ANOTHER APPROACH TO GET DERIVATIVE OF ODD-POWER

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Abstract. Your abstract here.

Contents

1. Introduction and Main Results 1
2. Conclusions 4
References 4

1. Introduction and Main Results

This manuscript provides an exponential identity in terms of partial derivatives, extending the main idea explained in [Kol22] that gives polynomial identity in a form as follows

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

where $m, n \in \mathbb{N}$ and $\mathbf{A}_{m,r}$ are real coefficients defined recursively, see [Kol16]. Define the function f_y such that based on the identity (1) with the only difference that values of n, m in the right part of (1) appear to be parameters of the function f_y. In contrast to the equation (1), upper bound n of the sum $\sum_{k=1}^{n}$ turned into fixed function's parameter y as well, so that f_y defined as follows

Definition 1.1. *(Polynomial function f_y.)*

$$f_y(x, z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r \tag{2}$$

where $x, z \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. At first glance, equation (2) might look complex, so in order to clarify the function f and polynomials it produces let's show few examples.

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Substituting the values of $y = 1, 2, 3$ to the function f we get the following polynomials

$$\begin{aligned}
f_1(x,z) &= 3xz - 3z^2 + 3xz^2 - 2z^3\\
f_2(x,z) &= 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5\\
f_3(x,z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4\\
&\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7
\end{aligned}$$

According to the main topic of the current manuscript, it provides Another approach to get derivative of odd-power. Therefore, we define odd-power function we work in context of. Odd-power function g_y is a function defined as follows

Definition 1.2. *(Odd-power function g_y.)*

$$g_y(x) = x^{2y+1} \tag{3}$$

where $x \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. One more important thing is to conclude on partial derivative notation, more precisely the following notation for the partial derivative is used across the manuscript and remains unchanged

Notation 1.3. *(Partial derivative.) Let be a function $f(x_1, x_2, \ldots, x_n)$ defined over the real space $\mathbb{R}^n$. We denote partial derivative of the function f with respect to x_i as follows*

$$f'_{x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_i}$$

Derivative of the function f with respect to x_i evaluated in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is denoted as follows

$$f'_{x_i}(y_1, y_2, \ldots, y_n)$$

Therefore, the following identity in terms of partial derivatives shows the relation between exponential function g_y and polynomial function f_y

Theorem 1.4. *Derivative $(g_v)'_x$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluated at point u equals to derivative $(f_v)'_x$ evaluated at point (u,u) plus derivative $(f_v)'_z$ evaluated at point (u,u)*

$$(g_v)'_x(u) = (f_v)'_x(u,u) + (f_v)'_z(u,u) \tag{4}$$

Particularly, it follows that for every pair u, v an identity holds

$$(2v+1)u^{2v} = (f_v)'_x(u,u) + (f_v)'_z(u,u) \tag{5}$$

that is also an ordinary derivative of odd-power function t^{2v+1}, therefore

$$\frac{d}{dt} t^{2v+1}(u) = (f_v)'_x(u,u) + (f_v)'_z(u,u)$$

To summarize and clarify all above, we provide a few examples that show an identity (4) in action.

Example 1.5. Identity (4) example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 1.$ Consider the explicit form of the function $f_1(x,z)$ i.e

$$f_1(x,z) = 3xz - 3z^2 + 3xz^2 - 2z^3$$

Therefore, derivative of f_1 with respect to x equals to

$$(f_1)_x^{'}(x,z) = \lim_{d \to 0} \frac{3dz + 3dz^2}{d} = 3z + 3z^2$$

Consider derivative of the function f_1 with respect to z, that is

$$\begin{aligned}
(f_1)_z^{'}(x,z) &= \lim_{d \to 0} \left[\frac{-3d^2 - 2d^3 + 3dx + 3d^2x - 6dz - 6d^2z + 6dxz - 6dz^2}{d} \right] \\
&= \lim_{d \to 0} \left[-3d - 2d^2 + 3x + 3dx - 6z - 6dz + 6xz - 6z^2 \right] \\
&= 3x - 6z + 6xz - 6z^2
\end{aligned}$$

Combining both $(f_1)_x^{'}$ and $(f_1)_z^{'}$ evaluated at point (u,u) we get

$$\begin{aligned}
(f_1)_x^{'} + (f_1)_z^{'} &= 3x - 3z + 6xz - 3z^2 \\
[(f_1)_x^{'} + (f_1)_z^{'}](u,u) &= 3u^2
\end{aligned}$$

that is derivative of the polynomial u^3.

Example 1.6. Identity (4) example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 2.$ Consider the explicit form of the function $f_2(x,z)$ i.e

$$f_2(x,z) = 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5$$

Therefore, derivative of f_2 with respect to x equals to

$$\begin{aligned}
(f_2)_x^{'} &= \lim_{d \to 0} \left[5dz + 10xz - 15z^2 + 15dz^2 + 30xz^2 - 30z^3 + 10dz^3 + 20xz^3 - 15z^4 \right] \\
&= 10xz - 15z^2 + 30xz^2 - 30z^3 + 20xz^3 - 15z^4
\end{aligned}$$

Consider derivative of the function f_2 with respect to z, that is

$$(f_2)_z^{'} = 5x^2 - 30xz + 30x^2z + 30z^2 - 90xz^2 + 30x^2z^2 + 60z^3 - 60xz^3 + 30z^4$$

Combining both $(f_2)_x^{'}(x,z)$ and $(f_2)_z^{'}(x,z)$ evaluated at point (u,u) we get

$$(f_2)_x^{'} + (f_2)_z^{'} = 5x^2 - 20xz + 30x^2z + 15z^2 - 60xz^2 + 30x^2z^2 + 30z^3 - 40xz^3 + 15z^4$$

$$[(f_2)_x^{'} + (f_2)_z^{'}](u,u) = 5u^4$$

that is derivative of the polynomial u^5.

Example 1.7. Identity (4) example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 3.$ Consider the explicit form of the function $f_3(x,z)$ i.e

$$\begin{aligned} f_3(x,z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\ &\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7 \end{aligned}$$

Therefore, derivative of f_3 with respect to x equals to

$$\begin{aligned} (f_3)'_x &= -7z + 28xz - 42z^2 + 105x^2z^2 - 280xz^3 + 210x^2z^3 + 175z^4 - 420xz^4 \\ &\quad + 105x^2z^4 + 210z^5 - 168xz^5 + 70z^6 \end{aligned}$$

Consider derivative of the function f_2 with respect to z, that is

$$\begin{aligned} (f_3)'_z &= -7x + 14x^2 + 14z - 84xz + 70x^3z + 84z^2 - 420x^2z^2 + 210x^3z^2 + 700xz^3 \\ &\quad - 840x^2z^3 + 140x^3z^3 - 350z^4 + 1050xz^4 - 420x^2z^4 - 420z^5 + 420xz^5 - 140z^6 \end{aligned}$$

Combining both $(f_3)'_x(x,z)$ and $(f_3)'_z(x,z)$ evaluated at point (u,u) we get

$$\begin{aligned} (f_3)'_x + (f_3)'_z &= -7x + 14x^2 + 7z - 56xz + 70x^3z + 42z^2 - 315x^2z^2 + 210x^3z^2 \\ &\quad + 420xz^3 - 630x^2z^3 + 140x^3z^3 - 175z^4 + 630xz^4 - 315x^2z^4 - 210z^5 \\ &\quad + 252xz^5 - 70z^6 \\ [(f_3)'_x + (f_3)'_z](u,u) &= 7u^6 \end{aligned}$$

that is derivative of the polynomial u^7.

2. Conclusions

Conclusions of your manuscript.

References

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