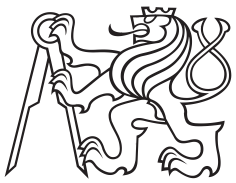
Bachelor's Thesis



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Distributed signal processing in radio communication networks

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Acknowledgement / Declaration

Thank you all.

I announce I made this thing alone.
In Prague, 3. 12. 2016

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Abstrakt / Abstract

Pokus napsat bakalarku a take ji obhajit.

Klíčová slova: konsensus algoritmus

Fireflyes are really amazing.

Keywords: averaging consensus algorithm

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Chapter 1
Introduction

To begin with an idea of an average consensus algorithm, let's make a thought experiment. We are looking for an average quantity, for example an average temperature in a room, with a group of wireless communication devices, that can exchange informations, provided they are in range to reach each other. We deploy these thermometers in the room randomly, with no special requirements on a topology. Next, let's consider, that for each pair of the thermometers we can decide, whether they can exchange information or not - meaning we know all neighbours of all devices, that are mutually in range to communicate.

Now, we can encode our experiment settings to a graph. A very natural way to represent this graph is drawing it. To do so, we simply take all thermometers as different vertices. Of course, every vertex always knows a result of its own measurement. By an edge between two vertices we mark the situation, that these two nodes can exchange informations. Which means, every node knows also the value that measures its neighbour. This ought to be only a very simple illustration how to transfer a physicaly realisable experiment to the terms of graphs.

Finally, as we shall see, if we fulfill some basic convergence conditions on the properties of this graph, the average consensus algorithm acts like this: We synchronously update the value in each node by some increment, which depends only on the old value in this node and the values of its direct neighbours in the graph. By doing this long enough, you obtain in each node a value, which goes in limit to the average of all initially measured values.

1.1 Outline

A Graph theory provides a very elegant way to represent informations encoded by graphs as matrices. In the first chapter we will provide some basic definitions to the Graph theory and properties of these important matrices. Using matrices, we will also briefly mention some very useful results from Matrix analysis, because also a serious object of our interest will be topic of eigenvalues of matrices. We will define a Laplacian of a graph and show some of its basic properties.

In next chapter, we will in detail provide the average consensus algorithm description and show some examples with graphically illustrated solution.

In last part of this thesis, we will try to implement this algorithm to easily solve some typical problems in area of wireless digital communication - time synchronisation or carrier frequency synchronisation. In a very simple case, we can also show how do the nonidealities change the result of algorithm (additive zero-mean noise).

Chapter 2
Graph theory

2.1 Motivation

It is commonly well known, that the elements of Graph theory were set by a mathematician Leonhard Euler in 1736. He solved a problem called Seven Bridges of Königsberg.

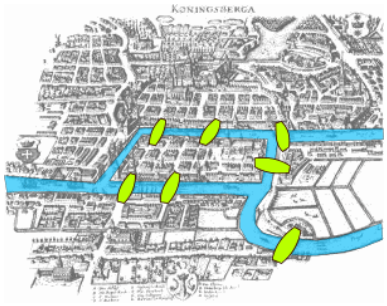

Figure 2.1. View on city Konigsberg with marked bridges [1] .

The problem was formulated like this: River Pregole flows through the city and creates two islands in there. These two islands are connected with the rest of the city by seven bridges (see figure above). The question was, whether it is possible to take a walk through the city in such a way, to pass each bridge exactly once. Euler described this problem as a graph, where the edges represented the seven bridges, and the vertices were the separated parts of the city.

Euler proved, that in this case, it is impossible to pass each bridge only once and showed, that the Eulerian trial (i.e. a trial, that contents every edge exactly once) exists if and only if every vertex of the graph is of an even degree.

