also be infinite intervals.

The end points of an interval make up the interval's boundary and are called boundary points. The remaining points make up the interval's interior and are called interior points. Closed intervals contain their boundary points, open intervals do not. Every point of an open interval is an interior point of the interval.

Definition: The variable x in a function $y = f(x)$ is called the independent variable or argument of the function. The variable y, whose value depends on x, is called the dependent variable.

Remark 1:

When we define functions, we never divide by 0. Suppose $y = \frac{1}{x}$, ”$x \neq 0$”. that is, zero is not in the domain of the function.

For

$$y = \frac{1}{(x - 2)}, \text{”} x \neq 2\text{”}.$$

This implies that 2 is not in the domain of the function.

Remark 2:

We deal exclusively with real-valued functions. Therefore, we restrict our domains when we have square roots or other even roots. If $y = \sqrt{1} - x2$, we should think that ”x^2”must be greater than 1. The domain must not extend beyond the interval $- \leqslant x \leqslant$.

Summary

A real-valued function of a real variable is a function whose values are real numbers. In other words , it is a function that assigns a real number to each member of its domain. Such functions are called real-valued functions of a real variable and are usually defined by formulas and equations.

An open interval does not include its end points and it is indicated with parenthesis, e.g (0, 1). A closed interval is an interval which includes all its endpoints, and it is denoted by square brackets, e.g [0, 1].

Self Assessment Questions

1. What is a real-valued function?
2. Give an example of a closed interval.
3. Give an example of an open interval.
4. Find the range of the function $y = \dfrac{1}{x^2 - 4}$

Tutor Marked Assignment

- Dene the real-valued function of a real variable.
- Graph the function $y = x^2$ over the interval $2 \leqslant x \leqslant 2$
- Find the domain of the function, $y = \sqrt{3^{2x-2} + 9^{x} - 10}$

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
- Hoffmann L.D. and Bradley G. L. (2004). Calculus for Business, Economics, and the Social Life Sciences. McGraw Hill Higher Education, Boston.
- Jackson, J.S.G (2002). Mathematical Methods for Mathematics, Science and Engineering students Volume 1, Published by Ghana University Press, Accra.
- Stroud K. A. and Dexter J. Booth (2007). Engineering Mathematics. Palgrave Macmillan. Houndsmill, Basingstoke, Hampshire RG21 6XS and 175 Fifth Avenue, New York.

Further Reading

- Erwin Kreyszig (2011) Advanced Engineering Mathematics, Wiley, New Jersey.
- Robert T. Smith and Roland B. Minton. (2011) Multivariable calculus, McGraw-Hill Education, London, United States, Europe.

UNIT 2

Graphs and Graphing

Introduction

Much of the research on mathematics learning and teaching has focused on the very earliest levels of mathematics content. Functions and graphs is one in which we can use symbolic systems to expand and understand functions.

Learning Outcomes

At the end of this unit, you should be able to:

1. define the graph of a function.
2. define the graph of an even function.
3. define the graph of an odd function.
4. define the graph of polynomials with odd or even powers.

Main Content

Graphs are used to define the mathematical concept of function. Neither functions nor graphs can be treated as isolated concepts. They are communicative systems, on the one hand, a construction and organization of mathematical ideas on the other hand. *A* function $y = f(x)$ is an even function of x if $f(-x) = f(x)$ for every x in the function's domain. It is an odd function of x if $f(-x) = -f(x)$ for every x in the domain of the function.

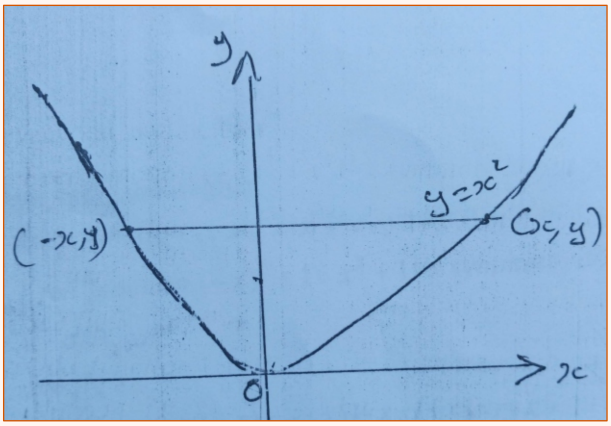
Activity 1

1. If y equals an even power of x, as in $y = x^2$ or $y = x^4$ it is an even function of x (because $(-x)\ 2 = x^2$ and $(-x)^4 = x^4$). If y equals an odd power of x, as in $y = x$ or $y = x^3$.

   It is an odd function of x (because $(-x)^1 = -x$ and $(-x)^3 = -x^3$)

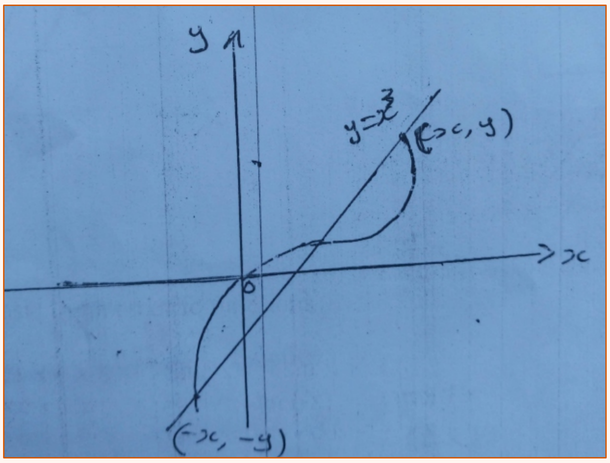
Activity 2

2. Saying that a function $y = f(x)$ is even is equivalent to saying that it's graph is symmetric about the y-axis. Since $f(-x) = f(x)$, the point $(-x, y)$ lies on the curve.



Activity 3

3. A function $y = f(x)$ is odd, it is equivalent to saying that its graph is symmetric with respect to the origin. Since $f(-x) = -f(x)$, the point (x, y) lies on the curve if the point $(-x, -y)$ lies on the curve as shown in the figure below



Activity 4

4. The graphs of polynomials with even powers of x are symmetric about the y-axis. The graphs of polynomials with odd power of x are symmetric about the origin.

Summary

In this unit, you have learnt how to represent functions with graphs. You also learnt how to construct odd and even functions.

Self Assessment Questions

1. Define the graph of a function.
2. Define the graph of an even function.
3. Define the graph of an odd function.
4. Define the graph of polynomials with odd or even powers.

Tutor Marked Assignment

- Graph the function $y = \sqrt{4 - x}$

Hint: The domain is the set of values for which $4 \quad x \geqslant 0$ or $x \leqslant 4$.

x	$\sqrt{4-x}$
4.0	0
3.75	0.5
2.0	$\sqrt{2} \approx 1.4$
0	2
-2	$\sqrt{6} \approx 2.4$

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
- Hoffmann L.D. and Bradley G. L. (2004). Calculus for Business, Economics, and the Social Life Sciences. Eight Edition, McGraw Hill Higher Education, Boston.
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Further Reading

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UNIT 3

Even and Odd functions

Introduction

In mathematics, even functions and odd functions are functions which satisfy particular symmetry relations with respect to taking additive inverses. They are important in many areas of mathematical analysis. They are named for the parity of the powers of the power functions which satisfy the given condition.

Learning Outcomes

At the end of this unit, you should be able to:

1. identify functions that are odd, even or neither; and
2. find the even and odd parts of a function when they exist.

Main Content

Evenness or oddness are generally considered for real functions, that is, real-valued functions of a real variable. However, the concept may be more generally defined for functions whose domains and co-domains have additive inverse. Functions are either even, odd or neither. *A* function $f(x)$ that satisfies the condition: $f(x) = x^n$ is an even function if n is an even integer. *A* function $f(x)$ is an odd function if n is an odd integer. Examples of these functions are given below

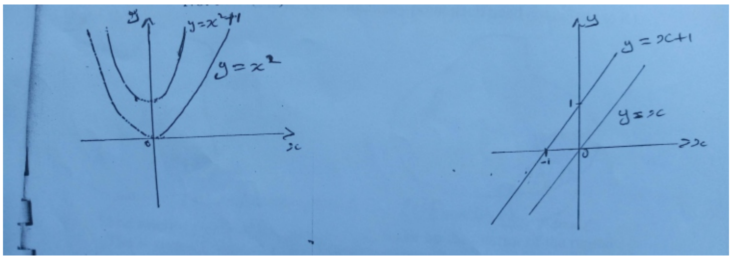
(I) $f(x) = x^2$ Even function $(-x)^2 = x^2\ \forall\ x$ symmetryabout y-axis

(ii) $f(x) = x^2 + 1$ Even function $(-x)^2 + 1 = x^2 + 1\ \forall\ x$ symmetry about y-axis

(iii) $f(x) = x$ Odd function $(-x) = -(x)\ \forall\ x$ symmetry about theorigin

(iv) $f(x) = x + 1$ Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. Not even $(-x) + 1 \neq x + 1$

The function is neither even nor odd.



Given below are the basic trigonometric functions.

Even	Odd
$\cos(-x) = \cos x \quad \forall \quad x$	$\sin(-x) = -\sin x \quad \forall \quad x$
$\sec(-x) = \sec x \quad \forall \quad x$	$\tan(-x) = -\tan x \quad \forall \quad x$
$cosec(-x) = -cosec(x) \quad \forall x$	
$\cot(-x) = -\cot(x) \quad \forall x$	

Definition:

Suppose D is a set of n-tuples of real values $(x_1, x_2, ..., x_n)$. A real-valued function f on D is a rule that assigns a real number $w = f(x_1, x_2, ..., x_n)$ to each element in D. The set D is the function's domain. The set B of w-values taken on by f is the function's range. The symbol w is the dependent variable of f and f is said to be a function of the n independent variables $x_1, ..., x_n$. We call the x's the function's input variables and call w the function's output variable.

Activity 1

The value of the function

$f(x,y.z) = \sqrt{x^2 + y^2 + z^2}$ at the points $(3,0,4)$ is

$f(3,0,4) = \sqrt{3 + 0^2 + 4^2}$

$= \sqrt{25} = 5$

Definitions

1. The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y) = c$ is called a level curve of f.

2. The set of all points $(x, y, f(x, y))$ in space for (x, y) in the domain of f, is called the graph of f. The graph of f is also called the surface $z = f(x,y)$.

