Formalizing Theory of Approximating the Traveling-Salesman Problem

Interdisciplinary Project, Mathematics

Lukas Koller

Thursday 29th June, 2023

Traveling-Salesman Problem

The TRAVELING-SALESMAN PROBLEM (TSP) is a well known combinatorial optimization problem.

Definition (TRAVELING-SALESMAN PROBLEM)

Input: undirected graph G with edge weights $c : E(G) \to \mathbb{R}_+$.

Task: Find a Hamiltonian cycle in G with minimum total weight.

Theorem (15.43, Korte and Vygen, 2018)

The decision problem of TSP *is* NP*-hard.*

$\implies$ We can't solve TSP optimally in polynomial time, unless $\mathrm{P} = \mathrm{NP}$.

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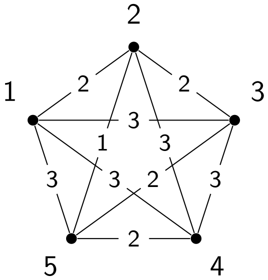

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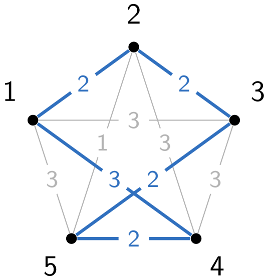

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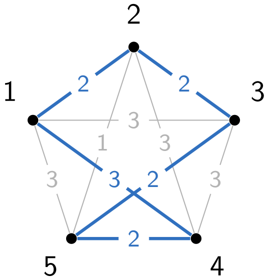

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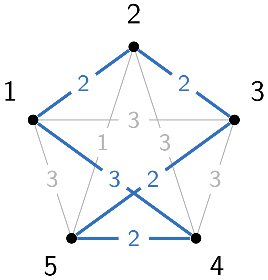

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Approximation Algorithms for TSP

- An *approximation algorithm* is a polynomial-time algorithm for an optimization problem that returns an approximate solution.
- The *approximation ratio* bounds the distance between the approximate solution and the optimal solution.

Negative result for TSP:

Theorem (Sahni and Gonzalez, 1976)

There is no constant-factor approx. algorithm for TSP, *unless* $\mathrm{P} = \mathrm{NP}$.

We simplify the TSP to the $\textsc{Metric}$ TSP by assuming:

(i) the graph G is complete and

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(i) I have formalized a proof for the equivalence of the Maximum Matching Problem and the Minimum Matching-Perfect Problem.

I have formalized parts of the section 21.1, *Approximation Algorithms for the TSP* from (Korte and Vygen, 2018).

(ii) I have formally verified two approx. algorithms for Metric TSP.
- DoubleTree and Christofides-Serdyukov algorithm

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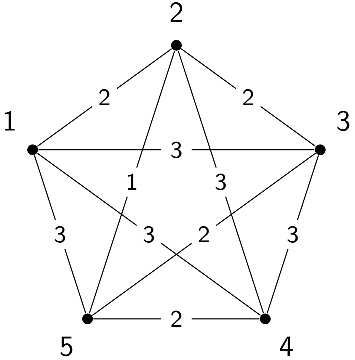

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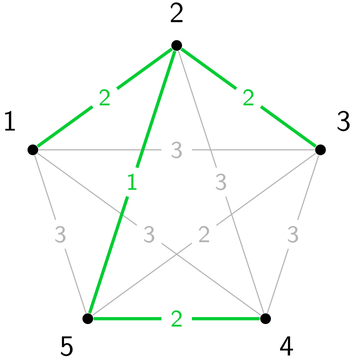

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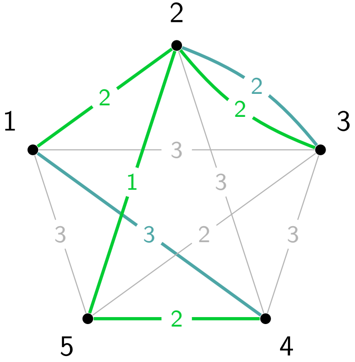

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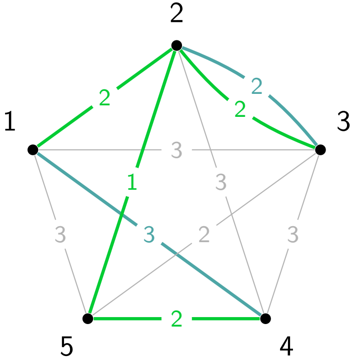

