A CURIOSITY ABOUT POLYNOMIAL INTERPOLATION

PETRO KOLOSOV

Abstract. Interpolation of cubes expected to be

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

but got

$$n^3 = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} k^1 (n-k)^1$$

Contents

1. Introduction

Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

At that time, in 2016, I was a first-year mechanical engineering undergraduate, so that due to lack of knowledge and perspective of view I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. *All the mathematical laws and*

Date: January 10, 2025.

2010 *Mathematics Subject Classification.* 26E70, 05A30.

Key words and phrases. Keyword1, Keyword2 .

Sources: https://github.com/kolosovpetro/ACuriosityAboutPolynomialInterpolation

relations exist from the very beginning, we only reveal and describe them, I thought. That mindset truly inspired me, so my own mathematical journey has been started. Consider finite differences of cubes n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

The problem of interpolation of polynomials is a classical problem in mathematics and has been widely studied in literature. For instance, Concrete mathematics [2] gives interpolation of cubes by using Newton's interpolation formula

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

because

$$f(x) = \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots + f(0)\binom{x}{0} = \sum_{r=0}^{d} \Delta^{d-r} f(0)\binom{x}{r}$$

However, interpolation of cubes can be also done in a different way. The key point that interpolation formula above iterates over the finite difference of order d, instead it is clear that n^3 can be reached as a sum of finite difference Δ^1 of first order

$$n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3 = \sum_{k=0}^{n-1} \Delta k^3$$

We know that $\Delta^3 n^3 = 6$ is the constant for each n. The second difference of cubes $\Delta^2 n^3$ is a linear relation in terms of third order finite difference $\Delta^3 n^3$.

$$\Delta^2 n^3 = (n+1)\Delta^3 n^3 = 6(n+1)$$

Finally, the first order finite difference Δn^3 is the following relation in terms of second order finite difference

$$\Delta n^3 = \Delta 0^3 + \Delta^2 (n-1)^3 = 1 + \Delta^2 (n-1)^3 = 1 + 6(n-1)$$

Meaning that

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

Finally reaching its generic form

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k$$

Because

$$\Delta(n^3) = \Delta(n-1)^3 + \Delta^2 (n-1)^3$$

Having the relation $n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3$, we get

$$n^3 = [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2]$$
$$+ \cdots + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)]$$

By rearranging the terms of the equation above, we get summation in terms of $k(n-k)$

$$n^3 = n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2]$$
$$+ \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)]$$

Giving an identity for cubes n^3

$$n^3 = \sum_{k=1}^{n} 6k(n-k)+1 \tag{1.1}$$

2. Conclusions

Conclusions of your manuscript.

References

[1] Meijering, Erik. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, 2002. https://infoscience.epfl.ch/record/63085/files/meijering0201.pdf.

[2] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, and Stanley Liu. Concrete mathematics: a foundation for computer science. *Computers in Physics*, 3(5):106–107, 1989.

Version: `Local-0.1.0`

Software Developer, DevOps Engineer

Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io