Not every function is either even or odd, but many can be written as the sum of an even part and an odd part. If given $f(x)$ where $f(-x)$ is also defined then:

$$f_e(x) = \frac{f(x) + f(-x)}{2} \text{ is even and } f_o(x) = \frac{f(x) - f(-x)}{2} \text{ is odd}$$

Furthermore, $f_e(x)$ is called the even part of $f(x)$ and $f_o(x)$ is called the odd part of $f(x)$.

Activity 1

If $f(x) = 2x^2 - 4x + 2$ then,

$f(-x) = 2(-x)^2 - 4(-x) + 2 = 2x^2 + 4x + 2$

So that the even and odd parts of $f(x)$ are:

$$f_e(x) = \frac{(2x^2 - 4x + 2) + (2x^2 + 4x + 2)}{2} = 2x^2 + 2 \quad \text{and}$$

$$f_o(x) = \frac{(2x^2 - 4x + 2) \quad (2x^2 + 4x + 2)}{2} = -4x$$

Summary

In this unit, you have been taught how to determine odd, even and numbers and numbers that are neither odd nor even. Even functions consist of only even powers and odd functions consist of only odd powers. Consequently, the even part of $f(x)$ consists of even powers only and the odd part of $f(x)$ consists of odd powers only.

Self Assessment Questions

1. When is a function $f(x)$ said to be even?
2. When is a function $f(x)$ said to be odd?
3. Find the odd and even parts of $f(x) = 3x^3 - 3x^2 + 3$.
4. Is $f(x) = x^2 + 1$ even or odd?

Tutor Marked Assignment

Are the following functions odd, even or neither odd nor even?

- $x + x^2$, $|x|$, e^x, e^{x^2}, $x \sin x$
- Find the even and odd parts of (i) $x^4 - 3x^3 + 2x^2 + 1$ (ii) $y^2 - 3^y + 5$

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
- Hoffmann L.D. and Bradley G. L. (2004). Calculus for Business, Economics, and the Social Life Sciences. Eight Edition, McGraw Hill Higher Education, Boston.
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Further Reading

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UNIT 4

Interior, Exterior and Boundary Points

Introduction

In this unit, you will learn the region of points.

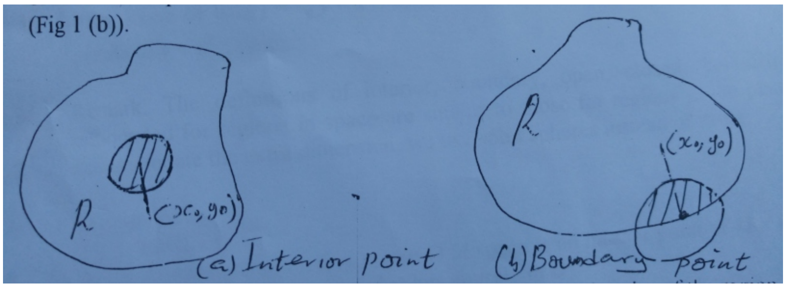
Learning Outcomes

At the end of this unit, you should be able to:

1. define the interior point of a region;
2. define the boundary points of a region;
3. state if a region is bounded or unbounded; and
4. determine if a region is closed or open.

Main Content

A point (x_0,y_0) in a region (set) R in the xy plane is an interior point of R if it is the center of the disk that lies entirely in R. A point (x_0,y_0) is a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R). (See the figure below).



The interior points of a region, as a set, make up the interior of the region. The region's boundary points make up its boundary. A region is open if it consists entirely of interior points. A region is closed if it contains all of its boundary points

Definition:

A region in the plane is bounded if it lies inside a disk of fixed radius. A region is unbounded if it is not bounded.

Activity 1

Bounded sets in the plane: line segment, triangles, rectangles, disks. Unbounded sets in the plane: lines, coordinate axes, graphs of functions defined on infinite intervals, quadrants, half-planes, the plane itself.

Activity 2

The domain of the function $f(x, y) = \sqrt{y - x^2}$ is closed and unbounded (fig below) The parabola $y = x^2$ is the boundary of the domain. The points inside the parabola make up the domain's interior.

The domain of $f(x, y) = \sqrt{y - x^2}$ consists of the shaded region and its boundary parabola $y = x^2$.

Remark:

The definitions of interior, boundary, open, closed, bounded, and unbounded for region in space are similar to those for regions in the plane. To accommodate the extra dimension, we use solid spheres instead of disks.

Definitions

1. Interior points: A point (x_0, y_0, z_0) in a region D in space is an interior point of D if it is the center of a solid sphere that lies entirely in D.

2. A point (x_0, y_0, z_0) is a boundary point of D if every spare centered at (x_0, y_0, z_0) encloses points that lie outside of D as well as points that lie inside D. The interior of D is the set of interior points of D. A region D is open if it consists entirely of interior points. A region is closed if it contains its boundary points.

Summary

As with interval of real numbers, some regions in plane are neither open nor closed. If you start with the open disk and add to it some but not all of its boundary points, the resulting set is neither open nor closed. The boundary points that are there keep the set from being open, the absence of the remaining boundary points keep the set from being closed.

Self Assessment Questions

1. Define an interior point in space R.
2. Define the boundary points of a region.

Tutor Marked Assignment

- State the condition for a region to be bounded and for it to be unbounded.
- State the condition for a region to be closed and for it to be open.

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)

- Hoffmann L.D. and Bradley G. L. (2004). Calculus for Business, Economics, and the Social Life Sciences. Eight Edition, McGraw Hill Higher Education, Boston.

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Further Reading

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- Robert T. Smith and Roland B. Minton. (2011) Multivariable calculus, McGraw-Hill Education, London, United States, Europe.

Module 2

Review of Differentiation

Units

Unit 1 - Differentiation
Unit 2 - Differentiation of Trigonometric Functions
Unit 3 - Applications of Differentiation
Unit 4 - Maximum & Minimum values

UNIT 1

Differentiation

Introduction

In this unit, you are going to learn differentiation and its applications. You we also learn the sum and difference rule, the product, the quotient rule, power rule for negative integer power, derivative of a function of a function and so on.

Learning Outcomes

At the end of this unit, you should be able to:

1. define differentiation;
2. find the differentials of functions;
3. state the derivatives of trigonometric functions; and
4. apply the rules of differentiation to solve mathematical problems.

Main Content

The process of finding the gradient function $\frac{dy}{dx}$ or $f'(x)$ of a function $f(x)$ is called the differentiation of the function $f(x)$. $\frac{dy}{dx}$ is called the derivative of y with respect to x. It is sometimes called the differential coefficient of y with respect to x. $\frac{dy}{dx}$ measures the rateof change of y as compared with the rate of change of x. Thus, the process of finding the derivative of a function is called differentiation. Given a function $y = f(x)$, we can write its derivative in the following ways:

1. $f'(x)$ reads "f prime of x".

2. $\frac{dy}{dy}$ read "dee-y, dee-x".

Differentiation is done with more ease when the rules are duly followed. Here is an outline of the basic rules of differentiation.

1. The derivative of a constant is zero.

   If c is a constant, then $\frac{d}{dx}(c) = 0$

2. If is n a positive integer, then

   $\frac{d}{dx}(x^n) = nx^{n-1}$

Activity 1

$$\frac{d}{dx}(x^2) = 2x$$

3. If u is a differentiable function of x and c is a constant then

$$\frac{d}{dx}(cx^n) = c\ \frac{d}{dx}(x^n) = cnx^{n-1}$$

Activity 2

$$\frac{d}{dx}(3x^2) = 3.2x = 6x$$

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1.u) = -1\frac{d}{dx}(u) = -\frac{du}{dx}$$

4. The sum and difference rule: If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are both differentiable. At such points

$$(1.) \quad \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$(2.) \quad \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

Similar equations hold for more than 2 differentiable functions, as long as the number of functions involved is finite.

Activity 3

$y = x^4 + 12x$

$\frac{dy}{dx} = 4x^3 + 12$

$y = x^3 + 3x^2 - 5x + 1$

$\frac{dy}{dx} = 3x^2 + 6x - 5$

Activity 4

The first 4 derivatives of $y = x^3 - 3x^2 + 2$ are

1st order $y' = 3x^2 - 6x$

2nd order $y'' = 6x - 6$

3rd order $y''' = 6$

4th order $y^{iv} = 0$

5. The product rule: Product of two differentiable functions u and v is differentiable, and

$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{du} + v\,\frac{du}{dv}$$

Activity 5

$y = (x^2 + 1)\,(x^3 + 3)$

$u = x^2 + 1 \qquad v = x^3 + 3$

$\frac{du}{dv}[(x^2 + 1)(x^3 + 3)] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$

$= 3x^4 + 3x^2 + 2x^4 + 6x$

$= 5x^4 + 3x^2 + 6x$

Activity 6

Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if $u(2) = 3, u'(2) = -4, v(2) = 1, v'(2) = 2$

Solution

$y' = (uv)' = uv' + vu'$

$y'(2) = u(2)v'(2) + v(2)u'(2) = (3)(2) + (1)(-4) = 6 - 4 = 2$

6. The quotient rule: At a point where $v \neq 0$ the quotient $y = \frac{u}{v}$ of two differentiable functions is differentiable and

Activity 7

Find the derivative of $y = \frac{x^2-1}{x^2+1}$

$u = x^2 - 1 \qquad v = x^2 + 1$

$$\frac{dy}{dx} = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2}$$

$$= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}.$$

7. Power rule for negative integer powers of x

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Activity 8

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

Summary

The rules of differentiation are guidelines to finding the derivative of a given function have been taught in this unit.

Self Assessment Questions

1. List the dierent rules of dierentiation.
2. Dierentiate (i) $3x^3$.(ii) $15x^4$

Tutor Marked Assignment

- Find the derivative of the following

1. $2x^2 + 5x + 4$
2. $\frac{3}{4}x^2$

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
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UNIT 2

Differentiation of Trigonometric Functions

Introduction

In this unit, you will be learning the differentials of trigonometric functions.

Learning Outcomes

At the end of this unit, you should be able to:

1. differentiate trigonometric functions; and
2. apply the rules of differentiation to trigonometric functions.

Main Content

So far, we have found the derivatives of polynomial functions using the standard derivative $\frac{d}{dx} = nx^{n-1}$. Derivative of trigonometric functions can be established by using trigonometrical formulae. We give an outline of these trigonometric functions and their derivatives.

y	$\frac{dy}{dx}$
$\sin x$	$\cos x$
$\cos x$	$\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x

As shown in the previous unit, to differentiate a product of two functions, we leave the first(differentiate the second)+ leave the second (differentiate the first).

