A CURIOSITY ABOUT POLYNOMIAL INTERPOLATION

PETRO KOLOSOV

ABSTRACT. Interpolation of cubes expected to be

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

but got

$$n^3 = \sum_{k=1}^{n} \mathbf{A}_{m,0}k^0(n-k)^0 + \mathbf{A}_{m,1}k^1(n-k)^1$$

CONTENTS

1. INTRODUCTION

Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

At that time, in 2016, I was a first-year mechanical engineering undergraduate, so that due to lack of knowledge and perspective of view I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. *All the mathematical laws and relations exist from the very beginning, we only reveal and describe them*, I thought. That mindset truly inspired me, so my own mathematical journey has been started. Consider finite differences of cubes n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

The problem of interpolation of polynomials is a classical problem in mathematics and has been widely studied in literature. For instance, Concrete mathematics [2] gives interpolation of cubes by using Newton's interpolation formula

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

because

$$f(x) = \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots + f(0)\binom{x}{0} = \sum_{r=0}^{d} \Delta^{d-r} f(0)\binom{x}{r}$$

However, interpolation of cubes can be also done in a different way. The key point that interpolation formula above iterates over the finite difference of order d, instead it is clear that n^3 can be reached as a sum of finite difference Δ^1 of first order

$$n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3 = \sum_{k=0}^{n-1} \Delta k^3$$

We know that $\Delta^3 n^3 = 6$ is the constant for each n. The second difference of cubes $\Delta^2 n^3$ is a linear relation in terms of third order finite difference $\Delta^3 n^3$.

$$\Delta^2 n^3 = (n+1)\Delta^3 n^3 = 6(n+1)$$

Finally, the first order finite difference Δn^3 is the following relation in terms of second order finite difference

$$\Delta n^3 = \Delta 0^3 + \Delta^2 0^3 + \Delta^2 1^3 + \cdots + \Delta^2(n-1)^3 = 1 + \sum_{k=0}^{n-1} 6(k+1)$$

Altering summation bounds yields

$$\Delta n^3 = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6\sum_{k=0}^{n} k$$

Therefore, we are able to express first order finite difference of cubes in form of sums as follows

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

Now it is time to assemble all the results above to get the polynomial n^3. Having the relation $n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3$, we get

$$n^3 = [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2]$$

$$+ \cdots + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)]$$

By rearranging the terms of the equation above, we get summation in terms of $k(n-k)$

$$n^3 = n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2]$$

$$+ \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)]$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term n in the sum above can be moved under sigma notation, because there is exactly n iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$$

By inspecting the expression $6k(n-k) + 1$ we iterate under summation, we can notice that it is symmetric over k, let be $T(n, k) = 6k(n-k) + 1$, then $T(n, k) = T(n, n-k)$. This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

Curiously enough that although $\sum_{k=0}^{n-1} 6k(n-k) + 1$ and $\sum_{k=1}^{n} 6k(n-k) + 1$ both simplify to n^3, they produce different closed forms. Let be $P(n, q) = \sum_{k=0}^{q-1} 6k(n-k) + 1$ and $Q(n, q) = \sum_{k=1}^{q} 6k(n-k) + 1$, then

$$P(n, q) = \begin{cases} q = 1 : & 1 \\ q = 2 : & -4 + 6n \\ q = 3 : & -27 + 18n \end{cases}$$

$$Q(n, q) = \begin{cases} q = 1 : & -5 + 6n \\ q = 2 : & -28 + 18n \\ q = 3 : & -81 + 36n \end{cases}$$

Now let's take a deep breath and summarize all the results we got so far. So, we have successfully interpolated the polynomial n^3 using discrete set of finite differences data points, by expressing its first order finite difference as $\Delta n^3 = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n$, so that n^3 is a sum of Δn^3. By rearranging the terms we got $n^3 = n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] + \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)]$.

2. Conclusions

Conclusions of your manuscript.

References

[1] Meijering, Erik. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, 2002. https://infoscience.epfl.ch/record/63085/files/meijering0201.pdf.

[2] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, and Stanley Liu. Concrete mathematics: a foundation for computer science. *Computers in Physics*, 3(5):106–107, 1989.

Version: `Local-0.1.0`