

# ANOTHER APPROACH TO GET DERIVATIVE OF ODD-POWER

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ABSTRACT. This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives of the polynomial function  $f_y$  defined as

$$f_y(x, z) = \sum_{k=1}^z \sum_{r=0}^y \mathbf{A}_{y,r} k^r (x - k)^r$$

where  $x, z \in \mathbb{R}$ ,  $y$  is fixed constant  $y \in \mathbb{N}$  and  $\mathbf{A}_{y,r}$  are real coefficients.

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## 1. INTRODUCTION AND MAIN RESULTS

This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives identity in terms of partial derivatives, extending the main idea explained in [Kol22a] that gives polynomial identity in a form as follows

$$n^{2m+1} = \sum_{k=1}^n \sum_{r=0}^m \mathbf{A}_{m,r} k^r (n - k)^r \quad (1)$$

where  $m$  is fixed constant  $m \in \mathbb{N}$ ,  $n \in \mathbb{N}$  and  $\mathbf{A}_{m,r}$  are real coefficients defined recursively, see [Kol16]. Define the function  $f_y$  such that based on the identity (1) with the only difference that values of  $n, m$  in the right part of (1) appear to be parameters of the function  $f_y$ . In contrast to the equation (1), upper bound  $n$  of the sum  $\sum_{k=1}^n$  turned into fixed function's parameter  $y$  as well, so that  $f_y$  defined as follows

**Definition 1.1.** (*Polynomial function  $f_y$ .*)

$$f_y(x, z) = \sum_{k=1}^z \sum_{r=0}^y \mathbf{A}_{y,r} k^r (x - k)^r \quad (2)$$

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where  $x, z \in \mathbb{R}$  and  $y$  is constant  $y \in \mathbb{N}$ . At first glance, equation (2) might look complex, so in order to clarify the function  $f_y$  and polynomials it produces let's show few examples. Substituting the values of  $y = 1, 2, 3$  to the function  $f$  we get the following polynomials

$$\begin{aligned} f_1(x, z) &= 3xz - 3z^2 + 3xz^2 - 2z^3 \\ f_2(x, z) &= 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5 \\ f_3(x, z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\ &\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7 \end{aligned}$$

According to the main topic of the current manuscript, it provides Another approach to get derivative of odd-power. Therefore, we define odd-power function we work in context of. Odd-power function  $g_y$  is a function defined as follows

**Definition 1.2.** (*Odd-power function  $g_y$ .*)

$$g_y(x) = x^{2y+1} \quad (3)$$

where  $x \in \mathbb{R}$  and  $y$  is constant  $y \in \mathbb{N}$ . One more important thing is to conclude on partial derivative notation, more precisely the following notation for the partial derivative is used across the manuscript and remains unchanged

**Notation 1.3.** (*Partial derivative.*) Let be a function  $f(x_1, x_2, \dots, x_n)$  defined over the real space  $\mathbb{R}^n$ . We denote partial derivative of the function  $f$  with respect to  $x_i$  as follows

$$f'_{x_i} = \lim_{\Delta x_i \rightarrow 0} \frac{f(x_1, x_2, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_i}$$

Derivative of the function  $f$  with respect to  $x_i$  evaluated in point  $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$  is denoted as follows

$$f'_{x_i}(y_1, y_2, \dots, y_n)$$

Therefore, the following identity in terms of partial derivatives shows the relation between odd-power function  $g_y$  and polynomial function  $f_y$

**Theorem 1.4.** Let be a fixed value  $v \in \mathbb{N}$ , then derivative  $(g_v)'_x$  of the odd-power function  $g_v(x) = x^{2v+1}$  evaluated at point  $u$  equals to derivative  $(f_v)'_x$  evaluated at point  $(u, u)$  plus derivative  $(f_v)'_z$  evaluated at point  $(u, u)$

$$(g_v)'_x(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u) \quad (4)$$

Particularly, it follows that for every pair  $u, v$  an identity holds

$$(2v + 1)u^{2v} = (f_v)'_x(u, u) + (f_v)'_z(u, u) \quad (5)$$

that is also an ordinary derivative of odd-power function  $t^{2v+1}$ , therefore

$$\frac{d}{dt}t^{2v+1}(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u)$$

To summarize and clarify all above, we provide a few examples that show an identity (4) in action.