Activity 1

If $y = x^2 \cos x$

$$\frac{dy}{dx} = x^2(-\sin x) + \cos x(2x)$$

$$\frac{dy}{dx} = -x^2 \sin x + 2x\cos x$$

$$\frac{dy}{dx} = -x(2\cos x - x\sin x)$$

To differentiate a quotient of two functions, leave the bottom function (differentiate the top) - leave the top function (differentiate the bottom) all over the bottom squared.

Activity 2

If $y = \frac{\cos x}{x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2(-\sin x) - \cos x\ (2x)}{(x^2)^2}$$

$$\frac{dy}{dx} = \frac{x(x\sin x + 2\cos x)}{x^4}$$

Summary

In this unit, you have undoubtedly discovered that the rules of differentiation totally apply to trigonometric functions. You also learnt the standards of trigonometric functions.

Self Assessment Questions

1. What is the derivative of $\sin x$, $\cos y$ and $\tan z$?
2. Differentiate $x^4 \tan x$.

Tutor Marked Assignment

1. Dierentiate $\frac{x^3}{\cos x}$
2. Dierentiate $x^5 e^x$

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
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UNIT 3

Applications of Differentiation

Introduction

Differentiation finds application in many life experiences. A shipwreck occurred because the ship was not where the captain thought it should be. There was not a good enough understanding of how the earth, stars, and planets moved with respect to each other. Hence, the need for differentiation.

Learning Outcomes

At the end of this unit, you should be able to:

1. define velocity;
2. define rectilinear motion; and
3. apply differentiation to solve mathematical problems.

Main Content

There are various applications of differentiation which you will learn in this unit.

Rectilinear motion

This is the motion of a particle along a straight line. It is specified by the equation $s = f(t)$ where s is the distance of the particle from an initial point 0 at the time t.

Instantaneous velocity (velocity)

This is the derivative of position with respect to time. If the position function of a body moving along a line is $s = f(t)$, the body's velocity at time t is $v(t) = \frac{ds}{dt}$

Activity 1

A dynamic blast blows a heavy rock straight up with a launch velocity of $160 ft/sec$ (about $109 mph$). It reaches height of $s = 160t - 16t^2 ft$ after $t\ sec$.

(a) How high does the rock go?

(b) How fast is the rock traveling when it is $256 ft$ above the ground on the way up and on the way down?

Solution

(a) To find how high the rock goes, we find the values of s when the rock's velocity is zero.

The velocity v is $= \dfrac{ds}{dt} = \dfrac{d}{dt}(160t - 16t^2)$

$= (160 - 32t) ft/sec$

The velocity is zero when $160t - 32t = 0$ or $t = 5 sec$

The rock's height at $t = 5\ sec$ is

$S_{max} = s(5) = 160(5) - 16(5)^2$

$= 800 - 400 = 400 ft.$

(b) To find the rock's velocity at 256ft on the way up and again on the way down, we find the two values of t for which

$s(t) = 160t - 16t^2 = 256 \cdots (*)$

Solving $(*)$

$16t^2 - 160t + 256 = 0$

$16(t^2 - 10t + 16) = 0$

$t = 2sec, t = 8sec$

The rock is $256 f t$ above the ground 2 *seconds* after the explosion and again 8 seconds after the explosion.

The rock's velocities at these times

$v(2) = 160 - 32(2) = 160 - 64 = 96\ ft/sec$

$v(8) = 160 - 32(8) = 160 - 256 = -96\ ft/sec$

Remark:

Since s measures height from the ground up, changes in s are positive as the rock rises and negative as the rock falls.

Definition:

Speed is the absolute value of velocity

Definition:

Acceleration $= \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Activity 2

The acceleration of the rock in Activity 2 above is

$$a = \frac{dv}{dt}\frac{d}{dt}(160 - 32t) = 0 - 32 = 32ft/sec^2$$

Remark: This is a deceleration since the value is negative.

Activity 3

1. Find an equation for the tangent to the curve $y = x + \frac{2}{x}$ at the point $(1,3)$.

Solution

$$\frac{d}{dx} = \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) = 1 + 2\left(\frac{-1}{x^2}\right) = 1 - \frac{2}{x^2}$$

The slope at $x = 1$ is

$$\frac{dy}{dx}_{x=1} = \left(1 - \frac{2}{1^2}\right)_{x=1} = 1 - 2 = -1$$

the line through $(1,3)$ with slope $m = 1$ is

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -1(x - 1)$$
$$y - 3 = 1 - x$$
$$y = -x + 4$$

Activity 4

Find the lines that are tangent and normal to the curve $y = \tan x$ at the point $(\frac{\pi}{4}, 1)$

Solution

The slope of the curve $(\frac{\pi}{4}, 1)$ is the value of $\frac{dy}{dx} = sec^2 x$ at $x = \frac{\pi}{4}$:

$\frac{dy}{dx}\Big|_{x=\frac{\pi}{4}} = sec^2\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = 2.$

The tangent is the line $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ or $y = 2x - \frac{\pi}{2} + 1$

The slope of the normal is $= -\frac{1}{2}$, so the equation of the normal is

$y - 1 = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$ or

$y = -\frac{x}{2} + \frac{\pi}{8} + 1.$

Summary

Differentiation can help us solve many types of real -life problems as we have seen its use in motion, instant velocity, acceleration etc.

Self Assessment Questions

1. Define velocity and rectilinear motion.
2. Let $y = x^4 + 12x$. Find $\frac{dy}{dx}$
3. Find the first 4 derivative of $y = x^3 - 3x^2 + 2$.
4. Differentiate $y = (x^2 + 1)(x^3 + 3)$.
5. Find the derivative of the function $y = (5x + 3)^2$.
6. Find the derivatives of $(x + 1)^{\frac{1}{2}} / (x - 1)^{\frac{1}{3}}$.
7. Find the equation of the tangent to the curve $x - \frac{2}{x}$ at the point (1,3).
8. Find the line that is perpendicular to the curve $y = \tan x$ at the point $(\frac{\pi}{4}, 1)$.

Tutor Marked Assignment

- Find the equations of the tangent and of the normal to the curve $y = x^3 - 8$ at the points where it crosses y-axis.

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UNIT 4

Maximum & Minimum values

Introduction

In this unit, you will learn the maximum and minimum values of functions.

Learning Outcomes

At the end of this unit, you should be able to:

1. classify maximum and minimum points; and
2. test for stationary points.

Main Content

At the stationary point, $\frac{dy}{dx} = 0$

Tests for stationary points

1. For a maximum or minimum point, $\frac{dy}{dx} = 0$.
2. For a maximum point, $\frac{d^2y}{dx^2} < 0$, i.e, $\frac{d^2y}{dx^2}$ is negative
3. For a minimum point, $\frac{d^2y}{dx^2} > 0$, i.e, $\frac{d^2y}{dx^2}$ is positive
4. For point of inexion $\frac{d^2y}{dx^2} = 0$, i.e, $\frac{d^2y}{dx^2} = 0$

Activity 1

Find the stationary points of the function $y = x^3/3 + x^2 - 3x + 4$ and determine whether the stationary points are maximum and minimum. Also find the values of y at these points.

Solution

$\frac{dy}{dx} = x^2 + 2x - 3$

Stationary points occur at $\frac{dy}{dx} = 0$

$\Rightarrow (x^2 + 2x - 3) = 0$

$(x - 1)(x + 3) = 0$

$x = 1$ or $x = -3$

To determine the type of each stationary point:

$$\frac{d^2y}{dx^2} = 2x + 2$$

At $x = 1$, $\frac{d^2y}{dx^2} = 2(1) + 2 = 4$ which is positive $\therefore x = 1$ gives y_{min}

At $x = -3$, $\frac{d^2y}{dx^2} = 2(-3) + 2 = -4$ which is negative $\therefore x = -3$ gives y_{max}

$$y_{max} = \frac{-3^3}{3} + (-3^2) - 3(-3) + 4 = 13$$

$$y_{min} = \frac{1^3}{3} + (1^2) - 3(1) + 4 = 2\frac{1}{3}$$

Summary

You have been taught maximum and minimum points in this unit.

Self Assessment Questions

Find the stationary points of the function $y = 3x^3/2 + x^2 - 9x + 12$ and determine whether the stationary points are maximum and minimum. Also find the values of y at these points.

Tutor Marked Assignment

- Find the stationary points of the function $y = 2x^3/3 + 2x^2 - 6x + 8$ and determine whether the stationary points are maximum and minimum. Also find the values of y at these points.

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
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Module 3

A Review of Integration

Units

UNIT 1

Integration

Introduction

If a function f(x) is a derivative, then the set of all anti-derivatives of f is the indefinite integral. If f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)dx$$

The symbol $\int$ is an integral sign.

The function f is the integrand of the integral and x is the variable of integration.

Learning Outcomes

At the end of this unit, you should be able to:

1. apply the methods of integration; and
2. integrate composite functions.

Main Content

The symbol $\int f(x)dx$ denotes the integral of $f(x)$ w.r.t x. The expression $f(x)$ to be integrated is called the integrand and the result is called the integral.

Integral	*Formulae*
Indefinite integral	*Reverse derivative formulae*
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
2. $\int \sin kx dx = -\frac{\cos kx}{k} + c$	$\frac{d}{dx}\left(\frac{-\cos kx}{k}\right) = \sin kx$
3. $\int \cos kx dx = \frac{\sin kx}{k} + c$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4. $\int \sec^2 x dx = \tan x + c$	$\frac{d}{dx}(\tan x) = \sec^2 x$
5. $\int \csc^2 x dx = -\cot x + c$	$\frac{d}{dx}(-\cot x) = \csc^2 x$
6. $\int \sec x \tan x dx = \sec x + c$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
7. $\int \csc x \cot x dx = -\csc x + c$	$\frac{d}{dx}(-\csc x) = \csc x \cot x$

Activity 1

a. $\int x^5 dx = \frac{x^6}{6} + c$ (formula 1 with $n = 5$)

b. $\frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}} + c = 2\sqrt{x} + c$ (formula 1 with $n = -\frac{1}{2}$)

c. $\int \sin 2x dx = -\frac{\cos 2x}{2} + c$ (formula 2 with $k = 2$)

d. $\int \cos \frac{x}{2} dx = \int \cos \frac{1}{2} x dx = \frac{\sin(\frac{1}{2})x}{\frac{1}{2}} + c = 2 \sin \frac{x}{2} + c$, where c is a constant.

