ANOTHER APPROACH TO GET DERIVATIVE OF ODD-POWER

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Abstract. In this manuscript, we provide and discuss another approach to get derivative of odd-power such that is based on an identity in partial derivatives in terms of polynomial function f_y defined as

$$f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r$$

where $x, z \in \mathbb{R}$, y is fixed constant $y \in \mathbb{N}$ and $\mathbf{A}_{y,r}$ are real coefficients.

Contents

1. Introduction and Main Results

This manuscript provides another approach to get derivative of odd-power, that is an approach based on partial derivatives of the polynomial function $f_y(x,z)$ defined as

$$f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r$$

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where $x, z \in \mathbb{R}$, y is fixed constant $y \in \mathbb{N}$ and $\mathbf{A}_{y,r}$ are real coefficients. The essence of the approach we discuss is build on an identity in terms of sum of partial derivatives of the polynomial function f_y. The function f_y is defined by the main results of the manuscript [1] that explains an odd-power in a form as follows

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

where m is fixed constant $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $\mathbf{A}_{m,r}$ are real coefficients defined recursively, see [2]. We define the function f_y such that based on the identity (1) with the only difference that values of n, m in the right part of (1) appear to be parameters of the function f_y. In contrast to the equation (1), upper bound n of the sum $\sum_{k=1}^{n}$ turned into fixed function's parameter y as well. Let the function f_y be defined as follows

Definition 1.1. *(Polynomial function f_y.)*

$$f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r \tag{2}$$

where $x, z \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. Note that for every $x \in \mathbb{R}$ and constant $y \in \mathbb{N}$ the polynomial identity satisfies

$$f_y(x,x) = x^{2y+1}$$

At first glance, the definition (2) might look complex, so in order to clarify the function f_y and polynomials it produces, let there be a few examples. Substituting the values of $y = 1, 2, 3$ to the function f_y we get the following polynomials in x, z

$$f_1(x,z) = 3xz - 3z^2 + 3xz^2 - 2z^3$$

$$f_2(x,z) = 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5$$

$$\begin{aligned} f_3(x,z) = &-7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\ &- 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7 \end{aligned}$$

These polynomials are obtained by rearranging the sums in the definition (2) as

$$f_y(x,z) = \sum_{r=0}^{y} \mathbf{A}_{y,r} \left[\sum_{k=1}^{z} k^r (x-k)^r \right]$$

So that part $\sum_{k=1}^{z} k^r (x-k)^r$ is polynomial in x, z calculated using Faulhaber's formula [3]. According to the main topic of the current manuscript, it provides another approach to get derivative of odd-power. Therefore, we define odd-power function we work in the context of. The odd-power function g_y is a function defined as follows

Definition 1.2. *(Odd-power function g_y.)*

$$g_y(x) = x^{2y+1}$$

where $x \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. The Interesting part is that odd-power function $g_y(x)$ may be obtained as a partial case of the function f_y for $z = x$. Also, the ordinary derivative of odd-power $\frac{d}{dx} g_y$ evaluate in point $u \in \mathbb{R}$ may be obtained as a sum of partial derivatives of f_y evaluate in point (u, u). We explain this further in the manuscript. One more important thing remains to conclude is to define partial derivative's notation. More precisely, the following notation for partial derivatives is used across the manuscript and remains unchanged

Notation 1.3. *(Partial derivative.) Let be a function $\beta(x_1, x_2, \ldots, x_n)$ defined over the real space $\mathbb{R}^n$. We denote partial derivative of the function β with respect to x_i as follows*

$$\beta'_{x_i} = \lim_{\Delta x_i \to 0} \frac{\beta(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - \beta(x_1, x_2, \ldots, x_n)}{\Delta x_i}$$

Partial derivative of the function β_{x_i} with respect to x_i evaluate in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is denoted as follows

$$\beta'_{x_i}(y_1, y_2, \ldots, y_n)$$

Moreover, partial derivative β'_{x_i} evaluate in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ plus partial derivative β'_{x_j} evaluate in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is equivalent to the sum of partial derivatives

$\beta^{'}_{x_i} + \beta^{'}_{x_j}$ evaluate in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and to be denoted as

$$\beta^{'}_{x_i}(y_1, y_2, \ldots, y_n) + \beta^{'}_{x_j}(y_1, y_2, \ldots, y_n) = [\beta^{'}_{x_i} + \beta^{'}_{x_j}](y_1, y_2, \ldots, y_n)$$

So that now we can switch our focus back to the functions g_y and f_y. Therefore, the following theorem in terms of partial derivatives reflects the relation between the ordinary derivative of odd-power function g_y and function f_y

