Model Theory of Valued Fields: Midterm

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Exercise 12: *Show that for local rings $R_0 \sqsubseteq R$ we have $\boldsymbol{m}_0 = \boldsymbol{m} \cap R_0$.*

Proof. We are assuming that R_0 is a subring of R.

$\boldsymbol{m}_0 \subseteq \boldsymbol{m} \cap R_0$ follows by definition of $R_0 \sqsubseteq R$. For the reverse containment, since R_0 is local, it suffices to show that $\boldsymbol{m} \cap R_0$ is an ideal of R_0, and that $\boldsymbol{m} \cap R_0$ is not equal to R_0. To prove the second claim, observe that R_0 cannot be contained in contain $\boldsymbol{m}$, because as a subring of R_0, R_0 must contain the identity of R, which does not lie in $\boldsymbol{m}$ since $\boldsymbol{m}$ is a non-trivial ideal of R. Thus $\boldsymbol{m} \cap R_0 \neq R_0$.

To prove the first claim, observe that as rings, both $\boldsymbol{m}$ and R_0 are closed under subtraction. As an ideal of R, $\boldsymbol{m}$ is closed under multiplication with elements of R_0. R_0 is also clearly closed under multiplication with elements of R_0. Thus $\boldsymbol{m} \cap R_0$ is an ideal of R_0. □

Q3: *(Lemma 7.19) Let (R, F) be a valued field of characteristic 0. Let $P[x] \in F[x]$ and $a \in F$ be in henselian configuration with $F(a) \neq 0$. There is at most one $b \in F$ with $P(b) = 0$ and $v(a-b) \geq v(P(a)) - v(P'(a))$. If F is henselian then there is $b \in F$ with $F(b) = 0$ and $v(a-b) = v(P(a)) - v(P'(a))$.*

Proof. □

Q4: *(Exercise 18) If (R, F) is finitely ramified then for any $k \in \mathbb{N}$ the set $\{\gamma \in \Gamma : 0 \leq \gamma < v(k)\}$ is finite.*

Proof. Since $R/\boldsymbol{m}$ is a field of characteristic p, the set

$$\{\boldsymbol{m}, 1 + \boldsymbol{m}, \ldots, p - 1 + \boldsymbol{m}\}$$

is an additive subgroup of $R/\boldsymbol{m}$ generated by $1 + \boldsymbol{m}$. Since $v(k) = 0$ for all $k \in i + \boldsymbol{m}$ where $i \in \{1, \ldots, p-1\}$, so if p does not divide k, $v(k)$ must be 0, making the set $\{\gamma \in \Gamma : 0 \leq \gamma < v(k)\}$ trivially finite.

It remains to prove the claim for $k = pk'$. Now $v(k) = v(p) + v(k')$. So if k' is not a multiple of p, the claim would hold since (R, F) is finitely ramified. Thus we only need consider the cases when $k = p^n$ for some $n > 1$. In these cases we have $v(p^n) = n \cdot v(p)$. Now if

$$0 \leq x \leq n \cdot v(p),$$

then

$$-v(p) \leq x - v(p) \leq (n-1) \cdot v(p).$$

By induction on n, there are only finitely many elements between 0 and $(n-1) \cdot v(p)$, and by the base case there are also only finitely many elements between $-v(p)$ and 0. Therefore there are only finitely many possibilities for $x - v(p)$, implying there are only finitely many x that can lie between 0 and $n \cdot v(p)$, which completes the proof. □

Q5: *Let $R < K$ be domains and every $k \in K$ is integral over R. If K is a field, show that R is a field as well.*

Proof. Let $r \in R^*$. We want to show that $r^{-1} \in F$ also lies in R. Since r^{-1} is integral over R, there exists $r_0, \dots, r_{n-1} \in R$ such that

$$r^{-n} + r_{n-1}r^{-n+1} + \dots + r_1 r^{-1} + r_0 = 0.$$

Rearranging, we get

$$r(-r_0 r^{n-1} - r_1 r^{n-2} - \dots - r_{n-2}r - r_{n-1}) = 1.$$

So

$$-r_0 r^{n-1} - r_1 r^{n-2} - \dots - r_{n-2}r - r_{n-1} \in R$$

is the multiplicative inverse of r, and it lies in R, as required. $\square$

Q6: *Show that $\mathbb{Q}_p$ does not have proper immediate extensions.*

Proof. By Theorem 7.15, it suffices to show that $\mathbb{Q}_p$ is spherically complete. Let $B_{\geq n}(a)$ be a ball in $\mathbb{Q}_p$, where $n \in \mathbb{Z}$ and $a = p^{-d}(b_i)$ for some $(b_i) \in \mathbb{Z}_p$ and $d \in \mathbb{N}$. Observe that elements in $B_{\geq n}(a)$ are exactly those of the form $p^{-d}(c_i)$, where $c_i = b_i$ for the first $(n+d)$ indices. Thus $B_{\geq n}(a) = B_{\geq n}(a')$, where $a = p^{-d}(b'_i)$, and b'_i is defined as

$$b'_i := \begin{cases} b_i, & \text{if}, i \in \{0, 1, \dots, n+d-1\}, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore for any ball $B_{\geq n}(p^{-d}(b_i))$, we can assume b_i is zero for all indices after the first $n+d$ ones. Since there are only countably many balls of such form, any chain of balls must be countable. Observe further that given two balls $B_{\geq n_0}(p^{d_0}(a_i))$ and $B_{\geq n_1}(p^{d_1}(b_i))$, the first ball contains the second if and only $d_0 = d_1$, $n_0 \leq n_1$, and $a_i = b_i$ for the first $n_0 + d_0$ indices.

Thus, given a chain of $\mathcal{B} = \{B_i : i \in \omega\}$, we can assume $B_i = B_{\geq n_i}(p^d(a_{i,j}))$, where $d \in \mathbb{Z}$ is fixed, $n_i \geq n_{i+1}$, $a_{i,j} = 0$ for all $j \geq n_i + d$, and $a_{i,j} = a_{i+1,j}$ for the first $n_i + d$ indices. Then the element $p^d(a_j) \in \mathbb{Q}_p$ where

$$a_j := \lim_{i \in \omega} a_{i,j},$$

will be contained in every ball in $\mathcal{B}$, which completes the proof. $\square$

Q7: *(Corollary 9.6) For a fixed prime p the theory of p-adically closed fields in the language $\mathcal{L}_v$ is complete and model complete.*

Proof. Since T_p^c has QE in the expanded language $\mathcal{L}_p^c$ (Theorem 9.4), and therefore is model-complete with respect to $\mathcal{L}_p^c$. Since no new axioms were added in the expanded language, the models of T_p^c in the original language are the same as the models in the expanded one. Thus T_p^c is model-complete.

To prove completeness, fix an arbitrary model $\mathcal{M}$ of T_p^c. Then $\mathcal{M}$ must embed $\mathbb{Q}$ as a valued field with p-adic valuation, and thus also embeds $\mathbb{Q}_p^h$, the hensalization of $\mathbb{Q}$ in $\mathbb{Q}_p \cap \bar{\mathbb{Q}}$. So $\mathbb{Q}_p^h$ is a substructure of $\mathcal{M}$, thus by model-completeness, is an elementary substructure of $\mathcal{M}$. In particular, $\mathbb{Q}_p^h$ and $\mathcal{M}$ are elementary equivalent. Since $\mathcal{M}$ is an arbitrary model of T_p^c, T_p^c is the complete theory of $\mathbb{Q}_p^h$. $\square$