Remark:

Finding an integral formula can sometimes be difficult, but checking it, once found is relatively easy; differentiate the right-hand side. If the derivative is the integrand, the formula is correct; otherwise it is wron

Constant of Integration

$$\text{If} \quad \frac{d}{dx}(x^4) = 4x^3 \therefore \int 4x^3 dx = x^4 + c$$

$$\text{Also,} \quad \frac{d}{dx}(x^4 + 3) = 4x^3 \therefore \int 4x^3 dx = x^4 + c$$

$$\text{and,} \quad \frac{d}{dx}(x^4 - 2) = 4x^3 \therefore \int 4x^3 dx = x^4 + c$$

In these three examples, we know the expression from which the derivative $4x^3$ derived. But any constant term in the original expression becomes zero in the derivative and all trace of it is lost. So if we do not know the history of the derivative of $4x^3$, we have no evidence of the value of the constant term. The presence of a constant term is added to the result of integration.i.e $\int 4x^3 dx = x^4 + c$.

c is called the constant of integration and must always be included. Such an integral is called an indefinite integral. The value of c might be found if further information about the integral is available.

Standard Integrals

Every derivative written in reverse gives an integral.

If $\frac{d}{dx}[G(x)+c] = g(x)$, then $\int g(x)dx = G(x)+c$

1. $\frac{d}{dx}(\frac{x^{n+1}}{n+1}+c) = x^n \therefore \int x^n dx = \frac{x^{n+1}}{n+1}+c.$ provided $n \neq -1$. Similarly, $\frac{d}{dx}(\frac{ax^{n+1}}{n+1}+c) = ax^n \therefore \int ax^n dx = \frac{ax^{n+1}}{n+1}+c.$ $[n \neq -1]$

2. $\frac{d}{dx}(\sin x + c) = \cos x \therefore \int \cos x dx = \sin x + c.$

3. $\frac{d}{dx}(\cos x + c) = -\sin x \therefore \int -\sin x dx = \cos x + c$ or $\int \sin x dx = -\cos x + c.$

4. $\frac{d}{dx}(\tan x + c) = \sec^2 x \therefore \int \sec^2 x dx = \tan x + c.$

5. $\frac{d}{dx}(-\cot x + c) = cosec^2 x \therefore \int cosec^2 x dx = -\cot x + c.$

6. $\frac{d}{dx}(\sec x + c) = \sec x \tan x \therefore \int \sec x \tan x dx = \sec x + c.$

7. $\frac{d}{dx}(-cosecx + c) = cosecx \cot x \therefore \int cosecx \cot x dx = -cosecx + c.$

8. $\frac{d}{dx}(e^x + c) = e^x \therefore \int e^x dx = e^x + c.$

9. $\frac{d}{dx}(lnx + c) = \frac{1}{x} \therefore \int \frac{1}{x} dx = lnx + c.$

These results can be expressed in tabular form as:

$f(x)$	$\int f(x)dx$
ax^n	$\frac{ax^{n+1}}{n+1}+c$, $(n \neq -1)$
1	$x+c$
a	$ax+c$
$\sin x$	$\cos x + c$
$\cos x$	$-\sin x + c$
$\sec^2 x$	$\tan x + c$
$cosec^2 x$	$\cot x + c$
$\sec x \tan x$	$\sec x + c$
$cosecx \cot x$	$-cosecx + c$
e^x	$e^x + c$
$\frac{1}{x}$	$Inx + c$
a^x	$\frac{a^x}{Ina} + c$

Activity 2

Evaluate

$\int 5 \sec x \tan x dx = \int 5 \text{ R}\sec x \tan x dx$

$= 5(\sec x + c)$

$= 5 \sec x + 5c$

$= 5 \sec x + c'$

$= 5 \sec x + c$

Activity 3

$$\begin{aligned} \int (x^2 - 2x + 5)dx &= \frac{x^3}{3} - \frac{2x^2}{2} + 5x + c \\ &= \frac{x^3}{3} - x^2 + 5x + c \end{aligned}$$

The integration of $\sin^2 x$ and $\cos^2 x$

Activity 4

Evaluate

$$\begin{aligned} \int \sin^2 x dx &= \int \tfrac{1-\cos 2x}{2} dx \\ &= \tfrac{1}{2} \int (1 - \cos 2x)dx \\ &= \tfrac{1}{2} \int dx - \tfrac{1}{2} \int \cos 2x dx \\ &= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c \\ &= \frac{x}{2} - \frac{\sin 2x}{4} + c \end{aligned}$$

Activity 5

$$\begin{aligned}\int \cos^2 x dx &= \int \left(\frac{1+\cos 2x}{2}\right) dx \\ &= \frac{x}{2} + \frac{\sin 2x}{4} + c\end{aligned}$$

Activity 6

Evaluate the following, (i) $\int (x^5+1)dx$ (ii) $\int (x^2+2x+5)dx$ (iii) $\int \sqrt{x}dx$

Solution

(i.) $\frac{x^6}{6} + x + c$

(ii.) $\frac{x^3}{3} - \frac{2x^2}{2} + 5x + c$

$\frac{x^3}{3} - x^2 + 5x + c$

(iii.) $\int x^{1/2} = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + \text{c}.$

Activity 7

The gradient of a curve is $6x+ 2$ and it passes through the point (1, 3). Find the equation of the curve.

Solution

$\frac{dy}{dx} = 6x + 2.$

Integrating both sides with respect to x

$\int \frac{dy}{dx}.dx = \int (6x + 2)dx$

$y = 3x^2 + 2x + c$

Since the curve passes through the point (1, 3) $\Rightarrow c = -2$ Hence, the equation of the curve is $y = 3x^2 + 2x - 2$.

Summary

Integration is the reverse process of differentiation. When we integrate, we start with the derivative and then find the expression from which it has been derived. Recall that $\frac{d}{dx}(x^4) = 4x^3$. Therefore, the integral of $4x^3$ with respect to x is x^4. This is written : $\int 4x^3 = x^4$.

Rules of Algebra for Integration

1. Constant multiple rule $\int kf(x)dx = k\int f(x)dx$ (does not work if k varies with x)
2. Rules for negative: $\int -f(x)dx = -\int f(x)dx$ (Rule 1 for $k = -1$)
3. Sum and difference rule; $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.

Self Assessment Questions

1. Evaluate $\int \sin^2 x dx$
2. $\int \frac{1}{\sqrt{x}} dx$
3. $\int \frac{x^3+2x-7}{x} dx$
4. Verify that $F(x) = \frac{1}{3}x^3 + 5x + 2$ is an anti-derivative of $f(x) = x^2 + 5$
5. The process of finding anti-derivative is called ....

Tutor Marked Assignment

- Evaluate the folowing integrals

(1.) $\int 7dx$

(2.) $\int \frac{4}{x^2} dx$

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UNIT 2

Forms of Integration

Introduction

Integration takes several forms. In this unit, you will learn these forms and the rules that apply to each form.

Learning Outcomes

At the end of this unit, you should be able to:

1. Integrate functions of a linear function;
2. Integrate functions of the form $\int \frac{f'(x)}{f(x)}dx$; and
3. Integrate functions of the form $\int f(x).f'(x)dx$.

Main Content

Functions of a linear function

Suppose we have an integrand of the form

$$\int (ax+b)dx$$

. To solve this, we say let u = ax + b

$\frac{du}{dx} = a$

$dx = \frac{du}{a}$

$\therefore \int f(u).\frac{du}{a} = \frac{1}{a}\int f(u)du.$

Activity 1

Evaluate the following

(1.) $\int \sin 3x dx$

Solution

(1.) To integrate $\sin 3xdx$, we know that $\int f(u).\frac{du}{a} = \frac{1}{a}\int f(u)du.$

Let $u = 3x$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$\int \sin 3xdx = \int \sin u$

$\int \sin u = -\cos u$

$\int \sin 3xdx = \frac{1}{3}. -\cos u$

$\int \sin 3xdx = -\frac{1}{3}.\cos 3x + c$

Activity 2

Evaluate

$\int e^{5x+2}$

Solution

Let $u = 5x + 2$

$\frac{du}{dx} = 5$

$dx = \frac{du}{5}$

$\int e^{5x+2} = \int e^{u}.\frac{du}{5}$

$\int e^{5x+2} = \frac{1}{5}e^{5x+2} + c$

Integrals of the form $\int \frac{f'(x)}{f(x)}dx$ and $\int f(x).f'(x)dx$

Any integral in which the numerator is the derivative of the denominator, the result is simply the log of the denominator.

$$\int \frac{f'(x)}{f(x)}dx = In[f(x)] + c$$

Activity 3

$$\int \frac{(2x+3)}{x^2+3x-5}dx = In(x^2+3x-5)+c$$

Because if $u = x^2+3x-5$,

then $\frac{du}{dx} = 2x+3$

So, $dx = \frac{du}{2x+3}$

$\therefore \int \frac{2x+3}{u} \cdot \frac{du}{2x+3} = \int \frac{1}{u} du = Inu + c$

$= In(x^2+3x-5)+c$

Suppose we have an integral of the form $\int f(x) f'(x)dx$, for example $\tan x \, sec^2 \, xdx$. This is a product and not a quotient. Note that $sec^2 \, x$ is the derivative of the function $\tan x$.

We have a product when one factor is the derivative of the other.

Put z = tan x, dz = sec2 xdx ⇒ dx = $\frac{dz}{\sec^2 x}$

$$\int \tan x \sec^2 x dx = \int z dz = \frac{z^2}{2} + c.$$

$$\int \tan x \sec^2 x dx = \frac{\tan^2 x}{2} + c.$$

Activity 4

$$\int \sin x \cos x dx = \int \sin x d(\sin x)$$

If $z = \sin x$, then

$$\int \sin x \cos x dx = \int z dz = \frac{z^2}{z} + c$$

$$= \frac{\sin^2 x}{2} + c$$

Integration of products - Integration by part

If u and v are functions of x, then we know that

$$\frac{d}{du}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides wrt *x*.