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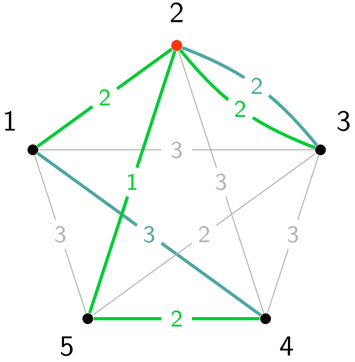

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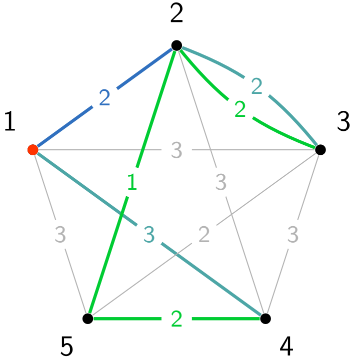

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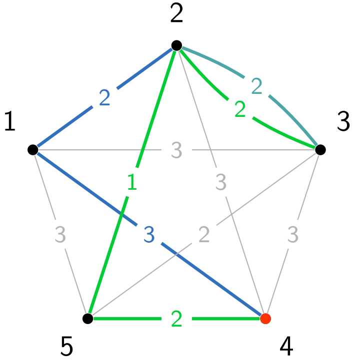

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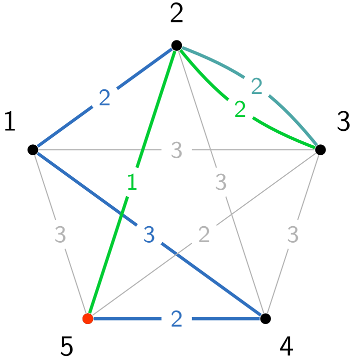

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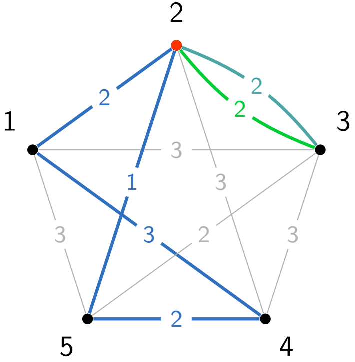

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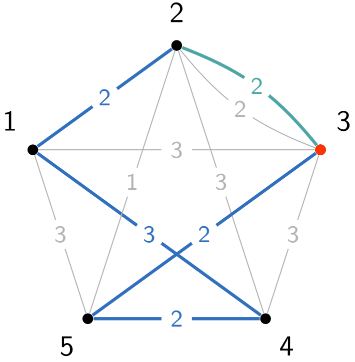

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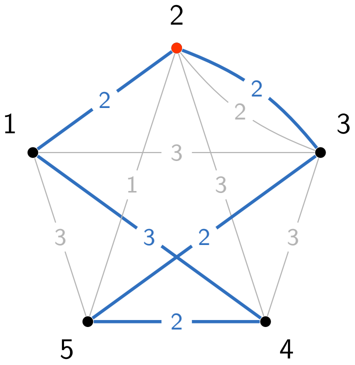

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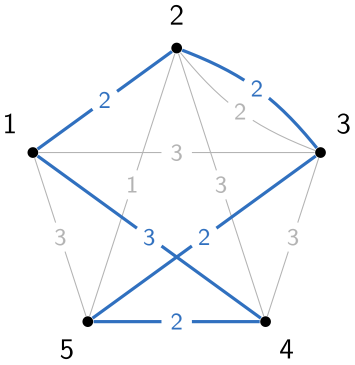

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Formalization of the Christofides-Serdyukov Algorithm

I have formalized and verified the Christofides-Serdyukov algorithm in Isabelle/HOL (Nipkow et al., 2002).

- I build on an exiting graph formalization by Abdulaziz (2020) for Berge's theorem (Berge, 1957).
- The Christofides-Serdyukov algorithm is formalized with the assumption of the existence of necessary algorithms.
  - minimum spanning-tree, minimum perfect-matching, and Eulerian tour.
- The feasibility and the approximation ratio of the Christofides-Serdyukov algorithm are proven.
- The definition of the Christofides-Serdyukov algorithm is refined to a `WHILE`-program using Hoare-Logic.

L-Reduction (Linear Reduction)

An L-reduction is a special type of reduction that is used to prove the $\textsc{MaxSNP}$-hardness of optimization problems.

