

Model Theory of Valued Fields: Midterm

Li Ling Ko
lko@nd.edu

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Exercise 12: Show that for local rings $R_0 \subseteq R$ we have $\mathfrak{m}_0 = \mathfrak{m} \cap R_0$.

Proof. We are assuming that R_0 is a subring of R .

$\mathfrak{m}_0 \subseteq \mathfrak{m} \cap R_0$ follows by definition of $R_0 \subseteq R$. For the reverse containment, since R_0 is local, it suffices to show that $\mathfrak{m} \cap R_0$ is an ideal of R_0 , and that $\mathfrak{m} \cap R_0$ is not equal to R_0 . To prove the second claim, observe that R_0 cannot be contained in \mathfrak{m} , because as a subring of R , R_0 must contain the identity of R , which does not lie in \mathfrak{m} since \mathfrak{m} is a non-trivial ideal of R . Thus $\mathfrak{m} \cap R_0 \neq R_0$.

To prove the first claim, observe that as rings, both \mathfrak{m} and R_0 are closed under subtraction. As an ideal of R , \mathfrak{m} is closed under multiplication with elements of R_0 . R_0 is also clearly closed under multiplication with elements of R_0 . Thus $\mathfrak{m} \cap R_0$ is an ideal of R_0 . \square

Q3: (Lemma 7.19) Let (R, F) be a valued field of characteristic 0. Let $P[x] \in F[x]$ and $a \in F$ be in henselian configuration with $F(a) \neq 0$. There is at most one $b \in F$ with $P(b) = 0$ and $v(a - b) \geq v(P(a)) - v(P'(a))$. If F is henselian then there is $b \in F$ with $P(b) = 0$ and $v(a - b) = v(P(a)) - v(P'(a))$.

Proof. \square

Q4: (Exercise 18) If (R, F) is finitely ramified then for any $k \in \mathbb{N}$ the set $\{\gamma \in \Gamma : 0 \leq \gamma < v(k)\}$ is finite.

Proof. Since R/\mathfrak{m} is a field of characteristic p , the set

$$\{\mathfrak{m}, 1 + \mathfrak{m}, \dots, p - 1 + \mathfrak{m}\}$$

is an additive subgroup of R/\mathfrak{m} generated by $1 + \mathfrak{m}$. Since $v(k) = 0$ for all $k \in i + \mathfrak{m}$ where $i \in \{1, \dots, p - 1\}$, so if p does not divide k , $v(k)$ must be 0, making the set $\{\gamma \in \Gamma : 0 \leq \gamma < v(k)\}$ trivially finite.

It remains to prove the claim for $k = pk'$. Now $v(pk') = v(p) + v(k')$. So if k' is not a multiple of p , the claim would hold since (R, F) is finitely ramified. Thus we only need consider the cases when $k = p^n$ for some $n > 1$. In these cases we have $v(p^n) = n \cdot v(p)$. Now if

$$0 \leq x \leq n \cdot v(p),$$

then

$$-v(p) \leq x - v(p) \leq (n - 1) \cdot v(p).$$

By induction on n , there are only finitely many elements between 0 and $(n - 1) \cdot v(p)$, and by the base case there are also only finitely many elements between $-v(p)$ and 0. Therefore there are only finitely many possibilities for $x - v(p)$, implying there are only finitely many x that can lie between 0 and $n \cdot v(p)$, which completes the proof. \square

Q5: Let $R < K$ be domains and every $k \in K$ is integral over R . If K is a field, show that R is a field as well.

Proof. Let $r \in R^*$. We want to show that $r^{-1} \in R$ also lies in R . Since r^{-1} is integral over R , there exists $r_0, \dots, r_{n-1} \in R$ such that

$$r^{-n} + r_{n-1}r^{-n+1} + \dots + r_1r^{-1} + r_0 = 0.$$

Rearranging, we get

$$r(-r_0r^{n-1} - r_1r^{n-2} - \dots - r_{n-2}r - r_{n-1}) = 1.$$

So

$$-r_0r^{n-1} - r_1r^{n-2} - \dots - r_{n-2}r - r_{n-1} \in R$$

is the multiplicative inverse of r , and it lies in R , as required. \square

Q6: Show that \mathbb{Q}_p does not have proper immediate extensions.

