

# A CURIOSITY ABOUT POLYNOMIAL INTERPOLATION

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ABSTRACT. Interpolation of cubes expected to be

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

but got

$$n^3 = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} k^1 (n-k)^1$$

## CONTENTS

<a href="#">1. Introduction</a>	1
<a href="#">References</a>	5

## 1. INTRODUCTION

Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton’s fundamental works, nowadays known as foundation of classical interpolation theory [\[1\]](#).

At that time, in 2016, I was a first-year mechanical engineering undergraduate, so that due to lack of knowledge and perspective of view I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. *All the mathematical laws and relations exist from the very beginning, we only reveal and describe them*, I thought. That mindset truly inspired me, so my own mathematical journey has been started. Consider finite differences of cubes  $n^3$

$n$	$n^3$	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

**Table 1.** Table of finite differences of the polynomial  $n^3$ .

The problem of interpolation of polynomials is a classical problem in mathematics and has been widely studied in literature. For instance, Concrete mathematics [2, p. 190] gives interpolation of cubes by using Newton's interpolation formula

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

because

$$f(x) = \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots + f(0)\binom{x}{0} = \sum_{r=0}^d \Delta^{d-r} f(0)\binom{x}{d-r}$$

However, interpolation of cubes can be also done in a different way. The key point that interpolation formula above iterates over the order  $d$  of finite difference. Alternatively, we can interpolate cubes  $n^3$  as a sum of first order finite difference  $\Delta$  as follows

$$n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3 = \sum_{k=0}^{n-1} \Delta k^3$$

We know that  $\Delta^3 n^3 = 6$  is the constant for each  $n$ . The second difference of cubes  $\Delta^2 n^3$  is a linear relation in terms of third order finite difference  $\Delta^3 n^3$

$$\Delta^2 n^3 = (n+1)\Delta^3 n^3 = 6(n+1)$$

Finally, the first order finite difference  $\Delta n^3$  is the following relation in terms of second order finite difference

$$\Delta n^3 = \Delta 0^3 + \Delta^2 0^3 + \Delta^2 1^3 + \cdots + \Delta^2 (n-1)^3 = 1 + \sum_{k=0}^{n-1} 6(k+1)$$

Altering summation bounds yields

$$\Delta n^3 = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^n k$$

Therefore, we are able to express first order finite difference of cubes in form of sums as follows

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

Now it is time to assemble all the results above to get the polynomial  $n^3$ . Having the relation in cubes  $n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta (n-1)^3$  we get

$$\begin{aligned} n^3 &= [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] \\ &\quad + \cdots + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

By rearranging the terms of the equation above, we get summation in terms of  $k(n-k)$

$$\begin{aligned} n^3 &= n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] \\ &\quad + \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)] \end{aligned}$$

By applying compact sigma sum notation yields an identity for cubes  $n^3$

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term  $n$  in the sum above can be moved under sigma notation, because there is exactly  $n$  iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$$

By inspecting the expression  $6k(n-k) + 1$  we iterate under summation, we can notice that it is symmetric over  $k$ , let be  $T(n, k) = 6k(n-k) + 1$ , then  $T(n, k) = T(n, n-k)$ . This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^n 6k(n-k) + 1$$

Curiously enough that although  $\sum_{k=0}^{n-1} 6k(n-k) + 1$  and  $\sum_{k=1}^n 6k(n-k) + 1$  both simplify to  $n^3$ , they produce different closed forms. Let be  $P(n, q) = \sum_{k=0}^{q-1} 6k(n-k) + 1$  and  $Q(n, q) = \sum_{k=1}^q 6k(n-k) + 1$ , then

$$P(n, q) = \begin{cases} q = 1 : & 1 \\ q = 2 : & -4 + 6n \\ q = 3 : & -27 + 18n \end{cases}$$

$$Q(n, q) = \begin{cases} q = 1 : & -5 + 6n \\ q = 2 : & -28 + 18n \\ q = 3 : & -81 + 36n \end{cases}$$

Now let's take a breath and briefly summarize all the results we got so far. So, we have successfully interpolated the polynomial  $n^3$  using discrete set of finite differences data points, by expressing its first order finite difference as  $\Delta n^3 = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \dots + 6 \cdot n$ , so that  $n^3$  is a sum of  $\Delta n^3$ . By rearranging the terms we got  $n^3 = n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] + \dots + [(n-k) \cdot 6 \cdot k] + \dots + [1 \cdot 6 \cdot (n-1)]$ , leading to identities in

terms of cubes

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1; \quad n^3 = \sum_{k=1}^n 6k(n-k) + 1$$

Indeed, summation bounds of above identities can be altered because  $T(n, k) = 6k(n-k) + 1$  is symmetric  $T(n, k) = T(n, n-k)$ .

## REFERENCES

- [1] Meijering, Erik. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, 2002. <https://infoscience.epfl.ch/record/63085/files/meijering0201.pdf>.
- [2] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. <https://archive.org/details/concrete-mathematics>.

**Version:** Local-0.1.0