Definition (L-Reduction, Papadimitriou and Yannakakis, 1991)

Let A and B be optimization problems with cost functions c_A and c_B. An L-Reduction from A to B is a pair of function f and g (computable in polynomial time) with $\alpha, \beta > 0$ s.t. for any instance x_A of A

(i) $f(x_A)$ is an instance of B and

$$\mathrm{OPT}(f(x_A)) \leq \alpha\, \mathrm{OPT}(x_A),$$

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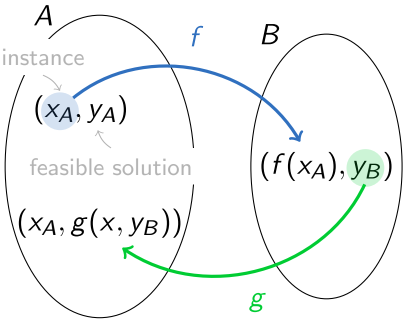

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VCP4[1]

Input: undir. graph G with maximum degree of 4.

Task: Find a vertex cover of G with minimum cardiality.

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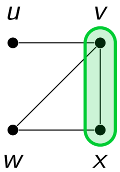

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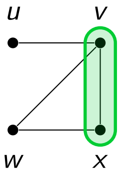

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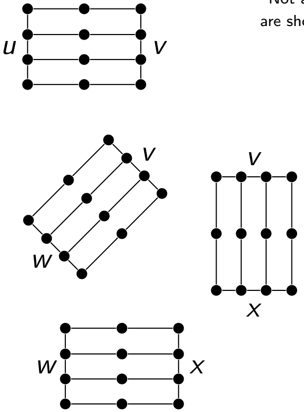

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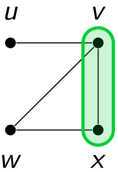

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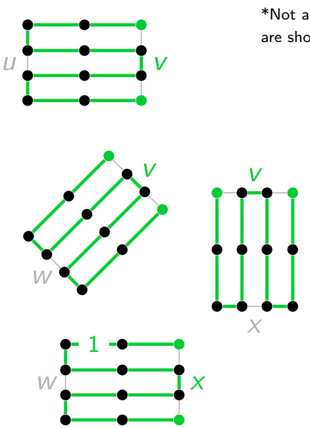

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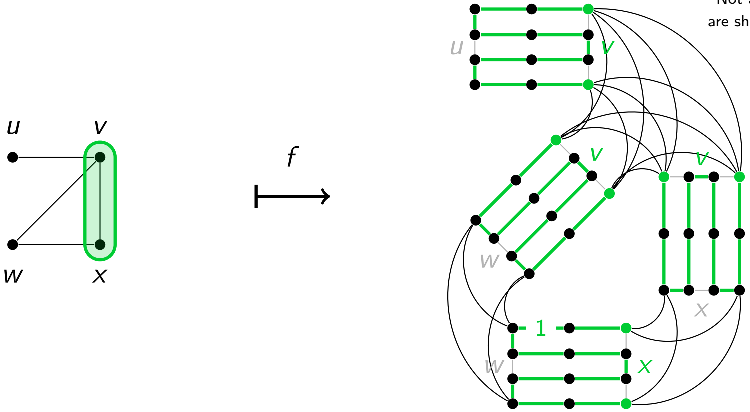

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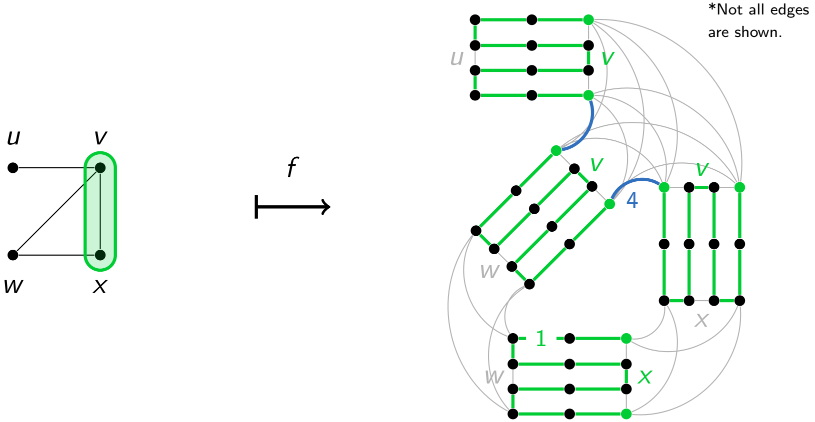

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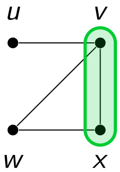
Input: undir. graph G with maximum degree of 4.

