

ANOTHER APPROACH TO GET DERIVATIVE OF ODD-POWER

PETRO KOLOSOV

ABSTRACT. This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives of the polynomial function f_y defined as

$$f_y(x, z) = \sum_{k=1}^z \sum_{r=0}^y \mathbf{A}_{y,r} k^r (x - k)^r$$

where $x, z \in \mathbb{R}$, y is fixed constant $y \in \mathbb{N}$ and $\mathbf{A}_{y,r}$ are real coefficients.

CONTENTS

1. Introduction and Main Results	1
2. Conclusions	4
References	4

1. INTRODUCTION AND MAIN RESULTS

This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives identity in terms of partial derivatives, extending the main idea explained in [Kol22a] that gives polynomial identity in a form as follows

$$n^{2m+1} = \sum_{k=1}^n \sum_{r=0}^m \mathbf{A}_{m,r} k^r (n - k)^r \quad (1)$$

where m is fixed constant $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $\mathbf{A}_{m,r}$ are real coefficients defined recursively, see [Kol16]. Define the function f_y such that based on the identity (1) with the only difference that values of n, m in the right part of (1) appear to be parameters of the function f_y . In contrast to the equation (1), upper bound n of the sum $\sum_{k=1}^n$ turned into fixed function's parameter y as well, so that f_y defined as follows

Definition 1.1. (*Polynomial function f_y .*)

$$f_y(x, z) = \sum_{k=1}^z \sum_{r=0}^y \mathbf{A}_{y,r} k^r (x - k)^r \quad (2)$$

Date: October 25, 2022.

2010 Mathematics Subject Classification. 32W50, 11C08.

Key words and phrases. Partial differential equations, PDE, Polynomials .

where $x, z \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. At first glance, equation (2) might look complex, so in order to clarify the function f_y and polynomials it produces let's show few examples. Substituting the values of $y = 1, 2, 3$ to the function f we get the following polynomials

$$\begin{aligned} f_1(x, z) &= 3xz - 3z^2 + 3xz^2 - 2z^3 \\ f_2(x, z) &= 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5 \\ f_3(x, z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\ &\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7 \end{aligned}$$

According to the main topic of the current manuscript, it provides Another approach to get derivative of odd-power. Therefore, we define odd-power function we work in context of. Odd-power function g_y is a function defined as follows

Definition 1.2. (*Odd-power function g_y .*)

$$g_y(x) = x^{2y+1} \quad (3)$$

where $x \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. One more important thing is to conclude on partial derivative notation, more precisely the following notation for the partial derivative is used across the manuscript and remains unchanged

Notation 1.3. (*Partial derivative.*) Let be a function $f(x_1, x_2, \dots, x_n)$ defined over the real space \mathbb{R}^n . We denote partial derivative of the function f with respect to x_i as follows

$$f'_{x_i} = \lim_{\Delta x_i \rightarrow 0} \frac{f(x_1, x_2, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_i}$$

Derivative of the function f with respect to x_i evaluated in point $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ is denoted as follows

$$f'_{x_i}(y_1, y_2, \dots, y_n)$$

Therefore, the following identity in terms of partial derivatives shows the relation between odd-power function g_y and polynomial function f_y

Theorem 1.4. Let be a fixed value $v \in \mathbb{N}$, then derivative $(g_v)'_x$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluated at point u equals to derivative $(f_v)'_x$ evaluated at point (u, u) plus derivative $(f_v)'_z$ evaluated at point (u, u)

$$(g_v)'_x(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u) \quad (4)$$

Particularly, it follows that for every pair u, v an identity holds

$$(2v + 1)u^{2v} = (f_v)'_x(u, u) + (f_v)'_z(u, u) \quad (5)$$

that is also an ordinary derivative of odd-power function t^{2v+1} , therefore

$$\frac{d}{dt}t^{2v+1}(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u)$$

To summarize and clarify all above, we provide a few examples that show an identity (4) in action.

Example 1.5. Identity (4) example for $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $y = 1$. Consider the explicit form of the function $f_1(x, z)$ i.e

$$f_1(x, z) = 3xz - 3z^2 + 3xz^2 - 2z^3$$

Therefore, derivative of f_1 with respect to x equals to

$$(f_1)'_x(x, z) = \lim_{d \rightarrow 0} \frac{3dz + 3dz^2}{d} = 3z + 3z^2$$

Consider derivative of the function f_1 with respect to z , that is

$$\begin{aligned} (f_1)'_z(x, z) &= \lim_{d \rightarrow 0} \left[\frac{-3d^2 - 2d^3 + 3dx + 3d^2x - 6dz - 6d^2z + 6dxz - 6dz^2}{d} \right] \\ &= \lim_{d \rightarrow 0} [-3d - 2d^2 + 3x + 3dx - 6z - 6dz + 6xz - 6z^2] \\ &= 3x - 6z + 6xz - 6z^2 \end{aligned}$$

Combining both $(f_1)'_x$ and $(f_1)'_z$ evaluated at point (u, u) we get

$$\begin{aligned} (f_1)'_x + (f_1)'_z &= 3x - 3z + 6xz - 3z^2 \\ [(f_1)'_x + (f_1)'_z](u, u) &= 3u^2 \end{aligned}$$

that is derivative of the polynomial u^3 .