Nowadays, a very modern and interesting example of usage of the Graph theory is visualisation and simulation of the communication networks, such as the Internet, mobile network etc. A typical task is to find the best route from a source to a destination location with respect to a given specific metric. This metric can depend on many paramaters, such as a number of intermediate devices (RIP), round-trip delay or a bandwidth of the connectivity (OSPF). These Graph algorithms are often based on a very efficient improvement of basic Depth-first search. (This is a case of the famous Dijkstra's algorithm used by OSPF routing protocol, to find the best way from each node to all the others with eges with a given cost.)

We also briefly mention the term of Spanning trees. In Ethernet-based communication is very important to avoid loops in a network, because they might cause a congestion of the network and failure of the service. A Spanning tree of a graph is a factor of this graph, which originates by removing some of its edges, in a way to preserve all vertices reachable, and removes all cycles in the graph. Later, in a chapter about Laplacian we shall see how to simply find a count of these Spanning trees.

To finish this motivation part, a very nice example of the usage of distributed algorithm is a Network time protocol (NTP). It is important to have globally synchronised time between server computers. Few of the servers are connected to the reference clocks and next, to them hiearchically connected servers, are averaging neighbour's time to obtain final value of time they use.

2.2 Definitions

A graph $G = (V, E)$ is described by a pair of its vertices V and edges E. $V = \{v_1, v_2, ..., v_N\}$ is a set of N vertices. By the vertices we understand the points connected by the edges. An edge (v_i, v_j) means a connection between vertices v_i and v_j.

2.2.1 Undirected graph

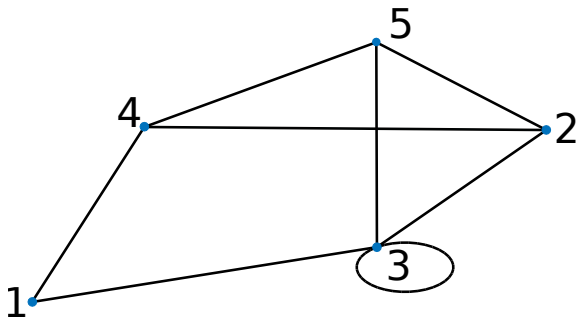

Figure 2.2. Example of a simple undirected graph to demonstrate basic definitions.

The graph $G = (V, E)$ above could be described by set of vertices $V = \{1; 2; 3; 4; 5\}$ and set of edges $E = \{(1, 4); (1, 3); (2, 4); (2, 5); (3, 2); (3, 3); (3, 5); (4, 5)\}$.

Set of neigbours $\mathcal{N}_i$ of a vertex v_i is $\mathcal{N}_i = \{v_j \in V | (v_i, v_j) \in E\}$. For example $\mathcal{N}_4 = \{1; 2; 5\}$. Degree of a node is $d_i = |\mathcal{N}_i|$.

2.2.2 Directed graph

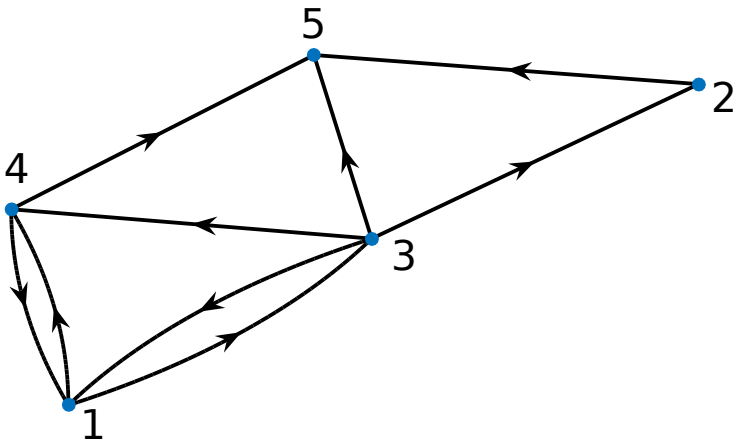

Figure 2.3. Example of a directed graph.

For directed graph holds the same as for undirected with only difference, we distinguish the edges (v_i, v_j) and (v_j, v_i). Then, for a degree of a node in directed graph, we have to consider only neighbours available via oriented edges, $d_i^{IN} = |\mathcal{N}_i|$. Drawing the figure, we distinguish the orientation of edges with arrows.

