Robust Standard Errors in Small Samples

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Description

This package implements small-sample degrees of freedom adjustments to robust and cluster-robust standard errors in linear regression, as discussed in Imbens and Kolesár [2016]. The implementation can handle models with fixed effects, and cases with a large number of observations or clusters [1].

```
library(dfadjust)
```

To give some examples, let us construct an artificial dataset with 11 clusters

```
set.seed(7)
d1 <- data.frame(y = rnorm(1000), x1 = c(rep(1, 3), rep(0,
    997)), x2 = c(rep(1, 150), rep(0, 850)), x3 = rnorm(1000),
    cl = as.factor(c(rep(1:10, each = 50), rep(11, 500))))
```

Let us first run a regression of `y` on `x1`. This is a case in which, in spite of moderate data size, the effective number of observations is small since there are only three treated units:

```
r1 <- lm(y ~ x1, data = d1)
## No clustering
dfadjustSE(r1)
#>
#> Coefficients:
#>              Estimate HC1 se HC2 se Adj. se     df p-value
#> (Intercept)   0.00266 0.0311  0.031  0.0311 996.00   0.932
#> x1            0.12940 0.8892  1.088  2.3743   2.01   0.916
```

We can see that the usual robust standard errors (`HC1 se`) are much smaller than the effective

[1] We thank Ulrich Müller for suggesting to us the lemma below.

standard errors (`Adj. se`), which are computed by taking the `HC2` standard errors and applying a degrees of freedom adjustment.

Now consider a cluster-robust regression of `y` on `x2`. There are only 3 treated clusters, so the effective number of observations is again small:

```
r1 <- lm(y ~ x2, data = d1)
# Default Imbens-Kolesár method
dfadjustSE(r1, clustervar = d1$cl)
#>
#> Coefficients:
#>             Estimate HC1 se HC2 se Adj. se   df p-value
#> (Intercept)  -0.0236 0.0135 0.0169  0.0222 4.94  0.2215
#> x2            0.1778 0.0530 0.0621  0.1157 2.43  0.0826
# Bell-McCaffrey method
dfadjustSE(r1, clustervar = d1$cl, IK = FALSE)
#>
#> Coefficients:
#>             Estimate HC1 se HC2 se Adj. se   df p-value
#> (Intercept)  -0.0236 0.0135 0.0169  0.0316 2.42  0.2766
#> x2            0.1778 0.0530 0.0621  0.1076 2.70  0.0731
```

Now, let us run a regression of `y` on `x3`, with fixed effects. Since we're only interested in `x3`, we specify that we only want inference on the second element:

```
r1 <- lm(y ~ x3 + cl, data = d1)
dfadjustSE(r1, clustervar = d1$cl, ell = c(0, 1, rep(0,
    r1$rank - 2)))
#>
#> Coefficients:
#>          Estimate HC1 se HC2 se Adj. se   df p-value
#> Estimate   0.0261 0.0463 0.0595  0.0928 3.23   0.688
dfadjustSE(r1, clustervar = d1$cl, ell = c(0, 1, rep(0,
    r1$rank - 2)), IK = FALSE)
#>
#> Coefficients:
#>          Estimate HC1 se HC2 se Adj. se   df p-value
#> Estimate   0.0261 0.0463 0.0595  0.0928 3.23   0.688
```

Finally, an example in which the clusters are large. We have 500,000 observations:

```
d2 <- do.call("rbind", replicate(500, d1, simplify = FALSE))
d2$y <- rnorm(length(d2$y))
r2 <- lm(y ~ x2, data = d2)
summary(r2)
#>
#> Call:
#> lm(formula = y ~ x2, data = d2)
#>
#> Residuals:
```

```
#>     Min      1Q  Median      3Q     Max
#> -5.073 -0.675  0.000  0.675  4.789
#>
#> Coefficients:
#>              Estimate Std. Error t value Pr(>|t|)
#> (Intercept) -0.000991   0.001535   -0.65     0.52
#> x2          -0.003590   0.003963   -0.91     0.37
#>
#> Residual standard error: 1 on 499998 degrees of freedom
#> Multiple R-squared:  1.64e-06,   Adjusted R-squared:  -3.59e-07
#> F-statistic: 0.821 on 1 and 5e+05 DF,  p-value: 0.365
# Default Imbens-Kolesár method
dfadjustSE(r2, clustervar = d2$cl)
#>
#> Coefficients:
#>              Estimate  HC1 se  HC2 se Adj. se   df p-value
#> (Intercept) -0.000991 0.00133 0.00168 0.00294 2.66   0.603
#> x2          -0.003590 0.00483 0.00568 0.00997 2.65   0.578
# Bell-McCaffrey method
dfadjustSE(r2, clustervar = d2$cl, IK = FALSE)
#>
#> Coefficients:
#>              Estimate  HC1 se  HC2 se Adj. se   df p-value
#> (Intercept) -0.000991 0.00133 0.00168 0.00315 2.42   0.607
#> x2          -0.003590 0.00483 0.00568 0.00984 2.70   0.577
```