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx.$$

Rearranging the terms, we have

$$u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

For convenience, this can be memorized as

$$\int udv = uv - \int vdu.$$

Activity 5

Evaluate $\int x^2 lnx dx$

Solution

$\int udv = uv - \int vdu$

let $u = ln\, x$ and $dv = x^2$

$\Rightarrow du = \frac{1}{x}$ and $v = \frac{x^3}{3}$

$$\int x^2 lnx dx = \frac{x^3}{3}lnx - \int \frac{x^3}{3}.\frac{1}{x}dx$$

$$\int x^2 lnx dx = \frac{x^3}{3}lnx - \int \frac{x^2}{3}dx$$

$$\int x^2 lnx dx = \frac{x^3}{3}lnx - \frac{1}{3}\int x^2 dx$$

$$\int x^2 lnx dx = \frac{x^3}{3}lnx - \frac{1}{3}\frac{x^3}{3}$$

$$\int x^2 lnx dx = \frac{x^3}{3}(lnx - \frac{1}{3}) + c$$

Summary

$$\int f(u).\frac{du}{a} = \frac{1}{a}\int f(u)du$$

$$\int \frac{f'(x)}{f(x)}dx{=}In\left[f(x)\right]+c$$

$$\int udv{=}uv \;\; - \int vdu.$$

Self Assessment Questions

1. Evaluate $\int lnx.\frac{1}{x}dx$.
2. Evaluate $\int \sin x \cos x dx$.

Tutor Marked Assignment

Evaluate

(1.) $\int \frac{2x}{x^2+1}$.

(2.) $\int x^2 e^x dx$.

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UNIT 3

Definite Integral

Introduction

In this unit, you will learn what definite integrals are. An integral is definite if it has an upper limit and a lower one. In other words, it is well defined.

Learning Outcomes

At the end of this unit, you should be able to:

1. define a definite integral;
2. state the properties of a definite integral; and
3. evaluate definite integrals.

Main Content

The integral $\int_a^b f(x)dx$ is called the definite integral of the function $f(x)$ with respect to x between the lower limit a and the upper limit b. The constant of integration c disappears in the subtraction. It appears in both brackets and disappears in subsequent working. The integral $\int_a^b f(x)dx$ geometrically represents the area bounded by the curve $y = f(x)$ and the lines $x = a$, $x = b$ and the axis. If

$$\frac{d}{dx}F(x) = f(x)$$

Then

$$\int_a^b f(x)dx = [F(x) + c]_a^b$$

$$\int_a^b f(x)dx = F(b) + c - (F(a) + c)$$

$$\int_a^b f(x)dx = F(b) + c - F(a) - c$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Properties of Definite Integral

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Activity 1

Evaluate $\int_0^1 2xdx$

Solution

$$\int_0^1 2xdx = \frac{2x^2}{2}\Big|_0^1 = x^2\Big|_0^1 = (1)^2 - (0)^2 = 1$$

Activity 2

Evaluate $\int_{-1}^1 e^t dt$

Solution

$$\int_{-1}^1 e^t dt = e^t\Big|_{-1}^1 = e^1 - e^{-1} = e - \frac{1}{e}.$$

Summary

When an integral has a lower limit and an upper limit, such an integral is a definite integral.

Self Assessment Questions

1. Define a definite integral.
2. List the properties of a definite integral.
3. Evaluate $\int_3^1 (x^2 - 1)dx$.

Tutor Marked Assignment

Evaluate

1. $\int_0^{\frac{\pi}{2}} \cos x dx$.

2. $\int_1^3 (3x^2 + 4x - 5)dx$.

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UNIT 4

Applications of Integration

Introduction

In this unit, you will learn how integration can be applied to physical activities.

Learning Outcomes

At the end of this unit, you should be able to:

1. find the area bounded by a curve;
2. define the volume of solids; and
3. define Kinematics.

Main Content

Area under a curve

To find the area bounded by the curve $y = f(x)$, the $x - axis$ and the coordinates at $x = a$ and $x = b$.

$$A = \int_a^b y dx$$

Activity 1

1. Find the area under the curve $y = x^2 - 2x + 3$ between $x = 1$ and $x = 2$

Solution

$$A = \int_a^b y dx = \int_1^2 (x^2 - 2x + 3)\, dx$$

$$A = \left[\frac{x^3}{3} - x^2 + 3x \right]_1^2 = [\frac{8}{3} - 4 + 6] - [\frac{1}{3} - 1 + 3]$$

$$A = 4\frac{2}{3} - 2\frac{1}{3}$$

$$A = 2\frac{1}{3} units^2$$

Volumes of solids of Revolution

A solid of revolution is formed when a region bounded by part of a curve is rotated about a straight line.

Let V be the volume of the solid generated

$$V = \pi \int_a^b y^2 dx \text{ – Rotation about } x-\text{axis. } y = f(x)$$

$$V = \pi \int_a^b x^2 dx \text{ – Rotation about } y-\text{axis. } x = f(y)$$

Activity 2

Find the volume of the solid generated by revolving the region bounded by $y = x$ the coordinate $x = 2, x = 4$ about the $x-$axis

Solution

Let V be the volume of solid generated,

Then $V = \pi\int_a^b y^2 dx$

$= \pi\int_2^4 x^2 dx$

$= \pi\left[\frac{x}{3}\right]_2^4$

$= \pi\left[\frac{64}{3} - \frac{8}{3}\right]$

$= \frac{56}{3}\pi \text{ unit}^3$

Kinematics

If the velocity is given as a function of time, then, the displacement is the integral of the velocity function with respect to the time variable. Thus, if

$$\frac{ds}{dt} = f(t) \text{ then } \quad s = \int f(t)dt \Rightarrow s = F(t) + c_1 .$$

Similarly, if the acceleration is a function of time, the velocity is the integral of the acceleration function.

Given that

$$\frac{dv}{dt} = g(t) \text{ then } v = \int g(t)dt \Rightarrow v = G(t) + c_2 .$$

Activity 3

The velocity v of a vehicle t sec after a certain instance is given by $v = (3t^2 + 4)m/s$. Determine how far it moves in the interval from $t = 1s$ to $t = 5s$.

Solution

$s = \int_1^5 (3t^2 + 4)\, dt$

$= [t^3 + 4t]_1^5$

$= (125 - 20) - (1 + 4) = 145 - 5$

$= 140\, m.$

Summary

You have learnt the applications of integration in this unit.

Self Assessment Questions

1. Find the area of the finite region enclosed by the two curves $y = 10x - x2$ and $y = 12x - x2$.
2. Define the volume of solids and Kinematics.

Tutor Marked Assignment

- Find the area of the finite region enclosed by the two curves $y = 5x - x^2$ and $y = 5x - x^2$.

- Find the volume of the solid generated by revolving the region bounded by $y = x - x^2$, the coordinates at $x = 0, x = 1$ about the x – axis.

- A particle is projected on a straight line from a point 0, with an initial velocity of $12m/s$. Its acceleration is later given by $(3t^2 + 2t)m/s^2$ Calculate (i) its velocity and (ii) its distance from 0 after $2secs$.

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UNIT 5

Initial Value Problems

Introduction

In the field of differential equations, an initial value problem, also called a Cauchy problem by some authors, is an ordinary differential equation together with a specified value called the initial condition of the unknown function, at a given point in the domain of the solution.

Learning Outcomes

At the end of this unit, you should be able to:

1. define an initial value problem; and
2. solve initial value problems.

Main Content

The problem of finding a function y of x when we know its derivative $\frac{dy}{dx} = f(x)$ and its value y_0 at a particular point x_0 is called an initial value problem. We solve such a problem in two steps. First, we integrate both sides of the equation with respect to x to find the indefinite integral of f.

$$y = f(x) + c \qquad (1)$$

Then we use the fact that $y = y_0$ when $x = x_0$ to find the right value of c.

In this case, $y_0 = f(x_0) + c$, so $c = y_0 - f(x)$.

This implies that $y = f(x) + [y_0 - f(x_0)]$

Activity 1

Find v if $\frac{dv}{dt} = 9.8m/sec^2$ with initial condition $v=0$, $t=0$ or $v(0) = 0$.

$v= \int 9.8dt= 9.8t+c$

$v= 9.8t+c$

Using the initial condition, $c = 0$

$v= 9.8t$

Activity 2

Find the curve whose slope at the point (x, y) is $3x^2$ if the curve is required to pass through the point $(1, -1)$.

Solution

Mathematically, we are asked to solve the initial value problem that consists of the differential equation $\frac{dy}{dx} = 3x^2$ with the initial condition; $y = -1$ when $x = 1$, (i.e, $y(1) = -1$).

$\int 3x^2dx=x^3+c$

Then, we substitute $x = 1$ and $y = -1$ to find c:

$y = x^3 + c$

$-1 = 1^3 + c$

$c = -2$

Hence, the curve is $y = x^3 - 2$

Summary

A solution to an initial value problem is the function $y = f(x)+[y_0 - f(x_0)]$

Self Assessment Questions

1. Define an initial value problem.
2. Solve the initial value problem: $\frac{dy}{dx} = 2x - 7, y = 0$ when $x= 26$

Tutor Marked Assignment

Solve the initial value problems

1. $\frac{dy}{dx} = 12x - 3, y = 0$ when $x = 6$.
2. $\frac{ds}{dt} = \cos t + \sin t, s = 1$ when $t = \pi$.

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UNIT 6

Extrema, Mean Value Theorem

Introduction

In this unit, I will introduce you to one of the most influential theorems in Calculus, the mean value theorem.

Learning Outcomes

At the end of this unit, you should be able to:

1. prove the first derivative theorem for local extreme values; and
2. find the value of c that satisfies the equation $\frac{f(b)-f(a)}{b-a} = f'(c)$ in the conclusion of mean-value theorem.

Main Content

One useful thing we do with derivatives is to find where functions take on their maximum and minimum values. Here, we lay the theoretical ground for finding these extreme values, as they are called.

Definitions

1. A function f has a local maximum value at an interior point c of its domain if $f(x) \leq f(c)$ $\forall$ x in some open interval I about c. The function has an absolute maximum value at c if $f(x) \leq f(c)$ $\forall$ x in the domain.

2. f has a local minimum value at an interior point c of the domain if $f(x) \geq f(c)$ $\forall$ x in and open interval I about c. The function has an absolute minimum value at an interior point c if $f(x) \geq f(c)$ $\forall$ x in the domain.

3. Points in the domain of f where f' is zero or fails to exist are called the critical points of f. Thus, the only point worth considering in a search for a function's extreme values are critical points and end points.

Theorem 1:

The first derivative theorem for local extreme values. If a function f has a local maximum or a local minimum value at an interior point c of an interval when it is defined, and if f' is defined at c, then $f'(c) = 0$.

