AN EXPONENTIAL IDENTITY IN TERMS OF PARTIAL DERIVATIVES

PETRO KOLOSOV

ABSTRACT. Your abstract here.

CONTENTS

1. INTRODUCTION

This manuscript provides an exponential identity in terms of partial derivatives, extending the main idea explained in [Kol22] that gives polynomial identity in a form as follows

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r, \quad (m,n) \in \mathbb{N}, \tag{1}$$

where $\mathbf{A}_{m,r}$ are real coefficients defined recursively, see [Kol16]. Define the function f such that based on the identity (1) with the only difference that values of n, m in its left part appear to be parameters of the function f, that is

Definition 1.1. *(Polynomial function f.)*

$$f(x,y,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r, \tag{2}$$

where definition space of (x,y,z) in (2) is $x \in \mathbb{R},\ y \in \mathbb{N},\ z \in \mathbb{R}$ so that the triple $(x,y,z) \in \mathbb{R} \times \mathbb{N} \times \mathbb{R}$. Important to note that upper bound of the sum $\sum_{k=1}^{z}$ in (2) is parameter of the function f in contrast to the equation (1) where upper bound of the sum is n. At first glance, equation (2) might look complex and not immediately understood, so in order to understand the function f and polynomials it produces. Substituting the values

Date: October 25, 2022.

2010 *Mathematics Subject Classification.* 32W50, 33B10.

Key words and phrases. Partial differential equations, PDE, Exponential function .

of $y = 1, 2, 3$ to the function f we get the following polynomials

$$\begin{aligned}
f(x,1,z) &= 3xz - 3z^2 + 3xz^2 - 2z^3 \\
f(x,2,z) &= 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5 \\
f(x,3,z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\
&\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7
\end{aligned}$$

According to the main topic of the current manuscript, it provides an odd-exponential identity in terms of partial derivatives. Therefore, define the exponential function we work in context of. Exponential function g is a function of two variables defined as follows

Definition 1.2. *(Exponential function.)*

$$g(x,y) = x^{2y+1}, \quad (x,y) \in \mathbb{R} \times \mathbb{N} \tag{3}$$

One more important thing is to conclude on partial derivative notation, more precisely the following notation for the partial derivative is used across the manuscript and remains unchanged

Notation 1.3. *(Partial derivative.) Let be a function $f(x_1, x_2, \ldots, x_n)$ defined over the real space $\mathbb{R}^n$. We denote partial derivative of the function f with respect to x_i, $\quad 1 \leq i \leq n$ as follows*

$$f'_{x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_i}$$

Derivative of the function f with respect to x_i, $\quad 1 \leq i \leq n$ evaluated in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is denoted as follows

$$f'_{x_i}(y_1, y_2, \ldots, y_n)$$

Therefore, the following identity in terms of partial derivatives shows the relation between exponential function g and polynomial function f

Theorem 1.4. *(Exponential identity in terms of partial derivatives.) Derivative of the exponential function $g'_x(u,v)$ with respect to x evaluated at point $(u,v) \in \mathbb{R}^2$ equals to derivative of the function f with respect to x evaluated at point $(u,v,u) \in \mathbb{R}^3$ plus derivative of the function f with respect to z evaluated at point $(u,v,u) \in \mathbb{R}^3$*

$$g'_x(u,v) = f'_x(u,v,u) + f'_z(u,v,u) \tag{4}$$

Particularly, it follows that for every pair $(u,v) \in \mathbb{R}$ an identity holds

$$(2v+1)u^{2v} = f'_x(u,v,u) + f'_z(u,v,u), \tag{5}$$

in its extended form

$$(2v+1)u^{2v} = \left(\sum_{k=1}^{z}\sum_{r=0}^{y}\mathbf{A}_{y,r}k^r(x-k)^r\right)'_x(u,v,u) + \left(\sum_{k=1}^{z}\sum_{r=0}^{y}\mathbf{A}_{y,r}k^r(x-k)^r\right)'_z(u,v,u)$$

that is also an ordinary derivative of odd-power function $f(u) = u^{2v+1}$. To summarize and clarify all above, we provide a few examples that show an identity (4) in action.

2. Conclusions

Conclusions of your manuscript.

References

[Kol16] Petro Kolosov. On the link between Binomial Theorem and Discrete Convolution of Polynomials. *arXiv preprint arXiv:1603.02468*, 2016. https://arxiv.org/abs/1603.02468.

[Kol22] Petro Kolosov. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. https://doi.org/10.1017/mag.2022.129.

Email address: kolosovp94@gmail.com

URL: https://razumovsky.me/