2.2.3 Adjacency matrix

Adjacency matrix is a very natural way of a full graph description. This matrix is $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ and for graph G with N vertices describes inner connectivity of the graph with information, to what all vertices goes an edge from a given vertex. Its values $a_{i,j}$ are defined as:

$$a_{i,j} = \begin{cases} 1 & \text{if there is the edge } (v_i, v_j), \\ 0 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Adjacency matrix of a graph from figure 2.2 reads

$$\boldsymbol{A}_{2.2} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

We can see, that Adjacency matrix of an undirected graph is symmetric.

And adjacency matrix of a graph from figure 2.3 is

$$\boldsymbol{A}_{2.3} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For directed graph the Adjacency matrix generally is not symmetric.

For undirected graphs allowing *Weighted graphs* means, that for each pair of vertices i, j we assign a certain weight $a_{i,j}$, that satisfies conditions: 1) $a_{i,j} = a_{j,i}$, 2) $a_{i,j} \geq 0$ and 3) $a_{ij} \neq 0$ if and only if vertices i and j are not connected by an edge. This is only a generalization of the Adjacency matrix definition above.

2.2.4 Degree matrix

A Degree matrix $\boldsymbol{D} \in \mathbb{R}^{N \times N}$ is a diagonal matrix bearing an information about degree of each vertex. Its diagonal elements are $d_i = \sum_{i \neq j} a_{i,j}$ and all nondiagonal elements are equal to 0. For example Degree matrix of undirected graph from figure 2.2 reads $\boldsymbol{D}_{2.2} = diag\{2, 3, 4, 3, 3\}$. Next, for the case of directed graph we have to consider only incoming edges. Which for graph from figure 2.3 means $\boldsymbol{D}_{2.3} = diag\{2, 1, 1, 2, 3\}$. And also let's define $\Delta = \max_i(d_i)$.

2.2.5 Incidence matrix

An Incidence matrix of a directed graph provides for each edge an information about an initial and terminal vertex. For a graph with N vertices and L edges the Incidence matrix $\boldsymbol{E} \in \mathbb{R}^{N \times L}$ elements $e_{i,j}$ are defined as:

$$e_{i,j} = \begin{cases} 1 & \text{if edge } j \text{ begins in the vertex } i, \\ -1 & \text{if edge } j \text{ ends in the vertex } i\ , \\ 0 & \text{otherwise.} \end{cases}$$

So Incidence matrix for graph on figure 2.3 reads

$$\boldsymbol{E}_{2.3} = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

We can see, this matrix is very rare. In each column contains only one pair of 1 and -1. The adjacency matrix provides same information but with typicaly smaller matrix.

2.3 Laplacian matrix

The folowing part about Laplacian is based mainly on [2].

2.3.1 Definitions

Now, in previous text we defined Adjacency and Degree matrix of a graph G. Next, we define the *Laplacian matrix* $\boldsymbol{L}(G)$ of a graph,

$$\boldsymbol{L}(G) = \boldsymbol{D}(G) - \boldsymbol{A}(G).$$

Matrix $\boldsymbol{L}(G)$ for a graph with N vertices is $\boldsymbol{L}(G) \in \mathbb{R}^{N\times N}$. From definitions of $\boldsymbol{D}(G)$ and $\boldsymbol{A}(G)$ implies, that loops in graph have no influence on $\boldsymbol{L}(G)$.

To make hold some important results from Linear algebra and Matrix analysis, we will next consider only undirected and loopless graphs. Which means, that the coresponding Adjacency matrix will be symmetric. Having symmetric $\boldsymbol{A}(G)$, the coresponding Laplacian matrix will be also a symmetric matrix.

