A CURIOSITY ABOUT POLYNOMIAL INTERPOLATION

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Abstract. Interpolation of cubes expected to be

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

but got

$$n^3 = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} k^1 (n-k)^1$$

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1. Introduction

Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

The first time I found interpolation interesting was in 2016 when I observed a table of finite differences of cubes. Back then, I was a first-year mechanical engineering undergraduate. Due to my lack of mathematical knowledge, I started re-inventing interpolation formulas myself, fueled by pure passion and a sense of mystery. *All the mathematical laws and relations exist from the very beginning; we only reveal and describe them*, I thought. That mindset truly inspired me, and thus, my own mathematical journey began.

Consider finite differences of cubes n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

The problem of interpolation of polynomials is a classical problem in mathematics and has been widely studied in literature. For instance, Concrete mathematics [2, p. 190] gives interpolation of cubes by using Newton's interpolation formula

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

because

$$f(x) = \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots + f(0)\binom{x}{0} = \sum_{r=0}^{d} \Delta^{d-r} f(0)\binom{x}{d-r}$$

However, interpolation of cubes can be also done in a different way. The key point that interpolation formula above iterates over the order d of finite difference. Alternatively, we can interpolate cubes n^3 as a sum of first order finite difference Δ as follows

$$n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3 = \sum_{k=0}^{n-1} \Delta k^3$$

We know that $\Delta^3 n^3 = 6$ is the constant for each n. The second difference of cubes $\Delta^2 n^3$ is a linear relation in terms of third order finite difference $\Delta^3 n^3$

$$\Delta^2 n^3 = (n+1)\Delta^3 n^3 = 6(n+1)$$

Finally, the first order finite difference Δn^3 is the following relation in terms of second order finite difference

$$\Delta n^3 = \Delta 0^3 + \Delta^2 0^3 + \Delta^2 1^3 + \cdots + \Delta^2 (n-1)^3 = 1 + \sum_{k=0}^{n-1} 6(k+1)$$

Altering summation bounds yields

$$\Delta n^3 = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + \cdots + 6\cdot n = 1 + 6\sum_{k=0}^{n} k$$

Therefore, we are able to express first order finite difference of cubes in form of sums as follows

$$\Delta(0^3) = 1 + 6\cdot 0$$

$$\Delta(1^3) = 1 + 6\cdot 0 + 6\cdot 1$$

$$\Delta(2^3) = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2$$

$$\Delta(3^3) = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3$$

Now it is time to assemble all the results above to get the polynomial n^3. Having the relation in cubes $n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3$ we get

$$n^3 = [1 + 6\cdot 0] + [1 + 6\cdot 0 + 6\cdot 1] + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2]$$
$$+ \cdots + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + \cdots + 6\cdot(n-1)]$$

By rearranging the terms of the equation above, we get summation in terms of $k(n-k)$

$$n^3 = n + [(n-0)\cdot 6\cdot 0] + [(n-1)\cdot 6\cdot 1] + [(n-2)\cdot 6\cdot 2]$$
$$+ \cdots + [(n-k)\cdot 6\cdot k] + \cdots + [1\cdot 6\cdot(n-1)]$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term n in the sum above can be moved under sigma notation, because there is exactly n iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$$

By inspecting the expression $6k(n-k)+1$ we iterate under summation, we can notice that it is symmetric over k, let be $T(n,k) = 6k(n-k)+1$, then $T(n,k) = T(n, n-k)$. This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

Curiously enough that although $\sum_{k=0}^{n-1} 6k(n-k)+1$ and $\sum_{k=1}^{n} 6k(n-k)+1$ both simplify to n^3, they produce different closed forms. Let be $P(n,q) = \sum_{k=0}^{q-1} 6k(n-k)+1$ and $Q(n,q) = \sum_{k=1}^{q} 6k(n-k)+1$, then

$$P(n,q) = \begin{cases} q=1: & 1 \\ q=2: & -4+6n \\ q=3: & -27+18n \end{cases}$$

$$Q(n,q) = \begin{cases} q=1: & -5+6n \\ q=2: & -28+18n \\ q=3: & -81+36n \end{cases}$$

2. Generalizations

Assume that our previously obtained identities $n^3 = \sum_{k=0}^{n-1} 6k(n-k)+1$ and $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$ have explicit form as follows

$$n^3 = \sum_{k} \mathbf{A}_{1,1} k^1 (n-k)^1 + \mathbf{A}_{1,0} k^0 (n-k)^0$$

where $\mathbf{A}_{1,1} = 6$ and $\mathbf{A}_{1,0} = 1$, respectively. Therefore, let be a conjecture

Conjecture 2.1. *For every $n \geq 1$, $n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that*

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

Note that conjecture above assumes the convention $0^0 = 1$, reader may found a comprehensive discussion of it in [3].

Long story short, above conjecture is true, so that real coefficients $\mathbf{A}_{m,r}$ are following

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 2. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [4, 5].

These coefficients $\mathbf{A}_{m,r}$ are defined via a recurrence relation involving Binomial coefficients and Bernoulli numbers

Definition 2.2. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

where B_t are Bernoulli numbers [6]. It is assumed that $B_1 = \frac{1}{2}$. Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$

- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

Proof of conjecture (2.1) as well as other discussions on topics above can be found in literature [7, 8, 9, 10, 11]. Few OEIS sequences were contributed as well [12, 13, 14, 15, 16].

Very well, let's wrap up this technical section and move on to the more engaging discussions.

References

[1] Meijering, Erik. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, 2002. https://infoscience.epfl.ch/record/63085/files/meijering0201.pdf.

[2] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. https://archive.org/details/concrete-mathematics.

[3] Knuth, Donald E. Two notes on notation. *The American Mathematical Monthly*, 99(5):403–422, 1992. https://arxiv.org/abs/math/9205211.

[4] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A302971, 2018.

[5] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A304042, 2018.

[6] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book company, 1953.

[7] Alekseyev, Max. MathOverflow answer 297916/113033. Published electronically at https://mathoverflow.net/a/297916/113033, 2018.

[8] Kolosov, Petro. History and overview of the polynomial P(m,b,x), 2024. https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP.

[9] Kolosov, Petro. On the link between binomial theorem and discrete convolution. *arXiv preprint arXiv:1603.02468*, 2016. https://arxiv.org/abs/1603.02468.

[10] Kolosov, Petro. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. https://doi.org/10.1017/mag.2022.129.

[11] Petro Kolosov. Polynomial identity involving Binomial Theorem and Faulhaber's formula. Published electronically at https://kolosovpetro.github.io/pdf/PolynomialIdentityInvolvingBTandFaulhaber.pdf, 2023.

[12] Petro Kolosov. Numerical triangle, row sums give third power, Entry A287326 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A287326, 2017.

[13] Petro Kolosov. Numerical triangle, row sums give fifth power, Entry A300656 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A300656, 2018.

[14] Petro Kolosov. The coefficients u(m, l, k), m = 3 defined by the polynomial identity, 2018. https://oeis.org/A316387.

[15] Petro Kolosov. The coefficients u(m, l, k), m = 2 defined by the polynomial identity, 2018. https://oeis.org/A316349.

[16] Petro Kolosov. The coefficients u(m, l, k), m = 1 defined by the polynomial identity, 2018. https://oeis.org/A320047.

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