Task: Find a vertex cover of G with minimum cardiality.

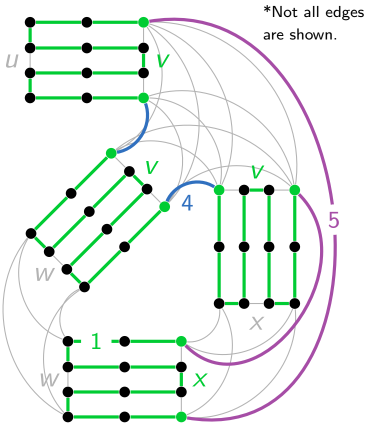
Metric TSP

Input: undir. & compl. graph G with edge weights $c : E(G) \to \mathbb{R}_+$ that satisfy the tri.-inequality.

Task: Find a Hamiltonian cycle in G with minimum total weight.



$\xmapsto{f}$



[1] Minimum Vertex Cover Problem with maximum degree of 4

L-Reduction for Metric TSP (Korte and Vygen, 2018) II

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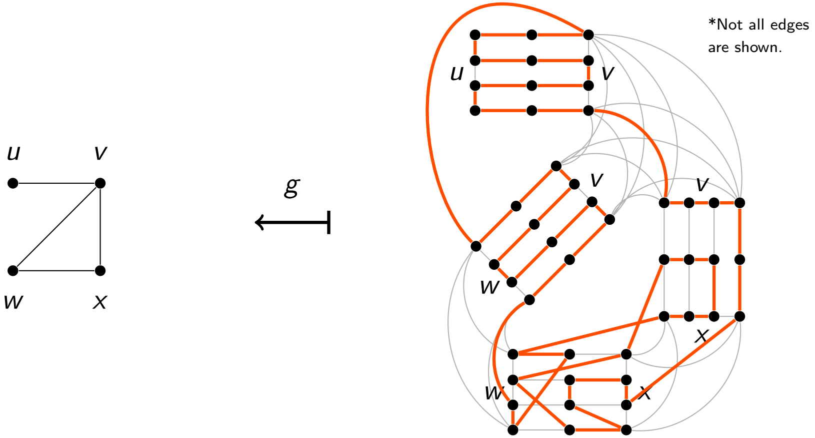

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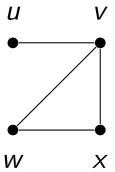
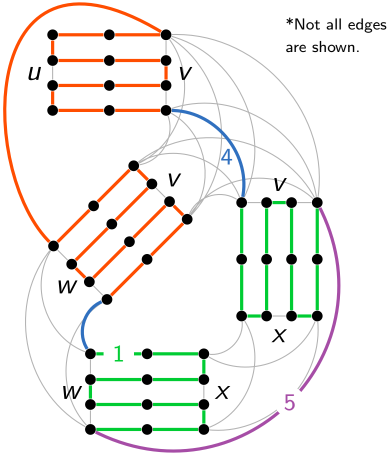


$\overset{g}{\longleftarrow\!\!\!|}$



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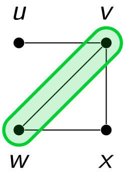

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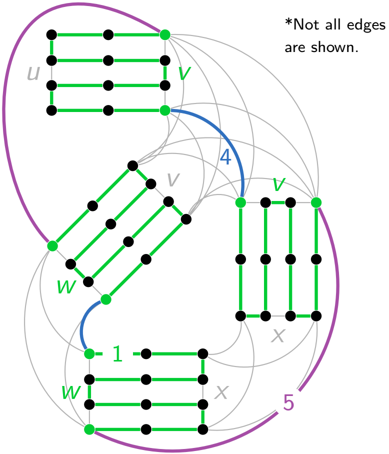

[1] Minimum Vertex Cover Problem with maximum degree of 4

Formalization of L-Reduction for Metric TSP

I have formalized the L-reduction in Isabelle/HOL.

- A locale provides an abstract executable graph representation based on an adjacency map.
- The locale assumes `fold`-functions to do computation on the graphs.
  - This approach is adapted from (Abdulaziz, 2022).
- Abstract versions of the reduction functions f and g are defined using the `fold`-functions.
- The feasibility of the function f and g and the linear inequalities are proven.
- The locale is instantiated with concrete implementations to obtain executable versions of the functions f and g.

