AN EXPONENTIAL IDENTITY IN TERMS OF PARTIAL DERIVATIVES

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Abstract. Your abstract here.

Contents

1. Introduction and Main Results

This manuscript provides an exponential identity in terms of partial derivatives, extending the main idea explained in [Kol22] that gives polynomial identity in a form as follows

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r, \quad (m,n) \in \mathbb{N}, \tag{1}$$

where $\mathbf{A}_{m,r}$ are real coefficients defined recursively, see [Kol16]. Define the function f such that based on the identity (1) with the only difference that values of n, m in the right part of (1) appear to be parameters of the function f. In contrast to the equation (1), upper bound n of the sum $\sum_{k=1}^{n}$ turned into function's parameter as well, so that f defined as follows

Definition 1.1. *(Polynomial function f.)*

$$f(x,y,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r, \tag{2}$$

where definition space of (x,y,z) in (2) is $x \in \mathbb{R},\ y \in \mathbb{N},\ z \in \mathbb{R}$ so that the triple $(x,y,z) \in \mathbb{R} \times \mathbb{N} \times \mathbb{R}$. At first glance, equation (2) might look complex and not immediately understood, so in order to understand the function f and polynomials it produces clearly

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let's show few examples. Substituting the values of $y = 1, 2, 3$ to the function f we get the following polynomials

$$\begin{aligned}
f(x,1,z) &= 3xz - 3z^2 + 3xz^2 - 2z^3 \\
f(x,2,z) &= 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5 \\
f(x,3,z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\
&\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7
\end{aligned}$$

According to the main topic of the current manuscript, it provides an odd-exponential identity in terms of partial derivatives. Therefore, define the exponential function we work in context of. Exponential function g is a function of two variables defined as follows

Definition 1.2. *(Exponential function g.)*

$$g(x,y) = x^{2y+1}, \quad (x,y) \in \mathbb{R} \times \mathbb{N} \tag{3}$$

One more important thing is to conclude on partial derivative notation, more precisely the following notation for the partial derivative is used across the manuscript and remains unchanged

Notation 1.3. *(Partial derivative.) Let be a function $f(x_1, x_2, \dots, x_n)$ defined over the real space $\mathbb{R}^n$. We denote partial derivative of the function f with respect to x_i, $\quad 1 \le i \le n$ as follows*

$$f'_{x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_i}$$

Derivative of the function f with respect to x_i, $\quad 1 \le i \le n$ evaluated in point $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ is denoted as follows

$$f'_{x_i}(y_1, y_2, \dots, y_n)$$

Therefore, the following identity in terms of partial derivatives shows the relation between exponential function g and polynomial function f

Theorem 1.4. *Derivative of the odd-exponential function $g(x,y) = x^{2y+1}$ with respect to x evaluated at point $(u,v) \in \mathbb{R} \times \mathbb{N}$ equals to derivative of the function f with respect to x evaluated at point $(u,v,u) \in \mathbb{R} \times \mathbb{N} \times \mathbb{R}$ plus derivative of the function f with respect to z evaluated at point $(u,v,u) \in \mathbb{R} \times \mathbb{N} \times \mathbb{R}$*

$$g'_x(u,v) = f'_x(u,v,u) + f'_z(u,v,u) \tag{4}$$

Particularly, it follows that for every triple $(u,v,u) \in \mathbb{R} \times \mathbb{N} \times \mathbb{R}$ an identity holds

$$(2v+1)u^{2v} = f'_x(u,v,u) + f'_z(u,v,u) \tag{5}$$

that is also an ordinary derivative of odd-power function $f(u) = u^{2v+1}$. To summarize and clarify all above, we provide a few examples that show an identity (4) in action.

2. Conclusions

Conclusions of your manuscript.

References

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