Proof. By Theorem 7.15, it suffices to show that \mathbb{Q}_p is spherically complete. Let $B_{\geq n}(a)$ be a ball in \mathbb{Q}_p , where $n \in \mathbb{Z}$ and $a = p^{-d}(b_i)$ for some $(b_i) \in \mathbb{Z}_p$ and $d \in \mathbb{N}$. Observe that elements in $B_{\geq n}(a)$ are exactly those of the form $p^{-d}(c_i)$, where $c_i = b_i$ for the first $(n+d)$ indices. Thus $B_{\geq n}(a) = B_{\geq n}(a')$, where $a = p^{-d}(b'_i)$, and b'_i is defined as

$$b'_i := \begin{cases} b_i, & \text{if } i \in \{0, 1, \dots, n+d-1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore for any ball $B_{\geq n}(p^{-d}(b_i))$, we can assume b_i is zero for all indices after the first $n+d$ ones. Since there are only countably many balls of such form, any chain of balls must be countable. Observe further that given two balls $B_{\geq n_0}(p^{d_0}(a_i))$ and $B_{\geq n_1}(p^{d_1}(b_i))$, the first ball contains the second if and only $d_0 = d_1$, $n_0 \leq n_1$, and $a_i = b_i$ for the first $n_0 + d_0$ indices.

Thus, given a chain $\mathcal{B} = \{B_i : i \in \omega\}$ of balls, we can assume $B_i = B_{\geq n_i}(p^d(a_{i,j}))$, where $d \in \mathbb{Z}$ is fixed, $n_i \geq n_{i+1}$, $a_{i,j} = 0$ for all $j \geq n_i + d$, and $a_{i,j} = a_{i+1,j}$ for the first $n_i + d$ indices. Then the element $p^d(a_j) \in \mathbb{Q}_p$ where

$$a_j := \lim_{i \in \omega} a_{i,j},$$

will be contained in every ball in \mathcal{B} , which completes the proof. \square

Q7: (Corollary 9.6) For a fixed prime p the theory of p -adically closed fields in the language \mathcal{L}_v is complete and model complete.

Proof. Since T_p^c has QE in the expanded language \mathcal{L}_p^c (Theorem 9.4), it is model-complete with respect to \mathcal{L}_p^c . Since no new axioms were added in the expanded language, the models of T_p^c in the original language are the same as the models in the expanded one. Thus T_p^c is also model-complete in the original language.

To prove completeness, consider \mathbb{Q}_p^h , defined as $\mathbb{Q}_p \cap \bar{\mathbb{Q}}$, considered as valued fields with the valuation of \mathbb{Q}_p . \mathbb{Q}_p^h is a maximal algebraic extension of \mathbb{Q} since $\bar{\mathbb{Q}}$ is a maximal algebraic extension of \mathbb{Q} . Also, \mathbb{Q}_p^h is an immediate extension of \mathbb{Q} . Therefore, \mathbb{Q}_p^h is a maximal immediate algebraic field extension of \mathbb{Q} , and is a model of T_p^c , and is thus a henselization of \mathbb{Q} (Proposition 7.26). Fix an arbitrary model \mathcal{M} of T_p^c . Then \mathcal{M} must embed \mathbb{Q} as a valued field with p -adic valuation, and therefore also embeds its henselization \mathbb{Q}_p^h (Definition 7.24). Then by model-completeness, \mathbb{Q}_p^h is an elementary substructure of \mathcal{M} . In particular, \mathbb{Q}_p^h and \mathcal{M} are elementary equivalent. Since \mathcal{M} is an arbitrary model of T_p^c , T_p^c is the complete theory of \mathbb{Q}_p^h . \square