Model Theory of Valued Fields: Midterm

Li Ling Ko
lko@nd.edu

October 20, 2017

Exercise 12: *Show that for local rings $R_0 \sqsubseteq R$ we have $\boldsymbol{m}_0 = \boldsymbol{m} \cap R_0$.*

Proof. We are assuming that R_0 is a subring of R.

$\boldsymbol{m}_0 \subseteq \boldsymbol{m} \cap R_0$ follows by definition of $R_0 \sqsubseteq R$. For the reverse containment, since R_0 is local, it suffices to show that $\boldsymbol{m} \cap R_0$ is an ideal of R_0, and that $\boldsymbol{m} \cap R_0$ is not equal to R_0. To prove the second claim, observe that R_0 cannot be contained in contain $\boldsymbol{m}$, because as a subring of R_0, R_0 must contain the identity of R, which does not lie in $\boldsymbol{m}$ since $\boldsymbol{m}$ is a non-trivial ideal of R. Thus $\boldsymbol{m} \cap R_0 \neq R_0$.

To prove the first claim, observe that as rings, both $\boldsymbol{m}$ and R_0 are closed under subtraction. As an ideal of R, $\boldsymbol{m}$ is closed under multiplication with elements of R_0. R_0 is also clearly closed under multiplication with elements of R_0. Thus $\boldsymbol{m} \cap R_0$ is an ideal of R_0. $\square$

Q5: *Let $R < K$ be domains and every $k \in K$ is integral over R. If K is a field, show that R is a field as well.*

Proof. Let $r \in R^*$. We want to show that $r^{-1} \in F$ also lies in R. Since r^{-1} is integral over R, there exists $r_0, \ldots, r_{n-1} \in R$ such that

$$r^{-n} + r_{n-1}r^{-n+1} + \ldots + r_1 r^{-1} + r_0 = 0.$$

Rearranging, we get

$$r(-r_0 r^{n-1} - r_1 r^{n-2} - \ldots - r_{n-2} r - r_{n-1}) = 1.$$

So

$$-r_0 r^{n-1} - r_1 r^{n-2} - \ldots - r_{n-2} r - r_{n-1} \in R$$

is the multiplicative inverse of r, and it lies in R, as required. $\square$

Q7: *(Corollary 9.6) For a fixed prime p the theory of p-adically closed fields in the language $\mathcal{L}_v$ is complete and model complete.*

Proof. Since T_p^c has QE in the expanded language $\mathcal{L}_p^c$ (Theorem 9.4), and therefore is model-complete with respect to $\mathcal{L}_p^c$. Since no new axioms were added in the expanded language, the models of T_p^c in the original language are the same as the models in the expanded one. Thus T_p^c is model-complete.

To prove completeness, fix an arbitrary model $\mathcal{M}$ of T_p^c. Then $\mathcal{M}$ must embed $\mathbb{Q}$ as a valued field with p-adic valuation, and thus also embeds $\mathbb{Q}_p^h$, the hensalization of $\mathbb{Q}$ in $\mathbb{Q}_p \cap \bar{\mathbb{Q}}$. So $\mathbb{Q}_p^h$ is a substructure of $\mathcal{M}$, thus by model-completeness, is an elementary substructure of $\mathcal{M}$. In particular, $\mathbb{Q}_p^h$ and $\mathcal{M}$ are elementary equivalent. Since $\mathcal{M}$ is an arbitrary model of T_p^c, T_p^c is the complete theory of $\mathbb{Q}_p^h$. $\square$