Proof

We want to show that $f'(c) = 0$. We show first, that $f'(c)$ cannot be positive and second, that $f'(c)$ cannot be negative either. Hence $f'(c) = 0$ because in the entire real number system, only one number is neither negative nor positive and that is zero. Suppose f has a local maximum value at $x = c$, so that $f(x) \leq f(c) \forall$ value x near c. Since c is an interior point of f''s domain, the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad (2)$$

defining $f'(c)$ is two sided. This means that the right hand and the left-hand limits both exist at $x = c$ and both equal to $f'(c)$.

When we examine these limits separately, we find that

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0 \qquad (2)$$

because immediately to the right of c, $f(x) \leq f(c)$ and $x - c > 0$. Similarly,

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0 \qquad (3)$$

because immediately to the left of c, $f(x) \leq f(c)$ and $x - c < 0$. Inequality (3) says that $f'(c)$ cannot be greater than zero, whereas (3) says that $f'(c)$ cannot be less than zero. So $f'(c) = 0$

Rolle's Theorem:

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b) = 0$, then there is at least one number c between a and b at which $f'(c) = 0$.

Rolle's theorem says that a smooth curve has at least one horizontal tangent between any two points where it crosses the x-axis.

The Mean Value Theorem

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one number between a and b at which $\frac{f(b) - f(a)}{b - a} = f'(c)$.

The hypothesis of the mean value theorem do not require f to be differentiable at either a or b. Continuity at a and b is enough.

Activity 1

The function $f(x) = x^2$ is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the mean value theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $\frac{4\ \ 0}{2\ \ 0} = 2$.

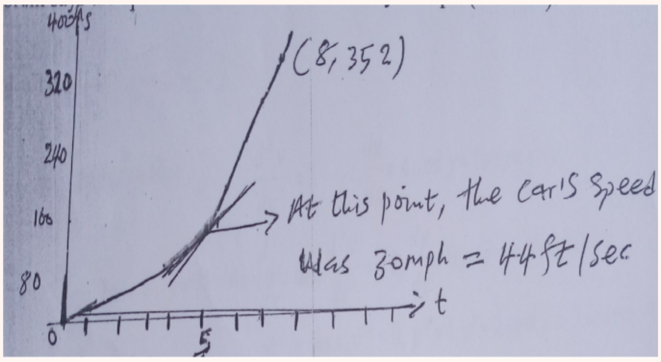
Since $f'(x) = 2x$, it implies that $f'(c) = 2c$

Therefore, from the mean value theorem,

$2 = 2c \Rightarrow c = 1$.

Activity 2

If a car accelerating from zero takes 8 *sec* to go 352*ft*, its average velocity for the 8 second interval is $\frac{352}{8} = 44ft/sec$ at some point during the acceleration; the mean value theorem says the speedometer must read exactly 30*mhs* (44*ft*/*sec*).



Distance vs elapsed time for the car in the above activity.

The mean value theorem for definite integrals If f is continuous on the closed interval $[a, b]$, then at some point c in the interval $[a, b]$,

$$f'(c) = \frac{1}{b - a} \int_a^b f(x)dx \qquad (4)$$

The number on the right-hand side of (4) is called the average value or mean value of f on $[a, b]$.

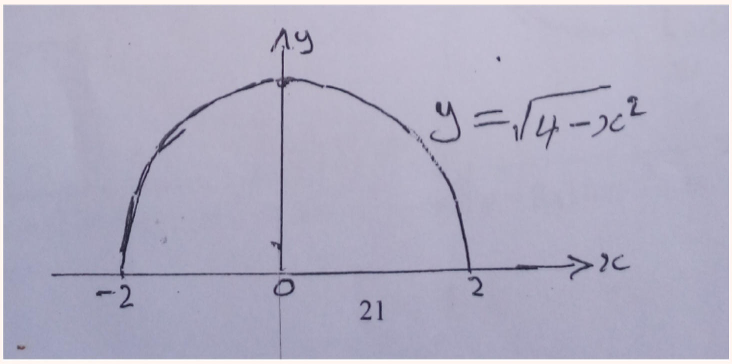
Definition

The average value (mean value) of an integrable function f on $[a, b]$ is

$$\frac{1}{b - a} \int_a^b f(x)dx$$

Activity 3

Evaluate $\int_{-2}^{2} \sqrt{4 - x^2}dx$ by regarding the integral as area under the graph of $f(x) = \sqrt{4 - x^2}$ from -2 to 2. Hence,n d its average.



Solution

The graph of f from -2 to 2 is the curve $y = \sqrt{4 - x^2}$, a semicircle of radius 2. The area below the semicircle is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 = \pi 2$$

Hence, the area is the value of the integral,

$$\int_{-2}^{2} \sqrt{4 - x^2}dx = 2\pi$$

Average value of f on $[-2,2] = \frac{1}{2-(-2)} \int_{-2}^{2} \sqrt{4-x^2} = \frac{1}{4}(2\pi) = \frac{\pi}{2}$

Summary

If a function $f(x)$ is a differentiable function over the closed interval $[a, b]$, then at some point between a and b, there is at least one number c such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Physical Interpretation

If we think of the number $\frac{f(b) - f(a)}{b - a}$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the mean value theorem says that at some interior point, the instantaneous change must equal the average change over the entire interval.

Differentiation implies that the function is also continuous. Rolle's theorem says that a differentiable curve has at least one horizontal tangent between any points where it crosses a horizontal line. It may have one or more curves.

Self Assessment Questions

(1.) $f(x) = x^2 + 2x - 1,\ 0 \leq x \leq 1$

(2.) $f(x) = x^{\frac{2}{3}},\quad 0 \leq x \leq 1$

(3.) Consider the function $f(x) = x^3 - 4x$ in the interval $[-3,3]$. Find the value of c.

Tutor Marked Assignment

1. Consider the function $f(x) = x^3 - 12x$ in the interval $[-2, 2]$. Find the value of c
2. If $f(x) = \sin 3x$ and $a = 0$, $b = \pi/6$, find c.
3. $f(x) = x + \frac{1}{x}$, $\frac{1}{2} \leq x \leq 2$, find c.
4. $f(x) = \int x - 1$, $1 \leq x \leq 3$, find c.

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UNIT 7

Taylor's Series and Maclaurin's Series

Introduction

In this unit, you will learn about Taylor's and Maclaurin's series. Maclaurin's series is the Taylor's series generated by a function f at the origin.

Learning Outcomes

At the end of this unit, you should be able to:

1. define the Taylor's series;
2. define the Maclaurin's series; and
3. find the Taylor and Maclaurin's series generated by a function.

Main Content

Let f be a function with derivative of all order throughout some interval containing as many a as an interior point, the Taylor series generated by f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f\,'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^n(a)}{n!}(x - a)^n + \ldots$$

The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(0)}(a)}{k!}x^k = f(a) + f\,'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^n(0)}{n!}x^n + \ldots$$

That is, the Taylor series generated by f at $x = 0$

Activity 1

Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\frac{1}{x}$?

Solution

We need to compute $f(2), f'(2), f''(2)$, and so on. Taking derivatives, we get

$f(x) = x^{-1}, \qquad f(2) = 2^{-1} = \frac{1}{2}$

$f'(x) = -x^{-2}, \qquad f'(2) = -\frac{1}{2^2}$

$f''(x) = 2!x^{-3}, \qquad f''(2) = 2^{-3} = \frac{1}{2^3}$

$f'''(x) = -3!x^{-4}, \qquad \frac{f'''(2)}{3!} = -\frac{1}{2^4}$

$\vdots f^{(n)}(x) = (-)^n n! x^{-(n+1)}, \qquad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$

The Taylor series is

$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \ldots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \ldots$

$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \ldots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \ldots$

This is a geometric series with the first term $\frac{1}{2}$ and ratio $r = \frac{(x-2)}{2}$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{a}{1-r} = \frac{\frac{1}{2}}{1 + \frac{(x-2)}{2}} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

The Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$ converges to $\frac{1}{x}$ for $|x-2| < 2$ or $0 < x < 4$.

Summary

Taylor's series and Maclaurins's series have been discussed in this unit. When a function f is generated at the origin, Taylor's series becomes Maclaurin's series.

Self Assessment Questions

1. State Taylor's series.

2. State Maclaurin's series.

Tutor Marked Assignment

Find the Taylor's series generated by $f(x) = \frac{2}{x}$ at $a = 2$.

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
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Module 4

Partial Derivatives

Units

UNIT 1

Differentiation of Functions of more than one variable

Introduction

Let $z = f(x, y)$ be defined on a domain D of the (x, y) plane. If we keep y constant, then the function $f(x, y_0)$ depends only on the variable x and it has a derivative with respect to x at x_0 which is computed in the usual way. The derivative is called the partial derivative of $f(x, y)$ with respect to x at x_0, y_0.

Learning Outcomes

At the end of this unit, you should be able to:

1. differentiate functions of more than one variable; and
2. differentiate higher order partial derivatives.

Main Content

Partial derivatives are what we get when we hold all but one of the independent variables of a function constant and differentiate with respect to (w.r.t) that one variable.

The partial derivative of $f(x, y)$ w.r.t. x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

provided the limit exists. (The letter ∂ is a special round d).

Partial derivative of $f(x, y)$ wrt y at point (x_0, y) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dx} f(x_0, y)\right|_{y=y_0} = lim_{k\to 0} \frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}$$

provided the limit exists.

Activity 1

Find the value of $\frac{\partial f}{\partial x}$ at the point $(4, -5)$ if $f(x, y) = x^2 + 3xy + y - 1$

Solution

We regard y as a constant and differentiate w.r.t x;

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3y + 0 - 0 = 2x + 3y$

At $(4, -5)$ $\quad \frac{\partial f}{\partial x} = 2(4) + 3(-5) = -7$

Activity 2

Find the value of $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if $f(x, y) = x^2 + 3xy + y - 1$

Solution

Here we regard x as a constant and differentiate w.r.t y;

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3x + 1 - 0 = 3x + 1$

At $(4, -5)$, $\frac{\partial f}{\partial y} = 3(4) + 1 = 13$

Activity 3

Find the value of $\frac{\partial f}{\partial y}$ if $f(x, y) = y \sin xy$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}(\sin xy) + \sin xy \frac{\partial}{\partial y}(y)$$

$$\frac{\partial f}{\partial y} = y \cos xy \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + sinxy$$

Activity 4

Find the value of f_x if $f(x,y) = \dfrac{2y}{y+\cos x}$

Solution

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$f_x = \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2}$$

$$f_x = \frac{2y\sin x}{(y+\cos x)^2}$$

Activity 5

Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x + y$ defines z as the dependent variable of the two independent variables x and y

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz-1}$$

Activity 6

If x, y, and z are independent variables and $f(x,y,z) = x\sin(y+3z)$.