**Example 1.5.** Identity (4) example for  $x \in \mathbb{R}$ ,  $z \in \mathbb{R}$  and  $y = 1$ . Consider the explicit form of the function  $f_1(x, z)$  i.e

$$f_1(x, z) = 3xz - 3z^2 + 3xz^2 - 2z^3$$

Therefore, derivative of  $f_1$  with respect to  $x$  equals to

$$(f_1)'_x = \lim_{d \rightarrow 0} \frac{3dz + 3dz^2}{d} = 3z + 3z^2$$

Consider derivative of the function  $f_1$  with respect to  $z$ , that is

$$\begin{aligned} (f_1)'_z &= \lim_{d \rightarrow 0} \left[ \frac{-3d^2 - 2d^3 + 3dx + 3d^2x - 6dz - 6d^2z + 6dxz - 6dz^2}{d} \right] \\ &= \lim_{d \rightarrow 0} [-3d - 2d^2 + 3x + 3dx - 6z - 6dz + 6xz - 6z^2] \\ &= 3x - 6z + 6xz - 6z^2 \end{aligned}$$

Combining both  $(f_1)'_x$  and  $(f_1)'_z$  evaluated at point  $(u, u)$  we get

$$\begin{aligned} (f_1)'_x + (f_1)'_z &= 3x - 3z + 6xz - 3z^2 \\ [(f_1)'_x + (f_1)'_z](u, u) &= 3u^2 \end{aligned}$$

that is derivative of the polynomial  $u^3$ .

**Example 1.6.** Identity (4) example for  $x \in \mathbb{R}$ ,  $z \in \mathbb{R}$  and  $y = 2$ . Consider the explicit form of the function  $f_2(x, z)$  i.e

$$f_2(x, z) = 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5$$

Therefore, derivative of  $f_2$  with respect to  $x$  equals to

$$\begin{aligned} (f_2)'_x &= \lim_{d \rightarrow 0} [5dz + 10xz - 15z^2 + 15dz^2 + 30xz^2 - 30z^3 + 10dz^3 + 20xz^3 - 15z^4] \\ &= 10xz - 15z^2 + 30xz^2 - 30z^3 + 20xz^3 - 15z^4 \end{aligned}$$

Consider derivative of the function  $f_2$  with respect to  $z$ , that is

$$(f_2)'_z = 5x^2 - 30xz + 30x^2z + 30z^2 - 90xz^2 + 30x^2z^2 + 60z^3 - 60xz^3 + 30z^4$$

Combining both  $(f_2)'_x(x, z)$  and  $(f_2)'_z(x, z)$  evaluated at point  $(u, u)$  we get

$$\begin{aligned} (f_2)'_x + (f_2)'_z &= 5x^2 - 20xz + 30x^2z + 15z^2 - 60xz^2 + 30x^2z^2 + 30z^3 - 40xz^3 + 15z^4 \\ [(f_2)'_x + (f_2)'_z](u, u) &= 5u^4 \end{aligned}$$

that is derivative of the polynomial  $u^5$ .

**Example 1.7.** Identity (4) example for  $x \in \mathbb{R}$ ,  $z \in \mathbb{R}$  and  $y = 3$ . Consider the explicit form of the function  $f_3(x, z)$  i.e

$$\begin{aligned} f_3(x, z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\ &\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7 \end{aligned}$$

Therefore, derivative of  $f_3$  with respect to  $x$  equals to

$$\begin{aligned} (f_3)'_x &= -7z + 28xz - 42z^2 + 105x^2z^2 - 280xz^3 + 210x^2z^3 + 175z^4 - 420xz^4 \\ &\quad + 105x^2z^4 + 210z^5 - 168xz^5 + 70z^6 \end{aligned}$$

Consider derivative of the function  $f_2$  with respect to  $z$ , that is

$$\begin{aligned} (f_3)'_z &= -7x + 14x^2 + 14z - 84xz + 70x^3z + 84z^2 - 420x^2z^2 + 210x^3z^2 + 700xz^3 \\ &\quad - 840x^2z^3 + 140x^3z^3 - 350z^4 + 1050xz^4 - 420x^2z^4 - 420z^5 + 420xz^5 - 140z^6 \end{aligned}$$

Combining both  $(f_3)'_x(x, z)$  and  $(f_3)'_z(x, z)$  evaluated at point  $(u, u)$  we get

$$\begin{aligned} (f_3)'_x + (f_3)'_z &= -7x + 14x^2 + 7z - 56xz + 70x^3z + 42z^2 - 315x^2z^2 + 210x^3z^2 \\ &\quad + 420xz^3 - 630x^2z^3 + 140x^3z^3 - 175z^4 + 630xz^4 - 315x^2z^4 - 210z^5 \\ &\quad + 252xz^5 - 70z^6 \\ [(f_3)'_x + (f_3)'_z](u, u) &= 7u^6 \end{aligned}$$

that is derivative of the polynomial  $u^7$ .

## 2. CONCLUSIONS

In this manuscript we have reviewed an approach to get derivative of odd-power using identity in partial derivatives of the function  $f$  evaluated at fixed point  $(u, u) \in \mathbb{R}^2$ . Main results of the manuscript can be validated using Mathematica programs available online at [Kol22b].

## REFERENCES

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