Example 1.6. Identity (4) example for $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $y = 2$. Consider the explicit form of the function $f_2(x, z)$ i.e

$$f_2(x, z) = 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5$$

Therefore, derivative of f_2 with respect to x equals to

$$\begin{aligned} (f_2)'_x &= \lim_{d \rightarrow 0} [5dz + 10xz - 15z^2 + 15dz^2 + 30xz^2 - 30z^3 + 10dz^3 + 20xz^3 - 15z^4] \\ &= 10xz - 15z^2 + 30xz^2 - 30z^3 + 20xz^3 - 15z^4 \end{aligned}$$

Consider derivative of the function f_2 with respect to z , that is

$$(f_2)'_z = 5x^2 - 30xz + 30x^2z + 30z^2 - 90xz^2 + 30x^2z^2 + 60z^3 - 60xz^3 + 30z^4$$

Combining both $(f_2)'_x(x, z)$ and $(f_2)'_z(x, z)$ evaluated at point (u, u) we get

$$\begin{aligned} (f_2)'_x + (f_2)'_z &= 5x^2 - 20xz + 30x^2z + 15z^2 - 60xz^2 + 30x^2z^2 + 30z^3 - 40xz^3 + 15z^4 \\ [(f_2)'_x + (f_2)'_z](u, u) &= 5u^4 \end{aligned}$$

that is derivative of the polynomial u^5 .

Example 1.7. Identity (4) example for $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $y = 3$. Consider the explicit form of the function $f_3(x, z)$ i.e

$$\begin{aligned} f_3(x, z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\ &\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7 \end{aligned}$$

Therefore, derivative of f_3 with respect to x equals to

$$\begin{aligned} (f_3)'_x &= -7z + 28xz - 42z^2 + 105x^2z^2 - 280xz^3 + 210x^2z^3 + 175z^4 - 420xz^4 \\ &\quad + 105x^2z^4 + 210z^5 - 168xz^5 + 70z^6 \end{aligned}$$

Consider derivative of the function f_2 with respect to z , that is

$$\begin{aligned} (f_3)'_z &= -7x + 14x^2 + 14z - 84xz + 70x^3z + 84z^2 - 420x^2z^2 + 210x^3z^2 + 700xz^3 \\ &\quad - 840x^2z^3 + 140x^3z^3 - 350z^4 + 1050xz^4 - 420x^2z^4 - 420z^5 + 420xz^5 - 140z^6 \end{aligned}$$

Combining both $(f_3)'_x(x, z)$ and $(f_3)'_z(x, z)$ evaluated at point (u, u) we get

$$\begin{aligned} (f_3)'_x + (f_3)'_z &= -7x + 14x^2 + 7z - 56xz + 70x^3z + 42z^2 - 315x^2z^2 + 210x^3z^2 \\ &\quad + 420xz^3 - 630x^2z^3 + 140x^3z^3 - 175z^4 + 630xz^4 - 315x^2z^4 - 210z^5 \\ &\quad + 252xz^5 - 70z^6 \\ [(f_3)'_x + (f_3)'_z](u, u) &= 7u^6 \end{aligned}$$

that is derivative of the polynomial u^7 .

2. CONCLUSIONS

In this manuscript we have reviewed an approach to get derivative of odd-power using identity in partial derivatives of the function f evaluated at fixed point $(u, u) \in \mathbb{R}^2$. Main results of the manuscript can be validated using Mathematica programs available online at [Kol22b].

REFERENCES

- [Kol16] Petro Kolosov. On the link between Binomial Theorem and Discrete Convolution of Polynomials. *arXiv preprint arXiv:1603.02468*, 2016. <https://arxiv.org/abs/1603.02468>.

- [Kol22a] Petro Kolosov. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. <https://doi.org/10.1017/mag.2022.129>.
- [Kol22b] Petro Kolosov. "Another approach to get derivative of odd-power" Source files. Published electronically at <https://github.com/kolosovpetro/AlternativeDerivative>, 2022.

Email address: kolosovp94@gmail.com

URL: <https://razumovsky.me/>