Then $\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}(x\sin(y+3z))$

$$\frac{\partial f}{\partial z} = x\frac{\partial}{\partial z}\sin(y+3z)$$

$$\frac{\partial f}{\partial z} = x\cos(y+3z)\frac{\partial}{\partial z}(y+3z)$$

$$\frac{\partial f}{\partial z} = 3x\cos(y+3z).$$

Summary

To differentiate partially, you keep the same variables constant and differentiate with one variable. The process continues by adding each of the results. The partial derivative of $F(x,y)$ with respect to x and y are denoted by

$$\frac{\partial F}{\partial x}\left[or\, F_x, F_x(x,y), \frac{\partial F}{\partial x}|_y\right] \qquad \frac{\partial F}{\partial x}\left[or\, F_y, or\, F_y(x,y)\, \frac{\partial F}{\partial x}|_x\right]$$

respectively, the latter notation being used when it is needed to emphasize which variables are held constant.

Self Assessment Questions

1. Let $f(x, y) = 2x^2y + 3xy^2$. Find fx.

2. Let $f(x, y) = 2x^2y + 3xy^2$. Find fy.

Tutor Marked Assignment

1. Let $f(x, y, z) = 2x^2y + 3xz^2$. Find f_x, f_y, f_z.
2. Let $f(x, y, z) = 7x^3y + 3yz^2 - 5z^2$. Find f_x, f_y, f$_z$.

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)

- Hoffmann L.D. and Bradley G. L. (2004). Calculus for Business, Economics, and the Social Life Sciences. Eight Edition, McGraw Hill Higher Education, Boston.

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UNIT 2

Second Order Partial Derivatives

Introduction

In this unit, partial derivatives shall be extended to higher orders.

Learning Outcomes

At the end of this unit, you should be able to:

1. find the partial derivatives of higher orders.

Main Content

If $f(x,y)$ has partial derivatives at each point (x,y) in a region, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are themselves functions of x and y, which may also have partial derivatives. The second derivatives are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = F_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = F_{yy}.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = F_{yx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = F_{xy}$$

Activity 1

If $f(x,y) = 2x^3 + 3xy^2$ Find $F_{xx},\ F_{yy},\ F_{xy},\ F_{yx}$

Also, find $F_{xx}(1,2),\ F_{yy}(1,2),\ F_{xy}(1,2),\ F_{yx}(1,2)$

Solution

$$F_x = \frac{\partial F}{\partial x} = 6x^2 + 3y^2$$

$$F_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}(6x^2 + 3y^2) = 12x$$

$$F_y = \frac{\partial F}{\partial y} = 6xy$$

$$F_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}(6xy) = 6x$$

$$F_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial y}(6x^2 + 3y^2) = 6y$$

$$F_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial x}(6xy) = 6y$$

$$F_{xx}(1,2) = 12 \quad F_{yy}(1,2) = 6 \quad F_{xy}(1,2) = F_{yx}(1,2) = 12$$

Summary

The second order partial derivatives are given as

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2} = F_{xx}. \qquad \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2} = F_{yy}.$$

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y} = F_{yx}. \qquad \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y \partial x} = F_{xy}.$$

Self Assessment Questions

1. If $f(x,y) = 5x^3 + 3x^2y + 4y^3$, find F_{xx}, F_{yy}, F_{xy}, F_{yx}. Also, find $F_{xx}(1,2)$, $F_{yy}(1,2)$, $F_{xy}(1,2)$, $F_{yx}(1,2)$.

2. If $f(x,y) = 3x^4 + 3xy^2 + 7xy$, find F_{xx}, F_{yy}, F_{xy}, F_{yx}.

Also, find $F_{xx}(1,3)$, $F_{yy}(1,3)$, $F_{xy}(1,3)$, $F_{yx}(1,3)$.

Tutor Marked Assignment

(1.) If $f(x,y) = x\cos y + ye^x$, find F_{xx}, F_{yy}, F_{xy}, F_{yx}.

(2.) If $f(x,y) = x\sin y + y^2e^x$, find F_{xx}, F_{yy}, F_{xy}, F_{yx}.

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- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
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UNIT 3

Chain Rule

Introduction

In this unit, we introduce the chain rule. If y is a differentiable function of u and u is a differentiable function of x, then, the chain rule is applicable.

Learning Outcomes

At the end of this unit, you should be able to:

1. explain the concept of the chain rule; and
2. solve problems using the chain rule.

Main Content

If y is a differentiable function of u, and u is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Suppose that fog is the composite of the differentiable functions $y = f(u)$ and $u = g(x)$, then fog is a differentiable function of x whose derivative at each value of x is $(fog)'(x) = f'(g(x))\,g'(x)$.

Activity 1

Suppose fog is a composite of the differentiable functions $y = f(u)$ and $u = g(x)$ and that $g'(2) = 4$ and $f'(g(2)) = -5$. Find (fog) at $x = 2$

Solution

$(fog)'(2) = f'(g(2)).\,g'(2) = (-5).4 = -20$

Activity 2

Find $\frac{dy}{dx}$ at $x = 0$ if $y = cos\left(\frac{\pi}{2} - 3x\right)$

Solution

Let y = cos u and u = $\frac{\pi}{2} - 3x$

$$\frac{dy}{dx} = \frac{dy}{du}\bigg|_{u=\frac{\pi}{2}} \cdot \frac{du}{dx}\bigg|_{x=0} \quad (u = \frac{\pi}{2} \text{ when } x = 0)$$

$$= -\sin u\big|_{u=\frac{\pi}{2}}(-3)$$

$$= 3\sin\frac{\pi}{2} = 3(1) = 3$$

Activity 3

Find $\frac{dy}{dx}$ if $y = \sin(x^2 - 4)$

Solution

Take $y = \sin u$, $u = x^2 - 4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 2x\cos u$$

$$= 2x\cos(x^2 - 4).$$

Remark:

The chain rule for functions of two independent variables $w = f(x, y)$, is differentiable. If x and y and are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Activity 4

Use the chain rule to find the derivative of $x = \cos t, y = \sin t$. What is the derivative at $= \frac{\pi}{2}$?

Solution

We evaluate the right hand side with $w = xy, x = \cos t, y = \sin t$

$$\frac{\partial w}{\partial x} = y = \sin t, \quad \frac{\partial w}{\partial y} = x = \cos t,$$

$$\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t = \cos 2t$$

$$\left(\frac{dw}{dt}\right)_{t=\frac{\pi}{2}} = \cos\left(2.\frac{\pi}{2}\right) = \cos \pi = 1.$$

Chain Rule for Functions of 3 Independent Variables

If $w = f(x, y, z)$ is differentiable and x, y and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Activity 5

Find $\frac{dw}{dt}$ if $xy{+}z{,}x{=}\cos t,\ y=\sin t,\ z{=}t$. What is the derivative's value at $t{=}\ 0$?

Solution

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt}=(y)(-\sin t)+(x)(\cos t)+(1)(1)$$

$$\frac{dw}{dt}=(\sin t)(-\sin t)+(\cos t)(\cos t)+(1)$$

$$\frac{dw}{dt}=-\sin^2 t+\cos^2 t+1$$

$$\frac{dw}{dt}=\left(\frac{dw}{dt}\right)_{t=0}=-\sin^2(0)+\cos^2(0)+1=1+1=2.$$

Summary

If $w=f(x,y,z)$ is differentiable and x,y and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$$

Self Assessment Questions

Given that $w=xy$ and $x=\cos t$ and $y=\sin t$. Find the derivative of w at $t=\pi/2$.

Tutor Marked Assignment

1. If $T = x^3 - xy + y^3$ and $x = \lambda \cos \theta$ and $y = \lambda \sin \theta$, find (a) $\frac{\partial T}{\partial}$ (b) $\frac{\partial T}{\partial}$.

2. If $u = z \sin y/x$ where $x = 3r^2+2s$ and $y = 4r-2s^3$ and $z = 2r^2-3s^2$,find (a) $\frac{\partial u}{\partial r}$ (b) $\frac{\partial u}{\partial s}$.

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UNIT 4

Lagrange Multipliers

Introduction

The Lagrange multiplier is a method for finding the maxima and minima of constrained functions. Today, the method is important in economics, engineering and mathematics itself.

Learning Outcomes

At the end of this unit, you should be able to:

1. define the Lagrange multiplier; and
2. state the method of Lagrange multiplier.

Main Content

The Lagrange multiplier method is used for finding the maxima and minima of constrained functions. That is, finding maximum and minimum of $f(x, y, z) = 0$ subject to $\phi(x, y, z) = 0$. This consists of forming an auxiliary function $G(x, y, z) = f + \lambda\phi$ or $\lambda f + \phi = 0$. The λ is called the Lagrange multiplier.

Activity 1

Find the point $P(x, y, z)$ on the plane: $2x + y - z - 5 = 0$ that lies closest to x, the origin.

Solution

The problem asks us to find the minimum value of the function

$\left|\overrightarrow{OP}\right| = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = \sqrt{x^2 + y^2 + z^2}$ subject to the constraint that $2x+y-z-5=0$

Since $\left|\overrightarrow{OP}\right|$ has a minimum value whenever the function

$f(x, y, z) = x^2 + y^2 + z^2$

has a minimum value, then we may solve the problem by finding the minimum value of $f(x,y,z)$ subject to the constraint $2x + y - z - 5 = 0$. If we regard x and y as the independent variables in this equation and write z as $z = 2x + y - 5$, the problem reduces to one of finding the points (X, Y) at which the function $h(x,y) = f(x,y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2$ has its minimum value or values. Since the domain of h is the entire xy-plane, any minima that h might have must occur at points where

$$h_x = 2x + 2(2x + y - 5)(2) = 0$$
$$h_y = 2y + 2(2x + y - 5) = 0$$

This leads to $5x + 2y = 10$

$2x + 2y = 5$

$x = \frac{5}{3},\ y = \frac{5}{6}$

$z = 2x + y - 5$

$= - 2(\frac{5}{3}) + \frac{5}{6} - 5 = -\frac{5}{6}$

Therefore, the closest point we seek is $P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$

Summary

Generally, the Lagrange multiplier method says that the extreme values of a function $f(x, y, z)$ whose variables are subject to constraint $g(x, y, z) = 0$ are to be found at the point where $\nabla f = \lambda \nabla g$ for some scalar λ (called a Lagrange multiplier)

Self Assessment Questions

Using the method of Lagrange multipliers, find the relative minimum of the function $f(x,y) = 2x^2 + y^2$ subject to the constraint $x+y = 1$.

