ANOTHER APPROACH TO GET DERIVATIVE OF ODD-POWER

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Abstract. This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives of the polynomial function f_y defined as

$$f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r$$

where $x, z \in \mathbb{R}$, y is fixed constant $y \in \mathbb{N}$ and $\mathbf{A}_{y,r}$ are real coefficients.

Contents

1. Introduction and Main Results

This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives identity in terms of partial derivatives, extending the main idea explained in [?] that gives polynomial identity in a form as follows

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

where m is fixed constant $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $\mathbf{A}_{m,r}$ are real coefficients defined recursively, see [?]. Define the function f_y such that based on the identity (**??**) with the only difference that values of n, m in the right part of (**??**) appear to be parameters of the function f_y. In contrast to the equation (**??**), upper bound n of the sum $\sum_{k=1}^{n}$ turned into fixed function's parameter y as well, so that f_y defined as follows

Definition 1.1. *(Polynomial function f_y.)*

$$f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r \tag{2}$$

where $x, z \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. At first glance, equation (**??**) might look complex, so in order to clarify the function f_y and polynomials it produces let's show few examples.

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Substituting the values of $y = 1, 2, 3$ to the function f we get the following polynomials

$$
\begin{aligned}
f_1(x,z) &= 3xz - 3z^2 + 3xz^2 - 2z^3 \\
f_2(x,z) &= 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5 \\
f_3(x,z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\
&\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7
\end{aligned}
$$

According to the main topic of the current manuscript, it provides Another approach to get derivative of odd-power. Therefore, we define odd-power function we work in context of. Odd-power function g_y is a function defined as follows

Definition 1.2. *(Odd-power function g_y.)*

$$g_y(x) = x^{2y+1} \tag{3}$$

where $x \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. One more important thing is to conclude on partial derivative notation, more precisely the following notation for the partial derivative is used across the manuscript and remains unchanged

Notation 1.3. *(Partial derivative.) Let be a function $f(x_1, x_2, \ldots, x_n)$ defined over the real space $\mathbb{R}^n$. We denote partial derivative of the function f with respect to x_i as follows*

$$f'_{x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_i}$$

Derivative of the function f with respect to x_i evaluated in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is denoted as follows

$$f'_{x_i}(y_1, y_2, \ldots, y_n)$$

Moreover, partial derivative f'_{x_i} evaluated at point $(y_1, y_2, \ldots, y_n)$ plus partial derivative f'_{x_j} evaluated at point $(y_1, y_2, \ldots, y_n)$ is equivalent to the sum of partial derivatives f'_{x_i}, f'_{x_j} evaluated at point $(y_1, y_2, \ldots, y_n)$ and to be denoted as

$$f'_{x_i}(y_1, y_2, \ldots, y_n) + f'_{x_j}(y_1, y_2, \ldots, y_n) = [f'_{x_i} + f'_{x_j}](y_1, y_2, \ldots, y_n)$$

Therefore, the following identity in terms of partial derivatives shows the relation between odd-power function g_y and polynomial function f_y

Theorem 1.4. *Let be a fixed value $v \in \mathbb{N}$, then derivative g'_v of the odd-power function $g_v(x) = x^{2v+1}$ evaluated at point u equals to derivative $(f_v)'_x$ evaluated at point (u,u) plus derivative $(f_v)'_z$ evaluated at point (u,u)*

$$g'_v(u) = (f_v)'_x(u,u) + (f_v)'_z(u,u) \tag{4}$$

Particularly, it follows that for every pair u, v an identity holds

$$\begin{aligned}(2v+1)u^{2v} &= (f_v)'_x(u,u) + (f_v)'_z(u,u)\\ &= [(f_v)'_x + (f_v)'_z](u,u)\end{aligned}$$

that is also an ordinary derivative of odd-power function t^{2v+1}, therefore

$$\begin{aligned}\frac{d}{dt}t^{2v+1}(u) &= (f_v)'_x(u,u) + (f_v)'_z(u,u)\\ &= [(f_v)'_x + (f_v)'_z](u,u)\end{aligned}$$

To summarize and clarify all above, we provide a few examples that show an identity (**??**) in action.