Taking the Incidence matrix of graph $\boldsymbol{E}(G)$, we can find the Laplacian matrix of graph G as

$$\boldsymbol{L}(G) = \boldsymbol{E}(G)\boldsymbol{E}^T(G).$$

Next, we mark $\mu(G, x)$ the characteristical polynom of $\boldsymbol{L}(G)$ defined as

$$\mu(G, x) = \det(\boldsymbol{L} - x\boldsymbol{I}).$$

Roots of this characteristical polynom are called *Laplacian eigenvalues* of G. As it is common in literature, we will denote them $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$, enumerated with lower indices in an increasing order with counting multiplicities. N denotes the number of vertices. The set $\{\lambda_1, \lambda_2, ..., \lambda_N\}$ is called the *spectrum* of $\boldsymbol{L}(G)$.

2.3.2 Basic properties

Theorem 2.1. If $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ is symmetric then $\boldsymbol{A}$ has real eigenvalues.

Proof: For example [3] , page number 92.

Theorem 2.2. Let G be an undirected graph without loops. Then 0 is an eigenvalue for the Laplacian matrix of G with an eigenvector $(1, 1, ..., 1)^T$.

Proof: Found in [4]. If G is an undirected graph then the sum of the entries in row i of Adjacency matrix $\boldsymbol{A}$ gives exactly the degree d_i of vertex i. So we can write:

$$\boldsymbol{A}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=\begin{pmatrix}d_1\\d_2\\\vdots\\d_N\end{pmatrix}.$$

And from that:

$$\boldsymbol{L}(G)\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=(\boldsymbol{D}(G)-\boldsymbol{A}(G))\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=\begin{pmatrix}d_1-d_1\\d_2-d_2\\\vdots\\d_N-d_N\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}=0\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}.$$

In which we easily recognize the relation holding for eigenvalues.

Theorem 2.3. The Laplacian matrix $\boldsymbol{L}(G)$ is positive semi-definite and singular.

Proof: Let λ be an eigenvalue and v its coresponding eigenvector. Then

$$\boldsymbol{L}v=\lambda v,$$

$$\lambda=v^T\boldsymbol{L}v=v^T\boldsymbol{E}\boldsymbol{E}^Tv=(v^T\boldsymbol{E})(\boldsymbol{E}^Tv)=(\boldsymbol{E}^Tv)^T(\boldsymbol{E}^Tv)=\|\boldsymbol{E}^Tv\|\geq 0.$$

$\boldsymbol{L}$ is singular, because sum of all elements in each column is zero.

We can come to the positive semidefiniteness also using the following quadratic form:

$$\langle\boldsymbol{L}x,x\rangle=\sum_{(u,v)\in E(G)}(x_u-x_v)^2,$$

which results will be always non-negative.

2.3.3 Bounds for eigenvalues

Theorem 2.4. *Gershgorin circle theorem.* Consider matrix $\boldsymbol{A}\in\mathbb{C}^{N\times N}$ and $i=1,2,...,N$. Let's denote

$$r_i=\sum_{j=1;\ i\neq j}^{N}=|a_{ij}|,\quad K_i=\{z\in\mathbb{C}\mid|z-a_{ii}|\leq r_i\}.$$

The K_i sets are called *Gershgorin circles*. It holds for all eigenvalues $\{\lambda_1,\lambda_2,...,\lambda_N\}$ of the matrix $\boldsymbol{A}$, that they are all localized in the union of *Gershgorin circles* $\{K_1\cup K_2\cup...\cup K_N\}$ in the Complex plane.

Proof: Let λ be an eigenvalue of $\boldsymbol{A}$ and its corresponding eigenvector $\boldsymbol{x}=(x_1,x_2,...,x_N)$. So holds $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$. Let x_k be the biggest absolute value number in vector $\boldsymbol{x}$. Then $\lambda x_k=\sum_{j=1}^{N}a_{kj}x_j$. Next move the $a_{kk}x_k$ summand from RHS to LHS. We obtain $x_k(\lambda-a_{kk})=\sum_{j=1;j\neq k}^{N}a_{kj}x_j$. Now we take an absolute value of this equation, divide by x_k and using Triangle inequality we go to:

$$|\lambda-a_{kk}|=\left|\frac{\sum_{j=1;j\neq k}^{N}a_{kj}x_j}{x_k}\right|\leq\sum_{j=1;j\neq k}^{N}\left|\frac{a_{kj}x_j}{x_k}\right|\leq\sum_{j=1;j\neq k}^{N}|a_{kj}|=r_k.$$