Theorem 1.4. *Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx}g_v(u)$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)^{'}_x(u, u)$ evaluate in point (u, u) plus partial derivative $(f_v)^{'}_z(u, u)$ evaluate in point (u, u)*

$$\frac{d}{dx}g_v(u) = (f_v)^{'}_x(u, u) + (f_v)^{'}_z(u, u) \tag{3}$$

In particular, it follows that for every pair $u \in \mathbb{R}, v \in \mathbb{N}$ an identity holds

$$\begin{aligned}(2v+1)u^{2v} &= (f_v)^{'}_x(u, u) + (f_v)^{'}_z(u, u) \\ &= [(f_v)^{'}_x + (f_v)^{'}_z](u, u)\end{aligned}$$

that is also an ordinary derivative of odd-power function t^{2v+1}, $v \in \mathbb{N}$, $v = \text{const}$ evaluate in point $u \in \mathbb{R}$, therefore

$$\begin{aligned}\frac{d}{dt}t^{2v+1}(u) &= (f_v)^{'}_x(u, u) + (f_v)^{'}_z(u, u) \\ &= [(f_v)^{'}_x + (f_v)^{'}_z](u, u)\end{aligned}$$

To summarize and clarify all about the theorem 1.4, we provide a few examples that show an application of it.

Example 1.5. Theorem 1.4 example for $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $y = 1$. Consider the explicit form of the function $f_1(x, z)$, that is

$$f_1(x, z) = 3xz - 3z^2 + 3xz^2 - 2z^3$$

Therefore, the partial derivative $(f_1)'_x$ with respect to x equals to

$$(f_1)'_x = \lim_{d\to 0} \frac{3dz + 3dz^2}{d} = 3z + 3z^2$$

Consider the partial derivative $(f_1)'_z$ with respect to z, that is

$$\begin{aligned}(f_1)'_z &= \lim_{d\to 0} \left[\frac{-3d^2 - 2d^3 + 3dx + 3d^2x - 6dz - 6d^2z + 6dxz - 6dz^2}{d}\right] \\ &= \lim_{d\to 0} \left[-3d - 2d^2 + 3x + 3dx - 6z - 6dz + 6xz - 6z^2\right] \\ &= 3x - 6z + 6xz - 6z^2\end{aligned}$$

Summing up both partial derivatives $(f_1)'_x$ and $(f_1)'_z$, we get

$$(f_1)'_x + (f_1)'_z = 3x - 3z + 6xz - 3z^2$$

Evaluating in point (u, u) yields

$$\frac{d}{dt}t^3(u) = [(f_1)'_x + (f_1)'_z](u, u) = 3u^2$$

That confirms the results of the theorem 1.4.

Example 1.6. Theorem 1.4 example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 2$. Consider the explicit form of the function $f_2(x, z)$, that is

$$f_2(x, z) = 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5$$

Therefore, the partial derivative $(f_2)'_x$ with respect to x equals to

$$\begin{aligned}(f_2)'_x &= \lim_{d\to 0} \left[5dz + 10xz - 15z^2 + 15dz^2 + 30xz^2 - 30z^3 + 10dz^3 + 20xz^3 - 15z^4\right] \\ &= 10xz - 15z^2 + 30xz^2 - 30z^3 + 20xz^3 - 15z^4\end{aligned}$$

Consider the partial derivative $(f_2)'_z$ with respect to z, that is

$$(f_2)'_z = 5x^2 - 30xz + 30x^2z + 30z^2 - 90xz^2 + 30x^2z^2 + 60z^3 - 60xz^3 + 30z^4$$

Summing up both partial derivatives $(f_2)'_x$ and $(f_2)'_z$, we get

$$(f_2)'_x + (f_2)'_z = 5x^2 - 20xz + 30x^2z + 15z^2 - 60xz^2 + 30x^2z^2 + 30z^3 - 40xz^3 + 15z^4$$

Evaluate in point (u, u) yields

$$\frac{d}{dt}t^5(u) = [(f_2)'_x + (f_2)'_z](u, u) = 5u^4$$

That confirms the results of the theorem 1.4.

Example 1.7. Theorem 1.4 example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 3$. Consider the explicit form of the function $f_3(x, z)$, that is

$$\begin{aligned} f_3(x, z) = &-7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\ &- 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7 \end{aligned}$$

Therefore, the partial derivative of $(f_3)'_x$ with respect to x equals to

$$\begin{aligned} (f_3)'_x = &-7z + 28xz - 42z^2 + 105x^2z^2 - 280xz^3 + 210x^2z^3 + 175z^4 - 420xz^4 \\ &+ 105x^2z^4 + 210z^5 - 168xz^5 + 70z^6 \end{aligned}$$