The L-reduction in (Thm. 21.6, Korte and Vygen, 2018) is incomplete.

(i) The proof for the linear inequalities fails if the optimal vertex cover has a cardinality of 1.

(ii) Not all cases are covered when constructing a vertex cover from a Hamiltonian cycle.

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Conclusion and Future Work

- I have formally verified two approx. algorithms for METRIC TSP.
- I have formalized a L-reduction from VCP4[1] to METRIC TSP, which proves the MAXSNP-hardness of METRIC TSP.

- Prove the existence of the necessary algorithms for the CHRISTOFIDES-SERDYUKOV algorithm.
  - Write an adaptor to the existing formalization of Prim's algorithm (Lammich and Nipkow, 2019).
  - Formalize and verify algorithms for minimum perfect-matching and Eulerian tour (Sec. 11.2 & 2.4, Korte and Vygen, 2018).
- Formalize an encoding of multigraphs with simple graphs.
- Prove the polynomial running time of reduction functions f and g.
  - Refine the functions f and g to `IMP-` and prove their running time.

[1] MINIMUM VERTEX COVER PROBLEM with maximum degree of 4

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Appendix – Equiv. Max. Matching and Min. Matching

Let G be an undirected graph with edge weights c.

(i) MaxMatch[1] solves MinMatch[2]. Assume f solves MaxMatch.

$$c'(e) := 1 + -c(e) + \sum_{e' \in E(G)} c(e')$$

- If matching $M := f(G, c')$ is perfect return M otherwise None.

(ii) MinMatch solves MaxMatch. Assume f solves MinMatch.

$$E_1 := \{e \times \{1\} \mid e \in E(G)\} \qquad E_2 := \{e \times \{2\} \mid e \in E(G)\}$$

$$E_V := \{\{(v,1),(v,2)\} \mid v \in V(G)\}$$

$$H := (V(G) \times \{1,2\}, E_1 \cup E_2 \cup E_V) \quad c'(e) := \begin{cases} 1 & \text{if } e \in E_V \\ -c(e) & \text{otherwise} \end{cases}$$

- Return matching $M := f(H, c') \cap E_1$.

[1] Maximum Matching Problem

[2] Minimum Matching-Perfect Problem

Appendix – Approximation Algorithms for Metric TSP

Let (G, c) be an instance of METRIC TSP.

Lemma (21.3, Korte and Vygen, 2018)

Given a connected Eulerian graph G' that spans $V(G)$, we can compute (in polynomial time) a Hamiltonian cycle for G with total weight of at most $c(E(G'))$.

Proof Idea.

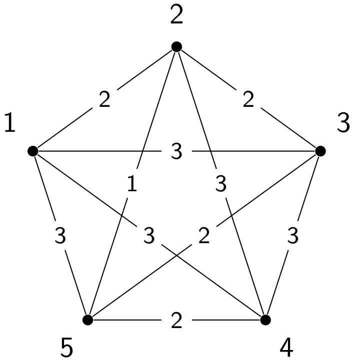
1. Compute a Eulerian tour in G'.
2. Remove duplicate vertices ("shortcutting").

$\Longrightarrow$ We only need to find Eulerian graph that spans $V(G)$ with the smallest total weight!

Appendix – DoubleTree Algorithm

Theorem (21.4, Korte and Vygen, 2018)

DOUBLETREE *is a 2-approximation algorithm for* METRIC TSP.



Algorithm:

1. compute minimum spanning-tree (MST)
2. compute Eulerian tour on doubled MST, e.g. 2,1,2,5,4,5,2,3,2
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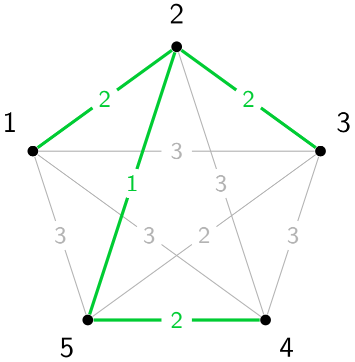