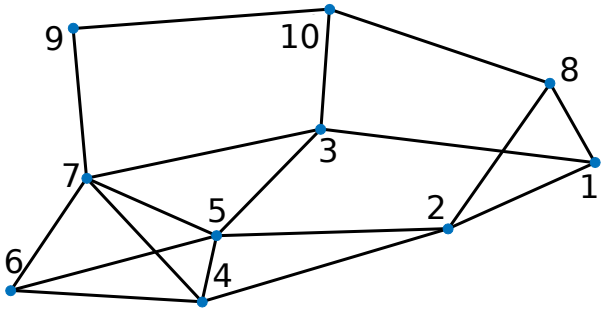

Figure 2.4. Graph to present Gershgorin theorem.

Example 2.5. To present Gershgorin theorem practicaly, consider the graph above:
With its Laplacian matrix:

$$\boldsymbol{L} = \begin{pmatrix} 3 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 4 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 4 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 4 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 5 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & 5 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 3 \end{pmatrix}.$$

And a characteristical polynomial: $\mu(G,x) = x^{10} - 36x^9 + 561x^8 - 4\,954x^7 + 27\,236x^6 - 96\,318x^5 + 218\,121x^4 - 303\,398x^3 + 233\,888x^2 - 75\,870x$. Note, that $\boldsymbol{L}$ is a symmetric matrix and 0 is clearly a root of $\mu(G,x)$.

Nummerically solving $\mu(G,x) = 0$ in Matlab we obtain the folowing eigenvalues (rounding for 3 decimal points):

$$\{0;\ 1,274;\ 1,416;\ 3,100;\ 3,233;\ 3,936;\ 4,826;\ 5,280;\ 6,458;\ 6,476\}.$$

All values are real and non-negative. Finally, plotting the graph with marked eigenvalues and Gershgorin circles. As expected, all eigenvalues are included in the circles.

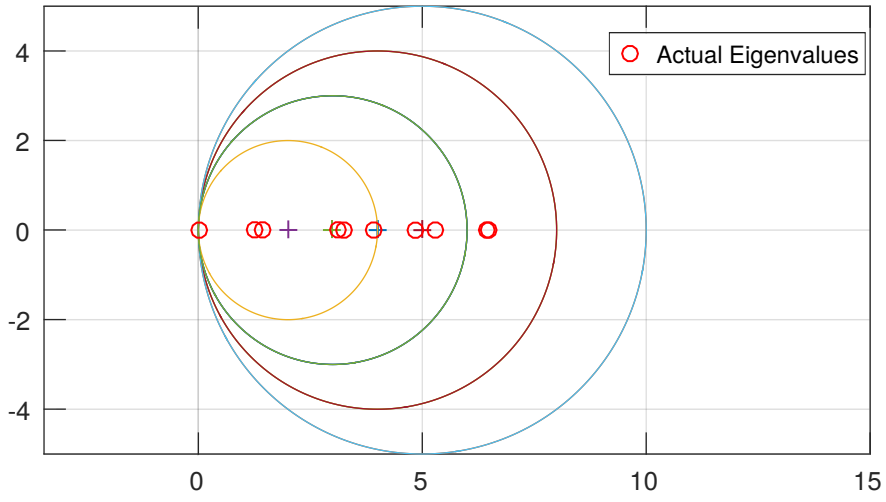

Figure 2.5. Plot of Eigenvalues and according Gershgorin circles.

