

# A CURIOSITY ABOUT POLYNOMIAL INTERPOLATION

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ABSTRACT. Interpolation of cubes expected to be

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

but got

$$n^3 = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} k^1 (n-k)^1$$

## CONTENTS

1. Introduction	1
2. Generalizations	5
3. Discussions	6
3.1. Interpolation	6
References	7

## 1. INTRODUCTION

Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton’s fundamental works, nowadays known as foundation of classical interpolation theory [1].

The first time I found interpolation interesting was in 2016 when I observed a table of finite differences of cubes. Back then, I was a first-year mechanical engineering undergraduate. Due to my lack of mathematical knowledge, I started re-inventing interpolation formulas myself, fueled by pure passion and a sense of mystery. *All the mathematical laws and relations exist*

*from the very beginning; we only reveal and describe them*, I thought. That mindset truly inspired me, and thus, my own mathematical journey began.

Consider finite differences of cubes  $n^3$

$n$	$n^3$	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

**Table 1.** Table of finite differences of the polynomial  $n^3$ .

The problem of interpolation of polynomials is a classical problem in mathematics and has been widely studied in literature. For instance, Concrete mathematics [2, p. 190] gives interpolation of cubes by using Newton's interpolation formula

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

because

$$f(x) = \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots + f(0)\binom{x}{0} = \sum_{r=0}^d \Delta^{d-r} f(0)\binom{x}{d-r}$$

However, interpolation of cubes can be also done in a different way. The key point that interpolation formula above iterates over the order  $d$  of finite difference. Alternatively, we can interpolate cubes  $n^3$  as a sum of first order finite difference  $\Delta$  as follows

$$n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3 = \sum_{k=0}^{n-1} \Delta k^3$$

We know that  $\Delta^3 n^3 = 6$  is the constant for each  $n$ . The second difference of cubes  $\Delta^2 n^3$  is a linear relation in terms of third order finite difference  $\Delta^3 n^3$

$$\Delta^2 n^3 = (n+1)\Delta^3 n^3 = 6(n+1) \quad (1)$$

Finally, the first order finite difference  $\Delta n^3$  is the following relation in terms of second order finite difference

$$\Delta n^3 = \Delta 0^3 + \Delta^2 0^3 + \Delta^2 1^3 + \cdots + \Delta^2 (n-1)^3 = 1 + \sum_{k=0}^{n-1} 6(k+1)$$

Altering summation bounds yields

$$\Delta n^3 = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^n k$$

Therefore, we are able to express first order finite difference of cubes in form of sums as follows

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

Now it is time to assemble all the results above to get the polynomial  $n^3$ . Having the relation in cubes  $n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3$  we get

$$\begin{aligned} n^3 &= [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] \\ &\quad + \cdots + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

By rearranging the terms of the equation above, we get summation in terms of  $k(n-k)$

$$\begin{aligned} n^3 &= n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] \\ &\quad + \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)] \end{aligned} \quad (2)$$

By applying compact sigma sum notation yields an identity for cubes  $n^3$

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term  $n$  in the sum above can be moved under sigma notation, because there is exactly  $n$  iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$$

By inspecting the expression  $6k(n-k) + 1$  we iterate under summation, we can notice that it is symmetric over  $k$ , let be  $T(n, k) = 6k(n-k) + 1$ , then  $T(n, k) = T(n, n-k)$ . This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^n 6k(n-k) + 1$$

Curiously enough that although  $\sum_{k=0}^{n-1} 6k(n-k) + 1$  and  $\sum_{k=1}^n 6k(n-k) + 1$  both simplify to  $n^3$ , they produce different closed forms. Let be  $P(n, q) = \sum_{k=0}^{q-1} 6k(n-k) + 1$  and  $Q(n, q) = \sum_{k=1}^q 6k(n-k) + 1$ , then

$$P(n, q) = \begin{cases} q = 1 : & 1 \\ q = 2 : & -4 + 6n \\ q = 3 : & -27 + 18n \end{cases}$$

$$Q(n, q) = \begin{cases} q = 1 : & -5 + 6n \\ q = 2 : & -28 + 18n \\ q = 3 : & -81 + 36n \end{cases}$$

## 2. GENERALIZATIONS

Assume that our previously obtained identities  $n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$  and  $n^3 = \sum_{k=1}^n 6k(n-k) + 1$  have explicit form as follows

$$n^3 = \sum_k \mathbf{A}_{1,1} k^1 (n-k)^1 + \mathbf{A}_{1,0} k^0 (n-k)^0$$

where  $\mathbf{A}_{1,1} = 6$  and  $\mathbf{A}_{1,0} = 1$ , respectively. Therefore, let be a conjecture

**Conjecture 2.1.** *For every  $n \geq 1$ ,  $n, m \in \mathbb{N}$  there are coefficients  $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$  such that*

$$n^{2m+1} = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \dots + \mathbf{A}_{m,m} k^m (n-k)^m$$

Note that conjecture above assumes the convention  $0^0 = 1$ , reader may found a comprehensive discussion of it in [3].

Long story short, above conjecture is true, so that real coefficients  $\mathbf{A}_{m,r}$  are following

$m/r$	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

**Table 2.** Coefficients  $\mathbf{A}_{m,r}$ . See OEIS sequences [4, 5].

These coefficients  $\mathbf{A}_{m,r}$  are defined via a recurrence relation involving Binomial coefficients and Bernoulli numbers

**Definition 2.2.** (*Definition of coefficient  $\mathbf{A}_{m,r}$ .*)

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1) \binom{2r}{r} & \text{if } r = m \\ (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

where  $B_t$  are Bernoulli numbers [6]. It is assumed that  $B_1 = \frac{1}{2}$ . Properties of the coefficients  $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$  for  $m < 0$  and  $r > m$
- $\mathbf{A}_{m,r} = 0$  for  $r < 0$
- $\mathbf{A}_{m,r} = 0$  for  $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$  for  $m \geq 0$
- $\mathbf{A}_{m,r}$  are integers for  $m \leq 11$
- Row sums:  $\sum_{r=0}^m \mathbf{A}_{m,r} = 2^{2m+1} - 1$

Proof of conjecture (2.1) as well as other discussions on topics above can be found in literature [7, 8, 9, 10, 11]. Few OEIS sequences were contributed as well [12, 13, 14, 15, 16].

Very well, let's wrap up this technical section and move on to the more engaging discussions.

### 3. DISCUSSIONS

**3.1. Interpolation.** Current manuscript starts from certain interpolation technique shown on base case of cubes, where the key identity is the tricky rearrangement terms of the sum  $\sum_k \Delta k^3$  in (2). This rearrangement was done instead of applying Faulhaber's formula on  $\Delta n^3 = 1 + 6 \sum_{k=1}^n k$  which leads to well-known result involving Binomial theorem:  $(n+1)^3 - n^3 = 1 + 6 \sum_{k=1}^n k = 1 + 6 \frac{1}{2}(n + n^2) = 1 + 3n^3 + 3n$ .

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**Version:** Local-0.1.0