A CURIOSITY ABOUT POLYNOMIAL INTERPOLATION

PETRO KOLOSOV

Abstract. Interpolation of cubes expected to be

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

but got

$$n^3 = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} k^1 (n-k)^1$$

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1. Introduction

Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

The first time I found interpolation interesting was in 2016 when I observed a table of finite differences of cubes. Back then, I was a first-year mechanical engineering undergraduate. Due to my lack of mathematical knowledge, I started re-inventing interpolation formulas myself, fueled by pure passion and a sense of mystery. *All the mathematical laws and relations exist from the very beginning; we only reveal and describe them*, I thought. That mindset truly inspired me, and thus, my own mathematical journey began.

Consider finite differences of cubes n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

The problem of interpolation of polynomials is a classical problem in mathematics and has been widely studied in literature. For instance, Concrete mathematics [2, p. 190] gives interpolation of cubes by using Newton's interpolation formula

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

because

$$f(x) = \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots + f(0)\binom{x}{0} = \sum_{r=0}^{d} \Delta^{d-r} f(0)\binom{x}{d-r}$$

However, interpolation of cubes can be also done in a different way. The key point that interpolation formula above iterates over the order d of finite difference. Alternatively, we can interpolate cubes n^3 as a sum of first order finite difference Δ as follows

$$n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3 = \sum_{k=0}^{n-1} \Delta k^3$$

We know that $\Delta^3 n^3 = 6$ is the constant for each n. The second difference of cubes $\Delta^2 n^3$ is a linear relation in terms of third order finite difference $\Delta^3 n^3$

$$\Delta^2 n^3 = (n+1)\Delta^3 n^3 = 6(n+1)$$

Finally, the first order finite difference Δn^3 is the following relation in terms of second order finite difference

$$\Delta n^3 = \Delta 0^3 + \Delta^2 0^3 + \Delta^2 1^3 + \cdots + \Delta^2 (n-1)^3 = 1 + \sum_{k=0}^{n-1} 6(k+1)$$

Altering summation bounds yields

$$\Delta n^3 = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k$$

Therefore, we are able to express first order finite difference of cubes in form of sums as follows

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

Now it is time to assemble all the results above to get the polynomial n^3. Having the relation in cubes $n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3$ we get

$$\begin{aligned} n^3 &= [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] \\ &+ \cdots + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

By rearranging the terms of the equation above, we get summation in terms of $k(n-k)$

$$\begin{aligned} n^3 &= n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] \\ &+ \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)] \end{aligned}$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term n in the sum above can be moved under sigma notation, because there is exactly n iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k)+1$$

By inspecting the expression $6k(n-k)+1$ we iterate under summation, we can notice that it is symmetric over k, let be $T(n,k) = 6k(n-k)+1$, then $T(n,k) = T(n,n-k)$. This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^{n} 6k(n-k)+1$$

Curiously enough that although $\sum_{k=0}^{n-1} 6k(n-k)+1$ and $\sum_{k=1}^{n} 6k(n-k)+1$ both simplify to n^3, they produce different closed forms. Let be $P(n,q) = \sum_{k=0}^{q-1} 6k(n-k)+1$ and $Q(n,q) = \sum_{k=1}^{q} 6k(n-k)+1$, then

$$P(n,q) = \begin{cases} q=1: & 1 \\ q=2: & -4+6n \\ q=3: & -27+18n \end{cases}$$

$$Q(n,q) = \begin{cases} q=1: & -5+6n \\ q=2: & -28+18n \\ q=3: & -81+36n \end{cases}$$

References

[1] Meijering, Erik. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, 2002. https://infoscience.epfl.ch/record/63085/files/meijering0201.pdf.

[2] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. https://archive.org/details/concrete-mathematics.

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