Theorem 2.6. Let G be a graph with N vertices. Then holds:

- $\lambda_2 \leq \frac{N}{N-1} \min_i\{d(v_i)|v_i \in V(G)\}$,
- $\lambda_N \leq \max_i\{d(v_i) + d(v_j)|\,(v_i, v_j) \in E(G)\}$,
- If G is a simple graph, then $\lambda_N \leq N$,
- $\sum_{m=1}^{N} \lambda_m = 2|E(G)| = \sum_{v_i} d(v_i)$,
- $\lambda_N \geq \frac{N}{N-1} \max_i\{d(v_i)|v_i \in V(G)\}$.

These may be found very usefull for example when using numerical methods for solving the roots of $\mu(G, x)$. It is well known, that we don't have exact analytical formulas to obtain roots of polynoms with higher degree than 5. Using these bounds, we know where the roots must be, respective where they can not be.

2.3.4 Matrix tree theorem

$\boldsymbol{L}(G)$ may be also refferred to as Kirchhoff matrix due to the following theorem. A *tree* is a connected, acyclic graph. A *spanning tree* of graph G is a tree which origins as a subgraph, preserving the set of vertices $V(G)$ and removing some of its edges to avoid cycles. It is clear, that a spanning tree may be found only for connected graphs.

An (i, j)-cofactor of a matrix is a determinant of a submatrix formed by deleting the i-th row and the j-th column.

Theorem 2.7. *Kirchhoff's Matrix-Tree Theorem.* Let G be a connected graph with its Laplacian matrix $\boldsymbol{L}(G)$. Then all $\boldsymbol{L}(G)$ cofactors are equal and this common value is the number of spanning trees of G.

Proof: Ommited. Is based on decomposing the Laplacian matrix into product of Incidednce matrix and its transpose and then usage of Cauchy-Binet formula.

Example 2.8. For graph from Figure 2.4 we could so find 7587 spanning-trees.

2.3.5 Eigen value λ_2

We call graph G with N vertices connected if there is a path from any vertex v_i to any other vertex v_j , $\forall i, j \in \{1, 2, ..., N\}$.

Eigen value λ_2 is also called *graph connectivity*. This eigenvalue is probably the most important from the whole spectrum. Holds, that $\lambda_2 > 0$ if and only if the graph is connected. Moreover, the multiplicity of 0 as an eigenvalue of $\boldsymbol{L}(G)$ is the number of connected components.

Diameter of a graph G, $diam(G)$, is the biggest number of edges we have to pass, to get from one vertex to another.

In [2] are in detail described some interesting properties and bounds for λ_2. Very interesting and easily interpratable, in context of the connectivity term, is the following one. Let's consider graph G with N vertices and diameter $diam(G)$. Then holds:

$$diam(G) \geq \frac{4}{N\lambda_2}.$$

2.3.6 Operations with disjoint graphs

Very detailed reading about this part of Laplacian topic may be found beside in [2] also in [5]. Let's now briefly mention what happens with Laplacian of a graph, that is not connected.

Considering the definition of the Laplacian $\boldsymbol{L}(G) = \boldsymbol{D}(G) - \boldsymbol{A}(G)$, we are not surprised, that Laplacian matrix of a graph consisting of k mutually disjoint sets of vertices will have block diagonal form obtained from matrices $\boldsymbol{L}(G_1), \boldsymbol{L}(G_2), ..., \boldsymbol{L}(G_k)$.

Theorem 2.9. Let G be a graph created as a union of disjoint graphs $G_1, G_2, ..., G_k$. Then holds:

$$\mu(G, x) = \prod_{i=1}^{k} \mu_i(G_i, x).$$

Example 2.10.

Let's take a graph from the following figure, consisting of two disjoint components with vertices $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$.

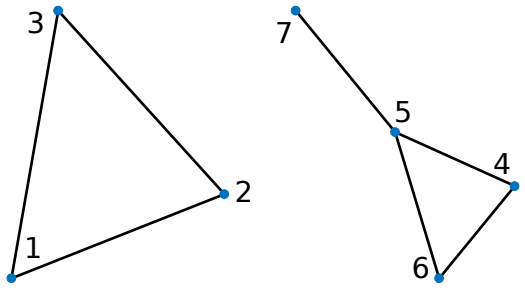

Figure 2.6. Example of a graph with two disconnected components.