Consider the partial derivative $(f_3)'_z$ with respect to z, that is

$$\begin{aligned} (f_3)'_z = &-7x + 14x^2 + 14z - 84xz + 70x^3z + 84z^2 - 420x^2z^2 + 210x^3z^2 + 700xz^3 \\ &- 840x^2z^3 + 140x^3z^3 - 350z^4 + 1050xz^4 - 420x^2z^4 - 420z^5 + 420xz^5 - 140z^6 \end{aligned}$$

Summing up both partial derivatives $(f_3)'_x(x,z)$ and $(f_3)'_z(x,z)$, we get

$$\begin{aligned}(f_3)'_x + (f_3)'_z &= -7x + 14x^2 + 7z - 56xz + 70x^3z + 42z^2 - 315x^2z^2 + 210x^3z^2 \\ &+ 420xz^3 - 630x^2z^3 + 140x^3z^3 - 175z^4 + 630xz^4 - 315x^2z^4 - 210z^5 \\ &+ 252xz^5 - 70z^6\end{aligned}$$

Evaluate in point (u,u) yields

$$\frac{d}{dt}t^3(u) = [(f_3)'_x + (f_3)'_z](u,u) = 7u^6$$

That confirms the results of the theorem 1.4.

2. Antiderivatives

So far we have successfully reviewed and discussed an identity in terms of partial derivatives evaluated in particular points $(v = u, z = u)$ that equals to the ordinary derivative of odd-power function t^{2u+1}, see (1.4). However, what is about the inverse process of differentiation, i.e. antiderivatives? Would theorem (1.4) have an analog in terms of antiderivatives? Well, consider an example. Heaving $m = 2$ antiderivatives of $f_y(x,z)$ are

$$\begin{aligned}\int f_y(x,z)\,dx &= \frac{1}{6}xz(-45xz(1+z)^2 + 10x^2(1+3z+2z^2) + 6z^2(10+15z+6z^2)) + C_1 \\ \int f_y(x,z)\,dz &= \frac{1}{2}z^2(5x^2(1+z)^2 + z^2(5+6z+2z^2) - xz(10+15z+6z^2)) + C_2\end{aligned}$$

Let be

$$\begin{aligned}F &= \int f_y(x,z)\,dx + C_1 \\ G &= \int f_y(x,z)\,dz + C_2\end{aligned}$$

Then, evaluating F and G in points (u,u) yields

$$F(u,u) + G(u,u) = \frac{1}{6}25u^4 + \frac{1}{2}11u^5 + \frac{1}{3}7u^6 + C_1 + C_2$$

Possible analog of theorem (1.4) assumes that

$$F(u,u)+G(u,u)\equiv u^5$$

Considering $C_1 = C_2 = C$ we get

$$F(u,u)+G(u,u)=\frac{1}{6}25u^4+\frac{1}{2}11u^5+\frac{1}{3}7u^6+2C$$

Thus, integration constant C is evaluated as

$$C=\frac{1}{12}(-25u^4-27u^5-14u^6)$$

Thus, antiderivative case of theorem (1.4) is a whole new direction for further research.

3. Conclusions

In this manuscript, we have reviewed an approach to get ordinary derivative of odd-power using an identity in partial derivatives of the function f_y evaluate in fixed point $(u,u)\in\mathbb{R}^2$., that is described by the theorem 1.4. The main results of the manuscript can be validated using Mathematica programs available online at [4].

4. Verification of the results

As it is stated in conclusions, there is a possibility to validate the main results of this manuscript using Wolfram Mathematica. Therefore, a complete guide to validate the main results and formulae is attached as well. Mathematica package source file is available online under the folder `mathematica`, see [4]. The following expressions could be verified:

- The function $f_y(x,z)$ for any constant argument $y\in\mathbb{N}$ using mathematica method `f[x, y, z]` e.g.

$$\texttt{f[x, 1, z]}=3xz-3z^2+3xz^2-2z^3$$

- Partial derivative $(f_y)'_x$ for any constant argument $y \in \mathbb{N}$ using mathematica method `DerivativeFByX[x, y, z]`

$$\texttt{DerivativeFByX[x, 1, z]} = 3z + 3z^2$$

- Partial derivative $(f_y)'_z$ for any constant argument $y \in \mathbb{N}$ using mathematica method `DerivativeFByZ[x, y, z]`

$$\texttt{DerivativeFByZ[x, 1, z]} = 3x - 6z + 6xz - 6z^2$$

- Theorem 1.4 for any constant argument $y \in \mathbb{N}$

$$\texttt{DerivativeFByX[x, 1, z]} + \texttt{DerivativeFByZ[x, 1, z]} = 3x - 3z + 6xz - 3z^2$$

$$\texttt{DerivativeFByX[u, 1, u]} + \texttt{DerivativeFByZ[u, 1, u]} = 3u^2$$

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