Methods

This section describes the implementation of the Imbens and Kolesár [2016] and Bell and McCaffrey [2002] degrees of freedom adjustments.

There are S clusters, and we observe n_s observations in cluster s, for a total of $n = \sum_{s=1}^{S} n_s$ observations. We handle the case with independent observations by letting each observation be in its own cluster, with $S = n$. Consider the linear regression of a scalar outcome Y_i onto a p-vector of regressors X_i,

$$Y_i = X_i'\beta + u_i, \qquad E[u_i \mid X_i] = 0.$$

We're interested in inference on $\ell'\beta$ for some fixed vector $\ell \in \mathbb{R}^p$. Let X, u, and Y denote the design matrix, and error and outcome vectors, respectively. For any $n \times k$ matrix M, let M_s denote the $n_s \times k$ block corresponding to cluster s, so that, for instance, Y_s corresponds to the outcome vector in cluster s. For a positive semi-definite matrix M, let $M^{1/2}$ be a matrix satisfying $M^{1/2\prime}M^{1/2} = M$, such as its symmetric square root or its Cholesky decomposition.

Assume that

$$E[u_s u_s' \mid X] = \Omega_s, \quad \text{and} \quad E[u_s u_t' \mid X] = 0 \quad \text{if } s \neq t.$$

Denote the conditional variance matrix of u by Ω, so that Ω_s is the block of Ω corresponding to cluster s. We estimate $\ell'\beta$ using OLS. In R, the OLS estimator is computed via a QR decomposition,

$X = QR$, where $Q'Q = I$ and R is upper-triangular, so we can write the estimator as

$$\ell'\hat{\beta} = \ell' \left(\sum_s X_s' X_s \right)^{-1} \sum_s X_s Y_s = \tilde{\ell}' \sum_s Q_s' Y_s, \qquad \tilde{\ell} = R^{-1\prime} \ell.$$

It has variance

$$V := \operatorname{var}(\ell'\hat{\beta} \mid X) = \ell' \left(X'X \right)^{-1} \sum_s X_s' \Omega_s X_s \left(X'X \right)^{-1} \ell = \tilde{\ell}' \sum_s Q_s' \Omega_s Q_s \tilde{\ell}.$$

Variance estimate

We estimate V using a variance estimator that generalizes the HC2 variance estimator to clustering. Relative to the LZ2 estimator described in Imbens and Kolesár [2016], we use a slight modification that allows for fixed effects:

$$\hat{V} = \ell'(X'X)^{-1} \sum_s X_s' A_s \hat{u}_s \hat{u}_s' A_s' X_s (X'X)^{-1} \ell = \ell' R^{-1} \sum_s Q_s' A_s \hat{u}_s \hat{u}_s' A_s' Q_s R'^{-1} \ell = \sum_{s=1}^{S} (\hat{u}_s' a_s)^2,$$

where

$$\hat{u}_s := Y_s - X_s \hat{\beta} = u_s - Q_s Q' u, \qquad a_s = A_s' Q_s \tilde{\ell},$$

and the matrix A_s is given by the symmetric square root of the inverse of $I - Q_s Q_s'$, or else its pseudo-inverse if it is singular, as is the case, for example, if X contains fixed effects. We do not need to insist on $I - Q_s Q_s'$ to be invertible, since, using the identity

$$\hat{V} = u \sum_s (I - QQ')_s' a_s a_s' (I - QQ')_s u,$$

one can verify by simple algebra that a sufficient condition for $\hat{V}$ to be unbiased under homoskedasticity is that $Q_s' A_s (I - Q_s Q_s') A_s Q_s = Q_s' Q_s$ (see, for example, Pustejovsky and Tipton [2018], for details).

