A NOVEL PROOF OF POWER RULE IN CALCULUS

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Abstract. The power rule for derivatives, typically proven through the limit definition of derivative in conjunction with the Binomial theorem. In this manuscript we present an alternative approach to proving the power rule, by utilizing a certain polynomial identity, such that expresses the function's growth.

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1. Introduction and main results

Power rule can be considered as the one of the most fundamental rules in calculus. Most of us remember this law from the first calculus course

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1}$$

where n is a constant. One of the common strategies to prove the power rule is by utilizing limit definition of derivative in conjunction with binomial theorem. Recall the limit form of derivative

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

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Sources: https://github.com/kolosovpetro/ANovelProofOfPowerRuleInCalculus

where $f(x)$ is defined over $\mathbb{R}$ and at least of smoothness class C^1. Let be $f(x) = x^n$ with constant n. Then its derivative is

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = \lim_{h \to 0} \left[\frac{(x+h)^n - x^n}{h} \right]$$

Notice that we can express the function's growth $(x+h)^n$ by using binomial theorem

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k$$

So that

$$\begin{aligned}
\frac{\mathrm{d}x^n}{\mathrm{d}x} &= \lim_{h \to 0} \left[\frac{1}{h} \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k \right] \\
&= \lim_{h \to 0} \left[\binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \binom{n}{3} x^{n-3} h^2 + \cdots + \binom{n}{n} x^0 h^n \right] \\
&= \binom{n}{1} x^{n-1}
\end{aligned}$$

However, is the binomial theorem the only polynomial identity to express the growth rate? Well, not really, we can utilize different approach to express polynomial growth. Consider the following identity for odd powers [1, 2, 3, 4]

$$(x-2a)^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=a+1}^{x-a} (k-a)^r (x-k-a)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{1.1}$$

where B_t are Bernoulli numbers [5] such that $B_1 = \frac{1}{2}$.

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