

Model Theory of Valued Fields: Midterm

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Exercise 12: Show that for local rings $R_0 \subseteq R$ we have $\mathfrak{m}_0 = \mathfrak{m} \cap R_0$.

Proof. We are assuming that R_0 is a subring of R .

$\mathfrak{m}_0 \subseteq \mathfrak{m} \cap R_0$ follows by definition of $R_0 \subseteq R$. For the reverse containment, since R_0 is local, it suffices to show that $\mathfrak{m} \cap R_0$ is an ideal of R_0 , and that $\mathfrak{m} \cap R_0$ is not equal to R_0 . To prove the second claim, observe that R_0 cannot be contained in \mathfrak{m} , because as a subring of R , R_0 must contain the identity of R , which does not lie in \mathfrak{m} since \mathfrak{m} is a non-trivial ideal of R . Thus $\mathfrak{m} \cap R_0 \neq R_0$.

To prove the first claim, observe that as rings, both \mathfrak{m} and R_0 are closed under subtraction. As an ideal of R , \mathfrak{m} is closed under multiplication with elements of R_0 . R_0 is also clearly closed under multiplication with elements of R_0 . Thus $\mathfrak{m} \cap R_0$ is an ideal of R_0 . \square

Q5: Let $R < K$ be domains and every $k \in K$ is integral over R . If K is a field, show that R is a field as well.

Proof. Let $r \in R^*$. We want to show that $r^{-1} \in R$ also lies in R . Since r^{-1} is integral over R , there exists $r_0, \dots, r_{n-1} \in R$ such that

$$r^{-n} + r_{n-1}r^{-n+1} + \dots + r_1r^{-1} + r_0 = 0.$$

Rearranging, we get

$$r(-r_0r^{n-1} - r_1r^{n-2} - \dots - r_{n-2}r - r_{n-1}) = 1.$$

So

$$-r_0r^{n-1} - r_1r^{n-2} - \dots - r_{n-2}r - r_{n-1} \in R$$

is the multiplicative inverse of r , and it lies in R , as required. \square