SMT: Equality Logic With Uninterpreted Functions

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Program Verification: OpenJML Example

- `https://www.rise4fun.com/OpenJMLESC/BinarySearch`

Program Verification: OpenJML Example

- How does OpenJML work?

Code + Pre- & Post- conditions $\xrightarrow{\text{OpenJML}}$ Formula (SMT-LIB) $\xrightarrow{\text{SMT solver (Z3, CVC4, \ldots)}}$ SATISFIABLE/ UNSATISFIABLE

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How do SMT solvers reason about equality?

What is SMT?

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- Satisfiability Modulo Theories (SMT) is a generalization of the Boolean satisfiability problem (SAT) for first-order logic

First-Order Theories

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Example (First-Order Theories)

- Theory of Equality Logic: $x_1 \neq x_2 \wedge x_1 = 4$
- Theory of Linear Arithmetic: $(3x^2 + 2x - 1 = 0) \wedge (0 < x)$
- Theory of Bit Vectors: $(a \gg 2) = c \wedge c \oplus d$

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- Solver for Theory of Equality Logic With Uninterpreted Functions (EUF)

Theory of Equality Logic With Uninterpreted Functions (EUF)

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- Introduces the equality predicate $(=)$

$$
\begin{aligned}
&\forall x.\ x = x && \text{(Reflexivity)}\\
&\forall x.\ \forall y.\ x = y \implies y = x && \text{(Symmetry)}\\
&\forall x.\ \forall y.\ \forall z.\ x = y \land y = z \implies x = z && \text{(Transitivity)}
\end{aligned}
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Example (Formula in EUF)

$$f(a, b) = a \land f(f(a, b), b) \neq a$$

Functional Congruence

Definition (Functional Congruence)

For each $n > 0$ and n-ary function f

$$\forall \bar{x}, \bar{y}.\ \bigwedge_{i=1}^{n} x_i = y_i \implies f(\bar{x}) = f(\bar{y})$$

- Ignore details and characteristics of a function

Functional Congruence

Example (Uninterpreted Functions & Commutativity)

The following formula is valid as $+$ is commutative.

$$x_1 = y_1 \land x_2 = y_2 \implies x_1 + x_2 = y_2 + y_1$$

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Abstracting $+$ with an uninterpreted function symbol f:

$$x_1 = y_1 \wedge x_2 = y_2 \implies f(x_1, x_2) = f(y_2, y_1)$$

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Additional constraint to keep the commutativity:

$$x_1 = y_1 \wedge x_2 = y_2 \implies f(x_1, x_2) = f(y_2, y_1) \vee f(x_1, x_2) = f(y_1, y_2)$$

Congruence Closure Algorithm

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Algorithm (Congruence Closure Algorithm)

$$F : (\bigwedge_{i=1}^{m} s_i = t_i) \wedge (\bigwedge_{j=m+1}^{n} s_j \neq t_j)$$

S: the set of all equalities and inequalities in F
T: set of all terms and subterms in F

A partition of T is constructed as follows:

(1) initial partition $\{\{t\} \mid t \in T\}$

(2) for all $1 \leq i \leq m$

  a. with $s_i = t_i$ merge the congruence classes of s_i and t_i
  b. propagate the new congruence with symmetry, transitivity, and functional congruence

Congruence Closure Algorithm

Constructed partition induces a *congruence relation* $\sim$ on T

- $\sim$ is reflexive, symmetric, & transitive (equivalence relation)
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Theorem

F is satisfiable $\iff \nexists s_i, t_i \in T$ such that $s_i \sim t_i$ and $(s_i \neq t_i) \in S$

Congruence Closure Algorithm: Example

Example (Congruence Closure Algorithm)

$$F : f(a, b) = a \wedge f(f(a, b), b) \neq a$$

- initial partition:

$\{a\}$ $\{b\}$

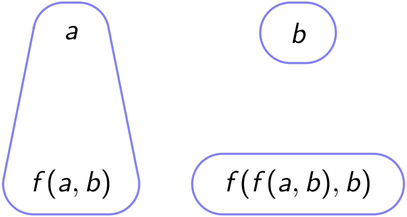
$\{f(a, b)\}$ $\{f(f(a, b), b)\}$

Congruence Closure Algorithm: Example

Example (Congruence Closure Algorithm)

$$F : f(a, b) = a \wedge f(f(a, b), b) \neq a$$

- with $f(a, b) = a$ merge the congruence classes of $f(a, b)$ and a:

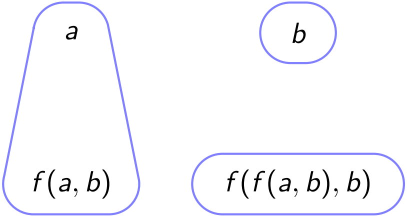


Congruence Closure Algorithm: Example

Example (Congruence Closure Algorithm)

$$F : f(a,b) = a \wedge f(f(a,b),b) \neq a$$

- $a \sim f(a,b)$, with functional congruence $f(a,b) \sim f(f(a,b),b)$:



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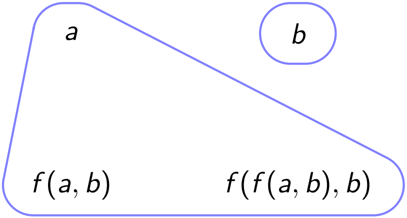

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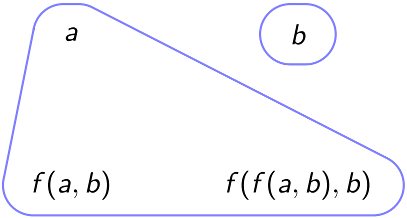

$\implies f(f(a, b), b) \sim a$, but $f(f(a, b), b) \neq a \implies F$ is UNSATISFIABLE

Satisfiability of Arbitrary EUF-Formulas

- Congruence Closure algorithm only for conjunctions of equalities and inequalities

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Algorithm (Satisfiability of Arbitrary EUF-Formulas)

1. Negate F & convert to DNF, yields F'
2. check unsatisfiability for each disjunct of F' with Congruence Closure algorithm
3. if all disjuncts of F' are unsatisfiable
   then return SATISFIABLE
   else return UNSATISFIABLE

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possible exponential blowup of DNF

Efficient Implementation of the Congruence Closure Algorithm

Constructed congruence relation $\sim$ on T

- $\sim$ is reflexive, symmetric, & transitive (equivalence relation)
- $\sim$ respects functional congruence

$\Longrightarrow$ Idea: Union-Find algorithm + efficient propagation of functional congruence

Efficient Implementation of the Congruence Closure Algorithm: Union-Find

General idea

- Membership to an equivalence class represented by reference to representative element

Example (Union-Find)

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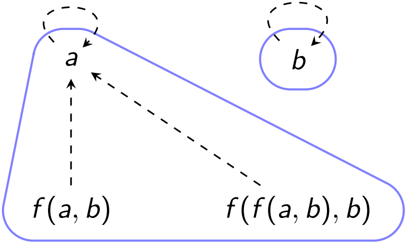

How to efficiently propagate functional congruence?

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$$\forall \bar{x}, \bar{y}. \bigwedge_{i=1}^{n} x_i = y_i \implies f(\bar{x}) = f(\bar{y})$$

$\implies$ functional congruence is only propagated from function arguments to function applications

$\implies$ *Directed-Acyclic-Graph* (DAG)

Efficient Implementation of the Congruence Closure Algorithm: DAG

Example (DAG)

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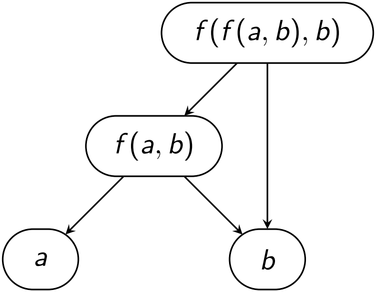

$\implies$ efficient propagation of functional congruence to predecessors

Efficient Implementation of the Congruence Closure Algorithm: Propagation of Functional Congruence

- P_s: all predecessors the congruence class that contains s
- P_t: all predecessors the congruence class that contains t
- new congruence $s \sim t$
- for any $(s', t') \in P_s \times P_t$ check if $s' \overset{?}{\sim} t'$