Tutor Marked Assignment

- Find the point $P(x, y, z)$ on the plane: $x + 2y - z - 7 = 0$ that lies closest to x the origin.

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)
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Module 5

Linearisation and Differentials

Units

Unit 1 - Linearisation

Unit 2 - Multiple integrals

UNIT 1

Linearisation

Introduction

The linearisation of a function $f\ (x, y)$ at a point (x_0, y_0) where f is differentiable is the function $\ell(x,y) = f(x_0,y_0) + fx(x_0,y_0)(x - x_0) + fy(x_0, y_0)(\ y - y_0)$. The approximation $\ell\ (x, y) = f\ (x, y)$ is the standard linear approximation of f at (x_0,y_0).

Learning Outcomes

At the end of this unit, you should be able to:

1. define the linearisation of a function;
2. linearize a function; and
3. linearize a function of more that two variables.

Main Content

A function $\omega = f\ (x, y, z)$ of three independent variables is differentiable at a point $P_0(x_0,y_0, z_0)$ if f_x, f_y, and f_z exist there and f satisfies an equation of the form $\nabla\omega - f_x\nabla_x + f_y\nabla_y + f_z\nabla_z + \in_1 \nabla_x + \in_2 \nabla_y + \in_3 \nabla \in_z$ in which f_x, f_y and f_z are evaluated at P_0 and $\in_1, \in_2, \in_3 \rightarrow 0$ as $\nabla_x, \nabla_y, \nabla_z \rightarrow 0$. The functions of two variables which arise in science and mathematics can be complicated and we sometimes need to replace them with simpler functions that give the accuracy required for specific application without being so hard to work with.

Definition:

The linearisation of a function $f(x,y)$ at a point (x_0,y_0) where f is differentiable is the function.

$L(x,y)= f(x_0,y_0)+ f(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$ The approximation $L(x,y) \approx f(x,y)$

Activity 1

Linearize $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at point $(3,2)$.

Solution

$$f(x_0, y_0) = (x^2 - xy + \frac{1}{2}y^2 + 3)_{(3,2)} = 8$$

$$f_x(x_0, y_0) = \frac{\partial}{\partial x}(x^2 - xy + \frac{1}{2}y^2 + 3)_{(3,2)}$$

$$= (2x - y)_{(3,2)} = 4$$

$$f_y(x_0, y_0) = \frac{\partial}{\partial y}(x^2 - xy + \frac{1}{2}y^2 + 3)_{(3,2)} = (-x + y)_{(3,2)} = -1,$$

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2$$

The linearisation of f at $(3,2)$ is $L(x,y) = 4x - y - 2$

Functions of more than two variables

The linearisation of a differentiable function $f(x,y,z)$ at a point $P_0(x_0, y_0, z_0)$ is $L(x,y,z) = f(p_0) + f_x(p_0)(x - x_0) + f_y(p_0)(y - y_0) + f_z(p_0)(z - z_0)$

Activity 2

Find $L(x, y, z)$ of $f(x, y, z) = x^2 - xy + 3 \sin z$ at the point $(x_0, y_0, z_0) = (2, 1, 0)$.

Solution

$f(2, 1, 0) = 2$

$f_x(2, 1, 0) = 3$

$f_y(2, 1, 0) = -2$

$f_z(2, 1, 0) = 3$

$L(x, y, z) = 2 + 3(x - 2) + (-2)(y - 1) + 3(z - 0) = 3x - 2y + 3z - 2$

Summary

In this unit, you have learnt the linearization method.

Self Assessment Questions

1. Find the linearisation of $L(x, y)(L(x, y, z))$ of the functions at each point.

i. $f(x, y) = x^2 + y^2 + 1$ at $(1, 1)$

ii. $f(x, y) = e^x \cos y$ at $(0, \frac{\pi}{2})$

iii. $f(x, y) = \exp(2y - x)$ at $(0, 0)$.

Tutor Marked Assignment

1. Find $L(x, y, z)$ of $f(x, y, z) = y + yz + xz$ at $(1, 1, 1)$.

2. Find $L(x, y, z)$ of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(1, 2, 2)$.

3. Find $L(x, y, z)$ of $f(x, y, z) = e^x \cos(y + z)$ at $(0, \frac{\pi}{2}, 0)$.

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)

- Hoffmann L.D. and Bradley G. L. (2004). Calculus for Business, Economics, and the Social Life Sciences. Eight Edition, McGraw Hill Higher Education, Boston.

- Jackson, J.S.G (2002). Mathematical Methods for Mathematics, Science and Engineering students Volume 1, Published by Ghana University Press, Accra.

- Stroud K. A. and Dexter J. Booth (2007). Engineering Mathematics. Palgrave Macmillan. Houndsmill, Basingstoke, Hampshire RG21 6XS and 175 Fifth Avenue, New York.

Further Reading

- Erwin Kreyszig (2011) Advanced Engineering Mathematics, Wiley, New Jersey
- Robert T. Smith and Roland B. Minton. (2011) Multivariable calculus, McGraw-Hill Education, London, United States, Europe.

UNIT 2

Multiple Integrals

Introduction

The basic theorem for evaluating double integrals says that every double integral can be evaluated in stages using the single integration methods already at our command. When two variables are involved, we shall integrate f(x, y) by holding one variable fixed and integrating the other.

Learning Outcomes

At the end of this unit, you should be able to:

1. define a line integral; and
2. integrate a continuous function f(x, y) over a bounded region in space.

Main Content

If a field exists in the $x - y$ plane, producing a force Ft on a particle at k, then F_t can be resolved into two components: P in the $x-$ direction, and Q in the $y-$ direction. Then the work done in moving the particle through a small distance δ_s from K to L along the curve is approximately $F_t\delta_s$. This can be stated as $P\,dx + Qdy$. So the total work done in moving a particle along the curve from A to B is given by

$$\int_{AB} F_t ds = \int_{AB} (Pdx + Qdy)$$

where P and Q are functions of x and y. Such an integral is called a line integral since integration is carried out along the path of the particular curve c joining A and B. Line integrals are integrals of function over curves.

If the curve is denoted by a single letter c for example, the notation for the integral is $\int_c f(x, y, z)ds$ We now show how to integrate a continuous function $f(x, y)$ over a bounded region in the xy-plane.

Activity 1

Evaluate $\int_c (x + 3y)dx$ from $A(0, 1)$ to $B(2, 5)$ along the curve $y = 1 + x^2$.

Solution

The line integral is of the form $I = \int_c (P\,dx + Qdy)$. where, in this case, $Q = 0$ and c is the curve $y = 1 + x^2$.

$$I = \int_c (Pdx + Qdy) = \int_c (x + 3y)dx = \int_o^2 (x + 3 + 3x^2)dx$$

$$= \left[\frac{x^2}{2} + 3x + x^3\right]_0^2$$

$$= 16$$

Activity 1

Find the volume of the prism whose base is the triangle in the xy-plane bounded by x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane. $z = f(x,y) = 3 - x - y$

Solution

For any x between 0 and 1, y may vary from $y = 0$ to $y = x$. Hence,

Hence, $V = \int_0^1 \int_0^x (3 - x - y)\, dydx = \int_0^1 \left(\int_0^x (3 - x - y)dy\right) dx.$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1$$

Or

$$V = \int_0^1 \int_y^1 (3 - x - y)dxdy$$

$$= \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2}y - 4y + \frac{3}{2}y^2\right) dy$$

$$= \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1$$

The two integrals are equal, as they should be.

Activity 3

Calculate $\int \int_R \frac{\sin x}{x} dA$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution

If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x} dy\right) dx = \int_0^1 \left(y\frac{\sin x}{x}\right)_{y=0}^{y=x} dx$$

$$= \int_0^1 \sin x \, dx = -\cos(1) + 1$$

$$\cong 0.0002.$$

If we reverse the order of integration and attempt to calculate $\int_0^1 \int_y^1 \frac{\sin x}{x} dxdy$ we are stopped by the fact that $\int \frac{\sin x}{x} dx$ can not be expressed in terms of elementary function.

There is no general rule for predicting which order of integration (if either) will be a good one in circumstances like these, so do not worry about how to start your integrations. Just forge ahead and if the order you choose first does not work, try the other.

Definition:

The volume of a closed bounded region D in space is the value of the integral volume of $D = \int\int\int dv$. Suppose we want to integrate a continuous function $F(x, y, z)$ over a region D, the integral of f over D is evaluated as

$$\int_D \int \int F(x,y,z)\,dv = \int_R \int \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)dzdydx.$$

Summary

Line integrals are integrals of functions over curves. The basic theorem for evaluating double integrals says that every double integral can be evaluated in stages, using the single- integration methods already at our command.

Self Assessment Questions

1. Define a line integral.

2. Define the volume of a bounded region.

3. Evaluate $I = \int c[(x^2 + y)dx + (x - y^2)dy]$ from $A(0, 2)$ to $B(2, 5)$ along the curve $y = 2 + x$.

Tutor Marked Assignment

1. Evaluate $I = \int_c [(x^2 + 2y)dx + xydy]$ from $A(0,0)$ to $B(1,4)$ along the curve $y = 4x^2$.

2. Evaluate $\int_{0,1}^{1,2} [(x^2 - y)dx + (y^2 + x)dy]$ along (i) a straight line from (0, 1) to (1, 2) (ii) straight lines from (0, 1) to (1, 1) and then from (1, 1) to (1, 2). (iii) the parabola $x = t, y = t^2 + 1$.

References

- Fadipe-Joseph O.A. (2011). Lecture notes on Mathematical Methods I, University of Ilorin, Ilorin, (unpublished)

- Jackson, J.S.G (2002). Mathematical Methods for Mathematics, Science and Engineering students Volume 1, Published by Ghana University Press, Accra.

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Further Reading

- Hoffman L.D. and Bradley G.L (2004). Calculus for Business,Economics,and the Social and life Sciences. English Edition, McGraw Hill Higher Education, Boston.