Laplacian matrix of the whole graph reads:

$$\boldsymbol{L}(G) = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

$\boldsymbol{L}(G)$ is a block diagonal matrix consisting of submatrices $\boldsymbol{L}(G_{\{1,2,3\}})$ and $\boldsymbol{L}(G_{\{4,5,6,7\}})$. For characteristic polynomoial holds:

$\mu(G, x) = \mu(G_{\{1,2,3\}}, x)\mu(G_{\{4,5,6,7\}}, x) = (x^3 - 6x^2 + 9x)(x^4 - 8^3 + 19x^2 - 12x) = x^7 - 14x^6 + 76x^5 - 198x^4 + 243x^3 - 108x^2$. Note, that 0 is clearly double root corresponding to the 2 components.

2.3.7 Laplacian matrix - notes

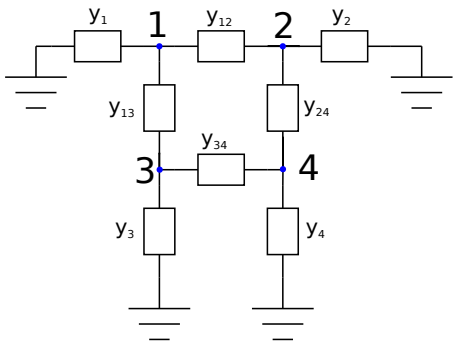

Figure 2.7. Example of admitance electrical circuit to be described with Laplacian matrix.

Laplacian matrix appears also in other fields, sometimes with other name. For example in Electrical circuits theory, Laplacian is called *Admitance matric.* In the Figure 2.7 is an example of such an electrical circuit described using admitance (i.e. reciprocal value of impedance) parameters. The aim in this case is to find all values of current I phasors in all edges and voltage U phasors in all nodes. These values may be placed in column vectors $\boldsymbol{I}$, $\boldsymbol{U}$ respectively. All the important information about the electrical circuit is encoded into the Admitance matrix $\boldsymbol{Y}$ and using Ohm's Law we are about to solve equation $\boldsymbol{Y\,I = U.}$

$$\boldsymbol{Y}(G) = \begin{pmatrix} y_1 + y_{12} + y_{13} & -y_{12} & -y_{13} & 0 \\ -y_{12} & y_2 + y_{12} + y_{14} & 0 & -y_{14} \\ -y_{13} & 0 & y_3 + y_{13} + y_{34} & -y_{34} \\ 0 & -y_{24} & -y_{34} & y_4 + y_{24} + y_{34} \end{pmatrix}.$$

To demonstrate relation between Laplacian matrix and continuous Laplacian operator Δ, let's consider a differential equation

$$\Delta z(x, y) + \lambda \Delta z(x, y) = 0,$$

with initial condition $z(x, y) = 0$ on a simple closed curve Γ in xy-plane. Solution of this task is determined by an infinite sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq ....$ An aproximation of this solution may be found by placing a grid over the area in plane curved by Γ. The matrix is of this finite, discretized problem, comes to be the Laplacian matrix.

Chapter 3
Distributed algorithms

A very good and step-by-step introduction to the theory of Distributed algorithms may be found in [6].

Chapter 4
Linear average consensus algorithm

In this chapter, let us consider an undirected and connected graph $G = (V, E)$ with N vertices and edges (v_i, v_j) between vertices i, j, where $i, j \in \{1, 2, ..., N\}$. We denote an initial value $x_i(0)$ the value assigned to the i-th vertex (node, agent) in time 0. Then $x_i(t)$ refers to the value in the i-th vertex in time t. Our goal is to, for $t \to \infty$, using local communication and computation, in all N vertices of the graph, obtain an average value of all these initial values. Based on a matrix-like description of graph G, our goal will be to construct matrix $\boldsymbol{Q}$, whose components q_{ij} will in fact bear this averaging algorithm, in a formalism of iterative matrix multiplication.