Efficient Implementation of the Congruence Closure Algorithm: Union-Find + DAG

Example (Union-Find + DAG)

$$G : f^5(a) = a \wedge f^3(a) = a \wedge f(a) \neq a$$

- DAG for G:

$$f^5(a) \rightarrow f^4(a) \rightarrow f^3(a) \rightarrow f^2(a) \rightarrow f(a) \rightarrow a$$

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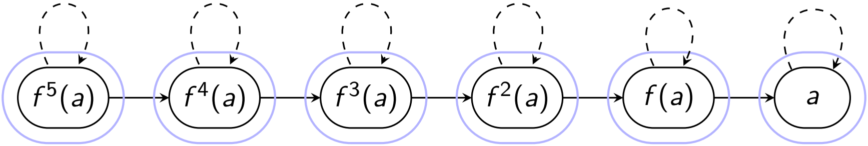
- Initial partition:



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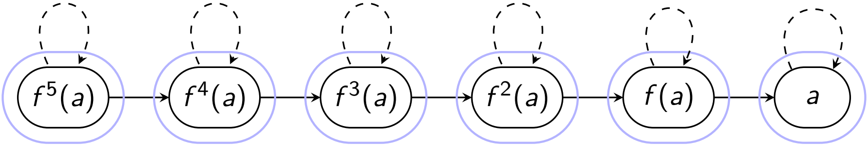

$P_{f^5(a)} \times P_a = \{\}$

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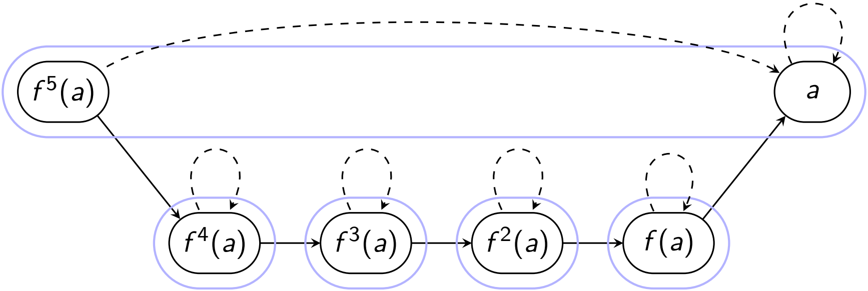

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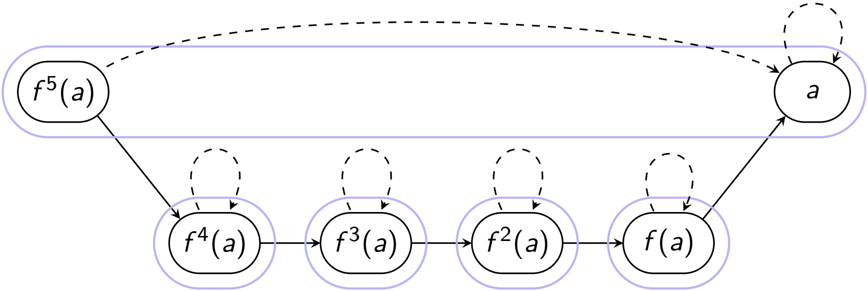

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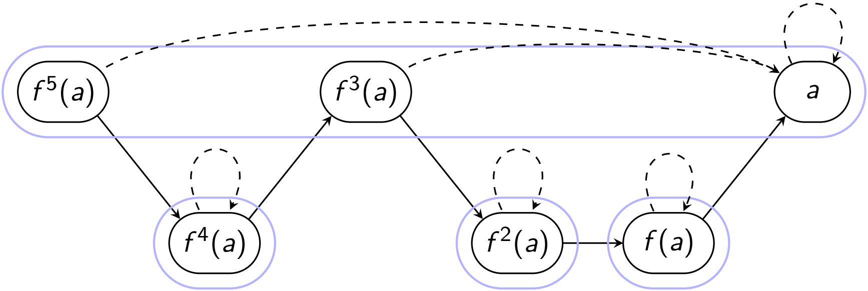