Example 1.5. Identity (**??**) example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 1$. Consider the explicit form of the function $f_1(x,z)$ i.e

$$f_1(x,z) = 3xz - 3z^2 + 3xz^2 - 2z^3$$

Therefore, derivative of f_1 with respect to x equals to

$$(f_1)'_x = \lim_{d\to 0}\frac{3dz + 3dz^2}{d} = 3z + 3z^2$$

Consider derivative of the function f_1 with respect to z, that is

$$\begin{aligned}(f_1)'_z &= \lim_{d\to 0}\left[\frac{-3d^2 - 2d^3 + 3dx + 3d^2x - 6dz - 6d^2z + 6dxz - 6dz^2}{d}\right]\\ &= \lim_{d\to 0}\left[-3d - 2d^2 + 3x + 3dx - 6z - 6dz + 6xz - 6z^2\right]\\ &= 3x - 6z + 6xz - 6z^2\end{aligned}$$

Combining both $(f_1)'_x$ and $(f_1)'_z$ evaluated at point (u,u) we get

$$(f_1)'_x + (f_1)'_z = 3x - 3z + 6xz - 3z^2$$

$$\frac{d}{dt}t^3(u) = [(f_1)'_x + (f_1)'_z](u,u) = 3u^2$$

that confirms results of the theorem **??**.

Example 1.6. Identity (**??**) example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 2$. Consider the explicit form of the function $f_2(x,z)$ i.e

$$f_2(x,z) = 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5$$

Therefore, derivative of f_2 with respect to x equals to

$$\begin{aligned}(f_2)'_x &= \lim_{d\to 0}\left[5dz + 10xz - 15z^2 + 15dz^2 + 30xz^2 - 30z^3 + 10dz^3 + 20xz^3 - 15z^4\right]\\ &= 10xz - 15z^2 + 30xz^2 - 30z^3 + 20xz^3 - 15z^4\end{aligned}$$

Consider derivative of the function f_2 with respect to z, that is

$$(f_2)'_z = 5x^2 - 30xz + 30x^2z + 30z^2 - 90xz^2 + 30x^2z^2 + 60z^3 - 60xz^3 + 30z^4$$

Combining both $(f_2)'_x(x,z)$ and $(f_2)'_z(x,z)$ evaluated at point (u,u) we get

$$\begin{aligned}(f_2)'_x + (f_2)'_z &= 5x^2 - 20xz + 30x^2z + 15z^2 - 60xz^2 + 30x^2z^2 + 30z^3 - 40xz^3 + 15z^4 \\ \frac{d}{dt}t^5(u) &= [(f_2)'_x + (f_2)'_z](u,u) = 5u^4\end{aligned}$$

that confirms results of the theorem **??**.

Example 1.7. Identity (**??**) example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 3$. Consider the explicit form of the function $f_3(x,z)$ i.e

$$\begin{aligned}f_3(x,z) = &-7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\ &- 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7\end{aligned}$$

Therefore, derivative of f_3 with respect to x equals to

$$\begin{aligned}(f_3)'_x = &-7z + 28xz - 42z^2 + 105x^2z^2 - 280xz^3 + 210x^2z^3 + 175z^4 - 420xz^4 \\ &+ 105x^2z^4 + 210z^5 - 168xz^5 + 70z^6\end{aligned}$$

Consider derivative of the function f_2 with respect to z, that is

$$\begin{aligned}(f_3)'_z = &-7x + 14x^2 + 14z - 84xz + 70x^3z + 84z^2 - 420x^2z^2 + 210x^3z^2 + 700xz^3 \\ &- 840x^2z^3 + 140x^3z^3 - 350z^4 + 1050xz^4 - 420x^2z^4 - 420z^5 + 420xz^5 - 140z^6\end{aligned}$$

Combining both $(f_3)'_x(x,z)$ and $(f_3)'_z(x,z)$ evaluated at point (u,u) we get

$$\begin{aligned}(f_3)'_x + (f_3)'_z = &-7x + 14x^2 + 7z - 56xz + 70x^3z + 42z^2 - 315x^2z^2 + 210x^3z^2 \\ &+ 420xz^3 - 630x^2z^3 + 140x^3z^3 - 175z^4 + 630xz^4 - 315x^2z^4 - 210z^5 \\ &+ 252xz^5 - 70z^6 \\ \frac{d}{dt}t^3(u) = &[(f_3)'_x + (f_3)'_z](u,u) = 7u^6\end{aligned}$$

that confirms results of the theorem **??**.

2. Conclusions

In this manuscript we have reviewed an approach to get derivative of odd-power using identity in partial derivatives of the function f evaluated at fixed point $(u,u) \in \mathbb{R}^2$. Main results of the manuscript can be validated using Mathematica programs available online at [**?**].

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