Model Theory of Valued Fields: Midterm

Li Ling Ko
lko@nd.edu

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Exercise 12: *Show that for local rings* $R_0 \sqsubseteq R$ *we have* $\boldsymbol{m}_0 = \boldsymbol{m} \cap R_0$.

Proof. We are assuming that R_0 is a subring of R.

$\boldsymbol{m}_0 \subseteq \boldsymbol{m} \cap R_0$ follows by definition of $R_0 \sqsubseteq R$. For the reverse containment, since R_0 is local, it suffices to show that $\boldsymbol{m} \cap R_0$ is an ideal of R_0, and that $\boldsymbol{m} \cap R_0$ is not equal to R_0. To prove the second claim, observe that R_0 cannot be contained in contain $\boldsymbol{m}$, because as a subring of R_0, R_0 must contain the identity of R, which does not lie in $\boldsymbol{m}$ since $\boldsymbol{m}$ is a non-trivial ideal of R. Thus $\boldsymbol{m} \cap R_0 \neq R_0$.

To prove the first claim, observe that as rings, both $\boldsymbol{m}$ and R_0 are closed under subtraction. As an ideal of R, $\boldsymbol{m}$ is closed under multiplication with elements of R_0. R_0 is also clearly closed under multiplication with elements of R_0. Thus $\boldsymbol{m} \cap R_0$ is an ideal of R_0. $\square$