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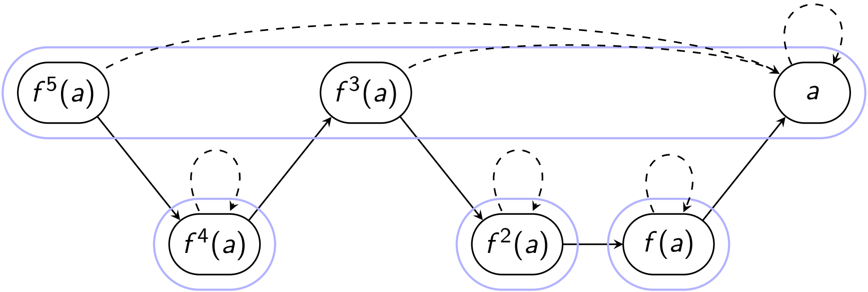

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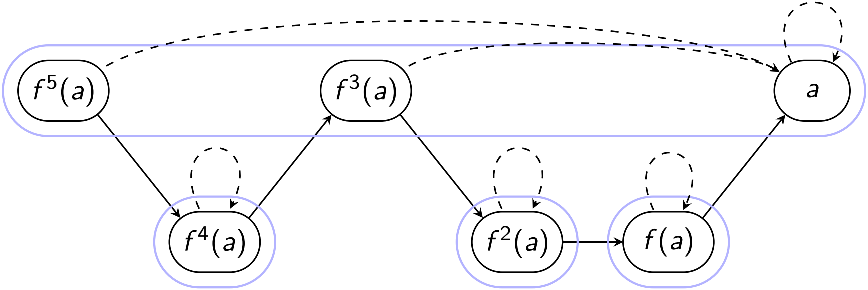

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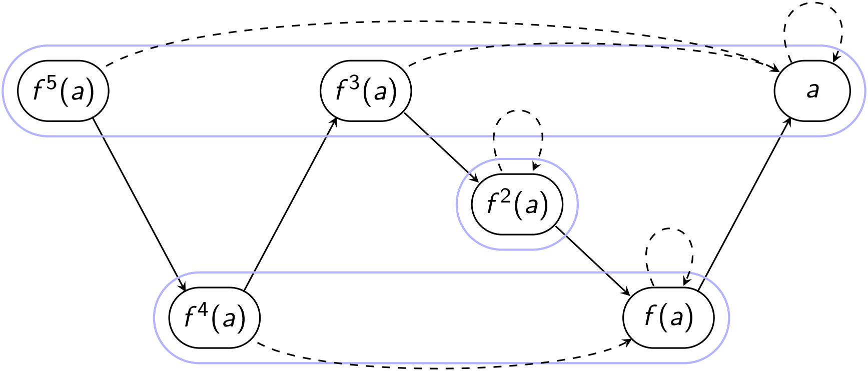

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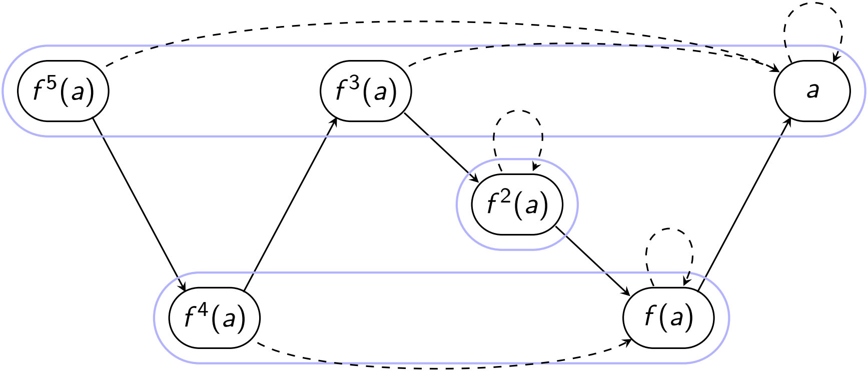