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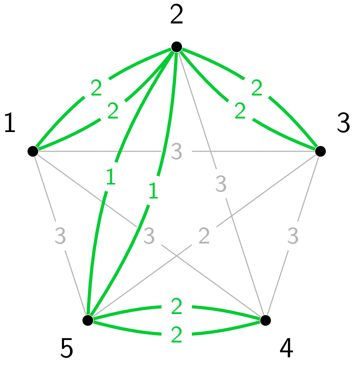

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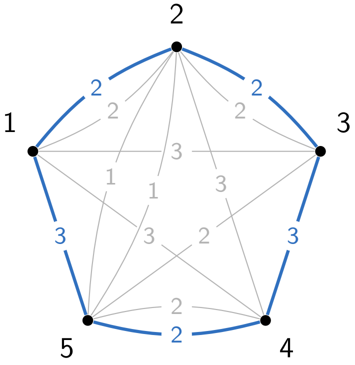

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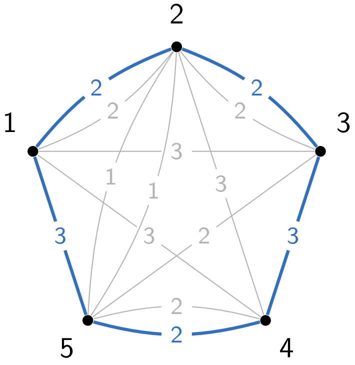

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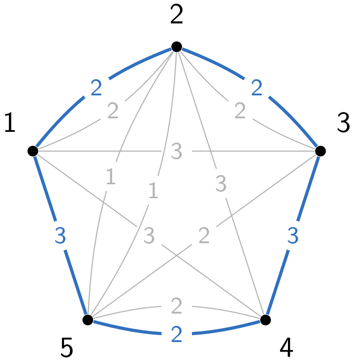

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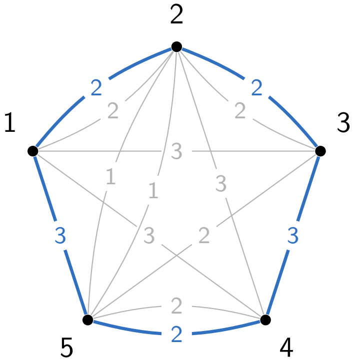

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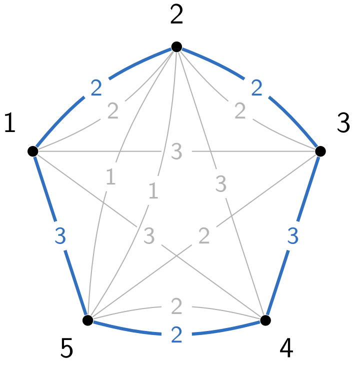

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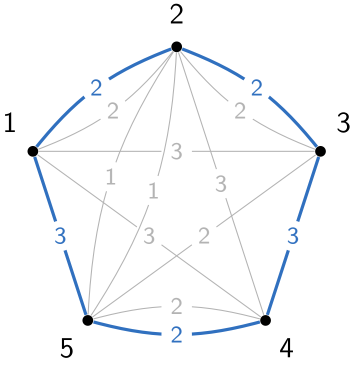

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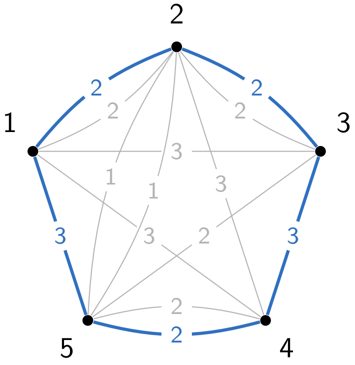

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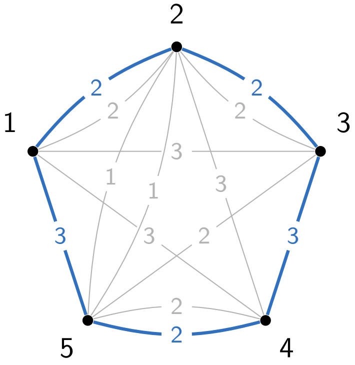

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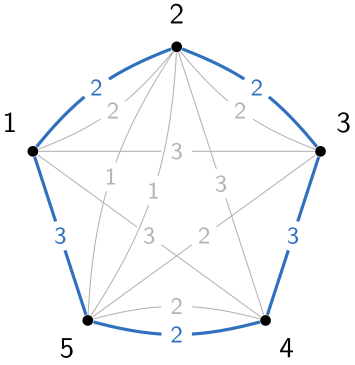

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