Model Theory of Valued Fields: Midterm

Li Ling Ko
lko@nd.edu

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Kunen I.9.6: *Derive the axioms of Infinity and Replacement from (2) of Lemma I.9.5. Hint. For Infinity, let A be the (possibly proper) class of all natural numbers, and let xRy iff $x = y + 1$.*

Proof. We follow the hint to derive Infinity. Let A be the possibly proper class of all natural numbers, and let xRy iff $x = y+1$. Formally, A is the class of sets n where n is transitive, well-ordered by $\in$, and contains only elements that are either the empty set or a successor. Also, relation xRy is defined by $x = S(y)$, where S denotes the formula for successor. Then R is set-like on A, because every natural number n has only one successor $S(n)$ which is also a natural number. Therefore $\text{pred}_{A,R}(n) = \{S(n)\}$, which is a set since $S(n)$ is a set and singletons of sets are sets by the Pairing Axiom. Thus from Lemma I.9.5 the transitive closure relation R^* is also set-like.

Consider the transitive closure B of $\emptyset$ in A. B is a set since R^* is set-like. Also, B will contain exactly all the natural numbers, and is therefore a witness for the Axiom of Infinity.

To derive Replacement, fix any set X and formula $\varphi(x, y, w)$ such that for a fixed set w, given any $x \in X$ there is a unique set y such that $\varphi(x, y, w)$ holds. We want to show that the range of φ, defined as $Y := \{y : (\exists x \in X)\ \varphi(x, y, w)\}$, is a set. We define a relation R on the possibly proper class $X \cup Y$ such that if $y \in Y$, then xRy iff $x \in X$ and $\neg\varphi(x, y, w)$. Otherwise if $x \in X$, then yRx iff $\varphi(x, y, w)$.

Now R is set-like on $X \cup Y$, because if $y \in Y$, then $\text{pred}_{X\cup Y,R}(y) = \{x \in X : \neg\varphi(x, y, w)\}$, which is a set by the axiom of Comprehension since it is a subset of X and can be defined by the formula $\neg\varphi(x, y, w)$. Also, if $x \in X$, then $\text{pred}_{X\cup Y,R}(x) = \{y\}$, where y is the unique set that $\varphi(x, y, w)$ holds; this will also be a set by the Pairing axiom since y is a set. Thus by Lemma I.9.5, the transitive closure relation R^* is also set-like.

Fix any $x \in X$, and consider its transitive closure $S = \text{pred}_{X\cup Y,R^*}(x)$. Then S is a set from Lemma I.9.5, and S is exactly $X \cup Y$ by definition. Then using Comprehension, we can extract Y as a set from S as follows:

$$Y = \{y \in S : \ (\exists x)\ \varphi(x, y, w)\}.$$

□