If the observations are independent, the vector of leverages $(Q_1' Q_1, \dots, Q_n' Q_n)$ can be computed directly using the `stats::hatvalues` function. In this case, use this function to compute $A_i = 1/\sqrt{1 - Q_i' Q_i}$ directly, and we then compute $a_i = A_i Q_i' \tilde{\ell}$ using vector operations. For the case with clustering, computing the spectral decomposition of $I - Q_s Q_s'$ can be expensive or even infeasible if the cluster size n_s is large. We therefore use the following result, suggested to us by Ulrich Müller, which allows us to compute a_s by computing a spectral decomposition of a $p \times p$ matrix.

- Let $Q_s' Q_s = \sum_{i=1}^{p} \lambda_{is} r_{is} r_{is}'$ be the spectral decomposition of $Q_s' Q_s$. Then $A_s = \sum_{i\colon \lambda_i \neq 1} (1 - \lambda_i)^{-1/2} Q_s r_{is} r_{is}' Q_s'$, satisfies $A_s (I - Q_s Q_s') A_s = I$.

  This follows from the fact that $I - Q_s Q_s'$ has eigenvalues $1 - \lambda_{is}$ and eigenvectors $Q_s r_{is}$, and hence its pseudoinverse is $\sum_{i\colon \lambda_i \neq 1} (1 - \lambda_i)^{-1} Q_s r_{is} r_{is}' Q_s'$.

Using the lemma, we can compute a_s efficiently as:

$$a_s = \sum_{i\colon \lambda_i \neq 1} (1 - \lambda_i)^{-1/2} Q_s r_{is} r_{is}' Q_s' Q_s \tilde{\ell} = Q_s D_s \tilde{\ell}, \qquad D_s = \sum_{i\colon \lambda_i \neq 1} \lambda_i (1 - \lambda_i)^{-1/2} r_{is} r_{is}'.$$

Degrees of freedom correction

Let G be an $n \times S$ matrix with columns $(I - QQ')_s' a_s$. Then the Bell and McCaffrey [2002] adjustment sets the degrees of freedom to

$$f_{\mathrm{BM}} = \frac{\operatorname{tr}(G'G)^2}{\operatorname{tr}((G'G)^2)}.$$

Since $(G'G)_{st} = a_s'(I - QQ')_s (I - QQ)_t' a_t = a_s(\mathbb{1}\{s = t\} - Q_s Q_t')a_t$, the matrix $G'G$ can be efficiently computed as

$$G'G = \operatorname{diag}(a_s' a_s) - BB' \qquad B_{sk} = a_s' Q_{sk}.$$

Note that B is an $S \times p$ matrix, so that computing the degrees of freedom adjustment only involves $p \times p$ matrices:

$$f_{\mathrm{BM}} = \frac{(\sum_s a_s' a_s - \sum_{s,k} B_{sk}^2)^2}{\sum_s (a_s' a_s)^2 - 2\sum_{s,k}(a_s' a_s) B_{sk}^2 + \sum_{s,t}(B_s' B_t)^2}.$$

If the observations are independent, we compute B directly as `B <- a*Q`, and since a_i is a scalar, we have

$$f_{\mathrm{BM}} = \frac{(\sum_i a_i^2 - \sum_{sk} B_{sk}^2)^2}{\sum_i a_i^4 - 2\sum_i a_i^2 B_i' B_i + \sum_{i,j}(B_i' B_j)^2}.$$

The Imbens and Kolesár [2016] degrees of freedom adjustment instead sets

$$f_{IK} = \frac{\operatorname{tr}(G'\hat{\Omega}G)^2}{\operatorname{tr}((G'\hat{\Omega}G)^2)},$$

where $\hat{\Omega}$ is an estimate of the Moulton [1986] model of the covariance matrix, under which $\Omega_s = \sigma_\epsilon^2 I_{n_s} + \rho \iota_{n_s} \iota_{n_s}'$. Using simple algebra, one can show that in this case,

$$G'\Omega G = \sigma_\epsilon^2 \operatorname{diag}(a_s' a_s) - \sigma_\epsilon^2 BB' + \rho(D - BF')(D - BF')',$$

where

$$F_{sk} = \iota_{n_s}' Q_{sk}, \qquad D = \operatorname{diag}(a_s' \iota_{n_s})$$

which can again be computed even if the clusters are large. The estimate $\hat{\Omega}$ replaces σ_ϵ^2 and ρ with analog estimates.

References

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