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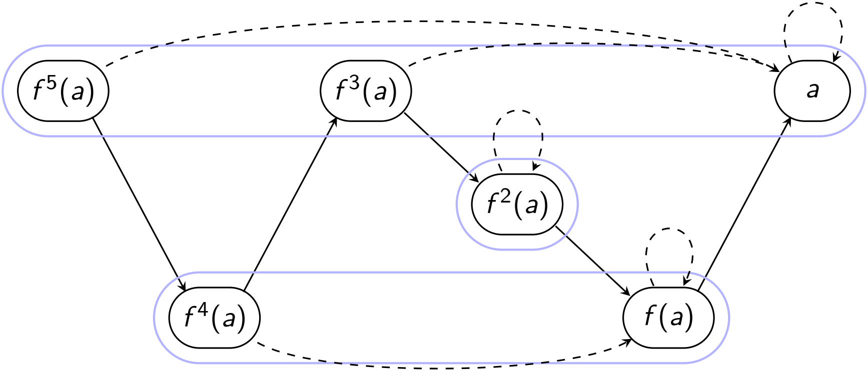

$$P_{f^5(a)} \times P_{f^2(a)} = \{(f^4(a), f^3(a)), (f(a), f^3(a))\}$$

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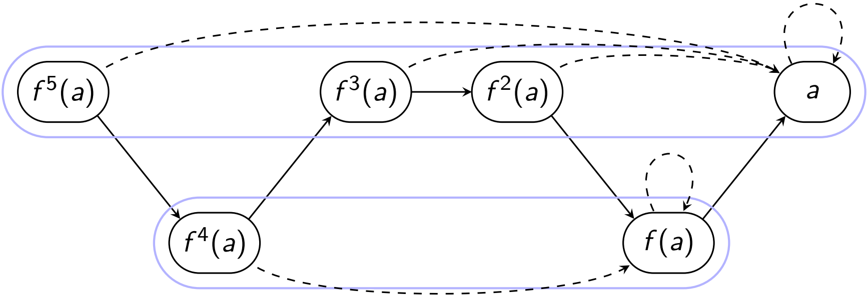

$P_{f^5(a)} \times P_{f^2(a)} = \{(f^4(a), f^3(a)), (f(a), f^3(a))\}$
$\implies f^4(a) \overset{?}{\sim} f^3(a)$ and $f(a) \overset{?}{\sim} f^3(a)$

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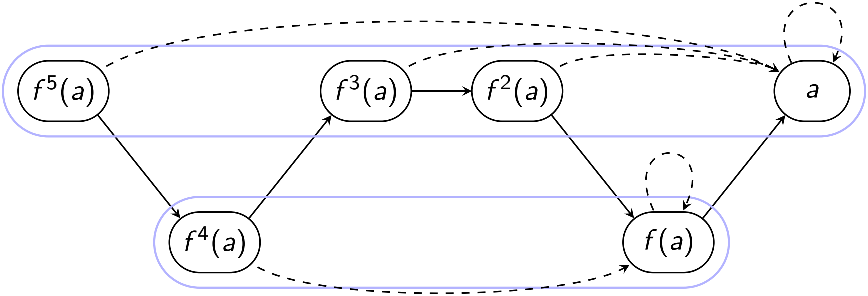

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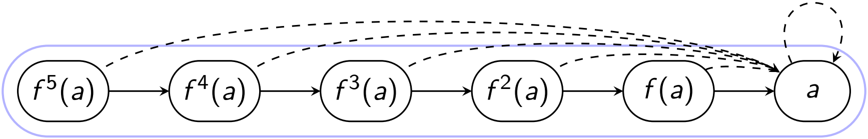

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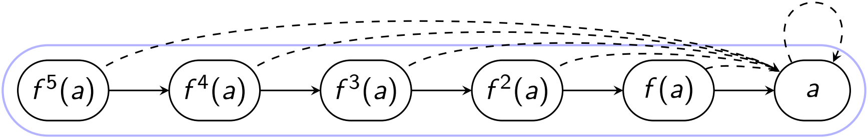

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- Any modern SMT Solver contains an efficient solver for EUF
- SMT solvers have numerous applications
  - Program & hardware verification $\implies$ OpenJML
  - Testcase-generation
  - Static analysis
  - ...

$\implies$ in almost all applications a SMT solver must be able to efficiently reason about equality