In this chapter, subject of our interest will be a linear, discrete-time consensus algorithm. A detailed description of the folowing is in [7], and it contains also rich references to other publications.

4.1 Introduction

Assume this *linear* update equation

$$x(t+1) = \boldsymbol{Q}(t)x(t),$$

where $x(t) = (x_1(t), x_2(t), ..., x_N(t))^T \in \mathbb{R}^N$ and for all values of t, $\boldsymbol{Q}(t) \in \mathbb{R}^{N \times N}$ is a *stochastic matrix*, i.e. $q_{ij}(t) \geq 0$ and $\sum_{j=1}^{N} q_{ij} = 1, \forall i, j \in 1, 2, ..., N$. Meaning, that all values in each row sum up to 1. The q_{ij} components are also often reffered to as *weights.*

Now, let's rewrite the equation above expanding a matrix multiplication:

$$x(t+1) = \sum_{j=1}^{N} q_{ij}(t)x_i(t) = x_i(t) + \sum_{j=1;j\neq i}^{N} q_{ij}(x_j(t) - x_i(t)).$$

This equation is for given $\boldsymbol{Q}(t)$ a general form of a *linear consensus algorithm*, that may be usually found in the literature. Frankly spoken, all the theory behind linear consensus algorithm aims to find the best matrix $\boldsymbol{Q}(t)$, such as the consensus is reached.

Formally defined, we say that $\boldsymbol{Q}(t)$ solves *consensus problem*, if for all i holds

$$\lim_{t\to\infty} x_i(t) = \alpha, \forall i.$$

Then, for a solution of the *average consensus problem* must be in an addition to the previous condition fulfilled also

$$\alpha = \frac{1}{N}\sum_{i=1}^{N} x_i(0).$$

Moreover, we call $\boldsymbol{Q}(t)$ *doubly stochastic*, if holds also $\sum_{j=1^N} q_{ij} = 1, \forall j \in 1, 2, ..., N$. So both, rows and columns sum up to 1. Note, that if $\boldsymbol{Q}(t)$ is stochastic and symetric, $\boldsymbol{Q}(t) = \boldsymbol{Q}(t)^T$, then $\boldsymbol{Q}(t)$ is doubly stochastic.

The $\boldsymbol{Q}(t)$ matrix may be considered as: 1) constant $\boldsymbol{Q}(t) = \boldsymbol{Q}$, 2) a deterministic time variable matrix, 3) randomly variable matrix. For simplicity, we will firsly concern only case 1). The others are behind scope of this text.

Now, let's have a look at a very useful theorem, that originaly comes from closely related topic of Markov Chains.

Theorem 4.1. (From [7].) Let us consider the sequence of constant matrices $\boldsymbol{Q}(t) = Q$. If the graph G Q ■ G sl and is rooted, then Q solves the consensus problem, and lim Q t = 1 ■ T t■■ where ■ ■ R N is the left eigenvector of Q for the eigenvalue one and has the prop- erties ■ i ■ 0 and 1 T ■ = 1. If G Q is strongly connected, then ■ i ¿ 0, ■i. If in addition Q is doubly-stochastic, then G Q is strongly connected and Q solves the average con- sensus problem, i.e. ■ = N 1 1. Moreover, in all cases the convergence is exponential and its rate is given by the essential spectral radius esr(Q).

4.2 Motivation

Let's think about an experiment, where few nodes aim to provide only one result of measurement based on many local measurments. For example we measure an average temperature in a room. Very acurate measure devices are expensive. We can generally try to replace small number of very good devices by some probably bigger number of less reliable devices whose benefit will be an interchange of information between near nodes.

They We want to replace a number of nodes, that exchange informatMany less accurate a reliable nodes as an alternative to very accurate and very reliable but also very expensive nodes.

4.3 Discrete and continuous time

4.4 Convergenece analysis

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